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On surfaces of general type with maximal Albanese dimension

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Abstract. Given a minimal surface S equipped with a generically finite map to an Abelian variety and $C \subset S$ a rational or an elliptic curve, we show that the canonical degree of C is bounded by four times the self-intersection of the canonical divisor of S. As a corollary, we obtain the finiteness of rational and elliptic curves with an optimal uniform bound on their canonical degrees on any surface of general type with two linearly independent regular one forms.

1. Introduction

The object of this paper is to give an effective bound on the canonical degrees of rational and elliptic curves on a minimal surface of general type with generically finite Albanese map linearly in terms of the canonical volume of the surface. Here, the canonical degree is the degree with respect to the canonical polarization of the surface and the self-intersection of this polarization is the canonical volume.

We will work in the complex analytic setting so that all varieties are complex analytic. We will assume the rudiments from the theory of classification of complex projective surfaces as found for example in [2]. We call a smooth projective surface to be of maximal Albanese dimension if its Albanese map is generically finite (N.B. This terminology is different from that introduced in [21]). We recall that any smooth projective surface S of maximal Albanese dimension admits a morphism to a minimal one whose canonical divisor pulls back to the positive part P of the Zariski decomposition of the canonical divisor K of S and the canonical volume of S is by definition $\mathrm{vol}(K) := P^2$. Our main theorem is as follows.

Theorem 1.1. *Let S be a smooth projective surface with maximal Albanese dimension and C a rational or an elliptic curve in S. Let P be the positive part in the Zariski decomposition of the canonical divisor K of S. Then*

$$PC \leqq 4\,\mathrm{vol}(K).$$

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Some remarks are in order:

• Yoichi Miyaoka has recently given effective bounds on the canonical degree of curves in surfaces with positive Segre class [16] generalizing the same given by him and the author [12]. It is easy to construct examples of surfaces with maximal Albanese dimension which do not have positive Segre class. Prior to the present article, all such canonical bounds were obtained via the powerful but highly nontrivial log-orbifold Miyaoka-Yau inequality on surfaces, for which we cite [4], [15], [22], [20], [12], [13], [11], [16]. But the bounds so obtained are neither simple nor are optimal. In this article, we introduce an elementary approach that relates geometrically the canonical degree of curves directly with the canonical class of the surface. In view of its simplicity and the resulting sharp bound, we expect it to have wider significance and applicability.

• A weaker noneffective version of our theorem has already been given by Noguchi-Winkelmann-Yamanoi [19] where the Albanese map is assumed to be finite and surjective, that is they exclude the possibility of exceptional fibers in their hypothesis although they do deal with the quasi-projective case at the same time. In that case, they bound the canonical degree of elliptic curves in terms of the degree of Gauss map of the canonical divisor of the surface, which they do not bound. Our bound gives a sharp effective bound for the degree of this Gauss map and we expect it to be sharp also for the canonical degree of elliptic curves on such a surface, if not extremely close to it, being by its very nature the best bound allowed by this approach.

• It is possible to obtain less effective (noneffective) bounds much more simply. But as all the details presented here are quite general in nature and have much wider applicability, we believe that they are worth the trouble for the optimal bound given.

Since a surface of general type cannot support a nontrivial family of rational or elliptic curves and a bound on the canonical degree of such curves puts them in a bounded family, our result implies that there are only a finite number of rational and elliptic curves on a surface of general type with generically finite Albanese map. In fact, our result is strictly stronger as it implies a global bound on such finiteness, in a smooth family of such surfaces for example. More generally, for a surface with irregularity at least two, if its Albanese map is not generically finite, then it admits a map to a hyperbolic curve by the structure theorem of Kawamata and Ueno [7]. Hence rational and elliptic curves lie on the fibers of such a map and are thus finite in number if the surface is of general type. It follows that there are only a finite number of rational and elliptic curves on any surface of general type with irregularity at least two. The following is an immediate corollary of this and of the main theorem of Noguchi-Winkelmann-Yamanoi in [18] giving the algebraic degeneracy of holomorphic curves in the remaining case of such surfaces.

Corollary 1.2. *Let S be a smooth projective surface of general type with irregularity two or more. Then S admits a proper Zariski subset that contains all nontrivial holomorphic images of $\mathbb{C}$.*

In general we have the following sweeping conjecture concerning the algebraic pseudo-hyperbolicity of varieties of general type. The conjecture is at least well indicated from the works of F. Bogomolov, B. Mazur, M. Green and P. Griffiths in the seventies and that of Y. Kawamata in the early eighties and made explicit by Serge Lang in 1986 [10]:

Conjecture 1.3. *An algebraic variety of general type admits a proper Zariski subset that contains all subvarieties not of general type.*

Aside from Bogomolov's result validating the conjecture for surfaces of positive Segre class [4], the best general evidence for such a conjecture up till now was given by Kawamata [8] validating the conjecture for subvarieties of Abelian varieties in characteristic zero. This result was later generalized to the case of semi-Abelian varieties by Noguchi [17] and to the case when the field of definition is an algebraically closed field of arbitrary characteristic [1]. However, none of these results is effective and, besides the above mentioned result of Noguchi-Winkelmann-Yamanoi, the conjecture seemed unknown in general for a variety with maximal Albanese dimension.

A short but already quite telling description of the proof of this main theorem is as follows. The theorem reduces easily to the case when S is minimal (so that $P = K$) and admits a surjective morphism α to an abelian surface A and to the case C is neither α-exceptional, so that $C_0 = \alpha(C)$ is an elliptic curve in A, nor contained in the ramification locus of α. In this case, $(\det d\alpha)$ is a canonical choice for K and, outside its common divisorial locus with $\hat{C} = \alpha^*(C_0)$, K intersects properly with $\hat{C}$ and the relevant local intersection numbers with C are dominated by the local intersection numbers of $\hat{C}$ with each such component of K. These local intersection numbers along a horizontal component of $D = K_{\mathrm{red}}$ can be interpreted in terms of the vanishing degree of the natural section of $K + D$ along D given by $(\alpha^* w_0) \wedge ds/s$ where w_0 is the nonzero holomorphic one form on A such that $TC_0 \in \ker w_0$. Since the intersection number of $\hat{C}$ with any vertical component of D or with any horizontal component of D in $\hat{C}$ is zero, finding a decomposition of K into parts having nonnegative intersection number with $K + D$ and summing yield the result.

As for the plan of the paper, section two deals with this decomposition of K while section three gives the key lemmas for the proof of the main theorem in section 4.

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2. A decomposition of divisors on surfaces

We will let S be a smooth projective surface and K a canonical divisor for S throughout. Let D, D' be two divisors in S. We will denote $D \leqq D'$ to mean that $D' - D$ is effective (possibly zero). By a component of a divisor, we will mean a prime component as opposed to a connected component of a divisor, so that it is reduced and irreducible.

Lemma 2.1. *Let D be a reduced divisor in S and C a connected divisor contained in D, i.e. $C \leqq D$. If $(K + D)C < 0$, then either $(K + D)C = -2$ or $(K + D)C = -1$. These two cases, which we will call case* 1 *and case* 2, *imply respectively*:

(1) $(K+C)C=-2$ *and* $(D-C)C=0$. *The latter means that* C *is isolated in* D, *i.e.* C *is not connected to the rest of* D.

(2) $(K+C)C=-2$ *and* $(D-C)C=1$. *The latter means that* C *intersects the rest of* D *at one point, and transversely.* (*We will denote by* C_0 *the component of* C *containing this intersection point.*)

Proof. Since C is connected, its arithmetic genus g is given by

$$1-\left(h^0(\mathcal{O}_C)-h^1(\mathcal{O}_C)\right)=h^1(\mathcal{O}_C)\geqq 0.$$

Hence by adjunction, $(K+C)C=2g-2$ is an even number that is at least -2. The rest follows from $(K+D)C=(K+C)C+(D-C)C$. $\square$

It is convenient at this point to introduce some notions from graph theory. Recall that a tree of curves is a connected reduced curve C which is disconnected by removing any of its component curves. Any component of such a tree is connected to any other component by a unique path (in its dual graph). A rooted tree of curves is a tree with a component curve singled out, called the *root*. Such a tree becomes directed by the partial ordering given by $C_i\leqq C_j$ if the unique path from the root to the component C_j passes through C_i, in which case we also say that C_j is a descendant of C_i. Note in particular that every component curve is a descendant of itself in a directed tree. A component curve C_k without a descendant beside itself is called a *leaf*. In a rooted tree of curves, the union of a component curve with all its descendants is called a *branch*. A tree without a root specified is called a free tree and such a tree becomes directed by either choosing a root curve or by choosing the unique transversal intersection of a pair of curves if exists, with the obvious ordering given as before. A disjoint collection of trees is called a forest.

Lemma 2.2. *In the previous lemma, a curve* C *in case* 1 *consists of an isolated* (*free*) *tree of smooth rational curves intersecting transversally and a curve* C *in case* 2 *consists of a rooted tree of smooth rational curves intersecting transversally, with the root given by* C_0. *If we direct the tree of case* 1 *by choosing a transversal intersection point of two curves if it exists, then every branch of the tree in either case satisfies the conditions of case* 2.

Proof. We will only show the lemma in case 2 since case 1 follows similarly. In this case, $(K+D)C=-1$ and C intersects the rest of D at one point p. Let C_0 be the component of C containing p and let $C_{01},\dots,C_{0k}$ be the connected components of $C-C_0$. Let $D'=D-\sum\limits_i C_{0i}$. By construction and by Lemma 2.1, $(K+D)C_{0i}\geqq -1$ and $C_{0i}(C-C_{0i})\geqq 1$ for all i and since C_0 is a connected nonisolated component of D', we have

$$\begin{aligned}-1\leqq (K+D')C_0&=(K+D)C_0-\sum_i C_{0i}C_0\\&=(K+D)C-\sum_i[(K+D)C_{0i}+C_{0i}(C-C_{0i})]\leqq (K+D)C=-1\end{aligned}$$

again by Lemma 2.1. This forces equality everywhere and so $(K+D)C_{0i}=-1$ for all i and $(K+D')C_0=-1$. This means that the C_{0i}'s are all curves belonging to case 1 connected to

C_0 and that C_0 is a smooth rational curve as $(K + C_0)C_0 = -2$ by the previous lemma. Hence we see that C is a rooted tree of rational curves by induction, with C_0 as a root. $\square$

We now set

$$F(D) = \{C \leqq D \mid C \text{ is connected and } (K + D)C < 0\}.$$

It follows that the union of the elements in $F(D)$ form a subset of D that is a forest. We will call this forest F_D. It is a disjoint union of maximal elements in $F(D)$ and the leaves in this forest are exactly the prime components of D belonging to $F(D)$.

We apply our result to the case $D = K_{\text{red}}$, the reduced canonical divisor, to obtain a positive partial decomposition of K. The following lemma is the key to this decomposition.

Recall that a curve C is called a Hirzebruch-Jung string if it is a rooted tree of curves with exactly one leaf so that we have actually a total ordering on C. In this case, we may write the prime decomposition of C as $C_0 + C_1 + \cdots + C_r$ where C_iC_j is one or zero according to whether $j = i + 1$ or not and C_0, C_r are the root and leaf of the string respectively. We will denote the subset of $F(D)$ consisting of elements that are not isolated in D and that are Hirzebruch-Jung strings by $F^{\text{HJ}}(D)$.

Lemma 2.3. *Let notation be as above but with $D = K_{\text{red}}$. Let $C \in F^{\text{HJ}}(D)$ with the prime decomposition $C_0 + C_1 + \cdots + C_r$ as given above. Let C_{-1} be the unique component of $D - C$ connected to C. Let $m_i > 0$ be the multiplicity of K on C_i. If S is minimal, then for $i = -1, 0, \ldots, r - 1$, we have*

$$m_i \geqq 2m_r.$$

Proof. Since each component C_i of C is a rational curve and S is minimal, we have, for each $i \geqq 0$,

$$C_i^2 = -2 - KC_i \leqq -2.$$

Let $\hat{C} = C_{-1} + C$. Then

$$(m_{r-1} + 1) + (m_r + 1)C_r^2 = (K + D)C_r = -1,$$

and we obtain $m_{r-1} + 2 = -(m_r + 1)C_r^2 \geqq 2(m_r + 1)$. Hence $m_{r-1} \geqq 2m_r$ and our lemma is true for $i = r - 1$. We now establish the lemma by showing that $m_{i-1} > m_i$ for all $i \geqq 0$. This being already established for $i = r$, we assume by induction that $m_i > m_{i+1}$ for an i strictly between -1 and r. For such an i, we have by Lemma 2.2 that

$$0 = (K + D)C_i = (m_{i-1} + 1) + (m_i + 1)C_i^2 + (m_{i+1} + 1) < m_{i-1} + 1 + (m_i + 1)(C_i^2 + 1),$$

and so $m_{i-1} > m_i$ and the result follows by induction. $\square$

We remark that if S is not minimal, the lemma would still hold with a weaker inequality.

Lemma 2.4. *With the notations as before and $D = K_{\mathrm{red}}$, let $C \in F(D)$ be a maximal element non-isolated in D. Let $C = \sum_{i \in I} C_i$ be the prime decomposition of C and let $K_C = \sum_{i \in I} m_i C_i$ where $m_i > 0$ is the multiplicity of K on C_i. If S is minimal and C is not a Hirzebruch-Jung string, then $(K + D)K_C \geqq 0$.*

Proof. We first introduce some terminology. Let $C \in F(D)$ be non-isolated in D. Let C_i be a component of C. Recall that $\bar{C}_i$ is the union of all the descendants of C_i. A component C_i of C is called split if $\bar{C}_i - C_i$ contains more than one connected components, at least one of which lies in $F^{\mathrm{HJ}}(D)$. In this case, we let $K_i = m_i C_i + \sum_j m_{ij} C_{ij}$ where the C_{ij}'s are the connected components of $\bar{C}_i - C_i$ lying in $F^{\mathrm{HJ}}(D)$, m_i is the multiplicity of K on C_i and m_{ij} is the multiplicity of K on the leaf of C_{ij}. Since $(K + D)\bar{C}_i = -1$, $(K + D)C_{ij} = -1$ for all j and $(K + D)(\bar{C}_i - C_{i1}) \leqq -1$ by Lemma 2.2 in this case, we obtain easily from Lemma 2.3 that

$$(K + D)K_i \geqq 0.$$

By assumption, C is not a Hirzebruch-Jung string. So every leaf of C is the descendant of a split component of C. Let C_i, $i \in I'$ be the collection of split components of C and K_i be as above. Now $K' = \sum_{i \in I'} K_i \leqq K_C$ by Lemma 2.3 and $K'' = K_C - K'$ has zero multiplicity on the leaves by construction. Hence K'' has only support on the C_j's which are not leaves and as $(K + D)C_j \geqq 0$ for such C_j's, $(K + D)K'' \geqq 0$. It follows that

$$(K + D)K_C = (K + D)K'' + \sum_{i \in I'} (K + D)K_i \geqq 0. \qquad \square$$

3. The key proposition

Let G be a reduced irreducible divisor on a smooth projective surface S and defined by the section $s \in H^0\big(S, \mathcal{O}(G)\big)$, i.e., $(s) = G$. A key point to our approach is the well known fact that ds/s gives a well defined nontrivial holomorphic section of $\Omega_S(G)$ over G. We will only need this fact in the following situation (which algebraic geometers should have no problem identifying with that of the adjunction formula both in its statement and in its proof).

Lemma 3.1. *With the data as given above, let w be a holomorphic one-form on S. Then $w \wedge ds/s$ gives a well defined holomorphic section of $\mathcal{O}(K + G)|_G$, i.e.,*

$$w \wedge ds \in H^0\big(G, \mathcal{O}(K + G)\big).$$

If w pulls back nontrivially to G, then $w \wedge ds$ is nowhere identically vanishing.

Proof. Given two local trivializations for $\mathcal{O}_G(G)$ on $G \cap U$ where U is a Stein neighborhood of a point $p \in G$, consider two trivializations of $\mathcal{O}_U(G)$ that extend them. Let s_1 and s_2 be the respective holomorphic functions on U representing s with respect to the trivializations. Then $t = s_1/s_2$ is a nonvanishing holomorphic function on U. Since $ds_1 = t\,ds_2 + s_2\,dt$ on U, we have $w \wedge ds_1 = tw \wedge ds_2$ on $G \cap U$. It follows that $w \wedge ds$ transforms as a holomorphic section of $\Omega_S^2(G) = \mathcal{O}(K+G)$ over G under different local trivializations of $\mathcal{O}_G(G)$. The last part of the lemma is clear. $\square$

An important remark for the application in general is that if s is only required to satisfy $G = (s)_{\mathrm{red}}$, then ds/s would give a nontrivial section of $\Omega_S(G)$ over G thereby effectively reducing the multiplicities of (s) to one.

We now apply this lemma to obtain the key proposition for the proof of our theorem.

Proposition 3.2. *Let S be a smooth complex surface, A a complex Abelian surface and $\alpha : S \to A$ a surjective morphism. We consider the following data on S:*

- $K = (\det d\alpha)$ *the canonical divisor of S determined by α,*
- $D = K_{\mathrm{red}}$ *the reduced canonical divisor,*
- C_0 *an elliptic curve in A considered as a reduced divisor,*
- $\hat{C} = \alpha^*(C_0)$ *the total transform of C_0,*
- C' *the sum of non-elliptic horizontal components of $\hat{C}$ on which $\hat{C}$ has multiplicity one,*
- $\bar{C} = \hat{C} - C' - C''$ *where C'' is some vertical part of $\hat{C}$, i.e., $C'' \leqq \hat{C}$ is α-exceptional.*

Let G be a horizontal component of D that is not a component of $\bar{C}$. Then we have

$$(G, \bar{C}) \leqq 2(G, K+D) - n_G,$$

where n_G is the intersection number of $D - G$ with the smooth part of G.

Proof. We note that, since $\hat{C}$ cannot be reduced on any horizontal component of D as α is ramified there, our assumption on G implies that it is not a component of $\hat{C}$. Hence, $G \cap (\hat{C})_{\mathrm{red}}$ is a finite set containing $Q = G \cap (\bar{C})_{\mathrm{red}}$.

We first assume that G is smooth for the proof.

Let w_0 be the constant holomorphic one-form that defines the direction of C_0, i.e., locally we can write $w_0 = df_0$ with $(f_0) = C_0$. By definition,

$$(G, \bar{C}) = \sum_{x \in Q} (G, \bar{C})_x \leqq \sum_{x \in Q} (G, \hat{C})_x.$$

Let $x \in Q$ and U a Stein neighborhood of $\alpha(x)$. Then $w_0|_U = df_0$ for a holomorphic function f_0 on U with $(f_0) = C_0|_U$. Let $V = \alpha^{-1}(U)$ and $f = \alpha^* f_0 = f_0 \circ \alpha \in \mathcal{O}_V$. Then $(G, \hat{C})_x$ is by definition the vanishing order at x of $f|_G$, i.e.,

$$(G, \hat{C})_x = \mathrm{ord}_x(f|_G).$$

Let $w = \alpha^* w_0$ and $\gamma_1 = w \wedge ds|_G \in H^0(G, K+G)$. Then $w|_V = df$ and so

$$\operatorname{ord}_x(\gamma_1) = \operatorname{ord}_x(df|_G) = (G, \hat{C})_x - 1.$$

Let G' be the reduced divisor $D - G$ considered also as a subset of S. Let r be a section of $\mathcal{O}(G')$ with $(r) = G'$ and let $\gamma = \gamma_1 \otimes r$. Then γ is a section of $\mathcal{O}(K + D)$ over G and

$$\text{(3.1)} \qquad \operatorname{ord}_x(\gamma) = \operatorname{ord}_x(\gamma_1) + (G, G')_x \geqq (G, D - G)_x.$$

We observe that if $x \notin G'$, then $(G, G')_x = 0$ and x lies in some horizontal component(s) of $\bar{C}$ away from any intersection point with vertical components of $\hat{C}$. In this case, either $\bar{C}$ is not reduced at x, in which case $(G, \hat{C})_x \geqq (G, \bar{C})_x \geqq 2$, or the components of $\bar{C}$ through x are horizontal elliptic curves. In the latter case, since the tangent directions of these elliptic curves cannot lie in the kernel of $d\alpha$ as no elliptic curve can ramify over an elliptic curve, $d\alpha$ has rank one at x and maps the tangent "directions" of C at x, and therefore also T_xS, to $T_{\alpha(x)}C_0 = \ker w_0(\alpha(x))$. Hence, in this case, $w = \alpha^* w_0$ is identically zero on T_xS and so $\operatorname{ord}_x(\gamma_1) \geqq 1$. It follows that if $x \notin G'$, then $\operatorname{ord}_x(\gamma) = \operatorname{ord}_x(\gamma_1) \geqq 1$ and so

$$(G, \hat{C})_x = \operatorname{ord}_x(\gamma_1) + 1 \leqq 2 \operatorname{ord}_x(\gamma_1) = 2 \operatorname{ord}_x(\gamma) - (G, D - G)_x.$$

On the other hand, if $x \in G'$, then $1 \leqq (G, G')_x$, and by (3.1) we have again

$$(G, \hat{C})_x = \operatorname{ord}_x(\gamma_1) + 1 \leqq \operatorname{ord}_x(\gamma) \leqq 2 \operatorname{ord}_x(\gamma) - (G, D - G)_x.$$

Since the inequality (3.1) is actually true for all $x \in G$ by the construction of γ, we have $2 \operatorname{ord}_x(\gamma) - (G, D - G)_x \geqq 0$ for all $x \in G$. It follows that

$$(G, \bar{C}) \leqq \sum_{x \in G} \big(2 \operatorname{ord}_x(\gamma) - (G, D - G)_x\big) = 2(G, K + D) - (G, D - G).$$

This proves the proposition in the case G is smooth.

In the case G is not smooth, let $\pi : \tilde{S} \to S$ be a minimal resolution of G. By replacing G with its strict transform $\tilde{G}$, $\bar{C}$ with its total transform $\pi^*\bar{C}$, K with the canonical divisor $\tilde{K} = \big(\det d(\alpha \circ \pi)\big)$ of $\tilde{S}$ and D with $(\tilde{K})_{\mathrm{red}} = (\pi^* D)_{\mathrm{red}}$ we find that all the assumptions of the proposition still hold and therefore

$$(G, \bar{C}) = (\tilde{G}, \pi^*\bar{C}) \leqq 2(\tilde{G}, \tilde{K} + \tilde{D}) - (\tilde{G}, \tilde{D} - \tilde{G}) \leqq 2(\tilde{G}, \tilde{K} + \tilde{D}) - n_G,$$

the last inequality owing to the fact that the intersection number n_G of $D - G$ with the smooth part of G is unaffected by π and so can be not greater than $(\tilde{G}, \tilde{D} - \tilde{G})$. The proposition now follows from the claim that $(\tilde{G}, \tilde{K} + \tilde{D}) \leqq (G, K + D)$ for which it is sufficient via induction to verify for the case π is a single blow-up. But this case follows directly from $\tilde{K} = \pi^*(K) + E$ and $\tilde{D} = \pi^*(D) - (m - 1)E$ where the multiplicity m of the point of D blown up is necessarily not less than two as D is singular there. (One can of course deduce this last inequality directly from the adjunction formula for singular divisors.) □

4. Proof of the main theorem

We first reduce the main theorem, Theorem 1.1, to the case when the surface S is minimal of general type admitting a surjective morphism to an Abelian surface.

Recall that the Albanese map of a smooth projective variety X is a morphism $\alpha_X : X \to A$ where A is an Abelian variety (i.e., A is a projective variety whose universal covering is $\mathbb{C}^n$ for some $n > 0$) and where the pair (α_X, A) is characterized by the following universal property: Any morphism $\beta : X \to A'$ where A' is an Abelian variety admits a unique factorization $\beta = j \circ \alpha_X$ where $j : A \to A'$ is a morphism of Abelian varieties. The pair (α, A) exists and is unique up to isomorphism for any smooth projective variety X. The Abelian variety A is called the Albanese torus of X, denoted by $\mathrm{Alb}(X)$. The universal property implies that $\alpha(X)$ generates A in the sense that it is not contained in any subabelian variety and, in particular, α induces an inclusion $H^0(\Omega_A) \subset H^0(\Omega_X)$.

Let S_0 be the minimal model of S and $\pi : S \to S_0$ the projection. Since π is a composition of blowups, its exceptional fibers consist of rational curves that are necessarily exceptional with respect to α. Hence $\alpha = \alpha_0 \circ \pi$ where α_0 is the Albanese map of S_0 and $\mathrm{Alb}(S) = \mathrm{Alb}(S_0)$ by the universal property of the Albanese. Since $P^2 = K_{S_0}^2 \geqq 0$ as S_0 is minimal and since the π-exceptional curves have zero degree with respect to $P = \pi^* K_{S_0}$ and all other irreducible curves in S are strict transforms of those in S_0, we see that it suffices to prove the theorem for $S = S_0$. So we may set $K = P = (\det d\alpha)$.

Now let A be an Abelian variety, and $\alpha : S \to A$ a generically finite morphism, i.e., $\dim \alpha(S) = 2$. Since S has maximal Albanese dimension, such a morphism exists. Hence, there is a nonvanishing holomorphic two form v on A such that $\alpha(S)$ is not contained in the codimensional two foliation defined by v and we may set the canonical divisor of S to be $K = (\alpha^* v)$. Now K is nef and contains all the exceptional divisors of α by construction. So any exceptional divisor of α satisfies the inequality given in the theorem. So it remains to prove the inequality of the theorem for elliptic curves C that are not α-exceptional as rational curves are necessarily α-exceptional. If C is such a curve and $\dim A > 2$, then $\alpha(C)$ is an elliptic curve in A and hence a translate of a one dimensional subgroup E of A. Let α_1 be the composition of α with the projection p_1 from A to the quotient abelian variety $A_1 = A/E$. If $\dim \alpha_1(S) = 2$, then C is α_1-exceptional and so, as all previous hypotheses on α are satisfied for α_1, C satisfies the conclusion of our theorem. Hence we may assume by induction that $\dim \alpha_1(S) = 1$. By the Poincaré reducibility theorem, see for example [5], Chap. 6, or [2], Theorem 5.3.5, there is an étale base change (i.e., an unramified covering map) $z : \tilde{A}_1 \to A_1$ for an Abelian variety such that $z^{-1}(A) = \tilde{A}_1 \times E$ and $z_1^{-1}(p_1) : \tilde{A}_1 \times E \to A_1$ is the projection π_1 to the first factor. This means that we have an étale covering $z : \tilde{A}_1 \times E \to A$ such that $p_1 \circ z = z_1 \circ \pi_1$. As our problem is unchanged by such an étale base change, we may assume that $\tilde{A}_1 = A_1$ and that $A = A_1 \times E$ so that π_1 is the quotient map of A by E. Let $\tilde{C}_1$ be the normalization of $C_1 = \alpha_1(S)$. Then the smooth surface $S' = \tilde{C}_1 \times E$ is the normalization of $\alpha(S)$ and, as S is smooth, α factors through S' by the Stein factorization theorem. So replacing A_1 by the Albanese torus of $\tilde{C}_1$, we may assume that C_1 is smooth. By construction, C lies in the pre-image of a point $p \in C_1$. Suppose $\dim A_1 > 1$. Then one can find a nonzero holomorphic one form u_1 on A_1 such that $T_p C = \ker u_1(p)$. It follows that $\alpha_1^* u_1$ vanishes at every point of C and therefore C lies in the zero locus of the section s of $\mathcal{O}_S(K) = \Omega_S^2$ defined by the wedge product of $\alpha_1^* u_1$ with the pull back of a one-form u_2 on E. Now the pull back of u_1 to C_1 is not identically vanishing as C_1 generates A_1 and so s is not identically vanishing. Hence the nef canonical divisor $K = (s)$ contains C giving again

$$KC \leqq K^2 \leqq 4K^2$$

and so C satisfies the conclusion of the theorem in this case. We are left with the case $C_1 = A_1$ so that α is a surjective morphism to A and so our main theorem reduces to the following proposition.

Proposition 4.1. *With the setup as in Proposition* 3.2, *assume further that S is a minimal surface. If C is a rational or an elliptic curve in S, then*

$$KC \leqq 4K^2.$$

Proof. We will classify a curve as being vertical or horizontal according whether it is α-exceptional or not and we first note that any element of $F(D)$ for a reduced divisor D in S is vertical, being a tree of rational curves by Lemma 2.2.

If C is vertical, then $C \leqq K$ and so $KC \leqq K^2 \leqq 4K^2$ by the nefness of K. As rational curves are necessarily vertical, it suffices to consider the case C is a horizontal reduced elliptic curve for our proposition. In this case, $C_0 = \alpha(C)$ is an elliptic curve in A. We will also consider C_0 as a reduced divisor in A. Let C' be the sum of the non-elliptic horizontal components of $\hat{C} = \alpha^* C_0$ on which $\hat{C}$ has multiplicity one and let $\bar{C} = \alpha^* C_0 - C'$. Then $C \leqq \bar{C} \leqq \alpha^* C_0$ and so

$$KC \leqq K\bar{C}$$

as K is nef. Let $\sum_i n_i D_i$ be the prime decomposition of K. We are reduced to bounding the intersection number

$$K\bar{C} = \sum_i n_i (D_i, \bar{C}).$$

Let D_i be a component of K that is either vertical or $D_i \leqq \bar{C}$ is horizontal, then

$$(D_i, \bar{C}) \leqq (D_i, \alpha^* C_0) = 0$$

by the definition of $\bar{C}$ (since C' is an effective horizontal divisor not containing D_i).

We now set $D = K_{\mathrm{red}}$ in what to follow and decompose $D = \sum_i D_i$ into three parts:

$$D = D^0 + D' + D'', \quad \text{where}$$

- D^0 consists of all the vertical components of D connected via vertical curves to a part of D that belongs to $F(D) \backslash F^{\mathrm{HZ}}(D)$ or to a part that is a maximal element of $F^{\mathrm{HJ}}(D)$ not attached to any horizontal component of D,

- D' consists of horizontal components of D not in $\bar{C}$ and of any element of $F^{\mathrm{HJ}}(D)$ that is attached to them,

- D'' consists of horizontal components of D in $\bar{C}$ and of any element of $F^{\mathrm{HJ}}(D)$ that is attached to them.

We first observe that there is no maximal element of $F(D)$ that is isolated as that would lead to an isolated surface singularity p sitting above a smooth surface but away from any ramification divisor, an impossibility by [3], theorem III.5.2. We observe also, by our last inequality above, that K_{D^0} and $K_{D''}$ have non-positive intersection with $\bar{C}$.

By definition, we can write $D^0 = \sum_i C_i^M + \sum_j (C_j^{\mathrm{HJ}} + C_j^+) + C'$ where the C_i^M's are the maximal elements in $F(D)$ not in $F^{\mathrm{HJ}}(D)$, the C_j^{HJ}'s are the maximal elements in $F^{\mathrm{HJ}}(D)$ not attached to any horizontal part of D, C_j^+ the vertical component of D attached to C_j^{HJ} and C' the rest of D^0. By construction, C' does not contain any leaf of F_D so that any component D_k of C' satisfies $(D_k, K+D) \geqq 0$. Hence $(K_{C'}, K+D) \geqq 0$, where we recall that $K_{C'}$ is the part of K supported on C'. We now note that, for all j, $\hat{C}_j = C_j^{\mathrm{HJ}} + C_j^+ \notin F(D)$ so that $(\hat{C}_j, K+D) \geqq 0$. Letting m_j be the multiplicity of K on the leaf of C_j^{HJ}, we have $m_j \hat{C}_j \leqq K$ by Lemma 2.3. As $K_{\hat{C}_j} - m_j \hat{C}_j$ consists of components not belonging to $F(D)$, we obtain $(K_{\hat{C}_j}, K+D) \geqq 0$ for all j. Finally, as $(K_{C_i^M}, K+D) \geqq 0$ by Lemma 2.4 and as $K_{D^0} = \sum_i K_{C_i^M} + \sum_j K_{\hat{C}_j} + K_{C'}$, we get

$$(K_{D^0}, \bar{C}) \leqq 0 \leqq (K_{D^0}.K+D) \leqq 2(K_{D^0}.K+D).$$

By the same token, $D'' \notin F(D)$. In fact, by definition, $D'' = \sum_l \left(D_l + \sum_{l'} D_{ll'} \right)$ where the D_l's are the horizontal components of D lying in $\bar{C}$ and, for each l, the $D_{ll'}$'s are the elements of $F^{\mathrm{HJ}}(D)$ attached to D_l and $E_l = D_l + \sum_{l'} D_{ll'} \notin F(D)$. Let m_l be the multiplicity of K on D_l. Then m_l is greater than the multiplicity $m_{ll'}$ of K on the leaf of $D_{ll'}$ for every l' by Lemma 2.3. We also have $\hat{D}_l = m_l D_l + \sum_{l'} m_{ll'} D_{ll'} \leqq K_{D''}$ by the same lemma. As $(m_l E_l, K+D) \geqq 0$ while $(D_{ll'}, K+D) < 0$ for all l', we see that $(\hat{D}_l, K+D) \geqq 0$ for all l. Since $K_{D''} - \sum_l \hat{D}_l$ is an effective divisor not having any component that belongs to $F(D)$, we obtain as before

$$(K_{D''}, \bar{C}) \leqq 0 \leqq (K_{D''}, K+D) \leqq 2(K_{D''}, K+D).$$

As for D', let $\{G_n\}_{n \in I}$ be the collection of horizontal components of D not lying in $\bar{C}$ and, for each $n \in I$, let $H_n = G_n + G_{nn'}$ where the $G_{nn'}$'s are the elements of $F^{\mathrm{HJ}}(D)$ attached to G_n. Then $D' = \sum_n H_n$ and we are interested in bounding $(K_{H_n}, \bar{C})$ for each n. Fix an n. We have by Proprosition 3.2 that

$$(G_n, \bar{C}) \leqq 2(G_n, K+D) - n_{G_n},$$

where n_{G_n} is the intersection number of the smooth part of G with the rest of D. Since $(G_{nn'}, G_n) = 1 = -(G_{nn'}, K+D)$, we have $n_{G_n} \geqq \left(\sum_{n'} G_{nn'}, G_n \right) = -\left(\sum_{n'} G_{nn'}, K+D \right)$ and so

$$(G_n, \bar{C}) \leqq 2(G_n, K+D) - n_{G_n} \leqq 2(G_n, K+D) + \left(\sum_{n'} G_{nn'}, K+D \right).$$

Let m_n be the multiplicity of K on G_n and $m_{nn'}$ that on the leaf of $G_{nn'}$. Then Lemma 2.3 implies that $2m_{nn'} \leqq m_n$ for all n' and that $\overline{H}_n = m_n G_n + \sum_{n'} m_{nn'} G_{nn'} \leqq K_{H_n}$. It follows from $(G_{nn'}, \overline{C}) \leqq 0$ and $(G_{nn'}, K + D) < 0$ that

$$(\overline{H}_n, \overline{C}) \leqq (m_n G_n, \overline{C}) \leqq \left(2m_n G_n + \sum_{n'} m_n G_{nn'}, K + D\right) \leqq (2\overline{H}_n, K + D) = 2(\overline{H}_n, K + D).$$

Since any component D_r of $K_{H_n} - \overline{H}_n$ is a vertical non-leaf component of D, we have $(D_r, \overline{C}) \leqq 0 \leqq 2(D_r, K + D)$ and so $(K_{H_n}, \overline{C}) \leqq 2(K_{H_n}, K + D)$. Summing over n gives

$$(K_{D'}, \overline{C}) \leqq 2(K_{D'}, K + D).$$

Finally, as $K = K_{D^0} + K_{D'} + K_{D''}$, we obtain

$$(K, \overline{C}) \leqq 2(K, K + D) \leqq 4K^2. \quad \square$$

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