A COMBINATORIAL RULE FOR (CO)MINUSCULE SCHUBERT CALCULUS

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ABSTRACT. We prove a root system uniform, concise combinatorial rule for Schubert calculus of *minuscule* and *cominuscule* flag manifolds G/P (the latter are also known as *compact Hermitian symmetric spaces*). We connect this geometry to the poset combinatorics of [Proctor '04], thereby giving a generalization of the [Schützenberger '77] *jeu de taquin* formulation of the Littlewood-Richardson rule that computes the intersection numbers of Grassmannian Schubert varieties. Our proof introduces *cominuscule recursions*, a general technique to relate the numbers for different Lie types. A discussion about connections of our rule to (geometric) representation theory is also briefly entertained.

CONTENTS

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1. INTRODUCTION

1.1. **Overview.** The goal of this paper is to introduce and prove a root-theoretically uniform generalization of the Littlewood-Richardson rule for intersection numbers of Schubert varieties in minuscule and cominuscule flag varieties.

Let G denote a complex, connected, reductive (e.g., semisimple) Lie group. Fix a choice of Borel and opposite Borel subgroups B, B_- and maximal torus $T = B \cap B_-$. Let W denote the Weyl group $N(T)/T$, $\Phi = \Phi^+ \cup \Phi^-$ the ordering of the roots into positives and negatives, and Δ the base of simple roots. Choosing a parabolic subgroup P canonically corresponds to a subset $\Delta_P \subseteq \Delta$; let $W_P := W_{\Delta_P}$ denote the associated parabolic subgroup of W. The **generalized flag variety** G/P is a union of B_--orbits whose closures $\mathcal{X}_w := \overline{B_- wP/P}$ with $wW_P \in W/W_P$ are the **Schubert varieties**. The Poincaré duals $\{\sigma_w\}$ of the Schubert varieties form the **Schubert basis** of the cohomology ring $H^\star(G/P) = H^\star(G/P; \mathbb{Q})$.

Among the simplest of the G/P's are the projective spaces and Grassmannians. However, as our results help demonstrate, the relative simplicity of their geometric and representation-theoretic features is shared by the wider settings of **minuscule** and **cominuscule flag varieties** (the latter are better known as **compact Hermitian symmetric spaces**). These are selected cases of G and its maximal parabolic subgroup P; see the precise definition in Section 2.1. (Actually there is little difference between the two settings. We focus on the latter, explaining the adjustments for the former as necessary.) These G/P's and their Schubert varieties are of significant and fundamental interest in geometry and representation theory, see, e.g., [BiLa00, Chapter 9], [Kos61] and the references therein, as well as, e.g., [Pe06, PuSo06] for more recent work.

The generalities of G/P specialize nicely to the (co)minuscule cases. Associated to G is the poset of positive roots $\Omega_G = (\Phi^+, \prec)$ defined by the transitive closure of the covering relation $\alpha \prec \gamma$ if $\gamma - \alpha \in \Delta$. For each (co)minuscule G/P let $\Delta \setminus \Delta_P = \{\beta(P)\}$ be the simple root corresponding to P. We study the subposet

$$\Lambda_{G/P} = \{\alpha \in \Phi^+ : \alpha \text{ contains } \beta(P) \text{ in its simple root expansion}\} \subseteq \Omega_G.$$

The (co)minuscule hypothesis assures that $\Lambda_{G/P}$ is self-dual and planar, see Section 2.2.

Moreover, rather than work with W/W_P-cosets directly, it is possible in (co)minuscule cases to view the Schubert basis as indexed by lower order ideals $\lambda \subseteq \Lambda_{G/P}$ (for a proof, see Proposition 2.1). We refer to these lower order ideals as **(straight) shapes**, and we call their elements **boxes**. Let $\mathbb{Y}_{G/P}$ be the lattice of these shapes, ordered by containment.

The **Schubert intersection numbers** $\{c_{\lambda,\mu}^{\nu}(G/P)\}$ are defined by

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu \in \mathbb{Y}_{G/P}} c_{\lambda,\mu}^{\nu}(G/P)\sigma_\nu. \tag{1}$$

These numbers count points of intersection of generically translated Schubert varieties and are therefore positive integers invariant under a natural S_3-action on the indices.

It is a longstanding goal in combinatorial algebraic geometry to discover a visibly positive combinatorial rule useful for understanding the numbers $c_{\lambda,\mu}^{\nu}(G/P)$. Few cases have complete solutions or conjectures, even for $G = GL_n(\mathbb{C})$. The archetypal Grassmannian case is solved by the Littlewood-Richardson rule; the first modern statement and proof is due to Schützenberger [Sc77] using the combinatorics of jeu de taquin. See, e.g., [Bu02, Co05, KnTa03, KnYo04, Kog01], and the references therein, for variations on generalized Littlewood-Richardson type formulas for $GL_n(\mathbb{C})$-Schubert calculus.

It is a natural problem is to find such a rule for (co)minuscule flag varieties.

This paper extends the jeu de taquin formulation of the Littlewood-Richardson rule to the Schubert calculus of (co)minuscule flag varieties. It provides the first uniform generalization of the Littlewood-Richardson rule that involves both classical and exceptional Lie types (see, e.g., [Pu06] for earlier efforts in this direction). This suggests the potential to extend alternative frameworks for the Littlewood-Richardson rule and its consequences and/or generalizations to the (co)minuscule setting, and beyond.

In particular, our rule may be interpreted in terms of emerging and classical connections between Schubert calculus and (geometric) representation theory: for example, in connection to the geometric Satake correspondence of Ginzburg, Mirković-Vilonen and others, see, e.g., [MiVi99], and separately, to Kostant's [Kos61] study of Lie algebra cohomology (see also [BelKu06]). See the remarks in Section 6.

1.2. **Statement of the main result.** If $\lambda \subseteq \nu$ are in $\mathbb{Y}_{G/P}$, their set-theoretic difference is the **skew shape** ν/λ. A **standard filling** of a (skew) shape ν/λ is a bijective assignment $\mathtt{label} : \nu/\lambda \to \{1, 2, \ldots, |\nu/\lambda|\}$ such that $\mathtt{label}(x) < \mathtt{label}(y)$ whenever $x \prec y$. The result of this assignment is a **standard tableau** T of **shape** $\nu/\lambda = \mathtt{shape}(T)$. The set of all standard tableaux is denoted $\mathrm{SYT}_{G/P}(\nu/\lambda)$.

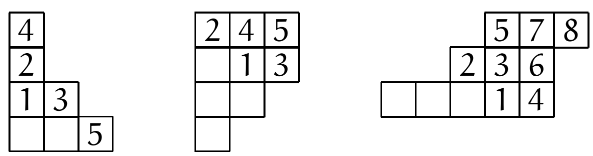


FIGURE 1. Standard tableaux of shape ν/λ for types A_{n-1}, C_n and E_6 respectively (the empty boxes are those of λ); see Sections 2 and 3 for context.

Given $T \in \mathrm{SYT}_{G/P}(\nu/\lambda)$ we now present **(co)minuscule jeu de taquin**. Consider $x \in \Lambda_{G/P}$ that is not in ν/λ, maximal in $\prec$ subject to the condition that it is below *some* element of ν/λ. We associate another standard tableau (of a different skew shape) $\mathtt{jdt}_x(T)$ arising from T called the **jeu de taquin slide** of T into x: Let y be the box of ν/λ with the smallest label, among those that cover x. Move $\mathtt{label}(y)$ to x, leaving y vacant. Look for boxes of ν/λ that cover y and repeat the process, moving into y the smallest label available among those boxes covering it. The tableau $\mathtt{jdt}_x(T)$ is outputted when no such moves are possible. (The result is a standard tableau; indeed, all the intermediate tableaux are.) The **rectification** of T is the result of an iteration of **jeu de taquin slides** until we terminate at a standard tableau $\mathtt{rectification}(T)$ of a (straight) shape.

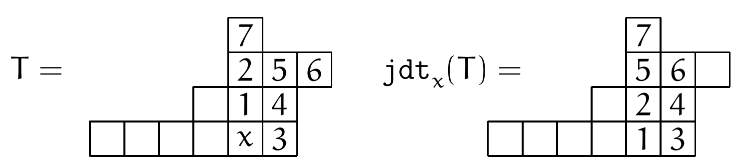


FIGURE 2. A standard tableau and a jeu de taquin slide, in type E_7

A novelty of this paper is the connection between (co)minuscule Schubert calculus and work of Proctor [Pro04]. That paper extends results of Schützenberger [Sc77], Sagan [Sa87] and Worley [Wo84]. It proves in the greater generality of "d-complete posets" that the rectification is independent of the order of jeu de taquin slides; see Section 4.

Define the statistic $\texttt{shortroots}$ on (skew) shapes to be the number of boxes of $\nu/\lambda \subseteq \Lambda_{G/P}$ that are short roots. We are now ready to state our combinatorial rule.

Main Theorem. *Let* $\lambda, \mu, \nu \in \mathbb{Y}_{G/P}$ *and fix* $T_\mu \in \mathrm{SYT}_{G/P}(\mu)$. *In the minuscule case, the Schubert intersection number* $c_{\lambda,\mu}^{\nu}(G/P)$ *equals the number of standard tableau of shape* ν/λ *whose rectification is* T_μ*; in the cominuscule case, multiply this by* $2^{\texttt{shortroots}(\nu/\lambda)-\texttt{shortroots}(\mu)}$.

The Main Theorem provides a root-theoretic generalization/reformulation of classical theorems in the subject. Besides Schützenberger's rule [Sc77] for Grassmannians, it generalizes the work of [Pra91, Wo84] for isotropic Grassmannians. Moreover, in the latter case, the power of 2 that appears in the product rule for Schur $Q-$ polynomials is given a new interpretation, via the $\texttt{shortroots}$ statistic. We emphasize that for the simply-laced root systems, the factor $2^{\texttt{shortroots}(\nu/\lambda)-\texttt{shortroots}(\mu)}$ always equals $2^{0-0} = 1$.

In Section 2, we give preliminaries about (co)minuscule flag varieties and associated combinatorics. We give examples of the Main Theorem in Section 3. Our proof, found in Sections 4–6, is a collaboration of combinatorial and geometric ideas. There, we introduce the ideas of the "infusion involution" and "cominuscule recursions". The latter are central to our (non-uniform) proof method of reducing the difficult *exceptional* Lie type cases to the *classical* Lie type orthogonal Grassmannian case; this argument is based on the geometric observation that certain Richardson varieties are isomorphic to Schubert varieties in smaller cominuscule flag manifolds. In fact, these ideas are introduced in greater generality than needed in our proofs. However, we believe they are interesting in their own right, and we attempted to describe them in a natural context. We conclude in Section 7 with a collection of remarks and problems.

The discovery of the Main Theorem exploited a number of computational tools: John Stembridge's `SF` and `Coxeter/Weyl` packages for `Maple`, Allen Knutson's algorithm [Kn03] (as implemented in [Yo06]). In addition, we wrote the `Maple` package `Cominrule` to aid the reader in exploring the properties of both the rule and (co)minuscule jeu de taquin. [1]

2. (CO)MINUSCULE FLAG VARIETIES AND THEIR COMBINATORICS

2.1. **Definition and classification.** Our main source for background on (co)minuscule flag varieties is [BiLa00, Chapter 9]. For a maximal parabolic subgroup P, interchangeably call it, its flag variety G/P or the root $\beta(P) \in \Delta$ (or, more properly, also the fundamental weight $\omega_{\beta(P)}$) **cominuscule** if whenever $\beta(P)$ occurs in the simple root expansion of $\gamma \in \Phi^+$, it does so with coefficient one. The cominuscule G/P's have been classified, see Table 1. In each case, the cominuscule $\beta(P) \in \Delta$ are marked in the Dynkin diagram. In case of choice, selecting either one leads to a (possibly isomorphic) cominuscule G/P.

A maximal parabolic subgroup P, G/P and $\beta(P) \in \Delta$ is **minuscule** if the associated fundamental weight $\omega_{\beta(P)}$ satisfies $\langle \omega_{\beta(P)}, \alpha^\star \rangle \leq 1$ for all $\alpha \in \Phi^+$ under the usual pairing between weights and coroots. The classification of minuscule flag varieties *almost* coincides with that of the cominuscules. In the conventions of Table 1, for the type B_n minuscule case we select node n rather than node 1. This is the **odd orthogonal Grassmannian** $OG(n, 2n+1)$, which is actually isomorphic to the even orthogonal Grassmannian

[1] Available at the authors' websites.

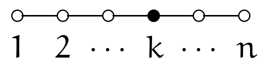
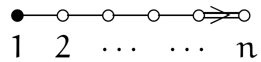
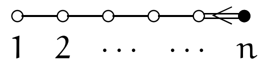
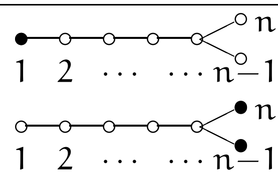
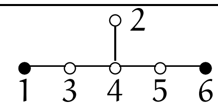
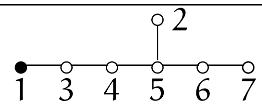
Root system	*Dynkin Diagram*	*Nomenclature*
A_n		Grassmannian $Gr(k, \mathbb{C}^n)$
B_n		Odd dimensional quadric $\mathbb{Q}^{2n-1}$
$C_n, n \geq 3$		Lagrangian Grassmannian $LG(n, 2n)$
$D_n, n \geq 4$		Even dimensional quadric $\mathbb{Q}^{2n-2}$ Orthogonal Grassmannian $OG(n+1, 2n+2)$ (for either choice of one of the nodes $n-1$ or n)
E_6		Cayley Plane $\mathbb{OP}^2$ (for either choice of one of the nodes 1 or 6)
E_7		(Unnamed) $G_\omega(\mathbb{O}^3, \mathbb{O}^6)$

TABLE 1. Classification of cominuscule G/P's

$OG(n+1, 2n+2)$. Consequently, their Schubert intersection numbers coincide. The type C_n minuscule case corresponds to selecting node 1 rather than node n.

2.2. **More specifics about cominuscule $\Lambda_{G/P}$.** We give two pictures of $\Lambda_{G/P}$: "thin" and "fattened" versions where elements of $\Lambda_{G/P}$ (i.e., boxes) are represented by either dots "$\bullet$" or squares "$\square$". The former are more convenient for illustrating the poset relations (smaller elements are lower in the diagram), and $\Lambda_{G/P}$'s position inside Ω_G. The latter are more convenient for manipulations in Section 3 and 4; for this reason, the fattened $\Lambda_{G/P}$ is rotated 45 degrees clockwise relative to the thin $\Lambda_{G/P}$.

The self-duality of $\Lambda_{G/P}$ is as follows. Let u_0 be the maximal length element of W_P. Note that u_0 preserves the subset $\Lambda_{G/P}$: the positive roots it makes negative are exactly those not in $\Lambda_{G/P}$, and thus if u_0 moved a root in $\Lambda_{G/P}$ outside, applying u_0 twice would send that root negative, contradicting the fact $u_0^2 = 1$. Let `rotate` denote this involution on $\Lambda_{G/P}$. In particular, this sends shapes to *upper* order ideals of $\Lambda_{G/P}$ and conversely.

We summarize features of each $\Lambda_{G/P}$ in Table 2. We proceed to analyze the specifics in each of the Lie types. Inside the fattened depictions, we draw a sample shape $\lambda = (\lambda_1, \lambda_2, \dots)$ where λ_i is the number of boxes in column i, as read from left to right.

Type A_{n-1}: Figure 3 depicts the case $k = 4$ and $n = 7$. Lower order ideals correspond to Young shapes (partitions) drawn in "conjugate French notation". The `rotate` involution is "rotate by 180 degrees". So, for example, $\texttt{rotate}(\nu)$ is the complement $(3,2)^c$ in $\Lambda_{Gr(k,\mathbb{C}^n)}$.

Type B_n: Classes are indexed by a single row of some length j, denoted $(1^j) = (1, 1, \dots, 1)$. Again `rotate` is "rotate by 180 degrees". See Figure 4. Here, $\texttt{rotate}(1^4) = (1^3)^c$.

Type C_n: Shapes are **strict partitions** $\lambda = (\lambda_1 > \lambda_2 > \dots)$ contained inside the "staircase". The boxes *not* on the **anti-diagonal** of the fattened diagram (i.e., those boxes not directly above "n" in the thin diagram) correspond to short roots. Here `rotate` corresponds to

G/P	*$\Lambda_{G/P}$ description*	*# boxes*	*Short root boxes*
$Gr(k, \mathbb{C}^n)$	$k \times (n-k)$ rectangle	$k(n-k)$	none
$\mathbb{Q}^{2n-1}$	$(2n-1)$-row	$2n-1$	middle box
$LG(n, 2n)$	n-step staircase	$\binom{n+1}{2}$	all *non*-anti-diagonal boxes
$\mathbb{Q}^{2n-2}$	double tailed diamond	$2n-2$	none
$OG(n+1, 2n+2)$	$n-1$-step staircase	$\binom{n}{2}$	none
$\mathbb{OP}^2$	irregular	16	none
$G_\omega(\mathbb{O}^3, \mathbb{O}^6)$	irregular	27	none

TABLE 2. Summary of facts about cominuscule $\Lambda_{G/P}$

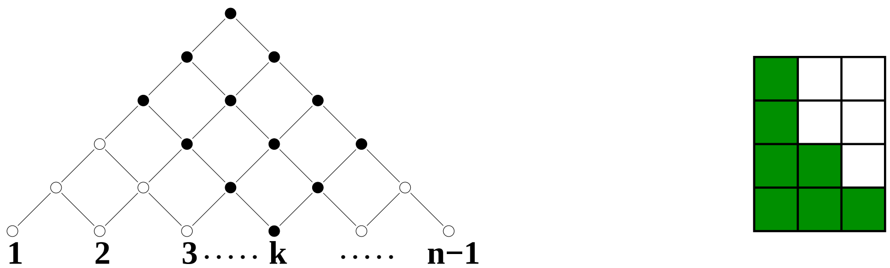


FIGURE 3. $\Lambda_{Gr(k,\mathbb{C}^n)}$, $\Omega_{GL_n}(\mathbb{C})$ and the shape $\nu = (4, 2, 1)$

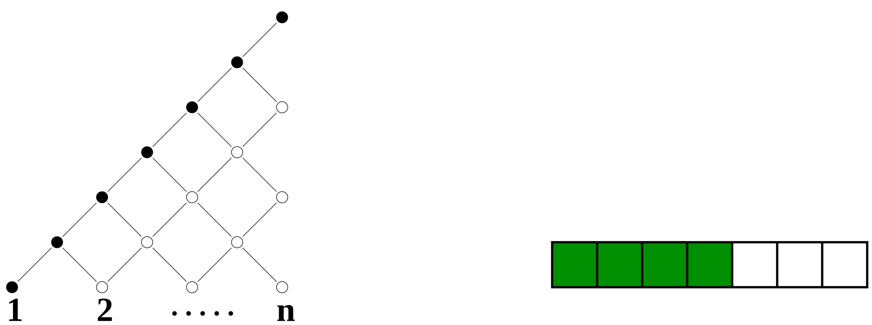


FIGURE 4. $\Lambda_{\mathbb{Q}^{2n-1}}$, $\Omega_{SO_{2n+1}(\mathbb{C})}$ and the shape $\nu = (1^4) = (1, 1, 1, 1)$

flipping across the diagonal line of symmetry in the fattened depiction. See Figure 5, where, e.g., the shape $(3, 1)$ involves two short roots, the shape $(4, 1)$ would involve three short roots. Here $\texttt{rotate}(3, 1) = (4, 2)^c$.

Type D_n*:* The Orthogonal Grassmannian $OG(n+1, 2n+2)$ comes from either choosing node $n-1$ or n. In either case, $\Lambda_{OG(n+1,2n+2)}$ is isomorphic (as a poset) to $\Lambda_{LG(n-1,2n-2)}$ (type C_{n-1}) drawn above. The shapes are shifted shapes, as in type C_{n-1} above. However, this time, none of the boxes correspond to short roots.

For the case of the even dimensional quadrics, see Figure 6. (For the Dynkin diagrams with "forks", the Ω_G becomes complicated to draw, so we omit them.) In this case the visualization of $\texttt{rotate}$ depends on the parity of n. For the n is odd case it is the 180 degree rotation, whereas if n is even the same is true except that the middle nodes stay fixed. For instance, if $n = 5$, $\texttt{rotate}(1^4)$ is $(1^4)^c$ whereas when $n = 6$, $\texttt{rotate}(1^5)$ is $(1, 1, 1, 2)^c$.

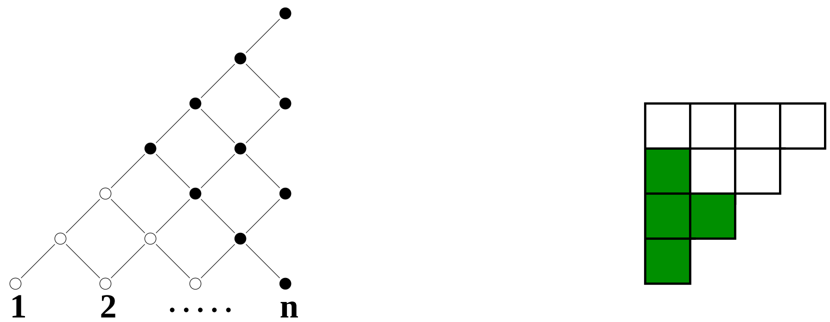


FIGURE 5. $\Lambda_{LG(n,2n)}$, $\Omega_{Sp_{2n}(\mathbb{C})}$ and the shape $\nu = (3, 1)$

FIGURE 6. $\Lambda_{\mathbb{Q}^{2n-2}}$ and the shape $\nu = (1, 1, 2, 1)$

Types E_6 *and* E_7*:* `rotate` is again 180 degree rotation. See Figures 7 and 8. In the depicted E_6 case, $\texttt{rotate}(\nu) = \nu^c$, whereas in the E_7 case $\texttt{rotate}(\nu) = (1, 1, 1, 2, 5, 5)^c$.

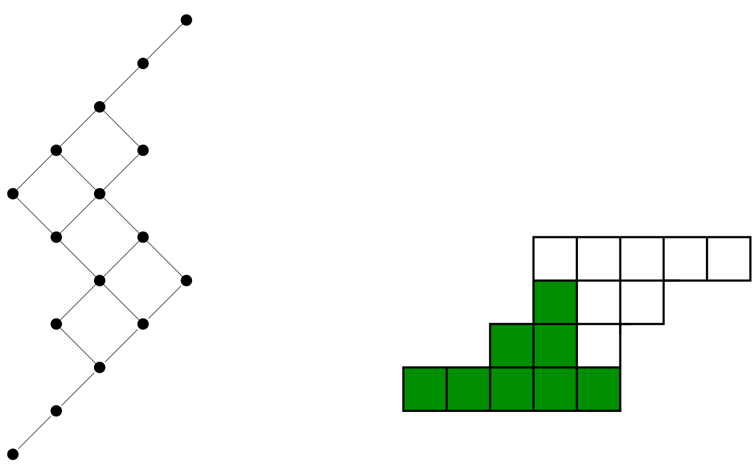


FIGURE 7. $\Lambda_{\mathbb{OP}^2}$ and the shape $\nu = (1, 1, 2, 3, 1)$

2.3. **Minuscule $\Lambda_{G/P}$.** As mentioned, the minuscule cases coincide with the cominuscules except in types B and C, which we discuss now.

Type B_n*:* This minuscule flag variety is isomorphic to $OG(n + 1, 2n + 2)$; its ambient poset is the same $\Lambda_{OG(n+1,2n+2)}$, see Figure 5. (Which nodes in the poset correspond to short roots is different, but, because we are now in the minuscule case of the Main Theorem,

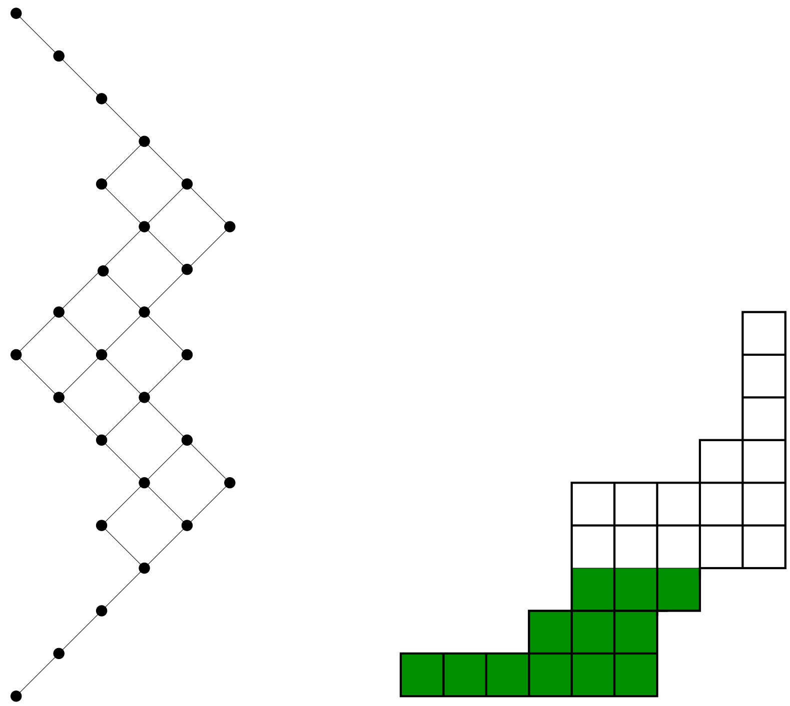


FIGURE 8. $\Lambda_{G_\omega(\mathbb{O}^3,\mathbb{O}^6)}$ and the shape $\nu = (1,1,1,2,3,3,1)$

this is irrelevant.) Note that u_0 in this case now acts differently on $\Lambda_{OG(n,2n+1)}$, so in this case we declare that `rotate` acts as it does on $\Lambda_{OG(n+1,2n+2)} \cong \Lambda_{LG(n-1,2n-2)}$.

Type C_n*:* This is the projective space $\mathbb{P}^{2n-1}$. Here $\Lambda_{G/P} \cong \Lambda_{\mathbb{Q}^{2n-1}}$ with the same shapes, see Figure 4. (Again, which nodes correspond to short and long roots differ, which is irrelevant to the Main Theorem.)

2.4. **Shapes index the Schubert basis.** For $w \in W$ the **inversion set** is $\mathcal{I}(w) = \{\alpha \in \Phi^+ : w \cdot \alpha \in \Phi^-\}$, using the standard action of W on Φ. Consider an irreducible rank two root system $\Phi_{\{\eta,\gamma\}} \subseteq \Phi$. It inherits from Φ a decomposition into positives and negatives, with $\eta, \gamma \in \Phi^+$ as its simple roots. If η and γ are of different lengths, let γ be the shorter of them. Order the positive roots of $\Phi_{\{\eta,\gamma\}}$

$$(\eta, \eta+\gamma, \gamma) \text{ or } (\eta, \eta+\gamma, \eta+2\gamma, \gamma) \tag{2}$$

depending on whether the root system is A_2 or B_2, respectively (G_2 has no cominuscule simple roots). A subset $S \subseteq \Phi^+$ is called **biconvex** if $S \cap \Phi_{\{\eta,\gamma\}}$ is either a beginning or ending subset of the positive roots, for all $\Phi_{\eta,\gamma} \subseteq \Phi^+$, with respect to (2). It is known that $S = \mathcal{I}(w)$ for some $w \in W$ if and only if S is biconvex [Bj83, BjEdZi90].

The natural projection $G/B \twoheadrightarrow G/P$ induces an inclusion $H^\star(G/P) \hookrightarrow H^\star(G/B)$ sending $\sigma_{wW_P} \in H^\star(G/P)$ to $\sigma_{w^P} \in H^\star(G/B)$, where $w^P \in W$ is the minimal length representative of wW_P. Alternatively, $\sigma_w \in H^\star(G/B)$ appears as the image of a Schubert class under the projection if and only if the **descents** of w are a subset of $\Delta \setminus \Delta_P$, i.e., $\ell(ws_\beta) < \ell(w)$ only when $\beta \in \Delta \setminus \Delta_P$. Here $\ell(w)$ denotes the **Coxeter length**, the minimal length of an expression for w in terms of the **simple reflections** s_β. When $\Delta \setminus \Delta_P = \{\beta(P)\}$ we call such a w **Grassmannian at** $\beta(P)$. Equivalently, these are the elements of W^P, the minimal length coset representatives of W/W_P.

The remaining facts in this section are essentially well-known. We include a proof of the following since it is basic to this paper.

Proposition 2.1. *Let* G/P *be a cominuscule flag variety. The Schubert classes in* $H^\star(G/P)$ *are in bijection with* $\lambda \in \mathbb{Y}_{G/P}$. *Specifically, for each* $w \in W$ *that is Grassmannian at* $\beta(P)$, *the inversion set* $\mathcal{I}(w)$ *is a lower order ideal in* $\Lambda_{G/P}$, *and conversely every lower order ideal in* $\Lambda_{G/P}$ *is the inversion set of a* $w \in W$ *which is Grassmannian at* $\beta(P)$.

Proof. Let $\lambda \in \mathbb{Y}_{G/P}$. Let $\Phi_{\{\eta,\gamma\}}$ be as above. If $\lambda \cap \Phi_{\{\eta,\gamma\}} \neq \emptyset$, then by (2) either η or γ involve $\beta(P)$ in their simple root expansions.

By the cominuscule assumption, any $\alpha \in \Phi^+$ involving $\beta(P)$ does so with coefficient one. From this and (2) combined we deduce that $\beta(P)$ appears in only one of η or γ, and in the nonsimply-laced case, that $\beta(P)$ appears in η. By symmetry, we may assume that $\beta(P)$ also appears in η in the simply-laced case. Thus $\lambda \cap \Phi_{\{\eta,\gamma\}}$ contains η and since λ is a lower order ideal, it is a beginning subset of the positive roots. Hence λ is biconvex and $\lambda = \mathcal{I}(w)$ for some $w \in W$. But the descents of w are $\Delta \cap \mathcal{I}(w) = \{\beta(P)\}$. So w is Grassmannian at $\beta(P)$ as desired.

Conversely, let w be Grassmannian at $\beta(P)$. Suppose $\alpha \in \mathcal{I}(w)$ but $\beta(P)$ does not occur in α. If $\alpha \notin \Delta$ then write $\alpha = \gamma_1 + \gamma_2$ where $\gamma_1, \gamma_2 \in \Phi^+$. Notice at least one of γ_1 or γ_2 is in $\mathcal{I}(w)$ (otherwise α would not be, a contradiction). Thus inductively, we reduce to the case $\alpha \in \Delta$ anyway. Thus α is a descent of w, contradicting our assumption that w is Grassmannian at $\beta(P)$. So α involves $\beta(P)$, and since $\beta(P)$ is cominuscule, it does so with coefficient one. Hence $\mathcal{I}(w) \subseteq \Lambda_{G/P}$.

Now, suppose $\gamma \in \mathcal{I}(w)$ and $\delta \prec \gamma$ in $\Lambda_{G/P}$. We may assume $\gamma - \delta = \rho \in \Delta$. Note $\rho \neq \beta(P)$. Since $\rho \notin \mathcal{I}(w)$ but $\gamma \in \mathcal{I}(w)$, then $w \cdot \delta = w \cdot \gamma - w \cdot \rho \in \Phi^-$. Thus $\delta \in \mathcal{I}(w)$, and so $\mathcal{I}(w)$ is a lower order ideal. ☐

For brevity, we omit the proof of the next fact.

Lemma 2.2. *Let* G/P *be a cominuscule flag variety and* $\lambda, \nu \in \mathbb{Y}_{G/P}$. *Then* $\lambda \subseteq \nu$ *if and only if* uW_P *is smaller than* vW_P *in the Bruhat order* W/W_P *where* $\lambda = \mathcal{I}(u)$ *and* $\nu = \mathcal{I}(v)$, *under the correspondence of Proposition 2.1.*

Corollary 2.3. (a) *If* $\lambda \not\subseteq \nu$ *then* $c^{\nu}_{\lambda,\mu}(G/P) = 0$ *for all shapes* $\mu \subseteq \Lambda_{G/P}$.

(b) *If* $|\lambda| + |\mu| = |\Lambda_{G/P}|$ *then* $\sigma_\lambda \cdot \sigma_\mu = \begin{cases} \sigma_{\Lambda_{G/P}} & \text{if } \lambda = \texttt{rotate}(\mu^c), \\ 0 & \text{otherwise,} \end{cases}$ *where* μ^c *is the complement of* $\mu \subseteq \Lambda_{G/P}$.

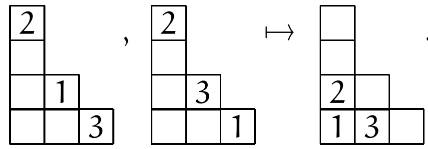
(c) *If* $\lambda \cap \texttt{rotate}(\mu) \neq \emptyset$ *then* $\sigma_\lambda \cdot \sigma_\mu = 0$.

Proof. By Lemma 2.2 and Proposition 2.1, combined with the discussion of the injection $H^\star(G/P) \hookrightarrow H^\star(G/B)$ given before Proposition 2.1, the assertions become well-known facts about the Schubert intersection numbers on G/B. ☐

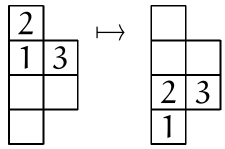
3. Examples of the combinatorial rule

In the examples, T_μ is the "consecutive" standard tableau, i.e., the one with $1, 2, 3, \ldots, \mu_1$ labeling the first column, followed by $\mu_1 + 1, \mu_1 + 2, \ldots$ labeling the second column etc.

The Grassmannian: Let us do the computation $c^{(4,2,1)}_{(3,1),(2,1)}(Gr(4,\mathbb{C}^7)) = 2$ (type A_7). Here $\nu/\lambda = (4,2,1)/(3,1)$. Of the 6 standard tableaux, only two rectify to $T_{2,1}$:

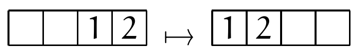


The isotropic Grassmannians: First we compute $c^{(4,2)}_{(2,1),(2,1)}(LG(4,8))$ (type C_4). Here $\nu/\lambda = (4,2)/(2,1)$ and $\mathrm{shortroots}(\nu/\lambda) = 3$, while $\mathrm{shortroots}(\mu) = 1$. There are two standard tableau of shape ν/λ, but only one rectifies to $T_{2,1}$:

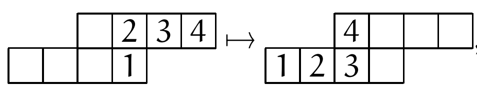


Hence $c^{(3,1)}_{(2,1),(2,1)}(LG(4,8)) = 1 \cdot 2^{3-1} = 4$. The same analysis shows $c^{(4,2)}_{(2,1),(2,1)}(OG(6,12)) = 1$ (type D_5), since $\Lambda_{OG(6,12)} \cong \Lambda_{LG(4,8)}$ but now there are no short roots.

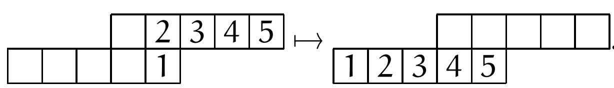
The quadrics: First let us compute $c^{(1^4)}_{(1^2),(1^2)}(\mathbb{Q}^7)$ (type B_4). Here $\nu/\lambda = (1^4)/(1^2)$ and $\mathrm{shortroots}(\nu/\lambda) = 1$ since this skew shape involves the middle box of $\Lambda_{\mathbb{Q}^7}$, while $\mathrm{shortroots}(\mu) = 0$. We have $c^{(1^4)}_{(1^2),(1^2)}(\mathbb{Q}^7) = 1 \cdot 2^{1-0} = 2$ since:



The even dimensional quadrics have a quirky dependency on the parity of n. When $n = 5$, we have $c^{\Lambda_{\mathbb{Q}^8}}_{(1,1,2),(1,1,2)}(\Lambda_{\mathbb{Q}^8}) = 1$ as witnessed by

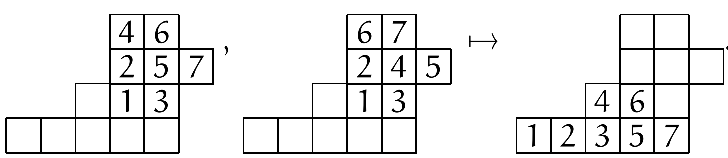


whereas $c^{\Lambda_{\mathbb{Q}^8}}_{(1,1,2),(1^4)}(\Lambda_{\mathbb{Q}^8}) = 0$. The similar $n = 6$ computation gives $c^{\Lambda_{\mathbb{Q}^{10}}(\mathbb{Q}^{10}}_{(1,1,1,2),(1^5)}) = 1$ because

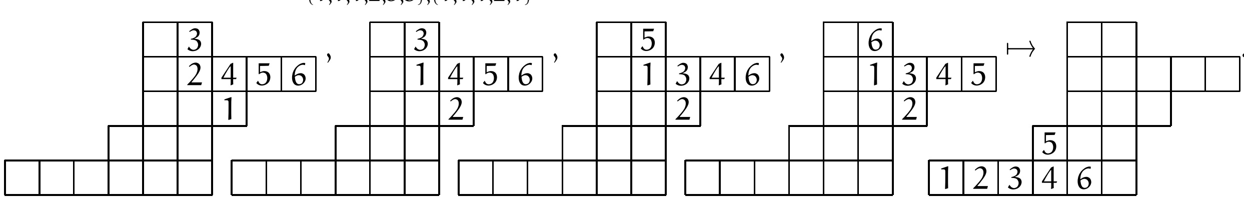


Thus cominuscule jeu de taquin properly detects the subtle definition of `rotate` in these cases: these calculations agree with Corollary 2.3(b).

The Cayley plane: We compute $c^{(1,1,2,4,4,1)}_{(1,1,2,1,1),(1,1,2,2,1)}(\mathbb{OP}^2) = 2$ as shown by



$G_\omega(\mathbb{O}^3,\mathbb{O}^6)$: Finally, $c^{(1,1,1,2,5,5,2,1,1)}_{(1,1,1,2,5,3),(1,1,1,2,1)}(G_\omega(\mathbb{O}^3,\mathbb{O}^6)) = 4$ since



4. Jeu de taquin methods

The basic result used is due to Robert Proctor [Pro04]. We develop consequences of it for our purposes.

Theorem 4.1. [Pro04] *Let $\nu/\lambda \subseteq \Lambda_{G/P}$ be a skew shape and $T \in \mathrm{SYT}_{G/P}(\nu/\lambda)$. Then the procedure* $\texttt{rectification}(T)$ *is independent of the order of applying jeu de taquin slides.*

In [Pro04], this theorem is proved for the more general class of "d-complete posets". Mark Haiman has raised the question of assigning a geometric context to these posets.

4.1. **Reversing.** Jeu de taquin is reversible. Given $T \in \mathrm{SYT}_{G/P}(\nu/\lambda)$, consider $x \in \Lambda_{G/P}$ not in ν/λ but *minimal* in $\prec$ subject to being above some element of ν/λ. The **reverse jeu de taquin slide** $\texttt{revjdt}_x(T)$ of T into x is defined similarly to a jeu de taquin slide, except we move into x the *largest* of the labels among boxes in ν/λ covered by x. The **reverse rectification** $\texttt{revrectification}(T)$ is the end result of the iterated application of these slides, ending with a standard filling of a rotated shape.

Proposition 4.2. *Fix a skew shape $\nu/\lambda \subseteq \Lambda_{G/P}$ and $T \in \mathrm{SYT}_{G/P}(\nu/\lambda)$.*

(a) *The analogue for reverse jeu de taquin of Theorem 4.1 holds.*
(b) *Suppose $y \in \Lambda_{G/P}$ is vacated by* $\texttt{jdt}_x(T)$, *then* $\texttt{revjdt}_y(\texttt{jdt}_x(T)) = T$.
(c) *Suppose $y \in \Lambda_{G/P}$ is vacated by* $\texttt{revjdt}_x(T)$, *then* $\texttt{jdt}_y(\texttt{revjdt}_x(T)) = T$.

Proof. Since $\Lambda_{G/P}$ is self-dual, reverse jeu de taquin slides is identified with ordinary jeu de taquin slides where the labels "i" play the usual role of the labels "$|\nu/\lambda| - i + 1$". This reduces (a) to Theorem 4.1. The claims (b) and (c) are easy inductions on the size of T. $\square$

4.2. **The infusion involution.** Given $U \in \mathrm{SYT}_{G/P}(\nu/\lambda)$, a specific choice of jeu de taquin slides usable to rectify U can be recorded as a tableau $T \in \mathrm{SYT}_{G/P}(\lambda)$. Suppose U rectifies to X with $\texttt{shape}(X) = \gamma$. It turns out that there is a natural choice of tableau Y of shape ν/γ which rectifies to T. In fact, the map taking (U, T) to (X, Y), which we will call $\texttt{infusion}$, and which we define below, is an involution.

Given $T \in \mathrm{SYT}_{G/P}(\lambda)$ and $U \in \mathrm{SYT}_{G/P}(\nu/\lambda)$, we define $\texttt{infusion}(T, U)$ to be a pair of tableau (X, Y) with $X \in \mathrm{SYT}_{G/P}(\gamma)$ and $Y \in \mathrm{SYT}_{G/P}(\nu/\gamma)$ (for some $\gamma \in \mathbb{Y}_{G/P}$ with $|\gamma| = |\nu/\lambda|$) as follows: place T and U inside $\Lambda_{G/P}$ according to their shapes. Now remove the largest label "m" that appears in T, say at box $x \in \lambda$. Since x necessarily lies next to ν/λ, apply the slide $\texttt{jdt}_x(U)$, leaving a "hole" at the other side of ν/λ. Place "m" in that hole and repeat moving the labels originally from U until all labels of T are exhausted. In particular, we declare that the labels placed in the created holes at each step never move for the duration of the procedure. The resulting straight shape tableau of shape γ and skew tableau of shape ν/γ are X and Y respectively.

Example 4.3. Let $G/P = Gr(3, \mathbb{C}^7)$, $\lambda = (2, 1)$ and $\nu = (3, 3, 2)$. The elements of T and U are depicted below, with the labels of the former are underlined and the labels of the latter are given in bold. We also compute $\texttt{infusion}(T, U)$ as a sequence of jeu de taquin slides, where at each stage the labels of (the eventual) Y are marked with "$\star$":

$$(T, U) = \begin{array}{|c|c|c|c|}\hline \mathbf{4} & \mathbf{5} & & \\ \hline \underline{3} & \mathbf{2} & \mathbf{3} & \\ \hline \underline{1} & \underline{2} & \mathbf{1} & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline \mathbf{4} & \mathbf{5} & & \\ \hline \mathbf{2} & \mathbf{3} & 3\star & \\ \hline \underline{1} & \underline{2} & \mathbf{1} & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline \mathbf{4} & \mathbf{5} & & \\ \hline \mathbf{2} & \mathbf{3} & 3\star & \\ \hline \underline{1} & \mathbf{1} & 2\star & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline \mathbf{4} & 1\star & & \\ \hline \mathbf{2} & \mathbf{5} & 3\star & \\ \hline \mathbf{1} & \mathbf{3} & 2\star & \\ \hline \end{array} = \texttt{infusion}(T, U) = (X, Y).$$

Hence $\gamma = (3, 2)$.

Theorem 4.4. *The procedure* `infusion` *defines an involution on*

$$\texttt{LockingSYT}(G/P) := \bigcup_{\nu,\gamma \in \mathbb{Y}_X} \mathrm{SYT}_{G/P}(\gamma) \times \mathrm{SYT}_{G/P}(\nu/\gamma).$$

In particular, let $\lambda, \mu, \nu \in \mathbb{Y}_{G/P}$ *and set*

$$\mathcal{I}^{\nu}_{\lambda,\mu}(G/P) = \{(T, U) : \texttt{shape}(T) = \lambda,\ \texttt{shape}(U) = \nu/\lambda,\ \texttt{infusion}(T, U) = (X, Y) \\ \textit{where } \texttt{shape}(X) = \mu\} \subseteq \texttt{LockingSYT}(G/P).$$

Then `infusion` *bijects* $\mathcal{I}^{\nu}_{\lambda,\mu}(G/P)$ *and* $\mathcal{I}^{\nu}_{\mu,\lambda}(G/P)$.

Proof. Consider the procedure `revinfusion`: given $T \in \mathrm{SYT}_{G/P}(\lambda)$, $U \in \mathrm{SYT}_{G/P}(\nu/\lambda)$ placed inside $\Lambda_{G/P}$ as in the above description of `infusion`, instead remove the smallest label "s" that appears in U, say at box $x \in \nu/\lambda$. Apply $\texttt{revjdt}_x(T)$, leaving a "hole" at the other side of λ. Place "s" in that hole and repeat until all labels of U are used. This is indeed the inverse map of `infusion`, as seen by inductively applying Proposition 4.2(b), i.e., $\texttt{revinfusion}(\texttt{infusion}(T, U)) = (T, U)$.

Thus, the "involution" assertion of the theorem amounts to showing that $\texttt{revinfusion}(T, U) = \texttt{infusion}(T, U)$ for all (T, U). View both procedures and the jeu de taquin slides as a sequence of "swaps" of the labels of adjacent boxes in $\Lambda_{G/P}$. We prove that for each label in T and U, those swaps involving the label are the same (i.e., in the same order and in the same position inside $\Lambda_{G/P}$, although possibly occurring at different times in the overall swap sequence). Then since the path taken by any label is the same in both procedures, the results are the same. In fact, it suffices to establish that T's labels undergo the same swaps in both operations, as the claim about U's labels is then implied.

It is easy to check from the definitions of `infusion` and `revinfusion` that the swaps involving the largest label "m" of T are the same in both procedures. Since after `infusion` moves "m" into its final position in $\Lambda_{G/P}$ it never moves again, for $|\lambda| \geq 2$ the remainder of the `infusion` procedure is an application of `infusion` to $(\widetilde{T}, \widetilde{U})$ where $\widetilde{T}$ is T with "m" removed and $\widetilde{U}$ is U after "m" has moved through it. By induction, $\texttt{revinfusion}(\widetilde{T}, \widetilde{U}) = \texttt{infusion}(\widetilde{T}, \widetilde{U})$ and all swaps involving labels of $\widetilde{T}$ and $\widetilde{U}$ are the same.

It remains to show that the swaps not using "m" are the same in both $\texttt{revinfusion}(T, U)$ and $\texttt{revinfusion}(\widetilde{T}, \widetilde{U})$. Define two sequences $T_0 := T, T_1, \ldots, T_{|\nu/\lambda|}$ and $\widetilde{T}_0 := \widetilde{T}, \widetilde{T}_1, \ldots \widetilde{T}_{|\nu/\lambda|}$, where T_i and $\widetilde{T}_i$ are the tableaux resulting from moving the labels "1" through "i" of U (respectively $\widetilde{U}$) through T during $\texttt{revinfusion}(T, U)$ (respectively $\texttt{revinfusion}(\widetilde{T}, \widetilde{U})$). Similarly define the pair of sequences $U_0 := U, U_1, \ldots, U_{|\nu/\lambda|}$ and $\widetilde{U}_0 := \widetilde{U}, \widetilde{U}_1, \ldots \widetilde{U}_{|\nu/\lambda|}$, which are derived from U and $\widetilde{U}$ respectively after the said moving of labels.

We show by induction on $i \geq 0$ that T_i is $\widetilde{T}_i$ with an added corner box containing "m" and $\widetilde{U}_i$ is obtained by applying to U_i a jeu de taquin slide into that box occupied by "m".

The base case $i = 0$ holds by construction. For the induction step, there are two cases to consider. If the "m" in T_{i-1} is not adjacent to the "i" in U_{i-1}, then "i" occupies the same (corner) box in both U_{i-1} and $\widetilde{U}_{i-1}$ since the jeu de taquin slide that makes up the difference between these two tableau does not affect the position of the label "i". Therefore the same moves will be made as we pass "i" through T_{i-1} and $\widetilde{T}_{i-1}$ (in particular, the label "m" never moves). Hence the desired conclusion for T_i and $\widetilde{T}_i$ holds. In addition, notice U_i and $\widetilde{U}_i$ only differ from U_{i-1} and $\widetilde{U}_{i-1}$ respectively by removing the label "i". Moreover, the jeu de taquin slide of U_{i-1} into the box labeled "m" has the same effect as the jeu de taquin slide of U_i into that box. Thus, U_i and $\widetilde{U}_i$ differ by a jeu de taquin slide into that box, as desired.

Otherwise, if the "m" in T_{i-1} is adjacent to the "i" in U_{i-1}, then to obtain T_i, the first swap that occurs is between the "m" and the "i". But after this initial swap, the "i" will be in the same location as the "i" in $\widetilde{U}_{i-1}$ and so the same swaps will be made as we pass to T_i and $\widetilde{T}_i$, so these latter two tableau satisfy the conclusion. Also, since the jeu de taquin slide that makes the difference between U_{i-1} and $\widetilde{U}_{i-1}$ involves swapping "i" and "m" as the first step, it is clear that U_i and $\widetilde{U}_i$ satisfy the conclusion. This completes the induction argument, and hence the theorem. $\square$

Corollary 4.5. *If* $\mu \in \mathbb{Y}_{G/P}$ *and* $U \in \mathrm{SYT}_{G/P}(\mu)$ *then* $\#\{W \in \mathrm{SYT}(\nu/\lambda) : \texttt{rectification}(W) = U\}$ *is independent of the choice of* $U \in \mathrm{SYT}_{G/P}(\mu)$.

Proof. First consider $\mathcal{I}^{\nu}_{\mu,\lambda}(G/P)$. Since choosing $U \in \mathrm{SYT}_{G/P}(\mu)$ amounts to a choice of sequence of jeu de taquin slides that rectify $V \in \mathrm{SYT}_{G/P}(\nu/\mu)$, and by Theorem 4.1, any choice leads to the same rectification, we conclude that the number of times a particular tableau $U \in \mathrm{SYT}_{G/P}(\mu)$ appears as the first component of a pair in $\mathcal{I}^{\nu}_{\mu,\lambda}(G/P)$ is independent of U. Specifically, it equals $\#\{V \in \mathrm{SYT}_{G/P}(\nu/\mu) : \texttt{shape}(\texttt{rectification}(V)) = \lambda\}$. On the other hand, the number of elements of $\mathcal{I}^{\nu}_{\lambda,\mu}(G/P)$ which are carried by $\texttt{infusion}$ to a pair of tableaux beginning with U is $f^{\lambda}(G/P) \cdot \#\{W \in \mathrm{SYT}_{G/P}(\nu/\lambda) : \texttt{rectification}(W) = U\}$. By Theorem 4.4, these two numbers are equal, so:

$$\#\{W \in \mathrm{SYT}_{G/P}(\nu/\lambda) : \texttt{rectification}(W) = U\} = \frac{\#\{V \in \mathrm{SYT}_{G/P}(\nu/\mu) : \texttt{shape}(\texttt{rectification}(V)) = \lambda\}}{f^{\lambda}(G/P)}$$

is independent of U. $\square$

Proposition 4.6. *Fix* $\mu \in \mathbb{Y}_{G/P}$. *The procedures* $\texttt{rectification}$ *and* $\texttt{revrectification}$ *are mutually inverse bijections between* $\mathrm{SYT}_{G/P}(\texttt{rotate}(\mu)) \overset{\sim}{\leftrightarrow} \mathrm{SYT}_{G/P}(\mu)$.

Proof. By Theorem 4.4 specialized to $\nu = \Lambda_{G/P}$, $\texttt{rectification}$ and $\texttt{revrectification}$ are mutually inverse bijections between the set of all fillings of rotated shapes and the set of all fillings of straight shapes. Given $T \in \mathrm{SYT}_{G/P}(\mu)$, let $\texttt{shape}(\texttt{revrectification}(T)) = \texttt{rotate}(\alpha)$ for some $\alpha \in \mathbb{Y}_{G/P}$. By Corollary 4.5, every tableau in $\mathrm{SYT}_{G/P}(\mu)$ appears as the rectification of some filling of shape $\texttt{rotate}(\alpha)$. By Corollary 4.5 applied to $\texttt{revrectification}$, any filling of shape $\texttt{rotate}(\alpha)$ can be obtained by reverse-rectifying some filling of shape μ, so any filling of shape $\texttt{rotate}(\alpha)$ rectifies to a filling of shape μ. Thus, $\texttt{rectification}$ and $\texttt{revrectification}$ are mutually inverse bijections between fillings of shape $\texttt{rotate}(\alpha)$ and of shape μ. We can therefore define a bijection Ψ which takes a shape α to the unique shape of the rectification of any standard filling of $\texttt{rotate}(\alpha)$. We now show Ψ is the identity map.

First, we show that Ψ is an automorphism of $\mathbb{Y}_{G/P}$. Suppose α consists of a shape β together with a single additional corner box x. It suffices to show that $\Psi(\beta) \subseteq \Psi(\alpha)$. Fix a filling T of $\texttt{rotate}(\alpha)$ in which $\texttt{rotate}(x)$ has label "1". Let the tableau $\widetilde{T}$ be T with the box labeled 1 removed. Clearly jeu de taquin methods apply to $\widetilde{T}$. Pick any sequence of jeu de taquin slides $T_0 = T$, $T_1 = \texttt{jdt}_{x_1}(T_0), \ldots, T_i = \texttt{jdt}_{x_i}(T_{i-1}), \ldots, \texttt{rectification}(T)$ rectifying T. We can define a parallel sequence of jeu de taquin slides $\{\widetilde{T}_i = \texttt{jdt}_{\widetilde{x}_i}(\widetilde{T}_{i-1})\}$ starting with $\widetilde{T}_0 = \widetilde{T}$ where $\widetilde{x}_i = x_i$ if the label of T_{i-1} moving into x_i is not "1" and $\widetilde{x}_i$ is the box in T_{i-1} with label "1" otherwise. It is easy to check that for each i, T_i with the "1" removed is $\widetilde{T}_i$. Also, when $T = T_k$ has been rectified, $\widetilde{T}_k$ is a filling of $\Psi(\alpha)/(1)$. Since a jeu de taquin slide removes an "outside corner" from a skew shape, then $\Psi(\alpha)$ is $\Psi(\beta)$ with such a corner added. Hence, Ψ takes covering relations to covering relations, as desired.

Let Q be an arbitrary finite poset, and let $D(Q)$ be the lattice of order ideals (down-closed sets) in Q. Let $\mathrm{Aut}(Q)$ and $\mathrm{Aut}(D(Q))$ denote the group of poset automorphisms of Q and $D(Q)$, respectively. There is a natural inclusion from $\mathrm{Aut}(Q)$ to $\mathrm{Aut}(D(Q))$. This inclusion is actually a group isomorphism: from $\phi \in \mathrm{Aut}(D(Q))$, the corresponding automorphism of Q can be recovered by restricting ϕ to the principal order ideals of $D(Q)$ (which form a poset canonically isomorphic to Q).

So Ψ induces a poset automorphism of $\Lambda_{G/P}$. For B_n ($n \geq 4$), C_n, D_n (for the Orthogonal Grassmannians), E_6 and E_7 the only poset automorphism is the identity. Hence Ψ is the identity in these cases. For the remaining cases, check that Ψ is the identity on principal lower order ideals of $\Lambda_{G/P}$. This is straightforward. $\square$

Let $\{e_{\lambda,\mu}^{\nu}(X)\}$ be computed by the rule of the Main Theorem.

Corollary 4.7. $e_{\lambda,\mu}^{\nu}(X) = e_{\mu,\lambda}^{\nu}(X) = e_{\texttt{rotate}(\nu^c),\mu}^{\texttt{rotate}(\lambda^c)}(X)$.

Proof. By Theorem 4.4 and Corollary 4.5, we have $e_{\lambda,\mu}^{\nu}(X) = |\mathcal{I}_{\lambda,\mu}^{\nu}(X)|/f^{\lambda}(X)f^{\mu}(X) = |\mathcal{I}_{\mu,\lambda}^{\nu}(X)|/f^{\mu}(X)f^{\lambda}(X) = e_{\mu,\lambda}^{\nu}(X)$. For the remaining equality, fix $U \in \mathrm{SYT}_X(\texttt{rotate}(\mu))$ and note that since $\texttt{rotate}(\nu/\lambda) = \texttt{rotate}(\lambda^c)/\texttt{rotate}(\nu^c)$, by Proposition 4.2:

$$e_{\texttt{rotate}(\nu^c),\mu}^{\texttt{rotate}(\lambda^c)}(X) = \#\{T \in \mathrm{SYT}_X(\nu/\lambda) : \texttt{revrectification}(T) = U\}. \tag{3}$$

By Theorem 4.1 and Proposition 4.2(b) combined, it follows that each $T \in \mathrm{SYT}_X(\nu/\lambda)$ in (3) satisfies $\texttt{rectification}(T) = \texttt{rectification}(U)$. By Proposition 4.6, $\texttt{shape}(\texttt{rectification}(U)) = \mu$. Hence $e_{\lambda,\mu}^{\nu}(X) \geq e_{\texttt{rotate}(\nu^c),\mu}^{\texttt{rotate}(\lambda^c)}(X)$. Reversing the roles of λ and ν above gives equality. $\square$

5. Proof of the main theorem

5.1. **Schubert like numbers.** Fix a cominuscule flag variety $X = G/P$ and a collection of real numbers $\{d_{\lambda,\mu}^{\nu}(X)\}$. It is useful to call $\{d_{\lambda,\mu}^{\nu}(X)\}$ **Schubert like** if (I)-(IV) below hold:

(I) (S_3-*symmetry*) $d_{\lambda,\mu}^{\nu}(X) = d_{\mu,\lambda}^{\nu}(X) = d_{\texttt{rotate}(\nu^c),\mu}^{\texttt{rotate}(\lambda^c)}(X) = d_{\lambda,\texttt{rotate}(\nu^c)}^{\texttt{rotate}(\mu^c)}(X) = d_{\texttt{rotate}(\nu^c),\lambda}^{\texttt{rotate}(\mu^c)}(X) = d_{\mu,\texttt{rotate}(\nu^c)}^{\texttt{rotate}(\lambda^c)}(X)$;

(II) (*Codimension*) $d_{\lambda,\mu}^{\nu}(X) = 0$ unless $|\lambda| + |\mu| = |\nu|$;

(III) (*Containment*) If $\lambda \not\subseteq \nu$ then $d_{\lambda,\mu}^{\nu}(X) = 0$ for all $\mu \in \mathbb{Y}_X$; and

(IV) (*Iterated box product*) $\sum_{|\gamma|=|\nu/\lambda|} f^{\gamma}(X) d_{\lambda,\gamma}^{\nu}(X) 2^{\texttt{shortroots}(\gamma)-\texttt{shortroots}(\nu/\lambda)} = f^{\nu/\lambda}(X)$, where $f^{\nu/\lambda}(X) = |\mathrm{SYT}_X(\nu/\lambda)|$ and $f^{\gamma}(X) = |\mathrm{SYT}_X(\gamma)|$.

Proposition 5.1. *For any cominuscule flag variety* X, $\{c_{\lambda,\mu}^{\nu}(X)\}$ *is Schubert like.*

The Monk-Chevalley formula states that for any $\beta \in \Delta$ and $w \in W$

$$\sigma_{s_\beta} \cdot \sigma_w = \sum_{\alpha\in\Phi^+,\ \ell(ws_\alpha)=\ell(w)+1} n_{\alpha\beta} \frac{(\beta,\beta)}{(\alpha,\alpha)}\, \sigma_{ws_\alpha}. \tag{4}$$

Here $n_{\alpha\beta}$ is the coefficient of β in the expansion of α into simple roots, and $(\bullet,\bullet)$ is the inner product defined on the span of Δ, as determined by the Cartan matrix.

Lemma 5.2. *Let $\beta = \beta(P)$ and $w \in W$ be Grassmannian at $\beta(P)$. Then in (4)*

(i) $n_{\alpha,\beta} \in \{0, 1\}$
(ii) *ws_α is Grassmannian at position $\beta(P)$ whenever $n_{\alpha\beta} \neq 0$*

Proof. (i) holds by the definition of $\beta = \beta(P)$ being cominuscule. (ii) uses our discussion about $H^\star(G/P) \hookrightarrow H^\star(G/B)$ in the paragraph before Proposition 2.1. □

Proposition 5.3. *For any cominuscule flag variety X,*

$$\sigma_\square \cdot \sigma_\lambda = \sum_{\mu\in\mathbb{Y}_X \text{ and } \mu/\lambda \text{ is a single box}} 2^{\mathtt{shortroots}(\mu/\lambda)}\sigma_\mu \tag{5}$$

Moreover, $\sigma_\square^i = \sum_{|\gamma|=i} f^\gamma(X) 2^{\mathtt{shortroots}(\gamma)}\sigma_\gamma$.

Proof. The first claim is proved by comparing (4) and (5) and applying Lemmas 2.2 and 5.2. The second holds by the definition of $f^\gamma(X)$, since each standard tableau is inductively built up by adding the box labeled "k" at the k^{th} step. □

Proof of Proposition 5.1: (I) and (II) hold by the geometric definition of $c_{\lambda,\mu}^\nu(X)$ while (III) holds by Corollary 2.3. For (IV), using the definition of $f^{\nu/\lambda}(X)$, Proposition 5.3, and Corollary 2.3 we have

$$\sigma_{\Lambda_X} f^{\nu/\lambda}(X) 2^{\mathtt{shortroots}(\nu/\lambda)} = \sigma_\lambda \sigma_\square^{|\nu/\lambda|}\sigma_{\mathtt{rotate}(\nu^c)} = \sum_{|\gamma|=|\nu/\lambda|} f^\gamma(X) c_{\lambda,\gamma}^\nu(X) 2^{\mathtt{shortroots}(\gamma)}\sigma_{\Lambda_X}. \quad \square$$

5.2. **Cominuscule recursions.** Let $\widetilde{X} = \widetilde{G}/\widetilde{P}$ be a second cominuscule flag variety. Define a **cominuscule recursion** to be a poset injection $\Theta : \Lambda_{\widetilde{X}} \hookrightarrow \Lambda_X$ such that Λ_X is a disjoint union of $\Theta(\Lambda_{\widetilde{X}})$ and of $L(\Theta)$ and $\Gamma(\Theta)$, which are subsets of Λ_X whose elements are all either incomparable with or below (respectively, above) every element of $\Theta(\Lambda_{\widetilde{X}})$. Clearly:

Definition-Lemma 5.4. *If $\lambda \in \mathbb{Y}_X$ then $\overline{\lambda} := \Theta^{-1}(\lambda)$ is in $\mathbb{Y}_{\widetilde{X}}$. Also, if $\gamma \in \mathbb{Y}_{\widetilde{X}}$ then $\hat{\gamma} := \Theta(\gamma) \cup L(\Theta)$ is in $\mathbb{Y}_X$.*

Fix $\lambda, \mu, \nu \in \mathbb{Y}_X$ such that

$$\lambda \subseteq \nu, L(\Theta) \subseteq \lambda \text{ and } \Gamma(\Theta) \subseteq \nu^c. \tag{6}$$

Then $d_{\lambda,\mu}^\nu(X)$ is **Θ-recursive** if

$$d_{\lambda,\mu}^\nu(X) = \sum_{\gamma\in\mathbb{Y}_{\widetilde{X}}} c_{\overline{\lambda},\gamma}^{\overline{\nu}}(\widetilde{X})\ d_{L(\Theta),\mu}^{\hat{\gamma}}(X). \tag{7}$$

A collection $\{d_{\lambda,\mu}^\nu(X)\}$ is Θ-recursive if each $d_{\lambda,\mu}^\nu(X)$ is, whenever (6) holds.

Recall that $\{e_{\lambda,\mu}^\nu(X)\}$ are the numbers computed by the rule of the Main Theorem, cf., Corollary 4.7.

Theorem 5.5. *Fix a cominuscule recursion* $\Theta : \Lambda_{\widetilde{X}} \to \Lambda_X$ *and assume* $e^{\nu}_{\lambda,\mu}(\widetilde{X}) = c^{\nu}_{\lambda,\mu}(\widetilde{X})$ *for all* $\lambda, \mu, \nu \in \mathbb{Y}_{\widetilde{X}}$*. Then* $\{e^{\nu}_{\lambda,\mu}(X)\}$ *is* Θ*-recursive.*

Proof. Construct standard fillings of ν/λ rectifying to a fixed $S \in \mathrm{SYT}_X(\mu)$ in two steps. First choose $\gamma \in \mathbb{Y}_{\widetilde{X}}$ and one of the $e^{\hat{\gamma}}_{L(\Theta),\mu}(X)$ tableaux $T \in \mathrm{SYT}_X(\hat{\gamma}/L(\Theta))$ rectifying to S. By Corollary 4.5, there are $c^{\overline{\nu}}_{\lambda,\gamma}(\widetilde{X})$ ways to fill ν/λ that rectify to T (i.e., rectify to T viewed as a standard tableau of $\gamma \in \mathbb{Y}_{\widetilde{X}}$). As this filling of ν/λ rectifies to S, "$\geq$" for (7) holds.

For "$\leq$", given a standard filling of ν/λ that rectifies to S, we want to show that it can be seen to arise as one the above standard fillings. By Theorem 4.1, we can start rectifying by exclusively choosing boxes $x \in \Theta(\Lambda_{\widetilde{X}})$ to slide into, until this is no longer possible. At that point we have a tableau in $\mathrm{SYT}_X(\hat{\gamma}/L(\Theta))$ for some $\gamma \in \mathbb{Y}_{\widetilde{X}}$. $\square$

We use Theorem 5.5 in the Main Theorem's proof. Once the latter is proved, we obtain:

Corollary 5.6. *For any cominuscule flag variety* X *and cominuscule recursion* Θ*,* $\{c^{\nu}_{\lambda,\mu}(X)\}$ *is* Θ*-recursive.*

5.3. The exceptionals $\mathbb{OP}^2$ and $G_\omega(\mathbb{O}^3, \mathbb{O}^6)$.

One has helpful Θ-recursions here:

- $\Theta_{E_6} : OG(6,12) \to \mathbb{OP}^2$ identifying $\Lambda_{OG(6,12)}$ with $(1,1,2,3,3,1)/(1)$
- $\Theta_{E_7(a)} : \mathbb{OP}^2 \to G_\omega(\mathbb{O}^3, \mathbb{O}^6)$ identifying $\Lambda_{\mathbb{OP}^2}$ with $(1,1,1,2,4,4,2,1)/(1)$
- $\Theta_{E_7(b)} : OG(7,14) \to G_\omega(\mathbb{O}^3, \mathbb{O}^6)$ identifying $\Lambda_{OG(7,14)}$ with $(1,1,1,2,5,5,3,3)/(1^6)$

Note the "twist" in how $\Lambda_{OG(6,12)}$ and $\Lambda_{OG(7,14)}$ are embedded; see Figure 9.

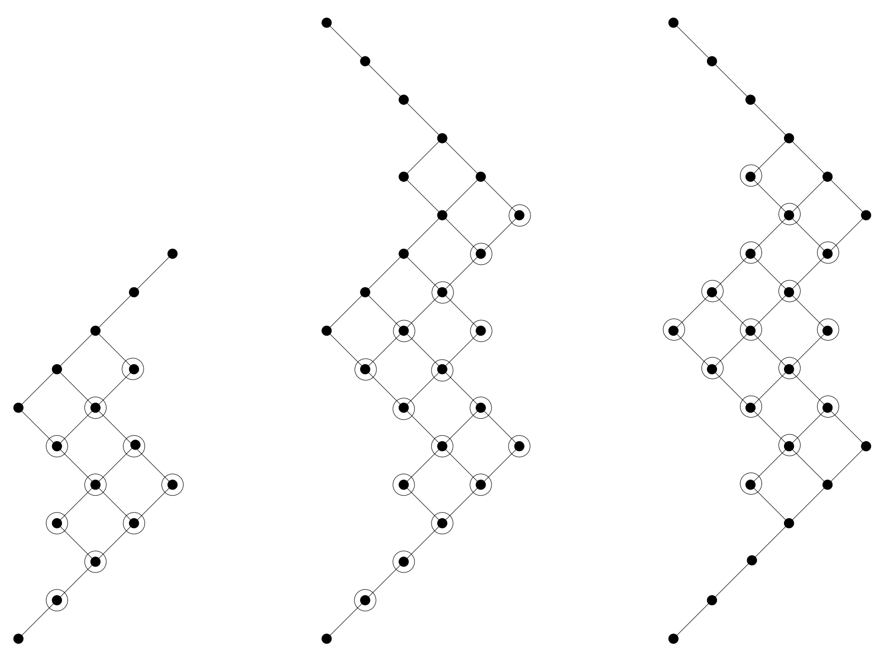


FIGURE 9. $\Theta_{E_6}, \Theta_{E_7(a)}$ and $\Theta_{E_7(b)}$ respectively (the circled nodes represent the image of the cominuscule recursion in each case)

The geometric proof of the following proposition is delayed until Section 6.

Proposition 5.7. *For* $X = \mathbb{OP}^2$ *and* $X = G_\omega(\mathbb{O}^3, \mathbb{O}^6)$*,* $\{c^{\nu}_{\lambda,\mu}(X)\}$ *is* Θ_{E_6}*-recursive and respectively* $\Theta_{E_7(a)}$ *and* $\Theta_{E_7(b)}$*-recursive.*

The strategy of the remainder of the proof is that any collection of Schubert like numbers $\{d^{\nu}_{\lambda,\mu}(X)\}$ satisfying the applicable Θ-recursions above are uniquely determined. Then since $\{c^{\nu}_{\lambda,\mu}(X)\}$ and $\{e^{\nu}_{\lambda,\mu}(X)\}$ have these properties, they are the same.

Lemma 5.8. *Suppose $\{d^{\nu}_{\lambda,\mu}(X)\}$ is Schubert like. Then $d^{\nu}_{\lambda,\mu}(X)$ is uniquely determined if any of the following hold:*

(i) $X = \mathbb{OP}^2$ *and* $d^{\nu}_{\lambda,\mu}(X)$ *is* Θ_{E_6}*-recursive; or*
(ii) $X = G_{\omega}(\mathbb{O}^3, \mathbb{O}^6)$ *and* $d^{\nu}_{\lambda,\mu}(X)$ *is* $\Theta_{E_7(a)}$*-recursive; or*
(iii) $X = \mathbb{OP}^2$ *or* $X = G_{\omega}(\mathbb{O}^3, \mathbb{O}^6)$ *and all but possibly one* $d^{\nu}_{\lambda,\gamma}(X)$ *with* $|\gamma| = |\nu| - |\lambda|$ *is uniquely determined; or*
(iv) $\mu = \emptyset$.

Proof. For (i) and (ii), since $L(\Theta_{E_6})$ and $L(\Theta_{E_7(a)})$ is a box, all $d^{\hat{\gamma}}_{L(\Theta),\mu}$ on the right-hand side of (7) are determined by (I), (II) and the case $|\nu/\lambda| = 1$ of (IV). For (iii), by (IV): $d^{\nu}_{\lambda,\mu}(X) = (f^{\nu/\lambda}(X) - \sum_{\gamma\neq\mu} f^{\gamma}(X) d^{\nu}_{\lambda,\gamma}(X))/f^{\mu}(X)$ and each of the $d^{\nu}_{\lambda,\gamma}(X)$ is determined, by (i) or (ii). Lastly, (iv) follows from (I) and the case $|\nu/\lambda| = 0$ of (IV). ☐

Corollary 5.9. *Assume $\{d^{\nu}_{\lambda,\mu}(G_{\omega}(\mathbb{O}^3, \mathbb{O}^6))\}$ is Schubert like and $\Theta_{E_7(a)}$ and $\Theta_{E_7(b)}$-recursive. Then $d^{\nu}_{\lambda,\mu}(G_{\omega}(\mathbb{O}^3, \mathbb{O}^6))$ is uniquely determined if either of the following hold:*

(i) $|\nu^c| \geq 14$*; or*
(ii) $\lambda = L(\Theta_{E_7(b)})$ *and* $\Gamma(\Theta_{E_7(b)}) \subseteq \nu^c$.

Proof. For (i), if $|\nu^c| \geq 18$ then $\Gamma(\Theta_{E_7(a)}) \subseteq \nu^c$, and apply Lemma 5.8(ii) or (iv). If $14 \leq |\nu^c| \leq 17$ then all shapes of that size contain $\mathtt{rotate}(\Gamma(\Theta_{E_7(a)}))$ except one, respectively: $(1,1,1,2,4,4,1)$, $(1,1,1,2,4,4,2)$, $(1,1,1,2,4,4,2,1)$, $(1,1,1,2,4,4,2,1,1)$, and then use Lemma 5.8(ii) or (iv), or (iii) applied to $d^{\mathtt{rotate}(\mu^c)}_{\lambda,\mathtt{rotate}(\nu^c)}(X) = d^{\nu}_{\lambda,\mu}(X)$.

For (ii) if $|\nu^c| \geq 14$ then apply (i). So assume $|\nu^c| \leq 13$. Thus $|\mu| \geq 8$. If $L(\Theta_{E_7(b)}) \subseteq \mu$ then $d^{\nu}_{\mu,L(\Theta_{E_7(b)})}$ is $\Theta_{E_7(b)}$-recursive and by (7):

$$d^{\nu}_{\mu,L(\Theta_{E_7(b)})}(G_{\omega}(\mathbb{O}^3, \mathbb{O}^6)) = \sum_{\gamma\in\mathbb{Y}_{OG(7,14)}} c^{\overline{\nu}}_{\mu,\gamma}(OG(7,14))\, d^{\hat{\gamma}}_{L(\Theta_{E_7(b)}),L(\Theta_{E_7(b)})}(G_{\omega}(\mathbb{O}^3, \mathbb{O}^6)).$$

and each nonzero $d^{\hat{\gamma}}_{L(\Theta_{E_7(b)}),L(\Theta_{E_7(b)})}(G_{\omega}(\mathbb{O}^3, \mathbb{O}^6))$ is known by (i) since $|\hat{\gamma}^c| = 15$. If $L(\Theta_{E_7(b)}) \not\subseteq \mu$ then $8 \leq |\mu| \leq 10$ and μ is the only shape of that size not containing $L(\Theta_{E_7(b)})$. Then $d^{\nu}_{\lambda,\mu}(G_{\omega}(\mathbb{O}^3, \mathbb{O}^6))$ is determined by the above argument and Lemma 5.8(iii). ☐

Proposition 5.10. *Let $X = \mathbb{OP}^2$ or $X = G_{\omega}(\mathbb{O}^3, \mathbb{O}^6)$. If $\{d^{\nu}_{\lambda,\mu}(X)\}$ are Schubert like and satisfy the applicable Θ-recursions, then each $d^{\nu}_{\lambda,\mu}(X)$ is uniquely determined (and by Propositions 5.1 and 5.7 thus equal to $c^{\nu}_{\lambda,\mu}(X)$).*

Proof. Suppose $X = \mathbb{OP}^2$. If $|\lambda|+|\mu|+|\nu^c| < 16 = |\Lambda_X|$ then use (II). Otherwise at least one of $|\lambda|$, $|\mu|$ or $|\nu^c|$ is at least 6. By (I) assume it is $|\nu^c|$. If $|\nu^c| \geq 9$, then $\Gamma(\Theta_{E_6}) \subseteq \nu^c$ and apply Lemma 5.8(i) or (iv). For $6 \leq |\nu^c| \leq 8$ there is only one shape of that size to which we cannot use Lemma 5.8(i) or (iv), namely $(1,1,2,2)$, $(1,1,2,3)$, and $(1,1,2,4)$ respectively. So we can use Lemma 5.8(iii) applied to $d^{\mathtt{rotate}(\mu^c)}_{\lambda,\mathtt{rotate}(\nu^c)}(X) = d^{\nu}_{\lambda,\mu}(X)$. This completes the proof for this case.

Now let $X = G_\omega(\mathbb{O}^3, \mathbb{O}^6)$. If $|\lambda|+|\mu|+|\nu^c| < 27 = |\Lambda_X|$ then use (II). Otherwise: by (I), assume $|\nu^c| \geq \max(|\lambda|, |\mu|, 9)$, by Corollary 5.9(i) assume $9 \leq |\nu^c| \leq 13$ and by Lemma 5.8(ii), assume that $d^{\gamma}_{\lambda,\mu}(X)$ is not $\Theta_{E_7(a)}$-recursive.

Hence, if $|\nu^c| \geq 10$ then $\Gamma(\Theta_{E_7(b)}) \subseteq \nu^c$. Since $10 \leq |\nu^c| \leq 13$, at least one of λ or μ has size at least 7. Suppose by (I) that it is λ. If $d^{\gamma}_{\lambda,\mu}(X)$ is $\Theta_{E_7(b)}$-recursive, we use (7) and Corollary 5.9(ii). Note that for the possible values of $|\lambda|$, at most one shape γ of that size has $L(\Theta_{E_7(b)}) \not\subseteq \gamma$. Thus, if $d^{\gamma}_{\lambda,\mu}(X)$ is not $\Theta_{E_7(b)}$-recursive, it is the unique $d^{\nu}_{\gamma,\mu}(X)$ with $|\gamma| = |\lambda|$ which is not, and we use Lemma 5.8(iii).

Finally, let $|\nu^c| = 9$. Thus $|\lambda| = |\mu| = 9$ also. If $\Gamma(E_7(b)) \subseteq \nu^c$ then apply the argument of the previous paragraph. Otherwise $\texttt{rotate}(\nu^c) = (1,1,1,2,4)$. By (I), we are done unless in fact $\lambda = \mu = \texttt{rotate}(\nu^c)$. So it remains to consider $d^{\texttt{rotate}((1,1,1,2,4)^c)}_{(1,1,1,2,4),(1,1,1,2,4)}(X) = d^{(1,1,1,2,5,4,2,1,1)}_{(1,1,1,2,4),(1,1,1,2,4)}$. By (IV): $f^{(1,1,1,2,5,4,2,1,1)/(1,1,1,2,4)}(X) = \sum_{|\gamma|=9} f^{\gamma}(X) d^{(1,1,1,2,5,4,2,1,1)}_{(1,1,1,2,4),\gamma}(X)$ which in turn equals $f^{(1,1,1,2,4)}(X) d^{(1,1,1,2,5,4,2,1,1)}_{(1,1,1,2,4),(1,1,1,2,4)}(X) + \sum_{\gamma \neq (1,1,1,2,4)} f^{\gamma}(X) d^{(1,1,1,2,5,4,2,1,1)}_{(1,1,1,2,4),\gamma}(X)$. In the latter summation, each γ contains $\texttt{rotate}(\Gamma(\Theta_{E_7(b)}))$. So $d^{(1,1,1,2,5,4,2,1,1)}_{(1,1,1,2,4),\gamma}$ is determined by (I) and the argument of the previous paragraph. The proposition follows. $\square$

Proposition 5.11. *For $X = \mathbb{OP}^2$ and $X = G_\omega(\mathbb{O}^3, \mathbb{O}^6)$, $\{e^{\nu}_{\lambda,\mu}(X)\}$ are Schubert like, satisfy the applicable Θ-recursions, and hence the Main Theorem holds in these cases.*

Proof. Clearly (II) and (III) are satisfied. (I) is immediate from Corollary 4.7. For (IV) we need to prove $\sum_{\gamma,|\gamma|=|\nu/\lambda|} f^{\gamma}(X) e^{\nu}_{\lambda,\gamma}(X) = f^{\nu/\lambda}(X)$. For each of the $f^{\gamma}(X)$ tableaux $T \in \mathrm{SYT}_X(\gamma)$ there are $e^{\nu}_{\lambda,\gamma}(X)$ tableaux $U \in \mathrm{SYT}_X(\nu/\lambda)$ such that $\texttt{rectification}(U) = T$. This proves "$\leq$". Conversely, since any $U \in \mathrm{SYT}_X(\nu/\lambda)$ rectifies to some $T \in \mathrm{SYT}_X(\gamma)$, equality holds. Finally, $e^{\nu}_{\lambda,\mu}(X)$ satisfies the stated recursions by Theorem 5.5. The remaining claim follows from Propositions 5.1, 5.10 and the fact that $c^{\nu}_{\lambda,\mu}(OG(6,12)) = e^{\nu}_{\lambda,\mu}(OG(6,12))$ and $c^{\nu}_{\lambda,\mu}(OG(7,14)) = e^{\nu}_{\lambda,\mu}(OG(7,14))$, which are shown in Section 5.5. $\square$

5.4. **The quadrics $\mathbb{Q}^{2n-1}$ and $\mathbb{Q}^{2n-2}$.** From Proposition 5.3, it is easy to check for $\mathbb{Q}^{2n-1}$ (type B_n) that

$$\sigma_{\square}^k = \begin{cases} \sigma_{(1^k)} & \text{if } 1 \leq k < n, \\ 2\sigma_{(1^k)} & \text{otherwise.} \end{cases}$$

Since $H^\star(\mathbb{Q}^{2n-1})$ is generated by $\sigma_\square$, the Main Theorem holds in this case.

The case $\mathbb{Q}^{2n-2}$ (type D_n), since $\sigma_\square$ does not generate $H^\star(\mathbb{Q}^{2n-2})$, we need to also use Corollary 2.3(c) and the following calculations using Proposition 5.3:

$$\sigma_{\square}^k = \begin{cases} \sigma_{(1^k)} & \text{if } 1 \leq k \leq n-2, \\ \sigma_{(1^{n-3},2)} + \sigma_{(1^{n-1})} & \text{if } k = n-1, \\ 2\sigma_{(1^{n-3},2,1)} & \text{if } k = n, \\ 2\sigma_{(1^{n-3},2,2,1^{k-n-1})} & \text{otherwise.} \end{cases}$$

and $\sigma_\square \cdot \sigma_{(1^{n-3},2)} = \sigma_\square \cdot \sigma_{(1^{n-1})} = \sigma_{(1^{n-3},2,1)}$.

5.5. **Conclusion of the proof of the Main Theorem; the minuscule cases.** For $Gr(k, \mathbb{C}^n)$, the result is a mild reformulation of [Sc77]. Similarly, for $LG(n, 2n)$ we have restated the work of [Pra91] and [Wo84, Theorem 7.2.2]. Now, it is known that the Schubert intersection numbers for $OG(n+2, 2n+4)$ differ from the $LG(n, 2n)$ case by a power of 2 plainly equal to $2^{\texttt{shortroots}(\nu/\lambda)-\texttt{shortroots}(\gamma)}$, see, e.g., [BerSo02, Section 3] and the references therein. This, combined with Proposition 5.11 proves the $\mathbb{OP}^2$ and $G_\omega(\mathbb{O}^3, \mathbb{O}^6)$ cases. Together with the analysis of the quadric cases in Section 5.4, this completes the cominuscule cases. For minuscules, the case of $OG(n, 2n+1)$ holds by the remarks in Sections 2.1 and 2.3, while for $\mathbb{P}^{2n-1}$ use Proposition 5.3, as done for the odd quadrics. □

6. Cominuscule recursions and Schubert/Richardson variety isomorphisms

6.1. **Proof of Proposition 5.7.** Fix lists of cominuscule Lie data $(G, B, T(G), P, \Phi(G), W(G))$ and $(H, C, T(H), Q, \Phi(H), W(H))$ as in Section 1.1, where $T(H) \subseteq T(G)$, $\Phi(H) \subseteq \Phi(G)$, $\widetilde{X} = H/Q$ and $X = G/P$. Let $\mathcal{Y}_w(X), \mathcal{X}_w(X) \subseteq X$ respectively be the Schubert cell and Schubert variety for $wW(G)_P \in W(G)/W(G)_P$. The **opposite Schubert cell** is $\mathcal{Y}^w(X) := B_{-}wP/P$ and the **opposite Schubert variety** is $\mathcal{X}^w(X) := \overline{\mathcal{Y}^w(X)}$. The **Richardson variety** $\mathcal{X}_u^v(X)$ is the reduced and irreducible scheme-theoretic intersection $\mathcal{X}_u(X) \cap \mathcal{X}^v(X)$. With obvious adjustments, we use this notation for the subvarieties of $\widetilde{X}$.

Below, we only refer to the cominuscule recursions Θ where $\Theta = \Theta_{E_6}, \Theta_{E_7(a)}$ or $\Theta_{E_7(b)}$. When unspecified, statements about Θ refer to all three choices. Also, $\beta(P)$ corresponds to the node 1 of the E_6 and E_7 Dynkin diagrams from Table 1. Set $\delta \in W(G)$ to be s_{β_1} if $\Theta = \Theta_{E_6}, \Theta_{E_7(a)}$ and $s_{\beta_7}s_{\beta_6}s_{\beta_5}s_{\beta_4}s_{\beta_3}s_{\beta_1}$ if $\Theta = \Theta_{E_7(b)}$.

There is a natural embedding of $W(H)$ into $W(G)$. Discussion of this, together with the proofs of the following proposition and lemma are briefly delayed until Section 6.2.

Proposition 6.1. *There is an embedding* $\eta : \widetilde{X} \hookrightarrow X$ *such that:*

(I) $\eta(\mathcal{X}^w(\widetilde{X})) = \mathcal{X}_\delta^{w\delta}(X)$;
(II) $\eta(\mathcal{X}_w(\widetilde{X})) = \mathcal{X}_{w\delta}^{w_{\max}\delta}(X)$, *where* $w_{\max} \in W(H)$ *is the maximal length element that is Grassmannian at* $\beta(Q)$.

Lemma 6.2. *Let* $w \in W(H)^Q$, *with* $\mathcal{I}(w) = \gamma$. *Then* $\mathcal{I}(w\delta) = \hat{\gamma}$.

Assuming Proposition 6.1 and Lemma 6.2, we now prove Proposition 5.7.

Corollary 6.3. $\eta(\mathcal{X}_{\overline{\lambda}}^{\overline{\nu}}(\widetilde{X})) = \mathcal{X}_\lambda^\nu(X)$ *and* $\eta_\star([\mathcal{X}_{\overline{\lambda}}^{\overline{\nu}}(\widetilde{X})]) = [\mathcal{X}_\lambda^\nu(X)] \in H_\star(X, \mathbb{Q})$.

Proof. Let $\mathcal{I}(u) = \overline{\lambda}$, $\mathcal{I}(v) = \overline{\nu}$. Now, $\eta(\mathcal{X}_{\overline{\lambda}}^{\overline{\nu}}(\widetilde{X})) = \eta(\mathcal{X}_u(\widetilde{X}) \cap \mathcal{X}^v(\widetilde{X}))$. The image is set-theoretically equal to $\mathcal{X}_{u\delta}^{w_{\max}\delta}(X) \cap \mathcal{X}_\delta^{v\delta}(X) = X_{u\delta}(X) \cap X_\delta(X) \cap X^{w_{\max}\delta}(X) \cap X^{v\delta}(X) = X_{u\delta}^{v\delta}(X) = X_\lambda^\nu(X)$. Since η is an embedding, this is the (scheme-theoretic) image. The statement about homology follows. □

Corollary 6.4. $\eta(\mathcal{X}^\gamma(\widetilde{X})) = \mathcal{X}_{L(\Theta)}^{\hat{\gamma}}(X)$ *and* $\eta_\star([\mathcal{X}^\gamma(\widetilde{X})]) = [\mathcal{X}_{L(\Theta)}^{\hat{\gamma}}(X)] \in H_\star(X, \mathbb{Q})$.

Proof. Specialize Corollary 6.3: $\nu = \hat{\gamma}$, $\lambda = L(\Theta)$. So $\eta(\mathcal{X}^\gamma(\widetilde{X})) = \eta(\mathcal{X}_\emptyset^\gamma(\widetilde{X})) = \mathcal{X}_{L(\Theta)}^{\hat{\gamma}}(X)$. □

Since Richardson varieties are homologous to scheme-theoretic unions of Schubert varieties (or equally, of opposite Schubert varieties), we have:

$$[\mathcal{X}_{\overline{\lambda}}^{\overline{\nu}}(\widetilde{X})] = \sum_{\gamma \in \mathbb{Y}_{\widetilde{X}}} c_{\overline{\lambda},\gamma}^{\overline{\nu}}(\widetilde{X})\ [\mathcal{X}^\gamma(\widetilde{X})] \in H_\star(\widetilde{X}, \mathbb{Q}). \tag{8}$$

Pushing forward on both sides of (8) gives, by Corollaries 6.3 and 6.4:

$$[\mathcal{X}_{\lambda}^{\gamma}(X)] = \sum_{\gamma \in \mathbb{Y}_{\widetilde{X}}} c_{\lambda,\gamma}^{\overline{\nu}}(\widetilde{X})\ [\mathcal{X}_{L(\Theta)}^{\hat{\gamma}}(X)] \in H_{\star}(X, \mathbb{Q}). \tag{9}$$

Expanding each $[\mathcal{X}_{L(\Theta)}^{\hat{\gamma}}(X)]$ into $\{[X^{\mu}(X)]\}$ and extracting coefficients on both sides of (9), we obtain: $c_{\lambda,\mu}^{\gamma}(X) = \sum_{\gamma \in \mathbb{Y}_{\widetilde{X}}} c_{\lambda,\gamma}^{\overline{\nu}}(\widetilde{X}) c_{L(\Theta),\mu}^{\hat{\gamma}}(X)$, proving Proposition 5.7.[2] □

6.2. **Proofs of Proposition 6.1 and Lemma 6.2.** We fix inclusions of root systems. When $\Theta = \Theta_{E_6}$ the inclusion identifies the nodes $1,2,3,4,5$ of D_5 respectively with $6,5,4,3,2$ of E_6. Similarly, for $\Theta = \Theta_{E_7(a)}$, identify $1,2,3,4,5,6$ of E_6 with $3,2,4,5,6,7$ while for $\Theta = \Theta_{E_7(b)}$ identify $1,2,3,4,5,6$ of D_6 with $1,3,4,5,6,2$ of E_7. This induces inclusions of the objects of the Lie data for H and G; we assume them for the duration of the paper. In particular, $W(H) \subseteq W(G)$ as a parabolic subgroup. Let $W(G)_P$ be the parabolic subgroup of $W(G)$ corresponding to omitting node 1, and let $W(H)_Q$ be the parabolic subgroup of $W(H) \subseteq W(G)$ omitting nodes 1 and 3 when $\Theta = \Theta_{E_6}, \Theta_{E_7(a)}$, and omitting nodes 2 and 7 when $\Theta = \Theta_{E_7(b)}$. The following is a straightforward (finite) check:

Lemma 6.5. *If* $\alpha \in \Lambda_{\widetilde{X}}$ *then* $\Theta(\alpha) = \delta^{-1}\alpha$. *If* $\alpha \in \Phi(H) \setminus (-\Lambda_{\widetilde{X}})$ *then* $\delta^{-1}\alpha \notin \Phi(G) \setminus (-\Lambda_X)$. *Also* $\mathcal{I}(\delta) = L(\Theta)$.

Lemma 6.6. *If* $w \in W(H)$, *then* $\ell(w\delta) = \ell(w) + \ell(\delta)$.

Proof. Check that $\mathcal{I}(\delta^{-1}) \cap \Phi^+(H) = \emptyset$. Since $\mathcal{I}(w) \subseteq \Phi^+(H)$, then $\mathcal{I}(\delta^{-1}) \cap \mathcal{I}(w) = \emptyset$. Thus if $\alpha \in \mathcal{I}(\delta)$, $\delta(\alpha) \in -\mathcal{I}(\delta^{-1}) \subseteq \Phi^-(G) \setminus \Phi^-(H) \subseteq \Phi^-(G) \setminus -\mathcal{I}(w)$. Thus, $\alpha \in \mathcal{I}(w\delta)$ as well. Hence by repeated application of [Hu90, Lemma 1.6(b)], $\ell(w\delta) = \ell(w) + \ell(\delta)$. □

Proof of Lemma 6.2. Lemma 6.6 implies $\mathcal{I}(w\delta) = \mathcal{I}(\delta) \cup \delta^{-1}\mathcal{I}(w)$. Now apply the first and third parts of Lemma 6.5. □

Corollary 6.7. *If* $w \in W(H)^Q$. *Then* $w\delta \in W(G)^P$.

Proof. By Lemma 6.2, $\mathcal{I}(w\delta) \in \mathbb{Y}_{G/P}$. By Proposition 2.1, $w\delta \in W(G)^P$. □

Lemma 6.8. $\delta^{-1}Q\delta \subseteq P$.

Proof. Let $U_\alpha, \alpha \in \Phi(H)$ denote the *root group*, see, e.g., [Hu75, Section 26.3]. It suffices to show $\delta^{-1}T(H)\delta \subseteq P$ and $\delta^{-1}U_\alpha\delta \subseteq P$ for $\alpha \in \Phi(H) \setminus (-\Lambda_{\widetilde{X}})$ since these subgroups generate Q. Now, $\delta^{-1}T(H)\delta \subseteq P$ since $\delta \in W(G) = N(T(G))/T(G)$. By [Hu75, Theorem 26.3], $\delta^{-1}U_\alpha\delta = U_{\delta^{-1}(\alpha)}$. By the second part of Lemma 6.5, $\delta^{-1}\alpha \in \Phi(G) \setminus (-\Lambda_X)$ so $U_{\delta^{-1}(\alpha)} \subseteq P$. □

Proof of Proposition 6.1: Pick a representative $\check{\delta} \in N(T(G))$ of δ. Consider the map $\upsilon : H \to G/P$ defined by the inclusion of H into G, right multiplying by $\check{\delta}$ and naturally projecting to G/P. This map descends to a well-defined set-theoretic map $\eta : H/Q \to G/P$: let $x, y \in H$ with $xQ = yQ$, i.e., $x = yq$ for some $q \in Q$. Now $\upsilon(x) = x\delta P/P = yq\delta P/P = y\delta\delta^{-1}q\delta P/P = y\delta P/P = \upsilon(y)$, where the second-last equality is by Lemma 6.8.

[2]It is worthwhile to point out that these ideas extend to equivariant K-theory $K_T(X)$.

Thus the fibers of υ are unions of cosets xQ. Since Q is a closed subgroup of H, the *universal mapping property* [Hu75, Theorem 12.1] shows υ factors uniquely as a morphism through H/Q and hence this latter morphism must be equal to η.

Let $U'_w(H) := \prod_{\alpha\in\mathcal{I}(w^{-1})} U_\alpha(H) \subseteq C$. There is a normal form for H/Q is given by:

$$H/Q = \coprod_{w\in W(H)^Q} U'_w(H)wQ/Q. \tag{10}$$

Here we have applied [Hu75, Theorem 28.4] together with the well-known identification of $\mathcal{Y}^w = CwQ/Q \subseteq H/Q$ with $CwC/C \subseteq H/C$, see, e.g., [Br05] (just before Example 1.2.3).

Consider η applied to a cell of H/Q: $\eta(U'_w(H)wQ/Q) = U'_w(H)w\delta P/P$. By Corollary 6.7, $w\delta \in W(G)^P$, so it indexes a cell in the decomposition of G/P similar to (10). The inclusion of H into G embeds $U'_w(H)$ into $U'_{w\delta}(G)$ (as varieties, both being affine spaces) since $\mathcal{I}(w^{-1}) \subseteq \mathcal{I}((w\delta)^{-1})$. Also, $H\check{\delta} \subseteq \overline{C_-Q\delta} \subseteq \overline{B_-\delta P}$ (again using Lemma 6.8). Thus $\eta(CwQ/Q)$ embeds into $Bw\delta P/P \cap \overline{B_-\delta P/P}$. Taking closures we get $\mathcal{X}^w(H/Q) \subseteq \mathcal{X}_\delta^{w\delta}(G/P)$. These are irreducible varieties of the same dimension, namely $\ell(w)$, so this inclusion is an isomorphism.

For (II), the argument are similar. First, η embeds $\mathcal{Y}_w(\widetilde{X}) = C_-wQ/Q$ into $B_-w\delta P/P = \mathcal{Y}_{w\delta}(X)$. Next, note that $\eta(H/Q) = \overline{\eta(Cw_{\max}Q/Q)} \subseteq \overline{Bw_{\max}\delta P/P}$. Thus, $\eta(\mathcal{Y}_w(\widetilde{X}))$ embeds into $\mathcal{X}_{w\delta}^{w_{\max}\delta}$. Taking closures, and observing that the varieties on both sides have the same dimension, namely $\ell(w_{\max}) - \ell(w)$, we see that the two varieties are isomorphic. $\square$

7. Final remarks and questions; (geometric) representation theory

Problem 7.1. *Find equivariant,* K*-theoretic and/or quantum analogues of the Main Theorem.*

This is a standard kind of question in the subject. That being said, in consultation with Allen Knutson and Terence Tao, we surmise that there is hope to obtain these generalizations in the cominuscule setting. For example, in the special case of Grassmannians, *puzzle* theorems/conjectures generalizing the Littlewood-Richardson rule exist in each of the three basic directions (as well as some combinations), see respectively, e.g., [KnTa03, Bu02, BuKrTa03] (the latter of which reduces the quantum problem to the 2-step flag manifold problem, which is computed by a conjecture of Knutson). Thus:

Problem 7.2. *Reformulate the Main Theorem via a uniform generalization of the puzzles of* [KnTa03].

In view of [Va05a, Appendix A], a geometric motivation for Problem 7.2 is that a solution might suggest appropriate degenerations of Richardson varieties giving a geometric version of the Main Theorem, in the spirit of [Va05a]. This geometry could yield interesting arithmetic consequences, see [Va05b].

We remark on a representation theoretic interpretation of the Main Theorem, as told to us by Allen Knutson. One special property of cominuscule flag manifolds that separates them from general G/P's is that they are also the only smooth Schubert varieties on the affine Grassmannian. The geometric Satake correspondence of Ginzburg, Mirković-Vilonen and others relates the geometry of the affine Grassmannian of G to the representation theory of the Langlands dual group $G^\vee$. This correspondence associates cominuscule flag manifolds to minuscule representations. See, e.g., [MiVi99]. Consequently, the Main

Theorem computes special cases of the natural action of the cohomology of the affine Grassmannian on the intersection homology of its Schubert varieties. This viewpoint suggests further avenues for potential generalization.

Finally, there is a connection between the cominuscule Schubert intersection numbers and tensor product multiplicities $m_{\lambda,\mu}^{\nu}(G)$ of the finitely many irreducible representations of Levi subgroups appearing in the exterior algebra of the subspace corresponding to $\Lambda_{G/P}$. See [Kos61, Section 8] and [BelKu06].

Problem 7.3. *When does* $c_{\lambda,\mu}^{\gamma}(G/P) = m_{\lambda,\mu}^{\gamma}(G)$*?*

It is known that the equality holds for the "single simple factor case" in $G = GL_n(\mathbb{C})$.

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