

BOUNDING THE DEGREES OF GENERATORS OF A HOMOGENEOUS DIMENSION 2 TORIC IDEAL

HUGH THOMAS

ABSTRACT. Let I be the toric ideal defined by a $2 \times n$ matrix of integers,

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

with $a_1 < a_2 < \dots < a_n$. We give a combinatorial proof that I is generated by elements of degree at most the sum of the two largest differences $a_i - a_{i-1}$. The novelty is in the method of proof: the result has already been shown by L'vovsky using cohomological arguments.

INTRODUCTION

Let $\mathcal{A} = (a_{ij})$ be a $d \times n$ matrix of integers. Let k be an arbitrary ground field. Let $R = k[z_1, \dots, z_n]$ and $S = k[y_1, \dots, y_d]$. From \mathcal{A} we get a ring homomorphism ψ from R to S by sending z_j to $y_1^{a_{1j}} y_2^{a_{2j}} \dots y_d^{a_{dj}}$. Let I be the kernel of this map. Ideals which arise in this way are called toric ideals. See [9] for a thorough introduction to the subject.

It is natural to try to determine the syzygies of such an ideal I , a problem pursued in [5, 6, 4], or, more restrictedly, to ask for a minimal set of generators for such an ideal, the approach taken in [1]. In our case, as in [2], we shall be interested in determining an upper bound for the degrees of a minimal generating set, in a special case, also singled out for consideration in [3, 8, 2], as follows: let $d = 2$, and let all the $a_{1j} = 1$. (It follows that the ideal I is the homogeneous ideal of a monomial curve in projective space, but we shall not adopt that point of view here.) For simplicity, we refer to a_{2j} as a_j . Without loss of generality let $a_1 < a_2 < \dots < a_n$.

The ring R has a $\mathbb{N} \times \mathbb{Z}$ grading, where z_j has degree $(1, a_j)$. It is easily seen that the ideal I is homogeneous with respect to this grading. Forgetting the \mathbb{Z} component of the grading, we recover the usual \mathbb{N} grading on R . We use the word “bidegree” to refer to degree in the $\mathbb{N} \times \mathbb{Z}$ grading, and “degree” to refer to degree in the usual \mathbb{N} grading.

This paper consists of a proof of the following theorem:

Theorem 1 (Main Theorem). *Let $r \geq s$ be the two largest successive differences $a_i - a_{i-1}$. Then I is generated by elements of degree no more than $r + s$.*

Simple examples show this bound is tight: let r and s be relatively prime integers, let $n = 3$, $a_1 = -r$, $a_2 = 0$, $a_3 = s$. Then I is the principal ideal generated by $z_2^{r+s} - z_1^s z_3^r$.

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Hugh Thomas is an Imperial Oil Post-doctoral Fellow at the University of Western Ontario.

Note that a given toric ideal I will arise from more than one choice of \mathcal{A} . Given a toric ideal I in R , to use the Main Theorem to obtain the best possible bound for the degrees of a minimal generating set for I , one should choose \mathcal{A} in such a way that the greatest common divisor of the successive differences $a_i - a_{i-1}$ is 1. Such a choice is always possible, and any such choice will yield the same bound.

In [8], L'vovsky used cohomological results from [7] to prove a stronger statement than our Main Theorem, bounding the regularity of I , which is to say, bounding the degrees of the i -syzygies of I for all i . Restricted to generators (0-syzygies), his result coincides with ours. It is not clear how the bound given by our Main Theorem compares with the bound obtained in [2].

In justification for this paper, aside from the intrinsic interest of a combinatorial proof of L'vovsky's bound, we hope that the techniques of this proof may extend to higher dimensional cases.

The Main Theorem follows easily from the following combinatorial result.

Theorem 2 (Connectedness Theorem). *Let $V \subset \mathbb{Z}$, not necessarily finite, with the sizes of gaps between successive elements bounded above. Let $r \geq s$ be the sizes of the two largest gaps between successive elements of V . For $(q, c) \in \mathbb{N} \times \mathbb{Z}$, let $\Pi_{(q,c)}$ be the collection of multisets with support in V , of cardinality q and sum c . Let $\Delta_{(q,c)}$ be the simplicial complex generated by the supports of the multisets in $\Pi_{(q,c)}$. Then if $q > r + s$, $\Delta_{(q,c)}$ is connected.*

TRANSLATION TO COMBINATORICS

A multiset is an unordered collection of elements, in which some elements may appear with multiplicity greater than one. We use $+$ and $-$ for addition and subtraction of multisets, and write $P = \{x_1, \dots, x_n\}_{\leq}$ to indicate that the elements of P are listed in non-decreasing order. $\sum P$ is the sum of the elements of P . We now give the proof of the Main Theorem assuming the Connectedness Theorem:

Proof of Main Theorem. Let $V = \{a_1, \dots, a_n\}_{<}$. Let $r \geq s$ be the two largest successive differences $a_i - a_{i-1}$. By the Connectedness Theorem, $\Delta_{(q,c)}$ is connected for $q > r + s$. The Main Theorem now follows from the following lemma:

Lemma 1 (Translation Lemma). *No minimal generating set for I has generators in bidegree (q, c) iff $\Delta_{(q,c)}$ is connected.*

Remark: This is a special case of a result of [6], which gives information about the degrees of minimal generators of I and also of all its i -syzygies, based on the homology of $\Delta_{(q,c)}$. In the interest of self-containedness, we give an elementary proof of the result we need.

Proof. (\Leftarrow) Let $I^{<}$ denote the ideal of R generated by the elements of I of bidegree (q', c') with $q' < q$. Then $I_{(q,c)}^{<}$, the (q, c) -bigraded part of $I^{<}$, is a sub-vector space of $I_{(q,c)}$. We wish to show that $I_{(q,c)} = I_{(q,c)}^{<}$.

For P a multiset with support in V , let z^P denote the monomial in R where the exponent of z_i is the multiplicity of a_i in P . $I_{(q,c)}$ is spanned as a k -vector space by elements of the form $z^P - z^{P'}$, for P and P' in $\Pi_{(q,c)}$.

Let P and P' be two elements of $\Pi_{(q,c)}$ with non-empty intersection, say A . Then $z^P - z^{P'} = z^A(z^{P-A} - z^{P'-A})$. But $z^{P-A} - z^{P'-A}$ is in I , in bidegree $(|P - A|, \sum(P - A))$. Thus $z^P - z^{P'} \in I_{(q,c)}^{<}$.

Now suppose P and P' are arbitrary elements of $\Pi_{(q,c)}$. Since $\Delta_{(q,c)}$ is connected, it follows that there exist $P = P_0, P_1, \dots, P_t = P'$ with each $P_i \in \Pi_{(q,c)}$, such that for each i , P_i and P_{i+1} have at least one element in common. By the previous argument, then, $z^{P_i} - z^{P_{i+1}} \in I_{(q,c)}^<$, from which it follows that $z^P - z^{P'} \in I_{(q,c)}^<$, as desired.

(\Rightarrow) $I_{(q,c)}^<$ is spanned by elements of the form $z^A(z^B - z^{B'})$ with A non-empty, or equivalently of the form $z^P - z^{P'}$ with P and P' having non-empty intersection. It follows that for any $z^P - z^{P'} \in I_{(q,c)}^<$, the supports of P and P' are in the same component of $\Delta_{(q,c)}$. Thus, if $\Delta_{(q,c)}$ has more than one component, $I_{(q,c)}^< \neq I_{(q,c)}$, as desired. \square

\square

COMBINATORIAL LEMMAS

We now develop the combinatorial tools to prove the Connectedness Theorem. Fix $V \subset \mathbb{Z}$, with r and s the sizes of the two largest gaps between successive elements of V . Fix $q > r + s$ and fix $c \in \mathbb{Z}$. Let $\Delta = \Delta_{(q,c)}$ and $\Pi = \Pi_{(q,c)}$.

Given a multiset $P = \{x_1, \dots, x_p\}_{\leq}$, define $m(P) = \sum_i ix_i$. Among multisets of the same sum and cardinality, the intuition is that m is a measure of how spread out P is — the more spread out P is, the greater $m(P)$ will be.

We now show that, for a suitable class of $P \in \Pi$, we can find another multiset $P' \in \Pi$ which is more spread out than P .

Lemma 2 (Expansion Lemma). *Let $P \in \Pi$, $P = A + C$. $|C| = r + s$, and C contains neither the greatest element nor the least element of V . Then there exists some $P' \in \Pi$, $P' = A + C'$, such that $m(P') > m(P)$.*

Note: the following proof owes its basic approach to the proof of Theorem 6.1 of [9].

Proof. Consider the following algorithm, which obtains a sequence of multisets C_i , with $C_0 = C$, where C_i is obtained from C_{i-1} by replacing one of the original elements of C by either the next larger or the next smaller element of V . When thinking about this algorithm, it's helpful to think of the elements of C as stones sitting on a number line, where the allowed positions for the stones are the numbers in V . C_i is obtained from C_{i-1} by jumping one stone which hasn't been moved yet to the next higher or lower allowed position.

Algorithm 1 (Expansion Algorithm).

$C_0 := C$.

Active := C . (These are element of C which haven't moved yet.)

$s_0 := 0$.

For $i := 1$ to $r + s$ do:

If $s_{i-1} \leq 0$,

Remove the largest element from Active, and call it x_i .

$C_i := C_{i-1} - \{x_i\} + \{\text{the least element of } V \text{ greater than } x_i\}$.

If $s_{i-1} > 0$,

Remove the smallest element from Active, and call it x_i .

$C_i := C_{i-1} - \{x_i\} + \{\text{the greatest element of } V \text{ less than } x_i\}$.

Let $s_i := \sum C_i - \sum C$.

We would now like to bound s_i . Observe that if $s_{i-1} \leq 0$ then $s_i > s_{i-1}$, and if $s_{i-1} > 0$, then $s_i < s_{i-1}$. Thus, the absolute value of s_i can be no greater than the largest jump possible on a single step, which is r . However, we can be a little more precise.

The elements of C which are increased by the algorithm are greater than or equal to all the elements which are decreased by the algorithm. Thus, any gap between successive elements of V can be jumped in only one direction in the course of running this algorithm (though it may be jumped more than one time).

Suppose there is a unique gap of size r , and it is jumped only in the increasing direction. Then it follows that $-s + 1 \leq s_i \leq r$ for all i . Symmetrically, suppose there is a unique gap of size r , and it is jumped only in the decreasing direction. Then $-r + 1 \leq s_i \leq s$. If there is a unique gap of size r which is not jumped, or there is more than one gap of size r (in which case $r = s$), then $-s + 1 \leq s_i \leq s$. In any case, we deduce that there are at most $r + s$ possible values for s_i . But there are $r + s + 1$ of the s_i . Thus, at least two of the s_i must be equal, say $s_j = s_l$, with $j < l$. Obtain C' from C by making the same jumps as were made in the algorithm on steps $j + 1$ through l . Then $\sum C' = \sum C + s_l - s_j = \sum C$. Let $P' = A + C'$. In going from P to P' , the elements which have been increased are all greater than or equal to the elements which have been decreased, and thus $m(P') > m(P)$. \square

Using only this lemma, we can prove the Connectedness Theorem with the additional assumption that V is not bounded below.

Proof of Connectedness Theorem assuming V is not bounded below. Let x and y be vertices of Δ . We want to show that they lie in the same component. Choose some $P \in \Pi$ with $x \in P$, and some $P' \in \Pi$, with $y \in P'$. Let x' be the maximum element of P , y' the maximum element of P' . If $x' = y'$ then we are done. So assume without loss of generality that $x' < y'$. Choose a subset A of P , whose size is $q - (r + s)$, and which contains x . Thus, we can write $P = A + C$, satisfying the hypotheses of the Expansion Lemma.

Now apply the Expansion Lemma recursively, with this choice of A fixed, obtaining a sequence of multisets $P = P_0, P_1, P_2, \dots, P_t$ until either $m(P_t) \geq (1 + 2 + \dots + q)y'$ or P_t contains the largest element of V . In either case, P_t clearly contains an element which is greater than or equal to y' . Thus, some previous P_i contains y' . Now by construction $P_i \in \Pi$, and x and y' are both contained in P_i , which finishes the proof. \square

Lemma 3 (Multiple Expansion Lemma). *Let $P, P' \in \Pi$. Suppose the largest and smallest elements of $P + P'$ occur in P' . Then there is a $Q \in \Pi$ containing at least one element of each of P and P' .*

Proof. The proof is essentially the argument given above, proving the Connectedness Theorem in the case where V is not bounded below (but we do not make the assumption that V is not bounded below). Fix some set A in P of size $q - (r + s)$. Apply the Expansion Lemma recursively with A fixed, producing a sequence of multisets $P = P_0, P_1, \dots$, until some P_t includes some element of P' . The Expansion Lemma can always be applied because at no stage before halting does P_i include the largest or smallest element of V , since prior to including one of these elements, it would include an element of P' . Then $Q = P_t$ satisfies the conditions in the statement of the lemma. \square

We now prove another lemma, similar to the Multiple Expansion Lemma, which we will need to prove the Connectedness Theorem in full generality.

Lemma 4 (Criss-Cross Lemma). *Let $P, P' \in \Pi$. Suppose the largest element of $P + P'$ occurs in P' , while the smallest element of $P + P'$ occurs in P . Then there exists a $Q \in \Pi$ which contains at least one element of each of P and P' .*

Proof. Assume, without loss of generality, that P and P' are disjoint. Pick $f \in P$, $f' \in P'$, with $f < f'$. Let $B = P - \{f\}$, $B' = P' - \{f'\}$.

Split B into two multisets, X and Y , where X consists of the elements of B less than all elements of B' , and Y is the remainder. Similarly, split B' into X' , the elements greater than all elements of B , and Y' , the remainder.

Lemma 5 (Size Lemma). *Either $|Y|$ is greater than the longest gap below B' or $|Y'|$ is greater than the longest gap above B .*

Proof. First, I claim that $|Y| + |Y'| > r + s$. Let $B = \{b_1, \dots, b_{q-1}\}_{\leq}$, $B' = \{b'_1, \dots, b'_{q-1}\}_{\leq}$. For $1 \leq i \leq |X|$, and for $q - |X'| \leq i \leq q - 1$, $b_i < b'_i$. So, if $|X| + |X'| \geq q - 1$, then $\sum B < \sum B'$. But $\sum B = \sum P - f > \sum P' - f' = \sum B'$, which is a contradiction. Thus, $|X| + |X'| < q - 1$, so $|Y| + |Y'| > q - 1 \geq r + s$.

Now, we can see that at least one of the following four cases holds:

1. There is a unique gap of size r below B' and $|Y| \geq r + 1$.
2. All the gaps below B' are of size no more than s and $|Y| \geq s + 1$.
3. There is a unique gap of size r above B and $|Y'| \geq r + 1$.
4. All the gaps above B are of size no more than s and $|Y'| \geq s + 1$.

If there is a unique gap of size r below B' then either $|Y| \geq r + 1$ (case (1)) or $|Y'| \geq s + 1$ (case (4)). Similarly, if there is a unique gap of size r above B , we are in case (2) or case (3). Otherwise, we are in case (2) or (4).

It is now easy to see that in cases (1) or (2), $|Y|$ is greater than the largest gap below B' , while in cases (3) or (4), $|Y'|$ is greater than the largest gap above B , proving the lemma. \square

Now, using the Size Lemma, by symmetry, we may assume without loss of generality that $|Y|$ is greater than the largest gap below B' . Let the size of this gap be g . Now consider the following algorithm.

Algorithm 2 (Criss-Cross Algorithm).

$B_0 := B$.

ActiveX := X . (These are the elements of X that haven't moved yet.)

ActiveY := Y . (And similarly for Y .)

$i := 0$.

$s_0 := 0$.

Repeat until $s_i = s_j$ for some $j < i$:

$i := i + 1$.

If $s_{i-1} \leq 0$, then

If *ActiveX* is non-empty,

Remove the largest element of *ActiveX* and call it x_i .

Otherwise,

Remove the largest element of *ActiveY* and call it x_i .

$B_i := B_{i-1} - \{x_i\} + \{\text{the least element of } V \text{ greater than } x_i\}$.

If $s_{i-1} > 0$,

Remove the smallest element of ActiveY and call it x_i .
 $B_i := B_{i-1} - \{x_i\} + \{\text{the greatest element of } V \text{ less than } x_i\}$.
Let $s_i = \sum B_i - \sum B$.

I first argue that there are always enough elements of ActiveX and ActiveY to run this algorithm. There are two potential problems:

First, suppose $s_{i-1} \leq 0$, ActiveX = \emptyset , ActiveY = \emptyset . Then $i \geq q$. So at least $r + s + 1$ values of s_j have been defined. But, as in the proof of the Expansion Lemma, no gap can get jumped in both directions, so there are at most $r + s$ possible values for the s_j , so the algorithm could never have reached this step, since there must already have been some value s_j which has occurred more than once.

Second, suppose $s_{i-1} > 0$, and ActiveY = \emptyset . If ActiveX = \emptyset , then $i \geq q$, and we have a contradiction as in the previous case. So assume that ActiveX $\neq \emptyset$.

Thus, every element of Y that have been removed from ActiveY was removed on a turn j with $s_j > 0$. Further, we can say that $g \geq s_j$ for all these j , since we have not yet jumped any elements of Y in the positive direction, so all the gaps that we have jumped in the positive direction are no longer than g . Since the algorithm has not yet terminated, all the s_j are different, and they have only g different possible positive values. Thus, the case $s_j > 0$ has been encountered at most g times. But since $|Y| \geq g + 1$, it follows that ActiveY $\neq \emptyset$, which contradicts the assumption that we had run out of ActiveY.

Thus, the algorithm runs successfully, and we obtain $i > j$ with $s_i = s_j$. Obtain \tilde{B} from B , by making the same jumps as were made on steps $j + 1$ through i of the algorithm. As in the proof of the Expansion Lemma, $\sum \tilde{B} = \sum B$.

Let $P_0 := P$, $P_1 := \{f\} + \tilde{B}$. If P_1 contains an element of B' , we are done with $Q = P_1$. Otherwise, we apply the algorithm recursively, keeping B' , f , and f' fixed. I claim that we eventually obtain a P_i which contains an element of B' .

Note first that, until we obtain a P_i which contains an element of B' , we stay in the same case of the Size Lemma, because no element of B can jump past an element of B' without hitting it, so X and Y will stay the same size. Thus, we can successfully repeat the algorithm until some P_i intersects B' .

For a multiset $D = \{x_1 \dots, x_p\}_{\leq}$, define $\tilde{m}(D) = \sum_i (p + 1 - i)x_i$. This function plays a similar role to m , except that as a set gets more spread out, \tilde{m} decreases.

Now, I claim that $\tilde{m}(Y)$ is decreased each time we run the algorithm, since the decreases to Y are at least as great as its increases, and the elements that are decreased precede the elements that are increased.

Thus, at some point, if the algorithm could run forever, $\tilde{m}(Y)$ would be smaller than $(1 + 2 + \dots + |Y|)$ times the smallest element of B' , but this would mean that Y contained elements smaller than every element of B' , which would be a contradiction.

So eventually, some P_i contains an element of B' . Then $Q = P_i$ satisfies the conditions of the statement of the lemma. \square

PROOF OF THE CONNECTEDNESS THEOREM

Now, we prove the connectedness of Δ in full generality.

Proof of Connectedness Theorem. We begin as in the case of V not bounded below. Let x and y be vertices of Δ . We want to show that x and y are in the same

component. Choose a set P in Π which contains x , and a set P' in Π which contains y .

If P and P' intersect, then we are done. So assume they do not. Suppose that the greatest and the least elements of $P + P'$ both occur in one of P or P' , without loss of generality, say P' . Now apply the Multiple Expansion Lemma to obtain a multiset $Q \in \Pi$ intersecting both P and P' . As in the proof of the Connectedness Theorem with V not bounded below, this shows that x and y are connected in Δ .

Now, suppose that the greatest and least elements of $P + P'$ do not both occur in either P or P' . Without loss of generality, let P' contain the greatest. This puts us in the position to apply the Criss-Cross Lemma, and again, the multiset Q which we obtain from it shows that x and y are connected in Δ . \square

FURTHER DIRECTIONS

First, it would be good to give a combinatorial proof of the entire result of L'vovsky (bounding the degrees of all i -syzygies of I , not just the generators). An argument might use the full strength of the result mentioned in the proof of the Translation Lemma from [6] to translate the problem into a combinatorial framework. The necessary combinatorial result might then be established by an induction argument with a version of the Connectedness Theorem as a base case, but so far we have been unable to accomplish this except in the case where V is not bounded below, which is of limited interest.

Also, as mentioned in the introduction, one might apply the techniques of this paper to prove degree bounds or regularity bounds in higher dimensions. However, this is considerably trickier. Our strategy for modifying a multiset P until it hits an element of P' is akin to a game of hide-and-seek in one dimension; the interested reader will see why the game is usually played in two dimensions.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, N6A
5B7 CANADA
E-mail address: `hthomas2@uwo.ca`