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A LÉVY INSURANCE RISK PROCESS WITH TAX

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Abstract. Using fluctuation theory, we identify the ruin probability of a general spectrally negative Lévy risk process with tax payments of loss carry forward type. We study arbitrary moments of the discounted total amount of tax payments and determine the surplus level to start taxation which maximizes the expected discounted aggregate income for the tax authority in this model. The results considerably generalize those for the Cramér-Lundberg risk model with tax.

1. Introduction

The classical risk model describes the surplus process of an insurance company by a stochastic process $U_0 = (U_0(t))_{t\geq 0}$ with

$$U_0(t) = u + ct - S(t),$$

where $S(t)$ is a compound Poisson process with jump intensity θ and jump distribution F (representing the aggregate claim payments up to time t), $u > 0$ denotes the initial surplus and $c > 0$ is a constant premium intensity. Usually it is assumed that the *net profit condition*

$$c > \theta\mu$$

holds, where μ denotes the expected value of the single claim size distribution F. This condition ensures that ruin will not occur almost surely. As a Lévy process, U_0 has a characteristic exponent given by

$$\Psi(\lambda) = -\ln \mathbb{E}\left[e^{i\lambda U_0(1)}\right] = -ic\lambda - \int_{-\infty}^{0} (e^{i\lambda z} - 1)\,\theta F(dz)$$

for $\lambda \in \mathbb{R}$.

One way to generalize the classical risk process is to consider an arbitrary spectrally negative Lévy process, i.e. a process $X = (X(t))_{t\geq 0}$ with independent and stationary increments and with characteristic exponent given by

$$\Psi(\lambda) = -ic\lambda + \frac{1}{2}\sigma^2\lambda^2 - \int_{-\infty}^{0} (e^{i\lambda z} - 1 - iz\mathbb{I}_{\{z>-1\}})\,\Pi(dz),$$

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for $\lambda \in \mathbb{R}$, $\sigma \geq 0$ and where Π is a measure on $(-\infty, 0)$ such that

$$\int_{-\infty}^{0} (1 \wedge z^2)\, \Pi(dz) < \infty.$$

Here, $c > 0$ again represents the constant premium density. The net profit condition for this Lévy insurance risk process now reads

$$\mathbb{E}[X(1)] > 0,$$

which is equivalent to $\lim_{t\to\infty} X(t) = \infty$ almost surely.

An interpretation of such Lévy risk processes for the surplus modelling of large insurance companies is for instance given in Klüppelberg and Kyprianou [12] and Kyprianou and Palmowski [15]. This model has recently attracted a lot of research interest, see e.g. also Furrer [8], Yang and Zhang [18], Huzak et al. [11], Klüppelberg et al. [13], Chiu and Yin [6] and Garrido and Morales [9].

In a recent paper, Albrecher and Hipp [1] investigated how tax payments (according to a loss carry forward system) affect the behaviour of a Cramér-Lundberg surplus process. In their model, taxes are paid at a fixed proportional rate γ whenever the company is in a *profitable situation*, defined as being at a running maximum of the surplus process. It turned out that in this model there is a strikingly simple relationship between ruin probabilities with and without tax and one can also get an explicit formula for the expected discounted sum of tax payments over the lifetime of the risk process.

In this paper we will embed this tax model into a general Lévy framework. Utilizing excursion theory and exploiting the structure of the model, we will establish the simple relation between ruin probabilities with and without tax in this more general class of models. Furthermore, expressions for arbitrary moments of discounted tax payments until ruin will be derived. It turns out that close connections of the distribution of tax payments to the distribution of dividend payments according to a horizontal barrier strategy, that were observed in the Cramér-Lundberg model, carry over to the Lévy setup.

The paper is organized as follows. In Section 2, we will review some preliminaries on spectrally negative Lévy processes that will be needed later on. Section 3 introduces the tax model under consideration and derives the ruin probability as well as moments of discounted tax payments until ruin. Finally, in Section 4 the problem of an optimal choice of a threshold surplus level for starting taxation to maximize the expected tax income will be addressed.

2. Preliminaries on spectrally negative Lévy processes

Let $X = (X(t))_{t\geq 0}$ be a spectrally negative Lévy process or, in other words, a Lévy process with no positive jumps (to avoid trivialities, we exclude the case where X is a negative subordinator or a deterministic drift). The law of X such that $X(0) = u \geq 0$ will be denoted by $\mathbb{P}_u$ and the corresponding expectation by $\mathbb{E}_u$ (for a general introduction to Lévy processes we refer to Bertoin [3] or Kyprianou [14]).
As the Lévy process X has no positive jumps, its Laplace transform is given by

$$\mathbb{E}_0\left[e^{\lambda X(t)}\right] = e^{t\psi(\lambda)}$$

for $\lambda \geq 0$ and $t \geq 0$, where $\psi(\lambda) = -\Psi(i\lambda)$. In this case, the Laplace exponent ψ is strictly convex and $\lim_{\lambda\to\infty}\psi(\lambda) = \infty$. Thus, there exists a function $\Phi\colon [0,\infty) \to [0,\infty)$ such that

$$\psi(\Phi(\lambda)) = \lambda, \quad \lambda \geq 0.$$

We now define the so-called scale functions $\{W_q; q \geq 0\}$ of the process X as in Bertoin [4]. For each q, $W_q\colon [0,\infty) \to [0,\infty)$ is the unique, strictly increasing and continuous function with Laplace transform

$$\int_0^\infty e^{-\lambda z} W_q(z)\,dz = \frac{1}{\psi(\lambda) - q},$$

for $\lambda > \Phi(q)$.

2.1. **Two-sided exit problem.**

Scale functions arise naturally when considering two-sided exit problems for spectrally negative Lévy processes. Indeed, let a be a positive real number and define

$$T_{(0,a)} = \inf\{t \geq 0 \mid X(t) \notin (0,a)\}.$$

When the process X starts within the interval (i.e. $X(0) = u \in (0,a)$), the random time $T_{(0,a)}$ is the first exit time of X from this interval. Since X has no positive jumps, it will hit the point a when exiting above, but it might jump below zero when exiting below. Its Laplace transform on the event where the process X leaves the interval at the upper boundary is given by

$$\mathbb{E}_u\left[e^{-qT_{(0,a)}}; X(T_{(0,a)}) = a\right] = \frac{W_q(u)}{W_q(a)}, \quad q \geq 0. \tag{1}$$

Consequently, when $q = 0$,

$$\mathbb{P}_u\left\{X(T_{(0,a)}) = a\right\} = \frac{W_0(u)}{W_0(a)}. \tag{2}$$

If X has a positive mean, we have that

$$\mathbb{P}_u\left\{\inf_{t\geq 0} X(t) \geq 0\right\} = \psi'(0+)W_0(u). \tag{3}$$

This result is of course related to the ruin and survival probabilities in insurance risk theory.

2.2. **Smoothness of the scale functions.** At several places in this paper, differentiability of the scale functions will be required. If the sample paths of X are of unbounded variation, then the scale functions W_q are continuously differentiable. When the sample paths of X are of bounded variation, then the scale functions are continuously differentiable if and only if Π has no atoms, or in other words if $\{x<0 \mid \Pi(\{x\})>0\}=\emptyset$. Note that if X has a Gaussian component, then its sample paths are of unbounded variation and, moreover, its scale functions are even twice continuously differentiable. Further, if the Lévy measure Π has a density, then the scale functions are always differentiable (see Doney [7] or Chan and Kyprianou [5] for more details).

3. The model

Let X be the underlying Lévy risk process with differentiable scale functions. Let $S^X=(S^X(t))_{t\geq 0}$ denote the running maximum of X, i.e. $S^X(t)=\max_{0\leq s\leq t} X(s)$. This process is continuous and, of course, increasing. Clearly, $S^X(0)=u$ as $X(0)=u$. For $0\leq\gamma\leq 1$, define a process $U_\gamma=(U_\gamma(t))_{t\geq 0}$ by

$$U_\gamma(t)=X(t)-\gamma(S^X(t)-X(0)).$$

One can think of U_γ as the surplus process of an insurance company that pays out taxes at a fixed rate γ whenever it is in a *profitable situation* (or, in other words, whenever the surplus is at a running maximum). When $\gamma=1$, this amounts to the situation where the company pays out as dividends any capital above its initial value.

3.1. **A fluctuation identity.** The following theorem generalizes both Theorem VII.8 in Bertoin [3] and Equation (1).

Theorem 3.1. *For any* $0<u<a$*, let* $\tau_a^+=\inf\{t>0\colon U_\gamma(t)>a\}$ *and* $\tau_0^-=\inf\{t>0\colon U_\gamma(t)<0\}$ *with the convention* $\inf\emptyset=\infty$*. If* $\gamma<1$*, then*

$$\mathbb{E}_u\left[e^{-q\tau_a^+}\mathbb{I}_{\{\tau_a^+<\tau_0^-\}}\right]=\left(\frac{W_q(u)}{W_q(a)}\right)^{1/1-\gamma}. \tag{4}$$

Proof. We only have to consider the case when X drifts to infinity (indeed, if X has no drift, then, akin to the proof of Theorem VII.8 in Bertoin [3], we can use an approximation by adding a small positive drift and if X has negative drift, then we can introduce a new probability measure under which X has a positive drift).

It is well-known that S^X is a local time at 0 for S^X-X. Then, let ϵ be the excursion process of S^X-X away from 0, let $\bar{\epsilon}$ be the excursion height process, and let n be the excursion measure. If X drifts to infinity, then ϵ is a Poisson point process and $\bar{\epsilon}$ is also a Poisson point process with characteristic measure ν given by $\nu(x,\infty)=W_0'(x)/W_0(x)$. By the definition of an excursion, the event $\{\tau_a^+<\tau_0^-\}$ is the same as

$$\{\bar{\epsilon}_s<u+(1-\gamma)s, \forall\, 0\leq s\leq (a-u)/(1-\gamma)\}.$$

Then, by the definition of a Poisson point process, we have that

$$\begin{aligned}\mathbb{P}_u\{\tau_a^+ < \tau_0^-\} &= \mathbb{P}\{N=0\} \\ &= \exp\left\{-\int_0^{\frac{a-u}{1-\gamma}} \frac{W_0'(u+(1-\gamma)s)}{W_0(u+(1-\gamma)s)}\, ds\right\} \\ &= \exp\left\{-\frac{1}{1-\gamma}\int_0^{a-u} \frac{W_0'(u+s)}{W_0(u+s)}\, ds\right\} \\ &= \left(\frac{W_0(u)}{W_0(a)}\right)^{1/1-\gamma}.\end{aligned}$$

where N is a Poisson distributed random variable with parameter

$$\int_0^{\frac{a-u}{1-\gamma}} n\left(\bar{\epsilon}_s \geq u+(1-\gamma)s\right)\, ds$$

that counts the number of Poisson points $(s, \bar{\epsilon}_s)$ in

$$\{(x,y)\in\mathbb{R}^2 \mid 0 \leq x \leq (a-u)/(1-\gamma),\ u+(1-\gamma)x \leq y\}.$$

When $q>0$, we can define the measure $\mathbb{P}_u^{\Phi(q)}$ on $\mathcal{F}_{\tau_a^+}$ with Radon-Nikodym derivative

$$\frac{d\mathbb{P}_u^{\Phi(q)}}{d\mathbb{P}} = e^{\Phi(q)(X(\tau_a^+)-u)-q\tau_a^+},$$

where $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration generated by X. Under $\mathbb{P}_u^{\Phi(q)}$, X is still a spectrally negative Lévy process, but now with $W_0^{\Phi(q)}$ as its scale functions, which are given by $e^{\Phi(q)x}W_0^{\Phi(q)}(x) = W_q(x)$; see Chapter 8 of Kyprianou [14] for details.

Observe that $X(\tau_a^+) = S^X(\tau_a^+)$ for $\tau_a^+ < \infty$. Since

$$a = U_\gamma(\tau_a^+) = X(\tau_a^+) - \gamma(S^X(\tau_a^+) - u)$$

for $\tau_a^+ < \infty$, we have

$$X(\tau_a^+)\mathbb{I}_{\{\tau_a^+<\infty\}} = \frac{a-\gamma u}{1-\gamma}\,\mathbb{I}_{\{\tau_a^+<\infty\}}.$$

We then further get that

$$\begin{aligned}\mathbb{E}_u\left[e^{-q\tau_a^+}\mathbb{I}_{\{\tau_a^+<\tau_0^-\}}\right] &= \mathbb{P}_u^{\Phi(q)}\left\{\tau_a^+ < \tau_0^-\right\}\exp\left\{-\Phi(q)\left(\frac{a-\gamma u}{1-\gamma} - u\right)\right\} \\ &= \left(\frac{W_0^{\Phi(q)}(u)}{W_0^{\Phi(q)}(a)}\right)^{1/1-\gamma}\exp\left\{-\frac{\Phi(q)(a-u)}{1-\gamma}\right\} \\ &= \left(\frac{e^{-\Phi(q)u}W_q(u)}{e^{-\Phi(q)a}W_q(a)}\right)^{1/1-\gamma}\exp\left\{-\frac{\Phi(q)(a-u)}{1-\gamma}\right\}.\end{aligned}$$

Therefore, the desired result follows readily. □

3.2. **The survival probability.** Let

$$\phi_\gamma(u) = \mathbb{P}_u \left\{ \inf_{t\geq 0} U_\gamma(t) \geq 0 \right\}$$

denote the survival probability in the risk model with tax rate γ and initial surplus u. Hence, $\phi_0(u)$ is the survival probability in the risk model without tax. For the compound Poisson risk model, Albrecher and Hipp [1] established a simple relation between the survival probability of a risk model with and without tax. We will now utilize Theorem 3.1 to generalize this result to spectrally negative Lévy risk processes.

Corollary 3.1. *If $\gamma < 1$, then*

$$\phi_\gamma(u) = (\phi_0(u))^{1/1-\gamma}.$$

Proof. From Theorem 3.1, we have that

$$\phi_\gamma(u) = \left(\psi'(0+) W_0(u)\right)^{1/1-\gamma},$$

since $\lim_{a\to\infty} W_0(a) = (\psi'(0+))^{-1}$. The result follows from Equation (3). □

Note that $\phi_\gamma(u) > 0$ if and only if $\phi_0(u) = \psi'(0+) > 0$, which is the case under the net profit condition $\mathbb{E}_0[X(1)] > 0$. However, the expectation need not be finite.

3.3. **The discounted tax payments.** Let us from now on assume that the net profit condition is fulfilled, i.e. $\mathbb{E}_u[X(1)] > 0$.
Let τ_γ be the time of ruin of the risk process with tax, i.e.

$$\tau_\gamma = \inf\left\{t \geq 0 \mid U_\gamma(t) < 0\right\}.$$

Let further

$$T(\gamma) = \gamma \int_0^{\tau_\gamma} e^{-\delta t}\, dD(t),$$

denote the present value of all tax payments until the time of ruin τ_γ, where $D(t) = S^X(t) - X(0)$ and $\delta \geq 0$ can be interpreted as the force of interest. Recall from Zhou [19] that

$$V_1(u,u) = \frac{W_\delta(u)}{W'_\delta(u)}, \tag{5}$$

where $V_1(u,u)$ is the expectation of the present value of all dividends paid until ruin when a horizontal barrier is at level u. Utilizing a methodology from Zhou [19] for horizontal barrier models, we will now compute $v_1^{(\gamma)}(u) = \mathbb{E}(T(\gamma))$.
Note that $v_1^{(1)}(u) = V_1(u,u)$ (so that the case $\gamma = 1$ is settled).

Theorem 3.2. *If $\gamma < 1$ and $\delta > 0$, then the expected discounted sum of tax payments until ruin is given by*

$$v_1^{(\gamma)}(u) = \frac{\gamma}{1-\gamma}\int_u^\infty \left(\frac{W_\delta(u)}{W_\delta(s)}\right)^{1/(1-\gamma)} ds. \tag{6}$$

Proof. For each $n \geq 1$, define an exit time T_n by

$$T_n = \inf\{t \geq 0 \mid X(t) \notin (\gamma/n, u+1/n)\}.$$

As X has no positive jumps, we have

$$v_1^{(\gamma)}(u) \geq \mathbb{E}_u\left[T(\gamma); X(T_n) = u+1/n\right].$$

T_n is strictly less than τ_γ on the event $\{X(T_n) = u+1/n\}$, using the integration by parts formula and the strong Markov property at time T_n, we get

$$\begin{aligned}\mathbb{E}_u\left[T(\gamma); X(T_n) = u+1/n\right] &\geq (\gamma/n)\mathbb{E}_u\left[e^{-\delta T_n}; X(T_n) = u+1/n\right] \\ &\quad + v_1^{(\gamma)}(u+(1-\gamma)/n)\mathbb{E}_u\left[e^{-\delta T_n}; X(T_n) = u+1/n\right].\end{aligned}$$

Hence,

$$v_1^{(\gamma)}(u) \geq \gamma\frac{W_\delta(u-\gamma/n)}{nW_\delta(u+(1-\gamma)/n)} + v_1^{(\gamma)}(u+(1-\gamma)/n)\frac{W_\delta(u-\gamma/n)}{W_\delta(u+(1-\gamma)/n)}.$$

In fact, one can show that

$$\begin{aligned}&v_1^{(\gamma)}(u) = \\ &\gamma\frac{W_\delta(u-\gamma/n)}{nW_\delta(u+(1-\gamma)/n)} + v_1^{(\gamma)}(u+(1-\gamma)/n)\frac{W_\delta(u-\gamma/n)}{W_\delta(u+(1-\gamma)/n)} + o(1/n),\end{aligned}$$

when n goes to infinity. Indeed, introducing, for each $n \geq 1$, the exit time T'_n defined by

$$T'_n = \inf\{t \geq 0 \mid X(t) \notin (0, u+1/n)\},$$

we get that

$$\begin{aligned}v_1^{(\gamma)}(u) &= \mathbb{E}_u\left[T(\gamma); X(T'_n) \leq 0\right] + \mathbb{E}_u\left[T(\gamma); X(T'_n) = u+1/n\right] \\ &\leq \mathbb{E}_u\left[\gamma\int_0^{T'_n} e^{-\delta t}\, dD(t); X(T'_n) \leq 0\right] \\ &\quad + \mathbb{E}_u\left[\gamma\int_0^{T'_n} e^{-\delta t}\, dD(t); X(T'_n) = u+1/n\right] \\ &\quad + \mathbb{E}_u\left[\gamma\int_{T'_n}^{\tau_\gamma \vee T'_n} e^{-\delta t}\, dD(t); X(T'_n) = u+1/n\right] \\ &\leq \gamma\frac{W_\delta(u)}{nW_\delta(u+1/n)} + v_1^{(\gamma)}(u+(1-\gamma)/n)\frac{W_\delta(u)}{W_\delta(u+1/n)} + o(1/n),\end{aligned}$$

where we have again used the integration by parts formula, the strong Markov property and the following two facts (cf. Zhou [19]):

$$\mathbb{E}_u\left[\int_0^{T'_n} e^{-\delta t}\, dD(t); X(T'_n) \le 0\right] = o(1/n);$$
$$\mathbb{E}_u\left[\int_0^{T'_n} e^{-\delta t} D(t)\, dt; X(T'_n) = u + 1/n\right] = o(1/n),$$

when n goes to infinity.

Consequently, using the continuity and the differentiability of the scale functions, we get that

$$v_1^{(\gamma)}(u) \lim_{n\to\infty} \frac{1 - \frac{W_\delta(u-\gamma/n)}{W_\delta(u+(1-\gamma)/n)}}{\gamma/n} - \lim_{n\to\infty} \frac{W_\delta(u-\gamma/n)}{W_\delta(u+(1-\gamma)/n)}$$
$$= \frac{1-\gamma}{\gamma} \lim_{n\to\infty} \frac{v_1^{(\gamma)}(u+(1-\gamma)/n) - v_1^{(\gamma)}(u)}{(1-\gamma)/n} \frac{W_\delta(u-\gamma/n)}{W_\delta(u+(1-\gamma)/n)}$$

and further

$$(v_1^{(\gamma)})'(u) = \frac{\gamma}{1-\gamma}\left(\frac{W'_\delta(u)}{\gamma W_\delta(u)} v_1^{(\gamma)}(u) - 1\right). \tag{7}$$

This is the analogue of Equation (14) in the Proof of Theorem 2 in Albrecher and Hipp [1]. Using the *integrating factor* technique for ordinary differential equations, we get that its solution is given by

$$v_1^{(\gamma)}(u) = \left(C - \frac{\gamma}{1-\gamma} U_2(u)\right) e^{U_1(u)/(1-\gamma)},$$

for some constant C, where

$$U_1(u) = \int_0^u \frac{W'_\delta(s)}{W_\delta(s)}\, ds\ , \qquad U_2(u) = \int_0^u e^{-U_1(s)/(1-\gamma)}\, ds.$$

We have that $W'_\delta(s)/W_\delta(s) \ge 0$ and

$$\lim_{s\to\infty} \frac{W'_\delta(s)}{W_\delta(s)} = \Phi(\delta).$$

The latter result can be found in Avram et al. [2] or in Zhou [20]. Hence, U_1 is unbounded because $\Phi(\delta) > 0$ for $\delta > 0$. Also, since $\tau_\gamma \to \infty$ as $u \to \infty$ (for any γ), with (5) we have that $\lim_{u\to\infty} v_1^{(\gamma)}(u) < \infty$. Thus,

$$\lim_{u\to\infty} U_2(u) = \frac{1-\gamma}{\gamma} C$$

and then

$$v_1^{(\gamma)}(u) = \frac{\gamma}{1-\gamma} e^{(1-\gamma)^{-1}\int_0^u \frac{W'_\delta(s)}{W_\delta(s)}\, ds} \int_u^\infty e^{-(1-\gamma)^{-1}\int_0^s \frac{W'_\delta(t)}{W_\delta(t)}\, dt}\, ds. \tag{8}$$

The statement follows from algebraic manipulations. □

Remark 3.1. *If X has a negative drift (i.e. $\mathbb{E}_u[X(1)] < 0$), then* (6) *also holds for $\delta = 0$.*

Remark 3.2. *Using Equation* (8)*, we can also write*

$$v_1^{(\gamma)}(u) = \frac{\gamma}{1-\gamma} e^{(1-\gamma)^{-1}\int_0^u (V_1(s,s))^{-1}\,ds} \int_u^\infty e^{-(1-\gamma)^{-1}\int_0^s (V_1(t,t))^{-1}\,dt}\,ds, \tag{9}$$

recovering Theorem 2 of Albrecher and Hipp [1] *in our more general Lévy setting.*

Remark 3.3. *Using L'Hôpital's rule, we recover the following interesting relation:*

$$\lim_{u\to\infty} v_1^{(\gamma)}(u) = \gamma \lim_{u\to\infty} V_1(u,u). \tag{10}$$

A direct probabilistic reasoning to obtain this identity goes as follows: in the absence of ruin, the only difference for the calculation of $v_1^{(\gamma)}(u)$ and $V_1(u,u)$ is that, whenever tax (dividend) payments start and last until the next deviation from the running maximum, in the tax case only the proportion γ of the income is paid whereas in the horizontal barrier case all the income is paid. The only further difference is then that the surplus level at the next payment stream is different, but the latter does not matter if the distance to the ruin boundary does not matter, which in the limit $u \to \infty$ is the case. Hence we immediately arrive at (10)*.*

3.4. **Higher moments.** We will now investigate higher moments of $T(\gamma)$. Let $v_k^{(\gamma)}(u)$ be the k-th moment of $T(\gamma)$ when the initial surplus is equal to u. Recall from Renaud and Zhou [17], and also from Kyprianou and Palmowski [15], that

$$V_k(u,u) = k! \prod_{i=1}^{k} \frac{W_{i\delta}(u)}{W'_{i\delta}(u)}, \tag{11}$$

where $V_k(u,u)$ is the k-th moment of the present value of all dividends paid until ruin when the horizontal barrier is at level u. Note that $v_k^{(1)}(u) = V_k(u,u)$. So we only need to address the case $\gamma < 1$:

Theorem 3.3. *If $\gamma < 1$ and $\delta > 0$, then the k-th moment of the present value of tax payments until ruin is related to the $(k-1)$-th moment by*

$$v_k^{(\gamma)}(u) = \frac{k\gamma}{1-\gamma} \int_u^\infty v_{k-1}^{(\gamma)}(s) \left(\frac{W_{k\delta}(u)}{W_{k\delta}(s)}\right)^{1/(1-\gamma)} ds. \tag{12}$$

Proof. First, proceeding as in the proof for Theorem 3.2 and using estimates from the proof of Proposition 1 in Renaud and Zhou [17], we have that

$$\begin{aligned} v_k^{(\gamma)}(u) = {} & k v_{k-1}^{(\gamma)}\Big(u + (1-\gamma)/n\Big) \frac{\gamma}{n} \frac{W_{k\delta}(u-\gamma/n)}{W_{k\delta}(u+(1-\gamma)/n)} \\ & + v_k^{(\gamma)}(u+(1-\gamma)/n) \frac{W_{k\delta}(u-\gamma/n)}{W_{k\delta}(u+(1-\gamma)/n)} + o(1/n). \end{aligned}$$

Further, we get that

$$(v_k^{(\gamma)})'(u) = \frac{\gamma}{1-\gamma}\left(\frac{W_\delta'(u)}{\gamma W_\delta(u)} v_k^{(\gamma)}(u) - k v_{k-1}^{(\gamma)}(u)\right).$$

Solving this ordinary differential equation leads to

$$v_k^{(\gamma)}(u) = \frac{k\gamma}{1-\gamma} e^{(1-\gamma)^{-1}\int_0^u \frac{W_{k\delta}'(s)}{W_{k\delta}(s)}\,ds} \int_u^\infty v_{k-1}^{(\gamma)}(s)\, e^{-(1-\gamma)^{-1}\int_0^s \frac{W_{k\delta}'(t)}{W_{k\delta}(t)}\,dt}\, ds.$$

Now the statement follows from simple algebraic manipulations. □

Remark 3.4. *From* (12)*, we get by L'Hôpital's rule*

$$\lim_{u\to\infty} v_k^{(\gamma)}(u) = k\gamma \lim_{u\to\infty} v_{k-1}^{(\gamma)}(u) \frac{W_{k\delta}(u)}{W_{k\delta}'(u)}.$$

With (11) *we can hence generalize the asymptotic relation* (10) *to arbitrary moments of tax and dividend payments, respectively:*

$$\lim_{u\to\infty} v_k^{(\gamma)}(u) = \gamma^k \lim_{u\to\infty} V_k(u,u). \tag{13}$$

The alternative probabilistic argument from Remark (3.3) *also carries over to explain formula* (13).

3.5. **Examples.**

3.5.1. *Cramér-Lundberg process with exponential claims.* If X is a compound Poisson process with exponential jumps (with Poisson parameter λ and exponential parameter α), then the scale functions are given by

$$W_\delta(x) = \frac{(\alpha+\rho)e^{\rho x}(1-\eta(x))}{c(\rho - r)}$$

(see e.g. Kyprianou & Palmowski [15]), where

$$\eta(x) = \frac{\alpha + r}{\alpha+\rho}\, e^{(r-\rho)x},$$

and ρ and r are the positive and negative, respectively, solution of the equation

$$cR^2 + (c\alpha - \lambda - \delta)R - \alpha\delta = 0.$$

Plugging this expression into formula (6), we eventually arrive at the explicit formula

$$\begin{aligned} v_1^{(\gamma)}(u) &= \frac{\gamma}{\rho}(1-\eta(u))^{1/1-\gamma} \\ &\quad \times {}_2F_1\left(\frac{1}{1-\gamma}, \frac{\rho}{(\rho-r)(1-\gamma)}, \frac{\rho}{(\rho-r)(1-\gamma)} + 1; \eta(u)\right), \end{aligned}$$

which was already derived in Albrecher and Hipp [1]. Here

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt$$

with $c > b > 0$ denotes the Gauss hypergeometric series.

3.5.2. *Brownian motion with drift.* Let $X(t) = mt + \sigma B(t)$ be a Brownian motion with drift (with $m \neq 0$ and $\sigma > 0$). As in this case $\psi(\lambda) = m\lambda + (1/2)\sigma^2\lambda^2$ and $\Phi(\alpha) = -\omega + \theta_\alpha$, one can verify that

$$W_\delta(x) = \frac{1}{\sigma^2\theta_\delta}\left(e^{(-\omega+\theta_\delta)x} - e^{-(\omega+\theta_\delta)x}\right),$$

where $\theta_\delta = \sqrt{m^2 + 2\delta\sigma^2}/\sigma^2$ and $\omega = m/\sigma^2$ (see also Avram et al. [2]). In particular, we have

$$W_0(x) = \frac{1}{m}\left(1 - e^{-\frac{2m}{\sigma^2}x}\right).$$

Thus,

$$v_1^{(1)}(u) = V_1(u,u) = \frac{\sigma^2}{2m}\left(e^{\frac{2m}{\sigma^2}u} - 1\right),$$

which recovers Equation (2.20) in Gerber and Shiu [10].
Also, if $\gamma < 1$ and if $\delta > 0$, then one obtains

$$v_1^{(\gamma)}(u) = \frac{\gamma}{1-\gamma}\left[e^{(\theta_\delta-\omega)u}(1 - e^{-2\theta_\delta u})\right]^{1/1-\gamma} \times \int_u^\infty \left[e^{(\theta_\delta-\omega)s}(1 - e^{-2\theta_\delta s})\right]^{-1/1-\gamma} ds.$$

Since $\theta_\delta > \omega$ when $\sigma > 0$ and $\delta > 0$, letting $r = \frac{e^{-2\theta_\delta s}}{e^{-2\theta_\delta u}}$ in the integral and simplifying yields

$$v_1^{(\gamma)}(u) = \frac{\gamma}{1-\gamma}\left[\frac{(1 - e^{-2\theta_\delta u})^{1/1-\gamma}}{\theta_\delta - \omega}\right] \times {}_2F_1\left((1-\gamma)^{-1}, \frac{\theta_\delta - \omega}{2\theta_\delta}, \frac{3\theta_\delta - \omega}{2\theta_\delta}; e^{-2\theta_\delta u}\right).$$

4. Optimality of the tax barrier

As tax payments stop at ruin, it is natural to ask whether the expected discounted tax payments over the lifetime of the process can be optimized when tax payments are only started after the surplus has reached a certain level M (see Albrecher and Hipp [1] for a corresponding study in the Cramér-Lundberg framework). Due to the strong Markov property we clearly have

$$v_{1,M}^{(\gamma)}(u) = \frac{W_\delta(u)}{W_\delta(M)}\, v_1^{(\gamma)}(M) \tag{14}$$

for $u < M$ and $v_{1,M}^{(\gamma)}(u) = v_1^{(\gamma)}(u)$ for $u \geq M$ (as then tax payments start right away). Hence the goal is to maximize (14) with respect to M.

Assumption 4.1. *In what follows, we assume that each scale function is three times differentiable and that its first derivative is a strictly convex function (so that $W_\delta''(u)$ changes its sign from negative to positive at most once).*

Assumption 4.1 is for instance fulfilled if the Lévy measure has a completely monotone density (see Loeffen [16] for the strict convexity of W_δ' and Chan and Kyprianou [5] for infinite differentiability). Among particular examples fulfilling Assumption 4.1 are Gamma process and the inverse Gaussian process (for more examples, see Loeffen [16]).

Differentiating Equation (14) with respect to M, one finds that M_0 is a critical point of $M \mapsto v_{1,M}^{(\gamma)}(u)$ if

$$v_1^{(\gamma)}(M_0) = V_1(M_0, M_0) \quad \text{or equivalently} \quad (v_1^{(\gamma)})'(M_0) = 1, \tag{15}$$

where (7) was used for the latter equivalence. To specify the nature of this critical point, we use the second derivative:

$$\left.\frac{\partial^2 v_{1,M}^{(\gamma)}(u)}{\partial M^2}\right|_{M=M_0} = \frac{\gamma}{1-\gamma}\frac{W_\delta(u)}{(W_\delta(M_0))^2} v_1^{(\gamma)}(M_0) W_\delta''(M_0). \tag{16}$$

Clearly, since $\lim_{M\to\infty} v_{1,M}^{(\gamma)}(u) = 0$ for any u, there is a point $M^\star \in [0,\infty)$ where the function $M \mapsto v_{1,M}^{(\gamma)}(u)$ reaches its global maximum.

Remark 4.1. *Note that $M \mapsto v_{1,M}^{(\gamma)}(u)$ can not have a local minimum in $[0,\infty)$. Indeed, if there existed a local minimum, then by virtue of $\lim_{M\to\infty} v_{1,M}^{(\gamma)}(u) = 0$, there would have to exist a local maximum for a larger value M. But in view of* (16) *and the strict convexity of W_δ', this can not occur.*

Similarly, we deduce that after a potential saddlepoint there can not be a local maximum.

Recall that

$$V_1'(s,s) = 1 - \frac{W_\delta(s) W_\delta''(s)}{(W_\delta'(s))^2}$$

and from Remark 3.3 that

$$\lim_{u\to\infty} v_1^{(\gamma)}(u) < \lim_{u\to\infty} V_1(u,u). \tag{17}$$

Remark 4.2. *From the above, it follows that $M \mapsto v_{1,M}^{(\gamma)}(u)$ also can not have a saddlepoint M_0 in $[0,\infty)$. Indeed, otherwise from $v_1^{(\gamma)}(M_0) = V_1(M_0, M_0)$ and $W_\delta''(M_0) = 0$, one can observe that*

$$V_1''(M_0, M_0) = \frac{-W_\delta(M_0) W_\delta'''(M_0)}{(W_\delta'(M_0))^2}$$

and $(v_1^{(\gamma)})''(M_0) = 0$. Hence, the function $s \mapsto v_1^{(\gamma)}(s) - V_1(s,s)$ reaches a local minimum value of 0 at this point M_0 (as $W_\delta'''(M_0) > 0$), implying that $v_1^{(\gamma)}$ is greater than V_1 in a neighbourhood of M_0, so that this saddlepoint would have to be followed by a maximum or another saddlepoint, which itself is excluded by the convexity of $W_\delta'(u)$.

As a consequence, Equation (15) has at most one positive solution M_0. If $V_1(0,0) \leq v_1^{(\gamma)}(0)$, then due to (17) such a solution $M_0 > 0$ exists and is the point of global maximum, i.e. $M^\star = M_0$.
If $V_1(0,0) > v_1^{(\gamma)}(0)$, then $M^\star = 0$ (i.e. tax payments start immediately), as a solution of (15), by (17), would have to be accompanied by a second one, which can not be the case.
Note that $M^\star$ is independent of the initial surplus u.

From the above discussion, we get the following final result which extends Theorem 3 in Albrecher and Hipp [1].

Theorem 4.1. *Suppose that the scale functions of X are three times differentiable and that their first derivatives are strictly convex functions. If $V_1(0,0) > v_1^{(\gamma)}(0)$, then the optimal height $M^\star$ is equal to 0. If $V_1(0,0) \leq v_1^{(\gamma)}(0)$, then the optimal height $M^\star$ is the unique positive solution of Equation (15). The maximum value is thus given by*

$$v_{1,M^\star}^{(\gamma)}(u) = \begin{cases} V_1(u, M^\star), & \text{if } u < M^\star; \\ v_1^{(\gamma)}(u), & \text{if } u \geq M^\star. \end{cases} \tag{18}$$

Proof. If $u < M^\star$, then

$$v_{1,M^\star}^{(\gamma)}(u) = \frac{W_\delta(u)}{W_\delta(M^\star)} v_1^{(\gamma)}(M^\star) = \frac{V_1(u, M^\star)}{V_1(M^\star, M^\star)} v_1^{(\gamma)}(M^\star) = V_1(u, M^\star).$$

Otherwise, we start to pay taxes right away and $v_{1,M^\star}^{(\gamma)}(u) = v_1^{(\gamma)}(u)$. □

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