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RICHNESS AND COMPLEXITY OF TEACHING DIVISION: PROSPECTIVE ELEMENTARY TEACHERS' ROLEPLAYING ON A DIVISION WITH REMAINDER

Caroline Lajoie & **Jean-François Maheux**

Université du Québec à Montréal

Division, one of the most difficult topics to teach in elementary mathematics, is rarely examined from the perspective of student teachers engaging with peers and making use of mathematical or pedagogical knowledge in teacher-pupil interaction. In this paper, we examine prospective elementary teachers preparing, realizing and discussing a role-play on division with remainder. Using the concept of "*structure of attention*"*, we offer a positive understanding of prospective teachers' performances in terms of navigating the richness and complexity of teaching division.*

INTRODUCTION

Division is known as one of the most difficult topics to teach in elementary mathematics (e.g. Salama, 1981; Silver *et al*., 1993). Thus, many studies examine preservice teachers' understandings of division (e.g Graeber & Tirosh, 1988; Harel, Behr, Post & Lesh, 1989; Ball, 1990; Simon, 1993; Campbell, 2002; Crespo & Nicol, 2006). Globally, researchers characterize those understandings in terms of actual or expected mathematical knowledge, stress on student teachers' difficulties, and occasionally offer alternatives to improve or extend their understandings of division and standard algorithms. However, those researches rarely examine student teachers as they engage with peers or make use of mathematical knowledge of division to teach (Simon, 1993).

Role-play involves staging a problematic situation with characters taking roles. It may be used to fulfill various objectives such as therapeutic objectives, personal and professional training objectives, or may be used as a pedagogical method (Mucchielli, 1983). The premise of role-play is to have persons, such as students, become active characters in a given situation. In one of our "mathematics method" courses, for example, students take the part of a teacher while others act as students, and they improvise around a mathematical task, a students' question or production, the use of teaching material, and so on (Lajoie & Pallascio, 2001; Lajoie, 2010). Since role-play asks student teachers to become active actors in different teaching situations, instead of simply imagining or analysing such situations, it provides an approach to research on mathematics teacher education that matches both of Simon (1993) demands.

In this paper, we report observations we made on a lesson in which student teachers prepared, performed and discussed a role-play on division with remainder. Bringing forth the richness and complexity of teaching division, we discuss how prospective teachers' *structure of attention* (Mason & Spence, 1993; Mason, 2003) played out in their actual use of resources such as mathematical knowledge, contextualization, or models such as the partitive and quotitive orientation to division.

BACKGROUND

Division with remainder presents a real mathematical challenge for prospective teachers. The division algorithm involves many arithmetical concepts, including positional, decimal numeral system, associativity, multiplication tables; while the remainder in itself opens to deeper questioning in terms of fractions, periodicity and infinite operations, and so on. In a somehow deficit perspective, previous research on the topic in the context of teacher education, giving attention to what might be missing rather than to the richness and potentiality of what is actually taking place, suggests that preservice teachers hardly make use of these concepts when confronted with division and its standard algorithm. More specifically, a gap is seen to exist between the conceptual and procedural levels (Silver, 1986) whereas:

> (…) the prospective teachers' conceptual knowledge was weak in a number of areas including the conceptual underpinnings of familiar algorithms, the relationship between partitive and quotitive division, the relationship between symbolic division and real-world problems, and identification of the units of quantities encountered in division computations (Simon, 1993, p. 233).

In this quotation, Simon also refers to two "primitive" models of division, the *partitive* (sharing) and the *quotitive* (grouping or measuring) models. As a way to student teachers' understanding of division, Fischbein *et al*. (1985) use these models to observe that even with a "solid formal-algorithmic training", students continue to be influenced by these intuitive models. Attachment to the models could explain a number of difficulties arising in unfamiliar problems, such as division by zero (Lajoie & Mura, 1998), because those models do not always constitute a solid conceptual basis (Simon, 1993). In addition, Kaput (1986) suggested an important mismatch between the dominant partitive experience of division and the quotitive approach generally used to teach division algorithms. Boulet (1998) confirmed this tendency in preservice teachers, noting that while the sharing model largely dominates when making sense of a division in real context, the measurement model clearly takes on when verbalising the algorithm. Finally, research pointed to student teachers' struggle to interpret remainders or the fractional part of the quotient (Silver, 1986; Simon, 1993). With a closer attention to teaching situations, some highlighted multiple possible treatments of the remainder, and turned our attention to the importance of being able to deal with children's ideas while making connections between the concrete, the symbolic and the algorithmic dimensions of the operation (Fang, Lee & Yang, 2012). Campbell (2002) observed that a quotitive disposition toward division also seems to impact future teachers' ability to make sense of the remainder, while confusions between the remainder and the quotient of a division may also appear when student teachers are confronted with unfamiliar tasks (e.g.

reconstituting the remainder from a calculator). As we can see, literature on the topic is rich in conceptual analysis in relation with division. But less is said about the ability actually to call on, articulate and make us of these underlying concepts and rooting metaphors.

ELEMENTS OF A CONCEPTUAL FRAMEWORK

Teaching involves various kinds of knowing (e.g. Shulman, 1987), among which "pedagogical" and "content knowledge" attracted much attention in the community. But beside knowledge "about" teaching and mathematical concepts, a growing attention is given to what some call "know-how" or "knowing-to act in the moment" (Mason et Spence, 1999): the ability to draw on various knowledge in response to actual situations. Mason and Spence (1999) suggested that such ability "depends on the structure of attention in the moment, depends of what one is aware of" (p. 135). "Structure of attention" is concerned with what is attended to and how (Mason, 2003), and its development can be conceptualized at two levels: learners need to gain access to increasingly sophisticated structures of awareness and attain greater flexibility to allow their attention to be multiply structured. Experiencing situations in which a rich network of triggers and connections come about and can be rendered explicit is considered fundamental to this development.

> Educating this awareness is most effectively done by labelling experiences in which powers have been exhibited, and developing a rich network of connections and triggers so that actions 'come to mind'. No-one can act if they are unaware of a possibility to act; no-one can act unless they have an act to perform (Mason & Spence, 1999, p. 135).

Hence, it is important for the students to be in the presence of someone, like an instructor, who is aware of the awarenesses (Mason & Spence, 1999).

> Knowing is not a simple matter of accumulation. It is rather a state of awareness, of preparedness to see in the moment. That is why it is so vital for students to have the opportunity to be in the presence of someone who is aware of the awareness that constitute their mathematical 'seeing' (Mason & Spence, 1999, p. 151).

As an element of a conceptual framework underdevelopment in order to understand the competencies future teachers develop in "maths method courses" (e.g. Lajoie, 2010; Maheux & Lajoie, 2011), the concept of "structure of attention" enables us to look into their ability to make use of various knowledge. In this paper, we exemplify this and the potential outcome of such approach in a case on division with remainder.

METHOD (RESEARCH CONTEXT AND DESIGN)

As part of our research program on teacher education, we decided to investigate division with remainder in a study involving 40 preservice teachers enrolled in a 45 hours undergraduate course (3h weekly) on *Didactique de l'arithmétique au primaire* [primary arithmetic maths method]. No selection of the students was conducted, beside a general appreciation of the group (as whole) as typical of those we worked with for the last decade. Students were in their second of a four-year program, and had completed a first mathematics course. The course used for this study was designed around ten different *role-plays* (Lajoie, 2010) on various topics including numeration, operations and algorithms, fractions and decimal numbers.

Each *role-play* is organized in four moments. First, we introduce the 'theme' on which students will need to improvise (introduction time). Students then have about 30 minutes to prepare in small groups (preparation time). Third comes the play itself, where students chosen by the instructor (and coming from different preparation teams), come in front of the classroom and improvise, in an informed way (Maheux & Lajoie, 2011), a teacher-pupil interaction (play time). Finally, we have a whole classroom discussion (discussion time). Importantly, students thus have a preparation time to consider what might happen between a teacher and a pupil, but the role-play is essentially improvisational since there is no script and students learn only minutes before the play if they will be performing that day, and what role they have to take. We designed, taught (first author) and videotaped (second author) the three hours role-plays using two cameras (one capturing the whole classroom setting and interactions, the other following some teams more specifically). In this paper, we will focus on the role-play presented in Figure 1, which was also transcribed.

Role play theme

Fifth grade students (10-11 years old) who worked on a problem involving division with a remainder are sharing their answers. You observe mathematical difficulties in their approaches. You wish to engage with them in such a way that you can build up on their mishaps and help them move toward correct answers. Here is the problem they have been working on:

> A student from the other class made a mistake while dividing 18181 by 9. Here is his answer. Can you help find his error?
>
> 18181 ÷ 9 = 22,111 remain 1.

Your students worked in team, and among them Team A and B arrived at the following explanations:

- Team A: The answer is 2020 remain 1/18181
- Team B: The answer is 2020 remain 0,111

Prepare yourselves to either play the role of a student from Team A or B, or that of the teacher who wants to help them starting from their reasoning.

Figure 1

That particular lesson, involving division with a remainder, took place midway into the term (when the students felt comfortable with role-play). As homework, the week before, they had to read Boulet's (1998) paper on the verbalisation of the division algorithm in terms of partitive or quotitive models, which were also discussed in class. The role-play (Figure 1) was designed to bring about multiple mathematical concepts potentially involved in (long hand) division with a remainder. This includes the fact that no context was provided for the division itself, so that preservice teachers would not be directed towards a partitive or quotitive model. To collect data, the moving camcorder mostly follows a team of four, part of which Justine will end up playing the role of the teacher. We thus focus on the interactions taking place in that team (during preparation time), on the play itself (play time), and on the whole group discussions (discussion time).

ANALYSIS

During preparation time, Justine's team quickly engaged in figuring out the correct mathematical answer to the division problem under investigation, and then turned to what should be done in regard with Team A's solution:

> *Cindy*. We could ask them what they did, and they may answer that they have 1 remaining unit over the 18181. But that's not really it, because the unit, you have to divide it in 9, but you can't divide it in 9. (...)
>
> *Bella*. When.. if you only look at this… it is part of the 18181… from the moment you decide to divide by 9… actually, when you see it written here it is still a unit part of your 18181, but when you write the result of the division, you can decide that it will stay as a unit, or you decide to write it as a remainder of 1 that you are going to divide by 9.
>
> *Justine*. If you write 1, its because you decide not to divide it. You keep the whole unit.

Attending to the operation as presented, the students here stay in the symbolic domain, focusing on what appears as problematic in Team A's answer: their writing of the remainder. Inasmuch, students in this discussion did not *explicitly* make use of the sharing/partitive or grouping/quotitive models despite the classroom discussion that proceeded. More so, they discuss the problem indistinctively using the "dividing in" and "dividing by" expressions which might refer to two different views on the division and its remainder. That is, as we can anticipate, exposure to the models through literature or classroom discussion does not spontaneously translate into resources for designing a teaching intervention.

When prompted by the second author (holding the camcorder), who asked them to explain what the remainder could represent, the students moved from the symbolic to the contextual level. They gave flesh to the problem in terms of "sharing candies between 9 children", clearly leaning toward a partitive orientation. If this contextualisation seemed to help them to confirm a relationship between the dividend (18181) and the remainder (1), Justine's team, however, did not anticipate the challenge of drawing on this model to verbalize the actual division. Nor did they consider where it would lead them, were they to continue the division.

As a result, during play time, when the role-play took place and Dominique – the student teacher chosen to play the pupil – tried using a similar context to explain her division, Justine did not react to her problematic verbalisation of the algorithm:

> *Dominique*. So I want to divide 1 candy between 9 friends, but I can't [writes 0 in the quotient]. So I'll take the 8, and now I have 18 candies to split between 9. Each will have 2. Now I subtract and I take the 1. Now if I want to divide it again between 9… well I place a 0 here [in the quotient] and I'll put my 1 there. Now again I take the 8, 18 candies between 9 friends gives them 2 each. 18-18 is 0. I have 1 left to take. If I want to divide it

> between 9 friends it doesn't work [adds another 0 to the quotient, now 02020]. I have 1 candy left over the 18181, so 1/18181 [writes it next to the quotient and add "remain" between the two].
>
> *Justine*. Ok, so what you are telling me… If we draw this like… Wait! Actually, what you are telling me is that I'm left with 2020… no, 2020 remaining 1 over the whole 18181. So your reasoning is good, but we'll illustrate that another way. […] Say you have a chocolate bar left. What did you do all the way through? You were dividing by?

Focusing on the remainder, she did not pick up on Dominique successively treating all "1" and "8" as units (she calls the first 1 in <u>1</u>8181 "one candy" instead of 1 group of 10000 candies, etc.). That is, she did not make use of the model, the context, nor the mathematical concepts at play to notice and act upon the problematic, highly procedural verbalisation of the algorithm. Justine responded to the student in the way she explored the problem in her group, the structure of her attention (to use Mason and Spence (1999)' expression for the condition of knowing-to act in the moment) being directed toward the problem of the remainder.

On the other hand, the context and partitive orientation to the problem are made use of when Justine guides Dominique to give meaning to 1 as a remainder. But when the time comes to consider 1/9 in relation with the quotient and what was previously the remainder, Justine's attention is all to the possibility of dividing this remainder which was at the center of her team's preparation. Hence, she does not pick up on Dominique (and her own!) confusion, calling 1/9 the remainder:

> *Dominique*. So I have a chocolate bar and I divide it in 9 equal parts [drawing a 3x3 rectangle].
> *Justine*. Yeah. I have… I have one left, one part … over nine. (…)
> *Dominique*. Before I had 1/18181 but now I'm at 1/9. A remainder of 1/9.
> *Justine*. So your remainder is 1 on 9, and could I… You agree that we divided the chocolate bar we had left in 9, so its like 1 divided by 9. (…)

This is particularly interesting because Justine actually faced the very same problem during preparation time, some time *before* they contextualized the problem. When the fraction 1/9 first came about, the second author, observing the team, asked Justine if 1/9 was the remainder. After a hesitation, she worked it out on paper and concluded that this was not the case: "there's nothing left, no remainder".

The question of the remainder came back minutes later, after Justine and her teammates move to Team B's solution ("2020 remain 0,111"). But first, we can see how confrontation with the decimal development took them to a procedural approach:

> *Justine*: Here the students say that they have a remainder of 1, and because they cannot divide it anymore, they add a period because they are no more in the integer, they are in the decimal. They say "I add a zero, it give us one tenths. Like 10, I divide it by 9, it fits in only once."

In contrast, the first part of their preparation, with its concerns about the relations between the "1" and the "18181", now appears quite conceptual! We also see students moving back to the symbolic domain, trying to make sense of the division in algorithmic terms. And although they had just worked out a context (sharing candies) that could give meaning to their work on the decimals part (turning the remaining unit into ten tenth to be shared among 9 friends), they did not use it. And when asked to do so, Bella explained "You divide the candy in 9, and there is a little bit left". This curious formulation (was the candy divided in 9 or in 10?), rephrased for her by the camera-holder as "divide it into *10*, what you have left is 0,1", then led to an actual explanation of the residue: "You are always left with something, but maybe its one tenth, maybe one thousandth…".

This brings to light not only the difficult task to linking the algorithm with a concrete situation, but also the particular challenge of articulating the remainder, the fractional part of the quotient, its decimal expression, and presence of a "remaining part" yet to be divided. This complexity was at the core of the role-play setup we presented to the students (during introduction time), involving many of the possible confusions, including dividing by 9 and dividing in 10 (so that sharing or grouping in 9 becomes possible). And indeed, when Justine plays with Dominique, she soon comes to a point where she asks her "pupil" to transform the fractional part of the quotient (1/9) in its decimal expression. As Dominique struggles, Justine offers to try and divide 100 by 90 instead: "you have bigger numbers but it's the same". Dominique then finds her way to mechanically proceed to the division (finding "1.11"), which Justine uses to emphasise the infinite development of the decimal part. Linking it with the answer, Dominique as a pupil was supposed to be answering to ("18181 ÷ 9 = 22,111 remain 1", see Figure 1), Justine offers an interpretation of the "remain 1". As she explains to Dominique, "there is still something to divide", but "we don't have a chocolate bar anymore": the decimal part is thus something already divided, it belongs with the quotient (in contrast with Team B's answer). After what she concludes to Dominique's original division: "it's the same here."

Along the conversation, Justine thus again, passes up the algorithm to keep focus on the decimal development of 100 ÷ 90, so much so that she also cuts through the fact that its quotient is actually ten times bigger than the one they were initially looking for. This could be analysed in terms of a lack of mathematical understanding, or as an effect of insufficient understanding of the division, its algorithm, of the pedagogical importance of making sense of computations, and so on. However, we must appreciate Justine's consistency in attending to what emerges from the task at hand in the moment. The sensitivity to notice, the instinct to call upon, and the flexibility to make use while keeping track of one's intention: those are complex, intricate dimension of attending that can only slowly develop. It seems possible, however, to correlate this development with the very design of the situations student teachers are presented with. Here, a task in which they were confronted with "extremes" in terms of possible interpretation of the remainder of a division, took them to an extended sense of the relations at play. During discussion time, when asked how they felt about the role-played interaction, students said “it was very good”, but agreed (Justine included) that there was something missing in the part where she began dealing with the 1/9 and the decimal part.

Summing up, we could say that Justine made use of mathematical understanding, the contextualisation and the intuitive models she actively experienced in preparation time, and also *as* they appeared in that moment. The problem as presented to the students (Figure 1) also seems to have largely oriented the students’ preparation. So much so that it rendered invisible even what might appear as obvious aspects to consider (contextualization of the problem, verbalisation of the algorithm, interpretation of fractional part of the quotient), all elements the students discussed previously in whole class discussions and read about in the article we gave them.

CONCLUSION

The analysis we conducted in this study allowed us to consider student teachers' ability to deal with the richness and complexity of the difficult subject in a positive way. That is, we came across the *multiple challenges* (prospective) teachers are presented with when preparing or actualizing teacher-pupils-like interactions: making use of models, contextualization and mathematical knowledge, making sense the remainder and navigating through various forms of expressing the fractional part of the quotient, or attending to and verbalising the algorithm.

We think this is an important nuance when comparing with, for example, Simon's (1993) analysis stating that prospective teachers “do not make use of their concrete conceptual knowledge” to make sense of division computation and rather seem to “search for meaning within a narrow procedural space, reiterating rules for the procedure” (Simon, 1993, p. 248). As we see it, student teachers work with/in situations that strongly contribute to a structuring of their attention, which leads to the observed behaviour. An interesting question would then be: how can mathematical understandings, contextualisation and models be made part of the situation so that student teachers actively use them in preparing or realizing an intervention? Giving the usual conditions of 3 hours x 15 weeks maths method courses, how readily available could those elements be so they play out as actual resources for students? How to ensure, in doing so, that those structuring resources do not become *too* constraining, leading student teachers to a technical view on preparing and performing mathematics instructional interactions?

It may feel easy to answer that student teachers should be *explicitly requested* to use various resources when they engage in preparation and the role-play. Although this might be the case, we find interesting to contrast what we observed in this study with similar claims. In a research design that used images and stories of real classroom teaching to help prospective teachers develop and practice the problem solving and decision-making skills they need, Crespo and Nicol (2006, p. 93) note:

> Although we acknowledge that this issue relates to the wording in this task, the preservice teachers' lack of mathematical exploration before designing a pedagogical response is concerning. In the context of teaching practice, mathematical challenges arise in much the same way as in this first version of the task; that is with no invitation or prompt to first explore and clarify one's own mathematical understanding.

In our case, although nothing was said about mathematically exploring the task, we could see how Justine's team is challenged right from the beginning of preparation time to develop an understanding of division with remainder. That is, before agreeing on a pedagogical response, the team took time to explore the situation mathematically. We suspect that the *goal* presented in the task (be ready to improvise on the theme, actually engaging with a "pupil") made the difference here. In other words, structure of attention may not only be only global awareness to the possibilities a situation offers: it is also motivated, purposeful, and probably goal *orienting* as well (situat*ing* as much as situat*ed*). If this is the case, greater attention might be needed to the understanding of how prospective teachers are supported in the process. To put it bluntly, on this one aspect illustratively, even if mathematical exploration went on, this was clearly not enough to resolve all the mathematical confusions that came up. More so, these confusions arose even though our students already took a mathematics course focused on elementary mathematics the previous year. And actually, we could feel that, at the end of the lesson, many of our students where left with more *new* questions than new answers. Now, we wonder, might not this be the result not of a mere lack of mathematical understanding, but of the expansion of those students' awareness of what is at stake? And may not this increasing awareness, fundamental to knowing to act in the moment, be the decisive outcome of teacher education? Another question being then: how can we provide, support, help students gaining means and confidence to navigate their queries?

REFERENCES

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, *21*, 132-144.

Boulet, G. (1998). La nature dichotomique de la division: une analyse didactique. *Bulletin de l'AMQ*, *38*(2), 14-22.

Campbell, S. R. (2002). Coming to Terms with Division: Preservice Teachers' Understanding. In S. R. Campbell & R. Zazkis (Eds.), *Learning and Teaching Number Theory*. Westport : Ablex Publishing.

Crespo, S., & Nicol, C. (2006). Challenging preservice teachers' mathematical understanding: The case of division by zero. *School Science and Mathematics*, *106*(2), 84-97.

Fang, Y., Lee, C. K.E. & Yang, Y. (2012). Developing curriculum and pedagogical resources for teacher learning: A lesson study video case of "Division with Remainder" from Singapore. *International Journal for Lesson and Learning Studies*, *1*, 65 - 84.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving problems in multiplication and division. *Journal for Research in Mathematics Education*, *16*, 3-17.

Graeber, A., & Tirosh, D. (1988). Multiplication and division involving decimals: Preservice elementary teachers' performance and beliefs. *Journal of Mathematical Behavior*, *7*, 263-280.

Harel, G., Behr, M., Post, T., & Lesh, R. (1989). Fischbein's theory: A further consideration. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the Thirteenth Psychology of Mathematics Education Conference* (vol. 2, pp. 52-59). Paris, France.

Kaput, J. (1986). *Word Problems Project, ETC Technical Report*. Cambridge, MA: Education-al Technology Center.

Lajoie, C. (2010). Les jeux de rôles : une place de choix dans la formation des maîtres du primaire en mathématiques à l'UQAM. In Proulx, J. & L. Gattuso (Eds.), *Formation des enseignants en mathématiques* (pp. 101-113). Sherbrooke : Éditions du CRP.

Lajoie, C. & Mura, R. (1998). The danger of being overly attached to the concrete : The case of division by zero. *Nordic Studies in Mathematics Education, 6*(1), 7-21.

Lajoie, C. & Pallascio, R. (2001). Role-play by pre-service elementary teachers as a means to develop professional competencies in teaching mathematics. *Proceedings of SEMT '01 - International Symposium Elementary Mathematics Teaching*. Prague, Czech Republic: Charles University.

Maheux, J.-F. &. Lajoie, C. (2011). On Improvisation in Teaching and Teacher Education. *Complicity*, *8*(2), 86-92.

Mason, J. (2003). On the structure of attention in the learning of mathematics. *Australian Mathematics Teacher, 59*(4), 17-2.

Mason, J. & Spence, M. (1999). Beyond Mere Knowledge of Mathematics : The Importance of Knowing to act in the moment. *Educational Studies in Mathematics*, *38*, 135–161.

Mucchielli, A. (1983) *Les jeux de rôles.* Paris: Presses universitaires de France, Que sais-je ? 126 pages.

Salama, H.A. (1981). *The Effect of the Place-value Method of Teaching Long Division upon the Teaching Ability of Prospective Elementary Teachers*. The Florida State University, Tallahassee, FL.

Silver, E. (1986). Using conceptual and procedural knowledge: A focus on relationships. In J. Hiebert (Ed.), *Conceptual and procedural knowledge* (pp. 181-198). Hillsdale, NJ: Lawrence Erlbaum Associates.

Shulman, L. (1987). Knowledge and Teaching: Foundations of the New Reform. Harvard Educational Review, *57* (1), 1–22.

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24*, 233-54.