UNIVERSITÉ DU QUÉBEC À MONTRÉAL

BILANS D'ÉNERGIE DE SIMULATIONS À L'ÉCHELLE RÉGIONALE AVEC LE MRCC5 : ANALYSE D'UN CYCLE D'ÉNERGIE CLIMATOLOGIQUE ET D'UN CYCLE D'ÉNERGIE DE TEMPÊTE

MÉMOIRE

PRÉSENTÉ

COMME EXIGENCE PARTIELLE

DE LA MAÎTRISE EN SCIENCES DE L'ATMOSPHÈRE

PAR

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NOVEMBRE 2015

REMERCIEMENTS

Mes plus sincères remerciements à mes deux directeurs de recherche, René Laprise et Oumarou Nikiéma; merci pour tout l'investissement de temps que vous avez mis avec moi dans ce projet, pour votre disponibilité, vos idées, vos solutions à tous mes problèmes.

Aux collègues du PK-6515, merci pour tous ces moments sérieux et intelligents passés ensembles.

Pour finir, un merci spécial à ces personnes qui n'ont aucune idée de quoi traite ce mémoire, mais sans qui il n'aurait peut-être pas été achevé : Olivier Savaria, Gabrielle Lalonde, Sylvie Robert, André Jr Clément et Julien Clément.

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LISTE DES ABRÉVIATIONS, SIGLES ET ACRONYMES

AE	Enthalpie disponible
APE	Énergie potentielle disponible
CRCM5	5[e] génération du modèle régional canadien du climat (MRCC5)
E	Type généralisé d'énergie atmosphérique
EAPE	Énergie potentielle disponible transitoire
EKE	Énergie cinétique transitoire
GCM	Modèle de circulation global (MCG)
IE	Énergie interne
IV	Variabilité inter-membre
KE	Énergie cinétique
PE	Énergie potentielle
RCM	Modèle régional du climat (MRC)
TPE	Énergie potentielle totale
UPE	Énergie potentielle non-disponible
ZAPE	Énergie potentielle disponible zonale

ZKE	Énergie cinétique zonale

LISTE DES SYMBOLES

a_{TV} Enthalpie disponible instantanée de variabilité temporelle

c_a Conversion instantanée d'enthalpie disponible de moyenne temporelle en enthalpie disponible de variabilité temporelle

c_k Conversion instantanée d'énergie cinétique de moyenne temporelle en énergie cinétique de variabilité temporelle

c_p Constante de chaleur spécifique à pression constante pour l'air sec

c_{TV} Conversion instantanée d'enthalpie disponible de variabilité temporelle en énergie cinétique de variabilité temporelle

d_{TV} Dissipation instantanée d'énergie cinétique de variabilité temporelle

$f_{a_{TV}}$ Transport instantané d'enthalpie disponible de variabilité temporelle

$f_{k_{TV}}$ Transport instantané d'énergie cinétique de variabilité temporelle

g_{TV} Production instantanée d'enthalpie disponible de variabilité temporelle

$h_{a_{TV}}$ Flux aux frontières instantané d'enthalpie disponible de variabilité temporelle

$h_{k_{TV}}$ Flux aux frontières instantané d'énergie cinétique de variabilité temporelle

j_{a1} Flux 1 instantané d'enthalpie disponible de variabilité temporelle

j_{a2} Flux 2 instantané d'enthalpie disponible de variabilité temporelle

j_{k1} Flux 1 instantané d'énergie cinétique de variabilité temporelle

k_{TV} Énergie cinétique instantanée de variabilité temporelle

p Pression

p_{00} Pression de surface moyennée sur le domaine

p_r Pression de référence

t Échantillons de temps utilisés pour les calculs

A Enthalpie disponible due à la température

A_{TM} Enthalpie disponible de moyenne temporelle

$A_{TM\,B}$ Composante barocline de l'enthalpie disponible de moyenne temporelle

$A_{TM\,S}$ Composante de stratification de l'enthalpie disponible de moyenne temporelle

A_{TV} Enthalpie disponible de variabilité temporelle

B Enthalpie disponible due à la pression

$C_{A\,B}$ Conversion de la composante barocline d'enthalpie disponible de moyenne temporelle en enthalpie disponible de variabilité temporelle

$C_{A\,S}$ Conversion de la composante de stratification d'enthalpie disponible de moyenne temporelle en enthalpie disponible de variabilité temporelle

C_K Conversion d'énergie cinétique de moyenne temporelle en énergie cinétique de variabilité temporelle

$C_{TM\,B}$ Conversion de la composante barocline d'enthalpie disponible de moyenne temporelle en énergie cinétique de moyenne temporelle

$C_{TM\,BS}$ Conversion de la composante de stratification d'enthalpie disponible de moyenne temporelle en composante barocline d'enthalpie disponible de moyenne temporelle

$C_{TM\,S}$ Conversion de la composante de stratification d'enthalpie disponible de moyenne temporelle en énergie cinétique de moyenne temporelle

C_{TV} Conversion d'enthalpie disponible de variabilité temporelle en énergie cinétique de variabilité temporelle

D_{TM} Dissipation d'énergie cinétique de moyenne temporelle

D_{TV} Dissipation d'énergie cinétique de variabilité temporelle

$F_{A_{TM\,B}}$ Transport de la composante barocline d'enthalpie disponible de moyenne temporelle

$F_{A_{TM\,S}}$ Transport de la composante de stratification d'enthalpie disponible de moyenne temporelle

$F_{A_{TV}}$ Transport d'enthalpie disponible de variabilité temporelle

F_B Transport d'enthalpie disponible due à la pression

$F_{K_{TM}}$ Transport d'énergie cinétique de moyenne temporelle

$F_{K_{TV}}$ Transport d'énergie cinétique de variabilité temporelle

$G_{TM\ B}$ Production de la composante barocline d'enthalpie disponible de moyenne temporelle

$G_{TM\ S}$ Production de la composante de stratification d'enthalpie disponible de moyenne temporelle

G_{TV} Production d'enthalpie disponible de variabilité temporelle

$H_{A_{TM\ B}}$ Flux aux frontières de la composante barocline d'enthalpie disponible de moyenne temporelle

$H_{A_{TM\ S}}$ Flux aux frontières de la composante de stratification d'enthalpie disponible de moyenne temporelle

$H_{A_{TV}}$ Flux aux frontières d'enthalpie disponible de variabilité temporelle

$H_{K_{TM}}$ Flux aux frontières d'énergie cinétique de moyenne temporelle

$H_{K_{TV}}$ Flux aux frontières d'énergie cinétique de variabilité temporelle

I_{AB} Conversion d'enthalpie disponible due à la pression en composante de stratification d'enthalpie disponible de moyenne temporelle

K Énergie cinétique

K_{TM} Énergie cinétique de moyenne temporelle

K_{TV} Énergie cinétique de variabilité temporelle

L_E Tendance temporelle du réservoir d'énergie E

R Constante des gaz parfaits

R_E Somme algébrique de tous les flux d'énergie agissant sur E

T Température

T_r Température de référence

$\vec{V}$ Vent horizontal

α Volume spécifique

α_r Volume spécifique de référence

τ Nombre d'échantillons de temps

ω Mouvement vertical en coordonnées de pression

Φ Hauteur géopotentielle

Ψ Type généralisé de variable atmosphérique

$\langle\Psi\rangle$ Opérateur de moyenne temporelle

Ψ' Opérateur de variabilité temporelle

$\overline{\Psi}$ Opérateur de moyenne isobare

$\Psi^{\times}$ Opérateur de déviation par rapport à la moyenne isobare

RÉSUMÉ

L'étude de l'énergétique atmosphérique permet de mieux comprendre la circulation générale dans l'atmosphère et les phénomènes qui assurent son maintien. L'énergie potentielle créée par le réchauffement différentiel de la planète par le soleil est convertie en énergie cinétique par les systèmes météorologiques puis dissipée par la friction, et ce, en continu. L'application de l'énergétique atmosphérique à un domaine à aire limitée a l'avantage de fournir de l'information d'une précision accrue sur la formation et le développement de systèmes météorologiques individuels d'une région précise.

L'ensemble d'équations développé par Nikiéma et Laprise (2013) pour le calcul d'un cycle d'énergie associé à la variabilité inter-membre basé sur l'enthalpie disponible et sur l'énergie cinétique a été ici adapté pour le calcul d'un cycle d'énergie atmosphérique. Deux analyses ont été faites dans le cadre de cette recherche : la première consiste en un cycle d'énergie climatologique et la deuxième, un cycle d'énergie d'une tempête. Ces analyses ont été réalisées avec une simulation du MRCC5 pilotée par les réanalyses pour le mois de décembre 2004 et pour une dépression ayant eu lieu durant ce même mois.

Pour le cycle d'énergie climatologique, les six réservoirs d'énergie et tous les flux d'énergie qui leurs sont associés (production, conversion, dissipation et flux aux frontières) ont été calculés. L'enthalpie disponible est principalement produite par la covariance des sources diabatiques et de la température, et l'énergie cinétique est principalement dissipée par l'effet de la friction dans la couche limite. Les importantes conversions d'énergie entre enthalpie disponible et énergie cinétique ont lieu dans la trajectoire de tempêtes, où le gradient de température et le vent sont importants et diffèrent largement de leurs états moyens. Pour le cycle d'énergie de tempête, un lien très évident a été noté entre l'évolution de la tempête et ses patrons d'énergie : l'enthalpie disponible et l'énergie cinétique suivent le gradient de température et le vent associés à la tempête. La production d'énergie se fait le long des fronts, et sa dissipation, là où les vents sont maximaux. Les termes de conversions sont directement liés à l'advection de température et aux mouvements verticaux créés par la tempête. Les deux cycles d'énergie ont été raisonnablement satisfaits.

Mots clés : cycle d'énergie, domaine à aire limitée, modèle régional du climat, climat de l'Amérique du Nord

INTRODUCTION

La circulation générale atmosphérique assure le transfert d'énergie entre les régions de basses et de hautes latitudes, gouvernant ainsi le comportement des masses d'air autour du globe. La circulation générale est principalement dictée par les lois de la thermodynamique, de la conservation de la masse et par les équations du mouvement d'Euler. Ce sont ces lois, sous forme d'équations, qui sont utilisées dans les modèles météorologiques et qui permettent, à l'aide d'observations atmosphériques, de faire des prévisions à courts, moyens et longs termes. Les observations atmosphériques proviennent de différentes variables qui caractérisent l'état de l'atmosphère : la pression atmosphérique, la température et humidité de l'air, sa vitesse horizontale et verticale, etc. L'étude de l'énergétique atmosphérique associée à la circulation générale fournit, quant à elle, de l'information fondamentale sur les transformations que l'énergie solaire incidente dans l'atmosphère subit.

Depuis les années 1950, plusieurs bilans d'énergie atmosphérique ont été calculés à l'échelle globale, c'est-à-dire pour l'ensemble de l'atmosphère, en utilisant des données d'analyse atmosphérique ou des données de simulation faite avec des modèles globaux. Margules (1905, voir également Tamura 1905), l'un des premiers à étudier l'énergétique atmosphérique, a identifié les différentes formes d'énergie présentes dans l'atmosphère : l'énergie cinétique (*KE*) associée aux vents, l'énergie potentielle gravitationnelle (*PE*) causée par la gravité et l'énergie interne (*IE*) due au mouvement brownien des molécules. Puisque lorsqu'intégrée sur une colonne d'atmosphère *IE* est directement proportionnelle à *PE*, Margules a introduit le concept d'énergie potentielle totale (*TPE*) correspondant à la somme de *IE* et *PE*. Le calcul d'un bilan énergétique quantitatif a cependant posé problème étant donné que dans l'atmosphère, les valeurs de *TPE* sont beaucoup plus importantes que celles de *KE*.

Lorenz (1955, 1967) a été le premier à développer un cycle d'énergie atmosphérique détaillé. Encore aujourd'hui, ses travaux sont considérés comme le classique de l'énergétique atmosphérique. Il a introduit un nouveau concept d'énergie : il a séparé la *TPE* en énergie potentielle indisponible (*UPE*) et en énergie potentielle disponible (*APE*). *UPE* correspond à l'énergie potentielle associée à un état atmosphérique de référence et *APE* correspond à l'énergie potentielle résultant de la différence entre *TPE* et *UPE*, donc à la déviation par rapport à cet état de référence. L'état de référence est un concept théorique qui correspond par définition à une redistribution adiabatique de la masse atmosphérique résultant en une même stratification horizontale de température potentielle sur tout le globe. L'état de référence ne permet donc aucune transformation de *UPE* en *KE*, car une stratification horizontale de température potentielle ne génère aucun gradient de pression, et donc aucun mouvement atmosphérique. Ainsi, c'est seulement la composante *APE* de *TPE* qui peut être transformée en *KE*. De plus, *APE* et *KE* sont du même ordre de grandeur. Lorenz a donc développé son cycle d'énergie basé sur *APE* et *KE*; ces deux réservoirs d'énergie étant plus pertinents pour traiter et comprendre l'énergétique l'atmosphérique.

Son cycle d'énergie se résume ainsi : le réchauffement différentiel de la planète par le soleil créé un gradient de température latitudinal qui génère *APE*. *APE* est ensuite convertie en *KE* par les systèmes météorologiques qui se développent et transportent la chaleur des basses vers les hautes latitudes, réduisant ainsi le gradient latitudinal de température. Finalement, *KE* est détruite par la friction. Ce cycle est continuel, car le gradient de température est constamment recréé par le soleil.

Afin de mieux comprendre le rôle des perturbations météorologiques dans l'énergétique atmosphérique, Lorenz a décomposé ses réservoirs en deux composantes, la première associée à la moyenne zonale et la seconde, aux déviations par rapport à cette moyenne. Séparer ainsi les champs d'énergie lui a permis d'obtenir des expressions d'énergie quadratiques et positives, ce qui a simplifié énormément l'interprétation physique des résultats. L'énergie potentielle disponible zonale (*ZAPE*) représente *APE* du champ moyen de température zonale, c'est-à-dire chaud aux basses latitudes et froid aux hautes latitudes. L'énergie cinétique zonale (*ZKE*) représente *KE* associée à la circulation moyenne zonale, c'est-à-dire des vents dominants tels que le courant-jet aux latitudes moyennes. L'énergie potentielle disponible de perturbation (*EAPE*) représente *APE* associée aux perturbations de température par rapport à *ZAPE*, c'est-à-dire la variance de la température sur un cercle de même latitude. L'énergie cinétique de perturbation (*EKE*) représente *KE* associée aux perturbations de vents par rapport à *ZKE*, c'est-à-dire la variance du vent sur un cercle de même latitude.

La vaste majorité de nos connaissances actuelles sur l'énergétique atmosphérique globale provient des travaux de Lorenz et de ceux qui se sont inspirés de son formalisme (ex : Oort 1964 a, b, Newell et al. 1972, 1974, Boer 1974, Peixoto and Oort 1992, etc.). Il est tout de même important de noter que plusieurs limitations sont associées à l'approche de Lorenz, principalement liées au fait que ses équations sont restreintes à une application globale et non régionale. En effet, afin de simplifier la définition de son état de référence, des approximations ont été introduites pour le calcul de *APE* qui ne sont valides que pour la totalité du globe. Ses concepts de *TPE*, *UPE* et *APE* ne sont donc applicables qu'à la totalité de l'atmosphère, et non à une zone restreinte, tel qu'un domaine à aire limitée (van Mieghem 1973, section 14.8). Pour établir *APE*, plusieurs manipulations ont requis une intégration sur la totalité de l'atmosphère, annulant conséquemment plusieurs termes par identités mathématiques : termes de divergence de flux, termes associés au mouvement vertical, etc. Or, sur un domaine à aire limitée, ces termes peuvent être très importants, car la masse n'est pas conservée. Ainsi, les calculs énergétiques sur des régions à aires limitées sont beaucoup plus complexes et difficiles à établir (ex : Spar 1950, Smith 1969 a, b).

Malgré ces difficultés, les avancées récentes d'analyses atmosphériques à haute résolution et de simulations régionales ont facilité l'accès aux données requises et ainsi motivé la recherche sur les bilans énergétiques régionaux. Avec une résolution de l'ordre de la dizaine de kilomètres, de tels bilans ont la possibilité de représenter avec une précision accrue l'état de l'atmosphère et ainsi d'augmenter la compréhension des processus physiques responsables de la formation et du développement de systèmes météorologiques individuels propres à une région précise.

Pearce (1978) a développé une approche distincte de celle de Lorenz en redéfinissant *APE* comme étant un écart par rapport à un état de référence atmosphérique isotherme. Ainsi, son *APE* est une fonction directe de la variance de la température, ce qui en facilite le calcul et l'interprétation physique. Comme Lorenz, Pearce a décomposé ses champs de *APE* et de *KE* en moyennes zonales et en perturbations. Il a cependant introduit une troisième composante à *APE*, qui est fonction de la distribution verticale de la température, permettant de tenir compte des changements de stabilité statique de l'atmosphère. Suite aux travaux de Pearce, Marquet (2003 a, b) a introduit un cycle d'énergie exact, c'est-à-dire sans aucune approximation mathématique, applicable à l'échelle régionale, mais pouvant également être appliqué à l'échelle globale. Sa formulation est basée sur *KE* et sur l'enthalpie disponible (*AE*) qui se sépare naturellement en contribution de température et de pression, permettant à son cycle d'être calculé pour n'importe quel niveau de pression voulu, tout en conservant l'avantage d'être une fonction directe de la température. Récemment, Nikiéma et Laprise (2013), noté « NL13 » pour la suite du mémoire, ont utilisé une approche similaire pour caractériser les fluctuations de la variabilité inter-membre dans des ensembles de simulations d'un model régional. Ils ont exploité le proche parallèle existant entre les conversions d'énergie inter-membre et les conversions d'énergie des systèmes météorologiques. Ils ont développé un cycle d'énergie approximatif où ils ont décomposé leurs champs en moyennes d'ensemble et en variabilité inter-membre.

Le but du projet présenté dans ce mémoire est premièrement de développer un cycle d'énergie atmosphérique applicable localement en utilisant une variante du formalisme de Pearce (1978) et Marquet (2003 a) puis exploité par NL13, et deuxièmement, d'utiliser ce cycle d'énergie pour calculer le bilan énergétique régional d'un domaine nord-américain. L'enthalpie disponible (*AE*) et l'énergie cinétique (ici notée *K*) seront décomposées en moyenne temporelle et en variabilité temporelle. Un bilan d'énergie climatologique sera calculé à partir d'une simulation de la 5^{è} génération du modèle régional canadien du climat (MRCC5) pilotée par les réanalyses pour le mois de décembre 2004 et un bilan d'énergie de tempête sera calculé pour une dépression ayant lieu durant ce même mois.

Le mémoire est organisé comme suit. Le chapitre 1 contient un article scientifique rédigé en anglais. La section 1.1 va présenter le contexte scientifique et les motivations. La section 1.2 va décrire la méthodologie utilisée pour calculer le cycle d'énergie climatologique. La section 1.3 va traiter du cadre expérimental et du climat du mois de décembre 2004. La section 1.4 va présenter les résultats et leur analyse pour le cycle d'énergie climatologique. La section 1.5 va décrire la méthodologie utilisée pour le cycle d'énergie de tempête et la section 1.6 va en présenter les résultats et leur analyse. La section 1.7 va résumer les résultats et conclure, ce qui va clore l'article scientifique en anglais. Une conclusion rédigée en français va suivre et faire le bilan des résultats principaux et des travaux futurs. Finalement, les appendices et figures seront respectivement présentées à la toute fin du mémoire.

CHAPITRE I

BILANS D'ÉNERGIE DE SIMULATIONS À L'ÉCHELLE RÉGIONALE AVEC LE MRCC5 : ANALYSE D'UN CYCLE D'ÉNERGIE CLIMATOLOGIQUE ET D'UN CYCLE D'ÉNERGIE DE TEMPÊTE

Ce chapitre, rédigé sous forme d'un article en anglais, présente deux analyses réalisées avec un cycle d'énergie applicable localement : un bilan d'énergie climatologique et un bilan d'énergie de tempête. La section 1.1 présente le contexte scientifique et les motivations justifiant cette recherche. Les sections 1.2 et 1.3 présentent la méthodologie employée pour la première analyse et les données utilisées, puis la section 1.4 en présente les résultats. Les sections 1.5 et 1.6 présentent respectivement la méthodologie et les résultats pour la deuxième analyse. La section 1.7 fait le retour sur les points importants et conclut.

Limited-Area Energy Budgets with the CRCM5: Analysis of an Energy Cycle Climatology and of a Storm Energy Cycle

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Abstract

The general circulation in the atmosphere and its maintenance can be studied by computing atmospheric energy cycles. The potential energy, generated by the differential heating of the planet by the Sun, is converted into kinetic energy by the weather systems and then dissipated by friction. The Sun replenishes constantly the planet's temperature gradient, making this cycle continual. Applying atmospheric energetics to a limited-area domain has the advantage of increasing the details of the analysis: the formation, development and dissipation of individual weather systems can then be studied.

The set of equations developed by Nikiéma and Laprise (2013) for the computation of an inter-member energy cycle based on available enthalpy and kinetic energy is here adapted for the computation of an atmospheric energy cycle. Two energetics analyses are made for this paper: an energy cycle climatology and a storm energy cycle. Those analyses are made with a CRCM5 simulation driven by reanalyses for the month of December 2004 and for a storm occurring during that same month.

For the energy cycle climatology, the energy reservoirs and all associated energy fluxes (generation, conversion, dissipation and boundary terms) are computed. The energy is mainly generated by the covariance of diabatic sources and temperature, and mainly dissipated by friction in the boundary layer. The important conversions of energy take place along the storm track, where the anomalies of temperature gradient and wind are important. For the storm energy cycle, the available enthalpy and kinetic energy follow the temperature gradient and the wind pattern associated with the storm: the generation of energy is maximum at the frontal lines and its dissipation where the wind is the strongest. The conversions of energy are directly related to the temperature advection and vertical movements produced by the passage of the storm. Both energy cycles are reasonably satisfied.

Keywords: Energy Budget, Limited-Area Domain, Regional Climate Model, North American Climate

1.1 Introduction

The study of atmospheric energetics provides fundamental information characterising several aspects of the physical behaviour and maintenance of the general circulation. Since the 1950's, many computations of atmospheric energy cycles have been made on the global scale using data from atmospheric analyses or from simulations of global climate models (GCM). These studies have provided detailed insights on the transformations and conversions of various forms of energy in the atmosphere.

Margules (1905; see also Tamura 1905) was one of the first to study the atmospheric energetics. He recognized the different forms of energy present in the atmosphere as being the kinetic energy (*KE*) associated with the wind, the gravitational potential energy (*PE*) due to the attraction by the Earth and the internal energy (*IE*) caused by the Brownian movement of the molecules. He noted that *IE* and *PE*, when integrated over a column of atmosphere, are proportional to one another and hence introduced the total potential energy (*TPE*) defined as the sum of *IE* and *PE*. He faced however practical difficulties in calculating quantitative energy budgets due to the *TPE* being much larger than *KE*, and the fluxes of *TPE* being huge but their convergence modest.

Lorenz (1955, 1967) was the first to develop a detailed atmospheric energy cycle and his work is still today considered as a classic in atmospheric energetics. He introduced the concept of available potential energy (*APE*). *TPE* is separated into unavailable potential energy (*UPE*), the *TPE* of an atmospheric reference state, and *APE*, the excess of *TPE* over the reference state. The reference state is defined as that which would exist if the total mass of the atmosphere were redistributed adiabatically to result in a horizontal density stratification everywhere. In such a notional barotropic atmosphere, *PE* could not be transformed into *KE* because there would be no pressure gradient to generate wind. Consequently, it is only the much smaller *APE* component of *TPE* that can be transformed into *KE*. Lorenz therefore developed an energy cycle based solely on *APE* and *KE* as the only two relevant forms of energy in the atmosphere.

His energy cycle goes as follows: *APE* is generated by the latitudinal temperature gradient caused by the differential heating of the Earth by the Sun; *APE* is converted into *KE* by the weather systems that transport heat from low to high latitudes, reducing the latitudinal temperature gradient and associated *APE*; *KE* is ultimately dissipated by friction. This cycle goes on infinitely as the temperature gradient is constantly replenished by the Sun. Much of our current understanding of global atmospheric energetics derives from Lorenz' seminal work (e.g. Oort 1964 a, b; Newell et al. 1972, 1974; Boer 1974; Peixoto and Oort 1992; to name just a few).

To account for the significant axisymmetric structure of the atmosphere and to better understand the role of weather systems in the atmospheric energetics, Lorenz further decomposed his energy fields into components associated with the zonal-mean atmospheric state and departures thereof, termed eddies. The zonal kinetic energy (*ZKE*) represents the *KE* of the zonal-mean wind, such as the westerly winds in mid-latitudes and the trade winds in the tropics, whereas the eddy kinetic energy (*EKE*) corresponds to the *KE* associated with the deviations from the zonal-mean wind (the variance of wind along latitude circles). The zonal available potential energy (*ZAPE*) represents the *APE* associated with the latitudinal gradient of the zonal-mean temperature field, cold at the poles and warm at the tropics, whereas the eddy available potential energy (*EAPE*) corresponds to the *APE* associated with deviations from the zonal-mean temperature field (the variance of temperature along latitude circles). A significant advantage of thus separating the atmospheric fields is that the resulting energy expressions are quadratic and positive definite quantities, which greatly simplifies their physical interpretation.

There are however noteworthy limitations associated with Lorenz' approach. An important one is that his concept of *TPE*, *UPE* and *APE* is only meaningful when applied to the entire atmosphere, and not for a portion of it (van Mieghem 1973, Section

14.8). Several mathematical manipulations used to establish *APE* and associated conversion terms require integrating over the entire atmosphere, causing several terms to vanish identically. There are further approximations introduced to simplify the definition of the reference state, to compute *APE* and to cope with topography. Hence, attempts at computing the energetics of a limited region have faced many challenges (e.g. Spar 1950; Smith 1969 a, b). First, the definition of a minimum-energy state for the *UPE* can only be done in a global framework. Second, a major consequence of carrying diagnostics over a limited domain is that flux-divergence terms do not cancel when integrated by part, as is the case over the whole atmosphere, and hence very large boundary fluxes exist. And third, the domain mass is not conserved and the mean vertical velocity does not vanish, generating additional terms in the energy budget.

Still, understanding the physical processes responsible for the formation and development of individual weather systems and the underlying energy conversions remains a long-standing scientific interest. The recent advent of high-resolution atmospheric analyses and regional model simulations facilitated the access to the required data and stimulated such undertaking. Pearce (1978) developed a distinct energy cycle approach by redefining *APE* as the departure from an isothermal reference state. His *APE* is therefore a direct function of the variance of the temperature, which facilitates its calculation and physical interpretation. Similarly to Lorenz, Pearce decomposed his fields of *APE* and *KE* into zonal-mean and eddy components. He also introduced a third component to *APE*, function of the vertical distribution of temperature, accounting for changes in static stability. As a result, his definition of *APE* is easier to interpret physically and better adapted to regional diagnostics in the atmosphere. Following Pearce's approach, Marquet (2003 a, b) established an exact local and complete energy cycle for a limited-area domain that works on the global scale as well. His formulation is based on available enthalpy (*AE*), which naturally separates into contributions from temperature and pressure, allowing this cycle to be used on any pressure level if desired. Recently, Nikiéma and Laprise (2013), noted

"NL13" hereafter, used an approximate form of this formalism to characterize the time fluctuations of inter-member variability (IV) in ensemble simulations of a nested Regional Climate Model (RCM). They decomposed their energy fields into ensemble-mean and inter-member departures.

A close parallel can be drawn between IV energy conversions and atmospheric energy conversions taking place in weather systems. Therefore, the purpose of this paper is first to formulate a limited-area atmospheric energy budget following closely the approach employed by NL13 for IV, and second, to apply this energy cycle to study the energetics of weather systems on a North American domain. Available enthalpy (*AE*) and kinetic energy (here noted *K*) will be decomposed into components relating to the time-mean state of the atmosphere and departures thereof, the time variability. Two budgets will be computed from a simulation of the fifth-generation of the Canadian Regional Climate Model (CRCM5) driven by reanalyses for the month of December 2004: an energy cycle climatology and a storm energy cycle.

The paper is organised as follows: Section 1.2 will describe the methodology used for the energy cycle climatology. Section 1.3 will describe the experimental design and the climate of December 2004. Section 1.4 will present the results and analysis for the energy cycle climatology. Section 1.5 will describe the methodology used for the storm energy cycle and Section 1.6 will present the associated results and their analysis. Finally, Section 1.7 will summarise the findings and conclude.

1.2 Methodology for the energy cycle climatology

1.2.1 Energy cycle averaged on pressure surfaces

Following closely the mathematical development of NL13 for the energetics of IV in RCM ensemble simulations, an atmospheric energy cycle applicable over a limited-area domain is here derived. The main difference in the approach is that atmospheric variable $\Psi \in \left\{\vec{F}, Q, T, \vec{V}(u,v), \alpha, \omega, \Phi\right\}$ is here decomposed as $\Psi = \langle\Psi\rangle + \Psi'$, where $\langle\Psi\rangle = \frac{1}{\tau}\sum_{t=1}^{\tau}\Psi_t$ is the time-mean state with τ the number of time samples, and $\Psi' = \Psi - \langle\Psi\rangle$ is its deviation (time variability), instead of as an ensemble-mean state and its IV as in NL13.

The available enthalpy (AE) is decomposed into a pressure-dependent component (B) and a temperature-dependent component (A). The component A is further decomposed into components associated with the time-mean state of the atmosphere (A_{TM}) and with its time variability (A_{TV}). The kinetic energy (K) is also decomposed into a time-mean state component (K_{TM}) and its time variability (K_{TV}). The five aforementioned energy reservoirs $(B, A_{TM}, A_{TV}, K_{TM}, K_{TV})$ are then horizontally averaged on isobaric surfaces. The horizontal average on isobaric surfaces (isobaric-mean) is noted $\overline{\Psi}$ and its deviation $\Psi^{\times} = \Psi - \overline{\Psi}$.

After applying the isobaric-mean operator to the energy tendency equations, the reservoir A_{TM} splits into two contributions: $\overline{A_{TM}} = \overline{A_{TM\,S}} + \overline{A_{TM\,B}}$ (see Appendix B, Section B.2). This separation provides a better representation of available enthalpy and is physically meaningful. The set of six approximate energy tendency equations is used in this work to compute an atmospheric energy cycle averaged on isobaric surfaces, valid for a limited-area domain.

The reservoir A_{TM} is the available enthalpy associated with the time-mean temperature. It represents the departure of the time-mean temperature from a reference temperature, chosen so that $1/T_r = \overline{\langle 1/T \rangle}$, with $\overline{\langle\ \rangle}$ the time and space average over the period and domain of interest (Marquet 1991). Here, $T_r = 260$ K.

The reservoir $A_{TM\,S}$ is the part of the time-mean available enthalpy A_{TM} due to the mean stratification of the atmosphere. It is the largest component of A_{TM} and its vertical structure reflects the specific choice of T_r as in Nikiéma and Laprise (2015) for IV energy cycle. The equation for $A_{TM\,S}$:

$$\frac{\partial \overline{A_{TM\,S}}}{\partial t} = \overline{G_{TM\,S}} + \overline{I_{AB}} - \overline{C_{TM\,S}} - \overline{C_{TM\,BS}} - \overline{C_{A\,S}} - \overline{F_{A_{TM}\,S}} - \overline{H_{A_{TM}\,S}} \tag{1.1}$$

where

$$\overline{A_{TM\,S}} = \frac{c_p}{2T_r}\langle \overline{T} - T_r \rangle^2$$

$$\overline{G_{TM\,S}} = \frac{l}{T_r}\left(\left(\overline{\langle T \rangle} - T_r\right)\overline{\langle Q \rangle}\right) \quad \text{with } l = \left\langle \frac{T_r}{T} \right\rangle \text{ a factor of order unity}$$

$$\overline{I_{AB}} = -\frac{RT_r}{p}\overline{\langle \omega \rangle}$$

$$\overline{C_{TM\,S}} = -\overline{\langle \omega \rangle \langle \alpha \rangle}$$

$$\overline{C_{TM\,BS}} = -\frac{c_p}{T_r}\overline{\left\langle \frac{\partial \overline{T}}{\partial p}\omega^{\times} \right\rangle \langle T^{\times} \rangle}$$

$$\overline{C_{A\,S}} = -\frac{c_p}{T_r}\left(\overline{\left\langle \vec{V}'T' \right\rangle \cdot \vec{\nabla}\langle T \rangle}\right) - \frac{c_p}{T_r}\left(\overline{\langle \omega' T' \rangle \frac{\partial \langle T \rangle}{\partial p}} + \overline{\left\langle \omega' T' \right\rangle \frac{\partial \langle T^{\times} \rangle}{\partial p}}\right)$$

$$\overline{F_{A_{TM}\,S}} = \vec{\nabla}\cdot\left(\overline{A_{TM\,S}\left\langle \vec{V} \right\rangle}\right) + \frac{\partial\left(\overline{A_{TM\,S}\langle \omega \rangle}\right)}{\partial p}$$

$$\overline{H_{A_{TM\,S}}} = \frac{c_p}{T_r}\left(\vec{\nabla}\bullet\left(\overline{\langle T_*\rangle\langle \vec{V}'T'\rangle} + \overline{\langle T^\times\rangle\langle \vec{V}'\overline{T'}\rangle}\right)\right) + \frac{c_p}{T_r}\left(\frac{\partial\left(\overline{\langle T_*\rangle\langle \omega'T'\rangle} + \overline{\langle T^\times\rangle\langle \omega'\overline{T'}\rangle}\right)}{\partial p}\right)$$

(See Appendix B, Subsection B.2.5 for the mathematical development)

The reservoir $A_{TM\,B}$ is the part of the time-mean available enthalpy A_{TM} due to the departure from the isobaric-mean temperature, hence due to the baroclinicity of the atmosphere. It generally has smaller values than the reservoir $A_{TM\,S}$ and is of the same order of magnitude as the time-mean kinetic energy reservoir K_{TM}, as in Nikiéma and Laprise (2015) for IV energy cycle. The value of $A_{TM\,B}$ decreases with height and the maximum energy is near the ground where the temperature deviation with respect to the isobaric-mean temperature is the largest. The equation for $A_{TM\,B}$:

$$\frac{\partial \overline{A_{TM\,B}}}{\partial t} = \overline{G_{TM\,B}} - \overline{C_{TM\,B}} + \overline{C_{TM\,BS}} - \overline{C_{A\,B}} - \overline{F_{A_{TM\,B}}} - \overline{H_{A_{TM\,B}}} \tag{1.2}$$

where

$$\overline{A_{TM\,B}} = \frac{c_p}{2T_r}\overline{\langle T^\times\rangle^2}$$

$$\overline{G_{TM\,B}} = \frac{1}{T_r}\overline{\langle T^\times\rangle\langle Q^\times\rangle}$$

$$\overline{C_{TM\,B}} = -\overline{\langle \omega^\times\rangle\langle \alpha^\times\rangle}$$

$$\overline{C_{TM\,BS}} = -\frac{c_p}{T_r}\overline{\left\langle \frac{\partial \overline{T}}{\partial p}\omega^\times\right\rangle\langle T^\times\rangle}$$

$$\overline{C_{A\,B}} = -\frac{c_p}{T_r}\left(\overline{\langle \vec{V}'T'^\times\rangle\bullet\vec{\nabla}\langle T\rangle}\right) - \frac{c_p}{T_r}\left(\overline{\langle \omega'T'^\times\rangle\frac{\partial\langle T^\times\rangle}{\partial p}}\right)$$

$$\overline{F_{A_{TM\,B}}} = \vec{\nabla}\bullet\left(\overline{A_{TM\,B}\langle \vec{V}\rangle}\right) + \frac{\partial\left(\overline{A_{TM\,S}\langle \omega\rangle}\right)}{\partial p}$$

$$\overline{H_{A_{TM\,B}}} = \frac{c_p}{T_r}\vec{\nabla}\bullet\left(\overline{\langle T^{\times}\rangle\left\langle \vec{V}'T'^{\times}\right\rangle}\right) + \frac{c_p}{T_r}\left(\overline{\frac{\partial\left(\langle T^{\times}\rangle\langle \omega' T'^{\times}\rangle\right)}{\partial p}}\right)$$

(See Appendix B, Subsection B.2.4 for the mathematical development)

The reservoir A_{TV} is the available enthalpy associated with the time variability of temperature. Its contribution becomes important according to the intensity of its anomaly with respect to the time-mean available enthalpy A_{TM}. The equation for A_{TV}:

$$\frac{\partial \overline{A_{TV}}}{\partial t} = \overline{G_{TV}} - \overline{C_{TV}} + \overline{C_{A\,S}} + \overline{C_{A\,B}} - \overline{F_{A_{TV}}} - \overline{H_{A_{TV}}} \tag{1.3}$$

where

$$\overline{A_{TV}} = \frac{c_p}{2T_r}\overline{\langle T'^2\rangle}$$

$$\overline{G_{TV}} = \frac{l}{T_r}\overline{\langle T'Q'\rangle}$$

$$\overline{C_{TV}} = -\overline{\langle \omega'\alpha'\rangle}$$

$$\overline{C_{A\,B}} = -\frac{c_p}{T_r}\left(\overline{\left\langle \vec{V}'T'^{\times}\right\rangle \bullet \vec{\nabla}\langle T\rangle}\right) - \frac{c_p}{T_r}\left(\overline{\langle \omega' T'^{\times}\rangle \frac{\partial \langle T^{\times}\rangle}{\partial p}}\right)$$

$$\overline{C_{A\,S}} = -\frac{c_p}{T_r}\left(\overline{\left\langle \vec{V}'T'\right\rangle \bullet \vec{\nabla}\langle T\rangle}\right) - \frac{c_p}{T_r}\left(\overline{\langle \omega' T'\rangle}\frac{\partial \overline{\langle T\rangle}}{\partial p} + \overline{\langle \omega' T'\rangle \frac{\partial \langle T^{\times}\rangle}{\partial p}}\right)$$

$$\overline{F_{A_{TV}}} = \vec{\nabla}\bullet\left(\overline{A_{TV}\left\langle \vec{V}\right\rangle}\right) + \frac{\partial\left(\overline{A_{TV}\langle \omega\rangle}\right)}{\partial p}$$

$$\overline{H_{A_{TV}}} = \frac{c_p}{2T_r}\left\langle \vec{\nabla}\bullet\left(\overline{\vec{V}'T'^2}\right) + \frac{\partial\left(\overline{\omega' T'^2}\right)}{\partial p}\right\rangle$$

(See Appendix B, Subsection B.2.6 for the mathematical development)

The reservoir K_{TM} is the kinetic energy associated with the time-mean wind. Its equation:

$$\frac{\partial \overline{K_{TM}}}{\partial t} = \overline{C_{TM\,S}} + \overline{C_{TM\,B}} - \overline{C_K} - \overline{D_{TM}} - \overline{F_{K_{TM}}} - \overline{H_{K_{TM}}} \tag{1.4}$$

where

$$\overline{K_{TM}} = \frac{1}{2}\overline{\langle\vec{V}\rangle \bullet \langle\vec{V}\rangle}$$

$$\overline{C_{TM\,S}} = -\overline{\langle\omega\rangle}\left(\overline{\langle\alpha\rangle} - \alpha_r\right)$$

$$\overline{C_{TM\,B}} = -\overline{\langle\omega^{\times}\rangle\langle\alpha^{\times}\rangle}$$

$$\overline{C_K} = -\overline{\left\langle \vec{V}' \bullet \left(\vec{V}' \bullet \vec{\nabla}\right)\langle\vec{V}\rangle\right\rangle} - \overline{\left\langle \vec{V}' \bullet \left(\omega' \frac{\partial\langle\vec{V}\rangle}{\partial p}\right)\right\rangle}$$

$$\overline{D_{TM}} = -\overline{\langle\vec{V}\rangle \bullet \langle\vec{F}\rangle}$$

$$\overline{F_{K_{TM}}} = \vec{\nabla} \bullet \left(\overline{K_{TM}\langle\vec{V}\rangle}\right) + \frac{\partial\left(\overline{K_{TM}\langle\omega\rangle}\right)}{\partial p}$$

$$\overline{H_{K_{TM}}} = \vec{\nabla} \bullet \left(\overline{\langle\vec{V}\rangle \bullet \langle\vec{V}'\vec{V}'\rangle}\right) + \vec{\nabla} \bullet \left(\overline{\langle\vec{V}\rangle\langle\Phi\rangle}\right) + \frac{\partial\left(\overline{\langle\vec{V}\rangle \bullet \langle\vec{V}'\omega'\rangle}\right)}{\partial p} + \frac{\partial\left(\overline{\langle\omega\rangle\langle\Phi\rangle}\right)}{\partial p}$$

(See Appendix B, Subsection B.3.8 for the mathematical development)

The reservoir K_{TV} is the kinetic energy associated with the time variability of the wind. Its contribution becomes important according to the intensity of its anomaly with respect to the time-mean kinetic energy K_{TM}. The equation K_{TV}:

$$\frac{\partial \overline{K_{TV}}}{\partial t} = \overline{C_{TV}} + \overline{C_K} - \overline{D_{TV}} - \overline{F_{K_{TV}}} - \overline{H_{K_{TV}}} \tag{1.5}$$

where

$$\overline{K_{TV}} = \frac{1}{2}\overline{\langle\vec{V}' \bullet \vec{V}'\rangle}$$

$$\overline{C_{TV}} = -\overline{\langle\omega'\alpha'\rangle}$$

$$\overline{C_K} = -\overline{\left\langle \vec{V}' \bullet \left(\vec{V}' \bullet \vec{\nabla}\right)\left\langle \vec{V} \right\rangle \right\rangle} - \overline{\left\langle \vec{V}' \bullet \left(\omega' \frac{\partial \left\langle \vec{V} \right\rangle}{\partial p} \right) \right\rangle}$$

$$\overline{D_{TV}} = -\overline{\left\langle \vec{V}' \bullet \vec{F}' \right\rangle}$$

$$\overline{F_{A_{TV}}} = \vec{\nabla} \bullet \left(\overline{K_{TV} \left\langle \vec{V} \right\rangle} \right) + \frac{\partial \left(\overline{K_{TV} \langle \omega \rangle} \right)}{\partial p}$$

$$\overline{H_{K_{TV}}} = \vec{\nabla} \bullet \overline{\left\langle \left(\frac{1}{2} \vec{V}' \vec{V}' + \Phi' \right) \vec{V}' \right\rangle} + \frac{\partial \overline{\left\langle \left(\frac{1}{2} \vec{V}' \vec{V}' + \Phi' \right) \omega' \right\rangle}}{\partial p}$$

(See Appendix B, Subsection B.3.9 for the mathematical development)

The reservoir B is the pressure-dependent part of available enthalpy and its variations reflect the fact that the mass is not constant over a limited-area domain. It represents the effect on AE of the departure from a reference pressure $p_r = p_{00}/e$, with the constant p_{00} approximately the mean value of the surface pressure. As the surface pressure does not vary a lot in time, B is very close to being constant. Its equation:

$$\frac{\partial \overline{B}}{\partial t} = -\overline{F_B} - \overline{I_{AB}} \tag{1.6}$$

where

$$\overline{B} = RT_r \overline{\left\langle \ln \frac{p}{p_r} \right\rangle}$$

$$\overline{F_B} = \vec{\nabla} \bullet \left(\overline{B \left\langle \vec{V} \right\rangle} \right) + \frac{\partial \left(\overline{B \langle \omega \rangle} \right)}{\partial p}$$

$$\overline{I_{AB}} = -\frac{RT_r}{p} \overline{\langle \omega \rangle}$$

(See Appendix B, Subsection B.2.7 for the mathematical development)

Eq. (1.1) to eq. (1.6) constitute an approximate set of equations describing the exchanges of energy between the time-mean and time variability states of available enthalpy and kinetic energy, appropriate for the study of energetics over a regional climate domain. There are six energy reservoirs corresponding to the six prognostic equations of $\overline{A_{TM\,S}}$, $\overline{A_{TM\,B}}$, $\overline{A_{TV}}$, $\overline{K_{TM}}$, $\overline{K_{TV}}$ and $\overline{B}$ defined over isobaric surfaces. These six equations are linked together, resulting in the energy cycle illustrated in Fig. 1.1. Boxes represent energy reservoirs and arrows indicate energy fluxes between reservoirs and with the environment (the magnitude and physical interpretation of each energy reservoir and associated fluxes will be discussed further in Section 1.4). It is noteworthy that the direction of the arrows is arbitrary and solely reflects the choice of sign used in the equations. An arrow pointing toward (away from) a reservoir indicates that the energy exchange is assigned as a source (sink) of energy for that reservoir, but only the calculations will reveal whether it has a positive or negative sign. As a result of this mathematical development, the four energy reservoirs $A_{TM\,B}$, A_{TV}, K_{TM} and K_{TV} are of the same order of magnitude and will therefore be easier to compare, as noted by Nikiéma and Laprise (2015) for IV energy cycle. The following paragraphs describe the energy flux terms present in eq. (1.1) to eq. (1.6) and illustrated in Fig. 1.1.

The terms I and all C represent conversions of energy between different reservoirs. They convert energy types from available enthalpy to kinetic energy and energy states from time-mean to time variability. The term I_{AB} converts available enthalpy from its pressure-dependant state B into the stratification component of its temperature-dependent state $A_{TM\,S}$. The term $C_{TM\,BS}$ represents a conversion of time-mean available enthalpy from its stratification component $A_{TM\,S}$ into its baroclinic component $A_{TM\,B}$. The terms $C_{TM\,S}$ and $C_{TM\,B}$ convert time-mean available enthalpy, respectively $A_{TM\,B}$ and $A_{TM\,S}$, into time-mean kinetic energy K_{TM}. The terms $C_{A\,S}$ and $C_{A\,B}$ convert time-mean available enthalpy, respectively $A_{TM\,B}$ and $A_{TM\,S}$, into time variability available enthalpy A_{TV}. The term C_K converts kinetic energy from its time-mean state K_{TM} into its time variability state K_{TV}. Finally, the term C_{TV} represents a baroclinic conversion from time variability available enthalpy A_{TV} into time variability kinetic energy K_{TV}.

The terms G represent generation of temperature-dependent available enthalpy, $A_{TM\,S}$, $A_{TM\,B}$ and A_{TV}. When the generation term G corresponds to a positive (negative) covariance of perturbations of temperature and diabatic heating (radiative energy, condensation and evaporation, convection, horizontal diffusion, etc.), it represents a source (sink) of available enthalpy.

The terms D represent dissipation of kinetic energy, K_{TM} and K_{TV}, due to friction and diffusion. The friction is mainly present in the boundary layer, where turbulence slows down wind. The maximum values are found where the surface stress is largest: over rough terrain and where the wind is the strongest.

The terms F are boundary fluxes that represent the transport of energy by the time-mean wind. Averaged over the domain of interest, these terms act as sinks (sources) of energy if more (less) energy leaves the domain than enters it. They would cancel out for an energy cycle computed over the whole globe.

The terms H are also boundary fluxes. As for the terms F, they would cancel out for an energy cycle computed over the whole globe.

1.3 Experimental design

1.3.1 Simulation condition, domain and computation

A one-year long simulation was made with the CRCM5 (Martynov et al. 2013) driven by ERA-interim reanalyses, available on a 1.5° grid (Kalnay et al. 1996), starting on

October 14, 2004. The month of December 2004 was chosen for the energy cycle calculations because of its important number of synoptic events. The simulation was carried out with a grid mesh of 0.3°, 56 terrain-following hybrid levels in the vertical, and a time step of 12 minutes. The free domain, excluding the 10 grid-point-wide lateral Davies' sponge zone and the 10 grid-point-wide semi-Lagrangian halo, contains 230 × 130 grid points.

The domain covers the east of the North American continent and a part of the Atlantic Ocean. It goes from 20.5° to 59.2° for the latitude and from 260.5° to 329.2° for the longitude. This domain was chosen for its extra tropical position, where the temperature gradient is important. It is indeed located on the storm track, where synoptical events are numerous and high in energy conversions. There are two important mountainous regions: the Appalachian Mountains and the North-of-Québec Central Plateau, as shown in the elevation map of Fig. 1.2. The sea-surface temperature and sea-ice cover are prescribed from reanalyses. The simulated data is archived at a three-hourly interval and the diagnostics are performed on variables interpolated on 18 pressure levels (1000, 975, 950, 925, 900, 850, 800, 700, 600, 500, 400, 300, 250, 200, 150, 100, 70 and 50 hPa).

The following computations are made for each variable for the 31 days of December 2004, on each grid point and pressure surfaces in the domain. Starting from the instantaneous values of an archived variable field, its time-mean state is computed by taking its time average. Then, its time deviations are calculated by subtracting the time-mean state from the instantaneous value of the field. The isobaric-mean state is computed by taking the domain average of the instantaneous value of the field on determined pressure levels. The isobaric deviations are calculated by subtracting the isobaric-mean state from the instantaneous value of the field. Finally, variances and covariance of time and/or isobaric perturbations are computed.

All vertical integrations or vertical averaging of fields in this work will be taken from the surface up to the 200 hPa level. This choice of upper integration limit will be discussed in Subsection 1.4.4

1.3.2 Climate of December 2004

Fig. 1.3 shows the vertically averaged time-mean air temperature as simulated by the CRCM5 for the month of December 2004. The temperature is quite warm and uniform in the south of the domain. There is an important latitudinal temperature gradient over the North American continent. Over the Atlantic Ocean, the Gulf Stream brings warm air from the south up to higher latitudes, where it gets mixed with the cold air carried from the Labrador and Greenland currents. The resulting temperature gradient is oriented south-southeastward.

Fig. 1.4 shows the mean atmospheric flow represented by the vertically averaged time-mean wind. The wind pattern resembles the upper tropospheric wind, which is well explained by the thermal wind relation, with the maximum wind being located over the largest horizontal temperature gradient. The flow is east-northeastward.

There are some semi-permanent high- and low-pressure systems that affect the weather in December 2004: the Icelandic Low to the northeast, the Canadian High in the Prairies, and the Bermuda High in the east of the domain, as shown in Figure 1.5.

1.4 Results and analysis for the energy cycle climatology

This section presents the results for the energy cycle climatology. Subsection 1.4.1 will confirm the validity of the energy tendency equations using averaged values of energy fluxes. Subsections 1.4.2 and 1.4.3 will present maps of the energy reservoirs and energy fluxes, respectively. Finally, subsection 1.4.4 will illustrate the vertical profiles of the fluxes acting on the different energy reservoirs.

1.4.1 Computation of the energy cycle contributions

The time tendency $L_E = \partial E/\partial t$ of every energy reservoir $E \in \{B, A_{TM\,S}, A_{TM\,B}, A_{TV}, K_{TM}, K_{TV}\}$ is computed as a centred time difference over a six-hourly interval, using archived values of E at three-hourly intervals. It is then compared to R_E, the sum of all sources and sinks of energy contributing to the tendency of E. In computing R_E, spatial derivatives are approximated as two-sided finite differences, and vertical integrals are computed with the trapezoidal rule after eliminating contributions below the surface. L_E is expected to be very small for long time averages, and formally R_E should be identical to L_E. In practice, due to the numerous approximations, the equality is not exactly satisfied; nevertheless $L_E \simeq R_E$.

The vertically integrated time- and isobaric-mean components of the energy cycle are shown in Fig. 1.1. R_E is computed by adding algebraically the fluxes contributing to E. An arrow pointing toward the reservoir is computed as a source of energy and therefore added, and an arrow pointing out of a reservoir is computed as a sink of energy and therefore subtracted. The following values of R_E are obtained for the month of December:

R_B	=	$-0.9\ \mathrm{W \cdot m^{-2}}$	$R_{A_{TM\,S}}$	=	$0.9\ \mathrm{W \cdot m^{-2}}$
$R_{A_{TM\,B}}$	=	$-0.6\ \mathrm{W \cdot m^{-2}}$	$R_{K_{TM}}$	=	$0.5\ \mathrm{W \cdot m^{-2}}$
$R_{A_{TV}}$	=	$-0.1\ \mathrm{W \cdot m^{-2}}$	$R_{K_{TV}}$	=	$0.2\ \mathrm{W \cdot m^{-2}}$

For all six energy reservoirs, R_E is very small and negligible compared to the energy fluxes acting upon E. This allows us to confirm the validity of the set of energy equations presented in Section 1.2 and to consider the limited-area energy cycle as being nearly satisfied.

1.4.2 Maps of the energy reservoirs

Fig. 1.6 and Fig. 1.7 show vertically integrated time-mean maps of energy reservoirs and energy fluxes, respectively. Upon taking the horizontal average on isobaric surfaces, all fields are reduced to a single value (see Fig. 1.1). The advantage of showing maps that are not yet isobarically averaged is that the results can be analysed with regard to the characteristics of the chosen domain: temperature, wind, surface pressure, surface relief, presence of the ocean, storm track, etc. The energy reservoir $A_{TM\,C}$ is not shown because it vanishes identically upon applying the isobaric-mean operator (see Appendix B, Section B.2). Therefore, the six energy reservoirs developed in Section 1.2 are presented in this Section.

Fig. 1.6 shows the energy reservoirs a) B, b) $A_{TM\,S}$, c) $A_{TM\,B}$, d) K_{TM}, e) A_{TV} and f) K_{TV}. The pressure-dependant energy reservoir B shows smaller values where the surface pressure is lower, such as over the mountainous regions and over the Icelandic Low (see Fig. 1.5). The available enthalpy reservoir $A_{TM\,S}$ reflects the mean temperature state of the atmosphere (see Fig. 1.3): cold at high latitudes (low available enthalpy) and warm in the tropics (high available enthalpy). The pattern of the available enthalpy reservoir $A_{TM\,B}$ is due to temperature deviations with respect to the isobaric temperature average: high latitudes are colder and low latitudes are warmer (high available enthalpy) than the isobaric-mean value, while the middle of the domain is close to the isobaric-mean temperature value (low available enthalpy). The same pattern as the mean flow (see Fig. 1.4) is noted for the kinetic energy reservoir K_{TM}: the maximum of kinetic energy is where the maximum wind is. Along the storm track, the maximum temperature and wind anomalies occur and contribute respectively to time variability energy reservoirs A_{TV} and K_{TV}.

1.4.3 Maps of the energy fluxes

Fig. 1.7 shows the fluxes of energy between reservoirs and with the environment for the whole energy cycle. The arrows have the same meaning as in Fig. 1.1: an arrow pointing toward (away from) a reservoir acts as a source (sink) of energy for that reservoir.

Generation terms $G_{TM\,S}$, $G_{TM\,B}$ and G_{TV} act on available enthalpy reservoirs $A_{TM\,S}$, $A_{TM\,B}$ and A_{TV}, respectively. The term $G_{TM\,S}$ is due to the covariance of diabatic sources and temperature. It is positive, which implies diabatic heating (cooling) in regions that are warmer (colder) than the reference temperature. The terms $G_{TM\,B}$ and G_{TV} are due to the covariance of diabatic sources and temperature deviations, which are small for winter months. As a result, they are both mostly negligible.

Dissipation terms D_{TM} and D_{TV} act on kinetic energy reservoirs K_{TM} and K_{TV}, respectively. They are both negative as they are destroying kinetic energy. The term D_{TV} is present over the Appalachian Mountains and over the storm track, where the surface stress due to important wind is maximum. The term D_{TM} is also present over the Appalachian Mountains but negligible elsewhere: the time-mean atmospheric state is not associated with important wind in the boundary layer.

The conversion term I_{AB} converts available enthalpy between reservoirs B and $A_{TM\,S}$, and is proportional to the time-mean vertical velocity. For the month of December 2004, the vertical velocity is mainly positive over the domain of interest; the term I_{AB} is therefore negative and converts available enthalpy from reservoir $A_{TM\,S}$ to reservoir B. It converts a remarkably high amount of energy due to the important value of RT_r.

Conversion terms $C_{A\,S}$ and $C_{A\,B}$ convert available enthalpy from reservoirs $A_{TM\,S}$ and $A_{TM\,B}$, respectively, into reservoir A_{TV}. They represent the effect of the covariance of wind and temperature anomalies acting to balance the mean temperature gradient. The temperature pattern of December 2004 (see Fig. 1.3) creates a transport of cold air eastward and a transport of warm air northward, making the terms $C_{A\,S}$ and $C_{A\,B}$ always positive. The energy conversions are maximum where the temperature gradient is the largest, along the storm track.

The conversion term C_{TV} converts time variability available enthalpy A_{TV} into time variability kinetic energy K_{TV}. It represents a baroclinic conversion due to the covariance of vertical velocity and temperature (specific volume) anomalies. This conversion is always positive over the period and domain of interest: a positive (negative) anomaly of temperature is associated with a negative (positive) anomaly of vertical velocity, i.e. an upward (downward) motion. Indeed, the term C_{TV} is maximum along the storm track, where fronts are responsible for warm air masses rising and cold air masses sinking. If we compare the term C_{TV} with the terms $C_{A\,S}+C_{A\,B}$, we note a great similarity between the patterns and intensity of fields, indicating that most of the energy brought by the terms $C_{A\,S}$ and $C_{A\,B}$ to available enthalpy reservoir A_{TV} is then converted by the term C_{TV} into kinetic energy K_{TV}.

The conversion term C_K converts kinetic energy between reservoirs K_{TM} and K_{TV}. It represents the effect of the variance and covariance of wind anomalies acting against the wind gradient to balance the mean horizontal wind. The field of the term C_K is almost always negative over the period and domain of interest, indicating that the conversion is from the reservoir K_{TV} into the reservoir K_{TM}. Indeed, the wind perturbations contribute to reinforce the time-mean wind.

The conversion term $C_{TM\,BS}$ converts available enthalpy between reservoirs $A_{TM\,S}$ and $A_{TM\,B}$. We see from the map that the field is negative: the conversion is from the reservoir $A_{TM\,B}$ into the reservoir $A_{TM\,S}$.

Conversion terms $C_{TM\,S}$ and $C_{TM\,B}$ convert available enthalpy $A_{TM\,S}$ and $A_{TM\,B}$, respectively, into kinetic energy K_{TM}. The terms $C_{TM\,S}$ and $C_{TM\,B}$ are both negative, indicating that kinetic energy K_{TM} is converted into available enthalpy $A_{TM\,S}$ and $A_{TM\,B}$. The time-mean wind hence contributes to increase the time-mean temperature gradient over the period and domain of interest. The term $C_{TM\,S}$ converts a remarkably high amount of energy due to the important value of RT_r.

The six terms F are boundary fluxes representing the transport of energy in and out of the limited-area domain. The term F acts as a source of energy (more energy enters in than leaves the domain) for the reservoir $A_{TM\,B}$ and as a sink of energy (more energy leaves than enters in the domain) for reservoirs B, $A_{TM\,S}$, K_{TM} and K_{TV}. The five terms H are also boundary fluxes. The term H acts as a source of energy for reservoirs $A_{TM\,B}$ and K_{TM} as a sink of energy for reservoirs $A_{TM\,S}$ and K_{TV}. Boundary fluxes $F_{A_{TV}}$ and $H_{A_{TV}}$ are negligible. Both terms F_B and $H_{K_{TM}}$ transport a remarkably high amount of energy due to the important values of B and of $\langle\Phi\rangle$, respectively. The values for terms F and H are consequences of the mean flow of the chosen period of time and domain. In fact, for different periods or different regions, these results could be completely different.

1.4.4 Vertical profiles

Fig. 1.8 shows the vertical profiles of the time- and isobaric-mean fluxes for the six energy reservoirs a) B, b) $A_{TM\,S}$, c) $A_{TM\,B}$, d) K_{TM}, e) A_{TV} and f) K_{TV}. The vertical profiles are shown from 1000 hPa to 50 hPa. As can be noted on most panels, the behaviour of fields above the 200 hPa level highly differs from their behaviour in the troposphere, where most of the weather phenomena occur. This justified our choice of restricting the vertical integration up to the 200 hPa level so as to keep the analyses focused upon meteorological events. The conversion terms are in dashed lines while the other terms are in full lines. The thick black lines represent R_E, the algebraic sum of all fluxes acting upon E. For the three reservoirs B, $A_{TM\,S}$ and K_{TM}, the four conversion and transport terms that carry remarkably higher values of energy, namely F_B, I_{AB}, $C_{TM\,S}$ and $H_{K_{TM}}$, balance each other almost perfectly and the other terms are negligible.

For the baroclinic component of time-mean available enthalpy $A_{TM\,B}$, several terms interact to balance each other. For the time variability available enthalpy A_{TV}, the conversion term $-C_{TV}$ is almost balanced by the conversion terms $C_{A\,S}+C_{A\,B}$ in the free atmosphere. The negative value of the generation term G_{TV} in the boundary layer is due to surface sensible heat flux and vertical diffusion. The other terms are negligible.

For the time variability kinetic energy K_{TV}, the conversion term C_{TV} is balanced by the boundary flux $-H_{K_{TV}}$ in the free atmosphere. In the boundary layer however, the balance is entirely different, with the dissipation term $-D_{TV}$ due to surface friction being balanced by the component of the term $-H_{K_{TV}}$ corresponding to the covariance of vertical motion and geopotential height perturbations $\partial\langle\Phi'\omega'\rangle/\partial p$. In low (high) pressure systems, surface friction induces convergence (divergence) of mass down (up) the gradient of geopotential, resulting in upward (downward) motion, a process called Ekman pumping (e.g. Holton 2004, Section 5.4). Hence, in the boundary layer, the energy lost by friction is balanced by the energy converted by Ekman pumping. The other terms in the budget for the reservoir K_{TV} are negligible.

1.5 Methodology for the storm energy cycle

1.5.1 Relevance of analysing a storm

The month of December 2004 was chosen for this energetic study due to the important number of synoptic events occurring in the domain. Following one low-pressure system allows us to study the time evolution of the energy cycle during the passage of a storm over a limited region and should provide us with quantitative physical interpretation of the generation, conversion, dissipation and transport of energy in and out of the different reservoirs.

1.5.2 Instantaneous energy equations

The methodology developed in Section 1.2 for an energy cycle climatology is here extended to allow the computation of an instantaneous energy cycle, applicable for the study of the life cycle of a storm. As the time-mean part of the energy cycle is not of much interest during the passage of a storm, only the time variability part will be considered for the following analysis. To obtain instantaneous energy tendency equations for the time variability energy cycle, a version of the equations for reservoirs A_{TV} and K_{TV} (eq. (1.3) and eq. (1.5)) before the time-mean operator was applied to them is developed (see Appendix A). The use of lower-case serves to differentiate instantaneous variables (lower case) from time-averaged ones (upper case).

The term a_{TV} represents the instantaneous available enthalpy associated with the time variability of temperature. Its equation:

$$\frac{\partial a_{TV}}{\partial t} = g_{TV} - c_{TV} + c_a - f_{a_{TV}} - h_{a_{TV}} - j_{a1} - j_{a2} \tag{1.7}$$

where

$$a_{TV} = \frac{c_p}{2T_r}\left(T'^2\right)$$

$$g_{TV} = l\left(\frac{T'}{T_r}Q'\right)$$

$$c_{TV} = -\omega'\alpha'$$

$$c_a = c_{a\,b} + c_{a\,s} = -\left(\vec{V}'\frac{c_pT'}{T_r}\right)\bullet\vec{\nabla}\langle T\rangle - \left(\omega'\frac{c_pT'}{T_r}\right)\frac{\partial\langle T\rangle}{\partial p}$$

$$f_{a_{TV}} = \vec{\nabla}\bullet\left(a_{TV}\left\langle\vec{V}\right\rangle\right) + \frac{\partial\left(a_{TV}\langle\omega\rangle\right)}{\partial p}$$

$$h_{a_{TV}} = \frac{c_p}{2T_r}\left(\vec{\nabla}\bullet\left(\vec{V}'T'^2\right) + \frac{\partial\left(\omega'T'^2\right)}{\partial p}\right)$$

$$j_{a1} = -\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'\right\rangle - \frac{c_pT'}{T_r}\frac{\partial\langle\omega'T'\rangle}{\partial p}, \qquad j_{a2} = -T'\left\langle\frac{Q}{T}\right\rangle$$

(See Appendix A, Subsection A.3.8 for the mathematical development)

The term k_{TV} represents the instantaneous kinetic energy associated with the time variability of wind. Its equation:

$$\frac{\partial k_{TV}}{\partial t} = c_{TV} + c_k - d_{TV} - f_{k_{TV}} - h_{k_{TV}} - j_{k1} \tag{1.8}$$

where

$$k_{TV} = \frac{1}{2}\vec{V}'\bullet\vec{V}'$$

$$c_{TV} = -\omega'\alpha'$$

$$c_k = -\vec{V}' \bullet \left(\vec{V}' \bullet \vec{\nabla}\right)\left\langle\vec{V}\right\rangle - \vec{V}' \bullet \left(\omega' \frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)$$

$$d_{TV} = -\vec{V}' \bullet \vec{F}'$$

$$f_{k_{TV}} = \vec{\nabla} \bullet \left(k_{TV}\left\langle\vec{V}\right\rangle\right) + \frac{\partial\left(k_{TV}\langle\omega\rangle\right)}{\partial p}$$

$$h_{k_{TV}} = \vec{\nabla} \bullet \left(\left(k_{TV} + \Phi'\right)\vec{V}'\right) + \frac{\partial\left(\left(k_{TV} + \Phi'\right)\omega'\right)}{\partial p}$$

$$j_{k1} = -\vec{V}'\left\langle\vec{V}' \bullet \vec{\nabla}\vec{V}'\right\rangle - \vec{V}'\left\langle\omega' \frac{\partial\vec{V}'}{\partial p}\right\rangle$$

(See Appendix A, Subsection A.4.12 for the mathematical development)

Eq. (1.7) and eq. (1.8) constitute an approximate set of equations describing the instantaneous exchanges of energy between the time variability reservoirs of the energy cycle, suitable for the study of a storm's energetics over a regional domain.

The terms c are instantaneous conversions of energy. They represent the same types of conversions as described in Section 1.2. For simplification, conversion terms $c_{a\,s}$ and $c_{a\,b}$ are combined into the conversion term c_a. The terms g_{TV} and d_{TV} are the instantaneous generation and instantaneous dissipation of energy, respectively. Terms f and h represent instantaneous boundary fluxes due to the domain being limited. Finally, terms j_{a1}, j_{a2} and j_{k1} identically vanish when the time-mean operator is taken on eq. (1.7) and eq. (1.8), hence their absence in eq. (1.3) and eq. (1.5). Fig. 1.9 shows the energy cycle that will be used for the analysis of the storm.

1.6 Results and analysis for the storm energy cycle

This section presents the results for the storm energy cycle. Subsection 1.6.1 will confirm the validity of the energy tendency equations using averaged values of energy fluxes, while Subsection 1.6.2 will justify the choice of the storm. Subsection 1.6.3 will present maps of energy reservoirs and energy fluxes and Subsection 1.6.4 will illustrate their time evolution for the duration of the storm.

1.6.1 Computation of the energy cycle contributions

L_E and R_E were computed for eq. (1.7) and eq. (1.8) for the month of December 2004 and were found have a correlation factor of $\mathrm{r} = 0.979$ for the energy reservoir a_{TV}, and $\mathrm{r} = 0.971$ for the energy reservoir k_{TV}. Their respective dispersion diagrams are shown in Fig. 1.10. Theses results imply that the fluxes capture very well the instantaneous time variations of energy.

1.6.2 Choice of storm

The storm chosen for the energetics analysis occurred from December 26 to December 29 2004 in the analyses, and the corresponding CRCM5-simulated fields are used for the quantitative energy diagnostics. The storm started developing in the Atlantic Ocean, off the coast of Florida, with an initial low-pressure centre of 1006 hPa. It intensified while making its way northeastward along the East Coast of the United States. The low-pressure centre deepened to 990 hPa as the depression reached Nova Scotia, and 970 hPa as it passed over Newfoundland. It reached its minimum pressure at 950 hPa in the Labrador Sea, while exiting the domain through the northeast boundary.

This specific storm was chosen because of its intensity (Fig. 1.11) and the fact that it developed and matured within the limited-area domain, making it an ideal case study of regional atmospheric energetics.

1.6.3 Maps of all terms

Three moments (t_1, t_2 and t_3) were chosen at a 24-hourly interval to illustrate the evolution of different terms through time:

	t_1 = 2004-12-26 2100 UTC	t_2 = 2004-12-27 2100 UTC	t_3 = 2004-12-28 2100 UTC
Location of storm	In the Atlantic, along the East Coast of the United States	In between Nova Scotia and Newfoundland	Crossing the Labrador Sea, exiting the domain
Central pressure value	1004 hPa	986 hPa	954 hPa

These three moments are illustrated in Fig. 1.12, Fig. 1.13 and Fig.1.14. The surface isobars are shown on all maps as a reminder of the storm's position. In the following discussion, if there is no mention of a pressure level, it means that the field is vertically integrated.

As a storm corresponds mathematically to a deviation from the time-mean atmospheric state, the temperature and wind patterns created by its passage are deviations from the time-mean temperature field and time-mean wind.

The storm is associated with an important cyclonic circulation (Fig. 1.12 a), where the wind on its west side is blowing southward, and the wind on its east side is blowing northward. The presence of the storm on the lower atmosphere's temperature (Fig. 1.12 b) is very apparent: cold air is carried southward on the west side of the storm and warm air, northward on its east side. The warm and cold fronts are very well defined: they correspond to zones of important temperature gradient and important vertical motion. There are two maxima of available enthalpy a_{TV} (Fig. 1.12 c): one located on the west side of the storm due to a cold air anomaly resulting from the advection of cold air southward, the other located on its east side due to a warm air anomaly resulting from the advection of warm air northward. The kinetic energy k_{TV} (Fig. 1.12 d) is maximum where the wind anomaly is the largest in magnitude. We note that the maximum of kinetic energy k_{TV} is centred in the middle of the depression where the low-level wind vanishes. We must however recall that the field of kinetic energy k_{TV} is vertically integrated; as the depression core tilts westward with height, the surface depression is overlaid by the upper-level jet stream that undergoes large time variations as the system grows and moves.

The generation term g_{TV} (Fig. 1.13 a) is produced by the covariance of temperature and diabatic sources perturbations. Indeed, the maximum for the term g_{TV} is aligned with the fronts, where the maximum convection and/or condensation occur. The dissipation term d_{TV} (Fig. 1.13 b) is maximum where the low-level wind is maximum (Fig. 1.12 b), which is around the storm where the isobars are the tightest. The conversion term c_a (Fig. 1.13 c) converts available enthalpy from reservoir a_{TM} into reservoir a_{TV} by destroying the latitudinal temperature gradient and generating gradients of temperature anomaly. There is a southward and downward advection on the west side of the storm and a northward and upward advection on the east of the storm. The conversion term c_{TV} (Fig. 1.13 d) converts available enthalpy a_{TV} into kinetic energy k_{TV} as a result of warm anomalies rising and cold anomalies sinking. It is noteworthy that there is a great similitude between the pattern and intensity of the terms c_a and c_{TV}: most of the energy brought by the term c_a to the available enthalpy reservoir a_{TV} is then converted by the term c_{TV} into kinetic energy k_{TV}. There are several positive and negative patches for the conversion term c_k (Fig. 1.13 e), with the negative areas dominating, indicating that the wind perturbations associated with the storm contribute to reinforce the time-mean wind oriented northeastward (see Fig. 1.4).

Transport terms $f_{a_{TV}}$ (Fig. 1.14 a) and $f_{k_{TV}}$ (Fig. 1.14 b) represent fluxes of energy for reservoirs a_{TV} and k_{TV}, respectively. Positive (negative) values are associated with the convergence (divergence) of energy by the time-mean wind. By comparing the transport terms with the energy terms (see Fig. 1.12 c for reservoir a_{TV} and Fig. 1.12 d for reservoir k_{TV}), we see that the border between a divergence-convergence pattern is centred on a maximum value of energy for both reservoirs a_{TV} and k_{TV}. This indicates that the energy is being transported from zones of divergence toward zones of convergence, and this transport follows the general direction of the time-mean wind. Terms $h_{a_{TV}}$ (Fig. 1.14 c) and $h_{k_{TV}}$ (Fig. 1.14 d) are perturbation fluxes for reservoirs a_{TV} and k_{TV}, respectively. They both mostly vanish when averaged over the domain. Finally, terms j_{a1}, j_{a2} and j_{k1} are not shown because they are negligible.

Fig. 1.9 shows the vertically integrated isobaric-mean and values of energy reservoirs and energy fluxes, which have been averaged over the two days of the storm. We see that most of the energy is converted from the time-mean state available enthalpy A_{TM} into its time variability state a_{TV} by the conversion term c_a, and simultaneously converted into time variability kinetic energy k_{TV} by the baroclinic conversion c_{TV}. Some of this kinetic energy k_{TV} is then converted to feed the time-mean kinetic energy K_{TM} by the barotropic conversion c_k, some is dissipated by the term d_{TV} and some is transported out of the domain by the term $f_{k_{TV}}$.

1.6.4 Time evolution of energy reservoirs and energy fluxes

Fig. 1.15 shows the time evolution for the duration of the storm of both energy reservoirs a_{TV} and k_{TV} and of the energy fluxes acting upon them. The results presented are the vertically integrated isobaric-mean fields going from December 25th 2004 to January 1st 2005. Fig 1.15 a shows the time evolution of the available enthalpy reservoir a_{TV} and kinetic energy reservoir k_{TV}. There is an increase of both energies that peak between the 28th and 29th of December, corresponding to the storm's maximum intensity. Indeed, it corresponds to the maximum temperature and wind anomalies (Fig. 1.12 a and Fig 1.12 b, respectively). For the available enthalpy reservoir a_{TV} (Fig. 1.15 b), both conversion terms c_a and $-c_{TV}$ balance each other at all times. Both conversion terms are at their highest intensity at the apex of the storm. For the kinetic energy reservoir k_{TV} (Fig 1.15 c), the energy brought by the baroclinic conversion term c_{TV} is mainly destroyed by the boundary flux term $-h_{k_{TV}}$, corresponding to Ekman pumping, and mainly converted by the barotropic conversion term c_k into time-mean kinetic energy K_{TM}. Throughout the passage of the storm, the kinetic energy is dissipated by the friction term $-d_{TV}$. The moment where the storm is advected out of the domain (December 28th - December 29th) corresponds to the maximum transport of kinetic energy $-f_{k_{TV}}$.

1.7 Conclusions

The aim of this paper was two-fold: (1) to develop a formalism suitable for the study of atmospheric energetics over a limited region, and (2) to illustrate its application to study the energetics of weather systems over a domain centred on the east of North America. The first goal was achieved through a relatively straightforward modification of the IV energy budget formulated by NL13. The second goal was realised through the application of the formalism to a simulation of the CRCM5 driven by reanalyses for the month of December 2004. First, an energy cycle for the whole month of December was computed to analyse the climatological winter energetics of a North American domain. Second, a rapidly developing storm was studied to further our understanding of the dynamical and physical processes responsible for the evolution of individual weather systems

1.7.1 Conclusions for the energy cycle climatology

The energy cycle climatology of December 2004 was studied over a domain covering the east of North America and the adjacent Atlantic Ocean region. The energy reservoirs and fluxes associated with the generation, conversion, dissipation and transport of energy between reservoirs were decomposed into their time-mean and time variability components.

Overall, the energy is mainly generated by the term $G_{TM\ S}$, the time-mean effect of diabatic sources and temperature, and mainly dissipated by the term D_{TV}, the time variability effect of friction in the boundary layer. Between the time-mean energy reservoirs, namely B, $A_{TM\ S}$, $A_{TM\ B}$ and K_{TM}, a high amount of energy is transported and converted without affecting the time variability reservoirs A_{TV} and K_{TV}. Available enthalpy is converted from its time-mean states $A_{TM\ S}$ and $A_{TM\ B}$ into its time variability state A_{TV} by the term C_A due to the effect of the wind transporting heat, thus reducing the temperature gradient between the tropics and the poles. This energy is simultaneously converted by the term C_{TV} into time variability kinetic energy K_{TV} by the covariance of temperature and vertical motion. Those two important conversions take place on the storm track, where the mid-latitudes weather systems serve to reduce the temperature gradient, and their circulation creates vertical movements. Boundary flux terms F and H act as sources or sinks of energy, depending on the energy reservoir.

1.7.2 Conclusions for the storm energy cycle

The equations for the time variability part of the energy cycle were employed to study the energetics of the life cycle (creation, development and transport out of the domain) of a specific storm. The chosen storm started in the Atlantic Ocean, off the shore of Florida, and followed the coastline up to the Labrador Sea, where it exited the domain through the northeast boundary.

The patterns of available enthalpy a_{TV} and kinetic energy k_{TV} clearly followed the temperature gradient anomaly created by the passage of the storm and the wind anomaly created by the circulation of air around it. The generation of energy g_{TV} followed the frontal lines, and the dissipation of energy d_{TV} followed the maximum intensity of the wind. The conversion terms c_a and c_{TV} had very similar fields and followed the storm track, advecting temperature to destroy the mean temperature gradient and causing warm (cold) air masses to rise (sink). The transport of energy terms $f_{a_{TV}}$ and $f_{k_{TV}}$ showed that energy is transported from zones of divergence toward zones of convergence, in the general direction of the time-mean wind.

1.7.3 Further studies

In the context of anticipated climate changes due to the continued release of greenhouse gases by anthropogenic activities, local energy budgets could be computed with future climate projections for the later part of this century for instance. The comparison of current and future energy budgets could provide a general overview of how the local energetics should be expected to change. Indeed, the location, intensity and frequency of individual storms would quite clearly be reflected in a storm's energetics itself and in the climatological energetics of that region.

CONCLUSION

Les cycles d'énergie atmosphériques permettent de comprendre comment est redistribué l'apport énergétique continuel du soleil. Le réchauffement différentiel qu'il produit sur Terre crée un gradient de température latitudinal qui génère de l'énergie potentielle disponible. Cette énergie est ensuite convertie en énergie cinétique par le biais des systèmes météorologiques qui se développent et transportent la chaleur des basses vers les hautes latitudes, réduisant ainsi le gradient de température. Finalement, l'énergie cinétique est détruite par la friction. Ces deux réservoirs d'énergie (énergie potentielle disponible et énergie cinétique) peuvent être décomposés en termes de moyenne temporelle et de déviations par rapport à cette moyenne pour une meilleure représentation de l'effet des systèmes météorologiques sur l'énergétique atmosphérique.

Cette recherche avait pour buts de d'abord adapter le formalisme de NL13 pour le calcul d'un cycle d'énergie atmosphérique applicable localement, puis, d'étudier l'énergétique des systèmes météorologiques sur une région centrée sur l'est du continent nord-américain. Le premier but a été atteint en modifiant le cycle d'énergie inter-membre de NL13 pour en faire un cycle basé sur la moyenne temporelle et la variabilité temporelle. Le deuxième but, quant à lui, a été atteint en appliquant ce nouveau formalisme à deux analyses énergétiques : un cycle d'énergie climatologique pour le mois de décembre 2004 et un cycle d'énergie de tempête pour une dépression ayant lieu durant ce même mois. Une simulation du CRCM5 pour le mois de décembre 2004, pilotée par les réanalyses, a fourni tous les champs nécessaires au calcul de ces bilans d'énergie. Ainsi, par le biais de cette recherche, il a été possible d'améliorer nos connaissances sur les procédés physiques impliqués dans le développement des systèmes météorologiques de l'est du continent nord-américain.

Six réservoirs d'énergie ont été considérés : l'enthalpie disponible due à la pression, la composante de stratification et la composante barocline de l'enthalpie disponible de moyenne temporelle due à la température, l'énergie cinétique de moyenne temporelle, l'enthalpie disponible de variabilité temporelle et l'énergie cinétique de variabilité temporelle (B, $A_{TM\,S}$, $A_{TM\,B}$, K_{TM}, A_{TV} et K_{TV}, respectivement). Les flux d'énergie suivants ont été considérés : conversion, production, dissipation, transport et flux aux frontières (I/C, G, D, F et H, respectivement). La tendance temporelle de chacun des six réservoirs d'énergie et la somme algébrique de tous ses flux ont été comparés et ont montrés un bon accord, confirmant la validité des ensembles d'équations utilisés.

L'énergie est principalement produite par $G_{TM\,S}$, l'effet combiné des sources diabatiques de chaleur et de la température, et principalement dissipée par D_{TV}, l'effet de la friction ralentissant puis détruisant les vents dans la couche limite. Dans la partie du cycle d'énergie associée aux réservoirs de moyennes temporelles $(B, A_{TM\,S}, A_{TM\,B} \text{ et } K_{TM})$, une très importante quantité d'énergie est transportée et convertie sans en affecter la partie de variabilité temporelle $(A_{TV} \text{ et } K_{TV})$. L'enthalpie disponible est convertie de ses états de moyenne temporelle $A_{TM\,S}$ et $A_{TM\,B}$ à son état de variabilité temporelle A_{TV} par C_A, l'effet du vent transportant la chaleur des basses vers les hautes latitudes de façon à réduire le gradient de température latitudinal. Cette énergie est simultanément convertie en énergie cinétique de variabilité temporelle K_{TV} par C_{TV}, l'effet de la covariance du mouvement vertical et de la température. Ces deux importantes conversions ont lieu sur la trajectoire de tempêtes, là où les systèmes météorologiques des latitudes moyennes réduisent le gradient de température latitudinal et où leur circulation génère d'importants mouvements verticaux. Les termes de flux F et H agissent comme sources ou puits d'énergie selon le réservoir, dépendamment de la quantité d'énergie qui entre et sort du domaine régional pour la période considérée.

L'ensemble d'équations du cycle d'énergie climatologique a été légèrement modifié pour l'analyse énergétique de tempête : il a été pris avant que la moyenne temporelle ne soit appliquée. Ainsi, un ensemble d'équations instantanées d'énergie a été obtenu, adapté pour l'étude de l'énergétique du cycle de vie d'une tempête spécifique. La tempête choisie s'est créée dans l'océan Atlantique, sur la côte de la Floride et a suivi le littoral en s'intensifiant jusqu'à la mer du Labrador où elle est sortie du domaine par la frontière nord-est.

Les patrons d'enthalpie disponible a_{TV} et d'énergie cinétique k_{TV} suivent respectivement l'anomalie de gradient de température créée par le passage de la tempête et l'anomalie de vent créée par sa circulation cyclonique. La production d'énergie g_{TV} suit les délimitations des fronts et la dissipation d'énergie d_{TV} suit le maximum d'intensité du vent. Les termes de conversion c_a et c_{TV} ont des champs très similaires et suivent la trajectoire de tempêtes. c_a agit de manière à réduire le gradient de température latitudinal tandis que c_{TV} provoque les mouvements verticaux des masses d'air. Les termes de transport d'énergie $f_{a_{TV}}$ et $f_{k_{TV}}$ montrent que a_{TV} et k_{TV} sont transportés de zones de divergence vers des zones de convergence, en se déplaçant dans la direction générale du vent moyen.

Pour faire suite à cette recherche, l'analyse de bilans d'énergie à l'échelle régionale pourrait être appliquée dans le contexte des changements climatiques : des bilans pourraient être calculés avec des simulations climatique futures. D'abord, la satisfaction de l'ensemble d'équations énergétiques servirait de validation pour les simulations. Puis, la comparaison entre les cycles d'énergie actuels et futurs permettrait de fournir un aperçu général des changements auxquels s'attendre. En effet, les différences de localisation, d'intensité et de fréquence de tempêtes individuelles transparaitraient assez clairement dans les bilans énergétiques de la région étudiée.

APPENDICE A

MATHEMATICAL DEVELOPMENT OF THE TIME TENDENCY EQUATIONS FOR ENERGY RESERVOIRS A_{TM}, a_{TV}, A_{TV}, B, K_{TM}, k_{TV} AND K_{TV}

A.1 Basic equations in the atmospheric system

Equations are expressed in spherical coordinates in the horizontal and in pressure in the vertical: λ, φ, p.

Momentum equation:

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \vec{\nabla} \vec{V} + \omega \frac{\partial \vec{V}}{\partial p} + f\hat{k} \times \vec{V} + \vec{\nabla} \Phi - \vec{F} = 0 \tag{A.1}$$

Approximate thermodynamics equation (see details in Section A.3):

$$\frac{\partial T}{\partial t} + \vec{V} \bullet \vec{\nabla} T + \omega \frac{\partial T}{\partial p} - \frac{R}{c_p p} \omega T_r - \frac{T_r}{c_p} \frac{Q}{T} = 0 \tag{A.2}$$

Continuity equation:

$$\vec{\nabla} \bullet \vec{V} + \frac{\partial \omega}{\partial p} = 0 \tag{A.3}$$

Hydrostatic equilibrium:

$$\frac{\partial \Phi}{\partial p} + \alpha = 0 \tag{A.4}$$

State law for ideal gas:

$$\alpha - \frac{RT}{p} = 0 \tag{A.5}$$

Where:

a : Average Earth radius

c_p : Specific heat at constant pressure

$f = 2\Omega \sin\varphi$: Coriolis parameter

$\vec{F}$: Horizontal momentum sources/sinks

Q : Total diabatic heating rate

R : Gas constant for dry air

T : Air temperature

$\vec{V}(u,v)$: Horizontal wind vector

$$u = a\cos\varphi \frac{\partial \lambda}{\partial t}$$

$$v = a \frac{\partial \varphi}{\partial t}$$

z : Altitude

α : Specific volume

λ : Longitude

φ : Latitude

$\omega = \frac{\partial p}{\partial t}$: Vertical motion

$\vec{\nabla}$: Lateral gradient

$\frac{\partial}{\partial t}$: Local time derivative

$\Phi = gz$: Geopotential height

Ω : Earth's rotation rate

A.2 Mathematical manipulations

A.2.1 Vector derivatives

For each atmospheric variable $\Psi \in \left\{ \vec{F}, Q, T, \vec{V}(u,v), \alpha, \omega, \Phi \right\}$:

$$\vec{\nabla}\Psi = \frac{1}{a\cos\varphi}\left(\frac{\partial\Psi}{\partial\lambda}\hat{\mathrm{i}} + \cos\phi\frac{\partial\Psi}{\partial\varphi}\hat{\mathrm{j}}\right)$$

$$\vec{V}\bullet\vec{\nabla}\Psi = \frac{u}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda} + \frac{v}{a}\frac{\partial\Psi}{\partial\varphi}$$

$$\vec{\nabla}\bullet\left(\vec{V}\Psi\right) = \frac{1}{a\cos\varphi}\left(\frac{\partial(u\Psi)}{\partial\lambda} + \frac{\partial(v\Psi\cos\varphi)}{\partial\varphi}\right)$$

A.2.2 Time-mean (TM) and time variability (TV) decomposition

Each atmospheric variable Ψ is decomposed as:

$\Psi = \langle \Psi \rangle + \Psi'$,

$\langle \Psi \rangle = \frac{1}{\tau}\sum_{t=1}^{\tau}\Psi_t$ is the time-mean state, with τ the number of time samples;

$\Psi' = \Psi - \langle \Psi \rangle$ is the deviation from the time-mean state.

A.2.3 Identities from Reynolds averaging rules

For each atmospheric variables Ψ and ψ :

$$\langle\langle \Psi \rangle\rangle = \langle \Psi \rangle \qquad \text{(A.i)}$$

$$\langle \Psi + \psi \rangle = \langle \Psi \rangle + \langle \psi \rangle \qquad \text{(A.ii)}$$

$$\langle \langle \Psi \rangle \psi \rangle = \langle \Psi \rangle \langle \psi \rangle \qquad \text{(A.iii)}$$

$$\left\langle \frac{\partial \Psi}{\partial x} \right\rangle = \frac{\partial \langle \Psi \rangle}{\partial x} \qquad \text{(A.iv)}$$

$$\left\langle \vec{\nabla} \Psi \right\rangle = \vec{\nabla} \langle \Psi \rangle \qquad \text{(A.v)}$$

$$\langle \Psi' \rangle = 0 \qquad \text{(A.vi)}$$

$$\langle \Psi \psi \rangle = \langle \Psi \rangle \langle \psi \rangle + \langle \Psi' \psi' \rangle \qquad \text{(A.vii)}$$

$$\Psi \psi - \langle \Psi \psi \rangle = \Psi' \langle \psi \rangle + \psi' \langle \Psi \rangle + \Psi' \psi' - \langle \Psi' \psi' \rangle \qquad \text{(A.viii)}$$

Where Ψ and ψ are scalars or vectors.

A.2.4 Time-mean basic atmospheric equations

Time-mean momentum equation:

$$\frac{\partial\langle\vec{V}\rangle}{\partial t}+\langle\vec{V}\bullet\vec{\nabla}\vec{V}\rangle+\left\langle\omega\frac{\partial\vec{V}}{\partial p}\right\rangle+f\hat{k}\times\langle\vec{V}\rangle+\vec{\nabla}\langle\Phi\rangle-\langle\vec{F}\rangle=0 \quad \text{(A.1a)}$$

Time-mean approximate thermodynamics equation:

$$\frac{\partial\langle T\rangle}{\partial t}+\langle\vec{V}\bullet\vec{\nabla}T\rangle+\left\langle\omega\frac{\partial T}{\partial p}\right\rangle-\frac{R}{c_p p}\langle\omega\rangle T_r-\frac{T_r}{c_p}\left\langle\frac{Q}{T}\right\rangle=0 \quad \text{(A.2a)}$$

Time-mean continuity equation:

$$\vec{\nabla}\bullet\langle\vec{V}\rangle+\frac{\partial\langle\omega\rangle}{\partial p}=0 \quad \text{(A.3a)}$$

Time-mean hydrostatic equilibrium:

$$\frac{\partial\langle\Phi\rangle}{\partial p}+\langle\alpha\rangle=0 \quad \text{(A.4a)}$$

Time-mean state law for ideal gas:

$$\langle\alpha\rangle-\frac{R}{p}\langle T\rangle=0 \quad \text{(A.5a)}$$

A.2.5 Time variability basic atmospheric equations

Time variability momentum equation:

$$\frac{\partial \vec{V}'}{\partial t}+\left(\vec{V}\bullet\vec{\nabla}\vec{V}-\left\langle\vec{V}\bullet\vec{\nabla}\vec{V}\right\rangle\right)+\left(\omega\frac{\partial\vec{V}}{\partial p}-\left\langle\omega\frac{\partial\vec{V}}{\partial p}\right\rangle\right)+f\hat{k}\times\vec{V}'+\vec{\nabla}\Phi'-\vec{F}'=0 \qquad \text{(A.1b)}$$

Time variability approximate thermodynamics equation:

$$\frac{\partial T'}{\partial t}+\left(\vec{V}\bullet\vec{\nabla}T-\left\langle\vec{V}\bullet\vec{\nabla}T\right\rangle\right)+\left(\omega\frac{\partial T}{\partial p}-\left\langle\omega\frac{\partial T}{\partial p}\right\rangle\right)-\frac{R}{c_p p}\omega' T_r-\frac{T_r}{c_p}\left(\frac{Q}{T}-\left\langle\frac{Q}{T}\right\rangle\right)=0 \qquad \text{(A.2b)}$$

Time variability continuity equation:

$$\vec{\nabla}\bullet\vec{V}'+\frac{\partial\omega'}{\partial p}=0 \qquad \text{(A.3b)}$$

Time variability hydrostatic equilibrium:

$$\frac{\partial\Phi'}{\partial p}+\alpha'=0 \qquad \text{(A.4b)}$$

Time variability state law for ideal gas:

$$\alpha'-\frac{R}{p}T'=0 \qquad \text{(A.5b)}$$

A.2.6 Flux and advection forms

Each atmospheric variable Ψ can be switched from its flux form to its advection form using eq. (A.3), eq. (A.3a) or eq. (A.3b) as follows:

$$\vec{\nabla}\cdot\left(\Psi\vec{V}\right)+\frac{\partial(\Psi\omega)}{\partial p}=\Psi\vec{\nabla}\cdot\vec{V}+\vec{V}\cdot\vec{\nabla}\Psi+\Psi\frac{\partial\omega}{\partial p}+\omega\frac{\partial\Psi}{\partial p}$$

$$=\Psi\cancel{\left(\vec{\nabla}\cdot\vec{V}+\frac{\partial\omega}{\partial p}\right)}+\vec{V}\cdot\vec{\nabla}\Psi+\omega\frac{\partial\Psi}{\partial p}$$

$$=\vec{V}\cdot\vec{\nabla}\Psi+\omega\frac{\partial\Psi}{\partial p}$$

$$\vec{\nabla}\cdot\left(\Psi\left\langle\vec{V}\right\rangle\right)+\frac{\partial\left(\Psi\langle\omega\rangle\right)}{\partial p}=\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\Psi+\langle\omega\rangle\frac{\partial\Psi}{\partial p}$$

$$\vec{\nabla}\cdot\left(\Psi\vec{V}'\right)+\frac{\partial(\Psi\omega')}{\partial p}=\vec{V}'\cdot\vec{\nabla}\Psi+\omega'\frac{\partial\Psi}{\partial p}$$

Using the inverse method, the equations can be switched from their advection forms to their flux forms.

A.3 Available enthalpy equations

$$a_h=\left(H-H_r\right)-T_r\left(S-S_r\right)$$

Where:

a_h : Available enthalpy

$H=c_pT$: Enthalpy

H_r : Reference enthalpy

$S=c_p\ln\left(\frac{\theta}{\theta_r}\right)+S_r$: Entropy

S_r : Reference entropy

θ : Potential temperature

θ_r : Reference potential temperature

Following Marquet (1991), a_h is split into separate contributions, depending on temperature and pressure:

$$a_h(T,p) = a_T(T) + a_p(p)$$

Exact temperature-dependent part of available enthalpy equation:

$$a_T(T) = c_p T_r \Im(\chi)$$
$$\Im(\chi) = \chi - \ln(1+\chi)$$
$$\chi = \frac{T - T_r}{T_r}$$

$$a_T(T) = c_p T_r \left[\left(\frac{T - T_r}{T_r} \right) - \ln\left(\frac{T}{T_r} \right) \right]$$

Where:

$T_r = \left\langle \frac{1}{\bar{T}} \right\rangle^{-1}$: Reference temperature.

$\bar{T}$: Domain-averaged value of T.

T_r is chosen to minimise the value of χ over the domain of interest. T deviates from T_r only by less than $\pm\ 20\%$. Hence χ is a small quantity that can be approximated by series expansion around $a_T(T_r)$.

Approximate temperature-dependent part of available enthalpy equation:

$$a_T(T) \simeq \frac{c_p T_r}{2}\left(\frac{T - T_r}{T_r}\right)^2 = \frac{c_p}{2T_r}(T - T_r)^2 \tag{A.6}$$

Approximate prognostic equation for the temperature-dependent part of available enthalpy:

$$\frac{da_T}{dt} = \frac{c_p}{T_r}(T - T_r)\frac{dT}{dt} \tag{A.7}$$

Approximate thermodynamics equation implied by the approximate prognostic equation:

$$\frac{\partial T}{\partial t} + \vec{V} \cdot \vec{\nabla} T + \omega \frac{\partial T}{\partial p} - \frac{R}{c_p p}\omega T_r - \frac{T_r}{c_p}\frac{Q}{T} = 0$$

Pressure-dependent part of available enthalpy equation:

$$a_p(p) = RT_r \ln\left(\frac{p}{p_r}\right) \tag{A.8}$$

Where:

$p_r = \frac{p_{00}}{e}$: Reference pressure

$p_{00} = \langle \overline{p}_s \rangle$

p_s : Surface pressure

$\overline{p}_s$: Domain-averaged value of p_s

Prognostic equation for the pressure-dependent part of available enthalpy:

$$\frac{da_p}{dt} = \frac{R\omega}{p} T_r \tag{A.9}$$

A.3.7 Time-mean temperature-dependent part of available enthalpy prognostic equation (A_{TM})

We express eq. (A.2a) in its flux form:

$$\frac{\partial \langle T \rangle}{\partial t} + \vec{\nabla} \bullet \left\langle T \vec{V} \right\rangle + \frac{\partial \langle T\omega \rangle}{\partial p} - \frac{R}{c_p p} \langle \omega \rangle T_r - \frac{T_r}{c_p} \left\langle \frac{Q}{T} \right\rangle = 0 \tag{A.10}$$

We multiply eq. (A.10) by $\frac{c_p}{T_r} \langle T - T_r \rangle$:

$$\begin{aligned} &\frac{c_p}{T_r} \langle T - T_r \rangle \frac{\partial \langle T \rangle}{\partial t} + \frac{c_p}{T_r} \langle T - T_r \rangle \vec{\nabla} \bullet \left\langle T \vec{V} \right\rangle + \frac{c_p}{T_r} \langle T - T_r \rangle \frac{\partial \langle T\omega \rangle}{\partial p} - \\ &\langle T - T_r \rangle \frac{R}{p} \langle \omega \rangle - \langle T - T_r \rangle \left\langle \frac{Q}{T} \right\rangle = 0 \end{aligned} \tag{A.11}$$

By definition, $T_* = T - T_r$, $A_{TM} \equiv \langle a_T \rangle = \frac{c_p}{2T_r} \langle T_* \rangle^2$ by eq. (A.6), and $\frac{\partial (A_{TM})}{\partial t} \equiv \frac{\partial \langle a_T \rangle}{\partial t} = \frac{c_p}{T_r} \langle T_* \rangle \frac{\partial T}{\partial t}$ by eq. (A.7), where A_{TM} is the time-mean available enthalpy. Eq. (A.11) becomes:

$$\frac{\partial A_{TM}}{\partial t} + \frac{c_p}{T_r} \langle T_* \rangle \vec{\nabla} \bullet \left\langle T \vec{V} \right\rangle + \frac{c_p}{T_r} \langle T_* \rangle \frac{\partial \langle T\omega \rangle}{\partial p} - \langle T_* \rangle \frac{R}{p} \langle \omega \rangle - \langle T_* \rangle \left\langle \frac{Q}{T} \right\rangle = 0 \tag{A.12}$$

We rewrite the second and third terms of eq. (A.12) in their advection forms and further decompose them with eq. (A.vii) as follows:

$$\begin{aligned}\vec{\nabla}\cdot\left\langle T\vec{V}\right\rangle+\frac{\partial\langle T\omega\rangle}{\partial p}&=\left\langle\vec{V}\cdot\vec{\nabla}T_*\right\rangle+\left\langle\omega\frac{\partial T_*}{\partial p}\right\rangle\\&=\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\langle T_*\rangle+\langle\omega\rangle\vec{\nabla}\langle T_*\rangle+\left\langle\vec{V}'\cdot\vec{\nabla}T'\right\rangle+\left\langle\omega'\frac{\partial T'}{\partial p}\right\rangle\end{aligned}\tag{A.13}$$

We insert eq. (A.13) in eq. (A.12):

$$\begin{aligned}&\frac{\partial A_{TM}}{\partial t}+\frac{c_p}{2T_r}\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\langle T_*\rangle^2+\frac{c_p}{2T_r}\langle\omega\rangle\frac{\partial\langle T_*\rangle^2}{\partial p}+\frac{c_p}{T_r}\langle T_*\rangle\left\langle\vec{V}'\cdot\vec{\nabla}T'\right\rangle+\\&\frac{c_p}{T_r}\langle T_*\rangle\left\langle\omega'\frac{\partial T'}{\partial p}\right\rangle-\frac{R}{p}\langle\omega\rangle\langle T_*\rangle-\langle T_*\rangle\left\langle\frac{Q}{T}\right\rangle=0\end{aligned}\tag{A.14}$$

We rewrite the second and third terms of eq. (A.14) in their flux forms:

$$\begin{aligned}&\frac{\partial A_{TM}}{\partial t}+\vec{\nabla}\cdot\left(\left\langle\vec{V}\right\rangle A_{TM}\right)+\frac{\partial(\langle\omega\rangle A_{TM})}{\partial p}+\frac{c_p}{T_r}\langle T_*\rangle\left(\left\langle\vec{\nabla}\cdot\left(\vec{V}'T'\right)+\frac{\partial(\omega'T')}{\partial p}\right\rangle\right)\\&-\frac{R}{p}\langle\omega\rangle\langle T_*\rangle-\langle T_*\rangle\left\langle\frac{Q}{T}\right\rangle=0\end{aligned}\tag{A.15}$$

The fourth term of eq. (A.15) is expanded as follows:

$$\begin{aligned}&\frac{c_p}{T_r}\langle T_*\rangle\left(\left\langle\vec{\nabla}\cdot\left(\vec{V}'T'\right)+\frac{\partial(\omega'T')}{\partial p}\right\rangle\right)=\frac{c_p}{T_r}\vec{\nabla}\cdot\left(\langle T_*\rangle\left\langle\vec{V}'T'\right\rangle\right)+\frac{c_p}{T_r}\frac{\partial(\langle T_*\rangle\langle\omega'T'\rangle)}{\partial p}\\&-\frac{c_p}{T_r}\left\langle\vec{V}'T'\right\rangle\cdot\vec{\nabla}\langle T\rangle-\frac{c_p}{T_r}\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}\end{aligned}\tag{A.16}$$

We insert eq. (A.16) in eq. (A.15):

$$\frac{\partial A_{TM}}{\partial t}+\vec{\nabla}\bullet\left(\left\langle\vec{V}\right\rangle A_{TM}\right)+\frac{\partial\left(\langle\omega\rangle A_{TM}\right)}{\partial p}+\frac{c_p}{T_r}\vec{\nabla}\bullet\left(\langle T_*\rangle\left\langle\vec{V}'T'\right\rangle\right)+\frac{c_p}{T_r}\frac{\partial\left(\langle T_*\rangle\langle\omega'T'\rangle\right)}{\partial p}$$
$$-\frac{c_p}{T_r}\left\langle\vec{V}'T'\right\rangle\bullet\vec{\nabla}\langle T\rangle-\frac{c_p}{T_r}\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}-\frac{R}{p}\langle\omega\rangle\langle T_*\rangle-\langle T_*\rangle\left\langle\frac{Q}{T}\right\rangle=0 \tag{A.17}$$

Using the small- χ approximation, the last term of eq. (A.17) is rewritten as:

$$\langle T_*\rangle\left\langle\frac{Q}{T}\right\rangle=\frac{T_r}{T_r}\langle T_*\rangle\left\langle\frac{Q}{T}\right\rangle\simeq\frac{1}{T_r}\frac{T_r}{\langle T\rangle}\langle T_*\rangle\langle Q\rangle=\frac{l}{T_r}\langle T_*\rangle\langle Q\rangle \tag{A.18}$$

with l an order unity factor.

We insert eq. (A.18) in eq. (A.17):

$$\frac{\partial A_{TM}}{\partial t}+\vec{\nabla}\bullet\left(\left\langle\vec{V}\right\rangle A_{TM}\right)+\frac{\partial\left(\langle\omega\rangle A_{TM}\right)}{\partial p}+\frac{c_p}{T_r}\vec{\nabla}\bullet\left(\langle T_*\rangle\left\langle\vec{V}'T'\right\rangle\right)+\frac{c_p}{T_r}\frac{\partial\left(\langle T_*\rangle\langle\omega'T'\rangle\right)}{\partial p}$$
$$-\frac{c_p}{T_r}\left\langle\vec{V}'T'\right\rangle\bullet\vec{\nabla}\langle T\rangle-\frac{c_p}{T_r}\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}-\frac{R}{p}\langle\omega\rangle\langle T_*\rangle-\frac{l}{T_r}\langle T_*\rangle\langle Q\rangle=0 \tag{A.19}$$

The final form for the time-mean temperature-dependent part of available enthalpy:

$$\frac{\partial A_{TM}}{\partial t}=\frac{l}{T_r}\langle T_*\rangle\langle Q\rangle+\frac{RT_r}{p}\langle\omega\rangle+\langle\omega\rangle\langle\alpha\rangle+\frac{c_p}{T_r}\left\langle\vec{V}'T'\right\rangle\bullet\vec{\nabla}\langle T\rangle+\frac{c_p}{T_r}\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}-$$
$$\vec{\nabla}\bullet\left(A_{TM}\left\langle\vec{V}\right\rangle\right)-\frac{\partial\left(A_{TM}\langle\omega\rangle\right)}{\partial p}-\frac{c_p}{T_r}\left(\vec{\nabla}\bullet\left(\langle T_*\rangle\left\langle\vec{V}'T'\right\rangle\right)+\frac{\partial\left(\langle T_*\rangle\langle\omega'T'\rangle\right)}{\partial p}\right) \tag{A.20}$$

$$\frac{\partial A_{TM}}{\partial t} = G_{TM} + I_{AB} - C_{TM} - C_A - F_{A_{TM}} - H_{A_{TM}} \tag{A.21}$$

Where:

$$A_{TM} = \frac{c_p}{2T_r}\langle T_* \rangle^2$$

$$G_{TM} = \frac{l}{T_r}\langle T_* \rangle \langle Q \rangle$$

$$I_{AB} = -\frac{RT_r}{p}\langle \omega \rangle$$

$$C_{TM} = -\langle \omega \rangle \langle \alpha \rangle$$

$$C_A = -\frac{c_p}{T_r}\left\langle \vec{V}'T' \right\rangle \bullet \vec{\nabla}\langle T \rangle - \frac{c_p}{T_r}\langle \omega' T' \rangle \frac{\partial \langle T \rangle}{\partial p}$$

$$F_{A_{TM}} = \vec{\nabla} \bullet \left(A_{TM} \left\langle \vec{V} \right\rangle \right) + \frac{\partial \left(A_{TM} \langle \omega \rangle \right)}{\partial p}$$

$$H_{A_{TM}} = \frac{c_p}{T_r}\left(\vec{\nabla} \bullet \left(\langle T_* \rangle \left\langle \vec{V}'T' \right\rangle \right) + \frac{\partial \left(\langle T_* \rangle \langle \omega' T' \rangle \right)}{\partial p} \right)$$

A.3.8 Instantaneous time variability temperature-dependent part of available enthalpy prognostic equation (a_{TV})

We express eq. (A.2b) in its flux form:

$$\frac{\partial T'}{\partial t} + \vec{\nabla} \bullet \left(T\vec{V} - \left\langle T\vec{V} \right\rangle \right) + \frac{\partial \left(T\omega - \langle T\omega \rangle \right)}{\partial p} - \frac{R}{c_p p}\omega' T_r - \frac{T_r}{c_p}\left(\frac{Q}{T} - \left\langle \frac{Q}{T} \right\rangle \right) = 0 \tag{A.22}$$

We apply eq. (A.viii) to the second and third terms of eq. (A.22) as follows:

$$\begin{aligned}&\left(T\vec{V}-\left\langle T\vec{V}\right\rangle\right)=T'\left\langle\vec{V}\right\rangle+\vec{V}'\langle T\rangle+T'\vec{V}'-\left\langle T'\vec{V}'\right\rangle\\&\left(T\omega-\langle T\omega\rangle\right)=T'\langle\omega\rangle+\omega'\langle T\rangle+T'\omega'-\langle T'\omega'\rangle\end{aligned}\tag{A.23}$$

We insert eq. (A.23) in eq. (A.22):

$$\begin{aligned}&\frac{\partial T'}{\partial t}+\vec{\nabla}\cdot\left(T'\left\langle\vec{V}\right\rangle+\vec{V}'\langle T\rangle+T'\vec{V}'-\left\langle T'\vec{V}'\right\rangle\right)+\\&\frac{\partial\left(T'\langle\omega\rangle+\omega'\langle T\rangle+T'\omega'-\langle T'\omega'\rangle\right)}{\partial p}-\frac{R}{c_p p}\omega'T_r-\frac{T_r}{c_p}\left(\frac{Q}{T}-\left\langle\frac{Q}{T}\right\rangle\right)=0\end{aligned}\tag{A.24}$$

We develop the second and third terms of eq. (A.24) and develop some of their expressions in their advection forms:

$$\begin{aligned}&\frac{\partial T'}{\partial t}+\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}T'+\langle\omega\rangle\frac{\partial T'}{\partial p}+\vec{V}'\cdot\vec{\nabla}\langle T\rangle+\omega'\frac{\partial\langle T\rangle}{\partial p}+\vec{\nabla}\cdot\left(\vec{V}'T'\right)+\\&\frac{\partial(\omega'T')}{\partial p}-\vec{\nabla}\cdot\left\langle\vec{V}'T'\right\rangle-\frac{\partial\langle\omega'T'\rangle}{\partial p}-\frac{RT_r}{c_p p}\omega'-\frac{1}{c_p}\left(\frac{T_r}{T}Q-\left\langle\frac{T_r}{T}Q\right\rangle\right)=0\end{aligned}\tag{A.25}$$

We multiply eq. (A.25) by T':

$$\begin{aligned}&\frac{1}{2}\frac{\partial\left(T'^2\right)}{\partial t}+\frac{1}{2}\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\left(T'^2\right)+\frac{1}{2}\langle\omega\rangle\frac{\partial\left(T'^2\right)}{\partial p}+\left(\vec{V}'T'\right)\cdot\vec{\nabla}\langle T\rangle+(\omega'T')\frac{\partial\langle T\rangle}{\partial p}+\\&T'\vec{\nabla}\cdot\left(\vec{V}'T'\right)+T'\frac{\partial(\omega'T')}{\partial p}-T'\vec{\nabla}\cdot\left\langle\vec{V}'T'\right\rangle-T'\frac{\partial\langle\omega'T'\rangle}{\partial p}-\frac{RT_r}{c_p p}\omega'T'-\\&\frac{T'}{c_p}\left(\frac{T_r}{T}Q-\left\langle\frac{T_r}{T}Q\right\rangle\right)=0\end{aligned}\tag{A.26}$$

We multiply eq. (A.26) by $\frac{c_p}{T_r}$:

$$\frac{c_p}{2T_r}\frac{\partial\left(T'^2\right)}{\partial t}+\frac{c_p}{2T_r}\left\langle\vec{V}\right\rangle\bullet\vec{\nabla}\left(T'^2\right)+\frac{c_p}{2T_r}\langle\omega\rangle\frac{\partial\left(T'^2\right)}{\partial p}+\frac{c_p}{T_r}\left(\vec{V}'T'\right)\bullet\vec{\nabla}\langle T\rangle+$$
$$\frac{c_p}{T_r}(\omega'T')\frac{\partial\langle T\rangle}{\partial p}+\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left(\vec{V}'T'\right)+\frac{c_pT'}{T_r}\frac{\partial(\omega'T')}{\partial p}-\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'\right\rangle- \quad \text{(A.27)}$$
$$\frac{c_pT'}{T_r}\frac{\partial\langle\omega'T'\rangle}{\partial p}-\frac{R}{p}\omega'T'-\frac{T'Q}{T}-T'\left\langle\frac{Q}{T}\right\rangle=0$$

By definition, $a_{TV}\equiv\frac{c_p}{2T_r}\left(T'^2\right)$, where a_{TV} is the instantaneous time variability available enthalpy. Eq. (A.27) becomes:

$$\frac{\partial(a_{TV})}{\partial t}+\left\langle\vec{V}\right\rangle\bullet\vec{\nabla}(a_{TV})+\langle\omega\rangle\frac{\partial(a_{TV})}{\partial p}+\left(\vec{V}'\frac{T'}{T_r}\right)\bullet\vec{\nabla}(c_p\langle T\rangle)+$$
$$\left(\omega'\frac{T'}{T_r}\right)\frac{\partial(c_p\langle T\rangle)}{\partial p}+\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left(\vec{V}'T'\right)+\frac{c_pT'}{T_r}\frac{\partial(\omega'T')}{\partial p}-\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'\right\rangle- \quad \text{(A.28)}$$
$$\frac{c_pT'}{T_r}\frac{\partial\langle\omega'T'\rangle}{\partial p}-\omega'\alpha'-\frac{T'Q}{T}-T'\left\langle\frac{Q}{T}\right\rangle=0$$

We express the second and third terms of eq. (A.28) in their flux forms:

$$\frac{\partial(a_{TV})}{\partial t}+\vec{\nabla}\bullet\left(a_{TV}\left\langle\vec{V}\right\rangle\right)+\frac{\partial(a_{TV}\langle\omega\rangle)}{\partial p}+\left(\vec{V}'\frac{T'}{T_r}\right)\bullet\vec{\nabla}(c_p\langle T\rangle)+$$
$$\left(\omega'\frac{T'}{T_r}\right)\frac{\partial(c_p\langle T\rangle)}{\partial p}+\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left(\vec{V}'T'\right)+\frac{c_pT'}{T_r}\frac{\partial(\omega'T')}{\partial p}-\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'\right\rangle- \quad \text{(A.29)}$$
$$\frac{c_pT'}{T_r}\frac{\partial\langle\omega'T'\rangle}{\partial p}-\omega'\alpha'-\frac{T'Q}{T}-T'\left\langle\frac{Q}{T}\right\rangle=0$$

The eleventh and ninth terms of eq. (A.29) are rewritten as follows:

$$\frac{T'Q}{T}=\frac{T'\langle Q\rangle}{T}+\frac{T'Q'}{T}=\frac{T'\langle Q\rangle}{T}+\frac{T_r}{T}\left(\frac{T'Q'}{T_r}\right)$$
$$\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left(\vec{V}'T'\right)+\frac{c_pT'}{T_r}\frac{\partial(\omega'T')}{\partial p}=\frac{c_p}{2T_r}\vec{\nabla}\bullet\left(\vec{V}'T'^2\right)+\frac{c_pT'}{T_r}\frac{\partial\left(\omega'T'^2\right)}{\partial p} \tag{A.30}$$

We insert eq. (A.30) in eq. (A.29):

$$\frac{\partial(a_{TV})}{\partial t}+\vec{\nabla}\bullet\left(a_{TV}\left\langle\vec{V}\right\rangle\right)+\frac{\partial(a_{TV}\langle\omega\rangle)}{\partial p}+\left(\vec{V}'\frac{T'}{T_r}\right)\bullet\vec{\nabla}\left(c_p\langle T\rangle\right)+$$
$$\left(\omega'\frac{T'}{T_r}\right)\frac{\partial\left(c_p\langle T\rangle\right)}{\partial p}+\frac{c_p}{2T_r}\vec{\nabla}\bullet\left(\vec{V}'T'^2\right)+\frac{c_p}{2T_r}\frac{\partial\left(\omega'T'^2\right)}{\partial p}-\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'\right\rangle- \tag{A.31}$$
$$\frac{c_pT'}{T_r}\frac{\partial\langle\omega'T'\rangle}{\partial p}-\omega'\alpha'-\frac{T'\langle Q\rangle}{T}-\frac{T_r}{T}\left(\frac{T'Q'}{T_r}\right)-T'\left\langle\frac{Q}{T}\right\rangle=0$$

The final form for the instantaneous time variability temperature-dependent part of available enthalpy:

$$\frac{\partial(a_{TV})}{\partial t}=\frac{T_r}{T}\left(\frac{T'Q'}{T_r}\right)+\omega'\alpha'-\frac{c_p}{T_r}\left(\vec{V}'T'\right)\bullet\vec{\nabla}\langle T\rangle-\frac{c_p}{T_r}(\omega'T')\frac{\partial\langle T\rangle}{\partial p}-$$
$$\vec{\nabla}\bullet\left(a_{TV}\left\langle\vec{V}\right\rangle\right)-\frac{\partial(a_{TV}\langle\omega\rangle)}{\partial p}-\frac{c_p}{2T_r}\vec{\nabla}\bullet\left(\vec{V}'T'^2\right)-\frac{c_p}{2T_r}\frac{\partial\left(\omega'T'^2\right)}{\partial p}+ \tag{A.32}$$
$$\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'\right\rangle+\frac{c_pT'}{T_r}\frac{\partial\langle\omega'T'\rangle}{\partial p}+T'\left(\left\langle\frac{Q}{T}\right\rangle+\frac{\langle Q\rangle}{T}\right)$$

$$\frac{\partial a_{TV}}{\partial t} = g_{TV} - c_{TV} + c_a - f_{a_{TV}} - h_{a_{TV}} - j_{a1} - j_{a2} \quad \text{(A.33)}$$

Where:

$$a_{TV} = \frac{c_p}{2T_r}\left(T'^2\right)$$

$$g_{TV} = \frac{T_r}{T}\left(\frac{T'Q'}{T_r}\right) = \frac{1}{T}(T'Q')$$

$$c_{TV} = -\omega'\alpha'$$

$$c_a = -\frac{c_p}{T_r}\left(\vec{V}'T'\right)\bullet\vec{\nabla}\langle T\rangle - \frac{c_p}{T_r}(\omega'T')\frac{\partial\langle T\rangle}{\partial p}$$

$$f_{a_{TV}} = \vec{\nabla}\bullet\left(a_{TV}\left\langle\vec{V}\right\rangle\right) + \frac{\partial\left(a_{TV}\langle\omega\rangle\right)}{\partial p}$$

$$h_{a_{TV}} = \frac{c_p}{2T_r}\left(\vec{\nabla}\bullet\left(\vec{V}'T'^2\right) + \frac{\partial\left(\omega'T'^2\right)}{\partial p}\right)$$

$$j_{a1} = -\frac{c_pT'}{T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'\right\rangle - \frac{c_pT'}{T_r}\frac{\partial\langle\omega'T'\rangle}{\partial p}, \qquad j_{a2} = -T'\left(\left\langle\frac{Q}{T}\right\rangle + \frac{\langle Q\rangle}{T}\right)$$

A.3.9 Time variability temperature-dependant part of available enthalpy prognostic equation $\left(A_{TV}\right)$

By definition, $A_{TV} \equiv \langle a_{TV}\rangle = \frac{c_p}{2T_r}\left\langle T'^2\right\rangle$, where A_{TV} is the time variability available enthalpy. We apply the time-mean operator on eq. (A.32):

$$\frac{\partial\left(A_{TV}\right)}{\partial t}=\frac{l}{T_r}\langle T'Q'\rangle+\langle\omega'\alpha'\rangle-\frac{c_p}{T_r}\left\langle\vec{V}'T'\right\rangle\bullet\vec{\nabla}\langle T\rangle-\frac{c_p}{T_r}\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}-$$
$$\vec{\nabla}\bullet\left(A_{TV}\left\langle\vec{V}\right\rangle\right)-\frac{\partial\left(A_{TV}\langle\omega\rangle\right)}{\partial p}-\frac{c_p}{2T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'^2\right\rangle-\frac{c_p}{2T_r}\frac{\partial\left\langle\omega'T'^2\right\rangle}{\partial p}+ \tag{A.34}$$
$$\cancel{\frac{c_p\langle T'\rangle}{T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'\right\rangle}+\cancel{\frac{c_p\langle T'\rangle}{T_r}\frac{\partial\langle\omega'T'\rangle}{\partial p}}+\cancel{\langle T'\rangle\left(\left\langle\frac{Q}{T}\right\rangle+\frac{\langle Q\rangle}{T}\right)}$$

The final form for the time variability temperature-dependent part of available enthalpy:

$$\frac{\partial\left(A_{TV}\right)}{\partial t}=\frac{l}{T_r}\langle T'Q'\rangle+\langle\omega'\alpha'\rangle-\frac{c_p}{T_r}\left\langle\vec{V}'T'\right\rangle\bullet\vec{\nabla}\langle T\rangle-\frac{c_p}{T_r}\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}-$$
$$\vec{\nabla}\bullet\left(A_{TV}\left\langle\vec{V}\right\rangle\right)-\frac{\partial\left(A_{TV}\langle\omega\rangle\right)}{\partial p}-\frac{c_p}{2T_r}\vec{\nabla}\bullet\left\langle\vec{V}'T'^2\right\rangle-\frac{c_p}{2T_r}\frac{\partial\left\langle\omega'T'^2\right\rangle}{\partial p} \tag{A.35}$$

$$\frac{\partial A_{TV}}{\partial t}=G_{TV}-C_{TV}+C_A-F_{A_{TV}}-H_{A_{TV}} \tag{A.36}$$

Where:

$$A_{TV}\equiv\left\langle a_{TV}\right\rangle=\frac{c_p}{2T_r}\left\langle T'^2\right\rangle$$

$$G_{TV}\equiv\left\langle g_{TV}\right\rangle=\left\langle\frac{T_r}{T}\left(\frac{T'Q'}{T_r}\right)\right\rangle\simeq\frac{l}{T_r}\langle T'Q'\rangle$$

$$C_{TV}\equiv\left\langle c_{TV}\right\rangle=-\langle\omega'\alpha'\rangle$$

$$C_A\equiv\left\langle c_a\right\rangle=-\frac{c_p}{T_r}\left\langle\vec{V}'T'\right\rangle\bullet\vec{\nabla}\langle T\rangle-\frac{c_p}{T_r}\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}$$

$$F_{A_{TV}}\equiv\left\langle f_{a_{TV}}\right\rangle=\vec{\nabla}\bullet\left(A_{TV}\left\langle\vec{V}\right\rangle\right)+\frac{\partial\left(A_{TV}\langle\omega\rangle\right)}{\partial p}$$

$$H_{A_{TV}} \equiv \langle h_{a_{TV}} \rangle = \frac{c_p}{2T_r}\left(\vec{\nabla} \bullet \left\langle \vec{V}' T'^2 \right\rangle + \frac{\partial \langle \omega' T'^2 \rangle}{\partial p} \right)$$

A.3.10 Time-mean pressure-dependent part of available enthalpy prognostic equation (B)

We express eq. (A.9) in its Eulerian form:

$$\frac{\partial a_p}{\partial t} + \vec{\nabla} \bullet \left(a_p \vec{V} \right) + \frac{\partial (a_p \omega)}{\partial p} - \frac{R\omega}{p} T_r = 0 \tag{A.37}$$

We apply the time-mean operator on eq. (A.37):

$$\frac{\partial \langle a_p \rangle}{\partial t} + \vec{\nabla} \bullet \left(\left\langle a_p \vec{V} \right\rangle \right) + \frac{\partial \left(\langle a_p \omega \rangle \right)}{\partial p} - \frac{RT_r}{p} \langle \omega \rangle = 0 \tag{A.38}$$

By definition, $B \equiv \langle a_p \rangle = a_p = RT_r \ln\left(\frac{p}{p_r} \right)$ by eq. (A.8), where B is the pressure-dependent part of available enthalpy. Eq. (A.38) becomes:

$$\frac{\partial B}{\partial t} + \vec{\nabla} \bullet \left(B \left\langle \vec{V} \right\rangle \right) + \frac{\partial \left(B \langle \omega \rangle \right)}{\partial p} - \frac{RT_r}{p} \langle \omega \rangle = 0 \tag{A.39}$$

The final form for the time-mean pressure-dependent part of available enthalpy:

$$\frac{\partial B}{\partial t} = -\vec{\nabla} \bullet \left(B \left\langle \vec{V} \right\rangle \right) - \frac{\partial \left(B \langle \omega \rangle \right)}{\partial p} + \frac{RT_r}{p} \langle \omega \rangle \tag{A.40}$$

$$\frac{\partial B}{\partial t} = -F_B - I_{AB} \tag{A.41}$$

Where:

$$B = RT_r \ln\left(\frac{p}{p_r}\right)$$

$$F_B = \vec{\nabla}\bullet\left(B\left\langle\vec{V}\right\rangle\right) + \frac{\partial\left(B\langle\omega\rangle\right)}{\partial p}$$

$$I_{AB} = -\frac{RT_r}{p}\langle\omega\rangle$$

There is no time variability part of pressure-dependent available enthalpy because $B' \equiv 0$.

A.4 Kinetic energy equation

A.4.11 Time-mean kinetic energy prognostic equation $\left(K_{TM}\right)$

We apply eq. (A.vii) to the second and third terms of eq. (A.1a) as follows:

$$\left\langle\vec{V}\bullet\vec{\nabla}\vec{V}\right\rangle = \left\langle\vec{V}\right\rangle\bullet\vec{\nabla}\left\langle\vec{V}\right\rangle + \left\langle\vec{V}'\bullet\vec{\nabla}\vec{V}'\right\rangle$$

$$\left\langle\omega\frac{\partial\vec{V}}{\partial p}\right\rangle = \langle\omega\rangle\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p} + \left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle \tag{A.42}$$

We insert eq. (A.42) in eq. (A.1a):

$$\frac{\partial\left\langle\vec{V}\right\rangle}{\partial t}+\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\left\langle\vec{V}\right\rangle+\langle\omega\rangle\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}+\left\langle\vec{V}'\cdot\vec{\nabla}\vec{V}'\right\rangle+$$
$$\left\langle\omega'\cdot\frac{\partial\vec{V}'}{\partial p}\right\rangle+f\hat{k}\times\left\langle\vec{V}\right\rangle+\vec{\nabla}\langle\Phi\rangle-\left\langle\vec{F}\right\rangle=0 \tag{A.43}$$

We multiply eq. (A.43) by $\left\langle\vec{V}\right\rangle$:

$$\frac{1}{2}\frac{\partial\left(\left\langle\vec{V}\right\rangle\cdot\left\langle\vec{V}\right\rangle\right)}{\partial t}+\frac{1}{2}\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\left(\left\langle\vec{V}\right\rangle\cdot\left\langle\vec{V}\right\rangle\right)+\frac{1}{2}\langle\omega\rangle\frac{\partial\left(\left\langle\vec{V}\right\rangle\cdot\left\langle\vec{V}\right\rangle\right)}{\partial p}+$$
$$\left\langle\vec{V}\right\rangle\cdot\left\langle\vec{V}'\cdot\vec{\nabla}\vec{V}'\right\rangle+\left\langle\vec{V}\right\rangle\cdot\left\langle\omega'\cdot\frac{\partial\vec{V}'}{\partial p}\right\rangle+\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\langle\Phi\rangle-\left\langle\vec{V}\right\rangle\cdot\left\langle\vec{F}\right\rangle=0 \tag{A.44}$$

By definition: $K_{TM}=\frac{1}{2}\left\langle\vec{V}\right\rangle\cdot\left\langle\vec{V}\right\rangle$, where K_{TM} is the time-mean kinetic energy. Eq. (A.44) becomes:

$$\frac{\partial(K_{TM})}{\partial t}+\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}(K_{TM})+\langle\omega\rangle\frac{\partial(K_{TM})}{\partial p}+\left\langle\vec{V}\right\rangle\cdot\left\langle\vec{V}'\cdot\vec{\nabla}\vec{V}'\right\rangle$$
$$+\left\langle\vec{V}\right\rangle\cdot\left\langle\omega'\cdot\frac{\partial\vec{V}'}{\partial p}\right\rangle+\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\langle\Phi\rangle-\left\langle\vec{V}\right\rangle\cdot\left\langle\vec{F}\right\rangle=0 \tag{A.45}$$

The second, third and sixth terms of eq. (A.45) are developed as follows:

$$\begin{aligned}
&\left\langle \vec{V} \right\rangle \bullet \vec{\nabla}\left(K_{TM}\right) + \left\langle \vec{V} \right\rangle \bullet \vec{\nabla}\left\langle \Phi \right\rangle + \left\langle \omega \right\rangle \frac{\partial \left(K_{TM}\right)}{\partial p} \\
&= \left\langle \vec{V} \right\rangle \bullet \vec{\nabla}\left(K_{TM}\right) + \left\langle \omega \right\rangle \frac{\partial \left(K_{TM}\right)}{\partial p} + \vec{\nabla} \bullet \left(\left\langle \vec{V} \right\rangle \bullet \left\langle \Phi \right\rangle \right) - \left\langle \Phi \right\rangle \vec{\nabla} \bullet \left\langle \vec{V} \right\rangle \\
&= \left\langle \vec{V} \right\rangle \bullet \vec{\nabla}\left(K_{TM}\right) + \left\langle \omega \right\rangle \frac{\partial \left(K_{TM}\right)}{\partial p} + \vec{\nabla} \bullet \left(\left\langle \vec{V} \right\rangle \bullet \left\langle \Phi \right\rangle \right) + \left\langle \Phi \right\rangle \frac{\partial \left\langle \omega \right\rangle}{\partial p} \\
&= \left\langle \vec{V} \right\rangle \bullet \vec{\nabla}\left(K_{TM}\right) + \left\langle \omega \right\rangle \frac{\partial \left(K_{TM}\right)}{\partial p} + \vec{\nabla} \bullet \left(\left\langle \vec{V} \right\rangle \bullet \left\langle \Phi \right\rangle \right) + \frac{\partial \left(\left\langle \omega \right\rangle \left\langle \Phi \right\rangle \right)}{\partial p} - \left\langle \omega \right\rangle \frac{\partial \left\langle \Phi \right\rangle}{\partial p} \\
&= \vec{\nabla} \bullet \left(\left\langle \vec{V} \right\rangle \bullet K_{TM} \right) + \frac{\partial \left(\left\langle \omega \right\rangle K_{TM} \right)}{\partial p} + \vec{\nabla} \bullet \left(\left\langle \vec{V} \right\rangle \bullet \left\langle \Phi \right\rangle \right) + \frac{\partial \left(\left\langle \omega \right\rangle \left\langle \Phi \right\rangle \right)}{\partial p} + \left\langle \omega \right\rangle \left\langle \alpha \right\rangle \\
&= \vec{\nabla} \bullet \left(\left(K_{TM} + \left\langle \Phi \right\rangle \right) \left\langle \vec{V} \right\rangle \right) + \frac{\partial \left(\left(K_{TM} + \left\langle \Phi \right\rangle \right) \left\langle \omega \right\rangle \right)}{\partial p} + \left\langle \omega \right\rangle \left\langle \alpha \right\rangle
\end{aligned} \tag{A.46}$$

We insert eq. (A.46) in eq. (A.45):

$$\begin{aligned}
&\frac{\partial \left(K_{TM}\right)}{\partial t} + \vec{\nabla} \left(\left(K_{TM} + \left\langle \Phi \right\rangle \right) \left\langle \vec{V} \right\rangle \right) + \frac{\partial \left(\left(K_{TM} + \left\langle \Phi \right\rangle \right) \left\langle \omega \right\rangle \right)}{\partial p} + \left\langle \omega \right\rangle \left\langle \alpha \right\rangle - \\
&\left\langle \vec{V} \right\rangle \bullet \left\langle \vec{F} \right\rangle + \left\langle \vec{V} \right\rangle \bullet \left\langle \vec{V}' \bullet \vec{\nabla} \vec{V}' \right\rangle + \left\langle \vec{V} \right\rangle \bullet \left\langle \omega' \frac{\partial \vec{V}'}{\partial p} \right\rangle = 0
\end{aligned} \tag{A.47}$$

We compute $\left\langle \vec{V} \right\rangle \bullet \left\langle \vec{V}' \text{eq. (3b)} \right\rangle$ as follows:

$$\left\langle \vec{V} \right\rangle \bullet \left\langle \vec{V}' \left(\vec{\nabla} \bullet \vec{V}' \right) \right\rangle + \left\langle \vec{V} \right\rangle \bullet \left\langle \vec{V}' \frac{\partial \omega'}{\partial p} \right\rangle = 0 \tag{A.48}$$

We add eq. (A.48) to eq. (A.47):

$$\frac{\partial(K_{TM})}{\partial t}+\vec{\nabla}\bullet\left((K_{TM}+\langle\Phi\rangle)\left\langle\vec{V}\right\rangle\right)+\frac{\partial\left((K_{TM}+\langle\Phi\rangle)\langle\omega\rangle\right)}{\partial p}+$$
$$\langle\omega\rangle\langle\alpha\rangle-\left\langle\vec{V}\right\rangle\bullet\left\langle\vec{F}\right\rangle+\left\langle\vec{V}\right\rangle\bullet\left\langle\left(\vec{V}'\bullet\vec{\nabla}\right)\vec{V}'+\vec{V}'\left(\vec{\nabla}\bullet\vec{V}'\right)\right\rangle+ \tag{A.49}$$
$$\left\langle\vec{V}\right\rangle\bullet\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}+\vec{V}'\frac{\partial\omega'}{\partial p}\right\rangle=0$$

We simplify eq. (A.49) as follows:

$$\frac{\partial(K_{TM})}{\partial t}+\vec{\nabla}\bullet\left((K_{TM}+\langle\Phi\rangle)\left\langle\vec{V}\right\rangle\right)+\frac{\partial\left((K_{TM}+\langle\Phi\rangle)\langle\omega\rangle\right)}{\partial p}+$$
$$\langle\omega\rangle\langle\alpha\rangle-\left\langle\vec{V}\right\rangle\bullet\left\langle\vec{F}\right\rangle+\left\langle\vec{V}\right\rangle\bullet\left(\vec{\nabla}\bullet\left\langle\vec{V}'\bullet\vec{V}'\right\rangle\right)+\left\langle\vec{V}\right\rangle\bullet\frac{\partial\left\langle\vec{V}'\omega'\right\rangle}{\partial p}=0 \tag{A.50}$$

We rewrite eq. (A.50) in its flux form:

$$\frac{\partial(K_{TM})}{\partial t}+\vec{\nabla}\bullet\left((K_{TM}+\langle\Phi\rangle)\left\langle\vec{V}\right\rangle\right)+\frac{\partial\left((K_{TM}+\langle\Phi\rangle)\langle\omega\rangle\right)}{\partial p}+$$
$$\langle\omega\rangle\langle\alpha\rangle-\left\langle\vec{V}\right\rangle\bullet\left\langle\vec{F}\right\rangle+\vec{\nabla}\bullet\left(\left\langle\vec{V}\right\rangle\bullet\left\langle\vec{V}'\bullet\vec{V}'\right\rangle\right)_{-}\frac{\partial\left(\left\langle\vec{V}\right\rangle\bullet\left\langle\vec{V}'\omega'\right\rangle\right)}{\partial p}- \tag{A.51}$$
$$\left\langle\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle-\left\langle\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle=0$$

We rearrange the second, third, sixth and seventh terms of eq. (A.51):

$$\frac{\partial (K_{TM})}{\partial t}+\vec{\nabla}\bullet\left(\left\langle \vec{V}\right\rangle K_{TM}\right)+\frac{\partial(\langle\omega\rangle K_{TM})}{\partial p}+\langle\omega\rangle\langle\alpha\rangle-\left\langle \vec{V}\right\rangle\bullet\left\langle \vec{F}\right\rangle+$$
$$\vec{\nabla}\bullet\left(\left\langle \vec{V}\right\rangle\bullet\left(\left\langle \vec{V}'\bullet\vec{V}'\right\rangle+\langle\Phi\rangle\right)\right)+\frac{\partial\left(\left\langle \vec{V}\right\rangle\bullet\left(\left\langle \vec{V}'\omega'\right\rangle+\langle\Phi\rangle\right)\right)}{\partial p}- \tag{A.52}$$
$$\left\langle \vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle \vec{V}\right\rangle\right\rangle-\left\langle \vec{V}'\bullet\left(\omega'\frac{\partial\left\langle \vec{V}\right\rangle}{\partial p}\right)\right\rangle=0$$

The final form for the time-mean kinetic energy:

$$\frac{\partial (K_{TM})}{\partial t}=-\langle\omega\rangle\langle\alpha\rangle+\left\langle \vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle \vec{V}\right\rangle\right\rangle+\left\langle \vec{V}'\bullet\left(\omega'\frac{\partial\left\langle \vec{V}\right\rangle}{\partial p}\right)\right\rangle+$$
$$\left\langle \vec{V}\right\rangle\bullet\left\langle \vec{F}\right\rangle-\vec{\nabla}\bullet\left(K_{TM}\left\langle \vec{V}\right\rangle\right)-\frac{\partial(K_{TM}\langle\omega\rangle)}{\partial p}- \tag{A.53}$$
$$\vec{\nabla}\bullet\left(\left(\left\langle \vec{V}'\bullet\vec{V}'\right\rangle+\langle\Phi\rangle\right)\bullet\left\langle \vec{V}\right\rangle\right)-\frac{\partial\left(\left(\left\langle \vec{V}'\omega'\right\rangle+\langle\Phi\rangle\right)\bullet\left\langle \vec{V}\right\rangle\right)}{\partial p}$$

$$\frac{\partial K_{TM}}{\partial t}=C_{TM}-C_K-D_{TM}-F_{K_{TM}}-H_{K_{TM}} \tag{A.54}$$

Where:

$$K_{TM}=\frac{1}{2}\left\langle \vec{V}\right\rangle\bullet\left\langle \vec{V}\right\rangle$$

$$C_{TM}=-\langle\omega\rangle\langle\alpha\rangle$$

$$C_K = -\left\langle \vec{V}' \bullet \left(\vec{V}' \bullet \vec{\nabla}\right)\left\langle \vec{V} \right\rangle \right\rangle - \left\langle \vec{V}' \bullet \left(\omega' \frac{\partial \left\langle \vec{V} \right\rangle}{\partial p} \right) \right\rangle$$

$$D_{TM} = -\left\langle \vec{V} \right\rangle \bullet \left\langle \vec{F} \right\rangle$$

$$F_{K_{TM}} = \vec{\nabla} \bullet \left(K_{TM} \left\langle \vec{V} \right\rangle \right) + \frac{\partial \left(K_{TM} \langle \omega \rangle \right)}{\partial p}$$

$$H_{K_{TM}} = \vec{\nabla} \bullet \left(\left(\left\langle \vec{V}' \bullet \vec{V}' \right\rangle + \langle \Phi \rangle \right) \left\langle \vec{V} \right\rangle \right) + \frac{\partial \left(\left(\left\langle \vec{V}' \omega' \right\rangle + \langle \Phi \rangle \right) \langle \omega \rangle \right)}{\partial p}$$

A.4.12 Instantaneous time variability kinetic energy prognostic equation $\left(k_{TV}\right)$

We apply eq. (A.viii) to the second and third terms of eq. (A.1b):

$$\begin{aligned} &\left(\vec{V} \bullet \vec{\nabla} \vec{V} - \left\langle \vec{V} \bullet \vec{\nabla} \vec{V} \right\rangle \right) = \left\langle \vec{V} \right\rangle \bullet \vec{\nabla} \vec{V}' + \vec{V}' \bullet \vec{\nabla} \left\langle \vec{V} \right\rangle + \vec{V}' \bullet \vec{\nabla} \vec{V}' - \left\langle \vec{V}' \bullet \vec{\nabla} \vec{V}' \right\rangle \\ &\left(\omega \frac{\partial \vec{V}}{\partial p} - \left\langle \omega \frac{\partial \vec{V}}{\partial p} \right\rangle \right) = \langle \omega \rangle \frac{\partial \vec{V}'}{\partial p} + \omega' \frac{\partial \left\langle \vec{V} \right\rangle}{\partial p} + \omega' \frac{\partial \vec{V}'}{\partial p} - \left\langle \omega' \frac{\partial \vec{V}'}{\partial p} \right\rangle \end{aligned} \tag{A.55}$$

We insert eq. (A.55) in eq. (A.1b):

$$\begin{aligned} &\frac{\partial \vec{V}'}{\partial t} + \left(\left\langle \vec{V} \right\rangle \bullet \vec{\nabla} \vec{V}' + \vec{V}' \bullet \vec{\nabla} \left\langle \vec{V} \right\rangle + \vec{V}' \bullet \vec{\nabla} \vec{V}' - \left\langle \vec{V}' \bullet \vec{\nabla} \vec{V}' \right\rangle \right) + \\ &\left(\langle \omega \rangle \frac{\partial \vec{V}'}{\partial p} + \omega' \frac{\partial \left\langle \vec{V} \right\rangle}{\partial p} + \omega' \frac{\partial \vec{V}'}{\partial p} - \left\langle \omega' \frac{\partial \vec{V}'}{\partial p} \right\rangle \right) + f\hat{k} \times \vec{V}' + \vec{\nabla} \Phi' - \vec{F}' = 0 \end{aligned} \tag{A.56}$$

We multiply eq. (A.56) by $\vec{V}'$:

$$\frac{1}{2}\frac{\partial\left(\vec{V}'\bullet\vec{V}'\right)}{\partial t}+\frac{1}{2}\left\langle\vec{V}\right\rangle\vec{\nabla}\bullet\left(\vec{V}'\bullet\vec{V}'\right)+\left(\vec{V}'\bullet\vec{V}'\right)\vec{\nabla}\bullet\left\langle\vec{V}\right\rangle+$$
$$\left(\vec{V}'\bullet\vec{V}'\right)\vec{\nabla}\bullet\vec{V}'-\vec{V}'\left\langle\vec{V}'\vec{\nabla}\bullet\vec{V}'\right\rangle+\frac{\langle\omega\rangle}{2}\frac{\partial\left(\vec{V}'\bullet\vec{V}'\right)}{\partial p}+\vec{V}'\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}+ \tag{A.57}$$
$$\vec{V}'\omega'\frac{\partial\vec{V}'}{\partial p}-\vec{V}'\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle+\vec{V}'\bullet\vec{\nabla}\Phi'-\vec{V}'\bullet\vec{F}'=0$$

By definition: $k_{TV}=\frac{1}{2}\left(\vec{V}'\bullet\vec{V}'\right)$, where k_{TV} is the instantaneous time variability kinetic energy. Eq. (A.57) becomes:

$$\frac{\partial(k_{TV})}{\partial t}+\left\langle\vec{V}\right\rangle\bullet\vec{\nabla}(k_{TV})+\langle\omega\rangle\frac{\partial(k_{TV})}{\partial p}+\left(\vec{V}'\bullet\vec{V}'\right)\vec{\nabla}\bullet\left\langle\vec{V}\right\rangle+$$
$$\left(\vec{V}'\bullet\vec{V}'\right)\vec{\nabla}\bullet\vec{V}'-\vec{V}'\left\langle\vec{V}'\vec{\nabla}\bullet\vec{V}'\right\rangle+\vec{V}'\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}+ \tag{A.58}$$
$$\vec{V}'\omega'\frac{\partial\vec{V}'}{\partial p}-\vec{V}'\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle+\vec{V}'\bullet\vec{\nabla}\Phi'-\vec{V}'\bullet\vec{F}'=0$$

The fourth and seventh terms from eq. (A.58) are simplified as follows:

$$\begin{aligned}\left(\vec{V}'\bullet\vec{V}'\right)\vec{\nabla}\bullet\vec{V}'+\vec{V}'\omega'\frac{\partial\vec{V}'}{\partial p}&=\frac{1}{2}\vec{V}'\vec{\nabla}\bullet\left(\vec{V}'\bullet\vec{V}'\right)+\frac{1}{2}\omega'\vec{\nabla}\bullet\left(\vec{V}'\bullet\vec{V}'\right)\\&=\vec{V}'\bullet\vec{\nabla}(k_{TV})+\omega'\frac{\partial(k_{TV})}{\partial p}\end{aligned} \tag{A.59}$$

We insert eq. (A.59) in eq. (A.58):

$$\frac{\partial(k_{TV})}{\partial t}+\left\langle\vec{V}\right\rangle\bullet\vec{\nabla}(k_{TV})+\langle\omega\rangle\frac{\partial(k_{TV})}{\partial p}+\vec{V}'\bullet\vec{\nabla}(k_{TV})+\omega'\frac{\partial(k_{TV})}{\partial p}+\left(\vec{V}'\bullet\vec{V}'\right)\vec{\nabla}\bullet\left\langle\vec{V}\right\rangle-$$
$$\vec{V}'\left\langle\vec{V}'\vec{\nabla}\bullet\vec{V}'\right\rangle+\vec{V}'\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}-\vec{V}'\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle+\vec{V}'\bullet\vec{\nabla}\Phi'-\vec{V}'\bullet\vec{F}'=0 \tag{A.60}$$

We express the second, third, fourth and fifth terms of eq. (A.60) in their flux forms:

$$\frac{\partial(k_{TV})}{\partial t}+\vec{\nabla}\bullet\left(k_{TV}\left\langle\vec{V}\right\rangle\right)+\frac{\partial(k_{TV}\langle\omega\rangle)}{\partial p}+\vec{\nabla}\bullet\left(k_{TV}\vec{V}'\right)+\frac{\partial(k_{TV}\omega')}{\partial p}+\left(\vec{V}'\bullet\vec{V}'\right)\vec{\nabla}\bullet\left\langle\vec{V}\right\rangle$$
$$-\vec{V}'\left\langle\vec{V}'\vec{\nabla}\bullet\vec{V}'\right\rangle+\vec{V}'\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}-\vec{V}'\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle+\vec{V}'\bullet\vec{\nabla}\Phi'-\vec{V}'\bullet\vec{F}'=0 \tag{A.61}$$

We transform the tenth term of eq. (A.61) as follows:

$$\begin{aligned}-\vec{V}'\bullet\vec{\nabla}\Phi' &= -\vec{\nabla}\bullet\left(\Phi'\vec{V}'\right)+\Phi'\vec{\nabla}\bullet\vec{V}'\\ &= -\vec{\nabla}\bullet\left(\Phi'\vec{V}'\right)-\Phi'\frac{\partial\omega'}{\partial p}\\ &= -\vec{\nabla}\bullet\left(\Phi'\vec{V}'\right)-\frac{\partial(\Phi'\omega')}{\partial p}+\omega'\frac{\partial\Phi'}{\partial p}\\ &= -\vec{\nabla}\bullet\left(\Phi'\vec{V}'\right)-\frac{\partial(\Phi'\omega')}{\partial p}-\frac{R}{p}\omega'T'\\ &= -\vec{\nabla}\bullet\left(\Phi'\vec{V}'\right)-\frac{\partial(\Phi'\omega')}{\partial p}-\omega'\alpha'\end{aligned} \tag{A.62}$$

We insert eq. (A.62) in eq. (A.61):

$$\frac{\partial(k_{TV})}{\partial t}+\vec{\nabla}\bullet\left(k_{TV}\left\langle\vec{V}\right\rangle\right)+\frac{\partial(k_{TV}\langle\omega\rangle)}{\partial p}+\vec{\nabla}\bullet\left(k_{TV}\vec{V}'\right)+$$
$$\frac{\partial(k_{TV}\omega')}{\partial p}+\left(\vec{V}'\bullet\vec{V}'\right)\vec{\nabla}\bullet\left\langle\vec{V}\right\rangle-\vec{V}'\left\langle\vec{V}'\vec{\nabla}\bullet\vec{V}'\right\rangle+ \tag{A.63}$$
$$\vec{V}'\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}-\vec{V}'\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle=-\vec{\nabla}\bullet\left(\vec{V}'\Phi'\right)-\frac{\partial(\Phi'\omega')}{\partial p}-\omega'\alpha'+\vec{V}'\bullet\vec{F}'$$

The final form for the instantaneous time variability kinetic energy:

$$\frac{\partial(k_{TV})}{\partial t}=-\omega'\alpha'-\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle-\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)+\vec{V}'\bullet\vec{F}'-$$
$$\vec{\nabla}\bullet\left(k_{TV}\left\langle\vec{V}\right\rangle\right)-\frac{\partial(k_{TV}\langle\omega\rangle)}{\partial p}-\vec{\nabla}\bullet\left(k_{TV}\vec{V}'\right)-\frac{\partial(k_{TV}\omega')}{\partial p}- \tag{A.64}$$
$$\vec{\nabla}\bullet\left(\Phi'\vec{V}'\right)-\frac{\partial(\Phi'\omega')}{\partial p}+\vec{V}'\left\langle\vec{V}'\bullet\vec{\nabla}\vec{V}'\right\rangle+\vec{V}'\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle$$

$$\frac{\partial k_{TV}}{\partial t}=c_{TV}+c_k-d_{TV}-f_{k_{TV}}-h_{k_{TV}}-j_{k1} \tag{A.65}$$

Where:

$$k_{TV}=\frac{1}{2}\vec{V}'\bullet\vec{V}'$$

$$c_{TV}=-\omega'\alpha'$$

$$c_k = -\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle - \vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)$$

$$d_{TV} = -\vec{V}'\bullet\vec{F}'$$

$$f_{k_{TV}} = \vec{\nabla}\bullet\left(k_{TV}\left\langle\vec{V}\right\rangle\right) + \frac{\partial\left(k_{TV}\langle\omega\rangle\right)}{\partial p}$$

$$h_{k_{TV}} = \vec{\nabla}\bullet\left(\left(k_{TV}+\Phi'\right)\vec{V}'\right) + \frac{\partial\left(\left(k_{TV}+\Phi'\right)\omega'\right)}{\partial p}$$

$$j_{k1} = -\vec{V}'\left\langle\vec{V}'\bullet\vec{\nabla}\vec{V}'\right\rangle - \vec{V}'\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle$$

A.4.13 Time variability kinetic energy prognostic equation $\left(K_{TV}\right)$

By definition, $K_{TV} = \langle k_{TV}\rangle = \frac{1}{2}\left\langle\vec{V}'\bullet\vec{V}'\right\rangle$, where K_{TV} is the time variability kinetic energy. We apply the time-mean operator on eq. (A.64):

$$\frac{\partial\left(K_{TV}\right)}{\partial t} = -\langle\omega'\alpha'\rangle - \left\langle\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle - \left\langle\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle + \left\langle\vec{V}'\bullet\vec{F}'\right\rangle -$$
$$\vec{\nabla}\bullet\left(K_{TV}\left\langle\vec{V}\right\rangle\right) - \frac{\partial\left(K_{TV}\langle\omega\rangle\right)}{\partial p} - \vec{\nabla}\bullet\left\langle k_{TV}\vec{V}'\right\rangle - \frac{\partial\left\langle k_{TV}\omega'\right\rangle}{\partial p} - \vec{\nabla}\bullet\left\langle\Phi'\vec{V}'\right\rangle - \tag{A.65}$$
$$\frac{\partial\left\langle\Phi'\omega'\right\rangle}{\partial p} + \cancel{\left\langle\vec{V}'\right\rangle\left\langle\vec{V}'\bullet\vec{\nabla}\vec{V}'\right\rangle} + \cancel{\left\langle\vec{V}'\right\rangle\left\langle\omega'\frac{\partial\vec{V}'}{\partial p}\right\rangle}$$

The final form for the time variability kinetic energy:

$$\frac{\partial(K_{TV})}{\partial t}=-\langle\omega'\alpha'\rangle-\left\langle\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle-\left\langle\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle+\left\langle\vec{V}'\bullet\vec{F}'\right\rangle-$$
$$\vec{\nabla}\bullet\left(K_{TV}\left\langle\vec{V}\right\rangle\right)-\frac{\partial(K_{TV}\langle\omega\rangle)}{\partial p}-\vec{\nabla}\bullet\left\langle k_{TV}\vec{V}'\right\rangle-\frac{\partial\langle k_{TV}\omega'\rangle}{\partial p}-\vec{\nabla}\bullet\left\langle\Phi'\vec{V}'\right\rangle-\frac{\partial\langle\Phi'\omega'\rangle}{\partial p} \tag{A.66}$$

$$\frac{\partial K_{TV}}{\partial t}=C_{TV}+C_K-D_{TV}-F_{K_{TV}}-H_{K_{TV}} \tag{A.67}$$

Where:

$$K_{TV}\equiv\langle k_{TV}\rangle=\frac{1}{2}\left\langle\vec{V}'\bullet\vec{V}'\right\rangle$$

$$C_{TV}\equiv\langle c_{TV}\rangle=-\langle\omega'\alpha'\rangle$$

$$C_K\equiv\langle c_k\rangle=-\left\langle\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle-\left\langle\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle$$

$$D_{TV}\equiv\langle d_{TV}\rangle=-\left\langle\vec{V}'\bullet\vec{F}'\right\rangle$$

$$F_{K_{TV}}\equiv\left\langle f_{k_{TV}}\right\rangle=\vec{\nabla}\bullet\left(K_{TV}\left\langle\vec{V}\right\rangle\right)+\frac{\partial(K_{TV}\langle\omega\rangle)}{\partial p}$$

$$H_{K_{TV}}\equiv\left\langle h_{k_{TV}}\right\rangle=\vec{\nabla}\bullet\left\langle(k_{TV}+\Phi')\vec{V}'\right\rangle+\frac{\partial\langle(k_{TV}+\Phi')\omega'\rangle}{\partial p}$$

APPENDICE B

MATHEMATICAL DEVELOPMENT OF THE TIME-TENDENCY EQUATIONS FOR ENERGY RESERVOIRS $\overline{A_{TM\,S}}$, $\overline{A_{TM\,B}}$, $\overline{A_{TV}}$, $\overline{B}$, $\overline{K_{TM}}$ AND $\overline{K_{TV}}$

This Appendix follows directly the mathematical development of Appendix A.

B.1 Mathematical manipulations

B.1.1 Isobaric-mean and isobaric deviation decomposition

Each atmospheric variable $\Psi \in \left\{\vec{F}, Q, T, \vec{V}(u,v), \alpha, \omega, \Phi\right\}$ is further decomposed as:

$$\Psi = \overline{\Psi} + \Psi^{\times},$$

$\overline{\Psi}$ is the isobaric-mean state, obtained by computing the surface average on a pressure level;

$\Psi^{\times}$ is the deviation from the isobaric-mean state.

B.1.2 Identities from Reynolds averaging rules

For each atmospheric variables Ψ and ψ :

$$\overline{\overline{\Psi}} = \overline{\Psi} \tag{B.i}$$

$$\overline{\Psi + \psi} = \overline{\Psi} + \overline{\psi} \tag{B.ii}$$

$$\overline{\overline{\Psi}\psi} = \overline{\Psi}\,\overline{\psi} \tag{B.iii}$$

$$\overline{\frac{\partial \Psi}{\partial p}} = \frac{\partial \overline{\Psi}}{\partial p} \tag{B.iv}$$

$$\overline{\vec{\nabla}\Psi} = \vec{\nabla}\overline{\Psi} \tag{B.v}$$

$$\Psi^{\times} = \Psi - \overline{\Psi} \tag{B.vi}$$

$$\overline{\Psi^{\times}} = 0 \tag{B.vii}$$

$$\overline{\Psi\psi} = \overline{\Psi}\,\overline{\psi} + \overline{\Psi^{\times}\psi^{\times}} \tag{B.viii}$$

$$\Psi\psi - \overline{\Psi\psi} = \Psi^{\times}\overline{\psi} + \psi^{\times}\overline{\Psi} + \Psi^{\times}\psi^{\times} - \overline{\Psi^{\times}\psi^{\times}} \tag{B.ix}$$

$$\Psi'_{*} = \Psi_{*} - \langle \Psi_{*} \rangle = (\Psi - \Psi_{r}) - \langle \Psi - \Psi_{r} \rangle = \Psi - \langle \Psi \rangle = \Psi' \tag{B.x}$$

$$\Psi^{\times}_{*} = \Psi_{*} - \overline{\Psi_{*}} = (\Psi - \Psi_{r}) - \overline{\Psi - \Psi_{r}} = \Psi - \overline{\Psi} = \Psi^{\times} \tag{B.xi}$$

$$\langle \overline{\Psi} \rangle = \overline{\langle \Psi \rangle} \tag{B.xii}$$

$$(\overline{\Psi})' = (\Psi - \Psi^{\times})' = \Psi' - (\Psi^{\times})' = \Psi' - \Psi'^{\times} = \overline{\Psi'} \tag{B.xiii}$$

$$(\overline{\Psi})' = (\Psi - \Psi^{\times})' = \Psi' - (\Psi^{\times})' = \Psi' - \Psi'^{\times} = \overline{\Psi'} \tag{B.xiv}$$

$$\langle \Psi \rangle^{\times} = (\Psi - \Psi')^{\times} = \Psi^{\times} - (\Psi')^{\times} = \Psi^{\times} - \Psi'^{\times} = \langle \Psi^{\times} \rangle \tag{B.xv}$$

$$\Psi'^{\times} = (\Psi - \langle \Psi \rangle)^{\times} = \Psi^{\times} - \langle \Psi \rangle^{\times} = \Psi^{\times} - \langle \Psi^{\times} \rangle = (\Psi^{\times})' \tag{B.xvi}$$

Where Ψ and ψ are scalars or vectors. $\Psi_* = \Psi - \Psi_r$, Ψ_r is a constant: reference value of Ψ.

B.2 Isobaric-mean available enthalpy equations

Following Marquet (1991), we further decompose our approximate expression of a_T into separate components due to mean atmospheric stratification and deviations thereof.

We rewrite eq. (A.6) as follows:

$$A_{TM} = \frac{c_p}{2T_r}\left\langle\left(T-\overline{T}\right)+\left(\overline{T}-T_r\right)\right\rangle^2 = \frac{c_p}{2T_r}\left(\left\langle T-\overline{T}\right\rangle+\left\langle\overline{T}-T_r\right\rangle\right)^2 \tag{B.1}$$

We develop eq. (B.1) as follows:

$$A_{TM} = \frac{c_p}{2T_r}\left(\left\langle T-\overline{T}\right\rangle^2+\left\langle\overline{T}-T_r\right\rangle^2+2\left\langle T-\overline{T}\right\rangle\left\langle\overline{T}-T_r\right\rangle\right) \tag{B.2}$$

Where:

$$A_{TM\,B} = \frac{c_p}{2T_r}\left\langle T^\times\right\rangle^2, \text{ where } T^\times = T-\overline{T} \tag{B.3}$$

$$A_{TM\,S} = \frac{c_p}{2T_r}\left\langle\overline{T}-T_r\right\rangle^2 \tag{B.4}$$

$$A_{TM\,C} = \frac{c_p}{T_r}\left\langle T^\times\right\rangle\left\langle\overline{T}-T_r\right\rangle \tag{B.5}$$

By applying the isobaric-mean operator on eq. (B.5), we get:

$$\overline{A_{TM\ C}} = \frac{c_p}{T_r}\overline{\langle T^{\times}\rangle\langle \overline{T} - T_r\rangle}$$

$$= \frac{c_p}{T_r}\overline{\langle T^{\times}\rangle\left(\langle \overline{T}\rangle - T_r\right)}$$

$$= \frac{c_p}{T_r}\overline{\langle T^{\times}\rangle\overline{\langle T\rangle}} - \overline{\langle T^{\times}\rangle T_r} \text{ , by eq. (Bii) and eq. (B.xii)}$$

$$= \frac{c_p}{T_r}\overline{\langle T^{\times}\rangle}\langle \overline{T}\rangle - \overline{\langle T^{\times}\rangle}T_r$$

$$\equiv 0$$

Since $A_{TM} = A_{TM\ B} + A_{TM\ S} + A_{TM\ C}$, by applying the isobaric-mean operator on A_{TM}, we have that $\overline{A_{TM}} = \overline{A_{TM\ B}} + \overline{A_{TM\ S}} + \overline{A_{TM\ C}} = \overline{A_{TM\ B}} + \overline{A_{TM\ S}}$.

Hence, we can compute $\overline{A_{TM\ S}}$ as: $\overline{A_{TM\ S}} = \overline{A_{TM}} - \overline{A_{TM\ B}}$

B.2.3 Time- and isobaric-mean temperature-dependent part of available enthalpy prognostic equation $\left(\overline{A_{TM}}\right)$

We start from eq. (A.20):

$$\frac{\partial A_{TM}}{\partial t} = \frac{l}{T_r}\langle T_*\rangle\langle Q\rangle + \frac{RT_r}{p}\langle \omega\rangle + \langle \omega\rangle\langle \alpha\rangle + \frac{c_p}{T_r}\left\langle \vec{V}'T'\right\rangle \cdot \vec{\nabla}\langle T\rangle + \frac{c_p}{T_r}\langle \omega' T'\rangle\frac{\partial\langle T\rangle}{\partial p} - \vec{\nabla}\cdot\left(A_{TM}\left\langle\vec{V}\right\rangle\right) - \frac{\partial\left(A_{TM}\langle\omega\rangle\right)}{\partial p} - \frac{c_p}{T_r}\left(\vec{\nabla}\cdot\left(\langle T_*\rangle\left\langle \vec{V}'T'\right\rangle\right) + \frac{\partial\left(\langle T_*\rangle\langle \omega' T'\rangle\right)}{\partial p}\right) \qquad \text{(B.6)}$$

We apply the isobaric-mean operator on eq. (B.6). The final form for the time- and isobaric-mean temperature-dependent part of available enthalpy:

$$\frac{\partial \overline{A_{TM}}}{\partial t} = \frac{l}{T_r}\overline{\langle T_*\rangle\langle Q\rangle} + \frac{RT_r}{p}\overline{\langle\omega\rangle} + \overline{\langle\omega\rangle\langle\alpha\rangle} + \frac{c_p}{T_r}\overline{\left\langle \vec{V}'T'\right\rangle \bullet \vec{\nabla}\langle T\rangle} + \frac{c_p}{T_r}\overline{\langle\omega' T'\rangle\frac{\partial\langle T\rangle}{\partial p}} - \vec{\nabla}\bullet\left(\overline{A_{TM}\left\langle\vec{V}\right\rangle}\right) - \frac{\partial\left(\overline{A_{TM}\langle\omega\rangle}\right)}{\partial p} - \frac{c_p}{T_r}\left(\vec{\nabla}\bullet\left(\overline{\langle T_*\rangle\left\langle\vec{V}'T'\right\rangle}\right) + \frac{\partial\left(\overline{\langle T_*\rangle\langle\omega' T'\rangle}\right)}{\partial p}\right) \qquad \text{(B.7)}$$

$$\frac{\partial \overline{A_{TM}}}{\partial t} = \overline{G_{TM}} + \overline{I_{AB}} - \overline{C_{TM}} - \overline{C_A} - \overline{F_{A_{TM}}} - \overline{H_{A_{TM}}} \qquad \text{(B.8)}$$

Where:

$$\overline{A_{TM}} = \frac{c_p}{2T_r}\overline{\langle T_*\rangle^2}$$

$$\overline{G_{TM}} = \frac{l}{T_r}\overline{\langle T_*\rangle\langle Q\rangle}$$

$$\overline{I_{AB}} = -\frac{RT_r}{p}\overline{\langle\omega\rangle}$$

$$\overline{C_{TM}} = -\overline{\langle\omega\rangle\langle\alpha\rangle}$$

$$\overline{C_A} = -\frac{c_p}{T_r}\overline{\left\langle \vec{V}'T'\right\rangle \bullet \vec{\nabla}\langle T\rangle} - \frac{c_p}{T_r}\overline{\langle\omega' T'\rangle\frac{\partial\langle T\rangle}{\partial p}}$$

$$\overline{F_{A_{TM}}} = \vec{\nabla}\bullet\left(\overline{A_{TM}\left\langle\vec{V}\right\rangle}\right) + \frac{\partial\left(\overline{A_{TM}\langle\omega\rangle}\right)}{\partial p}$$

$$\overline{H_{A_{TM}}} = \frac{c_p}{T_r}\left(\vec{\nabla}\bullet\left(\overline{\langle T_*\rangle\left\langle\vec{V}'T'\right\rangle}\right) + \frac{\partial\left(\overline{\langle T_*\rangle\langle\omega' T'\rangle}\right)}{\partial p}\right)$$

B.2.4 Baroclinic component of time- and isobaric-mean temperature-dependent part of available enthalpy prognostic equation $\left(\overline{A_{TM\ B}}\right)$

We apply the isobaric-mean operator on eq. (A.2):

$$\frac{\partial \overline{T}}{\partial t}+\overline{\vec{V}\bullet\vec{\nabla}T}+\overline{\omega\frac{\partial T}{\partial p}}-\frac{R}{c_p p}\overline{\omega}T_r-\frac{T_r}{c_p}\overline{\left(\frac{Q}{T}\right)}=0 \qquad \text{(B.9)}$$

We subtract eq. (B.9) from eq. (A.2):

$$\frac{\partial T^{\times}}{\partial t}+\left(\vec{V}\bullet\vec{\nabla}T-\overline{\vec{V}\bullet\vec{\nabla}T}\right)+\left(\omega\frac{\partial T}{\partial p}-\overline{\omega\frac{\partial T}{\partial p}}\right)-\frac{R}{c_p p}T_r\omega^{\times}-\frac{T_r}{c_p}\left(\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right)=0 \qquad \text{(B.10)}$$

We add the expression $\left(-\omega\frac{\partial \overline{T}}{\partial p}+\omega\frac{\partial \overline{T}}{\partial p}\right)=0$ to eq. (B.10):

$$\frac{\partial T^{\times}}{\partial t}+\vec{V}\bullet\vec{\nabla}T^{\times}+\omega\frac{\partial T^{\times}}{\partial p}-\overline{\vec{V}\bullet\vec{\nabla}T}-\overline{\omega\frac{\partial T}{\partial p}}+$$
$$\omega\frac{\partial \overline{T}}{\partial p}-\frac{R}{c_p p}T_r\omega^{\times}-\frac{T_r}{c_p}\left(\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right)=0 \qquad \text{(B.11)}$$

We apply the time-mean operator on eq. (B.11):

$$\frac{\partial\langle T^{\times}\rangle}{\partial t}+\left\langle\vec{V}\bullet\vec{\nabla}T^{\times}\right\rangle+\left\langle\omega\frac{\partial T^{\times}}{\partial p}\right\rangle-\left\langle\overline{\vec{V}\bullet\vec{\nabla}T}\right\rangle-\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle+$$
$$\left\langle\omega\frac{\partial \overline{T}}{\partial p}\right\rangle-\frac{R}{c_p p}T_r\langle\omega^{\times}\rangle-\frac{T_r}{c_p}\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=0 \qquad \text{(B.12)}$$

We multiply eq. (B.12) by $\frac{c_p\langle T^\times\rangle}{T_r}$:

$$\frac{c_p\langle T^\times\rangle}{T_r}\frac{\partial\langle T^\times\rangle}{\partial t}+\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\vec{V}\cdot\vec{\nabla}T^\times\right\rangle+\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\omega\frac{\partial T^\times}{\partial p}\right\rangle-\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\overline{\vec{V}\cdot\vec{\nabla}T}\right\rangle-$$
$$\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle+\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\omega\frac{\partial\overline{T}}{\partial p}\right\rangle-\langle T^\times\rangle\frac{R}{p}\langle\omega^\times\rangle-\langle T^\times\rangle\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=0 \tag{B.13}$$

The second and third terms of eq. (B.13) are expanded with eq. (A.vii) as follows:

$$\frac{c_p\langle T^\times\rangle}{T_r}\left(\left\langle\vec{V}\cdot\vec{\nabla}T^\times\right\rangle+\left\langle\omega\frac{\partial T^\times}{\partial p}\right\rangle\right)=\frac{c_p\langle T^\times\rangle}{T_r}\begin{pmatrix}\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\langle T^\times\rangle+\langle\omega\rangle\frac{\partial\langle T^\times\rangle}{\partial p}+\\ \left\langle\vec{V}'\cdot\vec{\nabla}T'^\times\right\rangle+\left\langle\omega'\frac{\partial T'^\times}{\partial p}\right\rangle\end{pmatrix} \tag{B.14}$$

We insert eq. (B.14) in eq. (B.13):

$$\frac{c_p\langle T^\times\rangle}{T_r}\left(\frac{\partial\langle T^\times\rangle}{\partial t}+\left\langle\vec{V}\right\rangle\cdot\vec{\nabla}\langle T^\times\rangle+\langle\omega\rangle\frac{\partial\langle T^\times\rangle}{\partial p}+\left\langle\vec{V}'\cdot\vec{\nabla}T'^\times\right\rangle+\left\langle\omega'\frac{\partial T'^\times}{\partial p}\right\rangle\right)-$$
$$\frac{c_p\langle T^\times\rangle}{T_r}\left(\left\langle\overline{\vec{V}\cdot\vec{\nabla}T}\right\rangle+\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle\right)+\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\omega\frac{\partial\overline{T}}{\partial p}\right\rangle-\langle T^\times\rangle\frac{R}{p}\langle\omega^\times\rangle- \tag{B.15}$$
$$\langle T^\times\rangle\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=0$$

We rewrite eq. (B.15) as follows:

$$
\begin{aligned}
&\frac{c_p}{2T_r}\left(\frac{\partial\langle T^{\times}\rangle^2}{\partial t}+\langle\vec{V}\rangle\bullet\vec{\nabla}\langle T^{\times}\rangle^2+\langle\omega\rangle\frac{\partial\langle T^{\times}\rangle^2}{\partial p}\right)+\\
&\frac{c_p\langle T^{\times}\rangle}{T_r}\left(\left\langle\vec{V}'\bullet\vec{\nabla}T'^{\times}\right\rangle+\left\langle\omega'\frac{\partial T'^{\times}}{\partial p}\right\rangle\right)-\frac{c_p\langle T^{\times}\rangle}{T_r}\left(\left\langle\overline{\vec{V}\bullet\vec{\nabla}T}\right\rangle+\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle\right)+\\
&\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\omega\frac{\partial\overline{T}}{\partial p}\right\rangle-\langle T^{\times}\rangle\frac{R}{p}\langle\omega^{\times}\rangle-\langle T^{\times}\rangle\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=0
\end{aligned}
\tag{B.16}
$$

By definition: $A_{TM\,B}=\frac{c_p}{2T_r}\langle T^{\times}\rangle^2$, where $A_{TM\,B}$ is the baroclinic component of time-mean available enthalpy. Eq. (B.16) becomes:

$$
\begin{aligned}
&\left(\frac{\partial A_{TM\,B}}{\partial t}+\langle\vec{V}\rangle\bullet\vec{\nabla}A_{TM\,B}+\langle\omega\rangle\frac{\partial A_{TM\,B}}{\partial p}\right)+\\
&\frac{c_p\langle T^{\times}\rangle}{T_r}\left(\left\langle\vec{V}'\bullet\vec{\nabla}T'^{\times}\right\rangle+\left\langle\omega'\frac{\partial T'^{\times}}{\partial p}\right\rangle\right)-\frac{c_p\langle T^{\times}\rangle}{T_r}\left(\left\langle\overline{\vec{V}\bullet\vec{\nabla}T}\right\rangle+\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle\right)+\\
&\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\omega\frac{\partial\overline{T}}{\partial p}\right\rangle-\langle T^{\times}\rangle\frac{R}{p}\langle\omega^{\times}\rangle-\langle T^{\times}\rangle\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=0
\end{aligned}
\tag{B.17}
$$

We express the second and third terms of eq. (B.17) in their flux forms:

$$
\begin{aligned}
&\frac{\partial A_{TM\,B}}{\partial t}+\vec{\nabla}\bullet\left(\langle\vec{V}\rangle A_{TM\,B}\right)+\frac{\partial\left(\langle\omega\rangle A_{TM\,B}\right)}{\partial p}+\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\vec{V}'\bullet\vec{\nabla}T'^{\times}\right\rangle\\
&+\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\omega'\frac{\partial T'^{\times}}{\partial p}\right\rangle-\frac{c_p\langle T^{\times}\rangle}{T_r}\left(\left\langle\overline{\vec{V}\bullet\vec{\nabla}T}\right\rangle+\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle\right)+\\
&\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\omega\frac{\partial\overline{T}}{\partial p}\right\rangle-\langle T^{\times}\rangle\frac{R}{p}\langle\omega^{\times}\rangle-\langle T^{\times}\rangle\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=0
\end{aligned}
\tag{B.18}
$$

We develop the fourth and fifth terms of eq. (B.18) as follows:

$$
\begin{aligned}
&\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle \vec{V}'\bullet\vec{\nabla}T'^{\times}\right\rangle+\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle \omega'\frac{\partial T'^{\times}}{\partial p}\right\rangle \\
&=\frac{c_p\langle T^{\times}\rangle}{T_r}\vec{\nabla}\bullet\left\langle \vec{V}'T'^{\times}\right\rangle+\frac{c_p\langle T^{\times}\rangle}{T_r}\frac{\partial\langle\omega'T'^{\times}\rangle}{\partial p} \\
&=\frac{c_p}{T_r}\vec{\nabla}\bullet\left(\langle T^{\times}\rangle\left\langle \vec{V}'T'^{\times}\right\rangle\right)+\frac{c_p}{T_r}\frac{\partial\left(\langle T^{\times}\rangle\langle\omega'T'^{\times}\rangle\right)}{\partial p}- \\
&\frac{c_p}{T_r}\left\langle \vec{V}'T'^{\times}\right\rangle\vec{\nabla}\bullet\langle T^{\times}\rangle-\frac{c_p}{T_r}\langle\omega'T'^{\times}\rangle\frac{\partial\langle T^{\times}\rangle}{\partial p}
\end{aligned}
\tag{B.19}
$$

We insert eq. (B.19) in eq. (B.18):

$$
\begin{aligned}
&\frac{\partial A_{TM\,B}}{\partial t}+\vec{\nabla}\bullet\left(\left\langle\vec{V}\right\rangle A_{TM\,B}\right)+\frac{\partial\left(\langle\omega\rangle A_{TM\,B}\right)}{\partial p}+\frac{c_p}{T_r}\vec{\nabla}\bullet\left(\langle T^{\times}\rangle\left\langle\vec{V}'T'^{\times}\right\rangle\right)+ \\
&\frac{c_p}{T_r}\frac{\partial\left(\langle T^{\times}\rangle\langle\omega'T'^{\times}\rangle\right)}{\partial p}-\frac{c_p}{T_r}\left\langle\vec{V}'T'^{\times}\right\rangle\vec{\nabla}\bullet\langle T^{\times}\rangle- \\
&\frac{c_p}{T_r}\langle\omega'T'^{\times}\rangle\frac{\partial\langle T^{\times}\rangle}{\partial p}-\frac{c_p\langle T^{\times}\rangle}{T_r}\left(\left\langle\overline{\vec{V}\bullet\vec{\nabla}T}\right\rangle+\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle\right)+ \\
&\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\omega\frac{\partial\overline{T}}{\partial p}\right\rangle-\langle T^{\times}\rangle\frac{R}{p}\langle\omega^{\times}\rangle-\langle T^{\times}\rangle\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=0
\end{aligned}
\tag{B.20}
$$

We rewrite the tenth term of eq. (B.20) with eq. (B.vi) as follows:

$$
\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\omega\frac{\partial\overline{T}}{\partial p}\right\rangle=\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\overline{\omega}\frac{\partial\overline{T}}{\partial p}\right\rangle+\frac{c_p\langle T^{\times}\rangle}{T_r}\left\langle\omega^{\times}\frac{\partial\overline{T}}{\partial p}\right\rangle
\tag{B.21}
$$

We insert eq. (B.21) in eq. (B.20):

$$\begin{aligned}&\frac{\partial A_{TM\,B}}{\partial t}+\vec{\nabla}\bullet\left(\left\langle\vec{V}\right\rangle A_{TM\,B}\right)+\frac{\partial\left(\langle\omega\rangle A_{TM\,B}\right)}{\partial p}+\frac{c_p}{T_r}\vec{\nabla}\bullet\left(\langle T^\times\rangle\left\langle\vec{V}'T'^\times\right\rangle\right)+\\&\frac{c_p}{T_r}\frac{\partial\left(\langle T^\times\rangle\langle\omega'T'^\times\rangle\right)}{\partial p}-\frac{c_p}{T_r}\left\langle\vec{V}'T'^\times\right\rangle\vec{\nabla}\bullet\langle T^\times\rangle-\frac{c_p}{T_r}\langle\omega'T'^\times\rangle\frac{\partial\langle T^\times\rangle}{\partial p}-\\&\frac{c_p\langle T^\times\rangle}{T_r}\left(\left\langle\overline{\vec{V}\bullet\vec{\nabla}T}\right\rangle+\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle\right)+\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\overline{\omega}\frac{\partial\overline{T}}{\partial p}\right\rangle-\\&\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\omega^\times\frac{\partial\overline{T}}{\partial p}\right\rangle-\langle\omega^\times\rangle\langle\alpha^\times\rangle-\langle T^\times\rangle\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=0\end{aligned}\tag{B.22}$$

Using the small- χ approximation, the last term of eq. (B.22) is rewritten as:

$$\langle T^\times\rangle\left\langle\frac{Q}{T}-\overline{\left(\frac{Q}{T}\right)}\right\rangle=\langle T^\times\rangle\left\langle\frac{Q}{T}\right\rangle-\langle T^\times\rangle\left\langle\overline{\frac{Q}{T}}\right\rangle\simeq\frac{l}{T_r}\langle T^\times\rangle\langle Q^\times\rangle-\langle T^\times\rangle\left\langle\overline{\frac{Q}{T}}\right\rangle\tag{B.23}$$

with l an order unity factor.

We insert eq. (B.23) in eq. (B.22):

$$\begin{aligned}&\frac{\partial A_{TM\,B}}{\partial t}+\vec{\nabla}\bullet\left(\left\langle\vec{V}\right\rangle A_{TM\,B}\right)+\frac{\partial\left(\langle\omega\rangle A_{TM\,B}\right)}{\partial p}+\frac{c_p}{T_r}\vec{\nabla}\bullet\left(\langle T^\times\rangle\left\langle\vec{V}'T'^\times\right\rangle\right)+\\&\frac{c_p}{T_r}\frac{\partial\left(\langle T^\times\rangle\langle\omega'T'^\times\rangle\right)}{\partial p}-\frac{c_p}{T_r}\left\langle\vec{V}'T'^\times\right\rangle\vec{\nabla}\bullet\langle T^\times\rangle-\frac{c_p}{T_r}\langle\omega'T'^\times\rangle\frac{\partial\langle T^\times\rangle}{\partial p}-\\&\frac{c_p\langle T^\times\rangle}{T_r}\left(\left\langle\overline{\vec{V}\bullet\vec{\nabla}T}\right\rangle+\left\langle\overline{\omega\frac{\partial T}{\partial p}}\right\rangle\right)+\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\overline{\omega}\frac{\partial\overline{T}}{\partial p}\right\rangle-\\&\frac{c_p\langle T^\times\rangle}{T_r}\left\langle\omega^\times\frac{\partial\overline{T}}{\partial p}\right\rangle-\langle\omega^\times\rangle\langle\alpha^\times\rangle-\frac{l}{T_r}\langle T^\times\rangle\langle Q^\times\rangle+\langle T^\times\rangle\left\langle\overline{\frac{Q}{T}}\right\rangle=0\end{aligned}\tag{B.24}$$

We apply the isobaric-mean operator on eq. (B.24):

$$
\begin{aligned}
&\frac{\partial \overline{A_{TM\,B}}}{\partial t}+\vec{\nabla}\bullet\left(\overline{\left\langle\vec{V}\right\rangle A_{TM\,B}}\right)+\frac{\partial\left(\overline{\langle\omega\rangle A_{TM\,B}}\right)}{\partial p}+\frac{c_p}{T_r}\vec{\nabla}\bullet\left(\overline{\left\langle T^{\times}\right\rangle\left\langle\vec{V}'T'^{\times}\right\rangle}\right)+\\
&\frac{c_p}{T_r}\frac{\partial\left(\overline{\left\langle T^{\times}\right\rangle\left\langle\omega'T'^{\times}\right\rangle}\right)}{\partial p}-\frac{c_p}{T_r}\overline{\left\langle\vec{V}'T'^{\times}\right\rangle\vec{\nabla}\bullet\left\langle T^{\times}\right\rangle}-\frac{c_p}{T_r}\overline{\left\langle\omega'T'^{\times}\right\rangle\frac{\partial\left\langle T^{\times}\right\rangle}{\partial p}}-\\
&\cancel{\overline{\frac{c_p\left\langle T^{\times}\right\rangle}{T_r}\left(\left\langle\vec{V}\bullet\vec{\nabla}T\right\rangle+\left\langle\omega\frac{\partial T}{\partial p}\right\rangle\right)}}+\cancel{\overline{\frac{c_p\left\langle T^{\times}\right\rangle}{T_r}\left\langle\omega\frac{\partial T}{\partial p}\right\rangle}}-\\
&\overline{\frac{c_p\left\langle T^{\times}\right\rangle}{T_r}\left\langle\omega^{\times}\frac{\partial\overline{T}}{\partial p}\right\rangle}-\overline{\left\langle\omega^{\times}\right\rangle\left\langle\alpha^{\times}\right\rangle}-\frac{l}{T_r}\overline{\left\langle T^{\times}\right\rangle\left\langle Q^{\times}\right\rangle}+\cancel{\overline{\left\langle T^{\times}\right\rangle\left\langle\frac{Q}{T}\right\rangle}}=0
\end{aligned}
\tag{B.25}
$$

The final form for the baroclinic component of time- and isobaric-mean temperature-dependant part of available enthalpy:

$$
\begin{aligned}
&\frac{\partial \overline{A_{TM\,B}}}{\partial t}=-\vec{\nabla}\bullet\left(\overline{\left\langle\vec{V}\right\rangle A_{TM\,B}}\right)-\frac{\partial\left(\overline{\langle\omega\rangle A_{TM\,B}}\right)}{\partial p}-\frac{c_p}{T_r}\vec{\nabla}\bullet\left(\overline{\left\langle T^{\times}\right\rangle\left\langle\vec{V}'T'^{\times}\right\rangle}\right)-\\
&\frac{c_p}{T_r}\frac{\partial\left(\overline{\left\langle T^{\times}\right\rangle\left\langle\omega'T'^{\times}\right\rangle}\right)}{\partial p}+\frac{c_p}{T_r}\overline{\left\langle\vec{V}'T'^{\times}\right\rangle\vec{\nabla}\bullet\left\langle T^{\times}\right\rangle}+\frac{c_p}{T_r}\overline{\left\langle\omega'T'^{\times}\right\rangle\frac{\partial\left\langle T^{\times}\right\rangle}{\partial p}}+\\
&\overline{\frac{c_p\left\langle T^{\times}\right\rangle}{T_r}\left\langle\omega^{\times}\frac{\partial\overline{T}}{\partial p}\right\rangle}+\overline{\left\langle\omega^{\times}\right\rangle\left\langle\alpha^{\times}\right\rangle}+\frac{l}{T_r}\overline{\left\langle T^{\times}\right\rangle\left\langle Q^{\times}\right\rangle}=0
\end{aligned}
\tag{B.26}
$$

$$
\frac{\partial \overline{A_{TM\,B}}}{\partial t}=\overline{G_{TM\,B}}-\overline{C_{TM\,B}}+\overline{C_{TM\,BS}}-\overline{C_{A\,B}}-\overline{F_{A_{TM\,B}}}-\overline{H_{A_{TM\,B}}}
\tag{B.27}
$$

Where:

$$
\overline{A_{TM\,B}}=\frac{c_p}{2T_r}\overline{\left\langle T^{\times}\right\rangle^{2}}
$$

$$\overline{G_{TM\,B}} = \frac{l}{T_r}\overline{\langle T^{\times}\rangle\langle Q^{\times}\rangle}$$

$$\overline{C_{TM\,B}} = -\overline{\langle \omega^{\times}\rangle\langle \alpha^{\times}\rangle}$$

$$\overline{C_{TM\,BS}} = -\frac{c_p}{T_r}\overline{\langle T^{\times}\rangle\left\langle \omega^{\times}\frac{\partial \overline{T}}{\partial p}\right\rangle}$$

$$\overline{C_{A\,B}} = -\frac{c_p}{T_r}\overline{\left\langle \vec{V}'T'^{\times}\right\rangle\vec{\nabla}\bullet\langle T\rangle} - \frac{c_p}{T_r}\overline{\langle \omega' T'^{\times}\rangle\frac{\partial\langle T^{\times}\rangle}{\partial p}}$$

$$\overline{F_{A_{TM\,B}}} = \vec{\nabla}\bullet\left(\overline{\left\langle\vec{V}\right\rangle A_{TM\,B}}\right) + \frac{\partial\left(\overline{\langle\omega\rangle A_{TM\,B}}\right)}{\partial p}$$

$$\overline{H_{A_{TM\,B}}} = \frac{c_p}{T_r}\vec{\nabla}\bullet\left(\left(\overline{\langle T^{\times}\rangle\left\langle\vec{V}'T'^{\times}\right\rangle}\right) + \frac{\partial\left(\overline{\langle T^{\times}\rangle\langle\omega' T'^{\times}\rangle}\right)}{\partial p}\right)$$

B.2.5 Stratification component of time- and isobaric-mean temperature-dependent part of available enthalpy prognostic equation $\left(\overline{A_{TM\,S}}\right)$

To compute $\frac{\partial\overline{A_{TM\,S}}}{\partial t} = \frac{\partial\overline{A_{TM}}}{\partial t} - \frac{\partial\overline{A_{TM\,B}}}{\partial t}$, we subtract eq. (B.26) from eq. (B.7):

$$\begin{aligned}\frac{\partial\overline{A_{TM\,S}}}{\partial t} &= \overline{G_{TM}} + \overline{I_{AB}} - \overline{C_{TM}} - \overline{C_A} - \overline{F_{A_{TM}}} - \overline{H_{A_{TM}}} - \\ &\left(\overline{G_{TM\,B}} - \overline{C_{TM\,B}} + \overline{C_{TM\,BS}} - \overline{C_{A\,B}} - \overline{F_{A_{TM\,B}}} - \overline{H_{A_{TM\,B}}}\right) \\ &= \left(\overline{G_{TM}} - \overline{G_{TM\,B}}\right) + \overline{I_{AB}} - \left(\overline{C_{TM}} - \overline{C_{TM\,B}}\right) - \overline{C_{TM\,BS}} - \\ &\left(\overline{C_A} - \overline{C_{A\,B}}\right) - \left(\overline{F_{A_{TM}}} - \overline{F_{A_{TM\,B}}}\right) - \left(\overline{H_{A_{TM}}} - \overline{H_{A_{TM\,B}}}\right) \\ &= \overline{G_{TM\,S}} + \overline{I_{AB}} - \overline{C_{TM\,S}} - \overline{C_{TM\,BS}} - \overline{C_{A\,S}} - \overline{F_{A_{TM\,S}}} - \overline{H_{A_{TM\,S}}}\end{aligned} \tag{B.28}$$

Where:

$$\overline{A_{TM\,S}} = \overline{A_{TM}} - \overline{A_{TM\,B}} = \frac{c_p}{2T_r}\overline{\langle T_* \rangle^2} - \frac{c_p}{2T_r}\overline{\langle T^\times \rangle^2}$$
$$= \frac{c_p}{2T_r}\overline{\langle T \rangle^2 - T_r^2 - \langle T^\times \rangle^2} = \frac{c_p}{2T_r}\overline{\langle T - T^\times - T_r \rangle^2} \quad \text{(B.29)}$$
$$= \frac{c_p}{2T_r}\overline{\langle \overline{T} - T_r \rangle^2}$$

$$\overline{G_{TM\,S}} = \overline{G_{TM}} - \overline{G_{TM\,B}} = \frac{l}{T_r}\overline{\langle T_* \rangle\langle Q \rangle} - \frac{l}{T_r}\overline{\langle T^\times \rangle\langle Q^\times \rangle}$$
$$= \frac{l}{T_r}\left(\overline{(\langle T \rangle - T_r)\langle Q \rangle} - \overline{\left(\langle T \rangle - \langle \overline{T} \rangle\right)\left(\langle Q \rangle - \langle \overline{Q} \rangle\right)}\right)$$
$$= \frac{l}{T_r}\left(\overline{\langle T \rangle\langle Q \rangle} - \overline{T_r\langle Q \rangle} - \overline{\langle T \rangle\langle Q \rangle} + \overline{\langle T \rangle\langle \overline{Q} \rangle} - \overline{\langle \overline{T} \rangle\langle Q \rangle} + \overline{\langle \overline{T} \rangle\langle \overline{Q} \rangle}\right) \quad \text{(B.30)}$$
$$= \frac{l}{T_r}\left(-T_r\overline{\langle Q \rangle} + \overline{\langle \overline{T} \rangle\langle \overline{Q} \rangle}\right) = \frac{l}{T_r}\left(-T_r\langle \overline{Q} \rangle + \langle \overline{T} \rangle\langle \overline{Q} \rangle\right) = \frac{l}{T_r}\langle \overline{T}_* \rangle\langle \overline{Q} \rangle$$

$$\overline{C_{TM\,S}} = \overline{C_{TM}} - \overline{C_{TM\,B}} = -\overline{\langle \omega \rangle\langle \alpha \rangle} + \overline{\langle \omega^\times \rangle\langle \alpha^\times \rangle}$$
$$= -\overline{\langle \omega \rangle\langle \alpha \rangle} + \overline{\langle \omega - \overline{\omega} \rangle\langle \alpha - \overline{\alpha} \rangle}$$
$$= -\overline{\langle \omega \rangle\langle \alpha \rangle} + \overline{\langle \omega \rangle\langle \alpha \rangle} - \overline{\langle \omega \rangle\langle \overline{\alpha} \rangle} - \overline{\langle \overline{\omega} \rangle\langle \alpha \rangle} + \overline{\langle \overline{\omega} \rangle\langle \overline{\alpha} \rangle} \quad \text{(B.31)}$$
$$= -\langle \overline{\omega} \rangle\langle \overline{\alpha} \rangle$$

$$\overline{F_{A_{TM\,S}}} = \overline{F_{A_{TM}}} - \overline{F_{A_{TM\,B}}}$$
$$= \vec{\nabla}\bullet\left(\overline{A_{TM}\left\langle \vec{V} \right\rangle}\right) + \frac{\partial\left(\overline{A_{TM}\langle \omega \rangle}\right)}{\partial p} - \vec{\nabla}\bullet\left(\overline{A_{TM\,B}\left\langle \vec{V} \right\rangle}\right) - \frac{\partial\left(\overline{A_{TM\,B}\langle \omega \rangle}\right)}{\partial p}$$
$$= \vec{\nabla}\bullet\left(\overline{A_{TM}\left\langle \vec{V} \right\rangle} - \overline{A_{TM\,B}\left\langle \vec{V} \right\rangle}\right) + \frac{\partial\left(\overline{A_{TM}\langle \omega \rangle} - \overline{A_{TM\,B}\langle \omega \rangle}\right)}{\partial p} \quad \text{(B.32)}$$
$$= \vec{\nabla}\bullet\left(\overline{A_{TM\,S}\left\langle \vec{V} \right\rangle}\right) + \frac{\partial\left(\overline{A_{TM\,S}\langle \omega \rangle}\right)}{\partial p}$$

$$
\begin{aligned}
\overline{C_{AS}} &= \overline{C_A} - \overline{C_{AB}} \\
&= -\frac{c_p}{T_r}\overline{\left\langle \vec{V}'T' \right\rangle \bullet \vec{\nabla}\langle T\rangle} - \frac{c_p}{T_r}\overline{\langle \omega' T'\rangle \frac{\partial \langle T\rangle}{\partial p}} + \\
&\quad \frac{c_p}{T_r}\overline{\left\langle \vec{V}'T'^{\times} \right\rangle \vec{\nabla}\bullet\langle T\rangle} + \frac{c_p}{T_r}\overline{\left\langle \omega' T'^{\times}\right\rangle \frac{\partial \left\langle T^{\times}\right\rangle}{\partial p}} \\
&= -\frac{c_p}{T_r}\left(\overline{\left\langle \vec{V}'T' \right\rangle \bullet \vec{\nabla}\langle T\rangle - \left\langle \vec{V}'T'^{\times} \right\rangle \vec{\nabla}\bullet\langle T\rangle}\right) - \\
&\quad \frac{c_p}{T_r}\left(\overline{\langle \omega' T'\rangle \frac{\partial \langle T\rangle}{\partial p} - \left\langle \omega' T'^{\times}\right\rangle \frac{\partial \left\langle T^{\times}\right\rangle}{\partial p}}\right) \\
&= -\frac{c_p}{T_r}\left(\overline{\left\langle \vec{V}'T' - \vec{V}'\left(T' - \overline{T'}\right) \right\rangle \bullet \vec{\nabla}\langle T\rangle}\right) - \\
&\quad \frac{c_p}{T_r}\left(\overline{\langle \omega' T'\rangle \frac{\partial \langle T\rangle}{\partial p} - \left\langle \omega'\left(T' - \overline{T'}\right)\right\rangle \frac{\partial \left\langle T - \overline{T}\right\rangle}{\partial p}}\right) \\
&= -\frac{c_p}{T_r}\left(\overline{\left\langle \vec{V}'\overline{T'} \right\rangle \bullet \vec{\nabla}\langle T\rangle}\right) - \\
&\quad \frac{c_p}{T_r}\left(\overline{\langle \omega' T'\rangle \frac{\partial \langle T\rangle}{\partial p} - \left\langle \omega' T' - \omega'\overline{T'}\right\rangle \left(\frac{\partial \langle T\rangle}{\partial p} - \frac{\partial \left\langle \overline{T}\right\rangle}{\partial p}\right)}\right) \\
&= -\frac{c_p}{T_r}\left(\overline{\left\langle \overline{T'}\vec{V}' \right\rangle \bullet \vec{\nabla}\langle T\rangle}\right) - \\
&\quad \frac{c_p}{T_r}\left(\overline{\left\langle \omega'\overline{T'}\right\rangle \frac{\partial \langle T\rangle}{\partial p} + \langle \omega' T'\rangle \frac{\partial \left\langle \overline{T}\right\rangle}{\partial p} - \left\langle \omega'\overline{T'}\right\rangle \frac{\partial \left\langle \overline{T}\right\rangle}{\partial p}}\right) \\
&= -\frac{c_p}{T_r}\left(\overline{\left\langle \overline{T'}\vec{V}' \right\rangle \bullet \vec{\nabla}\langle T\rangle}\right) - \frac{c_p}{T_r}\left(\overline{\left\langle \omega'\overline{T'}\right\rangle \frac{\partial \left\langle T^{\times}\right\rangle}{\partial p} + \langle \omega' T'\rangle \frac{\partial \left\langle \overline{T}\right\rangle}{\partial p}}\right)
\end{aligned}
\tag{B.33}
$$

$$\begin{aligned}
\overline{H_{A_{TM}\,S}} &= \overline{H_{A_{TM}}} - \overline{H_{A_{TM}\,B}} \\
&= \frac{c_p}{T_r}\left(\vec{\nabla}\bullet\left(\overline{\langle T_*\rangle\left\langle\vec{V}'T'\right\rangle}\right)+\frac{\partial\left(\overline{\langle T_*\rangle\langle\omega'T'\rangle}\right)}{\partial p}\right)- \\
&\quad \frac{c_p}{T_r}\left(\vec{\nabla}\bullet\left(\overline{\langle T^{\times}\rangle\left\langle\vec{V}'T'^{\times}\right\rangle}\right)+\frac{\partial\left(\overline{\langle T^{\times}\rangle\langle\omega'T'^{\times}\rangle}\right)}{\partial p}\right) \\
&= \frac{c_p}{T_r}\vec{\nabla}\bullet\left(\overline{\langle T_*\rangle\left\langle\vec{V}'T'\right\rangle-\langle T^{\times}\rangle\left\langle\vec{V}'T'^{\times}\right\rangle}\right)+ \\
&\quad \frac{c_p}{T_r}\frac{\partial\left(\overline{\langle T_*\rangle\langle\omega'T'\rangle-\langle T^{\times}\rangle\langle\omega'T'^{\times}\rangle}\right)}{\partial p} \\
&= \frac{c_p}{T_r}\vec{\nabla}\bullet\left(\overline{\langle T-T_r\rangle\left\langle\vec{V}'T'\right\rangle-\langle T-\overline{T}\rangle\left\langle\vec{V}'\left(T'-\overline{T'}\right)\right\rangle}\right)+ \\
&\quad \frac{c_p}{T_r}\frac{\partial\left(\overline{\langle T-T_r\rangle\langle\omega'T'\rangle-\langle T-\overline{T}\rangle\left\langle\omega'\left(T'-\overline{T'}\right)\right\rangle}\right)}{\partial p} \\
&= \frac{c_p}{T_r}\vec{\nabla}\bullet\left(\overline{-T_r\left\langle\vec{V}'T'\right\rangle+\langle T\rangle\left\langle\vec{V}'\overline{T'}\right\rangle+\langle\overline{T}\rangle\left\langle\vec{V}'T'\right\rangle-\langle\overline{T}\rangle\left\langle\vec{V}'\overline{T'}\right\rangle}\right)+ \\
&\quad \frac{c_p}{T_r}\frac{\partial\left(\overline{-T_r\langle\omega'T'\rangle+\langle T\rangle\left\langle\omega'\overline{T'}\right\rangle+\langle\overline{T}\rangle\langle\omega'T'\rangle-\langle\overline{T}\rangle\left\langle\omega'\overline{T'}\right\rangle}\right)}{\partial p} \\
&= \frac{c_p}{T_r}\vec{\nabla}\bullet\left(\overline{\langle\overline{T_*}\rangle\left\langle\vec{V}'T'\right\rangle+\langle T^{\times}\rangle\left\langle\vec{V}'\overline{T'}\right\rangle}\right)+\frac{c_p}{T_r}\frac{\partial\left(\overline{\langle\overline{T_*}\rangle\langle\omega'T'\rangle+\langle T^{\times}\rangle\left\langle\omega'\overline{T'}\right\rangle}\right)}{\partial p}
\end{aligned} \qquad \text{(B.34)}$$

We insert eq. (B.30), eq. (B.31), eq. (B.32), eq. (B.33) and eq. (B.34) in eq. (B.27). The final form for the stratification component of time- and isobaric-mean temperature-dependant part of available enthalpy:

$$\frac{\partial \overline{A_{TM\,S}}}{\partial t} = \frac{l}{T_r}\left\langle \overline{T_*} \right\rangle \left\langle \overline{Q} \right\rangle - \frac{RT_r}{p}\overline{\langle\omega\rangle} + \left\langle\overline{\omega}\right\rangle\left\langle\overline{\alpha}\right\rangle + \frac{c_p}{T_r}\overline{\left\langle T^{\times}\right\rangle\left\langle \omega^{\times}\frac{\partial \overline{T}}{\partial p}\right\rangle} + \frac{c_p}{T_r}\left(\overline{\left\langle \overline{T'\vec{V}'}\right\rangle \bullet \vec{\nabla}\langle T\rangle}\right) + \frac{c_p}{T_r}\left(\overline{\left\langle \omega'\overline{T'}\right\rangle \frac{\partial\left\langle T^{\times}\right\rangle}{\partial p} + \left\langle \omega' T'\right\rangle \frac{\partial\left\langle \overline{T}\right\rangle}{\partial p}}\right) - \vec{\nabla}\bullet\left(\overline{A_{TM\,S}\left\langle\vec{V}\right\rangle}\right) - \frac{\partial\left(\overline{A_{TM\,S}\langle\omega\rangle}\right)}{\partial p} - \frac{c_p}{T_r}\vec{\nabla}\bullet\left(\overline{\left\langle\overline{T_*}\right\rangle\left\langle\vec{V}'T'\right\rangle + \left\langle T^{\times}\right\rangle\left\langle\vec{V}'\overline{T'}\right\rangle}\right) - \frac{c_p}{T_r}\frac{\partial\left(\overline{\left\langle\overline{T_*}\right\rangle\left\langle\omega'T'\right\rangle + \left\langle T^{\times}\right\rangle\left\langle\omega'\overline{T'}\right\rangle}\right)}{\partial p} \tag{B.35}$$

$$\frac{\partial \overline{A_{TM\,S}}}{\partial t} = \overline{G_{TM\,S}} + \overline{I_{AB}} - \overline{C_{TM\,S}} - \overline{C_{TM\,BS}} - \overline{C_{A\,S}} - \overline{F_{A_{TM\,S}}} - \overline{H_{A_{TM\,S}}} \tag{B.36}$$

Where:

$$\overline{A_{TM\,S}} = \frac{c_p}{2T_r}\overline{\left\langle \overline{T} - T_r\right\rangle^2}$$

$$\overline{G_{TM\,S}} = \frac{l}{T_r}\left\langle\overline{T_*}\right\rangle\left\langle\overline{Q}\right\rangle$$

$$\overline{I_{AB}} = -\frac{RT_r}{p}\overline{\langle\omega\rangle}$$

$$\overline{C_{TM\,S}} = -\left\langle\overline{\omega}\right\rangle\left\langle\overline{\alpha}\right\rangle$$

$$\overline{C_{TM\,BS}} = -\frac{c_p}{T_r}\overline{\langle T^\times\rangle\left\langle \omega^\times \frac{\partial \overline{T}}{\partial p}\right\rangle}$$

$$\overline{C_{A\,S}} = -\frac{c_p}{T_r}\left(\overline{\left\langle \overline{T'\vec{V}'}\right\rangle\cdot\vec{\nabla}\langle T\rangle}\right) - \frac{c_p}{T_r}\left(\overline{\langle\omega'\overline{T'}\rangle\frac{\partial\langle T^\times\rangle}{\partial p}+\langle\omega' T'\rangle\frac{\partial\langle\overline{T}\rangle}{\partial p}}\right)$$

$$\overline{F_{A_{TM\,S}}} = \vec{\nabla}\cdot\left(\overline{A_{TM\,S}\left\langle\vec{V}\right\rangle}\right)+\frac{\partial\left(\overline{A_{TM\,S}\langle\omega\rangle}\right)}{\partial p}$$

$$\overline{H_{A_{TM\,S}}} = \frac{c_p}{T_r}\vec{\nabla}\cdot\left(\overline{\langle \overline{T_*}\rangle\left\langle\vec{V}'T'\right\rangle+\langle T^\times\rangle\left\langle\vec{V}'\overline{T'}\right\rangle}\right)+\frac{c_p}{T_r}\frac{\partial\left(\overline{\langle\overline{T_*}\rangle\langle\omega'T'\rangle+\langle T^\times\rangle\langle\omega'\overline{T'}\rangle}\right)}{\partial p}$$

B.2.6 Isobaric-mean time variability temperature-dependent part of available enthalpy prognostic equation $\left(\overline{A_{TV}}\right)$

We start from eq. (A.37):

$$\frac{\partial\left(A_{TV}\right)}{\partial t} = \frac{l}{T_r}\langle T'Q'\rangle+\langle\omega'\alpha'\rangle-\frac{c_p}{T_r}\left\langle\vec{V}'T'\right\rangle\cdot\vec{\nabla}\langle T\rangle-\frac{c_p}{T_r}\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}- \\ \vec{\nabla}\cdot\left(A_{TV}\left\langle\vec{V}\right\rangle\right)-\frac{\partial\left(A_{TV}\langle\omega\rangle\right)}{\partial p}-\frac{c_p}{2T_r}\vec{\nabla}\cdot\left\langle\vec{V}'T'^2\right\rangle-\frac{c_p}{2T_r}\frac{\partial\left\langle\omega'T'^2\right\rangle}{\partial p} \tag{B.37}$$

We apply the isobaric-mean operator on eq. (B.37):

$$\frac{\partial\left(\overline{A_{TV}}\right)}{\partial t} = \frac{l}{T_r}\overline{\langle T'Q'\rangle}+\overline{\langle\omega'\alpha'\rangle}-\frac{c_p}{T_r}\overline{\left\langle\vec{V}'T'\right\rangle\cdot\vec{\nabla}\langle T\rangle}-\frac{c_p}{T_r}\overline{\langle\omega'T'\rangle\frac{\partial\langle T\rangle}{\partial p}}- \\ \vec{\nabla}\cdot\left(\overline{A_{TV}\left\langle\vec{V}\right\rangle}\right)-\frac{\partial\left(\overline{A_{TV}\langle\omega\rangle}\right)}{\partial p}-\frac{c_p}{2T_r}\vec{\nabla}\cdot\overline{\left\langle\vec{V}'T'^2\right\rangle}-\frac{c_p}{2T_r}\frac{\partial\overline{\left\langle\omega'T'^2\right\rangle}}{\partial p} \tag{B.38}$$

From eq. (B.33), we replace $\overline{C_A}$ by $\overline{C_{AS}}+\overline{C_{AB}}$ and insert their expressions in eq. (B.38). The final form for the isobaric-mean time variability temperature-dependant part of available enthalpy:

$$\frac{\partial\left(\overline{A_{TV}}\right)}{\partial t}=\frac{l}{T_r}\overline{\langle T'Q'\rangle}+\overline{\langle\omega'\alpha'\rangle}--\frac{c_p}{T_r}\left(\overline{\left\langle T'\vec{V}'\right\rangle\bullet\vec{\nabla}\langle T\rangle}\right)-$$
$$\frac{c_p}{T_r}\left(\overline{\left\langle\omega'\overline{T}'\right\rangle\frac{\partial\left\langle T^{\times}\right\rangle}{\partial p}+\left\langle\omega'T'\right\rangle\frac{\partial\left\langle\overline{T}\right\rangle}{\partial p}}\right)-$$
$$\frac{c_p}{T_r}\overline{\left\langle\vec{V}'T'^{\times}\right\rangle\vec{\nabla}\bullet\langle T\rangle}-\frac{c_p}{T_r}\overline{\left\langle\omega'T'^{\times}\right\rangle\frac{\partial\left\langle T^{\times}\right\rangle}{\partial p}}\vec{\nabla}\bullet\left(\overline{A_{TV}\left\langle\vec{V}\right\rangle}\right)- \tag{B.39}$$
$$\frac{\partial\left(\overline{A_{TV}\langle\omega\rangle}\right)}{\partial p}-\frac{c_p}{2T_r}\vec{\nabla}\bullet\overline{\left\langle\vec{V}'T'^2\right\rangle}-\frac{c_p}{2T_r}\frac{\partial\overline{\left\langle\omega'T'^2\right\rangle}}{\partial p}$$

$$\frac{\partial\overline{A_{TV}}}{\partial t}=\overline{G_{TV}}-\overline{C_{TV}}+\overline{C_{AS}}+\overline{C_{AB}}-\overline{F_{A_{TV}}}-\overline{H_{A_{TV}}} \tag{B.40}$$

Where:

$$\overline{A_{TV}}=\frac{c_p}{2T_r}\overline{\left\langle T'^2\right\rangle}$$

$$\overline{G_{TV}}=\frac{l}{T_r}\overline{\langle T'Q'\rangle}$$

$$\overline{C_{TV}}=-\overline{\langle\omega'\alpha'\rangle}$$

$$\overline{C_{AS}}=-\frac{c_p}{T_r}\left(\overline{\left\langle\overline{T}'\vec{V}'\right\rangle\bullet\vec{\nabla}\langle T\rangle}\right)-\frac{c_p}{T_r}\left(\overline{\left\langle\omega'\overline{T}'\right\rangle\frac{\partial\left\langle T^{\times}\right\rangle}{\partial p}+\left\langle\omega'T'\right\rangle\frac{\partial\left\langle\overline{T}\right\rangle}{\partial p}}\right)$$

$$\overline{C_{AB}}=-\frac{c_p}{T_r}\overline{\left\langle\vec{V}'T'^{\times}\right\rangle\vec{\nabla}\bullet\langle T\rangle}-\frac{c_p}{T_r}\overline{\left\langle\omega'T'^{\times}\right\rangle\frac{\partial\left\langle T^{\times}\right\rangle}{\partial p}}$$

$$\overline{F_{A_{TV}}} = \vec{\nabla}\bullet\left(\overline{A_{TV}\left\langle\vec{V}\right\rangle}\right) + \frac{\partial\left(\overline{A_{TV}\langle\omega\rangle}\right)}{\partial p}$$

$$\overline{H_{A_{TV}}} = \frac{c_p}{2T_r}\left(\vec{\nabla}\bullet\overline{\left\langle\vec{V}'T'^2\right\rangle} + \frac{\partial\overline{\langle\omega'T'^2\rangle}}{\partial p}\right)$$

B.2.7 Time- and isobaric-mean pressure-dependent part of available enthalpy prognostic equation $\left(\overline{B}\right)$

We start from eq. (A.39):

$$\frac{\partial B}{\partial t} = -\vec{\nabla}\bullet\left(B\left\langle\vec{V}\right\rangle\right) - \frac{\partial\left(B\langle\omega\rangle\right)}{\partial p} + \frac{RT_r}{p}\langle\omega\rangle \qquad \text{(B.41)}$$

We apply the isobaric-mean operator on eq. (B.41). The final form for the time- and isobaric-mean pressure-dependant part of available enthalpy:

$$\frac{\partial\overline{B}}{\partial t} = -\vec{\nabla}\bullet\left(\overline{B\left\langle\vec{V}\right\rangle}\right) - \frac{\partial\left(\overline{B\langle\omega\rangle}\right)}{\partial p} + \frac{RT_r}{p}\overline{\langle\omega\rangle} \qquad \text{(B.42)}$$

$$\frac{\partial\overline{B}}{\partial t} = -\overline{F_B} - \overline{I_{AB}} \qquad \text{(B.43)}$$

Where:

$$\overline{B} = RT_r\ln\left(\frac{p}{p_r}\right)$$

$$\overline{F_B} = \vec{\nabla}\bullet\left(\overline{B\left\langle\vec{V}\right\rangle}\right) + \frac{\partial\left(\overline{B\langle\omega\rangle}\right)}{\partial p}$$

$$I_{AB} = -\frac{RT_r}{p}\overline{\langle\omega\rangle}$$

B.3 Isobaric-mean kinetic energy equations

B.3.8 Time- and isobaric-mean kinetic energy prognostic equation $\left(\overline{K_{TM}}\right)$

We start from eq. (A.52):

$$\frac{\partial(K_{TM})}{\partial t} = -\langle\omega\rangle\langle\alpha\rangle + \left\langle \vec{V}' \cdot \left(\vec{V}' \cdot \vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle + \left\langle \vec{V}' \cdot \left(\omega' \frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle +$$
$$\left\langle\vec{V}\right\rangle \cdot \left\langle\vec{F}\right\rangle - \vec{\nabla} \cdot \left(K_{TM}\left\langle\vec{V}\right\rangle\right) - \frac{\partial\left(K_{TM}\langle\omega\rangle\right)}{\partial p} - \tag{B.44}$$
$$\vec{\nabla} \cdot \left(\left(\left\langle\vec{V}' \cdot \vec{V}'\right\rangle + \langle\Phi\rangle\right) \cdot \left\langle\vec{V}\right\rangle\right) - \frac{\partial\left(\left(\left\langle\vec{V}'\omega'\right\rangle + \langle\Phi\rangle\right) \cdot \left\langle\vec{V}\right\rangle\right)}{\partial p}$$

We apply the isobaric-mean operator on eq. (B.42):

$$\frac{\partial\left(\overline{K_{TM}}\right)}{\partial t} = -\overline{\langle\omega\rangle\langle\alpha\rangle} + \overline{\left\langle \vec{V}' \cdot \left(\vec{V}' \cdot \vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle} + \overline{\left\langle \vec{V}' \cdot \left(\omega' \frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle} +$$
$$\overline{\left\langle\vec{V}\right\rangle \cdot \left\langle\vec{F}\right\rangle} - \vec{\nabla} \cdot \left(\overline{K_{TM}\left\langle\vec{V}\right\rangle}\right) - \frac{\partial\left(\overline{K_{TM}\langle\omega\rangle}\right)}{\partial p} - \tag{B.45}$$
$$\vec{\nabla} \cdot \left(\overline{\left(\left\langle\vec{V}' \cdot \vec{V}'\right\rangle + \langle\Phi\rangle\right) \cdot \left\langle\vec{V}\right\rangle}\right) - \frac{\partial\left(\overline{\left(\left\langle\vec{V}'\omega'\right\rangle + \langle\Phi\rangle\right) \cdot \left\langle\vec{V}\right\rangle}\right)}{\partial p}$$

From eq. (B.30), we replace $\overline{C_{TM}}$ by $\overline{C_{TM\ S}}+\overline{C_{TM\ B}}$ and insert their expressions in eq. (B.45). The final form for the time- and isobaric-mean time kinetic energy:

$$\frac{\partial\left(\overline{K_{TM}}\right)}{\partial t}=-\overline{\langle\overline{\omega}\rangle\langle\overline{\alpha}\rangle}-\overline{\langle\omega^{\times}\rangle\langle\alpha^{\times}\rangle}+\overline{\left\langle\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle}+\overline{\left\langle\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle}+\overline{\left\langle\vec{V}\right\rangle\bullet\left\langle\vec{F}\right\rangle}-\vec{\nabla}\bullet\left(\overline{K_{TM}\left\langle\vec{V}\right\rangle}\right)-\frac{\partial\left(\overline{K_{TM}\langle\omega\rangle}\right)}{\partial p}-\vec{\nabla}\bullet\left(\overline{\left(\left\langle\vec{V}'\bullet\vec{V}'\right\rangle+\langle\Phi\rangle\right)\bullet\left\langle\vec{V}\right\rangle}\right)-\frac{\partial\left(\overline{\left(\left\langle\vec{V}'\omega'\right\rangle+\langle\Phi\rangle\right)\bullet\left\langle\vec{V}\right\rangle}\right)}{\partial p} \tag{B.46}$$

$$\frac{\partial\overline{K_{TM}}}{\partial t}=\overline{C_{TM\ S}}+\overline{C_{TM\ B}}-\overline{C_{K}}-\overline{D_{TM}}-\overline{F_{K_{TM}}}-\overline{H_{K_{TM}}} \tag{B.47}$$

Where:

$$\overline{K_{TM}}=\frac{1}{2}\overline{\left\langle\vec{V}\right\rangle\bullet\left\langle\vec{V}\right\rangle}$$

$$\overline{C_{TM\ S}}=-\langle\overline{\omega}\rangle\langle\overline{\alpha}\rangle$$

$$\overline{C_{TM\ B}}=-\overline{\langle\omega^{\times}\rangle\langle\alpha^{\times}\rangle}$$

$$\overline{C_{K}}=-\overline{\left\langle\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle}-\overline{\left\langle\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle}$$

$$\overline{D_{TM}}=-\overline{\left\langle\vec{V}\right\rangle\bullet\left\langle\vec{F}\right\rangle}$$

$$\overline{F_{K_{TM}}}=\vec{\nabla}\bullet\left(\overline{K_{TM}\left\langle\vec{V}\right\rangle}\right)+\frac{\partial\left(\overline{K_{TM}\langle\omega\rangle}\right)}{\partial p}$$

$$\overline{H_{K_{TM}}}=\vec{\nabla}\bullet\left(\overline{\left(\left\langle\vec{V}'\bullet\vec{V}'\right\rangle+\langle\Phi\rangle\right)\left\langle\vec{V}\right\rangle}\right)+\frac{\partial\overline{\left(\left\langle\vec{V}'\omega'\right\rangle+\langle\Phi\rangle\right)\langle\omega\rangle}}{\partial p}$$

B.3.9 Isobaric-mean time variability kinetic energy prognostic equation $\left(\overline{K_{TV}}\right)$

We start from eq. (A.65):

$$\frac{\partial\left(K_{TV}\right)}{\partial t}=-\langle\omega'\alpha'\rangle-\left\langle\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle-\left\langle\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle+\left\langle\vec{V}'\bullet\vec{F}'\right\rangle- \tag{B.48}$$
$$\vec{\nabla}\bullet\left(K_{TV}\left\langle\vec{V}\right\rangle\right)-\frac{\partial\left(K_{TV}\langle\omega\rangle\right)}{\partial p}-\vec{\nabla}\bullet\left\langle k_{TV}\vec{V}'\right\rangle-\frac{\partial\langle k_{TV}\omega'\rangle}{\partial p}-\vec{\nabla}\bullet\left\langle\Phi'\vec{V}'\right\rangle-\frac{\partial\langle\Phi'\omega'\rangle}{\partial p}$$

We apply the isobaric-mean operator on eq. (B.48). The final form for the isobaric-mean time variability kinetic energy:

$$\frac{\partial\left(\overline{K_{TV}}\right)}{\partial t}=-\overline{\langle\omega'\alpha'\rangle}-\overline{\left\langle\vec{V}'\bullet\left(\vec{V}'\bullet\vec{\nabla}\right)\left\langle\vec{V}\right\rangle\right\rangle}-\overline{\left\langle\vec{V}'\bullet\left(\omega'\frac{\partial\left\langle\vec{V}\right\rangle}{\partial p}\right)\right\rangle}+\overline{\left\langle\vec{V}'\bullet\vec{F}'\right\rangle}- \tag{B.49}$$
$$\vec{\nabla}\bullet\left(\overline{K_{TV}\left\langle\vec{V}\right\rangle}\right)-\frac{\partial\left(\overline{K_{TV}\langle\omega\rangle}\right)}{\partial p}-\vec{\nabla}\bullet\overline{\left\langle k_{TV}\vec{V}'\right\rangle}-\frac{\partial\overline{\langle k_{TV}\omega'\rangle}}{\partial p}-\vec{\nabla}\bullet\overline{\left\langle\Phi'\vec{V}'\right\rangle}-\frac{\partial\overline{\langle\Phi'\omega'\rangle}}{\partial p}$$

$$\frac{\partial \overline{K_{TV}}}{\partial t} = \overline{C_{TV}} + \overline{C_K} - \overline{D_{TV}} - \overline{F_{K_{TV}}} - \overline{H_{K_{TV}}} \tag{B.50}$$

Where:

$$\overline{K_{TV}} = \frac{1}{2}\overline{\left\langle \vec{V}' \bullet \vec{V}' \right\rangle}$$

$$\overline{C_{TV}} = -\overline{\langle \omega' \alpha' \rangle}$$

$$C_K = -\overline{\left\langle \vec{V}' \bullet \left(\vec{V}' \bullet \vec{\nabla} \right) \left\langle \vec{V} \right\rangle \right\rangle} - \overline{\left\langle \vec{V}' \bullet \left(\omega' \frac{\partial \left\langle \vec{V} \right\rangle}{\partial p} \right) \right\rangle}$$

$$\overline{D_{TV}} = -\overline{\left\langle \vec{V}' \bullet \vec{F}' \right\rangle}$$

$$\overline{F_{K_{TV}}} = \vec{\nabla} \bullet \left(\overline{K_{TV} \left\langle \vec{V} \right\rangle} \right) + \frac{\partial \left(\overline{K_{TV} \langle \omega \rangle} \right)}{\partial p}$$

$$\overline{H_{K_{TV}}} = \vec{\nabla} \bullet \overline{\left\langle \left(k_{TV} + \Phi' \right) \vec{V}' \right\rangle} + \frac{\partial \overline{\left\langle \left(k_{TV} + \Phi' \right) \omega' \right\rangle}}{\partial p}$$

FIGURES

Figure 1.15 Subsection 1.6.4: Time evolution of energy reservoirs and energy fluxes

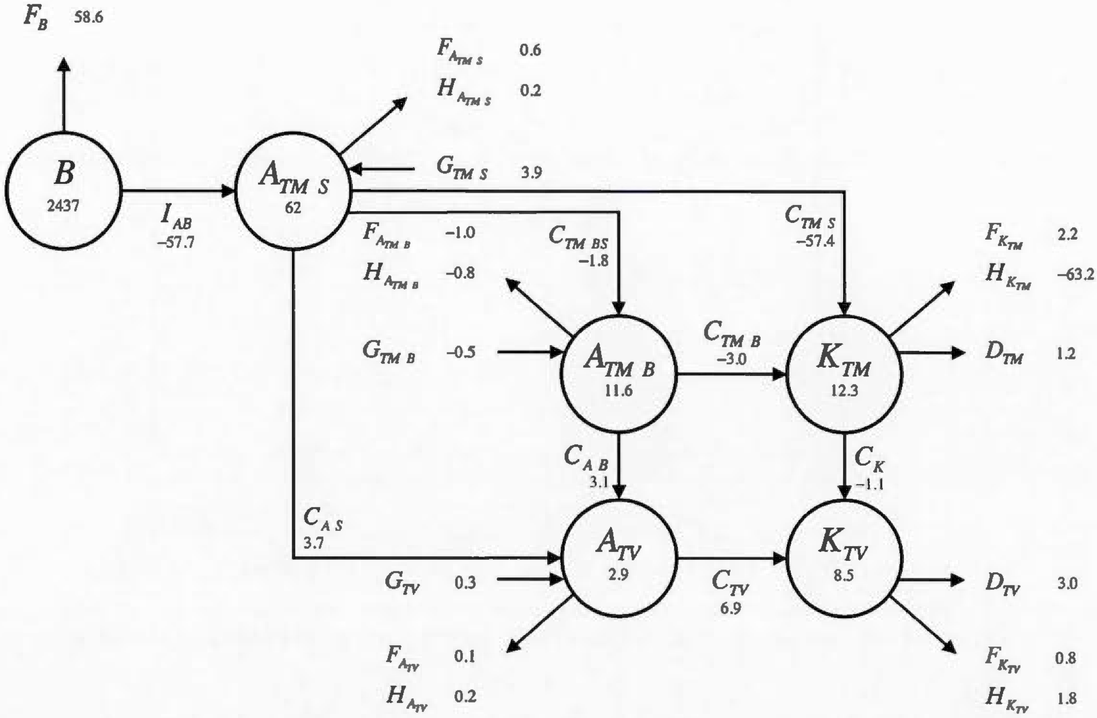


Figure 1.1 Vertically integrated time- and isobaric-mean energy cycle. Values are in $10^5\ \mathrm{J \cdot m^{-2}}$ for the energy reservoirs (boxes) and in $\mathrm{W \cdot m^{-2}}$ for the energy fluxes (arrows).

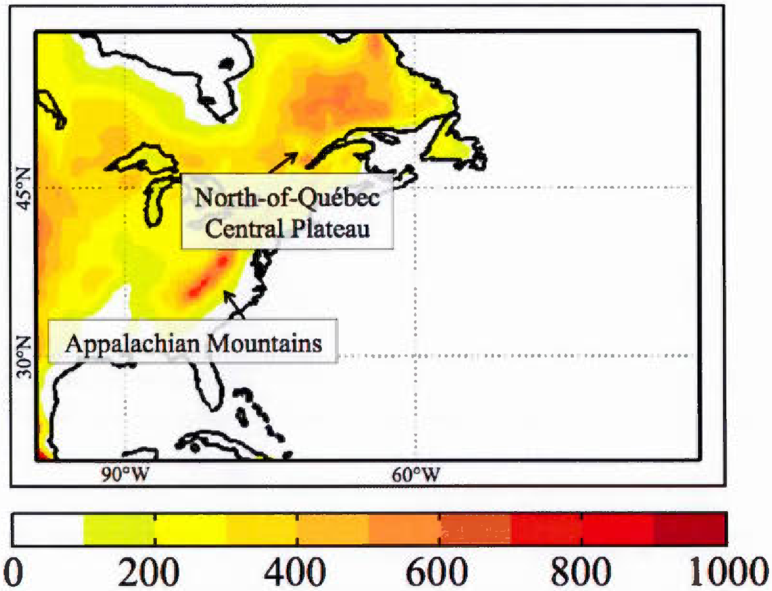


Figure 1.2 Terrain elevation map within the regional domain. Values are in m.

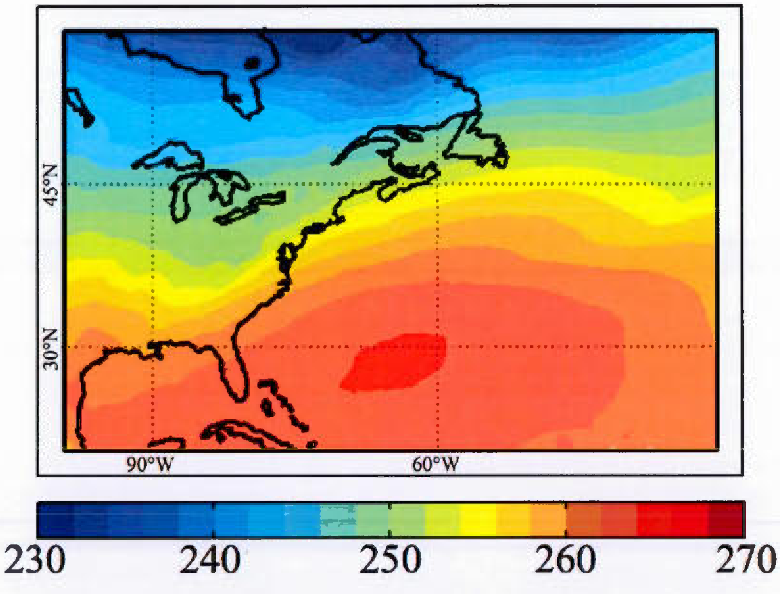


Figure 1.3 Vertically averaged time-mean temperature for the month of December 2004. Values are in K.

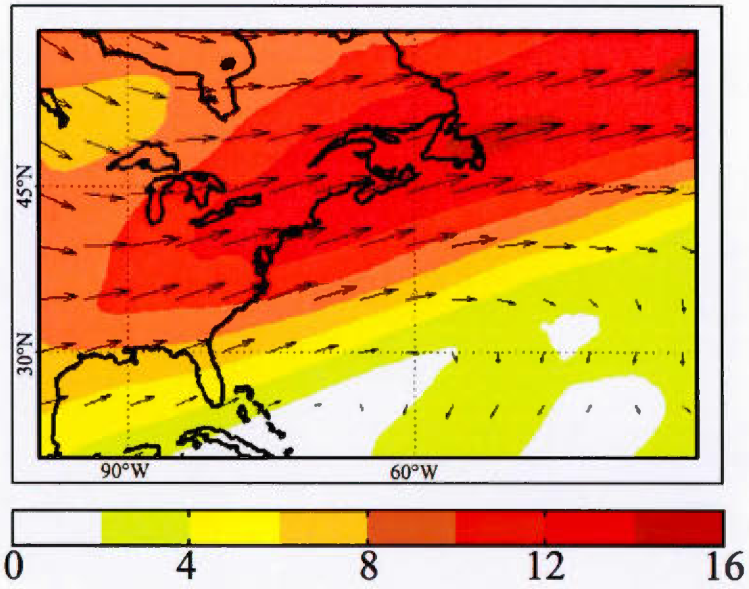


Figure 1.4 Vertically averaged time-mean wind for the month of December 2004. Values are in $m \cdot s^{-1}$.

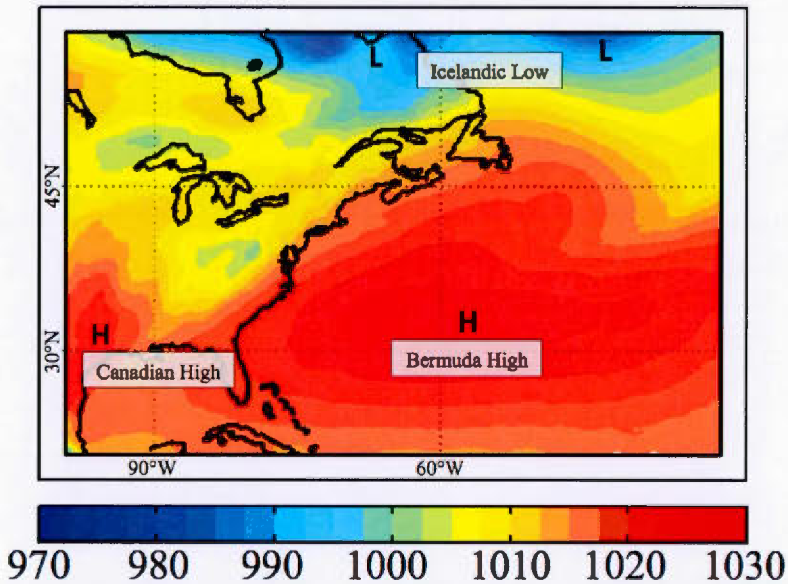


Figure 1.5 Time-averaged mean sea level pressure for the month of December 2004. Semi-permanent high- and low-pressure systems are indicated. Values are in hPa.

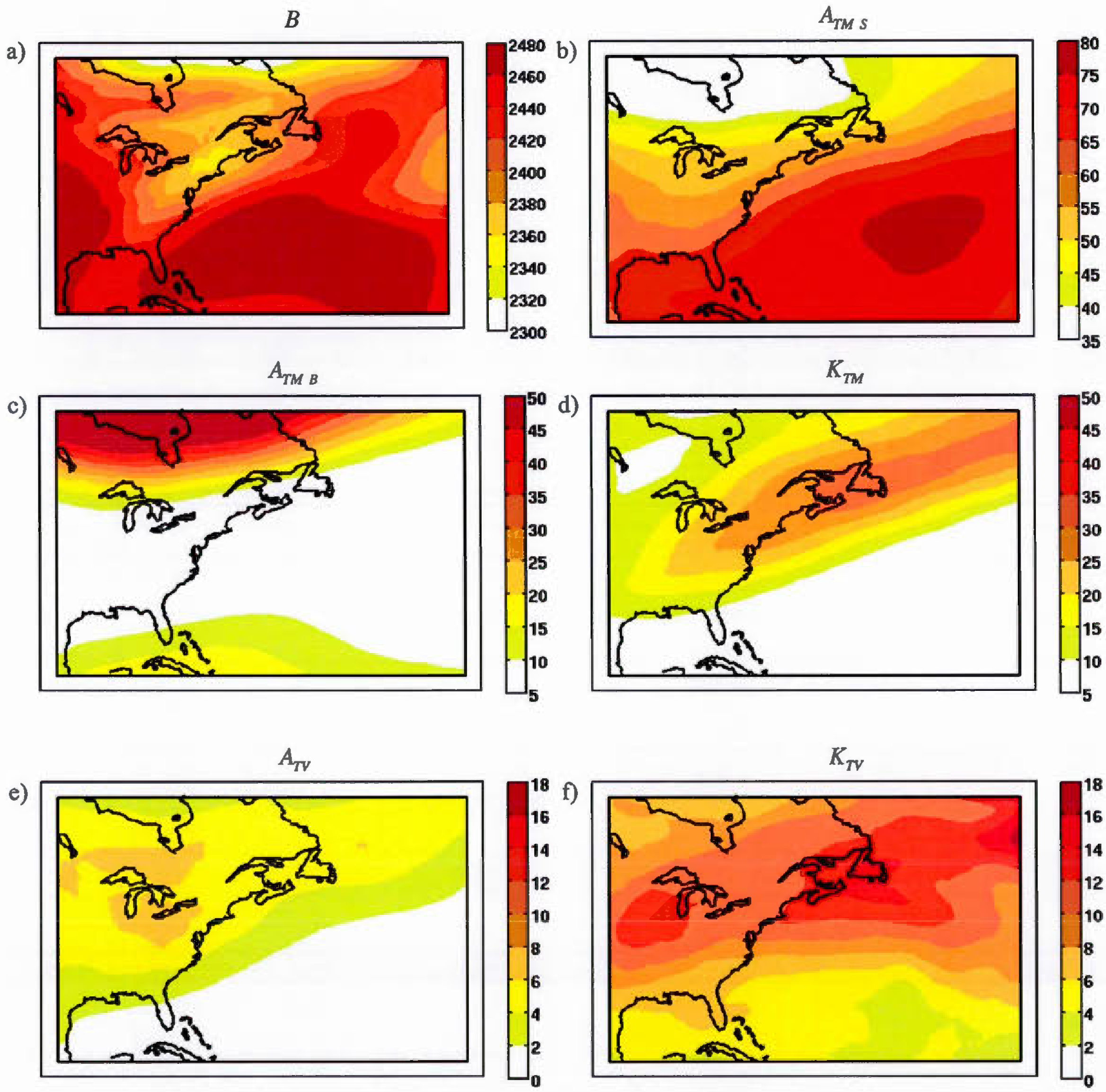


Figure 1.6 Maps of vertically integrated and time-mean reservoirs of energy. Panels a and b have different scales, panels c and d have the same scale and panels e and f have another scale. Values are in $10^5\ \mathrm{J \cdot m^{-2}}$.

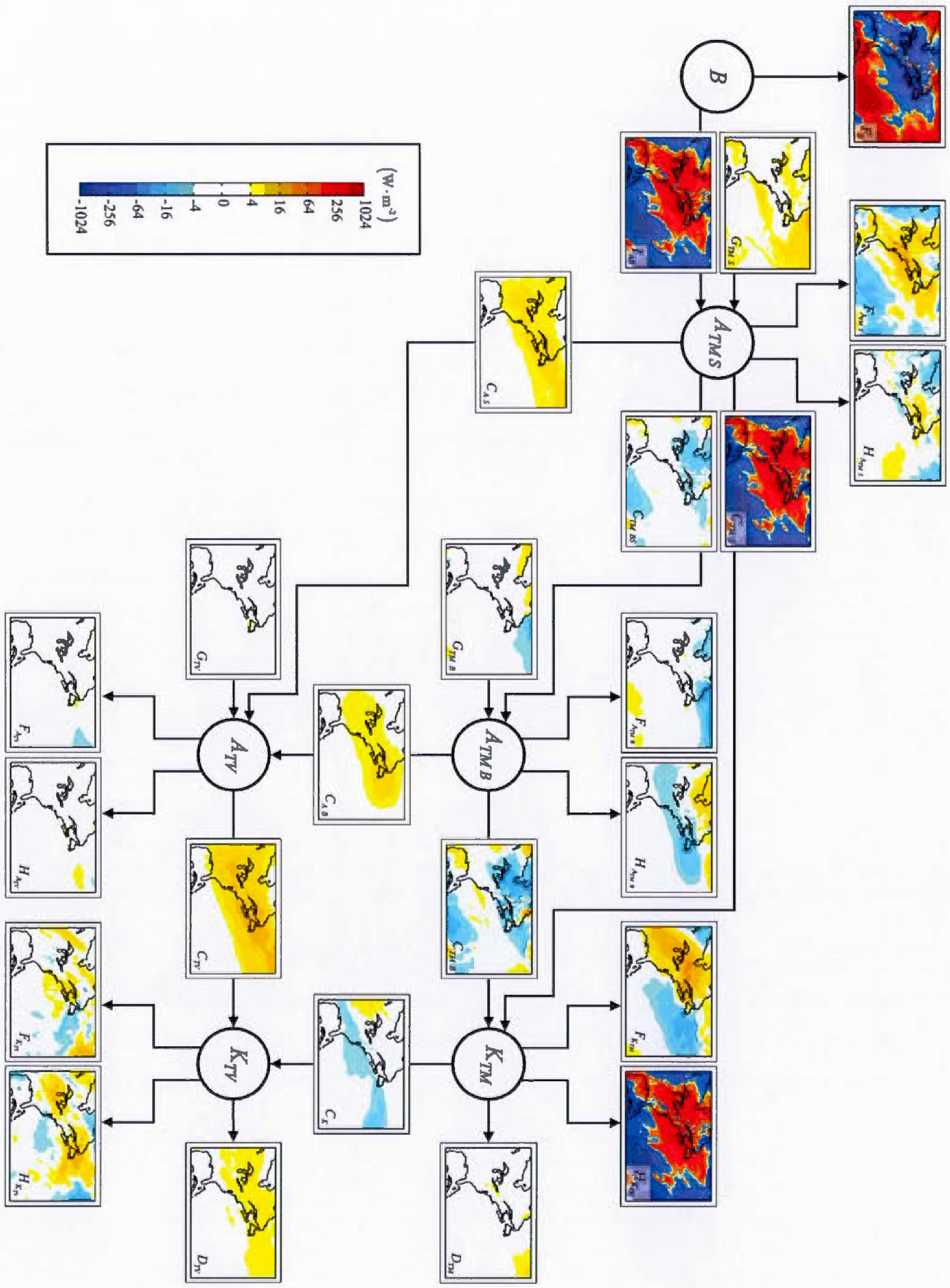


Figure 1.7 Maps of vertically integrated time-mean fluxes. The scale is non-linear. Values are in $W \cdot m^{-2}$. Note that the isobaric-mean value for each flux can be found in Fig. 1.1.

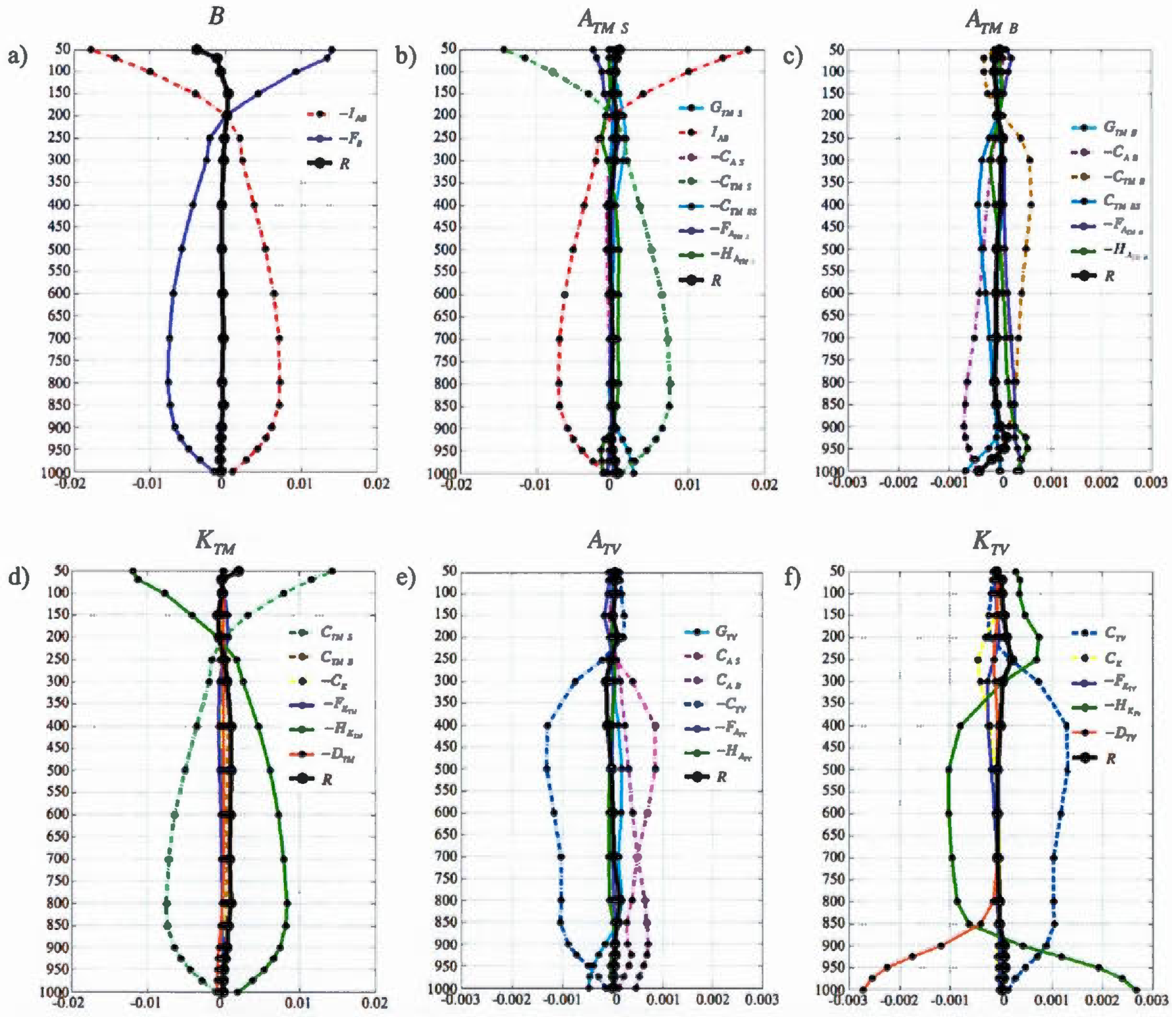


Figure 1.8 Vertical profiles of time- and isobaric-mean fluxes for the six energy reservoirs. The abscissa is the energy tendency in $W \cdot kg^{-1}$ and the ordinate is the pressure in hPa. Panels a, b and d have the same scale and panels c, e and f have another scale.

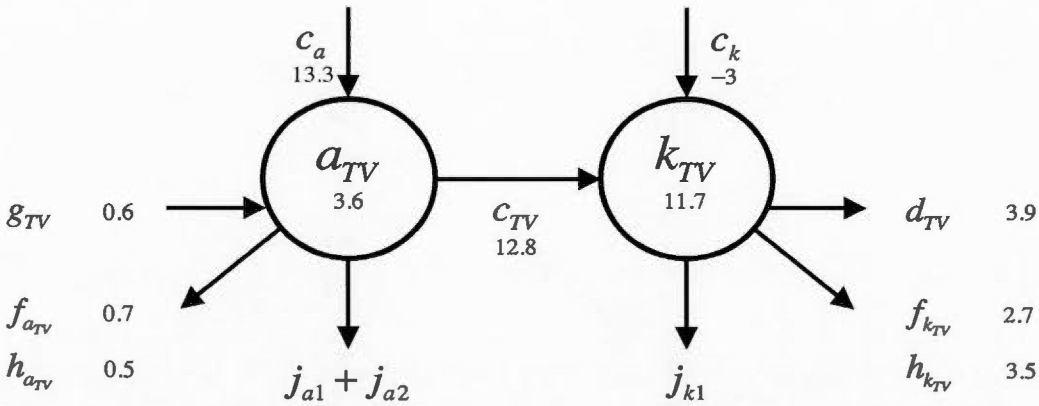
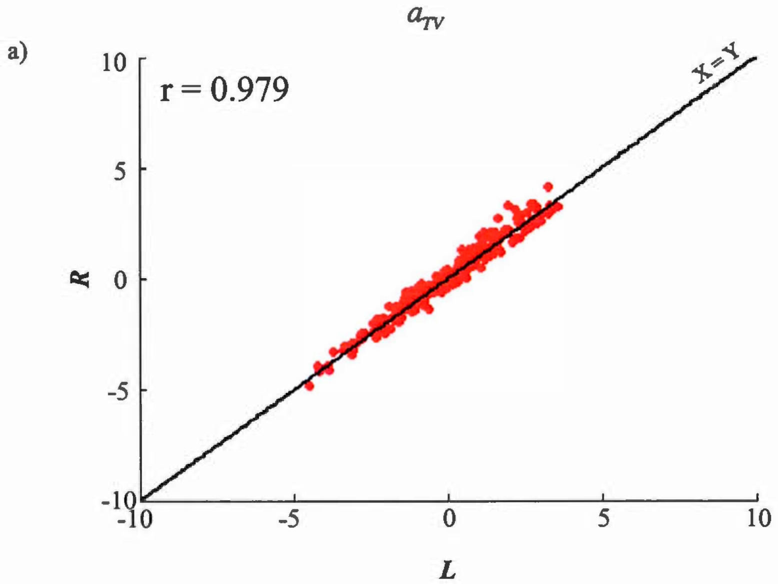


Figure 1.9 Vertically integrated time- and isobaric-mean storm energy cycle. Values are computed for the duration of the storm in $10^5\ \mathrm{J \cdot m^{-2}}$ for the energy reservoirs and in $\mathrm{W \cdot m^{-2}}$ for the energy fluxes.



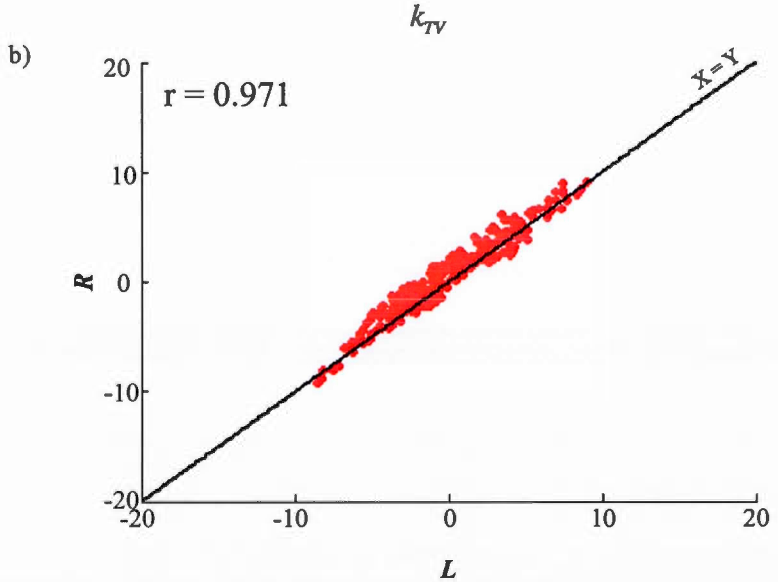


Figure 1.10 Dispersion diagrams of all time samples for the month of December 2004 to illustrate the comparison between L and R, both in $W \cdot m^{-2}$, for the a) time variability available enthalpy a_{TV} and b) time variability kinetic energy k_{TV}.

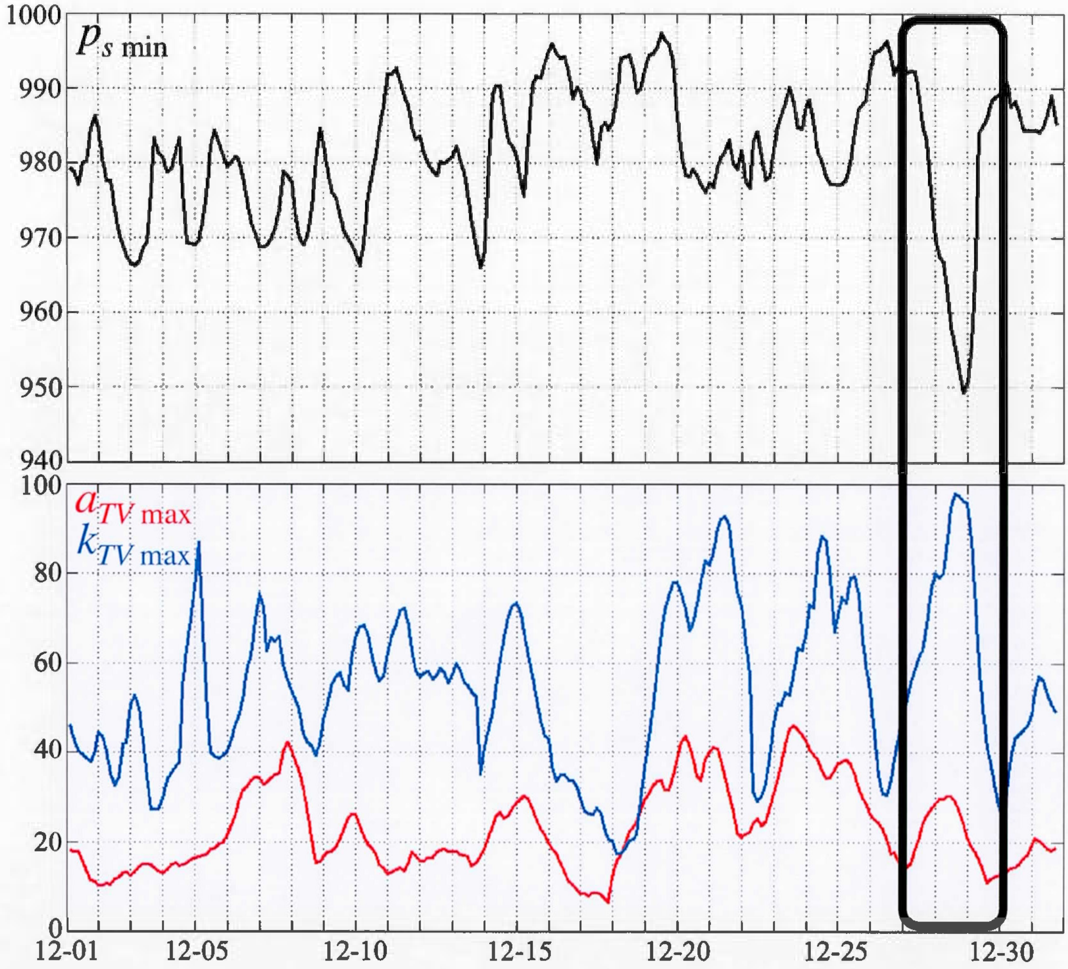


Figure 1.11 Time evolution of the minimum value of surface pressure in the domain $p_{s_{\min}}$ in hPa and of the maximum value of energy in the domain $a_{TV_{\max}} \times 10^5$ and $k_{TV_{\max}} \times 10^5$, both in $\mathrm{J \cdot m^{-2}}$. The period of occurrence of the storm studied is indicated by the black rectangle, corresponding to December 26 to December 29, 2004.

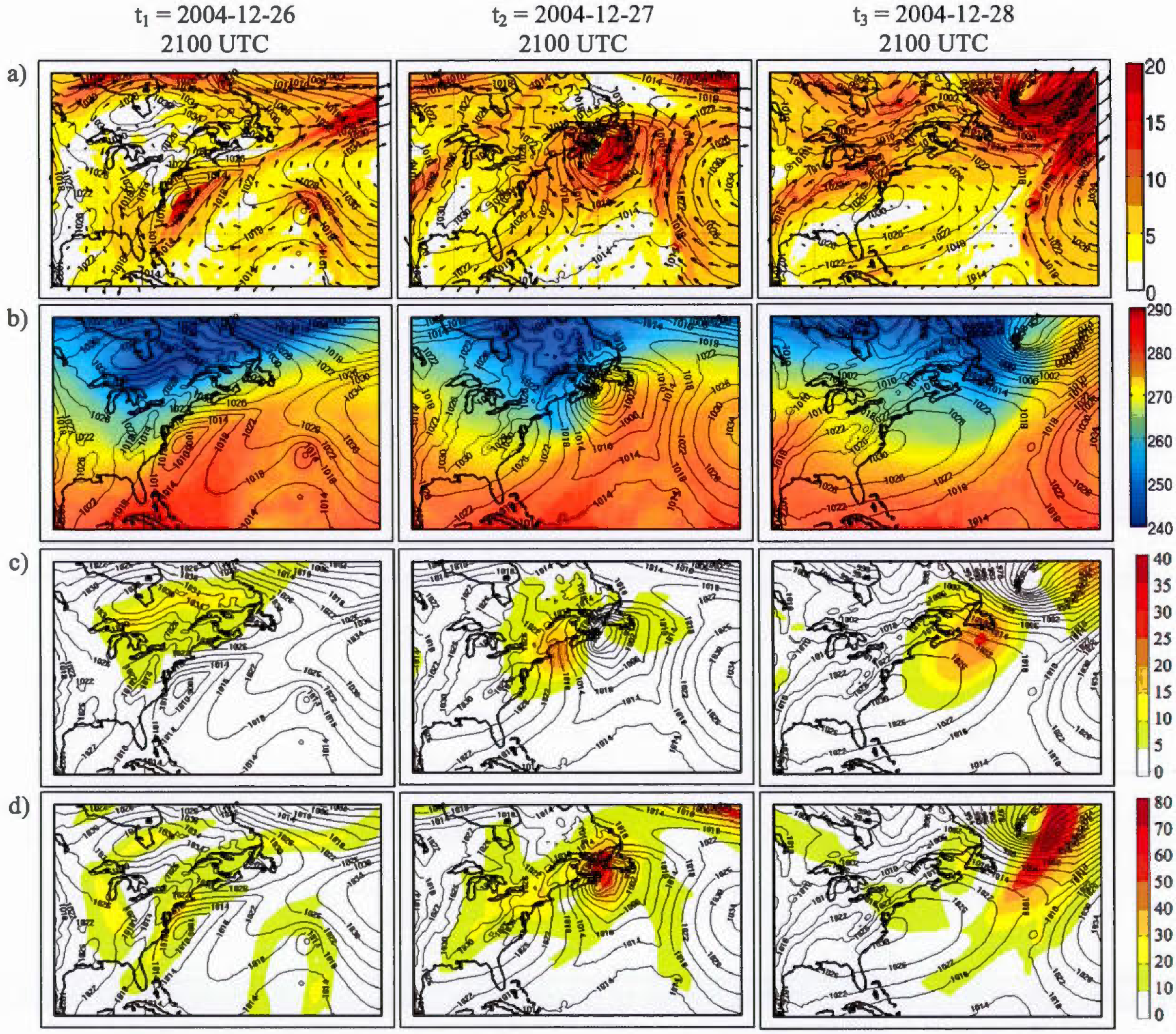


Figure 1.12 Maps of the a) horizontal wind at the 850 hPa pressure level $\vec{V}_{850}$ in $m \cdot s^{-1}$, b) air temperature at the 700 hPa pressure level T_{700} in K, c) instantaneous time variability $a_{TV} \times 10^5$ in $J \cdot m^{-2}$ and d) instantaneous kinetic energy $k_{TV} \times 10^5$ in $J \cdot m^{-2}$.

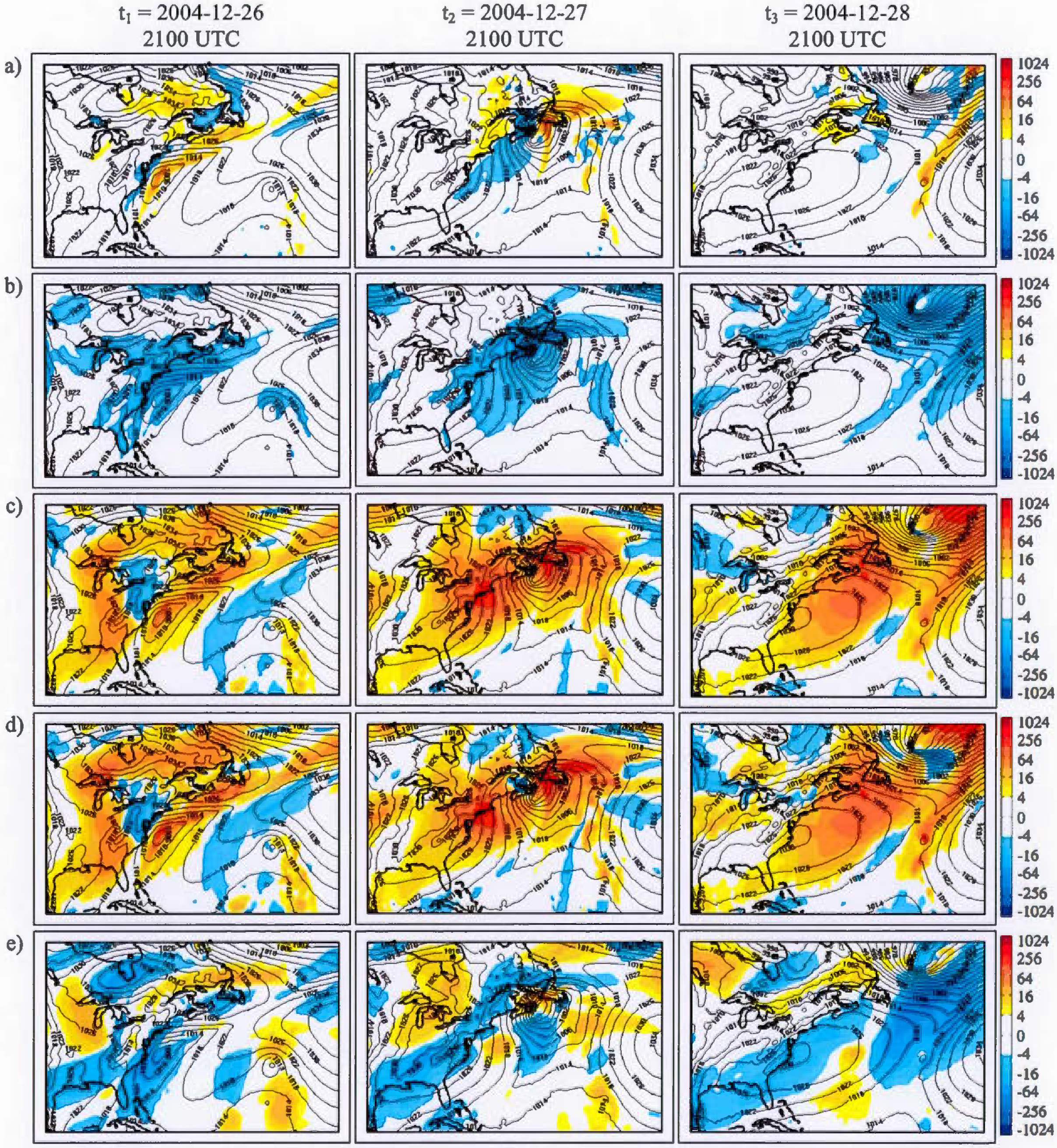


Figure 1.13 Maps of instantaneous energy fluxes of a) generation of time variability available enthalpy g_{TV}, b) dissipation of time variability kinetic energy $-d_{TV}$, c) conversion of time-mean into time variability available enthalpy c_a, d) conversion of time variability available enthalpy into time variability kinetic energy c_{TV} and e) conversion of time-mean into time variability kinetic energy c_k, all in $\mathrm{W \cdot m^{-2}}$.

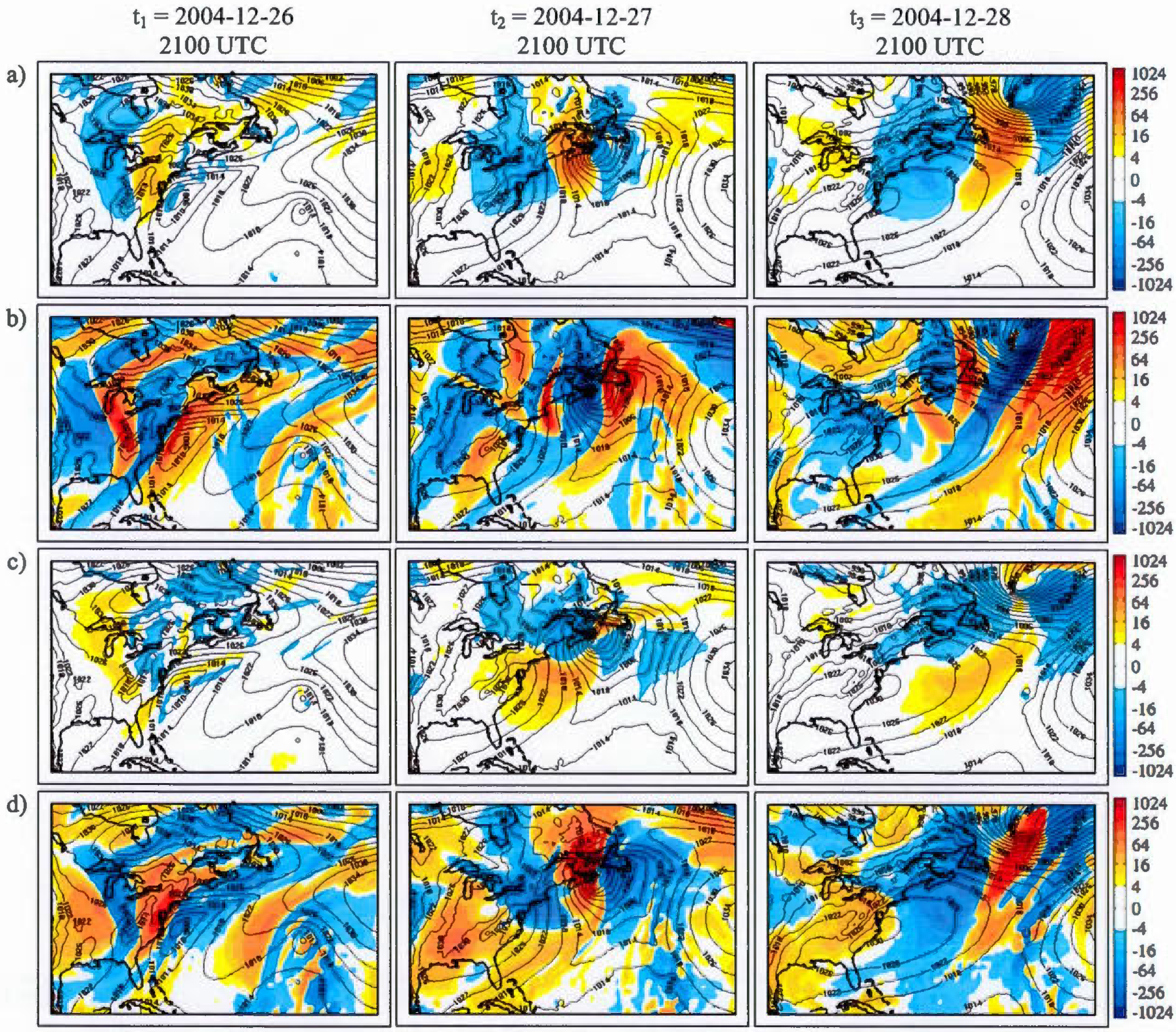


Figure 1.14 Maps of instantaneous energy fluxes of a) transport of time variability available enthalpy $-f_{a_{TV}}$, b) transport of time variability kinetic energy $-f_{k_{TV}}$, c) boundary flux of time variability available enthalpy $-h_{a_{TV}}$ and d) boundary flux of time variability kinetic energy $-h_{k_{TV}}$, all in $\mathrm{W \cdot m^{-2}}$.

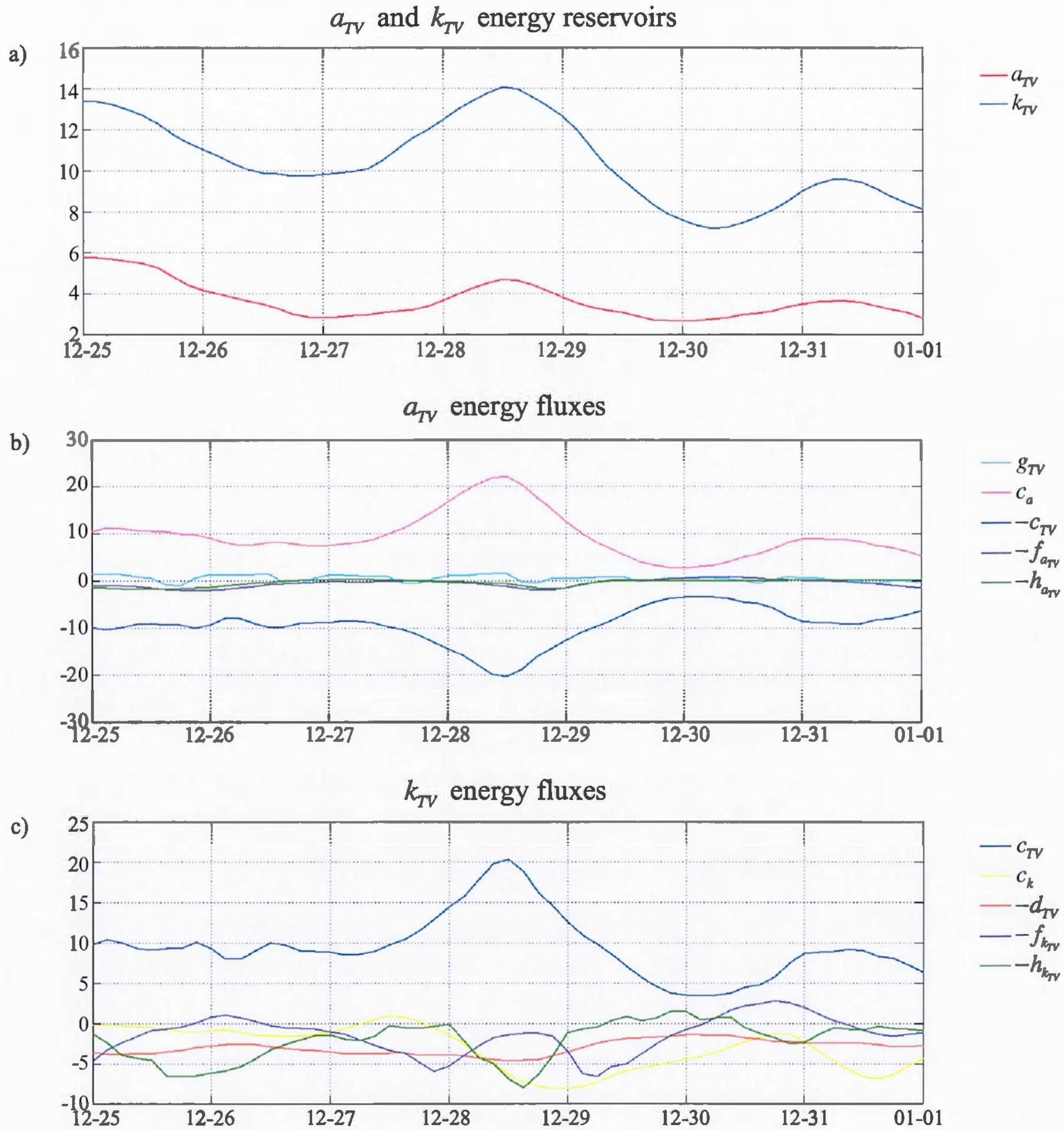


Figure 1.15 Time evolution for the duration of the storm of vertically integrated isobaric-mean values of a) available enthalpy $a_{TV} \times 10^5$ and kinetic energy $k_{TV} \times 10^5$, both in $\mathrm{J \cdot m^{-2}}$, b) energy fluxes acting on the available enthalpy reservoir a_{TV} in $\mathrm{W \cdot m^{-2}}$ and c) energy fluxes acting on the kinetic energy reservoir k_{TV} in $\mathrm{W \cdot m^{-2}}$.

RÉFÉRENCES

Boer GJ (1974) Zonal and eddy forms of the available potential energy equations in pressure coordinates. Tellus 27(5):433-442

Holton JR (2004) An Introduction to Dynamic Meteorology, Volume 1. Academic Press, 535

Kalnay E, Coauthors (1996) The NCEP–NCAR 40-Year Reanalysis Project. Bull Amer Meteor Soc 77:437–471

Lorenz EN (1955) Available potential energy and the maintenance of the general circulation. Tellus 7:157-167

Lorenz EN (1967) The nature and theory of the general circulation of the atmosphere. World Meteorological Organ 218 TP 115, 161 pp

Margules M (1905) Über die Energie der Stürme. In: Jahrbuch der Kaiserlich-Königliche Zentralanstalt für Meteorologie und Erdmagnetismus in Wien, NF, Bd. 42

Marquet P (1996) L'énergie utilisable de l'atmosphère. La Météorologie, 8e série 14:11-26

Marquet P (2003a) The available-enthalpy cycle. I: Introduction and basic equations. Q J R Meteorol Soc 129(593):2445–2466

Marquet P (2003b) The available-enthalpy cycle. II: Applications to idealized baroclinic waves. Q J R Meteorol Soc 129(593):2467–2494

Martynov A, Laprise R, Sushama L, Winger K, Šeparović L, Dugas B (2013) Reanalysis-driven climate simulation over CORDEX North America domain using the Canadian Regional Climate Model, version 5: Model performance evaluation. Clim Dyn 41(11-12):2973-3005

Newell RE, Kidson JW, Vincent DG, and Boer GJ (1972, 1974) The general circulation of the tropical atmosphere and interactions with extratropical latitudes. MIT Press, Vols 1 and 2

Nikiéma O, Laprise R (2011) Diagnostic budget study of the internal variability in ensemble simulations of the Canadian RCM. Clim Dyn 36(11):2313-2337

Nikiéma O, Laprise R (2013) An approximate energy cycle for inter-member variability in ensemble simulations of a regional climate model. Clim Dyn 41(3):831-852

Nikiéma O, Laprise R (2015) Energy cycle associated with inter-member variability in a large ensemble of simulations with the Canadian RCM (CRCM5). Clim Dyn http://link.springer.com/article/10.1007/s00382-015-2604-3

Oort AH (1964a) On estimates of the atmospheric energy cycle. Mon Wea Rev 92:483-493

Oort AH (1964b) On the energetics of the mean and eddy circulations in the lower stratosphere. Tellus 26(3):309-327

Pearce RP (1978) On the concept of available potential energy. Q J R Meteorol Soc 104:737-755

Peixoto JP, Oort AH (1992) Physics of climate. American Institute of Physics, College Park, 520 pp

Smith PJ (1969a) A computational study of the energetics of a limited region of the atmosphere. Tellus 21(2):195-201

Smith PJ (1969b) On the contribution of a limited region to the global energy budget. Tellus 21(2):202-207

Spar J (1950) Synoptic studies of the potential energy in cyclones. J Meteor 7:48-53

Tamura S (1905) Doctor Margules on the energy of storms. Mon Wea Rev 33:519-521

van Mieghem J (1973) Atmospheric Energetics. Oxford Univ Press, 306