On the Local Structure of $SL(2,\mathbb{C})$-Character Varieties at Reducible Characters

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Abstract

We describe a 1-cocycle condition that guarantees the smoothness of a reducible character in the $SL(2,\mathbb{C})$-character variety of a finitely generated group. This result is then applied to study fillings of one-cusped hyperbolic manifolds which yield Seifert fibred manifolds with Euclidean orbifolds.

1 Introduction

Throughout this paper Γ will denote a finitely generated group and M a compact, connected, irreducible, orientable 3-manifold whose boundary is an incompressible torus. We shall fix a base point in ∂M and therefore identify $\pi_1(\partial M)$ with a subgroup of $\pi_1(M)$.

The $SL(2,\mathbb{C})$-*representation variety* of Γ is a complex affine algebraic variety $R(\Gamma)$ whose points correspond to representations of Γ with values in $SL(2,\mathbb{C})$ [8]. Each $\rho \in R(\Gamma)$ determines a function

$$\chi_\rho : \Gamma \to \mathbb{C} \quad \chi_\rho(\gamma) = trace(\rho(\gamma)),$$

called its *character*, and the set of such characters $X(\Gamma) = \{ \chi_\rho \mid \rho \in R(\Gamma)\}$ admits the structure of a complex affine algebraic variety in such a way that the function

$$t : R(\Gamma) \to X(\Gamma), \quad \rho \mapsto \chi_\rho$$

is regular [8]. It turns out that t can be canonically identified with the algebro-geometric quotient of $R(\Gamma)$ by the natural action of $SL(2,\mathbb{C})$ [15, Theorem 3.3.5]. This means that $\mathbb{C}[X(\Gamma)]$ is isomorphic to the ring of invariants $\mathbb{C}[R(\Gamma)]^{SL(2,\mathbb{C})}$ and t corresponds to the inclusion $\mathbb{C}[R(\Gamma)]^{SL(2,\mathbb{C})} \subset \mathbb{C}[R(\Gamma)]$. The orbit of a representation ρ under this action will be denoted by $\mathcal{O}(\rho)$.

The set $R^{irr}(\Gamma)$ of irreducible representations in $R(\Gamma)$ is Zariski open, as is its image $X^{irr}(\Gamma) = t(R^{irr}(\Gamma))$ in $X(\Gamma)$ [8]. Their Zariski closures in $R(\Gamma)$ and $X(\Gamma)$ will be denoted by $\overline{R^{irr}(\Gamma)}$ and $\overline{X^{irr}(\Gamma)}$. Each representation $\rho \in R^{irr}(\Gamma)$ is *non-abelian*, that is its image in $SL(2,\mathbb{C})$ is a non-abelian group, but the converse does not hold.

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One of our main interests in this paper is the local behaviour of $X(\Gamma)$. André Weil realized ([19]) that the tangential structures of $R(\Gamma)$ and $X(\Gamma)$ are closely related to the cohomology of Γ. Thus for $\rho \in R(\Gamma)$, he showed that $T_\rho^{Zar} R(\Gamma)$, the Zariski tangent space of $R(\Gamma)$ at ρ, can be naturally identified with a subspace of $Z^1(\Gamma; sl_2(\mathbb{C})_{Ad\rho})$, the space of 1-cocycles determined by the composition $\Gamma \xrightarrow{\rho} SL(2,\mathbb{C}) \xrightarrow{Ad} Aut(sl_2(\mathbb{C}))$ (here Ad denotes the adjoint representation). We shall simplify notation by writing $Z^1(\Gamma; Ad\rho)$ for this set of 1-cocycles. If it turns out that $T_\rho^{Zar} R(\Gamma) = Z^1(\Gamma; Ad\rho)$, we call ρ *scheme reduced*.

A point x of a complex affine algebraic set V is called *simple* if $\dim_{\mathbb{C}} T_x^{Zar}(V) = \dim_x V$. It turns out that x is simple if and only if it is contained in a unique algebraic component of V and is a smooth point of that component (see eg. §II.2 of [17]).

The proof of the following theorem when ρ is irreducible is contained, at least implicitly, in that of the main result of [4]. The contribution we make here is in dealing with the more subtle case when ρ is reducible.

Theorem A *Suppose that $\rho \in R(\Gamma)$ is a non-abelian representation lying in $\overline{R^{irr}(\Gamma)}$ for which $Z^1(\Gamma; Ad\rho) \cong \mathbb{C}^4$. Then*
(1) *ρ is a simple point of $R(\Gamma)$ and the algebraic component of $R(\Gamma)$ which contains it is 4-dimensional.*
(2) *χ_ρ is a simple point of $\overline{X^{irr}(\Gamma)}$ and the algebraic component of $\overline{X^{irr}(\Gamma)}$ which contains it is a curve.*
(3) *there is an analytic 2-disk D, smoothly embedded in $R(\Gamma)$ and containing ρ, such that $t|D$ is an analytic isomorphism onto a neighbourhood of χ_ρ in $\overline{X^{irr}(\Gamma)}$.*
(4) *ρ is scheme reduced and there is a commutative diagram*

$$\begin{array}{ccccc} T_\rho^{Zar}\mathcal{O}(\rho) & \longrightarrow & T_\rho^{Zar}R(\Gamma) & \xrightarrow{dt} & T_{\chi_\rho}^{Zar}\overline{X^{irr}(\Gamma)} \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ B^1(\Gamma; Ad\rho) & \longrightarrow & Z^1(\Gamma; Ad\rho) & \longrightarrow & H^1(\Gamma; Ad\rho). \end{array}$$

The proof of this theorem depends on a detailed analysis (§2) of the set of reducible representations in $R(\Gamma)$ with a given character.

We are interested in applying Theorem A to the case $\Gamma = \pi_1(M)$ where M is a 3-manifold of the type described in the opening paragraph of this paper. This presupposes finding topologically interesting conditions on a representation $\rho : \pi_1(M) \to SL(2,\mathbb{C})$ which guarantee that $Z^1(\pi_1(M); Ad\rho) \cong \mathbb{C}^4$. One such condition, found in [13], states that if $\mathrm{rank}_{\mathbb{Z}} H_1(M) = 1$ and ρ corresponds to a simple root of the Alexander polynomial of M, then we often have $\rho \in \overline{R^{irr}(\pi_1(M))}$ and χ_ρ smooth on $\overline{X^{irr}(M)}$ (see also [18]). In order to describe another situation in which the group of cocycles is 4-dimensional, we develop a few notions.

Given a topological space W, we shall write $R(W)$ for $R(\pi_1(W))$ and $X(W)$ for $X(\pi_1(W))$. Research has shown that many topological properties of a 3-manifold M, as above, are encoded in the character variety $X(M)$, so it is of interest to study their basic properties.

A class in $\alpha \in H_1(\partial M)$ determines an element $\gamma(\alpha) \in \pi_1(M)$ through the Hurewicz isomorphism $H_1(\partial M) \cong \pi_1(\partial M) \subset \pi_1(M)$. In what follows, we shall abuse notation by writing α for $\gamma(\alpha)$. For instance the function $f_{\gamma(\alpha)}$ will be denoted by f_α.

A *slope* r on ∂M is a ∂M-isotopy class of essential, simple closed curves on ∂M. Any slope r on ∂M determines (and is determined by) a pair $\pm\alpha(r)$ of primitive elements of $H_1(\partial M)$ - the images in $H_1(\partial M)$ of the two generators of $H_1(C) \cong \mathbb{Z}$ where $C \subset \partial M$ is a representative curve for r.

As usual, $M(r)$ will denote the 3-manifold obtained by Dehn filling M along r. Note that if $\rho(\alpha(r)) = \pm I$ for some slope r, then $Ad\rho$ factors through a representation $\pi_1(M(r)) \to PSL(2,\mathbb{C})$. In particular, it makes sense to consider the cohomology group $H^1(M(r); Ad\rho)$.

Corollary B *Suppose that $\rho \in R(M)$ is a non-abelian representation for which $\rho \in \overline{R^{irr}(M)}$. If $\rho(\alpha(r)) \in \{\pm I\}$ but $\rho(\pi_1(\partial M))$ is not contained in $\{\pm I\}$, and $H^1(M(r); Ad\rho) = 0$, then $Z^1(M; Ad\rho) \cong \mathbb{C}^4$. Thus the conclusions of Theorem A hold for ρ.*

We complete the paper with some applications of the results above, and the main results of [1], to the case where r is a slope on ∂M such that $M(r)$ is a Seifert fibred space whose base orbifold $\mathcal{B}$ is Euclidean. The closed Euclidean orbifolds are:

- the torus: T.
- the Klein bottle: K.
- the projective plane with two cone points, each of order 2: $P^2(2,2)$.
- the 2-sphere with four cone points each of order 2: $S^2(2,2,2,2)$.
- the 2-sphere with 3 cone points whose orders form a Euclidean triple: $S^2(3,3,3)$, $S^2(2,4,4)$ and $S^2(2,3,6)$.

Theorem C *Let M be a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus. Suppose that r_1, r_2 are slopes on ∂M such that $M(r_1)$ is reducible while $M(r_2)$ is a Seifert fibred space whose base orbifold $\mathcal{B}$ is Euclidean. Suppose further that $M(r_1) \ncong S^1 \times S^2$. Then $\mathcal{B} \neq T$ and*

$$\Delta(r_1, r_2) \leq \begin{cases} 2 & \text{if } \mathcal{B} = K, P^2(2,2), S^2(2,2,2,2), \text{ or } S^2(2,4,4) \\ 3 & \text{if } \mathcal{B} = S^2(3,3,3) \text{ or } S^2(2,3,6). \end{cases}$$

Remark 1.1

1. According to [Oh] and [Wu], if M is hyperbolic and $M(r_2)$ contains an essential torus

then $\Delta(r_1, r_2) \leq 3$. This is certainly the case when $\mathcal{B} = K, P^2(2,2)$, or $S^2(2,2,2,2)$, as in these cases, $M(r_2)$ is the union of two twisted I-bundles over the Klein bottle. Note then that Theorem C improves this bound.

2. It is interesting to compare the estimates in Theorem 1.1 to those obtained when we replace the condition that M be hyperbolic with one where M is Seifert and atoroidal. In this case it is a simple matter to see that

$$\Delta(r_1, r_2) \leq \begin{cases} 1 & \text{if } \mathcal{B} = T \text{ or } K \\ 2 & \text{if } \mathcal{B} = P^2(2,2) \text{ or } S^2(2,2,2,2) \\ 3 & \text{if } \mathcal{B} = S^2(3,3,3) \\ 4 & \text{if } \mathcal{B} = S^2(2,4,4) \\ 6 & \text{if } \mathcal{B} = S^2(2,3,6). \end{cases}$$

Furthermore, these distances are sharp.

Next we consider surgery on hyperbolic knots $K \subset S^3$.

Theorem D *Suppose that K is a hyperbolic knot in S^3 with exterior M. There is at most one slope r on ∂M such that $M(r)$ is is a Seifert fibred space whose base orbifold is Euclidean, and if there is one, it is integral, though not longitudinal.*

In §2 we analyze the structure of the set of representations with a given reducible character. This leads to the proof of Theorem A in the case that ρ is a reducible representation given in §3. A key ingredient of this proof is found in Proposition 2.9, a basic result concerning the way the set of reducible representations sits in $\overline{R^{irr}(M)}$. The proofs of Theorems C and D necessitate the development of $PSL(2,\mathbb{C})$ versions of our results, and this is the subject of §4. Sections 5, 6 and 7 contain background results leading to the proof of Theorem C in §8. Finally Theorem D is dealt with in the last section.

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2 The set of reducible representations with a given character

Fix a reducible character $x \in X(\Gamma)$. Our goal in this section is to describe the structure of $R_x = t^{-1}(x)$. Much of what is contained here is already known, dating back to work of Burde [6], de Rham [9], and more recently Porti [16]. We do prove a new result, Proposition 2.9, which is of fundamental importance to our analysis of Theorem A in the case where the representation is reducible. We adopt the following notational conventions.

- $R^{red}(\Gamma)$, respectively $X^{red}(\Gamma)$, is the set of reducible representations, respectively characters, in $R(\Gamma)$, respectively $X(\Gamma)$.

- $U, U_P, D \subset SL_2(\mathbb{C})$ are respectively the upper-triangular matrices, the parabolic elements of U, and the diagonal matrices.

- $\mathcal{O}_U(\rho) = \{A\rho A^{-1} \mid A \in U\}$ is the U-*orbit* of a representation $\rho : \Gamma \to U$.

- The *first betti number* of Γ, is the quantity $b_1(\Gamma) = \text{rank}_{\mathbb{Z}}(H_1(\Gamma))$.

Fix a character $x \in X^{red}(\Gamma)$ and set

$$\begin{aligned} D_x &= \{\rho \in R_x \mid \rho(\Gamma) \subset D\}, \\ U_x &= \{\rho \in R_x \mid \rho(\Gamma) \subset U\}. \end{aligned}$$

Evidently $D_x \subset U_x \subset R_x = t^{-1}(x)$ and each representation in R_x is conjugate to an element of U_x, though not necessarily to one in D_x. Nevertheless each $\rho \in U_x$ gives rise to an element of D_x by postcomposition with the projection $U \to D$.

The set D_x is rather small. Indeed it is elementary to verify that the set of representations $a : \Gamma \to \mathbb{C}^*$ which satisfy $a(\gamma) + a(\gamma)^{-1} = x(\gamma)$ is of the form $\{a, a^{-1}\}$. Further, $a = a^{-1}$ if and only if $x(\Gamma) \subset \{\pm 2\}$. Call x *trivial* if $x(\Gamma) \subset \{\pm 2\}$, and *non-trivial* otherwise.

Fix a homomorphism $a : \Gamma \to \mathbb{C}^*$ as in the previous paragraph and set

$$\rho_a(\gamma) = \text{diag}(a(\gamma), a(\gamma)^{-1}) \qquad \rho_{a^{-1}}(\gamma) = \text{diag}(a(\gamma)^{-1}, a(\gamma)).$$

Each $\rho \in U_x$ is of the form

$$\rho = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \quad \text{or} \quad \rho = \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix}.$$

The following result is elementary.

Proposition 2.1 *Suppose that $x \in X^{red}(\Gamma)$.*
(1) *If x is trivial, then $D_x = \{\rho_a\}$. Furthermore, any representation in R_x conjugates into U_P, and so has abelian image.*
(2) *If x is nontrivial, then $D_x = \{\rho_a, \rho_{a^{-1}}\}$. These two representations are conjugate over $SL(2, \mathbb{C})$, but not over U. Finally, any representation ρ in R_x, which is not in the orbit of ρ_a or $\rho_{a^{-1}}$, has infinite, non-abelian image.* ♢

Proposition 2.2 *Suppose that $x \in X^{red}$ is trivial. Then U_x may be identified with $Hom(\Gamma, \mathbb{C})$, a complex vector space of dimension $b_1(\Gamma)$.*

Proof. From the triviality of x we see that there is a homomorphism $a : \Gamma \to \{\pm 1\}$ such that each $\rho \in U_x$ is of the form $\rho = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. The correspondence $U_x \to \mathrm{Hom}(\Gamma, \mathbb{C})$, $\rho \mapsto ab$ is bijective. $\diamondsuit$

Proposition 2.3 *Suppose that $x \in X^{red}(\Gamma)$ is nontrivial and that ρ_a and $\rho_{a^{-1}}$ are the two diagonal representations with character x. Then U_x has two topological components, U_x^a which contains ρ_a, and $U_x^{a^{-1}}$ which contains $\rho_{a^{-1}}$. Furthermore, both U_x^a and $U_x^{a^{-1}}$ are complex affine spaces of finite dimension.*

Proof. For $z \in \mathbb{C}^*$, let $\mathrm{diag}(z, z^{-1})$ denote the matrix with diagonal entries z and z^{-1}.

There is a continous map $\delta : U_x \to \{\rho_a, \rho_{a^{-1}}\}$ which sends $\rho \in U_x$ to the diagonal representation obtained by projecting to D. This is surjective so that U_x has at least two components. Since $\delta(\rho) = \lim_{n\to\infty} \mathrm{diag}(\frac{1}{n}, n)\ \rho\ \mathrm{diag}(\frac{1}{n}, n)^{-1}$, we see that $\delta(\rho) \in \overline{\mathcal{O}_U(\rho)}$, which is connected. Thus U_x has exactly two topological components, U_x^a which contains ρ_a, and $U_x^{a^{-1}}$ which contains $\rho_{a^{-1}}$.

To see that U_x^a is affine, note that any $\rho \in U_x^a$ is of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, and so is determined by the function $b : \Gamma \to \mathbb{C}$. This sets up a bijection

$$U_x^a \to \{b : \Gamma \to \mathbb{C} \mid b(\gamma_1\gamma_2) = a(\gamma_1)b(\gamma_2) + a_(\gamma_2)^{-1}b(\gamma_1) \text{ for each } \gamma_1, \gamma_2 \in \Gamma\}.$$

Since Γ is finitely generated, the range of this bijection is a complex affine space of finite dimension.

A similar analysis holds for $U_x^{a^{-1}}$. $\diamondsuit$

The previous two lemmas can be made slightly stronger. If we fix generators $\gamma_1, \gamma_2, ..., \gamma_n$ of Γ and consider $R(\Gamma) \subset \mathbb{C}^{4n}$ via the embedding $\rho \mapsto (\rho(\gamma_1), ..., \rho(\gamma_n))$, then U_x^a (and $U_x^{a^{-1}}$) correspond to affine subsets of $\mathbb{C}^{4n}$.

Clearly

$$R_x = R_x^a \cup R_x^{a^{-1}}$$

where

$$R_x^a = \{A\rho A^{-1} \mid A \in SL(2,\mathbb{C}), \rho \in U_x^a\}, \quad R_x^{a^{-1}} = \{A\rho A^{-1} \mid A \in SL(2,\mathbb{C}), \rho \in U_x^{a^{-1}}\}.$$

Proposition 2.4 *If x is a reducible character of Γ, then R_x^a and $R_x^{a^{-1}}$ are irreducible algebraic sets. Moreover,*
(1) *if x is trivial, then R_x is an irreducible algebraic set and*

$$dim\ R_x = \begin{cases} 0 & \text{if } b_1(\Gamma) = 0 \\ 1 + b_1(\Gamma) & \text{otherwise.} \end{cases}$$

(2) *if x is nontrivial, then*

(a) $R^a_x \cap R^{a^{-1}}_x = \mathcal{O}(\rho_a)$.

(b) *If R_x is irreducible then either $R^{a^{-1}}_x = \mathcal{O}(\rho_a)$ and $R_x = R^a_x$, or $R^a_x = \mathcal{O}(\rho_a)$ and $R_x = R^{a^{-1}}_x$.*

(c) *If R_x is reducible, then R^a_x and $R^{a^{-1}}_x$ are its two algebraic components.*

(d) *We have*

$$dim\ R^a_x = \begin{cases} 2 & \text{if } U^a_x \text{ is a point} \\ 1 + dim(U^a_x) & \text{otherwise.} \end{cases}$$

A similar dimension count holds for $R^{a^{-1}}_x$.

Proof. The affine sets U^a_x and $U^{a^{-1}}_x$ are smooth and connected, so they are irreducible. It follows that their $SL(2,\mathbb{C})$-orbits, R^a_x and $R^{a^{-1}}_x$ are irreducible as well. Since $R_x = R^a_x \cup R^{a^{-1}}_x$, it is irreducible when x is trivial.

To complete the proof we need to make the following observation: for any $\rho \in U_x \setminus \mathcal{O}(\rho_a)$ and $A \in SL(2,\mathbb{C})$ which satisfy $A\rho A^{-1} \in U_x$, we must have $A \in U$. When x is trivial the statement is the result of direct computation. When it is non-trivial the same is true, though we need to use the fact that in this case, any $\rho \in U_x \setminus \mathcal{O}(\rho_a)$ has a non-abelian image.

We show next that if x is non-trivial, then $\mathcal{O}(\rho_a) = R^a_x \cap R^{a^{-1}}_x$. Since ρ_a and $\rho_{a^{-1}}$ are conjugate over $SL(2,\mathbb{C})$, $R^a_x \cap R^{a^{-1}}_x \supset \mathcal{O}(\rho_a)$. On the other hand if $\rho \in R^a_x \cap R^{a^{-1}}_x \setminus \mathcal{O}(\rho_a)$, there is some $\rho_1 \in U^a_x \setminus \mathcal{O}(\rho_a)$ which is conjugate to some $\rho_2 \in U^{a^{-1}}_x \setminus \mathcal{O}(\rho_a)$. By the observation in the second paragraph of this proof, ρ_1 and ρ_2 are conjugate by an element of U. But this impossible as it implies that $a = a^{-1}$, i.e. x is trivial. Thus $R^a_x \cap R^{a^{-1}}_x = \mathcal{O}(\rho_a)$ as claimed. Parts (2)(ii), (iii) of the proposition are now easily seen to hold.

Finally we compute dimensions. In the case where x is a trivial character, ρ_a is a central representation so $\mathcal{O}(\rho_a) = \{\rho_a\}$. According to Proposition 2.2, U_x has dimension $b_1(\Gamma)$. Hence if $b_1(\Gamma) = 0$, $U_x = \{\rho_a\}$, and so $R_x = \{\rho_a\}$. On the other hand if $b_1(\Gamma) > 0$, then $U_x \setminus \mathcal{O}(\rho_a)$ is non-empty. By the observation above we have $\{A \in SL(2,\mathbb{C}) \mid AU_xA^{-1} \subset U_x\} = U$. Since U has codimension 1 in $SL(2,\mathbb{C})$ it follows that U_x has codimension 1 in R_x, i.e. dim $R_x = 1 + b_1(\Gamma)$. This completes the proof of part (1) of the lemma.

Assume next that x is a non-trivial character. If U^a_x is a point, then $R^a_x = \mathcal{O}(\rho_a)$ and so the dimension of R^a_x is 2 (remark that since x is non-trivial the centralizer of ρ_a is 1-dimensional). Otherwise there is some $\rho \in U^a_x \setminus \mathcal{O}(\rho_a)$ which, by Proposition 2.1 (2), is non-abelian. Hence it's centralizer is $\{\pm I\}$. It follows that U^a_x has codimension 1 in R^a_x, which was to be proved. $\diamondsuit$

The following useful concept was introduced by Porti in [16].

Definition 2.5 Let x be a reducible character of Γ and $\rho \in R_x^a(\Gamma) \setminus \mathcal{O}(\rho_a)$. The *defect* of ρ is the quantity

$$\mathrm{def}(\rho) = \dim R_x^a - 3.$$

Proposition 2.6 *Suppose that $\rho \in R_x \setminus \mathcal{O}(\rho_a)$. Then*
(1) $\overline{\mathcal{O}(\rho)} = \mathcal{O}(\rho) \cup \mathcal{O}(\rho_a)$.
(2) *If def$(\rho) = 0$, then $R_x^a = \mathcal{O}(\rho) \cup \mathcal{O}(\rho_a)$.*

Proof. (1) Without loss of generality we may suppose that $\rho \in U_x$, say

$$\rho = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Now $\rho_a = \lim_n \mathrm{diag}(\frac{1}{n}, n)\rho \mathrm{diag}(\frac{1}{n}, n)^{-1}$ so $\mathcal{O}(\rho) \cup \mathcal{O}(\rho_a) \subset \overline{\mathcal{O}(\rho)}$. Assume then that $\rho_0 \in \overline{\mathcal{O}(\rho)}$. There is a sequence $\{A_n\}$ in $SL(2, \mathbb{C})$ such that $\lim_n A_n \rho A_n^{-1}$ converges to a representation ρ_0. It is clear that $\chi_{\rho_0} = x$ so that $\rho_0 \in R_x$.

If there is a convergent subsequence of $\{A_n\}$, say $\lim_j A_{n_j} = A_0$, then $\rho_0 = \lim_j A_{n_j} \rho A_{n_j}^{-1} = A_0 \rho A_0^{-1} \in \mathcal{O}(\rho)$. Without loss of generality we shall suppose that $\{A_n\}$ does not lie in any compact portion of $SL(2, \mathbb{C})$. It follows that if we write

$$A_n = \begin{pmatrix} r_n & s_n \\ t_n & u_n \end{pmatrix},$$

then one of the sequences $\{|r_n|\}, \{|s_n|\}, \{|t_n|\}, \{|u_n|\}$ has a subsequence which tends to ∞.

Case 1. x is trivial.

Then $a(\gamma) \in \{\pm 1\}$ for all $\gamma \in \Gamma$ and $b : \Gamma \to \mathbb{C}$ is not identically zero. Then

$$A_n \rho A_n^{-1} = \begin{pmatrix} a(1 - r_n t_n b) & r_n^2 b \\ -t_n^2 b & a(1 + r_n t_n b) \end{pmatrix}.$$

Hence as $b \not\equiv 0$, both $\{|r_n|\}$ and $\{|t_n|\}$ converge. If they converge to zero then $\rho_0 = \rho_a \in \mathcal{O}(\rho_a)$, and we are done. Otherwise one of them, $|r_n|$ say, converges to an element $r_0 \in \mathbb{C}^*$. Since we can alter the sign of A_n without effecting the hypotheses, we may assume that $\lim_n r_n = r_0$. Consideration of the $(1,1)$-entry of $A_n \rho A_n^{-1}$ implies that the sequence $\{t_n\}$ converges. Now for $n \gg 1$ we have $r_n \neq 0$. For such n define $B_n = \begin{pmatrix} r_n & 1 \\ t_n & \frac{1+t_n}{r_n} \end{pmatrix} \in SL(2, \mathbb{C})$ and observe that $\{B_n\}$ converges while $A_n \rho A_n^{-1} = B_n \rho B_n^{-1}$. Thus $\rho_0 \in \mathcal{O}(\rho)$.

Case 2. x is non-trivial.

Since $\rho \in U_x$ is non-abelian, it may be conjugated by an element of U so that its image contains a non-central diagonal element, say $\rho(\gamma_0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ where $\lambda \neq \pm 1$. As

$$A_n \rho(\gamma_0) A_n^{-1} = \begin{pmatrix} \lambda + (\lambda - \lambda^{-1}) s_n t_n & -(\lambda - \lambda^{-1}) r_n s_n \\ (\lambda - \lambda^{-1}) t_n u_n & \lambda^{-1} - (\lambda - \lambda^{-1}) s_n t_n \end{pmatrix}$$

and $\lambda \neq \lambda^{-1}$, each of the sequences $\{s_n t_n\}, \{r_n s_n\}, \{t_n u_n\}, \{r_n u_n = 1 + s_n t_n\}$ converges.

If $\rho(\Gamma)$ contains no parabolics then it is isomorphic to its projection into D. Hence ρ is abelian, contrary to the fact that $\rho \in U_x \setminus \mathcal{O}(\rho_a)$. Thus $\rho(\Gamma)$ contains parabolics. Let $\gamma \in \rho^{-1}(U_P)$. The argument from Case 1 applied to $A_n \rho(\gamma) A_n^{-1}$ shows that we may suppose $\lim_n r_n = r_0$ and $\lim_n |t_n| = t_0$. If one of r_0, t_0 is non-zero, it also implies that we may suppose $\lim_n t_n = t_0$, and therefore since $\{s_n t_n\}, \{r_n s_n\}, \{t_n u_n\}$ and $\{r_n u_n\}$ converge, both $\{s_n\}$ and $\{u_n\}$ are convergent, i.e. $\lim_n A_n$ exists, contrary to our assumptions. Thus $\lim_n r_n = \lim_n t_n = 0$. Another appeal to the argument of Case 1 now shows that $\rho_0(\rho^{-1}(U_P)) \subset \{\pm I\}$. But then the surjective homomorphism

$$\rho(\Gamma) \to \rho_0(\Gamma) : \rho(\gamma) \mapsto \rho_0(\gamma)$$

factors through the abelian group $\Gamma/\rho^{-1}(U_P)$, and therefore ρ_0 is an abelian representation whose character is x. Hence $\rho_0 \in \mathcal{O}(\rho_a)$, which completes the proof of part (1) of the proposition.

(2) It follows from the definition of defect that R_x^a has dimension 3. Since it is irreducible and contains the 3-dimensional algebraic set $\overline{\mathcal{O}(\rho)} = \mathcal{O}(\rho) \cup \mathcal{O}(\rho_a)$, part(2) of the proposition follows. $\diamondsuit$

We shall an curve in $X(\Gamma)$ *non-trivial* if each of its algebraic components contatins the character of an irreducible representation.

Lemma 2.7 *Suppose that Γ is a finitely generated group and that $X_0 \subset X(\Gamma)$ is a non-trivial curve. The decompostion of $t^{-1}(X_0)$ into algebraic components is of the form*

$$t^{-1}(X_0) = R_0 \cup R_1 \cup ... \cup R_k$$

where
(a) each R_j is invariant under conjugation.
(b) dim $(R_0) = 4$ and $t(R_0) = X_0$.
(c) for each $j \in \{1, 2, ..., k\}$ there is a reducible character $x_j \in X_0$ such that $t(R_j) = x_j$.

Proof. Parts (i) and (ii) are proven in much the same way as the $PSL(2, \mathbb{C})$ case found in [3, Lemma 4.1], as is the fact that for $j \geq 1$, there is some $x_j \in X_0$ such that $t(R_j) = \{x_j\}$.

To complete the proof, suppose that some x_j is irreducible. If $\rho \in t^{-1}(x_j) \cap R_0$, then $R_j = t^{-1}(x_j) = \mathcal{O}(\rho)$, the latter equality being a consequence of the irreducibility of ρ. By parts (i) and (ii), $R_j = \mathcal{O}(\rho) \subset R_0$, contrary to the fact that R_j is a component of $t^{-1}(X_0)$. ♢

Proposition 2.8 *Suppose that Γ is a finitely generated group and that $X_0 \subset X(\Gamma)$ is a non-trivial curve. Let R_0 be the 4-dimensional subvariety of $R(M)$ with $t(R_0) = X_0$ described in Lemma 2.7.*

(1) *If $x \in X_0 \cap X^{red}(\Gamma)$ is trivial, then $b_1(\Gamma) \geq 2$ and each representation in $R_0 \cap R_x$ has abelian image.*

(2) *If x is nontrivial, then*

(*a*) *there is a representation $\rho \in R_0 \cap R_x$ whose image is nonabelian.*

(*b*) *$R_x \cap R_0 = \mathcal{O}(\rho_a) \cup \mathcal{O}(\rho_1) \cup \ldots \cup \mathcal{O}(\rho_n)$ where $\rho_1, \rho_2, \ldots, \rho_n$ are non-abelian.*

(*c*) *If R_0 is a component of $R(\Gamma)$ and the only component to contain some $\rho \in R_x \cap R_0 \setminus \mathcal{O}(\rho_a)$, then def$(\rho) = 0$.*

Proof. Since R_0 is 4-dimensional and contains irreducible representations, $R_x \cap R_0$ has dimension 3. Hence if x is trivial, Proposition 2.4 (1) implies that $b_1(\Gamma) \geq 2$ while Proposition 2.1 implies that each representation in $R_0 \cap R_x$ has abelian image.

Suppose then that x is nontrivial. It is shown in [7, §1.5] that there is a representation $\rho \in R_0 \cap R_x$ whose image is nondiagonalisable and since x is non-trivial, ρ must be nonabelian. By Proposition 2.6, $\mathcal{O}(\rho_a) \subset \overline{\mathcal{O}(\rho)} \subset R_0$. As $\mathcal{O}(\rho)$ is 3-dimensional, it follows that there is a finite collection $\rho_1, \ldots \rho_n \in R_x \cap R_0$ of non-abelian representations such that $R_x \cap R_0 = \mathcal{O}(\rho_a) \cup \mathcal{O}(\rho_1) \cup \ldots \cup \mathcal{O}(\rho_n)$. Finally assume R_0 is a component of $R(\Gamma)$ and the only component to contain some $\rho \in R_x \cap R_0 \setminus \mathcal{O}(\rho_a)$. Assume, without loss of generality, that $\rho \in R_x^a$. If def$(\rho) > 0$, then $\dim(R_x^a) \geq 4$ and since R_x^a consists entirely of reducible representations, $R_x^a \not\subset R_0$. But this contradicts the fact that R_0 is the unique component of $R(M)$ which contains ρ. Thus def $(\rho) = 0$. This completes the proof. ♢

If x is non-trivial, then it can be shown that U_x^+ and U_x^- have the same dimension when, for instance, $a(\Gamma) \subset S^1$ or Γ is the group of a knot in the 3-sphere [6], [9]. In general, though, there does not seem to be any reason for this to be the case. Interestingly enough, the next result shows that if x lies in $\overline{X^{irr}}$, one of U_x^+ and U_x^- is positive dimensional if and only if the other is.

Proposition 2.9 *Suppose that $\{\rho_n\}_{n\geq 1}$ is a sequence of irreducible representations in $R(\Gamma)$ which converge to a non-abelian reducible representation $\rho \in R_x^a \setminus \mathcal{O}(\rho_a)$. If χ_ρ is a nontrivial character, then there is a subsequence $\{\rho_{n_k}\}_{k\geq 1}$ of $\{\rho_n\}_{n\geq 1}$ and another sequence $\{\rho'_{n_k}\}_{k\geq 1}$ such that*

(a) ρ'_{n_k} *is conjugate to* ρ_{n_k} *for each* $k \geq 1$.

(b) $\{\rho'_{n_k}\}_{k\geq 1}$ *converges to a representation* $\rho' \in R_x^{a^{-1}} \setminus \mathcal{O}(\rho_a)$.

Proof. Without loss of generality we may assume that $\rho \in U_x^a \setminus \mathcal{O}(\rho_a)$, say

$$\rho = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Let $\{\gamma_1, \gamma_2, ..., \gamma_m\}$ be a generating set for Γ. As χ_ρ is nontrivial, we may assume that $\chi_\rho(\gamma_1) \neq \pm 2$. Our first goal is to show that we may assume that $\rho_n(\gamma_1)$ is diagonal for each $n \geq 1$. This takes up much of the proof.

Our hypotheses imply that there is an $a_1 \in \mathbb{C} \setminus \{\pm 1\}$ such that $\rho(\gamma_1) = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1^{-1} \end{pmatrix}$. Setting $A = \begin{pmatrix} 1 & \frac{b_1}{(a_1 - a_1^{-1})} \\ 0 & 1 \end{pmatrix}$ and replacing each ρ_n by $A\rho_n A^{-1}$ shows that we may assume

$$\rho(\gamma_1) = \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}.$$

Then if we write $\rho_n(\gamma_1) = \begin{pmatrix} a_{n1} & b_{n1} \\ c_{n1} & d_{n1} \end{pmatrix}$ we have

$$\lim_{n\to\infty} a_{n1} = a_1 \neq \pm 1 \qquad \lim_{n\to\infty} d_{n1} = a_1^{-1} \qquad \lim_{n\to\infty} c_{n1} = \lim_{n\to\infty} b_{n1} = 0.$$

Since $a_1 \neq \pm 1$, the first limit shows that for $n >> 0$, $a_{n1} \neq \pm 1$. Thus by passing to a subsequence, we may assume

$$a_{n1} \neq \pm 1 \text{ for each } n \geq 1.$$

Set

$$y_n = \begin{cases} \frac{-c_{n1}}{(a_{n1} - a_{n1}^{-1})} & \text{if } b_{n1} = 0 \\ \\ \frac{(a_{n1} - d_{n1}) - \sqrt{(a_{n1} - d_{n1})^2 + 4b_{n1}c_{n1}}}{2b_{n1}} & \text{if } b_{n1} \neq 0 \end{cases}$$

Lemma 2.10 $\lim_n y_n = 0$.

Proof. The limits calculated above show that $\lim_n \frac{-c_{n1}}{(a_{n1} - a_{n1}^{-1})} = 0$, so in what remains of the proof we shall suppose that for all values of n, $b_{n1} \neq 0$. Then $y_n = \frac{(a_{n1} - d_{n1}) - \sqrt{(a_{n1} - d_{n1})^2 + 4b_{n1}c_{n1}}}{2b_{n1}}$. Set $z_n = (a_{n1} - d_{n1}), w_n = 4b_{n1}c_{n1}$ and note

$$\lim_n z_n = a_1 - a_1^{-1} \neq 0 \qquad \lim_n w_n = 0.$$

Hence for $n \gg 0$, $|Re(\sqrt{1+(w_n/z_n^2)})| \geq 1/2$. Thus

$$|z_n + \sqrt{z_n^2 + w_i}| = |z_n||1 + \sqrt{1+(w_n/z_n^2)}| \geq |z_n||Re(1 + \sqrt{1+(w_n/z_n^2)})| \geq |z_n|/2.$$

Then $|w_n| = |z_n + \sqrt{z_n^2+w_n}||z_n - \sqrt{z_n^2+w_n}| \geq |z_n||z_n - \sqrt{z_n^2+w_n}|/2$ and therefore

$$|y_n| = (|z_n - \sqrt{z_n^2+w_n}|)/(2|b_{n1}|) \leq |w_n|/(|z_n||b_{n1}|) = 4|c_{n1}|/|z_n|.$$

The lemma now follows from the facts that $\lim_n c_{n1} = 0$ and $\lim_n z_n \neq 0$. $\diamondsuit$

Set $B_n = \begin{pmatrix} 1 & 0 \\ y_n & 1 \end{pmatrix}$. If replace ρ_n by $B_n \rho_n B_n^{-1}$, we obtain a conjugate sequence of irreducible representations which also converges to ρ (since $\lim_n y_n = 0$) and for which

$$B_n \rho_n(\gamma_1) B_n^{-1} = \begin{pmatrix} 1 & 0 \\ y_n & 1 \end{pmatrix} \begin{pmatrix} a_{n1} & b_{n1} \\ c_{n1} & d_{n1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y_n & 1 \end{pmatrix} = \begin{pmatrix} a_{n1} - b_{n1} y_i & b_{n1} \\ 0 & d_{n1} + b_{n1} y_i \end{pmatrix}.$$

Thus without loss of generality we may suppose that $c_{n1} = 0$ for each $n \geq 1$, and consequently $d_{n1} = a_{n1}^{-1}$. Set $x_n = b_{n1}/(a_{n1} - a_{n1}^{-1})$ and note that the numbers x_n tend to 0. Hence if we set $A_n = \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix}$, then the sequence $A_n \rho_n A_n^{-1}$ converges to ρ and satisfies

$$A_n \rho_n(\gamma_1) A_n^{-1} = \begin{pmatrix} a_{n1} & 0 \\ 0 & a_{n1}^{-1} \end{pmatrix}$$

for each $n \geq 1$. We shall therefore assume below that $\rho_n(\gamma_1)$ is diagonal for each value of n.

For $j = 1, 2, ..., m$ write

$$\rho_n(\gamma_j) = \begin{pmatrix} a_{nj} & b_{nj} \\ c_{nj} & d_{nj} \end{pmatrix}.$$

By construction,

$$b_{n1} = c_{n1} = 0 \text{ for all } n \geq 1$$

while for each $j = 1, 2, ..., m$ we have

$$\lim_{n\to\infty} a_{nj} = a_j \in \mathbb{C}^*, \quad \lim_{n\to\infty} d_{nj} = a_j^{-1}, \quad \lim_{n\to\infty} b_{nj} = b_j \in \mathbb{C}, \quad \lim_{n\to\infty} c_{nj} = 0.$$

Choose $j_0 \in \{2, 3, ..., m\}$ such that there are infinitely many values of n for which $|c_{nj_0}| \geq |c_{nj}|$ for all values of j. Now each ρ_n is irreducible, while by construction each $\rho_n(\gamma_1)$ is diagonal. Hence $c_{nj_0} \neq 0$ for the infinitely many values of n such that $|c_{nj_0}| \geq |c_{nj}|$, $j = 1, 2, \ldots, m$. Then by passing to a subsequence we may assume that

$$c_{nj_0} \neq 0 \text{ for each } n$$

$$|c_{nj}| \leq |c_{nj_0}| \text{ for all values of } n, j.$$

Now the second condition implies that for each n, $0 \leq |c_{nj}|/|c_{nj_0}| \leq 1$. Hence by again passing to a subsequence we may suppose that for each j, the sequence $\{c_{nj}/c_{nj_0}\}_{n\geq 1}$ converges to some $c_j \in \mathbb{C}$. Note that $c_1 = 0$ and $c_{j_0} = 1$.

Define

$$\rho_n' = \begin{pmatrix} 0 & \frac{i}{\sqrt{c_{nj_0}}} \\ i\sqrt{c_{nj_0}} & 0 \end{pmatrix} \rho_n \begin{pmatrix} 0 & \frac{i}{\sqrt{c_{nj_0}}} \\ i\sqrt{c_{nj_0}} & 0 \end{pmatrix}^{-1}.$$

and observe that

$$\lim_n \rho_n'(\gamma_j) = \lim_n \begin{pmatrix} d_{nj} & c_{nj}/c_{nj_0} \\ c_{nj_0} b_{nj} & a_{nj} \end{pmatrix} = \begin{pmatrix} a_j^{-1} & c_j \\ 0 & a_j \end{pmatrix}.$$

Thus the sequence $\{\rho_n'\}_{n\geq 1}$ is conjugate to the sequence $\{\rho_n\}_{n\geq 1}$, and converges to a representation in $U_x^{a^{-1}}$. Since $a_1 \neq \pm 1$, $c_1 = 0$ and $c_{j_0} = 1$, in fact it converges to a representation in $U_x^{a^{-1}} \setminus \mathcal{O}(\rho_a)$. This completes the proof. $\diamondsuit$

Corollary 2.11 *Let $X_0 \subset X(\Gamma)$ be a non-trivial curve which contains the character of an irreducible representation and $R_0 \subset R(\Gamma)$ the unique 4-dimensional subvariety such that $t(R_0) = X_0$. Suppose that $x \in X_0$ is reducible and non-trivial, and that $\rho_a \in D_x$. Then dim $(R_x^a \cap R_0) =$ dim $(R_x^{a^{-1}} \cap R_0) = 3$. In particular, $R_x^a \cap R_0 \setminus \mathcal{O}(\rho_a) \neq \emptyset$ and $R_x^{a^{-1}} \cap R_0 \setminus \mathcal{O}(\rho_a) \neq \emptyset$.*

Proof. First note that since R_0 contains irreducible representations, $\dim(R_x \cap R_0) = \dim(t|R_0)^{-1}(x) = 3$, so that in particular there is a some $\rho \in (R_x \cap R_0) \setminus \mathcal{O}(\rho_a)$, which is non-abelian by Proposition 2.1 (2). If $\rho \in R_x^a$ then Proposition 2.9 implies that there is some non-abelian $\rho' \in (R_x^{a^{-1}} \cap R_0) \setminus \mathcal{O}(\rho_a)$. Conversely if $\rho \in R_x^{a^{-1}}$ then it implies that there is some non-abelian $\rho' \in (R_x^a \cap R_0) \setminus \mathcal{O}(\rho_a)$. Either way, one of the two 3-dimensional sets $\mathcal{O}(\rho), \mathcal{O}(\rho')$ lies in $R_x^a \cap R_0$ and the other in $R_x^{a^{-1}} \cap R_0$. $\diamondsuit$

3 The proof of Theorem A

Assume the conditions of Theorem A. If ρ is irreducible, the proof of the theorem follows as in the proof of [4, Theorem 3], so we shall assume below that ρ is reducible. By hypothesis there is a sequence of irreducible representations $\{\rho_n\}$ in $R(\Gamma)$ which converges to ρ. After passing to a subsequence, we may suppose that the sequence is contained in a fixed algebraic component R_0 of $R(\Gamma)$. According to [CS, Proposition 1.1.1], R_0 is invariant under conjugation. Then since ρ is reducible, while the ρ_n are not, the dimension of R_0 is at least 4. But $T_\rho^{Zar} R(\Gamma)$ is a subspace of $Z^1(\Gamma; Ad\rho)$ ([19]) and therefore

$$4 \leq \dim T_\rho^{Zar} R_0 \leq \dim T_\rho^{Zar} R(\Gamma) \leq \dim Z^1(\Gamma; Ad\rho) = 4.$$

It follows that ρ is a simple point of $R(M)$, R_0 is of dimension 4, and ρ is scheme-reduced.

Set $x = \chi_\rho$ and observe that our hypothesis that ρ be non-abelian implies that x is non-trivial (Proposition 2.8 (1)). From [CS, 1.4.4, 1.5.3] we see that $X_0 = t(R_0)$ is a component of $X^{irr}(\Gamma)$ of dimension 1. Fix a homomorphism $a : \Gamma \to \mathbb{C}^*$ so that $\rho \in R_x^a$ and note that by Proposition 2.8 (2), the defect of ρ is zero. Proposition 2.6 (2) then implies that $R_x^a = \mathcal{O}(\rho) \cup \mathcal{O}(\rho_a) = \overline{\mathcal{O}(\rho)}$. Since $\mathcal{O}(\rho) \subset R_0$ it follows that

$$R_x^a = \overline{\mathcal{O}(\rho)} \subset R_0.$$

To see that X_0 is the only component of $\overline{X^{irr}(\Gamma)}$ to contain x, suppose that another does as well. Then there is an irreducible curve $X_1 \subset \overline{X^{irr}(\Gamma)}$, distinct from X_0, which contains x. Let $R_1 \subset R(\Gamma)$ be the unique 4-dimensional variety, invariant under conjugation, such that $t(R_1) = X_1$ (Lemma 2.7). Evidently $R_1 \neq R_0$. By Corollary 2.11, there is some $\rho_1 \in R_x^a \cap R_1 \setminus \mathcal{O}(\rho_a) = \mathcal{O}(\rho) \subset R_0$. But then $\rho \in R_0 \cap R_1$, which contradicts the fact that it is a simple point of $R(\Gamma)$.

It is shown in [16, Proposition 3.12(v)] how the equality $\mathrm{def}(\rho) = 0$ implies that the kernel of $T_\rho^{Zar} R(\Gamma) \xrightarrow{dt} T_x^{Zar} X(\Gamma)$ is 3-dimensional. Let $t_0 = t|R_0 : R_0 \to X_0$. From the commutative diagram

$$\begin{array}{ccc} T_\rho^{Zar} R_0 & \xrightarrow{dt_0} & T_x^{Zar} X_0 \\ = \big\downarrow & & \big\downarrow \\ T_\rho^{Zar} R(\Gamma) & \xrightarrow{dt} & T_x^{Zar} X(\Gamma) \end{array}$$

we see that the kernel of dt_0 is at most 3-dimensional. As it contains $T_\rho^{Zar} \mathcal{O}(\rho) \cong \mathbb{C}^3$, we conclude that $\ker(dt_0) = T_\rho^{Zar} \mathcal{O}(\rho)$.

Now $\mathcal{O}(\rho_a)$ is closed in $R(\Gamma)$, therefore $\mathcal{O}(\rho)$ is a closed subset of $R_0 \setminus \mathcal{O}(\rho_0)$ consisting of smooth points. Choose an analytic 2-disk $E \subset R_0 \setminus \mathcal{O}(\rho_0)$ which is transverse to $\mathcal{O}(\rho)$ at ρ. Since we have shown that $\ker(dt_0) = T_\rho^{Zar} \mathcal{O}(\rho)$, dt_0 is injective on the the tangent space to E at ρ. Hence a small neighbourhood E_0 of ρ in E is mapped by an analytic isomorphism into a smooth branch of X_0 at x. We shall prove that $t(E_0)$ is a neighbourhood of x in X_0. Let $B_1, B_2, \ldots, B_k$ be the branches of X_0 at x where $B_i \cap B_j = \{x\}$ if $i \neq j$. We may assume that $t(E_0) = B_1$. Our goal is to prove $k = 1$.

First observe that $\mathcal{O}(E_0) = \{A\phi A^{-1} \mid A \in SL(2,\mathbb{C}) \text{ and } \phi \in E_0\}$ is a neighbourhood of $\mathcal{O}(\rho)$ in R_0. For without loss of generality, each $\rho \in E_0$ is non-abelian and a smooth point of R_0. Then the image of the continous injection $PSL(2,\mathbb{C}) \times E_0 \to R_0, (\pm A, \phi) \mapsto A\phi A^{-1}$, lies in the smooth part of R_0. Hence invariance of domain implies that its image, $\mathcal{O}(E_0)$, is open in R_0.

There is a commutative diagram

$$\begin{array}{ccc} R_0^\nu & \xrightarrow{\;\;\nu\;\;} & R_0 \\ {\scriptstyle t^\nu}\downarrow & & \downarrow{\scriptstyle t} \\ X_0^\nu & \xrightarrow{\;\;\nu\;\;} & X_0 \end{array}$$

where the horizontal maps are normalizations, hence finite-to-one and surjective [17, Chapter II, §5]. In particular if Y is an algebraic subset of R_0, then $\dim \nu^{-1}(Y) = \dim Y$. Since X_0 has dimension 1, X_0^ν is a smooth affine curve ([17]).

Fix a branch B_j, a point $x_j \in X_0^\nu$, and an open disk neighbourhood $V \subset X_0^\nu$ of x_j for which $\nu(V) \subset B_j$. Since $(t^\nu)^{-1}(x_j)$ is 3-dimensional, while $\nu^{-1}(\mathcal{O}(\rho_a))$ is 2-dimensional, there is some $y \in (t^\nu)^{-1}(x_j) \setminus \nu^{-1}(\mathcal{O}(\rho_a))$. Choose a curve $C \subset R_0^\nu$ containing y such that $t^\nu(C) \cap V$ is a neighbourhood of x_j in X_0^ν. Then $\nu(t^\nu(C)) \cap B_j$ is a neighbourhood of x in B_j.

Select a sequence $\{y_n\}_{n\geq 1}$ in $(C \cap (t_\nu)^{-1}(V)) \setminus (t^\nu)^{-1}(x_j)$ which converges to y. Set $\rho_n = \nu(y_n)$ and $x_n = \chi_{\rho_n}$. Without loss of generality we may suppose that each $\rho_n \in R_0$ is irreducible. By construction, $\{\rho_n\}_{n\geq 1}$ is a sequence in R_0 which converges to $\nu(y) \in R_x \setminus \mathcal{O}(\rho_a)$ and $\{x_n\}_{n\geq 1}$ is a sequence in $B_j \setminus \{x\}$ which converges to x. There are two possibilities, either

- $\nu(y) \in R_x^a \setminus \mathcal{O}(\rho_a) = \mathcal{O}(\rho)$, or
- $\nu(y) \in R_x^{a^{-1}} \setminus \mathcal{O}(\rho_a)$.

Since $\mathcal{O}(E_0)$ is a neighbourhood of $\mathcal{O}(\rho)$ in R_0, if the first case arises it is clear that for $n \gg 1$, $\rho_n \in \mathcal{O}(E_0)$, and therefore $x_n \in B_1$. This implies $j = 1$, for otherwise $x_n \in B_1 \cap B_j = \{x\}$, contrary to our constructions. On the other hand, if $\nu(y) \in R_x^{a^{-1}} \setminus \mathcal{O}(\rho_a)$, then by Proposition 2.9, there is a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ and another sequence $\{A_k \rho_{n_k} A_k^{-1}\}$ which converges to an element of $R_x^a \setminus \mathcal{O}(\rho_a) = \mathcal{O}(\rho)$. Again we see that $x_{n_k} \in B_1$ for $k \gg 0$. Thus $j = 1$ so X_0 has only one branch through x. This implies that x is a smooth point of X_0.

The existence of the commutative diagram of the proposition follows from the work above, where we saw that there is an identification $T_\rho^{Zar} R(\Gamma) = Z^1(\Gamma; Ad\rho)$ under which $T_\rho^{Zar}\mathcal{O}(\rho)$ corresponds to $B^1(\Gamma; Ad\rho)$. Further $dt : T_\rho^{Zar} R^{irr}(\Gamma) \to T_{\chi_\rho}^{Zar} X^{irr}(\Gamma)$ is surjective with kernel $T_\rho^{Zar}\mathcal{O}(\rho)$. ◇

Proof of Corollary B.

Assume the conditions of Corollary B and let $M(r) = M \cup_{\partial M} V(r)$ where $V(r)$ is a solid torus whose meridian slope is identified with r in forming $M(r)$. The assumption that $H^1(M(r); Ad\rho) = 0$ combines with the Mayer-Vietoris sequence for cohomology with local coefficients to prove that $H^1(M; Ad\rho) \oplus H^1(V(r); Ad\rho) \cong H^1(\partial M; Ad\rho)$. Since $\rho(\pi_1(\partial M)) \not\subset \{\pm I\}$, we have $H^1(V(r); Ad\rho) \cong \mathbb{C}$ and $H^1(\partial M; Ad\rho) \cong \mathbb{C}^2$. Hence $H^1(M; Ad\rho) \cong \mathbb{C}$. Since ρ is non-abelian, this implies that $Z^1(M; Ad\rho) \cong \mathbb{C}^4$, and we are done. $\diamondsuit$

4 $PSL(2, \mathbb{C})$ characters and Culler-Shalen seminorms

With an eye on the applications we have in mind, it is necessary to rework the previous sections, replacing $SL(2, \mathbb{C})$ by $PSL(2, \mathbb{C})$. We content ourselves with stating the $PSL(2, \mathbb{C})$ analogues, leaving most of the details to the reader. A careful account of the theory of $PSL(2, \mathbb{C})$ representation and character varieties can be found in [3].

For a finitely generated group Γ we set $R_{PSL_2}(\Gamma) = \mathrm{Hom}(\Gamma, PSL(2, \mathbb{C}))$. As in the $SL(2, \mathbb{C})$ case described in the introduction, the natural action of $PSL_2(\mathbb{C})$ on $R_{PSL_2}(\Gamma)$ has an algebro-geometric quotient $X_{PSL_2}(\Gamma)$ and there is a surjective quotient map

$$t : R_{PSL_2}(\Gamma) \longrightarrow X_{PSL_2}(\Gamma)$$

which is constant on conjugacy classes of representations. For each $\gamma \in \Gamma$, the function $X_{PSL_2}(\Gamma) \to \mathbb{C}$ given by

$$f_\gamma : X_{PSL_2}(\Gamma) \to \mathbb{C}, \quad t(\rho) \mapsto trace(\rho(\gamma))^2 - 4.$$

lies in $\mathbb{C}[X_{PSL_2}(\Gamma)]$.

A representation $\rho \in R_{PSL_2}(\Gamma)$ is called *irreducible* if it is not conjugate to a representation whose image lies in $\{\pm \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{C}, a \neq 0\}$. Otherwise it is called *reducible.* Two points worth making are (i) the image of an irreducible representation in $PSL(2, \mathbb{C})$ is either non-abelian or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and (ii) any two subgroups of $PSL(2, \mathbb{C})$ abstractly isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ are conjugate. If ρ is irreducible, then $t^{-1}(t(\rho))$ is the orbit of ρ under conjugation [15, Corollary 3.5.2]. In analogy with the $SL(2, \mathbb{C})$ case, $X_{PSL_2}(\Gamma)$ is called the set of $PSL_2(\mathbb{C})$-*characters* of Γ and $t(\rho)$ is denoted by χ_ρ.

Let $N \subset PSL(2, \mathbb{C})$ denote the subgroup

$$N = \{\pm \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \pm \begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix} \mid z, w \in \mathbb{C}^*\}$$

of $PSL(2,\mathbb{C})$. One of the features which distinguishes the $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$ theories is that when an irreducible $PSL(2,\mathbb{C})$ representation ρ conjugates into N, $PSL(2,\mathbb{C})$ does not act freely on its orbit $\mathcal{O}(\rho)$. Indeed the isotropy group of ρ is $\mathbb{Z}/2$. This feature obliges us to add an extra case to the results of the previous sections.

Theorem 4.1 *Suppose that $\rho \in \overline{R^{irr}_{PSL_2}(\Gamma)}$ is a non-abelian representation or irreducible for which $Z^1(\Gamma; Ad\rho) \cong \mathbb{C}^4$. Then*
(1) *ρ is a simple point of $R_{PSL_2}(\Gamma)$ and the unique algebraic component of $R_{PSL_2}(\Gamma)$ which contains it is 4-dimensional.*
(2) *χ_ρ is a simple point of $\overline{X^{irr}_{PSL_2}(\Gamma)}$ and the algebraic component of $\overline{X^{irr}(\Gamma)}$ which contains it is a curve.*
(3) *there is an analytic 2-disk D, smoothly embedded in $R_{PSL_2}(\Gamma)$ and containing ρ, such that*

(a) *$t|D$ is an analytic isomorphism onto a neighbourhood of χ_ρ if ρ is not conjugate to an irreducible representation with image in N.*

(b) *$t|D$ is a $2-1$ cover, branched at ρ, otherwise.*

(4) *ρ is scheme reduced and*

(a) *if the image of ρ does not conjugate into N, then there is a commutative diagram*

$$\begin{array}{ccccc} T^{Zar}_\rho \mathcal{O}(\rho) & \longrightarrow & T^{Zar}_\rho R_{PSL_2}(\Gamma) & \xrightarrow{dt} & T^{Zar}_\rho \overline{X^{irr}_{PSL_2}(\Gamma)} \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ B^1(\Gamma; Ad\rho) & \longrightarrow & Z^1(\Gamma; Ad\rho) & \longrightarrow & H^1(\Gamma; Ad\rho). \end{array}$$

(b) *if the image of ρ does conjugate into N, then there is a commutative diagram*

$$\begin{array}{ccccccc} T^{Zar}_\rho \mathcal{O}(\rho) & \longrightarrow & T^{Zar}_\rho R_{PSL_2}(\Gamma) & & \longrightarrow & & T^{Zar}_\rho \overline{X^{irr}_{PSL_2}(\Gamma)} \\ \cong\downarrow & & \cong\downarrow & & & & \cong\downarrow \\ B^1(\Gamma; Ad\rho) & \longrightarrow & Z^1(\Gamma; Ad\rho) & \longrightarrow & H^1(\Gamma; Ad\rho) & \longrightarrow & H^1(\Gamma; Ad\rho)/\{\pm 1\}. \end{array}$$

$\diamondsuit$

Corollary 4.2 *Suppose that $\rho \in R_{PSL_2}(M)$ is a non-abelian or irreducible representation for which $\rho \in \overline{R^{irr}_{PSL_2}(M)}$. If $\rho(\alpha(r)) = \pm I$ but $\rho(\pi_1(\partial M)) \neq \{\pm I\}$, and $H^1(M(r); Ad\rho) = 0$, then $Z^1(M; Ad\rho) \cong \mathbb{C}^4$. Thus the conclusions of Theorem 4.1 hold for ρ.* $\diamondsuit$

Next consider a *non-trivial* curve $X_0 \subset X_{PSL_2}(M)$, that is a curve which contains the character of an irreducible representation. Assume that X_0 is irreducible. There is a unique 4-dimensional subvariety $R_0 \subset R_{PSL_2}(M)$ for which $t(R_0) = X_0$ ([3, Lemma 4.1]). The smooth projective model $\tilde{X}_0$ of X_0 decomposes as

$$X_0 \xleftarrow{\nu} X_0^\nu \xrightarrow{i} X_0^\nu \cup \mathcal{I} = \tilde{X}_0$$

where $\nu : X_0^\nu \to X_0$ is a surjective regular birational equivalence, i is an inclusion, and $\mathcal{I}$ is the finite set of ideal points of X_0. These maps induce an isomorphism between function fields:

$$\mathbb{C}(X_0) \to \mathbb{C}(\tilde{X}_0), \quad f \mapsto \tilde{f}.$$

We use $Z_x(\tilde{f}_\gamma)$ to denote the multiplicity of $x \in \tilde{X}_0$ as a zero of $\tilde{f}_\gamma$. By convention this means that $Z_x(\tilde{f}_\gamma) = \infty$ if $\tilde{f}_\gamma = 0$. The *Culler-Shalen* seminorm

$$\| \cdot \|_{X_0} : H_1(\partial M; \mathbb{R}) \to [0, \infty)$$

was introduced in [7] to study fillings of M with cyclic fundamental groups. It was adapted to the $PSL(2,\mathbb{C})$ setting in [3]. Roughly speaking, given a slope r, $\|\alpha(r)\|_{X_0}$ measures the number of characters in X_0 of representations which send $\alpha(r)$ to $\pm I$. As a consequence of Corollary B and [1, Theorem 2.1], we are often able to give an explicit calculation of $\|\alpha(r)\|_{X_0}$ when $M(r)$ is small Seifert, i.e. $M(r)$ admits the structure of a Seifert fibred space whose base orbifold is the 2-sphere with at most three cone points.

Theorem 4.3 [1, Theorem 2.3] *Let M be the exterior of a knot in a closed, connected, orientable 3-manifold W for which $Hom(\pi_1(W), PSL(2,\mathbb{C}))$ contains only diagonalisable representations. Suppose further that there is a non-boundary slope r for which $M(r)$ is a non-Haken small Seifert manifold. Fix a non-trivial curve $X_0 \subset X(M)$ for which $f_{\alpha(r)}|X_0$ is non-constant. Then*

$$\|\alpha(r)\|_{X_0} = m_0 + A + 2B$$

where

$$m_0 = \sum_{x \in \tilde{X}_0} min\{Z_x(\tilde{f}_\alpha) \mid \tilde{f}_\alpha | \tilde{X}_0 \not\equiv 0\},$$

while A, respectively B, is the number of irreducible characters $\chi_\rho \in X_0$ of representations ρ which conjugate, respectively do not conjugate, into N and such that $\rho(\alpha(r)) = \pm I$. $\diamondsuit$

This result will be used repeatedly in the following sections.

5 The $PSL(2,\mathbb{C})$-characters of Euclidean triangle groups

Let $2 \leq p \leq q$. In Example 2.1 of [3], it is shown that $X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ is a disjoint union of isolated points and $[\frac{p}{2}][\frac{q}{2}] \geq 1$ non-trivial curves, each isomorphic to a complex line. The curves are parameterised as follows.

Fix generators a of $\mathbb{Z}/p$ and b of $\mathbb{Z}/q$. For each pair (j,k), where $1 \leq j \leq [\frac{p}{2}]$ and $1 \leq k \leq [\frac{q}{2}]$, set $\lambda = e^{\pi i j/p}, \mu = e^{\pi i k/q}$, and $\tau = \mu + \mu^{-1}$. There is a curve $C_{p,q}(j,k) \subset$

$X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ whose points are the characters of the representations $\rho_z \in \mathrm{Hom}(\mathbb{Z}/p * \mathbb{Z}/q, PSL_2(\mathbb{C}))$, $z \in \mathbb{C}$, where

$$\rho_z(a) = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \rho_z(b) = \pm \begin{pmatrix} z & 1 \\ z(\tau - z) - 1 & \tau - z \end{pmatrix}.$$

The surjective, regular map

$$\Psi : \mathbb{C} \longrightarrow C_{p,q}(j,k), \quad z \mapsto \chi_{\rho_z}$$

is an isomorphism unless $j = p/2$ or $k = q/2$, in which case it is a $2-1$ map branched over the single point $\chi_{\rho_{\frac{\tau}{2}}}$. The characters on $C_{p,q}(j,k)$ corresponding to reducible representations are those given by the values $z = \mu, \mu^{-1}$ of the parameter. There are exactly two such characters if both $j \neq p/2$ and $k \neq q/2$, and one otherwise. We shall use this parameterization to calculate the $PSL_2(\mathbb{C})$-characters of Euclidean triangle groups

Let D_n denote the dihedral group of order $2n$ and T the tetrahedral group. There are isomorphisms $\Delta(2,2,n) \cong D_n$ and $\Delta(2,3,3) \cong T$.

We shall call a representation to $PSL(2,\mathbb{C})$ *dihedral*, respectively *tetrahedral*, if its image is isomorphic to some D_n, respectively T. The character of such a representation will also be referred to as dihedral or tetrahedral.

The following lemma is an easy consequence of the construction of the curves $C_{p,q}(j,k)$.

Lemma 5.1 *There is the character of a dihedral representation on $C_{p,q}(j,k)$ if and only if $p = 2$. Furthermore, when $p = 2$ and $q > 2$, there is a unique such character. If $p = q = 2$, there is a unique D_n-character for each $n \geq 1$.* $\diamondsuit$

The (l,m,n)-triangle group is given by the presentation

$$\Delta(l,m,n) \cong < a, b : a^l = b^m = (ab)^n = 1 > .$$

In particular it is a quotient of $\mathbb{Z}/l * \mathbb{Z}/m$ and so we may consider $X_{PSL_2}(\Delta(l,m,n)) \subset X_{PSL_2}(\mathbb{Z}/l * \mathbb{Z}/m)$. We will apply the description of $X_{PSL_2}(\mathbb{Z}/l * \mathbb{Z}/m)$ given above to determine $X_{PSL_2}(\Delta(l,m,n))$ when (l,m,n) is a Euclidean triple.

Proposition 5.2 $X_{PSL_2}(\Delta(3,3,3))$ *contains exactly one irreducible character corresponding to representation with image T, the tetrahedral group. Furthermore, if gcd(j,p) = gcd$(k,q) = 1$ and one of the curves $C_{p,q}(j,k)$ described above contains an irreducible $\Delta(3,3,3)$-character, then $\{p,q\} = \{2,3\}$ or $\{3,3\}$. Conversely,*
(1) *the curve $C_{2,3}(1,1)$ contains a unique tetrahedral character.*
(2) *the curve $C_{3,3}(1,1)$ contains a unique tetrahedral character.*

Proof. Consider the parameterisation of $X_{PSL_2}(\mathbb{Z}/3*\mathbb{Z}/3)$ given above. Set $\lambda = \mu = e^{\pi i/3}$ and $\tau = 1$. Then $X_{PSL_2}(\mathbb{Z}/3 * \mathbb{Z}/3)$ contains a unique curve $C_{3,3}(1,1)$ and there is an isomorphism $\Psi : \mathbb{C} \longrightarrow C_{3,3}(1,1)$, $\Psi : z \mapsto \chi_{\rho_z}$ where

$$\rho_z(a) = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \rho_z(b) = \pm \begin{pmatrix} z & 1 \\ z(\tau - z) - 1 & \tau - z \end{pmatrix}.$$

Furthermore, each irreducible character of $\mathbb{Z}/3 * \mathbb{Z}/3$ lies on $C_{3,3}(1,1)$. This curve contains exactly two reducible characters, corresponding to the values $z = \lambda, \lambda^{-1}$.

Suppose that $\chi_\rho \in X_{PSL_2}(\Delta(3,3,3)) \subset X_{PSL_2}(\mathbb{Z}/3 * \mathbb{Z}/3)$ is an irreducible character. Then there is a $z \in \mathbb{C}$ such that ρ is conjugate to $\Psi(z) = \rho_z$. Now $\rho_z(ab)^3 = \pm I$ and by irreducibility, $\rho_z(ab) \neq \pm I$. Thus $\rho_z(ab)$ has order 3 in $PSL_2(\mathbb{C})$, that is

$$1 = (trace(\rho(ab)))^2 = ((\lambda - \lambda^{-1})z + \lambda^{-1})^2.$$

It follows that there is an $\epsilon \in \{\pm 1\}$ such that

$$z = \frac{\epsilon - \lambda^{-1}}{\lambda - \lambda^{-1}} = \begin{cases} \frac{\lambda}{\lambda - \lambda^{-1}} & \text{if} \quad \epsilon = \quad 1 \\ \lambda & \text{if} \quad \epsilon = \quad -1. \end{cases}$$

When $\epsilon = -1$ we have a reducible character, while $\epsilon = 1$ corresponds to an irreducible one. Hence $X_{PSL_2}(\Delta(3,3,3))$ has a unique irreducible character. To see that it corresponds to a tetrahedral representation, recall that T is isomorphic to A_4, the group of even permutations on 4 letters. We construct a tetrahedral representation $\Delta(3,3,3) \rightarrow A_4 = T \subset SO(3) \subset PSL_2(\mathbb{C})$ by sending a to the permutation $(1,2,3)$ and b to $(1,4,2)$. Thus $\Delta(3,3,3)$ has a unique irreducible character and it is tetrahedral.

If $\gcd(j,p) = \gcd(k,q) = 1$, then any representation whose character lies on $C_{p,q}(j,k)$ has elements of order p and q in its image. Thus if $C_{p,q}(j,k)$ contains an irreducible $\Delta(3,3,3)$-character, then $\{p,q\} = \{2,3\}$ or $\{3,3\}$.

The analysis of the curve $C_{2,3}(1,1)$ is similar. We first note that any irreducible character of $\mathbb{Z}/2 * \mathbb{Z}/3$ lies on this curve, and there is at least one with image T, as T is generated by an element of order 2 and an element of order 3. To see that there are no more, let $x = \chi_{\rho_z} \in C_{2,3}(1,1)$ be a tetrahedral character. Now $\rho_z(ab) \in T$ cannot be $\pm I$ by irreducibility, and further cannot have order 2, for if $\rho_z(ab)$ had order 2, then ρ_z would factor through $\Delta(2,3,2) \cong D_3$, clearly an impossibility. Thus $\rho_z(ab)$ has order 3. One may apply the procedure of the previous paragraph to obtain $z = \frac{1}{2} \pm \frac{1}{2}i$. Now as we remarked above, the parameterisation $\Psi : \mathbb{C} \rightarrow C_{2,3}(1,1)$ is a 2-fold branched cover with $\Psi(z_1) = \Psi(z_2)$ if and only if $z_2 = 1 - z_1$. Thus $\Psi(\frac{1}{2} + \frac{1}{2}i) = \Psi(\frac{1}{2} - \frac{1}{2}i)$, and so there is a unique T-character on $C_{2,3}(1,1)$. $\diamondsuit$

The next two lemmas are proved in a similar manner.

Proposition 5.3 *$X_{PSL_2}(\Delta(2,4,4))$ contains exactly three irreducible characters, corresponding to representations with dihedral images D_2, D_4 and D_4. Furthermore, if gcd$(j,p) =$ gcd$(k,q) = 1$ and one of the curves $C_{p,q}(j,k)$ described above contains an irreducible $\Delta(2,4,4)$-character, then $\{p,q\} = \{2,2\}$ or $\{2,4\}$. Conversely,*
(1) *the curve $C_{2,2}(1,1)$ contains exactly one D_2-character one D_4-character;*
(2) *the curve $C_{2,4}(1,1)$ contains exactly one D_4-character and no D_2-character.* ♢

Proposition 5.4 *$X_{PSL_2}(\Delta(2,3,6))$ contains exactly two irreducible characters, one corresponding to a representations with image D_3, and the other to a representations with image T. Furthermore, if gcd$(j,p) =$ gcd$(k,q) = 1$ and one of the curves $C_{p,q}(j,k)$ described above contains an irreducible $\Delta(2,3,6)$-character, then $\{p,q\} = \{2,3\}$ or $\{3,3\}$. Conversely,*
(1) *the curve $C_{2,3}(1,1)$ contains exactly one T-character and one D_3-character.*
(2) *the curve $C_{3,3}(1,1)$ contains one tetrahedral character and no dihedral character.* ♢

Remark 5.5 As the reader will notice, the method of this section can be used to determine the number of irreducible $PSL(2,\mathbb{C})$ characters of an arbitrary triangle group $\Delta(p,q,r)$ where $p,q,r \geq 2$. Such a calculation is carried out in Lemma 3.2 of [1].

6 A result on curves in $X_{PSL_2}(M(r))$

We begin with a useful lemma.

Lemma 6.1 *Let Γ be a finitely generated group, $\gamma \in \Gamma$, and $X_0 \subset X(\Gamma)$ a curve. Then for each $n \in \mathbb{Z}$, there is some $g_n \in \mathbb{C}(X_0)$ such that $f_{\gamma^n}(x) = g_n(x) f_\gamma(x)$ for each $x \in X_0$. Furthermore if $\rho \in R_0$, then $g_n(\chi_\rho) = 0$ if and only if $\rho(\gamma)$ has finite order $d > 2$ where d divides $2n$.*

Proof. Let R_0 be the unique 4-dimensional subvariety of $R(M)$ for which $t(R_0) = X_0$. There is a finite extension F of $\mathbb{C}(R_0)$ and a *tautological representation* $P : \pi \rightarrow PSL_2(F)$ defined by

$$P(\zeta) = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the functions a, b, c and d satisfy the identity

$$\rho(\gamma) = \begin{pmatrix} a(\rho) & b(\rho) \\ c(\zeta) & d(\rho) \end{pmatrix}$$

for all $\rho \in R_0$ ([7]). In particular, for each $\rho \in R_0$ we have $f_\zeta(\chi_\rho) = (trace(P(\zeta))(\rho))^2 - 4$. By passing to an extension field of F if necessary, we may assume that $P(\gamma)$ is an upper-triangular matrix, say $P(\gamma) = \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$. Then for $n \in \mathbb{Z}$,

$$f_{\gamma^n} = (a^n - a^{-n})^2 = (a - a^{-1})^2(a^{(|n|-1)} + a^{(|n|-3)} + ... + a^{-(|n|-1)})^2.$$

If we set $g_n = a^{2(1-n)}(\frac{a^{2n}-1}{a^2-1}) = (a^{(|n|-1)} + a^{(|n|-3)} + ... + a^{-(|n|-1)})^2$, then $f_{\gamma^n} = g_n f_\gamma$. Further $g_n(\chi_\rho) = 0$ if and only if $a(\rho)^2 \neq 1$ and $a(\rho)^{2n} = 1$, which is what we set out to prove. ♢

For the remainder of this section we will assume that $X_0 \subset X_{PSL_2}(M(r)) \subset X_{PSL_2}(M)$ is an r-*curve*, that is a curve for which $\|\alpha(r)\|_{X_0} = 0$ but $\|\cdot\|_{X_0} \neq 0$. We denote by $R_0 \subset R_{PSL_2}(M(r)) \subset R_{PSL_2}(M)$ the unique 4-dimensional variety such that $t(R_0) = X_0$ ([3, Lemma 4.1]).

Let $\alpha(r) \in H_1(\partial M)$ be one of the primitive classes associated to r and $\gamma \in H_1(\partial M)$ any class for which $\{\alpha(r), \gamma\}$ forms a basis of $H_1(\partial M)$. For each $\alpha \in H_1(\partial M)$ set

$$\Delta(\alpha, \alpha(r)) = |\alpha \cdot \alpha(r)|$$

and observe that

$$\Delta(\alpha, \alpha(r)) = n \text{ if and only if } \alpha = \pm(m\alpha(r) + n\gamma)$$

for some $m \in \mathbb{Z}$. Hence as $\rho(\alpha(r)) = \pm I$ for each $\rho \in R_0$, we have

$$\rho(\alpha) = \rho(\gamma)^{\pm\Delta(\alpha,\alpha(r))},$$

the sign depending only on that of $\alpha \cdot \alpha(r)$.

Proposition 6.2 *Let $\alpha \in H_1(\partial M)$, $n = \Delta(\alpha, \alpha(r))$, and assume $n > 0$.*
(1) *Suppose that $x \in X_0^\nu$ and $\nu(x) = \chi_\rho$.*
(*a*) *If $Z_x(\tilde{f}_\alpha) > 0$, then $\rho(\pi_1(\partial M))$ is either parabolic or a finite cyclic group whose order divides n.*
(*b*) *$Z_x(\tilde{f}_\alpha) \geq Z_x(\tilde{f}_\gamma)$ with equality if and only if $\rho(\pi_1(\partial M))$ is parabolic or trivial.*
(2) *Suppose that $\tilde{f}_\gamma$ has poles at each ideal point of $\tilde{X}_0$. If $d > 1$ divides n, then there is a point $x \in X_0^\nu$ such that $Z_x(\tilde{f}_\alpha) > Z_x(\tilde{f}_\gamma)$ and if $\nu(x) = \chi_\rho \in X_0$, then $\rho(\pi_1(\partial M)) = \mathbb{Z}/d$.*

Proof. As we noted above, $\rho(\alpha) = \rho(\gamma)^{\pm n}$ for each $\rho \in R_0$, so $f_\alpha|X_0 = f_{n\gamma}|X_0$. Hence if $Z_x(\tilde{f}_\alpha) > 0$, then $f_{n\gamma}(\chi_\rho) = f_\alpha(\chi_\rho) = 0$. Therefore $\rho(\gamma)^n$ is either $\pm I$ or parabolic. Since $\rho(\alpha(r)) = \pm I$ it follows that in the former case $\rho(\pi_1(\partial M))$ is finite cyclic, while in the latter it is parabolic. This proves part (1) (i).

Next consider part (1) (ii). Since $f_\alpha|X_0 = f_{n\gamma}|X_0$, a $PSL(2,\mathbb{C})$ version of Lemma 6.1 implies that $f_\alpha|X_0 = g_n f_\gamma|X_0$ where $g_n : X_0 \to \mathbb{C}$ is a regular fuction which has a zero at χ_ρ if and only if $\rho(\gamma)$ has order $d > 1$ where d divides n. Therefore

$$Z_x(\tilde{f}_\alpha) = Z_x(\tilde{f}_\gamma) + Z_x(\tilde{g}_n).$$

It follows that $Z_x(\tilde{f}_\alpha) \geq Z_x(\tilde{f}_\gamma)$ with a strict inequality if and only if $\rho(\gamma)$ has order $d > 1$ where d divides n. Thus part (1) (ii) of the proposition holds.

Now for part (2). As $d > 1$ and $\tilde{f}_\gamma$ is infinite at all ideal points of $\tilde{X}_0$, there is a point $x \in X_0^\nu$ such that $\tilde{f}_\gamma(x) = (e^{\pi i/d} - e^{-\pi i/d})^2 \neq 0$. Note that if $\chi_\rho = \nu(x)$, then $trace(\rho(\gamma))^2 = (e^{\pi i/d} + e^{-\pi i/d})^2$, so $\rho(\gamma)$ has order d in $PSL_2(\mathbb{C})$. Since $\rho(\alpha(r)) = \pm I$, we have $\rho(\pi_1(\partial M)) = \mathbb{Z}/d$. Finally note that since d divides n we have $\rho(\alpha) = \rho(\gamma)^{\pm n} = \pm I$. Thus $Z_x(\tilde{f}_\alpha) > 0 = Z_x(\tilde{f}_\gamma)$. ♢

Corollary 6.3 *Suppose that $X_0 \subset X_{PSL_2}(M(r)) \subset X_{PSL_2}(M)$ is an r-curve which contains no trivial characters and that $M(r_2)$ is a Seifert fibred space with base orbifold $\mathcal{B} = S^2(l,m,n)$, where (l,m,n) is a Euclidean triple. Further assume that $f_{\alpha(r_2)}$ has poles at each ideal point of $\tilde{X}_0$. Then*
(1) $\Delta(r,r_2) \leq 3$ *if* $\mathcal{B} = S^2(3,3,3)$.
(2) $\Delta(r,r_2) \leq 4$ *and divides* 4 *if* $\mathcal{B} = S^2(2,4,4)$.
(3) $\Delta(r,r_2) \leq 6$ *and divides* 6 *if* $\mathcal{B} = S^2(2,3,6)$.

Proof. This is a simple application of Proposition 6.2. For instance, when $\mathcal{B} = S^2(3,3,3)$, we may assume, without loss of generality, that $n = \Delta(r,r_2) > 1$. Then by Proposition 6.2 (2), there is a representation $\rho \in R_0$ such that $\rho(\alpha(r_2)) = \pm I$ and $\rho(\pi_1(\partial M)) = \mathbb{Z}/n$. We may assume that ρ has nonabelian image, because its character is non-trivial. Thus it factors through the triangle group $\Delta(3,3,3)$. Then Proposition 5.2 implies that if ρ is irreducible, it has image T. In particular, $\rho(\pi_1(\partial M))$ must be $\mathbb{Z}/2$ or $\mathbb{Z}/3$. Thus $n \leq 3$. On the other hand if ρ is reducible we may assume that it is upper-triangular. Projecting to the diagonal elements only kills parabolics, so as we are only interested in $\rho(\pi_1(\partial M)) = \mathbb{Z}/n$, we may assume that ρ is diagonal. But then it factors through $H_1(\Delta(3,3,3)) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$. Hence again we have $n \leq 3$.

The two other cases are similar. ♢

7 Some topological results concerning Seifert fibred manifolds over Euclidean orbifolds

In this section we detail some results concerning Seifert fibred manifolds over Euclidean orbifolds which will be used in the proof of Theorem C.

Every Euclidean orbifold $\mathcal{B}$ is finitely covered by a torus and therefore every Seifert fibred space W whose base orbifold $\mathcal{B}$ is Euclidean is finitely covered by a circle bundle over the torus. It follows that the only closed, essential surfaces in W are tori.

Lemma 7.1 *Let M be a compact, connected, orientable, irreducible, atoroidal 3-manifold, whose boundary is a torus, and r be a slope on ∂M such that $M(r)$ is a Seifert fibred space whose base orbifold $\mathcal{B}$ is Euclidean. Then there is no closed essential surface in M which remains essential in $M(r)$.*

Proof. Let S be a closed, essential surface in M. Since M is irreducible and atoroidal, S has genus at least 2. But then as $M(r)$ is Seifert fibred with a Euclidean base orbifold, S cannot remain incompressible in $M(r)$. $\diamondsuit$

Proposition 7.2 *Let M be a compact, connected, orientable, irreducible, atoroidal 3-manifold, whose boundary is a torus, and r_1 is a slope such that $M(r_1)$ is not Haken. Further assume that $r_2 \neq r_1$ is a slope on ∂M such that $M(r_2)$ is a Seifert fibred space whose base orbifold $\mathcal{B}$ is Euclidean. Then*
(1) $b_1(M) = 1$*, and consequently $M(r_2)$ is not an S^1-bundle over the torus.*
(2) *If r_1 is a boundary slope, $M(r_1) \neq S^1 \times S^2$, and $\Delta(r_1, r_2) > 1$, then $M(r_1)$ is a connected sum of two nontrivial lens spaces.*
(3) *If r_2 is a boundary slope and $M(r_2)$ is not Haken, then $\Delta(r_1, r_2) \leq 1$.*

Proof. To prove part (1), assume that $b_1(M) > 1$. Let S be a closed surface in M which minimizes the Thurston norm of a nontrivial element of $H_2(M) \cong \mathbb{Z}^{b_1(M)-1}$. By Lemma 7.1, S compresses in $M(r_2)$, and so according to [11, Corollary], $M(r')$ is Haken for each $r' \neq r_2$ (indeed S remains incompressible in such $M(r')$). But this contradicts our hypotheses on $M(r_1)$. Thus we must have $b_1(M) = 1$. Since $1 \geq b_1(M) \geq b_1(M(r_2))$, $M(r_2)$ cannot be an S^1-bundle over the torus.

Parts (2) and (3) are consequences of [7, Theorem 2.0.3]. According to that result, if $b_1(M) = 1$ and r' is a boundary slope, then either

- $M(r')$ is Haken, or
- $M(r')$ is a connected sum of two non-trivial lens spaces, or
- there is a closed incompressible surface S in M which remains incompressible in $M(r'')$ as long as $\Delta(r', r'') > 1$, or
- $M(r') \cong S^1 \times S^2$.

Under the hypotheses of part (2), Lemma 7.1 implies that only the second possibility can arise. Under the hypotheses of part (3), only the third possibility can arise, and so if

$\Delta(r_1, r_2) > 1$, $M(r_1)$ contains an incompressible surface of positive genus. Since $M(r_1)$ is not Haken, it must be reducible and therefore r_1 is a boundary slope. Then part (2) of this proposition implies that $M(r_1)$ is either $S^1 \times S^2$ or a connected sume of lens spaces. But such manifolds do not contain incompressible surfaces of positive genus. Hence it must be that $\Delta(r, r_1) \leq 1$. ♢

Corollary 7.3 *Assume the hypotheses of the previous proposition. If r_1 is a boundary slope and $\Delta(r_1, r_2) > 1$, then each essential surface in M compresses in both $M(r_1)$ and in $M(r_2)$.* ♢

A similar argument to that in the proof of Proposition 7.2 gives the next result.

Corollary 7.4 *Let M be a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus, and r_2 a slope on ∂M such that $M(r_2)$ is a Seifert fibred space whose base orbifold $\mathcal{B}$ is Euclidean. Further assume that $\hat{M} \to M$ is a regular cover of finite degree and r_1 another slope on ∂M such that*

(i) the slopes r_1 and r_2 lift to slopes $\hat{r}_1$ and $\hat{r}_2$ on $\partial \hat{M}$,

(ii) $\partial \hat{M}$ is connected,

(iii) $\hat{M}(\hat{r}_1)$ is not Haken.

Then $b_1(\hat{M}) = 1$, and so in particular, $\hat{M}(r_2)$ is not an S^1-bundle over the torus.

Proof. As r_1, r_2 lift to slopes $\hat{r}_1, \hat{r}_2$ on $\partial \hat{M}$ and $\partial \hat{M}$ is connected, the cover $\hat{M} \to M$ extends to a finite cover $\hat{M}(\hat{r}_2) \to M(r_2)$. By hypothesis $\hat{M}(\hat{r}_1)$ is not Haken while $\hat{M}(\hat{r}_2)$ is Seifert with a Euclidean base orbifold. Now apply Proposition 7.2 (1). ♢

We note that Corollary 7.4 does not hold if we replace the condition that M be hyperbolic by the condition that M be an atoroidal Seifert manifold. This accounts for the difference in the distance estimates in Theorem C and those in Remark 1.1 (2).

As we mentioned above, each closed Seifert manifold W whose base orbifold is Euclidean is finitely covered by an S^1-bundle over T, which we shall denote by W. In fact there is such a k-fold cyclic cover $\hat{W} \to W$ with

$$k = \begin{cases} 1 & \text{if } \mathcal{B} = T \\ 2 & \text{if } \mathcal{B} = K, P^2(2,2) \text{ or } S^2(2,2,2,2) \\ 3 & \text{if } \mathcal{B} = S^2(3,3,3) \\ 4 & \text{if } \mathcal{B} = S^2(2,4,4) \\ 6 & \text{if } \mathcal{B} = S^2(2,3,6). \end{cases}$$

Corollary 7.5 *Let M be a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus. Suppose that r_1, r_2 are slopes on ∂M such that $M(r_1)$ is reducible while $M(r_2)$ is a Seifert fibred space whose base orbifold $\mathcal{B}$ is Euclidean. Let $\hat{M} \to M$ be the k-fold cyclic cover described immediately above. If $\Delta(r_1, r_2) \equiv 0 \pmod{k}$, then ∂M is not connected.*

Proof. Assume that $\hat{M} = \hat{T}$ is connected and let L be the image of $\pi_1(\hat{T})$ in $\pi_1(\partial M)$. Note that its index in $\pi_1(\partial M)$ is some number d which divides k, say $k = ld$. Note also that by construction, $\alpha(r_2) \in L$. Since $\alpha(r_2)$ is primitive, L is spanned by $\alpha(r_2)$ and $d\gamma$ where $\gamma \in H_1(\partial M)$ is any class which satisfies $|\gamma \cdot \alpha(r_2)| = 1$. As $|\alpha(r_1) \cdot \alpha(r_2)| = \Delta(r_1, r_2) = ak$ for some $a \in \mathbb{Z}$, it follows that there is an integer m such that $\alpha(r_1) = m\alpha(r_2) + ak\gamma = m\alpha(r_2) + al(d\gamma) \in L$. Thus $\alpha(r_1) \in L \subset \pi_1(\hat{M})$. In particular, r_1 lifts to a slope $\hat{r}_1$ on $\hat{T}$. The reducibility of $M(r_1)$ implies that of $\hat{M}(\hat{r}_1)$ [14]. But this is impossible as Proposition 7.4 would imply that $\hat{M}(\hat{r}_2)$ is not a circle bundle over the torus, contrary to our construction. Thus $\partial \hat{M}$ is connected. $\diamondsuit$

8 The proof of Theorem C

In this section we assume that M is a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus. Further r_1 is a a reducible filling slope on ∂M, $M(r_1) \neq S^1 \times S^2$, and $M(r_2)$ is a Seifert fibred space with Euclidean orbifold $\mathcal{B}$. We shall suppose throughout that $\Delta(r_1, r_2) > 1$. As r_1 is a boundary slope, we may apply Proposition 7.2 to conclude that

$$M(r_1) = L_p \# L_q,$$

a connected sum of two nontrivial lens spaces where $\pi_1(L_p) \cong \mathbb{Z}/p, \pi_1(L_q) \cong \mathbb{Z}/q$, and $2 \leq p \leq q$.

Identify $\pi_1(M(r_1))$ with $\mathbb{Z}/p * \mathbb{Z}/q$ and fix generators a and b of $\mathbb{Z}/p$ and $\mathbb{Z}/q$ respectively so that we can identify $X_{PSL_2}(M(r_1))$ with $X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$.

Lemma 8.1 *There is a non-trivial r_1-curve $X_0 \subset X_{PSL_2}(M(r_1)) = X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ which has the following properties.*
(1) *For each ideal point x of X_0, r_1 is the only slope r for which $f_{\alpha(r)}(x) \in \mathbb{C}$.*
(2) *X_0 is isomorphic to a complex line.*
(3) *If the character of $\rho \in R_{PSL_2}(M)$ lies in X_0, then $\rho(a)$ has order p and $\rho(b)$ has order q. In particular, each character in X_0 is non-trivial.*

Proof. Let X_0 be the curve $C_{p,q}(1,1)$ discussed in the §5. The discussion there shows that X_0 contains the character of an irreducible representation and that parts (2) and (3) of the lemma hold. Next we note that Corollary 7.3 and [Proposition 4.10, BZ2] imply that (1) holds, and therefore X_0 is an r_1-curve. $\diamondsuit$

Corollary 8.2 *X_0 is smooth and $\tilde{X}_0 = X_0 \cup \{\infty\}$.* $\diamondsuit$

Hence, in what follows, we may identify X_0^ν with X_0 and for $f \in \mathbb{C}(X_0)$, $\tilde{f}|X_0^\nu$ with f.

Recall that $M(r_2)$ has a k-fold cyclic cover, $\hat{M}(r_2)$, which is an S^1-bundle over the torus, where $k \in \{1,2,3,4,6\}$ is determined by the form of $\mathcal{B}$ (see the discussion following Corollary 7.4). According to Proposition 7.2, $\mathcal{B} \neq T$. Thus $k \neq 1$. Let $\hat{M}$ be the associated cover of M and $\hat{\pi}$ its fundamental group. Fix a component $\hat{T}$ of $\partial\hat{M}$. By construction, $\alpha(r_2) \in \hat{\pi} \subset \pi$.

Proposition 8.3 *Suppose that $x \in X_0$ is such that $Z_x(f_{\alpha(r_2)}) > Z_x(f_\beta)$ for some $\beta \in H_1(\partial M)$. Then $\rho \in t^{-1}(x)$ may be chosen so that one of the following three mutually exclusive possibilities occurs.*
(1) *$\rho(\hat{\pi})$ is parabolic, $\rho(\pi_1(\hat{T})) = \{\pm I\}$, and ρ is reducible with infinite, nonabelian image.*
(2) *$\rho(\hat{\pi}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Furthermore, if*
 (*a*) *$k = 2,4$, then the image of ρ is either D_2 and $\{p,q\} = \{2,2\}$, or D_4 and $\{p,q\} = \{2,4\}$.*
 (*b*) *$k = 3$, then the image of ρ is T and $\{p,q\} = \{2,3\}$ or $\{3,3\}$.*
 (*c*) *$k = 6$, then the image of ρ is T and $\{p,q\} = \{2,3\}$ or $\{3,3\}$.*
(3) *$\rho(\hat{\pi})$ is diagonalisable. Furthermore if $q > 2$, then $\rho(\pi) = D_q$, k is even, and $\{p,q\} = \{2,q\}$.*

Proof. Let R_0 be the unique 4-dimensional subvatiety of $R(M)$ for which $t(R_0) = X_0$ and suppose that $\rho \in t^{-1}(x) \cap R_0$. By assumption $Z_x(f_{\alpha(r_2)}) > Z_y(f_\beta)$ for some $\beta \in H_1(\partial M)$ and so if $\alpha(r_2)$ is one of the primitive elements of $H_1(\partial M) \cong \pi_1(\partial M) \subset \pi_1(M)$, then $\rho(\alpha(r_2)) = \pm I$ ([3, Proposition 4.8]). Hence ρ induces a representation $\pi_1(M(r_2)) \rightarrow PSL_2(\mathbb{C})$. By choice of X_0, x is a non-trivial character. Therefore we can assume that ρ is non-abelian or has image $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. On the other hand since $\hat{M}(\hat{r}_2)$ is an S^1-bundle over a torus, it is easy to argue that $\rho(\hat{\pi})$ is abelian. Hence it is either parabolic, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, or diagonalisable. Note also that by Proposition 6.2, $\rho(\pi_1(\partial M))$ is a nontrivial, finite, cyclic group. Further, $\rho(\hat{\pi})$ is a normal subgroup of $\rho(\pi)$.

Case 1. $\rho(\hat{\pi})$ is parabolic.

Say each element of $\rho(\hat{\pi})$ is upper-triangular. Then for each $\gamma \in \pi_1(\partial M)$ we have $\chi_\rho(\gamma) = \pm 2$. Since $\rho(\pi_1(\partial M))$ is a finite group, we must have $\rho(\pi_1(\hat{T})) \subset \{\pm I\}$. The normality of $\rho(\hat{\pi})$ in $\rho(\pi)$ shows that the image of ρ is contained in the normalizer of $\pm U_P$, which is the group of upper-triangular matrices. Thus ρ is reducible, and hence its image is non-abelian. This proves (1).

Case 2. $\rho(\hat{\pi}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Let $G = \rho(\pi)/\rho(\hat{\pi})$. There is a commutative diagram such that the right-hand vertical arrow is a surjection:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{\pi} & \longrightarrow & \pi & \longrightarrow & \mathbb{Z}/k & \longrightarrow & 1 \\ & & \rho| \downarrow & & \rho \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & \rho(\pi) & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Thus $G \cong \mathbb{Z}/j$ where j divides k. It follows that $\rho(\pi)$ is a subgroup of $PSL_2(\mathbb{C})$ of order $4j$ which contains $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ as a normal subgroup. Consideration of the finite subgroups of $PSL_2(\mathbb{C})$ shows that either

- $j = 1$ and ρ has image D_2, or
- $j = 2$ and ρ has image D_4, or
- $j = 3$ and ρ has image T, the tetrahedral group or
- $j = 6$ and ρ has image O, the octahedral group.

If the last possibility arises then $k = 6$ and so $\mathcal{B} = S^2(2,3,6)$. But according to Proposition 5.4, there is no representation $\rho : \pi \to PSL(2,\mathbb{C})$ with image O which factors through $\Delta(2,3,6)$. Thus $j \in \{1,2,3\}$.

Now assume $k = 2$ or 4 so that $j \in \{1,2\}$. If $j = 1$, then $\rho(\pi) = D_2$, and since by our choice of X_0, $\rho(a)$ has order p and $\rho(b)$ has order q, we see that $p = q = 2$. If $j = 2$, then $\rho(\pi) = D_4$. As $p \leq q$ and $\rho(a), \rho(b)$ generate this group, we must have $p = 2$ and $q = 4$.

If $k = 3$, then $\mathcal{B} = S^2(3,3,3)$ and so Proposition 5.2 shows that $j \neq 1$. Hence $j = 3$ and so $\rho(\pi) = T$. The values of p, q follow from the same lemma.

Finally assume that $k = 6$ so that $\mathcal{B} = S^2(2,3,6)$. Proposition 5.4 shows $j \neq 1$ and $j \neq 2$. Thus $j = 3$ and $\rho(\pi) = T$. The values of p, q follow from the same lemma.

Case 3. $\rho(\hat{\pi})$ is diagonalisable.

Since ρ is non-abelian, it has nondiagonalisable image. Thus as $\pi/\hat{\pi}$ is cyclic, $\rho(\hat{\pi}) \neq \{\pm I\}$. After conjugating, we can assume that $\rho(\hat{\pi})$ lies in D, the diagonal subgroup of $PSL(2,\mathbb{C})$. Then by the normality of $\hat{\pi}$ in π, $\rho(\pi)$ is a subgroup of

$$N = \{\pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \pm \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix} | \lambda, \mu \in \mathbb{C}^*\}.$$

Clearly one of $\rho(a)$ or $\rho(b)$ lie in $N \setminus D$ and hence has order 2. Thus $p = 2$ and if we assume that $q > 2$, then $\rho(b) \in D$ and so the image of ρ is D_q.

Finally observe that there is a commutative diagram in which the right-hand vertical arrow is surjective:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{\pi} & \longrightarrow & \pi & \longrightarrow & \mathbb{Z}/k & \longrightarrow & 1 \\ & & \rho| \downarrow & & \rho \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}/q & \longrightarrow & D_q & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1. \end{array}$$

Thus 2 divides k. $\diamondsuit$

Let $\alpha(r_1) \in H_1(\partial M)$ be one of the primitive classes associated to r_1 and $\gamma \in H_1(\partial M)$ any class such that $\{\alpha(r_1), \gamma\}$ is a basis of $H_1(\partial M)$. For each $\alpha \in H_1(\partial M) \setminus \{n\alpha(r_1) \mid n \in \mathbb{Z}\}$, define

$$Z(\alpha) = \{x \in \tilde{X}_0 \mid Z_x(f_\alpha) > 0\}$$

$$Z(\alpha)_+ = \{x \in Z(\alpha) \mid Z_x(f_\alpha) > Z_x(f_\gamma)\}$$

$$Z(\alpha)_0 = \{x \in Z(\alpha)_+ \mid \text{ if } x = \chi_\rho \text{ where } \rho \in R_0, \text{ then } \rho(\pi_1(\hat{T})) = \{\pm I\}\}.$$

Then $Z(\alpha)_0 \subset Z(\alpha)_+ \subset Z(\alpha) \subset X_0$, the last inclusion following from Lemma 8.1.

We shall divide the rest of the proof of Theorem C into the four cases $k = 2, 3, 4, 6$.

Case $\mathbf{k = 2}$. *If $k = 2$ and $\Delta(r_1, r_2) > 1$, then $\Delta(r_1, r_2) = 2$.*

Suppose first that $Z(\alpha(r_2))_0 = Z(\alpha(r_2))_+$. Then for each $\chi_\rho \in Z(\alpha(r_2))_+$, $\rho(\pi_1(\hat{T})) = \{\pm I\}$. Since $\pi_1(\hat{T})$ has index 1 or 2 in $\pi_1(\partial M)$, we have $\rho(\pi_1(\partial M)) \subset \mathbb{Z}/2$ and so by Proposition 6.2 (2), $\Delta(r_1, r_2) \leq 2$.

Assume then that there is some $\chi_\rho \in Z(\alpha(r_2))_+ \setminus Z(\alpha(r_2))_0$. Then $\rho(\pi_1(\hat{T}_1)) \neq \{\pm I\}$ and so one of possibilities (2) or (3) from Proposition 8.3 arises. As $k = 2$ and $q > 2$, $\rho(\pi) = D_q$. The proof of Proposition 8.3 shows that the quotient $\pi \rightarrow \pi/\hat{\pi} = \mathbb{Z}/2$ factors through ρ:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{\pi} & \longrightarrow & \pi & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1 \\ & & \rho| \downarrow & & \rho \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & \rho(\hat{\pi}) & \longrightarrow & D_q & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1. \end{array}$$

Thus $\ker(\rho) \subset \hat{\pi}$. Hence $\alpha(r_1), \alpha(r_2) \in \hat{\pi}$ and so Corollary 7.4 shows that $\partial \hat{M}$ cannot be connected. Thus it has two components. In particular, $\pi_1(\partial M) \subset \hat{\pi}$, and so by Proposition 6.2, $Z(\alpha(r_2))_0 = \emptyset$. Another application of Proposition 8.3 implies that each character in $Z(\alpha(r_2))_+$ is dihedral. Then by Lemma 5.1 we see that $Z(\alpha(r_2))_+ = \{\chi_\rho\}$. Hence by Theorem 4.3 we see that

$$\Delta(r_1, r_2)s = \|r_2\| \leq s + 1,$$

i.e. $\Delta(r_1, r_2) \leq 2$.

Case k = 3. *If* $k = 3$ *and* $\Delta(r_1, r_2) > 1$, *then* $\Delta(r_1, r_2) \leq 3$.

In this case $\mathcal{B} = S^2(3, 3, 3)$, and so Corollary 6.3, $\Delta(r_1, r_2) \leq 3$.

Case k = 4. *If* $k = 4$ *and* $\Delta(r_1, r_2) > 1$, *then* $\Delta(r_1, r_2) = 2$.

Here $\mathcal{B} = S^2(2, 4, 4)$, and so by our assumptions and Corollary 6.3 we see that $\Delta(r_1, r_2) = 2$ or 4. If the latter occurs, then Corollary 7.5 implies that $\partial \hat{M}$ is not connected. If it has four components, then as in the proof of the case $k = 2$, we have $Z(\alpha(r_2))_0$ is empty. Then by Proposition 8.3 and Lemma 5.1 we see that $Z(\alpha(r_2))_+ = \{\chi_\rho\}$ where ρ is dihedral. Hence $\Delta(r_1, r_2)s = \|r_2\| \leq s + 1$ (Theorem 4.3), which implies $\Delta(r_1, r_2) \leq 2$.

Finally assume that $\partial \hat{M}$ has two components. By Proposition 8.3, each reducible character which lies in $Z(\alpha(r_2))_+$ actually lies in $Z(\alpha(r_2))_0$. Since $\partial \hat{M}$ has two components, it follows that any such character lies in $Z(2\gamma)_+$. On the other hand, if $\chi_\rho \in Z(\alpha(r_2))_+ \setminus Z(\alpha(r_2))_0$ is irreducible, Proposition 8.3 implies that ρ is dihedral. Thus $Z(\alpha(r_2))_+ \subset Z(2\gamma)_+ \cup \{\chi_\rho\}$ where ρ is the unique dihedral character on X_0. Hence $\Delta(r_1, r_2)s = \|r_2\| \leq \|2\gamma\| + 1 = 2s + 1$. This implies $\Delta(r_1, r_2) \leq 3$, and as this distance divides 4, we obtain the desired bound.

Case k = 6. *If* $k = 6$ *and* $\Delta(r_1, r_2) > 1$, *then* $\Delta(r_1, r_2) \leq 3$.

This is proven much as the last case. Here $\mathcal{B} = S^2(2, 3, 6)$ and thus Corollary 6.3 shows that $\Delta(r_1, r_2) = 2, 3$ or 6. If the latter occurs, then Corollary 7.5 implies that $\partial \hat{M}$ is not connected, and hence has either $b = 2, 3$ or 6 components. In particular note that $Z(\alpha(r_2))_0 \subset Z(j\gamma)_+$ where $j = 6/b < 6$. Now consider any $x \in Z(\alpha(r_2))_+$. If x is reducible, then Proposition 8.3 shows that it lies in $Z(\alpha(r_2))_0$, and therefore in $Z(j\gamma)_+$. Hence if $\chi_\rho = x$, then $\rho(\pi_1(\partial M))$ is a subgroup of $\mathbb{Z}/j$ where $j < 6$. But if x is irreducible, then the same holds by Proposition 5.4. This contradicts Proposition 6.2. Hence $\Delta(r_1, r_2) = 2$ or 3.

This completes the proof of Theorem C. ♢

9 The proof of Theorem D

Consider a hyperbolic knot $K \subset S^3$ with exterior M and a slope r on ∂M such that $M(r)$ is a Seifert fibred manifold whose base orbifold is Euclidean. Orient the meridianal and longitudinal slopes of K and choose primitive classes $\alpha(\mu)$ and $\alpha(\lambda)$ accordingly,

Since $\mathbb{Z} \cong H_1(M) \to H_1(M(r)) \to H_1(\pi_1^{orb}(\mathcal{B}))$ is surjective, a straightforward calculation shows that the only possibility is for $\mathcal{B} = S^2(2, 3, 6)$, in which case $H_1(\pi_1^{orb}(\mathcal{B})) \cong \mathbb{Z}/6$. In particular, we may write

$$\alpha(r) = \pm(6p\alpha(\mu) + q\alpha(\lambda))$$

for some integers p, q.

If $|H_1(M(r))| = \infty$, then r is the longitudinal slope and $M(r)$ is a surface bundle over the circle [10, VI.13 and VI.34]. The fibre of this surface bundle must have genus 1 as $\mathcal{B}$ is Euclidean. Hence by [12, Corollary 8.19], K is a genus 1 fibred knot. But then it is either a trefoil or the figure-8 knot. The former is not hyperbolic while longitudinal surgery on the latter is not a Seifert fibred manifold. Thus we have proven

Lemma 9.1 *If r-surgery on a hyperbolic knot in S^3 yields a Seifert fibred manifold whose base orbifold is Euclidean, then*
(1) $\alpha(r) = \pm(6p\alpha(\mu) + q\alpha(\lambda))$ *for some integers p, q.*
(2) *r is not the longitudinal slope.*
(3) *$M(r)$ contains no closed, essential surfaces [10, VI.13].* ◇

We prove next that r is integral, i.e. $\Delta(r, \mu) = 1$ where μ is the meridinal slope of K.

If r is a boundary slope, then by [7, Theorem 2.0.3] there is a closed, essential surface $S \subset M$ which stays incompressible in $M(r')$ as long as $\Delta(r, r') > 1$. In particular, $\Delta(r, \mu) = 1$, that is, r is an integral slope. On the other hand if μ is a boundary slope, similar reasoning yields the same conclusion (cf. Lemma 9.1). We shall therefore assume that neither μ nor r is a boundary slope.

Let X_0 be a component of $X(M)$ which contains the character of a discrete, faithful representation. By §1.1 of [7], X_0 is a curve and $\| \cdot \|_{X_0}$ is a norm. Since μ is a cyclic filling slope which is not a boundary slope,

$$Z_x(\tilde{f}_\mu) \leq Z_x(\tilde{f}_\alpha)$$

for each $\alpha \in H_1(\partial M) \setminus \{0\}$ ([7, §1.1]). Hence setting $m_0 = \sum_{x \in \tilde{X}_0} \min\{Z_x(\tilde{f}_\alpha) \mid \alpha \in H_1(\partial M) \setminus 0\}$ we have $\|\mu\|_{X_0} = m_0$. According to Theorem C of [1],

$$\|\alpha(r)\|_{X_0} \leq m_0 + 2A$$

where A is the number of characters $\chi_\rho \in X_0$ of non-abelian representations ρ such that $\rho(\alpha(r)) = \pm I$. To determine A, we note that there is a central cyclic subgroup C of $\pi_1(M(r))$ and an exact sequence

$$1 \longrightarrow C \longrightarrow \pi_1(M(r)) \longrightarrow \pi_1^{orb}(S^2(2,3,6)) \longrightarrow 1$$

[10, VI.9]. Then any non-abelian ρ factors through a homomorphism $\pi_1^{orb}(S^2(2,3,6)) \cong \Delta(2,3,6) \to PSL(2,\mathbb{C})$. By Proposition 5.4, there are exactly two irreducible representations in $Hom(\Delta(2,3,6), PSL(2,\mathbb{C}))$, one with image T and the other with image D_3. A reducible representation in $Hom(\Delta(2,3,6), PSL(2,\mathbb{C}))$ may be conjugated to have image in U, and the associated diagonal representation factors through $H_1(\Delta(2,3,6)) \cong \mathbb{Z}/6$. It is shown in [6], [9] that if the image in $PSL(2,\mathbb{C})$ of $\alpha(\mu)$, under a diagonal representation, has order n, then the Alexander polynomial of the knot K has a root which is a primitive n^{th} root of unity. Since no root of the Alexander polynomial of a knot in S^3 can be a prime-power root of unity, the only possibility in our situation is for the image to be $\mathbb{Z}/6$. Up to conjugation, there are only two characters of diagonal representations with this image, thus $A \leq 5$ and it follows that

$$\|\alpha(r)\|_{X_0} \leq m_0 + 10.$$

Let B_0 denote the $\|\cdot\|_{X_0}$-ball of radius m_0. According to §7 of [2], if $x\alpha(\mu)+y\alpha(\lambda) \in \partial B_0$ then our hypotheses imply that

$$|y| \leq \begin{cases} 1 & \text{if } m_0 = 4 \\ \frac{6}{5} & \text{if } m_0 = 6 \\ 2 & \text{if } m_0 \geq 8. \end{cases}$$

Hence since $\alpha(r) = \pm(6p\alpha(\mu) + q\alpha(\lambda))$ and $\alpha(r) \in (1 + \frac{10}{m_0})B_0$, we obtain

- $m_0 = 4$ implies $|q| \leq \frac{7}{2}$.
- $m_0 = 6$ implies $|q| \leq \frac{16}{5}$.
- $m_0 \geq 8$ implies $|q| \leq \frac{9}{2}$.

As $\alpha(r)$ is primitive in $H_1(\partial M)$, $\gcd(q,6) = 1$. Thus the inequalities force us to conclude that $\Delta(r,\mu) = |q| = 1$, and so we are done.

Next we prove that there can be at most one slope on ∂M whose associated filling is a Seifert fibred manifold whose base orbifold is Euclidean. In order that we may apply the results of §13 of [5], we replace X_0 by C, its $H^1(M;\mathbb{Z}/2) \times Aut(\mathbb{C})$-orbit [5, §5].

Assume to the contrary that there are two slopes r_1 and r_2 whose associated fillings are Seifert fibred manifolds whose base orbifolds are Euclidean. Since these slopes are integral, Lemma 9.1 implies that we may write

$$\alpha(r_j) = \pm(6p_j\alpha(\mu) + \alpha(\lambda))$$

where $p_j \in \mathbb{Z}$ are distinct non-zero integers. In particular $\Delta(r_1, r_2) \geq 6$. Neither r_1 nor r_2 can be a boundary slope, for if, say, r_1 is, then [7, Theorem 2.0.3] implies that $M(r_2)$ admits a closed, incompressible surface, contrary to Lemma 9.1 (3). Thus neither r_1 nor r_2 is a boundary slope. It follows, as above, that

$$\|\alpha(r_j)\|_C \leq m_0 + 10.$$

Now fix a class $\beta \in H_1(\partial M)$ satisfying $\|\beta\|_C = s \geq m_0$ where s is defined to be

$$s = \min\{\ \|\alpha\|_C \mid \alpha \in H_1(\partial M) \setminus \{0\}\}.$$

At this stage, we cannot rule out the possibility that μ is a boundary slope and so it may be that $\beta \neq \pm\alpha(\mu)$. Let B_0 be the $\| \cdot \|_C$-ball of radius s. Clearly $\alpha(r_j) \in (1 + \frac{10}{s})B_0$ for $j = 1, 2$. Set $t = \frac{\|\beta\|_C}{\|\mu\|_C} \in (0, 1]$ and observe that since B_0 is convex and symmetric about 0, any line parallel to $\alpha(\mu)$ intersects it in a segment of length no greater than $2t$. Thus any such line intersects $(1 + \frac{10}{s})B_0$ in a segment of length no greater than $2t(1 + \frac{10}{s})$. But $\alpha(r_1), \alpha(r_2)$ lie in $(1 + \frac{10}{s})B_0$, so $2t(1 + \frac{10}{s}) \geq 6$, and therefore $s \leq 10/(\frac{3}{t} - 1) \leq 5$. Since s is even and at least 4 [5, Lemma 9.2] we deduce that $s = 4$.

The possible shapes for B_0 when $s = 4$ are described in §13 of [5]. If β is not a vertex of ∂B_0, then it is shown that β extends to an ordered basis $\{\beta, \gamma\}$ of $H_1(\partial M)$ in such a way that under the induced identification $H_1(\partial M; \mathbb{R}) \cong \mathbb{R}^2$, B_0 becomes a parallelogram with vertices $\pm(-1, \frac{2}{k+2}), \pm(1, \frac{2}{k+2})$ for some positive $k \equiv 2 \pmod 4$. As $\alpha(r_1), \alpha(r_2) \in (1 + \frac{10}{s})B_0$, the only possibility is for $k = 2$ and $\alpha(r_1), \alpha(r_2)$ to be, up to sign, the classes $-3\beta+\gamma$ and $3\beta+\gamma$. According to [5, Proposition 9.6 (2)], if i_* denotes the inclusion-induced homomorphism $H_1(\partial M; \mathbb{Z}/2) \to H_1(M; \mathbb{Z}/2)$, then under our present situation, $i_*(\beta) \neq 0$ while $i_*(\gamma) = 0$ (for the latter identity one must consider the Newton polygon of the A-polynomial of X_0, cf. [5, §13, Subcase I.1]). Then $0 \neq i_*(\beta) \pm 3i_*(\gamma) = i_*(\alpha(r_1)) = 6p_1 i_*(\alpha(r_1)) + i_*(\lambda) = 0$, a contradiction. Hence β must be a vertex of B_0. Examination of the possible shapes of B_0 listed in Subcase II of [5, §13] reveals that there are no pair of primitive classes in $(1+\frac{10}{s})B_0$ of distance at least 6. This final contradiction shows that there is at most one slope r on ∂M such that $M(r)$ is Seifert fibred with a Euclidean base orbifold. $\diamondsuit$

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