GRAPH MANIFOLDS $\mathbb{Z}$-HOMOLOGY 3-SPHERES AND TAUT FOLIATIONS

MICHEL BOILEAU AND STEVEN BOYER

Abstract. We show that a graph manifold which is a $\mathbb{Z}$-homology 3-sphere not homeomorphic to either S^3 or $\Sigma(2,3,5)$ admits a horizontal foliation. This combines with known results to show that the conditions of *not* being an L-space, of having a left-orderable fundamental group, and of admitting a co-oriented taut foliation, are equivalent for graph manifold $\mathbb{Z}$-homology 3-spheres.

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Throughout this paper we shall often use $\mathbb{Q}$-homology 3-sphere to abbreviate *rational homology 3-sphere* and $\mathbb{Z}$-homology 3-sphere to abbreviate *integer homology 3-sphere*.

Heegaard Floer theory is a package of 3-manifold homology invariants developed by Ozsváth and Szabó [OS3], [OS2] which provides relatively powerful tools to distinguish between manifolds. For a rational homology 3-sphere M, the simplest version of these invariants comes in the form of $\mathbb{Z}/2$-graded abelian groups $\widehat{HF}(M)$ whose Euler characteristic satisfies: $\chi(\widehat{HF}(M)) = |H_1(M)|$. In particular, rank $\widehat{HF}(M) \geq |H_1(M)|$.

Ozsváth and Szabó defined the family of *L-spaces* as the class of rational homology 3-spheres M for which the Heegaard Floer homology is as simple as possible. In other words, rank $\widehat{HF}(M) = |H_1(M)|$. Examples of L-spaces include the 3-sphere, lens spaces, and, more generally, manifolds admitting elliptic geometry. By Perelman's proof of the geometrisation conjecture, these are the closed 3-manifolds with finite fundamental group. Beyond these examples, Ozsváth and Szabó have shown that the 2-fold branched covering of any non-split alternating link is an L-space, thus providing infinitely many examples of hyperbolic L-spaces. None of these examples are integer homology 3-spheres, except for S^3 and the Poincaré sphere $\Sigma(2,3,5)$.

The last decade has shown that the conditions of *not* being an L-space, of having a left-orderable fundamental group, and of admitting a C^2 co-oriented taut foliation, are strongly correlated for an irreducible $\mathbb{Q}$-homology 3-sphere W:

- the three conditions are equivalent for non-hyperbolic geometric manifolds (cf. [BRW], [LS], [BGW]).

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- Ozsváth and Szábo have shown that if W admits a C^2 co-orientable taut foliation then it is not an L-space [OS1, Theorem 1.4].
- Calegari and Dunfield have shown that the existence of a co-orientable taut foliation on an atoroidal W implies that the commutator subgroup $[\pi_1(W), \pi_1(W)]$ is a left-orderable group [CD, Corollary 7.6].
- Boyer, Gordon and Watson have conjectured that W has a left-orderable fundamental group if and only if it is not an L-space and have provided supporting evidence in [BGW].
- Lewallen and Levine have shown that strong L-spaces do not have left-orderable fundamental groups [LL].

Recall that a *graph manifold* is a compact, irreducible, orientable 3-manifold whose Jaco-Shalen-Johannson (JSJ) pieces are Seifert fibred spaces. In this paper we focus on the case that W is an integer homology 3-sphere, and in particular one which is a graph manifold.

We begin with the statement of the *Heegaard-Floer Poincaré conjecture*, due to Ozsváth and Szábo.

Conjecture 0.1. (Ozsváth-Szábo) *An irreducible integer homology* 3*-sphere is an L-space if and only if it is either* S^3 *or the Poincaré homology* 3*-sphere* $\Sigma(2,3,5)$*.*

The truth of this striking conjecture would imply that among prime 3-manifolds, the 3-sphere is characterized by its Heegaard-Floer homology together with the vanishing of its Casson invariant (or even its μ invariant). It is known to hold in many instance, for example for integer homology 3-spheres obtained by surgery on a knot in S^3 [HW, Proposition 5]. It lends added interest to the questions:

- Which $\mathbb{Z}$-homology 3-spheres admit co-oriented taut foliations?
- Which $\mathbb{Z}$-homology 3-spheres have left-orderable fundamental groups?

We assume throughout this paper that foliations are C^2-smooth. The works of Eisenbud-Hirsh-Neumann [EHN], Jankins-Neumann [JN] and Naimi [Na] give necessary and sufficient conditions for a Seifert fibered 3-manifold to carry a horizontal foliation. It follows from their work that a Seifert manifold $\mathbb{Z}$-homology 3-sphere is an L-space if and only if it is either S^3 or the Poincaré homology 3-sphere $\Sigma(2,3,5)$ (cf. Proposition 2.2; see also [LS], [CM]). More recently, Clay, Lidman and Watson have shown that the fundamental group of a graph manifold $\mathbb{Z}$-homology 3-sphere is left-orderable if and only if it is neither S^3 nor $\Sigma(2,3,5)$ [CLW]. (By convention, the trivial group is *not* left-orderable.) The main result of this paper proves Ozsváth-Szábo conjecture for $\mathbb{Z}$-homology 3-spheres which are graph manifolds: we show that a graph manifold $\mathbb{Z}$-homology 3-sphere admits a co-oriented taut foliation if and only if it is neither S^3 nor $\Sigma(2,3,5)$. Before stating the precise version of our result, we need to introduce some definitions.

A *transverse loop* to a codimension one foliation $\mathcal{F}$ on a 3-manifold M is a loop in M which is everywhere transverse to $\mathcal{F}$. A codimension one foliation on a 3-manifold M is *taut* if each of its leaves meets a transverse loop.

A foliation is $\mathbb{R}$-*covered* if the leaf space of the pull-back foliation on the universal cover $\widetilde{M}$ of M is homeomorphic to the real line.

A foliation on a $\mathbb{Z}$-homology 3-sphere is always co-orientable.

We assume that the pieces of a graph manifold are equipped with a fixed Seifert structure. Note that this structure is unique up to isotopy when the graph manifold is a $\mathbb{Z}$-homology 3-sphere (cf. Proposition 1.1(2)).

A surface in a graph manifold W is *horizontal* if it is transverse to the Seifert fibres of each piece of W. It is *rational* if its intersection with each JSJ torus is a union of simple closed curves. A codimension 1 foliation of W is *horizontal*, respectively *rational*, if each of its leaves has this property. Horizontal foliations are obviously taut and they are known to be $\mathbb{R}$-covered [Br2, Proposition 7]. Rational foliations on graph manifold $\mathbb{Z}$-homology 3-spheres are necessarily horizontal (Lemma 2.1). Here is our main result.

Theorem 0.2. *Let W be a graph manifold which is a $\mathbb{Z}$-homology 3-sphere and suppose that W is neither S^3 nor $\Sigma(2,3,5)$. Then W admits a rational foliation.*

An action of a group G on the circle is called *minimal* if each orbit is dense.

A homomorphism $\rho : G \to \mathrm{Homeo}_+(S^1)$ is called *minimal* if the associated action on S^1 is minimal.

Corollary 0.3. *Let W be a graph manifold which is a $\mathbb{Z}$-homology 3-sphere and suppose that W is neither S^3 nor $\Sigma(2,3,5)$. Then*

(1) *W is not an L-space.*

(2) *$\pi_1(W)$ admits a minimal homomorphism ρ with values in Homeo$_+(S^1)$ whose image contains a nonabelian free group.*

(3) (Clay-Lidman-Watson [CLW]) *$\pi_1(W)$ is left-orderable.*

Proof. Since W is a $\mathbb{Z}$-homology 3-sphere, the taut foliation $\mathcal{F}$ given by Theorem 0.2 is co-orientable. Thus W cannot be an L-space [OS1, Theorem 1.4]. Assertion (3) is a consequence of the assertion (2); since $H^2(W) \cong \{0\}$, the homomorphism $\pi_1(W) \to \mathrm{Homeo}_+(S^1)$ lifts to a homomorphism $\pi_1(W) \to \widetilde{\mathrm{Homeo}_+(S^1)} \leq \mathrm{Homeo}_+(\mathbb{R})$ with non-trivial image. Theorem 1.1(1) of [BRW] now implies that $\pi_1(W)$ is left-orderable. (This also follows from the fact that $\pi_1(W)$ acts non-trivially on $\mathbb{R}$ by orientation-preserving homeomorphisms since $\mathcal{F}$ is co-oriented and $\mathbb{R}$-covered [Br2, Proposition 7].) Finally, assertion (2) follows from Lemma 0.4 below. $\square$

Lemma 0.4. *Let M be a $\mathbb{Z}$-homology 3-sphere which admits a taut foliation $\mathcal{F}$. Then $\pi_1(M)$ admits a minimal homomorphism $\rho : \pi_1(M) \to$ Homeo$_+(S^1)$ whose image contains a nonabelian free group.*

Proof. A theorem of Margulis [Gh, Corollary 5.15] shows that the image of a minimal representation $\rho : \pi_1(M) \to \text{Homeo}_+(S^1)$ is either abelian or contains a nonabelian free group. The former is not possible since $\pi_1(M)$ is perfect, so to complete the proof we must show that such a representation exists.

Since M is a $\mathbb{Z}$-homology 3-sphere, the co-orientability of $\mathcal{F}$ implies that it has no compact leaves ([Go, Proposition 2.1]. See also [God, Part II, Lemma 3.8]). Then by Plante's results [Pla, Theorem 6.3, Corollaries 6.4 and 6.5], every leaf of $\mathcal{F}$ has exponential growth, and thus $\mathcal{F}$ admits no non-trivial holonomy-invariant transverse measure. Hence Candel's uniformization theorem [CC1, Theorem 12.6.3] applies to show that there is a Riemannian metric on M such that $\mathcal{F}$ is leaf-wise hyperbolic. In this setting, Thurston's universal circle construction yields a homomorphism ρ_{univ} of $\pi_1(M)$ with values in $\text{Homeo}_+(S^1)$ [CD].

If L denotes the leaf space of the pullback $\widetilde{\mathcal{F}}$ of the foliation $\mathcal{F}$ to the universal cover $\widetilde{M}$ of M, then either L is Hausdorff and $\mathcal{F}$ is $\mathbb{R}$-covered or L has branching points. We treat these cases separately.

First suppose that $\mathcal{F}$ is $\mathbb{R}$-covered. Then Proposition 2.6 of [Fen] implies that after possibly collapsing at most countably many foliated I-bundles, we can suppose that $\mathcal{F}$ is a minimal foliation (i.e. each leaf is dense). If $\mathcal{F}$ is ruffled ([Ca1, Definition 5.2.1]), Lemma 5.2.2 of [Ca1] shows that the associated action of $\pi_1(M)$ on the universal circle of $\mathcal{F}$ is minimal, so we take $\rho = \rho_{univ}$. If $\mathcal{F}$ is not ruffled, it is uniform and so by [Ca1, Theorem 2.1.7], after possibly blowing down some pockets of leaves, we can suppose that $\mathcal{F}$ slithers over the circle ([Ca1, Definition 2.1.6]). Thus if $\widetilde{M}$ denotes the universal cover of M, there is a locally trivial fibration $\widetilde{M} \to S^1$ whose fibres are unions of leaves of the pull back of $\mathcal{F}$ to $\widetilde{M}$. Further, the deck transformations of the cover $\widetilde{M} \to M$ act by bundle maps and so determine a homomorphism of $\pi_1(M)$ with values in $\text{Homeo}_+(S^1)$. If this representation has a finite orbit, then a finite index subgroup of $\pi_1(M)$ acts freely and properly discontinuously on a fibre of the fibration $\widetilde{M} \to S^1$. This is impossible as each fibre is a surface and a finite index subgroup of $\pi_1(M)$ is the fundamental group of a closed 3-manifold. Therefore by [Gh, Propositions 5.6 and 5.8], the associated action on S^1 is semiconjugate to a minimal action $\rho : \pi_1(M) \to \text{Homeo}_+(S^1)$.

In the case that L branches, $\rho_{univ} : \pi_1(M) \to \text{Homeo}_+(S^1)$ is faithful. (See the last line of the first paragraph of [CD, §6.28].) If it branches in both directions, an application of [Ca3, Lemma 5.5.3] to any finite cover of M implies that $\rho_{univ}(\pi_1(M))$ has no periodic orbit. The conclusion then follows as above from [Gh, Propositions 5.7 and 5.8]. Thus we are left with the case where $\mathcal{F}$ has one-sided branching, say in the negative direction (cf. [Ca2]). As in the case of $\mathbb{R}$-covered foliations, we can suppose every leaf dense by [Ca2, Theorem 2.2.7]. We need only show that the action associated to the faithful representation $\rho_{univ} : \pi_1(M) \to \text{Homeo}_+(S^1)$ has no finite orbits as otherwise [Mat, Theorem 1.2] implies that ρ_{univ} is semiconjugate to an abelian representation, which is trivial since $\pi_1(M)$ is perfect. Hence the action of $\rho_{univ}(\pi_1(M))$ on S^1 has an uncountable compact set Σ of global fixed points. By [Ca2, Theorem 3.2.2] the image of Σ is dense in almost every circle at infinity of the leaves of $\widetilde{\mathcal{F}}$, and hence in S^1_{univ} by the construction of the universal circle, see [Ca2, Theorem 3.4.1]. This contradicts the faithfullness of ρ_{univ}. When M is hyperbolic, we can also obtain a contradiction to the existence of a finite orbit from that of topologically pseudo-Anosov elements of $\rho_{univ}(\pi_1(M))$ which have at most finitely many fixed points in S^1_{univ}, see [Ca2, Lemma 4.2.5]. This completes the proof of the lemma and therefore that of Corollary 0.3. □

The conclusion of Lemma 0.4 combines with the two questions above to motivate the following question:

Question 0.5. *For which aspherical $\mathbb{Z}$-homology 3-spheres M does $\pi_1(M)$ admit a minimal representation to $Homeo_+(S^1)$?*

Our discussion above yields the following corollary.

Corollary 0.6. *The following conditions are equivalent for W a graph manifold $\mathbb{Z}$-homology 3-sphere:*

(a) *$\pi_1(W)$ is left-orderable.*

(b) *W is not an L-space.*

(c) *W admits a rational foliation.* □

Sections 1 and 2 contain background material on, respectively, the pieces of graph manifold $\mathbb{Z}$-homology 3-spheres and strongly detected slopes on the boundaries of Seifert fibered $\mathbb{Z}$-homology solid tori. Theorem 0.2 is proven in §3.

1. Pieces of graph manifold $\mathbb{Z}$-homology 3-spheres

A torus T in a $\mathbb{Z}$-homology 3-sphere W splits W into two $\mathbb{Z}$-homology solid tori X and Y. Let λ_X and λ_Y be primitive classes in $H_1(T)$ which are trivial in $H_1(X)$ and $H_1(Y)$ respectively. The associated slopes on T, which we also denote by λ_X and λ_Y, are well-defined. We refer to these slopes as the *longitudes* of X and Y. A simple homological argument shows that $X(\lambda_Y)$ and $Y(\lambda_X)$ are $\mathbb{Z}$-homology 3-spheres while $X(\lambda_X)$ and $Y(\lambda_Y)$ are $\mathbb{Z}$-homology $S^1 \times S^2$'s.

Let K be a knot in a $\mathbb{Z}$-homology 3-sphere with exterior M_K. The *longitude* λ_K of K is the longitude of M_K. The *meridian* μ_K of K is the longitude of the tubluar neighbourhood $\overline{W \setminus M_K}$ of K. The pair μ_K, λ_K forms a basis for $H_1(\partial M_K)$.

Lemma 1.1. *Suppose that T is a torus in a $\mathbb{Z}$-homology 3-sphere W and let X, Y be the components of W cut open along T. Suppose that $Y = P \cup Y_0$ where $P \cap Y_0 = \partial P \setminus T$ and P is a Seifert manifold or than $S^1 \times D^2$ and $S^1 \times S^1 \times I$. Then*

(1) *the underlying space B of the base orbifold of P is planar, hyperbolic, and the multiplicities of the exceptional fibres in P are pairwise coprime;*

(2) *P has a unique Seifert structure;*

(3) *if ϕ is the P-fibre slope on T and P has an exceptional fibre, then $\phi \notin \{\lambda_X, \lambda_Y\}$.*

Proof. If B is non-orientable, or is orientable of positive genus, or has two exceptional fibres whose multiplicities are not coprime, then W admits a degree 1 map to a manifold with non-trivial first homology group, which is impossible. Thus (1) holds. Assertion (2) is a consequence of (1) and the classification of Seifert structures on 3-manifolds (cf. [Ja, §VI.16]). Finally observe that as $H_1(Y(\lambda_X)) \cong \{0\}$ and $H_1(Y(\lambda_Y)) \cong \mathbb{Z}$, neither $Y(\lambda_X)$ nor $Y(\lambda_Y)$ has a lens space summand. On the other hand, if P has an exceptional fibre, then $Y(\phi)$ does have such a summand. This completes the proof. □

2. Horizontal foliations and strongly detected slopes in Seifert fibred $\mathbb{Z}$-homology solid tori

The set $\mathcal{S}_{rat}(T)$ of (rational) slopes on a torus T is naturally identified with the subset $P(H_1(T;\mathbb{Q}))$ of the projective space $\mathcal{S}(T) = P(H_1(T;\mathbb{R})) \cong S^1$. We endow $\mathcal{S}_{rat}(T)$ with the induced topology as a subset of $\mathcal{S}(T)$. The projective class of an element $\alpha \in H_1(T;\mathbb{R})$ will be denoted by $[\alpha]$, though we sometimes abuse notation and write $\alpha \in \mathcal{S}_{rat}(T)$ for a non-zero class α in $H_1(T)$.

For a 3-manifold X whose boundary is a torus T, set $\mathcal{S}_{rat}(X) = \mathcal{S}_{rat}(T)$. We say that $[\alpha] \in \mathcal{S}_{rat}(X)$ is *strongly detected* by a taut foliation $\mathcal{F}$ on X if $\mathcal{F}$ restricts on T to a fibration of slope $[\alpha]$. In this case we call $[\alpha]$ the *slope of* $\mathcal{F}$.

When X is Seifert fibred and T is a boundary component of X, we say that $[\alpha] \in \mathcal{S}_{rat}(X)$ is *horizontal* if it is not the fibre slope.

Lemma 2.1. *Suppose that $\mathcal{F}$ is a co-oriented taut foliation on a $\mathbb{Z}$-homology 3-sphere W.*

(1) *If $\mathcal{F} \cap T$ is a fibration by simple closed curves for some boundary component T of a piece P of W, then the slope of T represented by these curves is horizontal.*

(2) *If $\mathcal{F}$ is rational, then it is horizontal.*

Proof. Suppose that $\mathcal{F} \cap T$ is a fibration by simple closed curves of vertical slope ϕ and let P' be the manifold obtained by the (T,ϕ)-Dehn filling P. Since P has base orbifold of the form $B(a_1,\dots,a_n)$ for a planar surface B (Lemma 1.1), P' is homeomorphic to $(\#_{i=1}^n L_{a_i})\#(\#_{j=1}^{r-1} S^1 \times D^2)$ where $r = |\partial P| - 1$. On the other hand, $\mathcal{F}$ extends to a co-oriented taut foliation $\mathcal{F}'$ on P' and so P' is either prime or $S^2 \times I$ (see e.g. [CC2, Corollary 9.1.9]). As the latter case does not arise, we have $n + (r-1) \leq 1$. Thus P is either a solid torus or $S^1 \times S^1 \times I$, which is impossible for a piece of W. Thus part (1) the lemma holds.

Next suppose that $\mathcal{F}$ is rational and let P be a piece of W. By part (1), for each boundary component T of P, $\mathcal{F} \cap T$ is a fibration by simple closed horizontal curves. Since the base orbifold of P is planar (Lemma 1.1), we can now argue as in the proof of [Br1, Proposition 3] to see that if $\mathcal{F}$ is not horizontal in P, it contains a vertical, separating leaf homeomorphic to a torus. This is impossible as it contradicts the assumption that $\mathcal{F}$ is co-oriented and taut ([Go, Proposition 2.1]). Thus part (2) holds. □

Here is a special case of our main theorem.

Proposition 2.2. *Let W be a Seifert fibred $\mathbb{Z}$-homology 3-sphere. Then the following conditions are equivalent:*

(a) $\pi_1(W)$ is left-orderable.

(b) W is not an L-space.

(c) W admits a co-oriented horizontal foliation.

Further, W satisfies these conditions if and only if it is neither S^3 nor $\Sigma(2,3,5)$.

Proof. Lemma 1.1 implies that the base orbifold $\mathcal{B}$ of W has underlying space S^2. In this case the equivalence of (a) and (c) was established in [BRW], while those of (b) and (c) was established in [LS] (see also [CM]).

Next suppose that W is either S^3 or $\Sigma(2,3,5)$. Then the fundamental group of W is finite so its fundamental group is not left-orderable, W is an L-space [OS4, Proposition 2.3] and therefore it does not admit a co-oriented horizontal foliation [OS1, Theorem 1.4].

Conversely suppose that $W \neq S^3, \Sigma(2,3,5)$. Equivalently, $\chi(\mathcal{B}) \leq 0$. If $\chi(\mathcal{B}) = 0$, $\mathcal{B}$ would support a Euclidean structure and would therefore be one of $S^2(2,3,6), S^2(2,4,4), S^2(3,3,3)$ or $S^2(2,2,2,2)$. But then $H_1(\mathcal{B}) \neq \{0\}$ contrary to the fact that $H_1(W) = \{0\}$. Thus $\chi(\mathcal{B}) < 0$, so $\mathcal{B}$ is hyperbolic. It follows that there is a discrete faithful representation $\pi_1(\mathcal{B}) \to PSL_2(\mathbb{R})$ and therefore a non-trivial homomorphism $\pi_1(W) \to PSL_2(\mathbb{R})$. As $H^2(W) = \{0\}$, this homomorphism factors through $\widetilde{SL_2} \leq \widetilde{Homeo_+}(S^1) \leq \text{Homeo}_+(\mathbb{R})$. Hence $\pi_1(W)$ is left-orderable (cf. [BRW, Theorem 1.1(1)]). It follows from the first paragraph of the proof that W is not an L-space and it admits a co-oriented horizontal foliation. □

Let X be a Seifert fibered $\mathbb{Z}$-homology solid torus and set

$$\mathcal{D}^{str}_{rat}(X) = \{[\alpha] \in \mathcal{S}_{rat}(X) : [\alpha] \text{ is strongly detected by a rational foliation on } X\}$$

Clearly $\mathcal{D}^{str}_{rat}(X)$ coincides with the set of slopes α on ∂X such that $X(\alpha)$ admits a horizontal foliation (cf. Lemma 2.1). The work of a number of people ([EHN], [JN], [Na]) shows that the latter set is completely determined by the Seifert invariants of $X(\alpha)$. In particular, we have the following result.

Proposition 2.3. *Let X be a Seifert manifold which is a $\mathbb{Z}$-homology solid torus with incompressible boundary. Then there is a connected open proper subset U of $\mathcal{S}(X)$ such that*

(1) $\mathcal{D}^{str}_{rat}(X) = U \cap \mathcal{S}_{rat}(X)$.

(2) If X is not contained in S^3 and $\Sigma(2,3,5)$, then U contains all the slopes α on ∂X such that $X(\alpha)$ is a $\mathbb{Z}$-homology 3-sphere.

Proof. The base orbifold of X is of the form $D^2(a_1, a_2, \ldots, a_n)$ where n and each a_i are at least 2. Since X is a $\mathbb{Z}$-homology solid torus, the a_i are pairwise coprime. We can assume that the

Seifert invariants $(a_1, b_1), \ldots (a_n, b_n)$ satisfy $0 < b_i < a_i$ for each i. Then

$$\pi_1(X) = \langle y_1, y_2, \ldots, y_n, h : h \text{ central}, \; y_1^{a_1} = h^{b_1}, y_2^{a_2} = h^{b_2}, \ldots, y_n^{a_n} = h^{b_n} \rangle$$

Further,

$$h^* = y_1 y_2 \ldots y_n$$

is a peripheral element of $\pi_1(X)$ *dual* to h. That is, $H_1(\partial X) = \pi_1(\partial X)$ is generated by h and h^*.

Set $\gamma_i = \frac{b_i}{a_i}$. If $\alpha = ah + bh^*$ is a slope on ∂X, then $X(\alpha)$ has Seifert invariants $(0; 0; \gamma_1, \ldots, \gamma_n, \frac{a}{b})$ and therefore also $(0; -\lfloor \frac{a}{b} \rfloor; \gamma_1, \ldots, \gamma_n, \{\frac{a}{b}\})$ where $\{\frac{a}{b}\} = \frac{a}{b} - \lfloor \frac{a}{b} \rfloor$. According to [EHN], [JN], [Na], $X(\alpha)$ admits a horizontal foliation if and only if one of the following conditions holds:

(1) $1 - n < \frac{a}{b} < -1$;

(2) $\lfloor \frac{a}{b} \rfloor = -1$ and there are coprime integers $0 < A < M$ and some permutation $(\frac{A_1}{M}, \frac{A_2}{M}, \ldots, \frac{A_{n+1}}{M})$ of $(\frac{A}{M}, \frac{M-A}{M}, \frac{1}{M}, \ldots, \frac{1}{M})$ such that $\gamma_i < \frac{A_i}{M}$ for $1 \leq i \leq n$ and $\{\frac{a}{b}\} < \frac{A_{n+1}}{M}$;

(3) $\lceil \frac{a}{b} \rceil = 1 - n$ and there are coprime integers $0 < A < M$ and some permutation $(\frac{A_1}{M}, \frac{A_2}{M}, \ldots, \frac{A_{n+1}}{M})$ of $(\frac{A}{M}, \frac{M-A}{M}, \frac{M-1}{M}, \ldots, \frac{M-1}{M})$ such that $\gamma_i > \frac{A_i}{M}$ for $1 \leq i \leq n$ and $\{\frac{a}{b}\} > \frac{A_{n+1}}{M}$.

Let $V \subset \mathbb{R}$ be the convex hull of the set of rationals $\frac{a}{b}$ determined these three conditions. We leave it to the reader to verify that V is an open interval if and only if $n > 2$ or $n = 2$ and $\gamma_1 + \gamma_2 \neq 1$ (cf. [BC, Proposition A.4]). On the other hand, our hypothesis that X is a $\mathbb{Z}$-homology solid torus rules out the possibility that $n = 2$ and $\gamma_1 + \gamma_2 = 1$. Thus if U is the connected proper subset of $\mathcal{S}(X)$ corresponding to V under the identification $\frac{a}{b} \leftrightarrow [ah + bh^*]$, then U is open and $\mathcal{D}^{str}_{rat}(X) = U \cap \mathcal{S}_{rat}(X)$, which proves (1). Part (2) then follows from Proposition 2.2. ∎

The case when X is contained in S^3 or $\Sigma(2,3,5)$ is dealt with in the following two propositions.

Proposition 2.4. *Let X be a (p,q) torus knot exterior where $p, q \geq 2$ and fix a meridian-longitude pair μ, λ for X such that the Seifert fibre of X has slope $pq\mu + \lambda$. Identify the non-meridional slopes on ∂X with $\mathbb{Q}$ in the usual way: $m\mu + n\lambda \leftrightarrow \frac{m}{n}$. Then there is a co-oriented horizontal foliation of slope $r \in \mathbb{Q}$ in X if and only if $r < pq - (p+q)$. In particular, the result holds for each $r < 1$.*

Proof. Fix integers a, b such that $1 = bp + aq$ and $0 < a < p$. Note that $b < 0$ but $p(q+b) > aq + pb = 1$, so $0 < b_0 = b + q < q$. There is a Seifert structure on X with base orbifold $D^2(p,q)$ where the two exceptional fibres have Seifert invariants (p, a) and (q, b). Hence if $r = \frac{n}{m} \neq pq$ is a reduced rational fraction where $m > 0$, the Dehn filling $X(r)$ of X is a Seifert fibred manifold with Seifert invariants $(0; 0; \frac{a}{p}, \frac{b}{q}, \frac{m}{n - mpq}) = (0; 0; \frac{a}{p}, \frac{b}{q}, \frac{1}{r - pq})$. Then $X(r)$ also has a Seifert structure with Seifert invariants $(0; 1 - \lfloor \frac{1}{pq - r} \rfloor; \frac{a'}{p}, -\frac{b}{q}, \{\frac{1}{pq-r}\})$ where $a' = p - a$. Assume that $\{\frac{1}{pq-r}\} \neq 0$. Then arguing as in the proof of Proposition 2.3, if $X(r)$ admits a horizontal foliation, we have $\lfloor \frac{1}{pq-r} \rfloor \in \{-1, 0\}$. If $\lfloor \frac{1}{pq-r} \rfloor = -1$, then $X(r)$ has Seifert invariants $(0; 1; \frac{a}{p}, \frac{b_0}{q}, 1 - \{\frac{1}{pq-r}\})$ and there are positive integers A_1, A_2 coprime with an integer

$M < A_1, A_2$ such that $\frac{a}{p} < \frac{A_1}{M}, \frac{b_0}{q} < \frac{A_2}{M}$ and $\frac{A_1+A_2}{M} \leq 1$. But this is impossible since then $\frac{A_1+A_2}{M} > \frac{a}{p} + \frac{b_0}{q} = 1 + \frac{1}{pq}$. Hence $\lfloor \frac{1}{pq-r} \rfloor = 0$ and therefore $0 < \frac{1}{pq-r} < 1$ and $X(r)$ has Seifert invariants $(0; 1; \frac{a'}{p}, -\frac{b}{q}, \{\frac{1}{pq-r}\})$. It follows that $r < pq - 1$. A straightforward, though tedious, calculation yields the bound stated in the proposition. This calculation can be avoided if we are willing to appeal to results from Heegaard-Floer theory. For instance, the (p, q) torus knot K is an L-space knot since $pq - 1$ surgery on K yields a lens space. Hence as the genus of K is $\frac{1}{2}(p-1)(q-1)$, $K(r)$ is an L-space if and only if $r \geq pq - (p+q)$ ([OS5, Proposition 9.5]. See also [Hom, Fact 2, page 221]). Hence, according to Proposition 2.2, $X(r)$ admits a horizontal foliation if and only if $r < pq - (p+q)$. □

Proposition 2.5. *Let X be a Seifert manifold which is the exterior of a knot K in $\Sigma(2,3,5)$, the Poincaré homology 3-sphere.*

(1) *K is a fibre in a Seifert structure on $\Sigma(2,3,5)$.*

(2) *X has base orbifold $D^2(2,3), D^2(2,5), D^2(3,5)$, or $D^2(2,3,5)$.*

(3) *Suppose that K has multiplicity $j \geq 1$. Then there is a choice of meridian μ and longitude λ of K such that X admits a horizontal foliation detecting the slope $a\mu + b\lambda$ if and only if*

$$\frac{a}{b} > -29 \text{ if } j = 1$$

and

$$\frac{a}{b} < \begin{cases} 7 & \text{if } j = 2 \\ 3 & \text{if } j = 3 \\ 1 & \text{if } j = 5 \end{cases}$$

In particular, there is a sequence of slopes α_n on ∂X which converge projectively to the meridian of K such that X admits a horizontal foliation of slope α_n for each n.

(4) *There is a unique slope on ∂X such that $X(\alpha) \cong \Sigma(2,3,5)$.*

Proof. The boundary of X is incompressible since the fundamental group of $\Sigma(2,3,5)$ is non-abelian. It follows from Lemma 1.1 that X has base orbifold of the form $D^2(a_1, a_2, \ldots, a_n)$ where each $a_i \geq 2$ and $n \geq 2$. Since $\Sigma(2,3,5)$ has no lens space summands, the meridian of K cannot be the fibre slope of X. Thus the Seifert structure on X extends to one on $\Sigma(2,3,5)$ in which K is a fibre. This implies assertions (1) and (2) of the proposition.

Next we deal with (3). Let K_j be a fibre of multiplicity j in $\Sigma(2,3,5)$ for $j = 1,2,3,5$ and let X_0 be the exterior of $K_1 \cup K_2 \cup K_3 \cup K_5$. Denote by T_j the boundary component of X_0 corresponding to K_j and by μ_j the meridional slope of K_j on T_j. Let ϕ_j be the fibre slope on T_j. Note that X_0 is a trivial circle bundle over a 4-punctured sphere Q. Orient Q. Since $\Sigma(2,3,5)$ has Seifert invariants $(0; -1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5})$, there is a section of this bundle with image $\widetilde{Q} \subset X_0$ such that if σ_j is the slope of $\widetilde{Q} \cap T_j$ oriented by the induced orientation from Q. Orient the fibre of X_0 so that for each j, $\sigma_j \cdot \phi_j = 1$.

There is a horizontal foliation on X_j detecting the slope $n\sigma_j + m\phi_j$ if and only if the $(n\sigma_j + m\phi_j)$-Dehn filling of X_j admits a horizontal foliation. The latter problem has been resolved in the papers [EHN], [JN], and [Na]. First we prove that X_j has a horizontal foliation if and only if $\frac{m}{n} \in (-1,0)$ for $j=1$ and $\frac{m}{n} \in (0, \frac{1}{j})$ for $j>1$.

The exterior X_j of K_j is obtained from X_0 by performing the (T_k, μ_k)-filling for $k \neq j$. It follows that the $(n\sigma_j + m\phi_j)$-Dehn filling of X_j has Seifert invariants

- $(0; -1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{m}{n})$ if $j=1$;
- $(0; -1, \frac{1}{3}, \frac{1}{5}, \frac{m}{n})$ if $j=2$;
- $(0; -1, \frac{1}{2}, \frac{1}{5}, \frac{m}{n})$ if $j=3$;
- $(0; -1, \frac{1}{2}, \frac{1}{3}, \frac{m}{n})$ if $j=5$.

Suppose first that $j=1$. If $n=0$, $X_1(n\sigma_1 + m\phi_1) = X_1(\phi_1)$ is a connected sum of lens spaces of orders $2, 3$, and 5 so does not admit a taut foliation (see e.g. [CC2, Corollary 9.1.9]). If $|n| = 1$, then $\Delta(n\sigma_1 + m\phi_1, \phi_1) = 1$, so $X_1(n\sigma_1 + m\phi_1)$ admits a Seifert structure with base orbifold $S^2(2,3,5)$. Hence it has a finite fundamental group and so does not admit a horizontal foliation. Assume then that $|n|>1$, and therefore $0 < \{\frac{m}{n}\} = \frac{m}{n} - \lfloor \frac{m}{n} \rfloor < 1$. In this case, $X_1(n\sigma_1 + m\phi_1)$ has Seifert invariants $(0; \lfloor \frac{m}{n} \rfloor - 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$. Theorem 2 of [JN] implies that when $\lfloor \frac{m}{n} \rfloor = -1$ there is a horizontal foliation for all values of $\{\frac{m}{n}\}$. In other words, whenever $\frac{m}{n} \in (-1, 0)$. It also shows that there is no horizontal foliation when $\lfloor \frac{m}{n} \rfloor < -2$ or $\lfloor \frac{m}{n} \rfloor > 0$

If $\lfloor \frac{m}{n} \rfloor = 0$, then $X_1(n\sigma_1 + m\phi_1)$ has Seifert invariants $(0; -1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$. Conjecture 2 of [JN] was verified in [Na] so in this case $X_1(n\sigma_1 + m\phi_1)$ has a horizontal foliation if and only if we can find coprime integers $0 < A < M$ such that for some permutation $\{\frac{a_1}{m_1}, \frac{a_2}{m_2}, \frac{a_3}{m_3}, \frac{a_4}{m_4}\}$ of $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\}\}$ satisfies $\frac{a_1}{m_1} < \frac{1}{M}, \frac{a_2}{m_2} < \frac{1}{M}, \frac{a_3}{m_3} < \frac{A}{M}$ and $\frac{a_4}{m_4} < \frac{M-A}{M}$. It is elementary to verify that there is no such pair A, M.

If $\lfloor \frac{m}{n} \rfloor = -2$, then $X_1(n\sigma_1 + m\phi_1)$ has Seifert invariants $(0; -3, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$ and therefore also $(0; -1, \frac{1}{2}, \frac{2}{3}, \frac{4}{5}, 1 - \{\frac{m}{n}\})$. As in the previous paragraph, $X_1(n\sigma_1 + m\phi_1)$ never admits a horizontal foliation on this case. We conclude that $X_1(n\sigma_1 + m\phi_1)$ admits a horizontal foliation if and only if $\frac{m}{n} \in (-1, 0)$.

We proceed similarly when $j=2$. As above we can rule out the cases $n=0$ and $|n|=1$. When $|n|>1$, so $0 < \{\frac{m}{n}\} = \frac{m}{n} - \lfloor \frac{m}{n} \rfloor < 1$, $X_2(n\sigma_2 + m\phi_2)$ has Seifert invariants $(0; \lfloor \frac{m}{n} \rfloor - 1, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$. By Theorem 2 of [JN], there is no horizontal foliation when $\lfloor \frac{m}{n} \rfloor < -1$ or $\lfloor \frac{m}{n} \rfloor > 0$. If $\lfloor \frac{m}{n} \rfloor = 0$, $X_2(n\sigma_2 + m\phi_2)$ has Seifert invariants $(0; -1, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$. Conjecture 2 of [JN] was verified in [Na] so in this case $X_2(n\sigma_2 + m\phi_2)$ has a horizontal foliation if and only if we can find coprime integers $0 < A < M$ such that for some permutation $\{\frac{a_1}{m_1}, \frac{a_2}{m_2}, \frac{a_3}{m_3}\}$ of $\{\frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\}\}$ satisfies $\frac{a_1}{m_1} < \frac{1}{M}, \frac{a_2}{m_2} < \frac{A}{M}$ and $\frac{a_3}{m_3} < \frac{M-A}{M}$. It is elementary to verify that there is a solution to this problem if and only if $\frac{m}{n} \in (0, \frac{1}{2})$. On the other hand, if $\lfloor \frac{m}{n} \rfloor = -1$, $X_1(n\sigma_1 + m\phi_1)$ has Seifert invariants $(0; -2, \frac{1}{3}, \frac{1}{5}, \{\frac{m}{n}\})$ and therefore $(0; -1, \frac{2}{3}, \frac{4}{5}, 1 - \{\frac{m}{n}\})$. As above, $X_2(n\sigma_2 + m\phi_2)$ never admits a horizontal foliation on this case. We conclude that $X_2(n\sigma_2 + m\phi_2)$ admits a horizontal foliation if and only if $\frac{m}{n} \in (0, \frac{1}{2})$.

We leave the cases $j = 3, 5$ to the reader.

To complete the proof of (3) we must express the conclusions we have just obtained in terms of appropriately chosen meridians and longitudes for the knots K_j. We proceed as follows. The euler number of $X_j(n\sigma_j+m\phi_j)$ is given, up to sign, by the sum of its Seifert invariants. Further, since $H_1(X_j(\lambda_j)) \cong \mathbb{Z}$, we can solve for the coefficients n, m of λ_j. For instance for $j > 1$, set $\{j, p, q\} = \{2, 3, 5\}$. If $\lambda_j = n\sigma_j + m\phi_j$, then $0 = |e(X_j(n\sigma_j + m\phi_j))| = |-1 + \frac{1}{p} + \frac{1}{q} + \frac{m}{n}|$. Thus $\frac{m}{n} = \frac{pq-(p+q)}{pq}$. Since $\gcd(pq, pq-(p+q)) = 1$, we have

$$\lambda_j = -pq\sigma_j + (p+q-pq)\phi_j$$

Similarly for $j = 1$ we have $\frac{m}{n} = 1 - (\frac{1}{2} + \frac{1}{3} + \frac{1}{5}) = -\frac{1}{30}$. Hence

$$\lambda_1 = -30\sigma_1 + \phi_1$$

The μ_j Dehn filling of X_j yields $\Sigma(2,3,5)$ and it is known that $|e(\Sigma(2,3,5))| = \frac{1}{30}$. Combined with the identity $\Delta(\mu_j, \lambda_j) = 1$ we can solve for the coefficients of μ_j:

$$\mu_j = \begin{cases} \sigma_1 & \text{if } j = 1 \\ j\sigma_j + \phi_j & \text{if } j > 1 \end{cases}$$

With these choices, it is easy to verify that the set of detected slopes $a\mu_1 + b\lambda_1$ corresponds to the interval specified in (3).

To prove (4), let $\alpha = a\mu_j + b\lambda_j$ be a slope on ∂X_j such that $X_j(\alpha) \cong \Sigma(2,3,5)$. Since $\Sigma(2,3,5)$ is a $\mathbb{Z}$-homology 3-sphere, $1 = \Delta(\alpha, \lambda_j) = |a|$. Without loss of generality we can suppose that $a = 1$. On the other hand, the core of the filling torus in $X_j(\alpha)$ is K_j, so

$$j = \Delta(\alpha, \phi_j) = \begin{cases} \Delta(\mu_j + b\lambda_j, 30\mu_1 + \lambda_1) & \text{if } j = 1 \\ \Delta(\mu_j + b\lambda_j, pq\mu_j + j\lambda_j) & \text{if } j > 1 \end{cases}$$

$$= \begin{cases} |1 - 30b| & \text{if } j = 1 \\ |j - pqb| & \text{if } j > 1 \end{cases}$$

Hence there is an $\epsilon \in \{\pm 1\}$ such that $j\epsilon = \begin{cases} 1 - 30b & \text{if } j = 1 \\ j - pqb & \text{if } j > 1 \end{cases}$. It follows that $b = 0$ so that $\alpha = \mu_j$. This proves (4). □

Corollary 2.6. *Suppose that K is a knot in either S^3 or $\Sigma(2,3,5)$ whose exterior X is Seifert fibered and let U be the connected open subset of $\mathcal{S}(X)$ described in Proposition 2.3.*

(1) *If X is the trefoil exterior, then U contains all the slopes α on ∂X such that $X(\alpha)$ is a $\mathbb{Z}$-homology 3-sphere other than S^3 and $\Sigma(2,3,5)$. The two slopes yielding the latter two manifolds are the end-points of $\overline{U}$.*

(2) *If X is not the trefoil exterior, then U contains all the slopes α on ∂X such that $X(\alpha)$ is a $\mathbb{Z}$-homology 3-sphere other than the meridian of K, which is an end-point of $\overline{U}$.* □

3. Existence of rational foliations on aspherical graph $\mathbb{Z}$-homology 3-spheres

We prove Theorem 0.2 in this section by induction on the number of its JSJ pieces, the base case being dealt with in Proposition 2.2. We suppose below that W is a non-Seifert graph manifold $\mathbb{Z}$-homology 3-sphere.

Lemma 3.1. *Suppose that M is a graph manifold $\mathbb{Z}$-homology solid torus with incompressible boundary. If α and β are slopes on ∂M whose associated fillings are $\mathbb{Z}$-homology 3-spheres which are either $S^3, \Sigma(2,3,5)$ or reducible, then $\Delta(\alpha,\beta) \leq 1$.*

Proof. If M is Seifert fibred, it has base orbifold $D^2(a_1,\ldots,a_n)$ where n and each a_i are at least 2. Further, the a_i are pairwise coprime. In this case M admits no fillings which are simultaneously reducible and $\mathbb{Z}$-homology 3-spheres. Thus $M(\alpha)$ and $M(\beta)$ are either S^3 or $\Sigma(2,3,5)$. If α and β are distinct slopes, then $M(\alpha)$ and $M(\beta)$ cannot both be S^3 as torus knots admit unique S^3-surgery slopes. Similarly Proposition 2.5 implies that $M(\alpha)$ and $M(\beta)$ cannot both be $\Sigma(2,3,5)$. On the other hand, if one of $M(\alpha)$ and $M(\beta)$ is S^3 and the other $\Sigma(2,3,5)$, then M must be the trefoil knot exterior and $\Delta(\alpha,\beta) = 1$.

Next suppose that M is not Seifert fibred. If $M(\alpha)$ is reducible, then the main result of [GLu] combines with [BZ2, Theorem 1.2] to show that $\Delta(\alpha,\beta) \leq 1$. On the other hand, if $M(\alpha)$ and $M(\beta)$ are either S^3 or $\Sigma(2,3,5)$ and $\Delta(\alpha,\beta) \geq 2$, then [BZ1, Theorem 1.2(1)] implies that M has two pieces, one a cable space and the other a Seifert manifold M_0 with base orbifold a 2-disk with two cone points. The proof of [BZ1, Theorem 1.2(1)] (see §8 of [BZ1]) now implies that M_0 admits two Dehn fillings yielding S^3 or $\Sigma(2,3,5)$ whose slopes are of distance at least 8, which is impossible. (See the discussion which follows the statement of [BZ1, Theorem 1.2].) Thus $\Delta(\alpha,\beta) \leq 1$. □

Let X be a piece of W whose boundary is a torus. (Thus X corresponds to a leaf of the JSJ-graph of W.) If $Y = \overline{W \setminus X}$ is the exterior of X in W, then $T = X \cap Y$ is an essential torus. Let λ_X and λ_Y be the longitudes of X and Y. For slopes α and β on T we have

$$|H_1(X(\alpha))| = \Delta(\alpha,\lambda_X) \quad \text{and} \quad |H_1(Y(\beta))| = \Delta(\beta,\lambda_Y)$$

Hence as we noted in §1 that $\Delta(\lambda_X,\lambda_Y) = 1$, both $X(\lambda_Y)$ and $Y(\lambda_Y)$ are $\mathbb{Z}$-homology 3-spheres.

Let ϕ_X and ϕ_Y be primitive elements of $H_1(T)$ representing, respectively, the slopes of the Seifert fibre of X and that of the piece P of Y incident to T. Since X has exceptional fibres, $\pm\phi_X \notin \{\lambda_X,\lambda_Y\}$ (Lemma 1.1(3)). It follows that $X(\lambda_X)$ and $X(\lambda_Y)$ are irreducible Seifert manifolds (Lemma 1.1(1)).

Proof of Theorem 0.2. For an integer n, set

$$\alpha_n = \lambda_X + n\lambda_Y$$

and observe that $\lim_{|n|}[\alpha_n] = [\lambda_Y] \in \mathcal{S}_{rat}(T)$. Since $X(\lambda_Y)$ is a $\mathbb{Z}$-homology 3-sphere, α_n is strongly detected by a horizontal foliation in X for $n \gg 0$ or for $n \ll 0$ or for both (Proposition

2.3 and Corollary 2.6). To complete the proof it suffices to find a rational foliation of Y which strongly detects α_n for all large $|n|$.

Since $\Delta(\alpha_n, \lambda_Y) = 1$, the manifolds $Y(\alpha_n)$ are $\mathbb{Z}$-homology 3-spheres, and since Y is irreducible and $\Delta(\alpha_n, \alpha_m) = |n - m|$, there are at most two n such that $Y(\alpha_n)$ is either reducible, S^3 or $\Sigma(2,3,5)$, and if two, they are successive integers (Lemma 3.1). Thus for $|n|$ large, $Y(\alpha_n)$ is an irreducible graph manifold $\mathbb{Z}$-homology 3-sphere which is neither S^3 nor $\Sigma(2,3,5)$. Hence our inductive hypothesis implies that $Y(\alpha_n)$ admits a rational foliation $\mathcal{F}_n$ for large $|n|$. If $\lambda_Y \neq \phi_Y$, then as $\Delta(\alpha_n, \phi_Y) = |\alpha_n \cdot \phi_Y| \geq |n||\lambda_Y \cdot \phi_Y| - |\lambda_X \cdot \phi_Y|$, for large $|n|$ the JSJ pieces of $Y(\alpha_n)$ are $P(\alpha_n)$ and the JSJ pieces of $\overline{Y \setminus P}$. Thus $\mathcal{F}_n$ induces a rational foliation of slope α_n on Y, which completes the proof.

Suppose then that $\lambda_Y = \phi_Y$. Then Lemma 1.1(3) implies that P is a product $F \times S^1$ where F is a planar surface with $|\partial P| \geq 3$ boundary components. Since $\Delta(\alpha_n, \phi_Y) = \Delta(\alpha_n, \lambda_Y) = 1$, each $P(\alpha_n)$ is a product $\bar{F} \times S^1$ where $\bar{F}$ is a planar surface with $|\partial P| - 1 \geq 2$ boundary components. If $|\partial P| \geq 4$, the JSJ pieces of $Y(\alpha_n)$ are $P(\alpha_n)$ and the JSJ pieces of $\overline{Y \setminus P}$, so we can proceed as above.

Finally assume that $|\partial P| = 3$ and let Y_1, Y_2 be the components of $\overline{Y \setminus P}$. Denote the JSJ torus $Y_i \cap P$ by T_i, so $\partial P = \partial Y \cup T_1 \cup T_2$. For each n we have $P(\alpha_n) \cong S^1 \times S^1 \times I$, so $Y(\alpha_n) \cong Y_1 \cup Y_2 \not\cong S^3, \Sigma(2,3,5)$. By induction, there is a rational foliation $\mathcal{F}_n$ on $Y(\alpha_n)$. Since there is no vertical annulus in P which is cobounded by the Seifert fibres of the two pieces of Y incident to P, the reader will verify that there is at most one value of n for which there is an annulus in $P(\alpha_n)$ cobounded by these fibres. Thus for $|n| \gg 0$, $Y(\alpha_n)$ is a graph manifold $\mathbb{Z}$-homology 3-sphere whose pieces are the JSJ pieces of $\overline{Y \setminus P}$. Fix such an n and note that up to isotopy, we can suppose that $\mathcal{F}_n$ is a product fibration on $P(\alpha_n) \cong S^1 \times S^1 \times I$ whose fibre is an annulus. It follows that we can choose primitive classes $\beta_n^1 \in H_1(T_1)$ and $\beta_n^2 \in H_1(T_2)$ representing the slopes of $\mathcal{F}_n$ on T_1, T_2 and an integer k such that $k\alpha_n + \beta_n^1 + \beta_n^2 = 0$ in $H_1(P)$.

Let $p : P = F \times S^1 \to F$ be the projection and denote by $a, b_1, b_2 \in H_1(F)$ the classes associated to the boundary components of F, where a corresponds to $p(T)$, b_1 to $p(T_1)$, and b_2 to $p(T_2)$. We may assume that $a + b_1 + b_2 = 0$. Since $\Delta(\alpha_n, \phi_Y) = 1$, we can also assume that the projection $p : P \to F$ sends α_n to a. Fix integers k_1, k_2 so that $p_*(\beta_n^j) = k_j b_j$. Clearly $|k_j| = \Delta(\beta_n^j, \phi_j)$ where ϕ_j is the slope on T_j determined by the Seifert structure on P. Then we have

$$0 = p_*(k\alpha_n + \beta_n^1 + \beta_n^2) = ka + k_1 b_1 + k_2 b_2$$

in $H_1(F)$. This can only happen if $k = k_1 = k_2$. Thus if $k \neq 0$, the fibration in $P(\alpha_n)$ determined by $\mathcal{F}_n$ is horizontal in P and of slope α_n on T, so we are done.

Suppose then that $k = 0$, so $0 = |k_j| = \Delta(\beta_n^j, \phi_j)$. Thus $[\beta_n^1] = [\phi_1]$ and $[\beta_n^2] = [\phi_2]$ are vertical in P. By construction, $Y(\lambda_Y) = Y(\phi_Y) = Y_1(\phi_1) \# Y_2(\phi_2) = Y_1(\beta_n^1) \# Y_2(\beta_n^2)$ and as $\mathbb{Z} \cong H_1(Y(\lambda_Y)) = H_1(Y_1(\phi_1)) \oplus H_1(Y_2(\phi_2))$, we can suppose that $H_1(Y_1(\beta_n^1)) \cong \mathbb{Z}$ and $H_1(Y_2(\beta_n^2)) \cong \{0\}$. Thus $\phi_1 = \beta_n^1 = \lambda_{Y_1}$ and $\Delta(\phi_2, \lambda_{Y_2}) = \Delta(\beta_n^2, \lambda_{Y_2}) = 1$.

Fix $\delta_0 \in H_1(T_1)$ such that $1 = \Delta(\delta_0, \lambda_{Y_1}) = \Delta(\delta_0, \phi_1)$ and $p_*(\delta_0) = b_1$. Then $p_*(\lambda_X + \delta_0 + \lambda_{Y_2}) = a + b_1 + b_2 = 0 \in H_1(F)$ and therefore $\lambda_X + \delta_0 + \lambda_{Y_2} = j\phi_Y \in H_1(P)$ for some integer j. After replacing δ_0 by $\delta_0 - j\phi_1$ we can suppose that

$$\lambda_X + \delta_0 + \lambda_{Y_2} = 0 \in H_1(P)$$

With this choice, set $\delta_m = \delta_0 + m\phi_1$.

Claim 3.2. *For all but at most finitely many m, Y_1 admits a rational foliation of slope δ_m.*

Proof. Since $\Delta(\delta_m, \lambda_{Y_1}) = 1$ for all m, $Y_1(\delta_m)$ is a $\mathbb{Z}$-homology 3-sphere. Let ϕ_{Y_1} be the primitive element of $H_1(T_1)$ representing the slope of the Seifert fibre of the piece P_1 of Y_1 incident to $T_1 = \partial Y_1$, then $\Delta(\lambda_{Y_1}, \phi_{Y_1}) \geq 1$, since $\lambda_{Y_1} = \phi_Y$ and T_1 is a JSJ-torus of Y. Therefore our inductive hypothesis combines with Lemma 3.1 to show, as in the first part of the proof, that for all but at most fnitely many m, Y_1 admits a rational foliation of slope δ_m. □

Claim 3.3. *Y_2 admits a rational foliation of slope $\gamma = p\lambda_{Y_2} + q\phi_2$ where p and q are relatively prime and non-zero.*

Proof. Let ϕ_{Y_2} be the primitive element of $H_1(T_2)$ representing the slope of the Seifert fibre of the piece P_2 of Y_2 incident to $T_2 = \partial Y_2$. If $\Delta(\lambda_{Y_2}, \phi_{Y_2}) \geq 1$, the assertion follows from the proof of Claim 3.2 by taking $\gamma = p\lambda_{Y_2} + \phi_2$, for some $|p|$ sufficiently large.

We consider now the case where $\lambda_{Y_2} = \phi_{Y_2}$. Let $E \subset S^3$ be the trefoil exterior, $\mu_E \in H_1(\partial E)$ its meridional slope and $\nu_E \in H_1(\partial E)$ the unique slope such that $E(\nu_E) \cong \Sigma(2,3,5)$. Then $\Delta(\mu_E, \nu_E) = 1$. Further, E does not admit a horizontal foliation of slope μ_E or ν_E. We build a $\mathbb{Z}$-homology 3-sphere $W_2 = E \cup Y_2$ by gluing E and Y_2 along their boundaries in such a way that the slope μ_E is identified with the slope λ_{Y_2} and the slope ν_E is identified with the slope ϕ_2. Since the fiber slope $\phi_{Y_2} = \lambda_{Y_2}$ is identified with the meridional slope μ_E, the Seifert fibrations on E and P_2 do not match up, and the torus $\partial Y_2 = \partial E$ is a JSJ-torus of W_2. Hence W_2 is a graph $\mathbb{Z}$-homology 3-sphere whose JSJ pieces are E and the JSJ pieces of Y_2. In particular, W_2 has fewer pieces than W. By the inductive hypothesis W_2 carries a rational foliation which intersects the JSJ torus $\partial Y_2 = \partial E$ in a circle fibration of some slope γ. Hence Y_2 admits a rational foliation of slope γ. Moreover $\Delta(\gamma, \lambda_{Y_2}) \geq 1$ and $\Delta(\gamma, \phi_2) \geq 1$ since E cannot admit a horizontal foliation of slope μ_E or ν_E. □

Now we complete the proof of Theorem 0.2.

For $|m|$ sufficiently large, let $\delta_m = \delta_0 + m\phi_1 \in H_1(T_1)$ be the slope of a rational foliation on Y_1 given by Claim 3.2, and $\gamma = p\lambda_{Y_2} + q\phi_2 \in H_1(T_2)$ the slope of a rational foliation on Y_2 given by Claim 3.3. Since $\lambda_Y = \phi_Y = \phi_1 = \phi_2$ and $\lambda_X + \delta_0 + \lambda_{Y_2} = 0$ in $H_1(P)$, the sum $\zeta_m + p\delta_m + \gamma = 0 \in H_1(P)$ where $\zeta_m = p\lambda_X - (pm+q)\lambda_Y \in H_1(T)$ is a primitive class. Thus there is a properly embedded, horizontal surface F_m in P with boundary curves of slope ζ_m, δ_m and γ. Hence P fibres over the circle with fibre F_m and Y admits a rational foliation of slope ζ_m for large $|m|$. Now, it is easy to verify that $\lim_{|m|}[\zeta_m] = [\lambda_Y]$ and that for large $|m|$, reversing the sign of m sends $[\zeta_m]$ from one side of $[\lambda_Y]$ to the other. Since $X(\lambda_Y)$ is a $\mathbb{Z}$-homology 3-sphere, Proposition 2.3 and Corollary 2.6 imply that X admits a horizontal foliation of slope

δ_m for $m \gg 0$ or for $m \ll 0$ or for both. This completes the induction and the proof of Theorem 0.2. □

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Institut de mathétiques de Toulouse, UMR 5219 et Institut Universitaire de France, Université Paul Sabatier 31062 Toulouse Cedex 9, France.

E-mail address: boileau@math.univ-toulouse.fr

Département de Mathématiques, Université du Québec à Montréal, 201 avenue du Président-Kennedy, Montréal, QC H2X 3Y7.

E-mail address: boyer.steven@uqam.ca

URL: http://www.cirget.uqam.ca/boyer/boyer.html