THEORIES OF MODULES CLOSED UNDER DIRECT PRODUCTS

ROGER VILLEMAIRE

Abstract. We generalize to theories of modules (complete or not) a result of U. Felgner stating that a complete theory of abelian groups is a Horn theory if and only if it is closed under products. To prove this we show that a reduced product of modules $\prod_{i \in I} M_i$ is elementarily equivalent to a direct product of ultraproducts of the modules $M_i (i \in I)$.

§1. Introduction. Let $L$ be some first-order language. By an $L$-theory we mean any consistent set of $L$-sentences. Furthermore we say that a theory is closed under some operation on models if this is the case for its class of models. For example, a theory $T$ is closed under direct products if for any models $M_i$ of $T (i \in I)$, $\prod_{i \in I} M_i$ (the cartesian product) is a model of $T$. We will use the book [1] of Chang and Keisler as a general reference on model theory.

One may ask if there is any relationship between the fact of being closed under direct products and the fact of being closed under binary products, i.e. if $M$ and $N$ are both models then $M \times N$ is also a model. This question is answered by the following classical theorem of Vaught (see [1, Theorem 6.3.14])

**Theorem 1.1 (Vaught).** A theory $T$ is closed under direct products if and only if it is closed under binary products.

The following class of formulas plays an important role in the analysis of theories closed under direct products.

**Definition.** Let $L$ be some first-order language. The set of Horn formulas of $L$ is the smallest set of formulas containing every disjunction of finitely many negations of atomic formulas with at most one atomic formula, which is closed under conjunction and both quantifiers. Furthermore a theory is said to be a Horn theory if it is axiomatized by Horn sentences.

It has been proved by Horn that any Horn theory is closed under direct products. Unfortunately the converse is not true. Chang and Morel showed (see [1, Example 6.2.3]) that the theory of Boolean algebras having at least one atom is closed under direct products but that it is not a Horn theory. Nevertheless Horn theories are exactly the theories closed under reduced products, an algebraic operation which we will now define.

**Definition.** Let $L$ be any first-order language, and let $M_i (i \in I)$ be some $L$-structures. For $F$ a filter over $I$ we define the reduced product $\prod_F M_i$ to be as follows.
The universe of the reduced product is the cartesian product \( \prod_i M_i \) modulo the equivalence relation \( \sim \), where \( m \sim n \) if \( \{ i \in I; m(i) = n(i) \} \in F \) (here \( m(i) \) is the \( i \)th component of \( m \)).

In the same way we say that a relation or an operation is satisfied in the reduced product if the set of indices of \( I \) where it is satisfied is in \( F \).

The following result was proved by Keisler in [4, Result A, p. 307] using the continuum hypothesis. Galvin in [3, Theorem 6.1] showed that the continuum hypothesis was not necessary for the result to hold. The next theorem can also be seen as a consequence of \([1, \text{Lemma 6.2.5 and 6.2.5'\} using the fact, which follows from the existence for any formula of an autonomous set containing it (see [1, p. 426] for the definition of autonomous set and [1, Theorem 6.3.6(i)] for a proof of this fact), that a theory closed under reduced products is axiomatized by reduced products formulas, i.e. formulas which by themselves form theories closed under reduced products.

**Theorem 1.2** (Galvin and Keisler). Let \( L \) be any first-order language and let \( T \) be an \( L \)-theory. The following conditions are equivalent.

(a) \( T \) is a Horn theory.

(b) \( T \) is closed under reduced products.

Hence to show that the theory of Boolean algebras with at least one atom is not a Horn theory Chang and Morel considered a reduced product of an atomic Boolean algebra over the Fréchet filter on the natural numbers, i.e. the filter of cofinite set. As it is easily shown, this reduced product has no atom; hence the theory of Boolean algebras with at least one atom is not a Horn theory.

In the following section we will show that the situation is much simpler for modules, namely that the theory of modules is closed under products if and only if it is a Horn theory.

**§2. Theories of modules closed under direct products.** In the remainder of this paper, let \( L \) be the language of the theory of modules over some fixed ring. Every module that we will consider in this section will be over this fixed ring. As a general reference on model theory of modules we use the book [5] of M. Prest.

**Definition.** A positive primitive formula is an \( L \)-formula of the form

\[
\exists \bar{y}
\left( \bigwedge_i \varphi_i(\bar{x}, \bar{y}) \right)
\]

where the conjunction is finite and \( \varphi_i(\bar{x}, \bar{y}) \) is an atomic \( L \)-formula.

Let us first recall that direct products and direct sums are elementarily equivalent (see [5, Lemma 2.24(a)]). Hence a theory of modules is closed under direct products if and only if it is closed under direct sums.

Let \( M \) be a module. The structures \( M \) and \( M \oplus M \) are elementarily equivalent if and only if each of the Baur-Monk invariants of \( M \) are either equal to 1 or infinite (see [5, Corollary 2.18]). U. Felgner noticed this fact, and he furthermore proved the following result.

**Theorem 2.1** [2, Theorem 2.1]. Let \( T \) be a complete theory of abelian groups. The following conditions are equivalent.

(a) \( T \) is closed under direct products.
(b) $T$ is a Horn theory.
(c) For some (any) model $M$ of $T$, we have that $M$ and $M \oplus M$ are elementarily equivalent.

This result is somewhat surprising since, as mentioned in §1, it is not the case for all first-order languages that every theory closed under direct products is a Horn theory.

We will now show that this result generalizes to any theory of modules (complete or not). To prove this we will show that a reduced product of modules $\prod F M_i (i \in I)$ is elementarily equivalent to a product of ultraproducts of the modules $M_i (i \in I)$.

**Definition.** Let $\prod F M_i (i \in I)$ be a reduced product. The Boolean algebra $\mathcal{P}(I)/F$, where $\mathcal{P}(I)$ is the power set of $I$, is called the quotient of this reduced product.

We will first show that a reduced product is elementarily equivalent to the direct sum of a reduced product with atomless quotient and products of ultraproducts.

**Definition.** Let $A$ and $B$ be two modules. A homomorphism $\alpha$ of $A$ in $B$ is said to be a pure embedding if for any tuple $\bar{a} \in A$ and any positive primitive formula $\varphi(\bar{x})$, we have that $\varphi(\bar{a})$ is true in $A$ whenever $\varphi(\alpha(\bar{a}))$ is true in $B$.

**Definition.** Let $A$, $B$ and $C$ be modules, and let $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$ be homomorphisms. The sequence

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

is said to be pure exact if $\alpha$ is a pure embedding, $\beta$ is surjective and the kernel of $\beta$ is equal to the image of $\alpha$.

**Notation.** Let $I$ be a set, and let $F$ be a filter over $I$. Let $E$ be a subset of $I$. The equivalence class of $E$ modulo $F$ will be written $E/F$. It is clear that for subsets $E$ and $E'$ of $I$, $E/F = E'/F$ if and only if $(E \Delta E')^c$ (the complement of the symmetric difference of $E$ with $E'$) is in $F$.

**Definition.** Let $F$ be a filter over a set $I$, and let $E$ and $E'$ be subsets of $I$. We say that $E$ and $E'$ are $F$-disjoint if $(E \cap E')/F = \emptyset/F$.

In the following proofs we will work with representatives in $\prod I M_i$ of elements of $\prod F M_i$. The support of an element $m$ of $\prod I M_i$ is the subset of $I$ for which the component of $m$ is nonzero. For a tuple of elements $\bar{m}$ it is the union of the supports of the elements. Furthermore for some $m$ of $\prod I M_i$ and some subset $X$ of $I$, the restriction $m|_X$ is the element of $\prod I M_i$ which is equal to $m$ on $X$ and equal to $0$ outside $X$; for a tuple it is the tuple of the restrictions. As before, we will write $m(i)$ for the $i$th component of $m$, for some $m$ in $\prod I M_i$. Furthermore if $\bar{m} \in \prod I M_i$, then $\bar{m}/F$ will be the canonical image of $\bar{m}$ in $\prod F M_i$.

**Lemma 2.2.** Let $\prod F M_i$ be a reduced product of modules and let $\bigwedge \eta_i (\bar{x}, \bar{y})$ be a finite conjunction of atomic formulas. If $\bigwedge \eta_i (\bar{m}/F, \bar{m}'/F)$ for some $\bar{m}/F, \bar{m}'/F \in \prod F M_i$, then there exists $\bar{m}''$ in $\prod I M_i$ such that $\bigwedge \eta_i (\bar{m}/F, \bar{m}''/F)$ and the support of $\bar{m}''$ is included in the support of $\bar{m}$.

**Proof.** Suppose $\bigwedge \eta_i (\bar{m}/F, \bar{m}'/F)$ is satisfied in $\prod F M_i$. Take $X$ to be the support of $\bar{m}$. Let $\bar{m}'' = \bar{m}'|_X$. Let me show that $\bigwedge \eta_i (\bar{m}/F, \bar{m}''/F)$ is satisfied in $\prod F M_i$. Let $Y = \{ i \in I ; \bigwedge \eta_i (\bar{m}(i), \bar{m}''(i)) \}$. By definition of the reduced product, $Y \in F$. Now $\{ i \in I ; \bigwedge \eta_i (\bar{m}(i), \bar{m}''(i)) \} = X^c \cup (X \cap Y)$ since on $X^c$ both $\bar{m}(i)$ and $\bar{m}''(i)$ are $0$ ($\bigwedge \eta_i (0, 0)$ is always true), and $\bar{m}(i)$ and $\bar{m}''(i)$ are equal on $X$. Hence since $Y \subseteq X^c \cup (X \cap Y)$ it follows that $X^c \cup (X \cap Y) \in F$, and the result is proved.
LEMMA 2.3. Let $F$ be a filter over a set $I$, $E \subseteq I$, and $F'$ the filter generated by \{a \cap E; a \in F\}. If $E/F$ is an atom in $\mathcal{P}(I)/F$, then $F'$ is an ultrafilter.

PROOF. Let $X$ be any subset of $I$. Since $E/F$ is an atom, it follows that either $(X \cap E)/F = E/F$ or $(X \cap E)/F = \emptyset/F$. Suppose we are in the first case; then $((X \cap E) \Delta E)^c$ is in $F$. Therefore since $((X \cap E) \Delta E)^c \cap X \subseteq X$ it follows that $X$ is in $F'$. In the second case we have that $((X \cap E) \Delta \emptyset)^c = (X \cap E)^c$ is in $F'$; hence $(X \cap E)^c \cap E$ is in $F'$. Therefore since $(X \cap E)^c \cap E \subseteq X^c$ it follows that $X^c$ is in $F'$, proving that $F'$ is an ultrafilter.

LEMMA 2.4. Let $\prod F M_i (i \in I)$ be a reduced product of modules. Let \{E_j; j \in J\} be a maximal set of pairwise $F$-disjoint subsets of $I$ such that $E_j/F$ is an atom in $\mathcal{P}(I)/F$ for every $j \in J$ (there is such a set by Zorn's lemma). Let $F_j$ be the filter generated by \{a \cap E_j; a \in F\} (j \in J). Then the $F_j$ (j \in J) are ultrafilters on $E$, and there exists a homomorphism $\alpha$ such that the following sequence is pure-exact:

$$0 \to \bigoplus_{j \in J} \left[ \prod_{F_j} M_i \right] \xrightarrow{\beta} \prod_{F} M_i \xrightarrow{\alpha} \prod_{F'} M_i \to 0,$$

where $F'$ is the filter generated by $F$ and the set \{E_j; j \in J\} of complements of the $E_j$ (j \in J), and the mapping $\beta$ is canonical.

Furthermore, $\mathcal{P}(I)/F'$ is atomless.

REMARK. This result should be seen as a slight generalization of the well-known fact that for any index set $J$ and modules $N_j (j \in J)$ the following sequence is pure exact:

$$0 \to \bigoplus_{j \in J} N_j \to \prod_{J} N_j \to \prod_{Fr} N_j \to 0,$$

where all mappings are canonical. Furthermore the $N_j$ in the direct sum is thought of as the ultraproduct over the principal ultrafilter $(j)$ and $Fr$ is the Fréchet filter of cofinite sets over $J$. In this case it is clear that $\mathcal{P}(J)/Fr$ is atomless.

PROOF OF LEMMA 2.4. The $F_j$ (j \in J) are ultrafilters by Lemma 2.3.

Let us now define $\alpha$. Let $\alpha_j$ be the map from $\prod F M_i$ to $\prod F M_i$ which sends an element with representative $m$ to $m|_{E_j}$. This map is well defined. To show this, suppose that $m$ and $m'$ represent the same element in $\prod F_j M_i$, and let $X$ be the set of elements of $I$ such that $m(i) = m'(i)$. Hence $X$ is in $F_j$. Therefore $X$ contains $a \cap E_j$ for some $a$ in $F$. Now the set $Y$ of $i \in I$ such that $m|_{E_j}(i) = m'|_{E_j}(i)$ contains also $a \cap E_j$. Furthermore, $E_j \subseteq Y$ by the definition of restriction. Hence $a \subseteq Y$ and $Y$ is in $F$. This shows that $\alpha_j$ is well defined. It is clear that $\alpha_j$ is also a homomorphism. We now define $\alpha$ to be the sum over $j \in J$ of the various $\alpha_j$.

We can now prove that $\alpha$ and $\beta$ possess the properties stated. First it is clear, since $\prod F M_i$ is a quotient of $\prod F M_i$, that $\beta$ is surjective.

Let us now show that the kernel of $\beta$ is equal to the image of $\alpha$. Let $m$ be a representative of an element of the image of $\alpha$. The support of $m$ is included in a finite union $E_{i_1} \cup \cdots \cup E_{i_k}$. Therefore this element is sent in $\prod F M_i$ to 0. Hence the image of $\alpha$ is included in the kernel of $\beta$.

Now let $m$ be a representative which is sent by $\beta$ to 0. Hence its support is included in a finite union $a^c \cup E_{i_1} \cup \cdots \cup E_{i_k}$, where $a \in F$. Let $m_{i_1}, \ldots, m_{i_k}$ be such that they
coincide with \( m \) on \( E_{i_1} \setminus (E_{i_2} \cup \cdots \cup E_{i_k}) \), \( E_{i_3} \setminus (E_{i_2} \cup \cdots \cup E_{i_k}) \), \ldots , E_{i_k} \), respectively, and are 0 elsewhere. Let us show that \( m/F \) and \((m_{i_1} + \cdots + m_{i_k})/F \) are equal. The representatives \( m \) and \( m_{i_1} + \cdots + m_{i_k} \) coincide in every \( i \)th component, except maybe for \( i \in \alpha \). Hence they coincide in a set containing \( \alpha \in F \); hence \( m/F \) and \( m_{i_1} + \cdots + m_{i_k}/F \) are equal, proving that \( m \) represents an element of the image of \( \alpha \). Hence the kernel of \( \alpha \) is included in the image of \( \alpha \).

We will now show that \( \alpha \) is a pure embedding. Let \( \bar{m} \) be the representative of a tuple of elements of the image of \( \alpha \). Suppose \( \bar{m}/F \) satisfies some positive primitive formula \( \phi(x) = \exists \bar{y} \bigwedge_i \eta_i(x, \bar{y}) \) in \( \prod_i M_i \). Then there exists \( \bar{y}/F \) such that \( \bigwedge_i \eta_i(\bar{m}/F, \bar{y}/F) \) is satisfied, and by Lemma 2.2 we can suppose that the support of \( \bar{y} \) is included in the support of \( \bar{m} \). Since the support of \( \bar{m} \) is included in some finite union \( E_{i_1} \cup \cdots \cup E_{i_k} \), let as before \( \bar{y}_{i_1}, \ldots , \bar{y}_{i_k} \) be such that they coincide with \( \bar{y} \) on \( E_{i_1} \setminus (E_{i_2} \cup \cdots \cup E_{i_k}) \), \( E_{i_2} \setminus (E_{i_3} \cup \cdots \cup E_{i_k}) \), \ldots , \( E_{i_k} \), respectively, and are 0 elsewhere. Hence \( \bar{y} = \bar{y}_{i_1} + \cdots + \bar{y}_{i_k} \), and it follows that \( \bar{y} \) is in the image of \( \alpha \). Hence \( \alpha \) is a pure embedding.

It is now left to show that \( \mathcal{P}(I)/F' \) is atomless. Suppose a subset \( X \) of \( I \) was the representative of an atom in \( \mathcal{P}(I)/F' \). If \( X' = X \setminus \bigcup_{j \in J} E_j \) was not equal to \( \emptyset/F' \), then \( X'/F' \) would also be an atom; hence \( X'/F' \) also would be an atom, contradicting the maximality of \( \{E_i; j \in J\} \). Hence \( X'/F' = \emptyset/F' \). Let \( X'' = X \cap \bigcup_i E_i \). Suppose that there are only finitely many \( E_{i_1}, \ldots , E_{i_k} \) for which \( (X'' \cap E_{i_j})/F \) is different from \( \emptyset/F \). Then \( (X'' \setminus E_{i_1} \cup \cdots \cup E_{i_k})/F \) would be equal to \( \emptyset/F \) and \( X''/F' = \emptyset/F' \), a contradiction to the fact that \( X''/F' \) is an atom. Therefore there is an infinite subset \( J' \subseteq J \) such that \( X'' \cap E_i/F \neq \emptyset/F \) for all \( j \in J' \). Write \( J' \) as a disjoint union \( J_1 \cup J_2 \) of two infinite subsets. I now claim that \( (X'' \cap \bigcup_{j \in J_1} E_j)/F' \) and \( (X'' \cap \bigcup_{j \in J_2} E_j)/F' \) are both different from \( \emptyset/F' \). Suppose this was not the case, i.e. suppose that \( (X'' \cap \bigcup_{j \in J_1} E_j)/F' = \emptyset/F' \). Then \( Y'' = (X'' \cap \bigcup_{j \in J_1} E_j) \subseteq \alpha \cap E_{i_1} \cup \cdots \cup E_{i_k} \) for some \( \alpha \) in \( F \) and \( i_1, \ldots , i_k \) of \( J \). But \( Y'' \cap E_i \subseteq \alpha \cap E_{i_1} \cup \cdots \cup E_{i_k} \) and for \( l \in J \) different from \( i_1, \ldots , i_k \) it follows from \( E_{i_l} \cap E_i/F = \emptyset/F \) that \( Y'' \cap E_i \subseteq \alpha \cap a_{i_1} \cup \cdots \cup a_{i_k} \) for \( a_{i_1}, \ldots , a_{i_k} \) in \( F \). Hence \( Y'' \cap E_i/F = \emptyset/F \), which is a contradiction to the fact that there are infinitely many \( E_i \) such that \( X'' \cap E_i/F \neq \emptyset/F \). In the same way one can show that \( (X'' \cap \bigcup_{j \in J_2} E_j)/F' \) is different from \( \emptyset/F' \). Furthermore \( X'' \cap \bigcup_{j \in J_1} E_j/F' \) and \( X'' \cap \bigcup_{j \in J_2} E_j/F' \) are disjoint, which contradicts the fact that \( X''/F' = X/F' \) is an atom. Hence \( \mathcal{P}(I)/F' \) is atomless.

**Corollary 2.5.** Under the same hypothesis \( \prod_i M_i \) is elementarily equivalent to

\[
\bigoplus_{j \in J} \left[ \prod_{F_j} M_i \right] + \prod_{F} M_i.
\]

**Proof.** It follows from Theorem 2.4 using the fact (see [5, Lemma 2.23]) that for any pure exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) the modules \( A \oplus C \) and \( B \) are elementarily equivalent.

We will now prove that a reduced product with atomless quotient \( \prod_i M_i (i \in I) \) is elementarily equivalent to a product of ultraproducts of the modules \( M_i (i \in I) \).

**Lemma 2.6.** Let \( \prod_i M_i (i \in I) \) be a reduced product of modules, and let \( \phi(x) \) be some positive primitive formula. For any \( m/F \in \prod_i M_i \) we have that \( \prod_i M_i \models \phi(\bar{m}/F) \) if and only if \( \{i \in I; M_i \models \phi(\bar{m}(i))\} \in F \).
PROOF. Let \( \varphi(\bar{x}) = \exists \bar{y} \land \eta_j(\bar{x}, \bar{y}) \), where \( \eta_j(\bar{x}, \bar{y}) \) is atomic. Hence

\[
\prod F M_i \models \varphi(\bar{m}/F)
\]

if and only if there exists a \( \bar{y}/F \) in \( \prod F M_i \) such that \( \prod F M_i \models \land j \eta_j(\bar{m}/F, \bar{y}/F) \). Since \( \prod F M_i \models \land j \eta_j(\bar{m}/F, \bar{y}/F) \) is equivalent to \( \{ i \in I; M_i \models \land j \eta_j(\bar{m}(i), \bar{y}(i)) \} \in F \), it follows that \( \prod F M_i \models \varphi(\bar{m}/F) \) if and only if \( \{ i \in I; M_i \models \exists \bar{y} \land j \eta_j(\bar{m}(i), \bar{y}) \} \in F \).

We can now prove the following result.

THEOREM 2.7. Let \( \prod F M_i \) be a reduced product with an atomless quotient. Then for any positive primitive formulas \( \varphi(x) \) and \( \psi(x) \) we have that Inv\((\prod F M_i, \varphi, \psi)\) is either 1 or infinite.

PROOF. Let \( \mathcal{P}(I)/F \) be the quotient of \( \prod F M_i \), and let \( X/F \) be a nonzero element. Let \( F_x = \{ a \cap X; a \in F \} \). It is clear that \( \mathcal{P}(X)/F_x \) is also an atomless Boolean algebra. Furthermore, since \( \mathcal{P}(X)/F_x \) is infinite, the structure \( \langle \mathcal{P}(X)/F_x, \cup, \cap, 0/F_x \rangle \), where \( \cup \) is the symmetric difference, is an infinite abelian group of exponent 2.

Now let \( \varphi \) and \( \psi \) be two positive primitive formulas such that Inv\((\prod F M_i, \varphi, \psi)\) > 1. Let \( m \) be an element of \( \prod F M_i \) representing an element \( m/F \) of \( \varphi(\prod F M_i) \) which is not in \( \psi(\prod F M_i) \). Let \( X = \{ i; M_i \models \neg \psi(m(i)) \} \). Since \( \neg \psi(\prod F M_i) \) holds by hypothesis, it follows that \( X/F \) is a nonzero element. By the above argument \( \langle \mathcal{P}(X)/F_x, \cup, \cap, 0/F_x \rangle \) is an infinite abelian group of exponent 2, hence an infinite-dimensional vector space over the two-element field. Let \( X_i/F (i \in S) \) be a basis of this space, and let \( m_i = m|_{X_i} \).

By Lemma 2.6 we know that for any \( m'/F \) of \( \prod F M_i \) and any positive primitive formula \( \eta, \eta(m'/F) \) holds in \( \prod F M_i \) if and only if the set of components of \( m' \) satisfying \( \eta \) is in \( F \). Hence for any \( i \in S \) the formula \( \varphi(m_i/F) \) holds. Let \( i \) and \( j \) be in \( S \). Now since \( m_i + m_j \) is equal to \( m \) on \( X_i \cup X_j \) and since \( (X_i \cup X_j)/F \neq \cap /F \) (because \( X_i/F \) and \( X_j/F \) are linearly independent in \( \langle \mathcal{P}(X)/F_x, \cup, \cap, 0/F_x \rangle \)), it follows that \( \psi(m_i/F + m_j/F) \) does not hold. Hence Inv\((\prod F M_i, \varphi, \psi)\) is infinite.

PROPOSITION 2.8. Let \( \prod F M_i \) be some reduced product of modules. For any invariant Inv\((- , \varphi, \psi) \) such that Inv\((\prod F M_i, \varphi, \psi) > 1 \) there exists an ultrafilter \( U \) containing \( F \) such that Inv\((\prod U M_i, \varphi, \psi) > 1 \).

PROOF. Let Inv\((\prod F M_i, \varphi, \psi) > 1 \), and let \( m/F \) be an element of \( \varphi(\prod F M_i) \) which is not in \( \psi(\prod F M_i) \). Let \( X \) be the set of \( i \in I \) such that \( \psi(m(i)) \) does not hold in \( M_i \). Since \( m/F \) is not in \( \psi(\prod F M_i) \), the set \( X^c \) cannot be in \( F \). Take \( U \) to be an ultrafilter extending \( F \) and containing \( X \). Hence \( \varphi(m/U) \), but \( \psi(m/U) \) does not hold since \( X \in U \).

COROLLARY 2.9. If the quotient of a reduced product \( \prod F M_i \) is atomless, then \( \prod F M_i \) is elementarily equivalent to \( (\prod_{i \in J} \prod U_j M_i) \) (the countable direct power), where \( \{ U_j; j \in J \} \) is the set of all ultrafilters extending \( F \).

PROOF. It is sufficient to show that the Baur-Monk invariants of \( (\prod_{i \in J} \prod U_j M_i) \) and \( \prod F M_i \) are equal. By Theorem 2.7 each invariant of \( \prod F M_i \) is either 1 or infinite. Since \( (\prod_{i \in J} \prod U_j M_i) \) is an infinite direct product, here also each invariant is either 1 or infinite. Furthermore, for each ultrafilter \( U \) extending \( F \) there is a canonical projection \( \pi_U: \prod F M_i \to \prod U M_i \). I claim that the kernel of this projection is pure in \( \prod F M_i \). This follows from the fact that a tuple \( \bar{m} \) of \( \prod_{i \in I} M_i \) represents an element of the kernel of \( \pi_U \) if and only if its support is
THEORIES OF MODULES CLOSED UNDER DIRECT PRODUCTS

equal to $\emptyset$ modulo $U$. Now if $\bar{m}$ satisfies a positive primitive formula $\exists \bar{y} \land_{i} \eta_{i}(\bar{x}, \bar{y})$ in $\prod_{F} M_{i}$, then by Lemma 2.2 it is possible to find a $\bar{y}/F$ such that $\land_{i} \eta_{i}(\bar{m}/F, \bar{y}/F)$ and such that the support of $\bar{y}$ is included in the support of $\bar{m}$; hence $\bar{y}/F$ is also in the kernel of $\pi_{U}$. Since the kernel of $\pi_{U}$ is pure in $\prod_{F} M_{i}$, it follows that (see [5, Lemma 3.23(a)]) $\text{Inv}(\prod_{U} M_{i}, \varphi, \psi) \leq \text{Inv}(\prod_{F} M_{i}, \varphi, \psi)$ for every pair of positive primitive formulas $\varphi$ and $\psi$.

Therefore to show that $\prod_{F} M_{i}$ and $((\prod_{i \in J} \prod_{U_{i}} M_{i}))^{\sigma}$ are elementarily equivalent it is sufficient to show that for any Baur-Monk invariant greater than 1 in $\prod_{F} M_{i}$ there exists an ultrafilter $U$ extending $F$ such that this invariant is also greater than 1 in $\prod_{U} M_{i}$. This is exactly the statement of Proposition 2.8.

**Theorem 2.10.** Any reduced product $\prod_{F} M_{i}$ ($i \in I$) is elementarily equivalent to a direct product of ultraproducts of the modules $M_{i}$ ($i \in I$).

**Proof.** Let $\prod_{F} M_{i}$ be a reduced product of modules. By Corollary 2.5 it is elementarily equivalent to a direct sum of ultraproducts of the modules $M_{i}$ ($i \in I$) and of a reduced product with an atomless quotient. Since direct sums and direct products are elementarily equivalent (see [5, Lemma 2.24(a)]), it is sufficient to prove the result for reduced products with atomless quotient. Now Corollary 2.9 completes the proof.

**Theorem 2.11.** Let $T$ be a theory of modules. The following conditions are equivalent.

(a) $T$ is closed under direct product.

(b) $T$ is a Horn theory.

**Proof.** The second condition implies the first by Theorem 1.2. By Theorem 2.10, if the first condition is satisfied, i.e. if $T$ is closed under direct products, then it is also closed under reduced products; hence the second condition holds again by Theorem 1.2.

**Acknowledgments.** I would like to thank Professor Ulrich Felgner for suggesting this research theme and for many stimulating discussions (not only on this topic) as I was studying under his direction in Tübingen. This paper was written as I held a postdoctoral fellowship of the Conseil de Recherches en Sciences Naturelles et en Génie du Canada at McGill University.

**References**


**Département de Mathématiques et Informatique**

**Université du Québec à Montréal**

**Montréal, Québec H3C 3P8, Canada**

**E-mail:** villem@math.uqam.ca