A generalisation of plactic-coplactic equivalences
and Kazhdan-Lusztig cells

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Abstract

In this article, we propose a generalisation of plactic (Knuth) and
coplactic (dual-Knuth) equivalences to finitely generated Coxeter groups.
We obtain a decomposition of the left cells by coplactic classes which
agrees with two induction properties of the cells.

In the case of simply laced Coxeter groups, we give a crochet procedure
to study bijections between plactic and coplactic classes.

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1 Introduction

1.1 Main Results

In [12], Kazhdan and Lusztig have constructed their famous cells and left cells representations.

In symmetric groups, they have shown that the left cells representations are irreducible [12]. To prove this result, they use a combinatorial description of cells given by the Robinson-Schensted correspondence [20]. More precisely, they have observed that the left cells are precisely the coplastic (dual-Knuth) classes ([12, 26, 11], see also [1]); and these classes describe the Robinson-Schensted correspondence with subsets of symmetric groups (see [19]).

In this article, we take the “Coxeter groups point of view” of the definition of plactic and coplastic equivalences in symmetric groups; and we take it for a general definition in a finitely generated Coxeter system $(W, S)$.

We denote by $e$ the identity of $W$ and by $\ell(w)$ the length of $w$ as a word in the elements of $S$, for any $w \in W$. If $n = \ell(w)$ and $w = s_1 \ldots s_n$ say that $s_1 \ldots s_n$ is a reduced expression of $w$. Denote by $\leq$ the Bruhat order on $W$, that is, $u \leq v$ if $u$ is obtained from $v$ as a subexpression of a reduced expression of $v$. It is readily seen that $u \leq v$ implies $\ell(u) \leq \ell(v)$ (see [10]). Let $w \in W$, the set

$$D(w) = \{ s \in S | \ell(ws) < \ell(w) \} = \{ s \in S | ws \leq w \}$$

is called the descent set of $w$. For $J \subseteq S$, denote by $D_{\geq J}$ the set of elements of $W$ such that $D(w) = J$.

Definition Let $g, h \in W$, then $g$ is a plactic neighborhood of $h$, denoted $g \sim_P h$, if there is $s \in S$ such that:

a) $h = gs$,

b) $D(g) \not\subseteq D(h)$ and $D(h) \not\subseteq D(g)$.

Observe that if we take the condition $g \leq h$, (b) is simply $D(g) \not\subseteq D(h)$, since $s \in D(h)$ and $s \not\in D(g)$. We say that $g$ is a coplastic neighborhood of $h$, denoted $g \sim_C h$, if $g^{-1} \sim_P h^{-1}$. In symmetric groups, this definition is equivalent to the definition of Knuth and dual-Knuth relations (see [1]). The name plactic has been introduced by Lascoux and Schützenberger in [13].

The plactic equivalence $\sim_P$ (resp. coplastic equivalence $\sim_C$) is the reflexive and transitive closure of the plactic (resp. coplastic) neighborhood. The equivalence class

$$P(g) = \{ w \in W | w \sim_P g \}$$

is called a plactic classes. We define similarly the coplastic class $C(g)$. It is immediate from definitions that $P(e) = \{ e \}$;

$$P(g)^{-1} = \{ w | w^{-1} \in P(g) \} = C(g^{-1}), \quad \forall g \in W;$$
and if \( u \in W \) is an involution, \( P(u) = C(u)^{-1} \). That is, to study coplastic classes, we study plactic classes by taking inverses.

Finally, let \( \sim_{GP} \) the smallest equivalence on \( W \) containing both \( \sim_P \) and \( \sim_C \). We call carpets the equivalence classes for \( \sim_{GP} \). Observe that a carpet is a disjoint union of plactic (resp. coplastic) classes. These definitions follow the terminology of symmetric groups (see [4, Chapter 10]).

Let \( I \subset S \) and \( W_I \) be the parabolic subgroup of \( W \) generated by \( I \). It is well-known that if \( W_I \) is finite, there is a unique element (an involution) of maximal length, denoted by \( w_{0,I} \). Therefore, if \( W \) is finite, \( P(w_0) = \{w_0\} \), with \( w_0 = w_{0,S} \). We obtain the following link with Kazhdan-Lusztig cells (for a definition of cells, see Section 2.1).

**Theorem 1.1.** Let \( W \) be a Coxeter group then

1. each left cell of \( W \) is a disjoint union of coplastic classes;
2. each right cell of \( W \) is a disjoint union of plactic classes;
3. each two sided cell of \( W \) is a disjoint union of carpets.

Moreover, this decomposition agrees with some properties of the cells: Let \( I \subset S \) and \( C_I \) be a coplastic class of \( W_I \), then:

1. If \( x, y \in C_I \) and \( a \in X_I, xa^{-1} \) and \( ya^{-1} \) are in the same coplastic class of \( W \) (first induction property);
2. \( X_I \cdot C_I \) is an union of coplastic classes of \( W \) (second induction property);
3. If \( W_I \) is finite, \( w_{0,I} C_I \) and \( C_I w_{0,I} \) are two coplastic classes of \( W_I \).

**Remark.** 1): The above properties have been discovered, in the case of cells, by: (a) Lusztig [18]; (b) Geck [9]; and (c) Kazhdan and Lusztig [12].

2): A left-connected set is a subset \( X \) in \( W \) such that for all \( w, g \in X \), there are \( x_1, \ldots, x_n \in X \) such that \( x_1 = w, x_n = g \) and \( x_i x_{i+1}^{-1} \in S \). By definition, the coplastic classes are left-connected. Lusztig has conjectured [14] that each left cells are left connected (it is done in symmetric groups and affine Weyl groups of type \( A \) [24]. The above theorem gives a decomposition of any left cell as left-connected sets.

3) A question: Could we construct representations from coplastic classes which decompose left cells representations which agrees with induction properties.

**Geometrical plactic and coplastic relations:** We represent a plactic neighborhood \( g \sim_P h \) by an horizontal bar \( g \longrightarrow h \). As example, consider \( S_4 \) the symmetric group generated by the simple transposition \( s_i = (i \ i + 1) \), \( i = 1, \ldots, 3 \). Then

\[
\begin{align*}
  s_1 & \longrightarrow s_1 s_2 \longrightarrow s_1 s_2 s_3 .
\end{align*}
\]
In the same way, we represent a coplactic neighborhood \( g \sim_C h \) by a vertical bar

\[
\begin{array}{c}
g \\
\mid \\
h
\end{array}
\]

As example, with \( S_4 \) as above, we have

\[
\begin{array}{c}
s_1 \\
\mid \\
s_2 \\
\mid \\
s_3 s_2 s_1
\end{array}
\]

The case of simply laced Coxeter groups: In the Kazhdan-Lusztig theory, it is useful to study the relation between two left cells taken in a same two-sided cell. In symmetric groups, all left cells representations associated to left cells in a same two-sided cell are isomorphic, since coplactic classes in a same carpet are in bijection, with respect to plactic equivalence. In the case of simply laced Coxeter groups, we give a way to study the relation between two plactic (or coplactic) classes taken in a same carpet: the crochet procedure.

In symmetric groups, the crochet procedure has been introduced by Blessehol and Jölleneck in [3], to find a realisation of the Robinson-Schensted correspondence with subsets of the group. We generalise it to all simply laced Coxeter groups, with adding a conjugation property of involutions.

**Theorem 1.2 (Crochet procedure).** Let \( W \) be a simply laced Coxeter group and \( x, y, g \in W \).

i) If \( x \sim_P g \) and \( y \sim_C g \) then there is a unique \( h \in W \), \( h \neq g \), such that \( x \sim_C h \) and \( y \sim_P h \).

ii) If \( g \) is an involution and \( x = y^{-1} \), then \( h \) is an involution conjugate to \( g \).

The geometrical interpretation shows the “to weave a carpet” aspect of the crochet procedure:

\[
\begin{array}{c}
g \\
\mid \\
\vdots \\
y \\
\mid \\
h
\end{array}
\]

*The crochet procedure*

The crochet procedure is a useful tool to work on the combinatorial structure of a carpet, let us give the following first application.
Corollary 1.3. Let $u, v \in W$ such that $u \sim_C v$, then there is a unique bijection $\theta_{u,v} : P(u) \rightarrow P(v)$ verifying the following properties:

1. $\theta_{u,v}(u) = v$;
2. $\theta_{u,v}(x) \sim_C x$, for all $x \in P(u)$.

$$
\begin{align*}
  u &= g_1 \quad g_2 \quad g_3 \quad \cdots \quad g_n = x \\
  v &= h_1 \quad \vdots \quad \vdots \quad \vdots \\
  h_n &= \theta_{u,v}(x)
\end{align*}
$$

Geometrical illustration of Corollary 1.3

The following theorem is a generalisation of the above corollary.

Theorem 1.4. Let $T$ be a carpet in $W$.

i) $P \cap C \neq \emptyset$, for any plactic class $P$ in $T$ and any coplactic class $C$ in $T$.

ii) Let $P_1, P_2$ be plactic classes in $T$. Let $u \in P_1$. We choose $v \in C(u) \cap P_2$ and $u_1, \ldots, u_n \in T$ such that

$$
u = u_1 \sim_C u_2 \sim_C \cdots \sim_C u_n = v.$$

Then there is a bijection $\theta_{u,v} : P_1 \rightarrow P_2$, depending on the $u_i$, and verifying the following properties:

(a) $\theta_{u,v} = \theta_{u,v} \circ \theta_{u_2,u_1} \circ \cdots \circ \theta_{u,n,u_1}$, with $\theta_{u_i+1,u_i}$ as in Corollary 1.3;

(b) $\theta_{u,v}(u) = v$;

(c) $\theta_{u,v}(x) \sim_C x$, for all $x \in P_1$;

(d) $\theta_{u,v}(P \cap C) = P_2 \cap C$, for any coplactic class $C$ in $T$.

iii) For all plactic classes $P_1, P_2$ in $T$ and all coplactic classes $C_1, C_2$ in $T$, the sets $P_1 \cap C_1$ and $P_2 \cap C_2$ are in bijection.

Remark. If $P \cap C$ is a singleton, for any plactic class $P$ in $T$ and any coplactic class $C$ in $T$, then we have a “generalised” Robinson-Schensted correspondence

$$
\begin{align*}
  W &\twoheadrightarrow \bigcup_{T \text{plax}} \{ (P, C), P \subseteq C \} \\
  w &\mapsto (P(w), C(w)).
\end{align*}
$$

But, there is coplactic class which contains more than one element (see Section 1.2 where we give a coplactic class $C$ with two involutions, that is, $C \cap C^{-1}$ contains two involutions).

If a carpet contains an involution, an other application of the crochet procedure is the following theorem.
**Theorem 1.5.** Let $\mathcal{T}$ be a carpet which contains an involution $u$.

i) All plactic classes in $\mathcal{T}$ contain an involution conjugate to $u$.

ii) The map $w \mapsto w^{-1}$ is a self-preserving bijection on $\mathcal{T}$ which sends a plactic class on a coplactic class. In particular, any plactic class $P$ in $\mathcal{T}$ and any coplactic class $C$ in $\mathcal{T}$ are in bijection.

iii) Let $P_1, P_2$ be plactic classes in $\mathcal{T}$ and $\text{Inv} W$ be the set of involutions in $W$. There is a bijection

$$\Phi : P_1 \cap \text{Inv} W \to P_2 \cap \text{Inv} W$$

such that $\Phi(x)$ is conjugated to $x$, for all $x \in P_1 \cap \text{Inv} W$.

**Remark.** If there is no involution in a plactic class, or a coplactic class, the above theorem implies that there is no involution in the carpet which contains this class.

The proof of Theorem 1.1 is given in Section 2.1. As second part of this section, we have added a study of descent sets by means of an equivalence relation for which they are the equivalence classes. This is a generalisation, to finitely generated $W$, of a result of Atkinson [2] in the case of finite Coxeter groups.

Section 3 is allotted to the proof of the crochet procedure and its applications. These proofs use some useful tools like a distance between two plactic classes with respect to coplactic equivalence and a standard square which is a loop to work on a piece of a carpet. Finally, as an application of the conjugacy property in the crochet procedure, we give a new proof of a result of Schützenberger in symmetric groups: all involutions in a carpet are conjugated.

### 1.2 An example in $D_4$.

Consider $W$ the simply laced Coxeter group of type $D_4$, that is, its Coxeter diagram is

$$s_2
\begin{cases} 
  \downarrow \\
  s_3 \\
  \downarrow \\
  s_1
\end{cases}
\quad s_4$$

Using the GAP part of CHEVIE (see [22] and [7]), we obtain the following left cell of $D_4$ and its decomposition into two coplactic classes. This implies that coplactic classes are not the left cells in general (but it is done in symmetric groups), and that there is coplactic class with no involution or more that one involution. This left cell contains 14 elements and two coplactic classes, the first one contains 12 elements and 2 involutions ($s_1s_3s_1$ and $s_1s_3s_4s_3s_1s_3$); the second one contains 2 elements and no involution. The cell is left connected, indeed, $s_1s_3s_4s_3s_1s_3 = s_2s_4s_3s_1s_3 = s_4s_2s_3s_1s_3$ is connected to $s_4s_3s_1s_3$.
and \( s_2 s_3 s_1 s_3 \).

We begin and we end the following “snake” geometrical representation with the same element.

\[
\begin{array}{c|cc}
& s_1 s_3 s_2 s_4 s_3 s_1 s_3 & s_3 s_4 s_3 s_1 s_3 \\
\hline
s_1 s_3 s_4 s_2 s_3 s_1 s_2 s_3 & \quad & \quad \\
& s_4 s_1 s_3 s_4 s_2 s_3 s_1 s_3 & \quad \\
& \quad & s_4 s_3 s_1 s_3 \\
& s_3 s_4 s_2 s_3 s_1 s_2 s_3 & \quad & \quad \\
& \quad & s_3 s_4 s_3 s_1 s_3 \\
& s_4 s_2 s_3 s_1 s_2 s_3 & \quad & \quad \\
& \quad & s_3 s_3 s_1 s_3 \\
& s_2 s_3 s_1 s_2 s_3 & = & s_3 s_4 s_3 s_1 s_3 \\
& \quad & s_2 s_4 s_3 s_1 s_3 \\
& \quad & \quad \\
& s_2 s_3 s_1 s_3 & = & s_3 s_4 s_3 s_1 s_3 \\
\end{array}
\]

\[
\begin{array}{c|c}
& s_1 s_2 s_4 \quad 3 \quad 1 \\
\hline
s_3 s_1 s_2 s_4 & \quad \\
\end{array}
\]

\[
\begin{array}{c|c}
s_3 s_1 s_2 s_4 & \quad \\
\end{array}
\]

2 The general case

2.1 Proof of Theorem 1.1

Kazhdan-Lusztig cells: Our basic references for the work of Kazhdan and Lusztig is [12] (see also [5]).

Let \( \mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}] \) where \( q^{1/2} \) is an indeterminate. Let \( \mathcal{H} \) be the Hecke algebra over \( \mathcal{A} \) corresponding to \( W \). Let \( (T_w)_{w \in S_n} \) be the standard basis of \( \mathcal{H} \) and \( (\tilde{T}_w)_{w \in W} \) the basis defined as follow:

\[
\tilde{T}_w = q^{-l(w)/2} T_w.
\]

In [12, Theorem 1.1], Kazhdan and Lusztig have shown that there is a basis \((b_w)_{w \in S_n}\) of \( \mathcal{H} \), called the Kazhdan-Lusztig basis, such that

\[
b_w = \sum_{y \leq w} (-1)^{l(w)-l(y)} q^{l(w)-l(y)/2} P_{y,w}(q^{-1}) \tilde{T}_y,
\]

where \( P_{y,w} \in \mathcal{A} \) are polynomials, called the Kazhdan-Lusztig polynomials.

For \( y, w \in W \), we write \( y \prec w \) if \( y \leq w \) and \( \mu(y, w) \neq 0 \), where \( \mu(y, w) \) is the coefficient of \( q^{l(w)-l(y)-1}/2 \) in the Kazhdan-Lusztig polynomial \( P_{y,w} \). In this case, we define \( \mu(w, y) = \mu(y, w) \).
We define the elementary relation $<_L$ on $W$ as follow: $y <_L w$ if and only if $y \leq w$, or $w \leq y$, $D(y^{-1}) \not\subseteq D(w^{-1})$ and $\mu(w, y) \neq 0$. The preorder $\preceq_L$ is the reflexive and transitive closure of $<_L$. The equivalence relation defined by $\preceq_L$ is denoted by $\sim_L$, that is, $y \sim_L x$ if and only if $y \preceq_L w$ and $w \preceq_L y$. An equivalence class for $\sim_L$ is called a left cell.

Write $y \preceq_R w$ if and only if $y^{-1} \preceq_L w^{-1}$. Denote $\sim_R$ the equivalence relation defined by $\preceq_R$ and $\sim_{LR}$ the smallest equivalence containing both $\sim_L$ and $\sim_R$. An equivalence class for $\sim_R$ is called a right cell and for $\sim_{LR}$ is called a two-sided cell.

**Definition** Write $x \sim_L w$, say $x$ is a left cell neighborhood of $w$, if $w <_L x$ and $x <^c_L w$ are elementary relations. We denote $-L$ the reflexive and transitive closure of $\sim_L$, that is, if $x, w \in W$ such that $x \sim_L w$ there are $x_1, \ldots, x_n \in W$ such that

$$x = x_1 \sim_L x_2 \sim_L \ldots \sim_L x_n = w.$$ 

Similarly, we define $-R$ and $-LR$.

We have seen, in Example 1.2, that these equivalences do not coincide with $\sim_L$, $\sim_R$ and $\sim_{LR}$ in general. The first part of Theorem 1.1 is a direct consequence of the following proposition.

**Proposition 2.1.** Let $W$ be a finitely generated Coxeter group, then $w \sim_C x$ if and only if $w \sim_L x$.

In particular, the equivalence classes of $-L$ (resp. $-R$) are the coplactic (resp. plastic) classes of $W$; and the carpets of $W$ are the equivalence classes of $-LR$.

**Remark.** In particular, as Knuth-plactic classes are the right cells in symmetric groups (see [1]), the following proposition shows that the above equivalences coincide in symmetric groups, in other words, Knuth elementary relations can be seen with Kazhdan-Lusztig terminology. In Example 1.2, we see that $\sim_C$ and $-L$ do not coincide in a more general case.

For the proof of $w \sim_C x$ implies $w \sim_L x$, we follow the symmetric group case one. The proposition is a direct consequence of the the next lemma.

**Lemma 2.2.** Let $w, g \in W$ then:

$$w \sim_C g \iff w \sim_L g.$$ 

Recall this multiplication rule of the Kazhdan-Lusztig base [12]: Let $s \in S$ and $w \in W$ then

$$b_s b_w = \begin{cases} b_{sw} + \sum_{y \prec w, sy \preceq y} \mu(y, w) b_y & \text{if } sw > w \\ (q + q^{-1}) b_w & \text{if } sw < w \end{cases}.$$ 

8
Using this result and following [5], we have \( y \lessdot_L w \) if and only if \( y \neq w \) and \( b_y \) appears with a nonzero coefficient in \( b_sh \) for some \( s \in S \) [5, Lemma 5.3].

**Proof.** If \( w \prec_L g \) then, by definition, \( w \lessdot_L g \) and \( g \lessdot_L w \), and \( D(w^{-1}) \not\subseteq D(g^{-1}) \) and \( D(g^{-1}) \not\subseteq D(w^{-1}) \). One just has to show the following:

\( (\diamond) \) there is \( s \in S \) such that \( w = sg \).

step 1 As \( w \lessdot_L g \), \( g \neq w \) and there is \( s \in S \) such that \( b_w \) appears with a nonzero coefficient in \( b_sh \). Thus \( g \leq sg \). By \((*)\), either \( w = sg \) or \( w \prec g \) and \( sw \leq w \). In the first case, \( (\diamond) \) done.

step 2 As \( g \lessdot_L w \), there is \( t \in S \) such that \( g \neq w \) and \( b_g \) appears with a nonzero coefficient in \( b_tw \). Thus \( w \leq tw \). By \((*)\) again, either \( g = tw \) or \( g \prec w \) and \( tg \leq g \). In the first case, \( (\diamond) \) also done and the second case implies \( w = sg \) in the first step.

If \( w \precC g \), then there is \( s \in W \) such that \( w = sg \). Therefore \( w \leq g \) or \( g \leq w \). By definition of coplactic neighborhood, \( D(g^{-1}) \not\subseteq D(w^{-1}) \) and \( D(w^{-1}) \not\subseteq D(g^{-1}) \). Assume that \( w \leq g \), the other case is symmetric. As \( \ell(w) = \ell(g) - 1 \), \( \mu(w, g) = 1 \neq 0 \). Hence \( w \lessdot_L g \) and \( g \lessdot_L w \), and then \( w \precC g \).

**End of the proof of Theorem 1.1** We need the following lemmas. The first one gives a useful characterisation of plactic neighborhood.

**Lemma 2.3.** Let \( g, h \in W \). Then the following propositions are equivalent:

i) \( h \prec_P g \) if and only if there are \( u \in W \) and \( s, t \in S \) such that

\[ (*) \quad u \leq ut \leq uts \leq utst \quad \text{and} \quad \{g, h\} = \{ut, uts\}; \]

and either \( g = ut \) and \( h = uts \) if \( g \leq h \), or \( g = uts \) and \( h = ut \) if \( h \leq g \).

ii) \( h \precC g \) if and only if there are \( u \in W \) and \( s, t \in S \) such that

\[ u \leq tu \leq stu \leq tstu \quad \text{and} \quad \{g, h\} = \{tu, stu\}; \]

and either \( g = tu \) and \( h = stu \) if \( g \leq h \), or \( g = stu \) and \( h = tu \) if \( h \leq g \).

**Proof.** Assume that \( g \leq h \), the other case is symmetric.

If \( h \prec_P g \), there is \( s \in S \) such that \( h = gs \) and \( D(g) \not\subseteq D(h) \). Hence there is \( t \in D(g) \) such that \( h \leq ht \). Write \( u = gt \) then

\[ u \leq ut = g \leq uts = h \leq ht = utst. \]

Conversely assume that \((*)\) holds. Then, as \( g \leq h \), \( g = ut \) and \( h = uts \). As \( s \in D(h) \) and \( g \leq gs \) one has \( D(h) \not\subseteq D(g) \). In the same way, with \( t \in D(g) \), one shows that \( D(g) \not\subseteq D(h) \), hence \( h \prec_P g \). For coplactic classes, one takes inverses. \( \square \)
\textbf{Lemma 2.4.} \textit{Let }$W$\textit{ be finite, then}
\[ x \prec p xs \iff w_0x \prec p w_0xs \iff xw_0 \prec p xsw_0 \]
and
\[ x \prec_C sx \iff w_0x \prec_C w_0sx \iff xw_0 \prec_C sxw_0. \]

\textit{Proof.} Recall that $\ell(w_0w) = \ell(w_0) - \ell(w)$ and that $u \leq w$ if and only if $w_0w \leq w_0u$, for all $u, w \in W$. One has $x \prec_p xs$ one assumes that $x \leq xs$ (the other case is symmetric). By Lemma 2.3, there is $t \in S$ such that
\[ xt \leq x \leq xs \leq xt. \]
Thus
\[ (w_0x)st \leq (w_0x)s \leq (w_0x)t. \]
Therefore $w_0x \prec_p w_0xs$ by Lemma 2.3. The equivalence follows from $w_0^2 = e$. On the other hand, as the conjugation by $w_0$ is a bijection on $S$, there are $s', t' \in S$ such that $sw_0 = w_0s'$ and $tw_0 = w_0t'$. Thus
\[ xsw_0 = (xw_0)s't' \leq xsu_0 = (xw_0)s' \leq (xw_0)le \leq xtw_0 = (xw_0)t'. \]
Therefore $xw_0 \prec_p xw_0s' = xsw_0$ by Lemma 2.3 again. The equivalence follows from $w_0^2 = e$. For coplactic neighborhoods, one takes inverses and the lemma is proved. \hfill \Box

We recall here the useful Deodhar’s Lemma ([6] or see [8, Lemma 2.1.2]).

\textbf{Lemma 2.5 (Deodhar’s Lemma).} \textit{Let }$I \subset S$, $x \in X_I$ \textit{and }$s \in S$ \textit{then}
\begin{enumerate}[(i)]
\item if $s \leq x$, $sx \in X_I$;
\item if $x \leq sx$, either $sx \in X_I$ or $sx = xr$ with some $r \in I$.
\end{enumerate}

\textit{Proof of the second part of Theorem 1.1.} (a) follows from definition and Lemma 2.3.
(c) is a direct consequence of Lemma 2.4.

For (b): Let $u = u't' \in X_I \cdot C_I$, one just has to show that if $v \prec_C u$ then $v \in X_I \cdot C_I$. As $v \prec_C u$, there is $s \in S$ such that $u = sv$. One uses parabolic components: By Deodhar’s Lemma, $sv' \in X_I$ or $sv' = v'r$ for some $r \in I$. In the first case, by uniqueness of the parabolic components, $sv' = u'$ and $v_I = u_I'$; Therefore $u, v \in X_I\{e\}$ and \{e\} is a coplactic class of $W_I$.

In the second case, $u' = v'$ and $u_I = rv_I$. It remains to show that $v_I \in C_I$. As $v \leq u$ and $u' = v'$, $v_I \leq rv_I = u_I$ in $W_I$. Hence $D(u_I'^{-1}) \not\subseteq D(v_I^{-1})$. As $u \prec_C v$, there is $t \in S$ such that $tv \leq u$ and $u \leq tu$. If $tu' = u't'$ with $t' \in I$ then $u_I \leq t'u_I$ and $t'v_I \leq v_I$ since $u_I = v_I$. Therefore, $D(u_I'^{-1}) \not\subseteq D(u_I'^{-1})$ and $v_I \in C_I$. If $tu_I \in X_I$, then on the one hand, $tv = tu_Iv_I \leq v = v'v_I$ hence $tu_I \leq v'I$. On the other hand, $u'u_I \leq tu'I$ thus $u = u'I \leq tu'I$ contradicting $u'I = v'I$. \hfill \Box

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2.2 Descent Classes

In [2], the author gives a new proof of the Solomon result [25] using descent sets. We extend these results to \( W \) to define a equivalence relation related to descent set. This point of view will be useful in Section 3.

**Definition** Let \( g, h \in W \), then \( g \) is a descent neighborhood of \( h \), denoted \( g \sim_D h \), if there is \( s \in S \) such that \( h = sg \) and there is no \( t \in S \) such that \( sg = gt \). As easily seen, the relation \( \sim_D \) is symmetric.

The smallest equivalence \( \sim_D \) on \( W \) refining the descent neighborhood is called the descent equivalence. That is, if \( g \sim_D h \), there are \( g_1, \ldots, g_n \in W \) such that

\[
g = g_1 \sim_D g_2 \sim_D \cdots \sim_D g_n = h .
\]

Let \( g \in W \), the equivalence class

\[
[g]_D = \{ h \in W | h \sim_D g \}
\]

is called the descent class of \( g \). Observe that descent classes are left-connected.

The terminology descent is explained in the following result:

**Proposition 2.6.** Let \( g, h \in W \) then

\[
g \sim_D h \iff D(g) = D(h) .
\]

**Corollary 2.7.** Let \( I \subseteq S \) and \( g \in D_{=I} \), then \( [g]_D = D_{=I} \).

Before proving the proposition, we need to recall some well-known results (see [8, Chapter 1, 2] for the two last Lemmas). Let \( I \subseteq S \), then the cross section of \( W/W_I \) consisting of the unique coset representatives of minimal lengths ([10, Section 5.12]) is given by

\[
X_I = \{ x \in W | \ell(xs) > \ell(x), \forall s \in I \}
\]

\[
= \{ x \in W | x \leq xs, \forall s \in I \} .
\]

Let \( w \in W \), then there is a unique \( (w^I, w_I) \in X_I \times W_I \) such that \( w = w^I w_I \). The couple \((w^I, w_I)\) is called the parabolic components of \( w \). Moreover, \( w^I \) is the unique element of smallest length in the coset \( wW_I \) and \( \ell(w) = \ell(w^I) + \ell(w_I) \). Observe that \( X_I \) is a disjoint union of \( D_{=I} \) for \( I \subseteq S \setminus J \).

**Lemma 2.8.** Let \( I \subseteq S \) and \( w \in W \) such that \( D(w) = I \), then \( W_I \) is finite. In particular, the longest element \( w_{0,1} \) of \( W_1 \) is well defined.

**Proof.** Write \( w = w^I w_I \). Thus for all \( s \in I \), \( w_I s \leq w_I \). Observe that if \( e \leq u \leq w_I \), for any \( u \in W_I \), then \( W_I \) is finite.

One proves that \( e \leq u \leq w_I \), for any \( u \in W_I \), by induction on \( \ell(u) \). Let \( u \in W_I \), if \( u = e \), the lemma done. Assume that \( \ell(u) > 0 \), then there is \( s \in I \) such that \( us \leq u \). By induction, one has \( us \leq w_I \). By [10, Proposition 5.9], one has \( u = (us)s \leq w_I \) or \( u \leq w_I s \). As \( w_I s \leq w_I \) for all \( s \in I \), \( u \leq w_I \) and \( w_I = w_{0,1} \).

\( \square \)
Lemma 2.9. Let $I \subset S$ and $w \in D_{=I}$. Then there is a unique $x \in X_I$ such that $w = x w_{0,I}$, where $w_{0,I}$ is the longest element of $W_I$.

Proof. Let $(x,y)$ be the parabolic components of $w$. If $y \neq w_{0,I}$, then there is $s \in I$ such that $\ell(ys) > \ell(y)$. Thus

$$\ell(ws) = \ell(xys) = \ell(x) + \ell(ys) > \ell(x) + \ell(y) = \ell(w).$$

Therefore $s \notin D(w) = I$, which is a contradiction. □

Corollary 2.10. Let $I \subset S$ then $w_{0,I}$ is the unique element of minimal length in $D_{=I}$.

We recall here the useful \textit{(right) exchange condition} (see [10, 5.8]). Let $w \in W$ and $w = s_1 \ldots s_n$ an expression of $w$, not necessarily reduced, with $s_i \in S$. For all $s \in D(w)$, there is an $1 \leq i \leq n$ such that

$$w = s_1 \ldots \hat{s}_i \ldots s_n,$$

where the symbol $\hat{s}_i$ denotes that $s_i$ is omitted. If the expression of $w$ is reduced, that is $n = \ell(w)$, then the index $i$ is uniquely determined, and this new expression of $w$ is also reduced. Observe that if $s \in D(w)$, there is a reduced expression of $w$ ending by $s$. In other words, $w = s_1 \ldots s_{n-1}s$ is a reduced expression. In the same way, we have a \textit{(left) exchange condition}.

Lemma 2.11. Let $I \subset S$ and $w = s_1 \ldots s_n \in X_I$ a reduced expression. Then

$$s_i \ldots s_n \in X_I \quad \forall 1 \leq i \leq n.$$

Proof. Assume there is $1 \leq i \leq n$ such that $g = s_i \ldots s_n \notin X_I$. Then there is $s \in I$ such that $\ell(gs) < \ell(g)$. Thus, by exchange condition, there are $i \leq j \leq n$ such that $g = s_i \ldots \hat{s}_j \ldots s_n$. Therefore

$$w = s_1 \ldots s_{i-1}g = s_1 \ldots \hat{s}_j \ldots s_n s ;$$

and this expression is reduced. Thus $\ell(ws) < \ell(w)$ which is a contradiction, since $w \in X_I$ and $s \in I$. □

Proof of Proposition 2.6. On one hand, if $D(g) = D(h) = I$, there is $x \in X_I$ such that $g = xw_{0,I}$, by Lemma 2.9. Without loss of generality, assume that $h = w_{0,I}$.

Let $x = s_1 \ldots s_n$ be a reduced expression. One proves by induction on $n = \ell(x)$ that

$$w_{0,I} \prec_D s_n w_{0,I} \prec_D \ldots \prec_D s_2 \ldots s_n w_{0,I} \prec_D g .$$

The case $n = 0$ is trivial. If $n > 0$ then $g = xw_{0,I} = s_1 \ldots s_n w_{0,I}$. Case $n = 1$: one has to show that $w_{0,I} \prec_D g = s_1 w_{0,I}$. Otherwise, there is $t \in S$ such that $g = s_1 w_{0,I} = w_{0,I}t$ which contradicts $D(g) = I$. Case $n > 1$: Let $x' = s_2 \ldots s_n$, then $x' \in X_I$ by Lemma 2.11. Thus $g' = s_1 g = x' w_{0,I}$. By induction one has

$$w_{0,I} \prec_D s_n w_{0,I} \prec_D \ldots \prec_D s_2 \ldots s_n w_{0,I} = g',$$

12
hence \( g' \sim_D w_{0,t} \). It remains to show that \( g' \prec_D g \). Otherwise, there is \( t \in S \) such that \( g' = s_1 g = gt \). As \( g = gt \) with \( \ell(g) = \ell(g') + 1 \), one has \( t \in D(g) = I \). Thus \( g = xw_{0,t} = x'w_{0,t} \). But \( w_{0,t} \in W_I \). Hence \( x = x' \) by uniqueness of the parabolic components, which is a contradiction. Thus \( g' \prec_D g \) and (*) is proved. In particular, \( g \sim_D g' \sim_D w_{0,t} \).

On the other hand, if \( g \sim_D h \) there are \( g_1, \ldots, g_n \in W \) such that

\[
g = g_1 \prec_D g_2 \prec_D \cdots \prec_D g_n = h.
\]

Thus one just has to show that \( g \prec_D h \) implies \( D(g) = D(h) \). As \( g \prec_D h \), there is \( s \in S \) such that \( g = sh \). If there is \( t \in D(g) \) and \( t \notin D(h) \), the exchange condition implies that \( sht = h \) or \( sh = ss_1 \ldots s_i \ldots s_k t \) where \( h = s_1 \ldots s_k \) is a reduced expression. The first case contradicts \( g \prec_D h \). The second case implies that \( h = s_1 \ldots s_i \ldots s_k t \) is a reduced expression, which contradicts \( t \notin D(h) \). Therefore \( D(g) \subseteq D(h) \). To prove the other inclusion, one proceeds similarly.

**Corollary 2.12 (of the proof).** Let \( I \subseteq S \), \( u \in D_{=I} \) and \( x \in X_I \) such that \( u = xw_{0,I} \). Let \( x = s_1 \ldots s_n \) be a reduced expression then

\[
w_{0,I} \prec_D s_nw_{0,I} \prec_D \cdots \prec_D s_2 \ldots s_nw_{0,I} \prec_D u.
\]

The next proposition can be obtain as a corollary of Theorem 1.1 and a well-known property linking left cells and descent sets [12]. We give here a direct proof using the descent equivalence relation.

**Proposition 2.13.** Let \( g, h \in W \) then \( g \sim_C h \implies D(g) = D(h) \).

**Proof.** One just has to show that \( g \sim_C h \implies g \sim_D h \) to prove (i). Without loss of generality, one assumes that \( g \leq h \). By Lemma 2.3, one may assume

\[
v \leq sv = g \leq tsv = h \leq stsv,
\]

with \( s, t \in S \) and \( v \in W \). Hence \( \ell(stsv) > \ell(g) \). If \( g \not\sim_D h \), there is \( r \in S \) such that \( tsv = svr \). Thus \( stsv = vr \) therefore \( \ell(stsv) = \ell(vr) \leq \ell(v) + 1 = \ell(g) \) which is a contradiction.

We end with the following observation, that gives a connection between \( \preceq_L \) and \( \sim_D \).

**Proposition 2.14.** Let \( I \subseteq S \) then for all \( u \in D_{=I} \) we have \( u \preceq_L w_{0,I} \).

**Proof.** Let \( x \in X_I \) such that \( u = xw_{0,I} \) and \( x = s_1 \ldots s_n \) be a reduced expression. Denote \( u_i = s_i \ldots s_nw_{0,I} \). By Lemma 2.12, \( u_{i+1} \prec_D u_i \). As \( u_{i+1} \leq u_i \) and \( \ell(u_i) = \ell(u_{i+1}) + 1 \), one has \( \mu(u_i, u_{i+1}) = 1 \) by definition. Moreover, \( D(u_{i+1}^{-1}) \notin D(u_{i+1})^{-1} \) since \( s_iu_i \leq u_i \) and \( u_{i+1} \leq s_iu_{i+1} \). Therefore \( u_i \preceq_L u_{i+1} \). The proposition follows since \( u_n \preceq_L w_{0,I} \).
3 The case of simply laced Coxeter groups

From now, $W$ is a simply laced Coxeter group, that is, for all $s, t \in S$ we have $1 \leq m_{s,t} \leq 3$.

3.1 Proof of The crochet procedure

The following lemma, with Lemma 2.3, gives a new characterisation of plastic and coplastic neighborhood in the case of simply laced Coxeter groups.

Lemma 3.1. Let $v \in W$ and $s, t \in S$. Then

\[ v \leq vs \leq vst \leq vsts \iff v \in X_{\{s,t\}} \text{ and } m_{s,t} = 3 \]

and

\[ v \leq sv \leq tsv \leq stsv \iff v \in X_{\{s,t\}}^{-1} \text{ and } m_{s,t} = 3 . \]

Proof. Observe that the lemma follows for the second equivalence by taking inverses.

If $v \in X_{\{s,t\}}$ and $m_{s,t} = 3$ then $\ell(vy) = \ell(v) + \ell(y)$, for all $y \in W_{\{s,t\}}$. As $s, st, sts$ are reduced expression and $v \in X_{\{s,t\}}$, 

\[ v \leq vs \leq vst \leq vsts . \]

Conversely, observe that $m_{s,t} > 2$. As $W$ is a simply laced Coxeter group, $m_{s,t} = 3$. By hypothesis, one has $\ell(vs) > \ell(v)$. Assume that $\ell(vt) < \ell(v)$. Then one writes $v = r_1 \ldots r_n t$ (reduced expression). Thus

\[ vst = r_1 \ldots r_n tst , \quad \text{reduced expression} \]

and

\[ vsts = r_1 \ldots r_n tsts = r_1 \ldots r_n st . \]

Therefore $\ell(vsts) \leq n + 2 < n + 3 = \ell(vst)$, which is a contradiction. Thus $v \leq vs$ and $v \leq vt$. Therefore $v \in X_{\{s,t\}}$ and the Lemma is proved. \hfill \Box

The following lemma gives a uniqueness property which strongly depends on the simply laced hypothesis.

Lemma 3.2 (Unicity Lemma). Let $x, g, h \in W$ such that $x \prec_P g$, $x \prec_P h$ and $g \sim_C h$. Then $g = h$.

Remark. This result is generally false for Coxeter groups which are not simply laced. As example, Coxeter groups of type $I_2(\bar{m})$, where $\bar{m} \geq 4$.

Proof. By definition, there are $s_1, s_2 \in S$ such that $g = x s_1$ and $h = x s_2$. As $g \sim_C h$, Proposition 2.13 and Proposition 2.6 imply that $D(g) = D(h) = I$. Then there is $x_1, x_2 \in X_I$ such that $g = x_1 w_{0,I}$ and $h = x_2 w_{0,I}$, by Lemma 2.9.

Case 1. If $s_1, s_2 \in I$ then $x = x_1 w_{0,I} s_1 = x_2 w_{0,I} s_2$. As $w_{0,I} s_i \in W_I$, one has
$x_1 = x_2$, by uniqueness of the parabolic components. Therefore $g = h$.

Case 2. If $s_1, s_2 \notin I$ then $s_1, s_2 \in D(x)$. Hence $\ell(h) = \ell(x) - 1$. Let $x = r_1 \ldots r_n s_1$ a reduced expression, $r_i \in S$. By exchange condition, with $s_2$, one has

$$x = \begin{cases} r_1 \ldots \hat{r}_i \ldots r_n s_1 s_2 \\
or \\
r_1 \ldots r_n s_2 \end{cases}$$

If $x = r_1 \ldots r_n s_2$ then $s_1 = s_2$. Therefore $g = h$. If $x = r_1 \ldots \hat{r}_i \ldots r_n s_1 s_2$ then $h = r_1 \ldots \hat{r}_i \ldots r_n s_1$ is a reduced expression, since $\ell(h) = \ell(x) - 1$. Thus $s_2 \in I$, which is a contradiction.

Case 3. If $s_1 \in I$ and $s_2 \notin I$, then $s_2 \in D(x)$ and $\ell(g) = \ell(h) + 2$ (this case is symmetric). As $s_2 \in D(x)$, there is a reduced expression $x = r_1 \ldots r_n s_2$. As $h = r_1 \ldots r_n = xs_2$ and $s_1 \in I$, one has by exchange condition

$$h = r_1 \ldots \hat{r}_i \ldots r_n s_1,$$

which is a reduced expression. Therefore

$$g = xs_1 = hs_2 s_1 = r_1 \ldots \hat{r}_i \ldots r_n s_1 s_2 s_1.$$

If $s_1 s_2 = s_2 s_1$ then $g = r_1 \ldots \hat{r}_i \ldots r_n s_2$ which contradicts $\ell(g) > \ell(h)$. As $W$ is simply laced, the only other case is $s_1 s_2 s_1 = s_2 s_1 s_2$. Hence

$$g s_2 = r_1 \ldots \hat{r}_i \ldots r_n s_2 s_1$$

thus $\ell(g s_2) \leq \ell(h) + 1 < \ell(g)$ which contradicts $s_2 \notin I$ and the lemma is proved.

In the following, we use a special terminology: we say that $w = u_1 u_2$ is a reduced expression of $w$ if $\ell(w) = \ell(u_1) + \ell(u_2)$. By induction, we say that $w = u_1 \ldots u_k$ is a reduced expression of $w$ if $\ell(w) = \ell(u_1) + \cdots + \ell(u_k)$, with $u_i \in W$. In particular, if we take a reduced expression for each $u_i$, we obtain a (usually) reduced expression for $w$. As example, in our terminology, the parabolic component $(w', w_1)$ gives a reduced expression of $w$.

Lemma 3.3. Let $u, v \in W$ and $s_1, s_2, t_1, t_2 \in S$ such that

$$u \leq s_1 u \leq t_1 s_1 u \leq s_1 t_1 s_1 u$$

and

$$u \leq us_2 \leq us_2 t_2 \leq us_2 t_2 s_2 .$$

i) If $s_1 u = vs_2$ then either $u = v$ and $t_1 u = vt_2$, or $t_1 v \in X_{\{s_1, u_2\}}$ and $ut_2 \in X^{-1}_{\{s_1, u_2\}}$.

ii) If $s_1 u = vs_2 t_2$ then $t_1 v \in X_{\{s_2, u_2\}}$ and $ut_2 \in X^{-1}_{\{s_1, u_2\}}$.

Proof. Case $s_1 u = vs_2$. Observe that $\ell(s_1 u) = \ell(vs_2) = \ell(u) + 1 = \ell(v) + 1$ hence

$$\ell(u) = \ell(v) .$$

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1. First, one proves that \( t_1v \in X_{s_1} \) and \( ut_2 \in X^{-1}_{s_1} \).  
If \( t_1v \not\in X_{s_1}, \ell(t_1uv) < \ell(t_1v) \). By Lemma 3.1, \( v \in X_{\{s_1,t_2\}} \subset X_{\{s_2\}} \). As \( v \leq t_1v \) and \( t_1v \not\in X_{s_2} \), one has \( t_1v = vs_2 \), by Deodhar's Lemma. This implies  
\[
t_1s_1ut_2s_2 = t_1vs_2t_2s_2 = vt_2s_2
\]
therefore \( v = t_1s_1u \). By Lemma 3.1, \( u \in X^{-1}_{\{s_1,t_1\}} \) thus \( \ell(u) < \ell(u) + 2 = \ell(v) \) which contradicts (\( \ast \)). Hence \( \ell(t_1uv) > \ell(t_1v) \). One proceeds similarly with \( ut_2 \).

2. On the other hand, as \( s_1u = vs_2 \), (\( \ast \)) implies  
\[
\ell(s_1uv) = \ell(u) < \ell(u) + 1 = \ell(u) + 1 = \ell(vs_2).
\]
Thus \( s_1uv \leq vs_2 \). Let \( v = r_1 \ldots r_n \) a reduced expression. By left exchange condition, one has  
\[
s_1uv = u = \begin{cases} v & \text{or} \\ vs_2 & \end{cases}
\]
with \( w = r_1 \ldots r_i \ldots r_n \), for some \( 1 \leq i \leq n \). Observe that the expression of \( w \) is reduced. One studies these two cases.

3. Case 1. If \( u = vs_2 \) then \( v = s_1uv = s_1w \) is also a reduced expression, by (\( \ast \)). As \( v = s_1w \in X_{\{s_2,t_2\}}, w \in X_{\{s_2,t_2\}} \) by Lemma 2.11. Now, observe that \( \ell(w) = \ell(vs_2) = \ell(s_1v) < \ell(v) = \ell(u) \), hence \( u = vs_2 \) is a reduced expression. By Lemma 2.11 again, \( w \in X_{\{s_1,t_1\}} \). One will show that  
\[
t_1v \in X_{\{s_2,t_2\}} \text{ and } ut_2 \in X^{-1}_{\{s_1,t_1\}}.
\]
By (1), one just has to prove that \( \ell(t_1vt_2) > \ell(t_1v) \) and \( \ell(t_1ut_2) > \ell(u) \). Assume that \( \ell(t_1vt_2) < \ell(t_1v) \). As in (1), \( t_1v = vt_2 \) by Deodhar's Lemma. This implies that \( t_1s_1w = s_1w \). As \( w \in X^{-1}_{\{s_1,t_1\}} \), one has  
\[
\ell(w) + 1 = \ell(wt_2) = \ell(s_1(s_1wt_2)) = \ell(s_1t_1s_1w) = \ell(w) + 3
\]
which is a contradiction. Therefore \( \ell(t_1vt_2) > \ell(t_1v) \). One proceeds similarly to show that \( \ell(t_1ut_2) > \ell(u) \).

4. Case 2. If \( u = v \) then \( u = v \in X^{-1}_{\{s_1,t_1\}} \cap X_{\{s_2,t_2\}} \), by Lemma 3.1. If \( t_1u \not\in X_{\{s_1,t_1\}} \), as above, \( t_1u = ut_2 \), by Deodhar's Lemma. 
If \( t_1u \in X_{\{s_2,t_2\}} \) then \( ut_2 \in X^{-1}_{\{s_1,t_1\}} \) or Deodhar’s Lemma imply \( ut_2 = t_1u \not\in X_{\{s_2,t_2\}} \subset X_{s_2} \), since \( t_1u \in X_{s_2} \) by (1).

Case \( s_1u = vs_2t_2 \): Observe that \( \ell(s_1u) = \ell(vs_2t_2) = \ell(u) + 1 = \ell(u) + 2 \), hence  
\[
(\phi) \quad \ell(u) = \ell(u) + 1.
\]
1. First, one proves \( ut_2 \in X_{(s_1, t_1)}^{−1} \).
   Assume that \( ut_2 \notin X_{\{s_1, t_1\}}^{−1} \). Thus \( s_1 u = ut_2 \) or \( t_1 u = ut_2 \), by Deodhar’s Lemma.

   If \( t_1 u = ut_2 \), one has
   \[ ut_2 = t_1 u = t_1 s_1 v s_2 t_2 \]
   thus \( s_1 t_1 u = v s_2 \). As \( u \in X_{\{s_1, t_1\}}^{−1} \) and \( v \in X_{\{s_2, t_2\}} \), one has \( \ell(u) + 1 = \ell(v) \)
   which contradicts (c).

   If \( s_1 u = ut_2 \), one has
   \[ v s_2 t_2 s_2 = s_1 u s_2 = u t_2 s_2 = s_1 v. \]

   Therefore one obtains the following contradiction
   \[ \ell(v s_2 t_2 s_2) = \ell(v) + 3 = \ell(s_1 v) \leq \ell(v) + 1. \]

2. Now, one proves \( t_1 v \in X_{\{s_2, t_2\}} \).
   Assume that \( t_1 v \notin X_{\{s_2, t_2\}} \). By Lemma 3.1, \( v \in X_{\{s_2, t_2\}} \). Thus \( t_1 v = v s_2 \)
   or \( t_1 v = v t_2 \), by Deodhar’s Lemma.

   If \( t_1 v = v s_2 \), one has
   \[ t_1 s_1 u = t_1 v s_2 t_2 = v t_2. \]

   As \( u \in X_{\{s_1, t_1\}}^{−1} \) and \( v \in X_{s_2, t_2} \),
   \[ \ell(t_1 s_1 u) = \ell(u) + 2 = \ell(v t_2) = \ell(v) + 1, \]
   therefore \( \ell(u) < \ell(v) \) which contradicts (c).

   If \( t_1 v = v t_2 \) then
   \[ s_1 t_1 s_1 u = s_1 t_1 v s_2 t_2 = s_1 v t_2 s_2 t_2 = u t_2 s_2 t_2 s_2 t_2 = u s_2, \]
   since \( s_2 t_2 s_2 = t_2 s_2 t_2 \) by Lemma 3.1. Therefore one obtains the following
   contradiction
   \[ \ell(s_1 t_1 s_1 u) = \ell(u) + 3 = \ell(u s_2) \leq \ell(u) + 1. \]

   The Lemma is proved.

\[ \square \]

**Proof of the Crochet Procedure 1.2, (i).** The uniqueness is given by Lemma 3.2.

As \( g \vartriangleright_C x \), \( D(x) = D(g) \) by Proposition 2.13. By definition \( D(x) \neq D(h) \).

Therefore, if \( h \) exists, \( D(h) \neq D(g) \) which implies \( h \neq g \).

**Existence:** As \( x \vartriangleright_C g \) and \( y \vartriangleright_C g \), by Lemma 2.3, there are \( u \in X_{\{s_1, t_1\}}^{−1} \),
\( v \in X_{\{s_2, t_2\}} \) and \( s_1, s_2, t_1, t_2 \in S \) such that
\[ u \leq s_1 u \leq t_1 s_1 u \leq s_1 t_1 s_1 u \quad \text{and} \quad \{g, x\} = \{s_1 u, t_1 s_1 u\} \]

and
\[ v \leq u s_2 \leq v s_2 t_2 s_2 \quad \text{and} \quad \{g, x\} = \{u s_2, u s_2 t_2\}. \]

Therefore, one has to consider four cases:

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1. \( g = s_1u = vs_2, \ x = vs_2t_2 \) and \( y = t_1s_1u \),

2. \( g = s_1u = vs_2t_2, \ x = vs_2 \) and \( y = t_1s_1u \),

3. \( g = t_1s_1u = vs_2, \ x = vs_2t_2 \) and \( y = s_1u \),

4. \( g = t_1s_1u = vs_2t_2, \ x = vs_2 \) and \( y = s_1u \).

Observe that the cases (3) and (4) are similar to the cases 2 and 1, by Lemma 2.4.

**Case 1.** By Lemma 3.3 (i), we have two possibilities.

a) If \( t_1u = ut_2 \) then \( g = s_1u = us_2, \ x = us_2t_2 = s_1t_1u \) and \( y = t_1s_1u = ut_2s_2 \). Let \( h = t_1u = ut_2 \), thus

\[
0 \leq h = t_1u \leq x = s_1t_1u \leq t_1s_1u
\]

and

\[
0 \leq h = ut_2 \leq y = ut_2s_2 \leq ut_2s_2t_2,
\]

since \( u = v \in X^{-1}_{\{t_1,t_1\}} \cap X_{\{s_2,t_2\}} \). Therefore \( x \sim_C h \) and \( y \sim_P h \) by Lemma 3.1.

b) If \( t_1v = X_{\{s_2,t_2\}} \) and \( ut_2 \in X^{-1}_{\{s_2,t_2\}} \). Let \( h = t_1vs_2t_2 = t_1s_1ut_2 \), then

\[
ut_2 \leq x = s_1ut_2 \leq h = t_1s_1ut_2 \leq s_1s_1ut_2
\]

and

\[
t_1v \leq y = t_1vs_2 \leq h = t_1vs_2t_2 \leq t_1vs_2t_2s_2 \cdot
\]

As above, \( x \sim_C h \) and \( y \sim_P h \) and this case is done.

**Case 2.** Observe that \( x = vs_2 = s_1ut_2 \) and \( y = t_1s_1u = t_1vs_2t_2 \).

Let \( h = t_1vs_2 = t_1s_1ut_2 \) then by Lemma 3.3 (ii)

\[
ut_2 \leq x = s_1ut_2 \leq h = t_1s_1ut_2 \leq s_1s_1ut_2
\]

and

\[
t_1v \leq h = t_1vs_2 \leq y = t_1vs_2t_2 \leq t_1vs_2t_2s_2 \cdot
\]

As above, \( x \sim_C h \) and \( y \sim_P h \) and the first part of the theorem is proved.

Now, let \( g \) be an involution and \( y = x^{-1} \) in the first part of the crochet procedure. As \( h \sim_C x \) and \( h \sim_P x^{-1} \), one has \( h^{-1} \sim_C x \) and \( h^{-1} \sim_P x^{-1} \). Hence

\[
h^{-1} \sim_P x^{-1} \sim_P h
\]

and therefore \( h \sim_C h^{-1} \). By Lemma 3.2, one has \( h = h^{-1} \). Therefore \( h \) is an involution.

The last part of the crochet procedure is a direct consequence of the following lemma.

**Lemma 3.4.** Let \( w, w' \in W \) be two distinct involutions and \( x \in W \) such that \( x \sim_P w \) and \( x \sim_C w' \). Then we have one of the following cases:
i) \( \ell(w') = \ell(w) + 2 \), and there is \( s \notin D(w) \) such that \( w' = sws \) and \( x = ws \);

ii) \( \ell(w') = \ell(w) - 2 \), and there is \( s \in D(w) \) such that \( w' = sws \) and \( x = ws \);

iii) \( \ell(w') = \ell(w) \), and there are:

\[
\begin{align*}
  & s \notin D(w) \text{ and } r \in D(w) \text{ such that } w' = rswrsr \text{ and } x = ws = rw', \\
  & \text{or } s \in D(w) \text{ and } r \notin D(w) \text{ such that } w' = rswrsr \text{ and } x = ws = rw'.
\end{align*}
\]

In particular, \( w \) and \( w' \) are conjugate.

Proof. By definitions of plastick and coplastick neighborhoods, there is \( s, r \in S \) such that \( x = ws = rw' \). Then \( x^{-1} = sw = w'r \) and therefore \( w = s'w'r = rw' \).

One begins to show the following properties

\[
(*) \quad ws \neq s \text{ and } rw' \neq w'r.
\]

If \( sw = ws \) then \( x = x^{-1} \). As \( x \succ_p w \) and \( x \succ_C w' \) there is a unique \( y \neq x \) such that \( y \succ_p w \) and \( y \succ_C w' \) (first part of crochet procedure). As \( x^{-1} \succ_C w \) and \( x^{-1} \succ_p w' \), \( y = x^{-1} \), by uniqueness of the crochet procedure, which contradicts \( y \neq x \). Proceeds similarly with \( rw' = w'r \).

Case A: \( s \notin D(w) \), that is, \( \ell(w') \geq \ell(w) \).

(A1) One has \( s \notin D(sw) \). Otherwise the exchange condition implies \( sw = ws \) or \( sw = s\tilde{w}s \), where \( \tilde{w} \) means that one has omitted a simple reflection in a reduced expression of \( w \). The case \( sw = ws \) contradicts \((*)\). Therefore \( sw = s\tilde{w}s \) and \( w = \tilde{w}s \). If we take a reduced expression of \( w \), then \( \tilde{w}s \) is a also reduced expression of \( w \). This forces \( s \in D(w) \) which contradicts \( s \notin D(w) \).

(A2) (A1) implies \( \ell(sws) = \ell(w) + 2 \). The case \( s = r \) implies (i) in the lemma.

(A3) If \( s \neq r \) we shall prove that \( r \in D(sw) \) and \( r \in D(w) \). Suppose that \( r \notin D(sw) \), then \( \ell(w') = \ell(w) + 2 \) since \( w' = swr = rws \) and \( s \notin D(w) \). Therefore, \( s, r \in D(w') \), hence \( r \notin D(w) \). By exchange condition with \( s \in D(w') \):

\[
\begin{align*}
  w' &= swr \begin{cases} 
  ws \quad (a) \\
  wrs \quad (b) \\
  \tilde{w}rs \quad (c)
\end{cases}.
\end{align*}
\]

(a) implies \( s = r \) which is a contradiction. (b) implies that \( sw = ws \) which contradicts \((*)\). Finally (c) implies \( w = \tilde{w}rsr \) (with \( \tilde{w} \) defined in (A1)). Considering \( \ell(w) \) and the reduction condition (see [8, Theorem 1.2.5]), one must delete two generators on the expression \( \tilde{w}rsr \) to obtain a reduced expression of \( w \). Hence, observe that there is \( s \) or \( r \) at the right end of a reduced expression of \( w \). Therefore, \( r \in D(w) \) or \( s \in D(w) \) which contradicts the hypotheses. This contradiction shows that \( r \notin D(w) \). Hence \( r \in D(sw) \). If \( r \notin D(w) \), the exchange condition with \( r \in D(sw) \) implies \( swr = w \) contradicting \( w' \neq w' = swr \).
(A4) By (A3), \( \ell(w) = \ell(w') \). Moreover, \( sr = rsr \). Otherwise \( sr = rs \) and
\[
w = sw'r = srsr = srs = sws = rwr.
\]
In this, (A2) implies:
\[
\ell(w) + 2 = \ell(sws) = \ell(rwr) \leq 1 + \ell(wr) = 1 + \ell(w) - 1 = \ell(w),
\]
since \( r \in D(w) \), by (A3), which is a contradiction. As \( W \) is simply laced, the only other case is \( srs = rsr \).

(A5) Take \( u = srs = rs = u^{-1} \) then \( w \) and \( w' \) are conjugate by \( u \) as follows:
\[
u^{-1}wu = (srs)w(rs) = sr(swrs)r = (swr)s = swr = w'.
\]
And the first part of (iii) is proved.

Case B: \( s \in D(w) \), that is, \( \ell(w') \leq \ell(w) \).

(B1) One has \( s \in D(sw) \). Indeed, otherwise \( \ell(sws) = \ell(w) \) and \( \ell(ws) = \ell(sw) \). Therefore, \( sw = ws \) ([8, Lemma 1.2.6]) which contradicts (x).

(B2) (B1) implies \( \ell(sws) = \ell(w) - 2 \). Case (ii) of the lemma follows if \( s = r \).

(B3) If \( s \neq r \) we shall prove that \( r \not\in D(sw) \) and \( r \not\in D(w) \). Suppose that \( r \in D(sw) \), then \( \ell(w') + 2 = \ell(sw) \) since \( w' = sw \) and \( s \in D(w) \). As \( sw' = wr \) and \( \ell(w) = \ell(w') + 2 \), one has \( r, s \in D(w) \). Now, one proceeds as in (A3) interchanging the role of \( w \) and \( w' \) to obtain a contradiction. Therefore \( r \not\in D(sw) \) and \( \ell(w') = \ell(w) \). If \( r \in D(w) \), one has \( \ell(sw) = \ell(w) \) and \( \ell(swrs) = \ell(w') = \ell(w) \). Therefore, \( w = rws = w' \) ([8, Lemma 1.2.6]) which is a contradiction.

(B4) (B3) implies \( \ell(w') = \ell(w) \) and \( srs = rsr \). Otherwise, if \( sr = rs \) then \( sws = rwr \) as above. Therefore, (B2) implies:
\[
\ell(w) - 2 = \ell(sws) = \ell(rwr) \geq \ell(wr) - 1 = \ell(w) + 1 - 1 = \ell(w),
\]
since \( r \not\in D(w) \), which is a contradiction.

(B5) Take \( u = srs = rs = u^{-1} \) then \( w \) and \( w' \) are conjugate by \( u \) as in (A5). The last part of (iii) is proved.

\[ \square \]

3.2 Proof of Corollary 1.3 and Theorem 1.4

Proof of Corollary 1.3. Let \( x \in P(u) \), then there are \( g_1, \ldots, g_n \in P(u) \) such that
\[
u = g_1 \se P, \ldots, g_n \se P, g_n = x .
\]
By crochet procedure 1.2, one constructs a unique family \( h_1, \ldots, h_n \in P(v) \) such that \( h_i \prec_C g_i \) and

\[
v = h_1 \prec_p h_2 \prec_p \cdots \prec_p h_n.
\]

Denotes \( y = h_n \). First, one shows that \( y \) does not depends of the choice of the \( g_i \). Let \( g'_1, \ldots, g'_k \in P(u) \) such that

\[
u = g'_1 \prec_p \cdots \prec_p g'_k = x,
\]

and the corresponding family \( h'_1, \ldots, h'_k \in P(v) \), as above. Denotes \( y' = h'_k \), then \( y' \prec_C x \prec_C y \) and \( y' \sim_p y \). Applying Lemma 3.2 to their inverses, one has \( y' = y \).

One defines \( \theta_{u,v} \) by \( \theta_{u,v}(u) = v \) and \( \theta_{u,v}(x) = y \). As the above construction is symmetric in \( P(u) \) and \( P(v) \), one obtains a unique bijection \( \theta_{u,v} : P(u) \to P(v) \).

\[
\begin{align*}
u = h_1 & \quad g_2 \quad g_3 \quad \cdots \quad g_n = x \\
| & \quad \vdots \quad \vdots \quad \vdots \\
v = h_1 & \quad h_2 \quad h_3 \quad \cdots \quad h_n = y
\end{align*}
\]

\textbf{Geometrical illustration of the proof}

\[\square\]

\textbf{Lemma 3.5.} Let \( u \in P_1 \). If \( P_1 \) and \( P_2 \) live in the same carpet \( T \), then there is \( v \in P_2 \) such that \( u \sim_C v \).

\textbf{Proof.} Let \( w \in P_2 \). By definition, there is \( u_1, \ldots, u_n \in T \) such that \( u = u_1, w = u_n \), and either \( u_i \not\prec_p u_{i+1} \) or \( u_i \prec_C u_{i+1} \), for all \( 1 \leq i \leq n - 1 \). One proceeds by induction on \( n \). The case \( n = 1 \) is trivial.

\textit{Case } \( n = 2 \): if \( u \not\prec w \), take \( v = w \). If \( u \prec w \), \( P_1 = P_2 \) and take \( v = u \).

\textit{Case } \( n = 3 \): if \( u \prec_C u_2 \prec_C w \), take \( v = w \). If \( u \not\prec_C u_2 \prec w \), take \( v = u_2 \).

Finally, if \( u \prec_C u_2 \prec_C w \), there is a bijection \( f \) from \( P_1 = P(u_2) \) to \( P_2 \) given by \( u_2 \prec_C w \), by Corollary 1.3. Therefore, one takes \( v = f(u) \).

\textit{Case } \( n > 3 \): if for all \( 1 \leq i \leq n - 1 \), \( u_i \prec_C u_{i+1} \), take \( v = w \). Otherwise, there is \( 1 \leq i \leq n - 1 \) such that \( u_i \prec u_{i+1} \). Denote \( P_i = P(u_i) = P(u_{i+1}) \) then, by induction, there is \( v_i \in P_i \) such that \( u \sim_C v_i \). By induction again, there is \( v \in P_2 \) such that \( v_i \sim_C v \). Therefore, \( u \sim_C v \sim_C v \).

\[\square\]

\textbf{Proof of Theorem 1.4. i) } is Lemma 3.5.

\textit{ii) } Observe that property (c) implies the property (d) in the theorem. Let \( v \in P_2 \) and \( u_1, \ldots, u_n \) such that

\[
u = u_1 \prec_C u_2 \prec_C \cdots \prec_C u_n = v \in P_2.
\]

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One shows ii) by induction on \( n \). If \( n = 0 \), \( u = v \) and there is nothing to prove. If \( n = 1 \), Corollary 1.3 implies that there is a bijection \( \theta_{u,v} \) from \( P_1 \) to \( P_2 \) verifying (b), (c) and (d) in the theorem. If \( n \geq 2 \), one defines

\[
\theta_{u,v} = \theta_{u,v-1,v} \circ \cdots \circ \theta_{u,u_2},
\]

which verifies the properties by induction.

iii) By ii), \( P_1 \cap C_1 \) and \( P_2 \cap C_1 \) are in bijection. Applying ii) to coplactic classes (by taking inverses) instead of plactic classes: one has a bijection between \( P_1 \cap C_i \) and \( P_2 \cap C_i \), where \( i = 1,2 \). Thus there is a bijection from \( P_1 \cap C_1 \) to \( P_2 \cap C_2 \).

\[ \square \]

**Remark.** In the proof of ii), \( \theta_{u,v} \) depends of the choice of the \( u_i \), that is, of the choosen path from \( u \) to \( v \).

**Distance between plactic classes.** Let \( P_1, P_2 \) be two plactic classes. The *distance from \( P_1 \) to \( P_2 \)*, denoted by \( d(P_1, P_2) \), is defined as follow: If \( P_1 \) and \( P_2 \) live in the same carpet, then for all \( u \in P_1 \), there is \( v \in P_2 \) such that \( u \sim_C v \), by Lemma 3.5. In other words, for all \( u \in P_1 \), there is \( u_1, \ldots, u_n \) such that

\[
(\circ) \quad u = u_1 \circ_C u_2 \circ_C \cdots \circ_C u_n = v \in P_2.
\]

Then \( d(P_1, P_2) \) is the minimal number \( n \) verifying \((\circ)\) for all \( u \in P_1 \) and \( v \in P_2 \).

As example, \( d(P_1, P_1) = 0 \). If \( P_1 \) and \( P_2 \) do not live in the same carpet, we take \( d(P_1, P_2) = -\infty \).

**Corollary 3.6.** Let \( P_1, P_2 \) be in the same carpet. For any \( u \in P_1 \), there is \( v \in P_2 \) and \( u_1, \ldots, u_n \) such that

\[
u = u_1 \circ_C u_2 \circ_C \cdots \circ_C u_n = v \in P_2,
\]

and \( n = d(P_1, P_2) \).

**Proof.** Take \( u', v' \) such that \((\circ)\) done. By Theorem 1.4, there is a bijection \( \theta_{u',v'} : P_1 \to P_2 \). Take \( v = \theta_{u',v'}(u) \). One concludes by applying ((ii), (a)) in Theorem 1.4 to \( \theta_{u',v'} \) and induction. \[ \square \]

### 3.3 Carpets and involutions: Proof of Theorem 1.5

Before the proof of the theorem, we introduce a useful object. Let \( T \) be a carpet that contains an involution \( u \). Let \( v \in P(u) \), then there are \( v_1, \ldots, v_n \in P(u) \) such that:

\[
u = v_1 \circ p v_2 \circ p \cdots \circ p v_n = v.
\]

Thus

\[
u = v_1 \circ v_2^{-1} \circ \cdots \circ v_n^{-1} = v^{-1}.
\]

For \( i = 1, \ldots, n \) we construct, by induction on \( i \), a family of sets

\[
P_i = \{u_{i,1}, \ldots, u_{i,n}\} \subset T
\]

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and 
\[ C_i = \{u_{1,i}, \ldots, u_{n,i}\} \subset T \]

such that

a) \( u_i^{-1} \in P_i \) and \( v_j \in C_j \);

b) \( |P_i| = |P_k| \), for all \( 1 \leq i, k \leq n \);

c) \( |C_j| = |C| \), for all \( 1 \leq j, l \leq n \);

d) \( u_{i,j} \sim_p u_{k,j+1} \) for all \( 1 \leq i \leq n \) and all \( 1 \leq j \leq n - 1 \);

e) \( u_{i,j} \sim_C u_{i+1,j} \) for all \( 1 \leq i \leq n - 1 \) and all \( 1 \leq j \leq n \).

[Initialize] Set \( i = 1 \). Denote
\[
P_i = \{u_{1,j} | j = 1, \ldots, n\},
\]

where \( u_{1,j} = v_j \), for \( j = 1, \ldots, n \).

[Iteration] Assume that \( P_i \) is constructed and consider the coplactic neighborhood \( v_i^{-1} = u_{i,1} \sim_C u_{i+1,1} = u_i^{-1} \). By Corollary 1.3, there is a bijection \( f_i \) from \( P(u_{i}) \) to \( P(u_{i+1}) \). Take \( u_{i+1,j} = f_i(u_{i,j}) \) and \( P_{i+1} = f_i(P_i) \). It is readily seen that \( P_{i+1} \) has the desired properties. Set \( i \leftarrow i + 1 \).

[Loop] Repeat until \( i > n \).

Denote \( C_j = \{u_{i,j} | i = 1, \ldots, n\} \). Observe that \( v_j = u_{1,j} \in C_j \) as desired.

**Definition** The **standard square** of \( u, v \) (which depends on the \( v_i \)) is the set
\[
T(u,v) = \bigcup_{i=1}^{n} P_i = \bigcup_{j=1}^{n} C_j.
\]

We call \( n \) the **square order** of \( T(u,v) \). Observe that if the square order \( n = d(P(u), P(u^{-1})) \), then \( |P_i| = |C_j| \), for any \( 1 \leq i, j \leq n \).

\[
\begin{array}{cccccc}
u & u_{1,1} & u_{1,2} & \ldots & u_{1,n} & = v \\
\uparrow & \uparrow & \uparrow & \cdots & \uparrow & \\
u_{1,1}^{-1} & u_{2,1} & u_{2,2} & \cdots & u_{2,n} & \\
\uparrow & \uparrow & \uparrow & \cdots & \uparrow & \\
\vdots & \vdots & \ddots & \vdots & \\
\uparrow & \uparrow & \uparrow & \cdots & \uparrow & \\
u_{1,n}^{-1} & u_{n,1} & u_{n,2} & \cdots & u_{n,n} & \\
\end{array}
\]

**A Standard square** \( T(u,v) \)

The following Proposition gives some important results about standard squares.
Proposition 3.7. Let $\mathcal{T}$ be a carpet, $u \in \mathcal{T}$ an involution, $v \in P(u)$ and $v_1, \ldots, v_n \in P(u)$ such that
\[ u = v_1 \succ_C v_2^{-1} \succ_C \ldots \succ_C v_n^{-1} = v^{-1}. \]
Take $\theta_{u,v^{-1}}$ in Theorem 1.4 which agrees with the $v_i^{-1}$. Then $\mathcal{T}(u,v)$ has the following properties:

i) $u_{i,j} = u_{i,j}^{-1}$, for all $1 \leq i, j \leq n$;

ii) For all $i = 1, \ldots, n$, $u_{i,i}$ is an involution conjugated to $u$;

iii) $u_{n,n} = \theta_{u,v^{-1}}(v) \sim_C v$.

Lemma 3.8. Let $\mathcal{T}(u,v)$ as in the proposition, then the set
\[ \mathcal{T}(u_{2,2}, u_{2,n}) = \{u_{i,j} | 2 \leq i, j \leq n\} \]
is a standard square contained in $\mathcal{T}(u,v)$. We say that $\mathcal{T}(u_{2,2}, u_{2,n})$ is a standard subquare of $\mathcal{T}(u,v)$.

Proof. One must prove that the set $\mathcal{T}(u_{2,2}, u_{2,n})$ has the following property:
\[ u_{i,2} = u_{2,i}^{-1}, \text{ for all } 2 \leq i \leq n. \]
Let $2 \leq i \leq n$, then by construction of $\mathcal{T}(u,v)$ one has $u_{1,i} \succ_C u_{2,i}$ and $u_{i,1} = u_{1,i}^{-1} \succ_P u_{i,2}$. Thus $u_{2,i}^{-1} \sim_P u_{i,1}^{-1} \sim_P u_{i,2}$, by taking inverses in the above coplastic neighborhood. As $u_{1,1}$ is an involution, the crochet procedure and the construction of $P_2$ implies that $u_{2,2}$ is an involution, $u_{i,2} \sim_C u_{2,2}$ and $u_{2,i} \sim_P u_{2,2}$. Taking inverses as above, one has $u_{2,i}^{-1} \sim_C u_{2,2}^{-1} = u_{2,i} \sim_C u_{i,2}$. Finally one has $u_{i,2}^{-1} \sim_P u_{1,2}^{-1} \sim_P u_{i,2}$ and $u_{2,i}^{-1} \sim_C u_{i,2}$. Hence $u_{i,2} = u_{2,i}^{-1}$, by Lemma 3.2.

Proof of Proposition 3.7. One proves (i) - (iii) by induction on $n$. (i) follows directly by induction on the square order $n$ of $\mathcal{T}(u,v)$, using $\mathcal{T}(u_{2,2}, u_{2,n})$ and Lemma 3.8. To prove (ii), observe that $u_{i,i} = u_{i,i}^{-1}$, by (i). The crochet procedure and the construction of $P_2$ implies that $u_{2,2}$ is an involution conjugated to $u_{1,1} = u$. The assertion follows again by induction on $n$ using the subquare $\mathcal{T}(u_{2,2}, u_{2,n})$, since it contains $u_{i,i}$. (iii) follows from $v = u_{1,n} \sim_C u_{n,n}$, the decomposition
\[ \theta_{u,v^{-1}} = \theta_{u_{n-1},v^{-1}} \circ \cdots \circ \theta_{u,v^{-1}}, \]
of Theorem 1.4 ((ii), (a)), and induction as above.

Proof of Theorem 1.5. (i): Let $\mathcal{T}$ be a carpet, $u \in \mathcal{T}$ be an involution and $C$ be a coplastic class in $\mathcal{T}$. Denote $P = P(u)$ the plactic class of $u$, then $P \cap C \neq \emptyset$ by Theorem 1.4. Let $v \in P \cap C$, then $u \sim_P v$ and $u \sim_C v^{-1}$. Construct the standard square $\mathcal{T}(u,v)$. By Proposition 3.7, there is an involution $g \in \mathcal{T}(u,v)$ such that $g \sim_C v$ and $g$ is conjugated to $u$. For plactic classes, consider inverses.
ii): Let $P$ be a plactic class in $\mathcal{T}$ and $w \in P$ an involution. Then $w \in C(w) = P^{-1} \subset \mathcal{T}$. This proves the first part of (ii). Let $C$ be a coplactic class in $\mathcal{T}$, then $C$ and $P^{-1}$ are in bijection, by Theorem 1.4. As $P^{-1}$ and $P$ are in bijection, the second part of (ii) done.

iii): Choose an involution $u \in P_1 \cap \text{Inv} \, W$ and $v \in P_2 \cap C(u)$. By Theorem 1.4, there is a bijection $\theta_{u,v}$ from $P_1$ to $P_2$. Consider the standard square $\mathcal{T}(u,v^{-1})$ which agrees with $\theta_{u,v}$. By Proposition 3.7, $\theta_{u,v}(v^{-1}) = g \in P_2 \cap \text{Inv} \, W$ is an involution conjugated to $u$.

Now, let $u_1 \in P_1 \cap \text{Inv} \, W$ and $v_1 = \theta_{u,v}(u)$. As above, by Proposition 3.7, $\theta_{u,v}(v_1^{-1}) = g \in P_2 \cap \text{Inv} \, W$ is an involution conjugated to $u$. By this way, one has constructed an injective map $\Phi$ from $P_1 \cap \text{Inv} \, W$ to $P_2 \cup \text{Inv} \, W$. As the above construction is symmetric, the Theorem is proved.

\[ \diamond \]

3.4 Applications: A new proof of a result of Schützenberger

In the symmetric group $S_n$, it is well-known that all plactic classes contain a unique involution. Hence Theorem 1.5 implies the following well-known result:

**Corollary 3.9.** In $S_n$, all involutions contained in a same carpet are conjugated.

The above corollary is well-known as a corollary of a result of Schützenberger [23, 4.4]. We give a new proof of this result using Theorem 1.5.

The carpets are indexed by partitions $\lambda$ of $n$ (see [4]). The set of fixed points of a permutation $w \in S_n$ is:

\[ \{1 \leq i \leq n \mid w(i) = i\}, \]

where $w$ is seen acting on the set $\{1, \ldots, n\}$. In [23, 4.4], Schützenberger has shown the following theorem, using Robinson-Schensted correspondence:

**Theorem 3.10.** Let $\lambda$ be a partition of $n$, $\mathcal{T}^\lambda$ its associated carpet and $w \in \mathcal{T}^\lambda$ be an involution of type, then the number of fixed points of $w$ is equal to the number of odd columns in a standard tableau of shape $\lambda$.

Another proof, using the crochet procedure, has been obtained independently by Bessenohl, Jöllenbeck and Schöcker [21]. They use the construction of the unique involution in each plactic class (see [4, Corollary 9.25]).

Recall some notations: let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 1)$ be a partition of $n$ and $w_\lambda$ be the involution of maximal length in the Young subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Therefore we have the following expression for $w_\lambda$, seen as a word on letters $1, \ldots, n$:

\[ w_\lambda = \lambda_1 \ldots 1 \lambda_1 + \lambda_2 \ldots \lambda_1 + 1 \ldots n \ldots \lambda_1 + \cdots + \lambda_k - 1 + 1. \]

**Lemma 3.11.** Let $\lambda$ be a partition, then the number of fixed points of $w_\lambda$ is equal to the number of odd parts of $\lambda$.

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Proof. One proceeds by induction on \( k \). If \( k = 1 \), \( \lambda = (n) \) and \( w_\lambda = w_0 \). It is well-known that the number of fixed points of \( w_0 \) is equal to 1 if \( n \) is odd, or 0 if \( n \) is even. If \( k > 1 \), one concludes since the cycle decomposition of \( w_\lambda \) is the product of the cycle decomposition of the \( w_\lambda \)’s.

Proof of Theorem 3.10. Let \( w \) be an involution in \( T^\lambda \). It is well-known that \( T^\lambda \) is the set of all permutations which are mapped by the Robinson-Schensted correspondence on a pair of tableaux of shape \( \lambda \). It is also well-known that all plactic classes in \( S_n \) contains a unique involution. Therefore, all involutions in \( T^\lambda \) are conjugated to \( w \), by Proposition 1.5. In particular, \( w_{\lambda^\dagger} \in T^\lambda \), where \( \lambda^\dagger \) denote the conjugated partition of \( \lambda \). One concludes with Lemma 3.11, since the number of fixed points is constant on each conjugacy class and that the number of odd columns in a standard tableau of shape \( \lambda \) is the number of odd parts in \( \lambda^\dagger \).

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References


