A generalisation of plactic-coplactic equivalences and Kazhdan-Lusztig cells

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Abstract

In this article, we propose a generalisation of plactic (Knuth) and coplactic (dual-Knuth) equivalences to finitely generated Coxeter groups. We obtain a decomposition of the left cells by coplactic classes which agrees with two induction properties of the cells.

In the case of simply laced Coxeter groups, we give a *crochet procedure* to study bijections between plactic and coplactic classes.

Contents

1 Introduction

1.1 Main Results

In [12], Kazhdan and Lusztig have constructed their famous cells and left cells representations.

In symmetric groups, they have shown that the left cells representations are irreducible [12]. To prove this result, they use a combinatorial description of cells given by the Robinson-Schensted correspondence [20]. More precisely, they have observed that the left cells are precisely the coplactic (dual-Knuth) classes ([12, 26, 11], see also [1]); and these classes describe the Robinson-Schensted correspondence with subsets of symmetric groups (see [19]).

In this article, we take the "Coxeter groups point of view" of the definition of plactic and coplactic equivalences in symmetric groups; and we take it for a general definition in a finitely generated Coxeter system (W, S).

We denote by e the identity of W and by $\ell(w)$ the length of w as a word in the elements of S, for any $w \in W$. If $n = \ell(w)$ and $w = s_1 \dots s_n$ say that $s_1 \dots s_n$ is a *reduced expression* of w. Denote by $\leq$ the *Bruhat order* on W, that is, $u \leq v$ if u is obtained from v as a subexpression of a reduced expression of v. It is readily seen that $u \leq v$ implies $\ell(u) \leq \ell(v)$ (see [10]). Let $w \in W$, the set

$$D(w) = \{s \in S \mid \ell(ws) < \ell(w)\} = \{s \in S \mid ws \leq w\}$$

is called the *descent set* of w. For $J \subset S$, denote by $D_{=J}$ the set of elements of W such that $D(w) = J$.

Definition Let $g, h \in W$, then g is a *plactic neighborhood* of h, denoted $g \smile_P h$, if there is $s \in S$ such that:

a) $h = gs$,

b) $D(g) \not\subset D(h)$ and $D(h) \not\subset D(g)$.

Observe that if we take the condition $g \leq h$, (b) is simply $D(g) \not\subset D(h)$, since $s \in D(h)$ and $s \notin D(g)$. We say that g is a *coplactic neighborhood* of h, denoted $g \smile_C h$, if $g^{-1} \smile_P h^{-1}$. In symmetric groups, this definition is equivalent to the definition of Knuth and dual-Knuth relations (see [1]). The name *plactic* has been introduced by Lascoux and Schützenberger in [13].

The *plactic equivalence* $\sim_P$ (resp. *coplactic equivalence* $\sim_C$) is the reflexive and transitive closure of the plactic (resp. coplactic) neighborhood. The equivalence class

$$P(g) = \{w \in W \mid w \sim_P g\}$$

is called a *plactic classes.* We define similarly the *coplactic class* $C(g)$. It is immediate from definitions that $P(e) = \{e\}$;

$$P(g)^{-1} = \{w \mid w^{-1} \in P(g)\} = C(g^{-1}), \quad \forall g \in W;$$

and if $u \in W$ is an involution, $P(u) = C(u)^{-1}$. That is, to study coplactic classes, we study plactic classes by taking inverses.

Finally, let $\sim_{CP}$ the smallest equivalence on W containing both $\sim_P$ and $\sim_C$. We call *carpets* the equivalence classes for $\sim_{CP}$. Observe that a carpet is a disjoint union of plactic (resp. coplactic) classes. These definitions follow the terminology of symmetric groups (se [4, Chapter 10]).

Let $I \subset S$ and W_I be the parabolic subgroup of W generated by I. It is well-known that if W_I is finite, there is a unique element (an involution) of maximal length, denoted by $w_{0,I}$. Therefore, if W is finite, $P(w_0) = \{w_0\}$, with $w_0 = w_{0,S}$. We obtain the following link with Kazhdan-Lusztig cells (for a definition of cells, see Section 2.1).

Theorem 1.1. *Let W be a Coxeter group then*

i) each left cell of W is a disjoint union of coplactic classes;

ii) each right cell of W is a disjoint union of plactic classes;

iii) each two sided cell of W is a disjoint union of carpets.

Moreover, this decomposition agrees with some properties of the cells: Let $I \subset S$ and C_I be a coplactic class of W_I, then:

a) If $x, y \in C_I$ and $a \in X_I$, xa^{-1} and ya^{-1} are in the same coplactic class of W (first induction property);

b) $X_I \cdot C_I$ is an union of coplactic classes of W (second induction property);

c) If W_I is finite, $w_{0,I}\, C_I$ and $C_I\, w_{0,I}$ are two coplactic classes of W_I.

Remark. 1): The above properties have been discovered, in the case of cells, by: (a) Lusztig [18]; (b) Geck [9]; and (c) Kazhdan and Lusztig [12].

2): A *left-connected set* is a subset X in W such that for all $w, g \in X$, there are $x_1, \ldots, x_n \in X$ such that $x_1 = w$, $x_n = g$ and $x_i x_{i+1}^{-1} \in S$. By definition, the coplactic classes are left-connected. Lusztig has conjectured [14] that each left cells are left connected (it is done in symmetric groups and affine Weyl groups of type $\tilde{A}$ [24]. The above theorem gives a decomposition of any left cell as left-connected sets.

3) A question: Could we construct representations from coplactic classes which decompose left cells representations which agrees with induction properties.

Geometrical plactic and coplactic relations: We represent a plactic neighborhood $g \smile_P h$ by an horizontal bar g — h. As example, consider S_4 the symmetric group generated by the simple transposition $s_i = (i\, i+1)$, $i = 1, \ldots, 3$. Then

$$s_1 \quad — \quad s_1 s_2 \quad — \quad s_1 s_2 s_3 \ .$$

In the same way, we represent a coplactic neighborhood $g \smile_C h$ by a vertical bar

$$\begin{array}{c} g \\ | \\ h \end{array} .$$

As example, with S_4 as above, we have

$$\begin{array}{c} s_1 \\ | \\ s_2 s_2 \\ | \\ s_3 s_2 s_1 \end{array}$$

The case of simply laced Coeter groups: In the Kazhdan-Lusztig theory, it is useful to study the relation between two left cells taken in a same two-sided cell. In symmetric groups, all left cells representations associated to left cells in a same two-sided cell are isomorphic, since coplactic classes in a same carpet are in bijection, with respect to plactic equivalence. In the case of simply laced Coxeter groups, we give a way to study the relation between two plactic (or coplactic) classes taken in a same carpet: the *crochet procedure.*

In symmetric groups, the crochet procedure has been introduced by Blessenhol and Jöllenbeck in [3], to find a realisation of the Robinson-Schensted correspondence with subsets of the group. We generalise it to all simply laced Coxeter groups, with adding a conjugation property of involutions.

Theorem 1.2 (Crochet procedure). *Let W be a simply laced Coxeter group and $x, y, g \in W$.*

i) If $x \smile_P g$ and $y \smile_C g$ then there is a unique $h \in W$, $h \neq g$, such that $x \smile_C h$ and $y \smile_P h$.

ii) If g is an involution and $x = y^{-1}$, then h is an involution conjugate to g.

The geometrical interpretation shows the "to weave a carpet" aspect of the crochet procedure:

$$\begin{array}{ccc} g & — & x \\ | & & \vdots \\ y & \dots & h \end{array}$$

The crochet procedure

The crochet procedure is a useful tool to work on the combinatorial structure of a carpet, let us give the following first application.

Corollary 1.3. *Let $u, v \in W$ such that $u \smile_C v$, then there is a unique bijection $\theta_{u,v} : P(u) \to P(v)$ verifying the following properties:*

1. $\theta_{u,v}(u) = v$;
2. $\theta_{u,v}(x) \smile_C x$, *for all* $x \in P(u)$.

$$\begin{array}{ccccccccc}
u = g_1 & — & g_2 & — & g_3 & — & \dots & — & g_n = x \\
| & & \vdots & & \vdots & & & & \vdots \\
v = h_1 & \dots & h_2 & \dots & h_3 & \dots & \dots & \dots & h_n = \theta_{u,v}(x)
\end{array}$$

Geometrical illustration of Corollary 1.3

The following theorem is a generalisation of the above corollary.

Theorem 1.4. *Let $\mathcal{T}$ be a carpet in W.*

i) $P \cap C \neq \emptyset$, *for any plactic class P in $\mathcal{T}$ and any coplactic class C in $\mathcal{T}$.*

ii) Let P_1, P_2 be plactic classes in $\mathcal{T}$. Let $u \in P_1$. We choose $v \in C(u) \cap P_2$ and $u_1, \dots, u_n \in \mathcal{T}$ such that

$$u = u_1 \smile_C u_2 \smile_C \dots \smile_C u_n = v.$$

Then there is a bijection $\theta_{u,v} : P_1 \to P_2$, depending of the u_i, and verifying the following properties:

(a)

$$\theta_{u,v} = \theta_{u_{n-1},v} \circ \theta_{u_{n-2},u_{n-1}} \circ \dots \circ \theta_{u_2,u_3} \circ \theta_{u,u_2},$$

with θ_{u_{i+1},u_i} as in Corollary 1.3;

(b) $\theta_{u,v}(u) = v$;

(c) $\theta_{u,v}(x) \sim_C x$, *for all* $x \in P_1$;

(d) $\theta_{u,v}(P_1 \cap C) = P_2 \cap C$, *for any coplactic class C in $\mathcal{T}$.*

iii) For all plactic classes P_1, P_2 in $\mathcal{T}$ and all coplactic classes C_1, C_2 in $\mathcal{T}$, the sets $P_1 \cap C_1$ and $P_2 \cap C_2$ are in bijection.

Remark. If $P \cap C$ is a singleton, for any plactic class P in $\mathcal{T}$ and any coplactic class C in $\mathcal{T}$, then we have a "generalised" Robinson-Schensted correspondence

$$\begin{array}{ccc}
W & \longrightarrow & \biguplus_{\mathcal{T} \text{ tapis}} \{(P, C),\ P, C \subset \mathcal{T}\} \\
w & \longmapsto & (P(w), C(w)).
\end{array}$$

But, there is coplactic class which contains more than one element (see Section 1.2 where we give a coplactic class C with two involutions, that is, $C \cap C^{-1}$ contains two involutions).

If a carpet contains an involution, an other application of the crochet procedure is the following theorem.

Theorem 1.5. *Let $\mathcal{T}$ be a carpet which contains an involution u.*

i) All plactic classes in $\mathcal{T}$ contain an involution conjugate to u.

ii) The map $w \mapsto w^{-1}$ is a self-preserving bijection on $\mathcal{T}$ which sends a plactic class on a coplactic class. In particular, any plactic class P in $\mathcal{T}$ and any coplactic class C in $\mathcal{T}$ are in bijection.

iii) Let P_1, P_2 be plactic classes in $\mathcal{T}$ and $Inv\, W$ be the set of involutions in W . There is a bijection

$$\Phi : P_1 \cap Inv\, W \to P_2 \cap Inv\, W$$

such that $\Phi(x)$ is conjugated to x, for all $x \in P_1 \cap Inv\, W$.

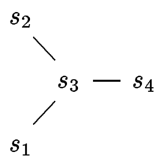
Remark. If there is no involution in a plactic class, or a coplactic class, the above theorem implies that there is no involution in the carpet which contains this class.

The proof of Theorem 1.1 is given in Section 2.1. As second part of this section, we have added a study of descent sets by means of an equivalence relation for which they are the equivalence classes. This is a generalisation, to finitely generated W, of a result of Atkinson [2] in the case of finite Coxeter groups.

Section 3 is alloted to the proof of the crochet procedure and its applications. These proofs use some useful tools like a *distance between two plactic classes* with respect to coplactic equivalence and a *standard square* which is a loop to work on a piece of a carpet. Finally, as an application of the conjugacy property in the crochet procedure, we give a new proof of a result of Schützenberger in symmetric groups: all involutions in a carpet are conjugated.

1.2 An example in D_4.

Consider W the simply laced Coxeter group of type D_4, that is, its Coxeter diagram is



Using the GAP part of CHEVIE (see [22] and [7]), we obtain the following left cell of D_4 and its decomposition into two coplactic classes. This implies that coplactic classes are not the left cells in general (but it is done in symmetric groups), and that there is coplactic class with no involution or more that one involution. This left cell contains 14 elements and two coplactic classes, the first one contains 12 elements and 2 involutions ($s_1s_3s_1$ and $s_1s_3s_2s_4s_3s_1s_3$); the second one contains 2 elements and no involution. The cell is left connected, indeed, $s_1s_2s_4s_3s_1 = s_2s_4s_3s_1s_3 = s_4s_2s_3s_1s_3$ is connected to $s_4s_3s_1s_3$

and $s_2s_3s_1s_3$.

We begin and we end the following "snake" geometrical representation with the same element.

$$
\begin{array}{ccccc}
s_1s_3s_2s_4s_3s_1s_3 & & & & \\
| & & & & \\
s_1s_3s_4s_2s_3s_1s_2s_3 & & s_3s_4s_3s_1s_3 & = & s_4s_3s_1s_4s_3 \\
\| & & | & & | \\
s_4s_1s_3s_2s_4s_3s_1s_3 & & s_4s_3s_1s_3 & & s_2s_4s_3s_1s_4s_3 \\
| & & | & & | \\
s_3s_4s_2s_3s_1s_2s_3 & & s_3s_1s_3 & & s_3s_2s_4s_3s_1s_4s_3 \\
| & & | & & | \\
s_4s_2s_3s_1s_2s_3 & & s_2s_3s_1s_3 & & s_1s_3s_2s_4s_3s_1s_4s_3 \\
| & & | & & \| \\
s_2s_3s_1s_2s_3 & = & s_3s_2s_3s_1s_3 & & s_2s_1s_3s_2s_4s_3s_1s_3 \\
 & & & & | \\
 & & & & s_1s_3s_2s_4s_3s_1s_3
\end{array}
$$

$$
\begin{array}{c}
s_1s_2s_4s_3s_1 \\
| \\
s_3s_1s_2s_4s_3s_1
\end{array}
$$

2 The general case

2.1 Proof of Theorem 1.1

Kazdhan-Lusztig cells: Our basic references for the work of Kazhdan and Lusztig is [12] (see also [5]).

Let $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ where $q^{1/2}$ is an indeterminate. Let $\mathcal{H}$ be the Hecke algebra over $\mathcal{A}$ corresponding to W. Let $(T_w)_{w \in S_n}$ be the standard basis of $\mathcal{H}$ and $(\widetilde{T}_w)_{w \in W}$ the basis defined as follow:

$$\widetilde{T}_w = q^{-\ell(w)/2} T_w.$$

In [12, Theorem 1.1], Kazhdan and Lusztig have shown that there is a basis$(b_w)_{w \in S_n}$ of $\mathcal{H}$, called the *Kazhdan-Lusztig basis*, such that

$$b_w = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} q^{(\ell(w)-\ell(y))/2} P_{y,w}(q^{-1}) \widetilde{T}_y,$$

where $P_{y,w} \in \mathcal{A}$ are polynomials, called the *Kazhdan-Lusztig polynomials.*

For $y, w \in W$, we write $y \prec w$ if $y \leq w$ and $\mu(y,w) \neq 0$, where $\mu(y,w)$ is the coefficient of $q^{(\ell(w)-\ell(y)-1)/2}$ in the Kazhdan-Lusztig polynomial $P_{y,w}$. In this case, we define $\mu(w,y) = \mu(y,w)$.

We define the *elementary relation* $\overset{e}{<}_L$ on W as follow: $y \overset{e}{<}_L w$ if and only if $y \leq w$, or $w \leq y$, $D(y^{-1}) \not\subset D(w^{-1})$ and $\mu(w,y) \neq 0$. The preorder $\leqq_L$ is the reflexive and transitive closure of $\overset{e}{<}_L$. The equivalence relation defined by $\leqq_L$ is denoted by $\sim_L$, that is, $y \sim_L x$ if and only if $y \leqq_L w$ and $w \leqq_L y$. An equivalence class for $\sim_L$ is called a *left cell.*

Write $y \leqq_R w$ if and only if $y^{-1} \leqq_L w^{-1}$. Denote $\sim_R$ the equivalence relation defined by $\leqq_R$ and $\sim_{LR}$ the smallest equivalence containing both $\sim_L$ and $\sim_R$. An equivalence class for $\sim_R$ is called a *right cell* and for $\sim_{LR}$ is called a *two-sided cell.*

Definition Write $x \smile_L w$, say x is a *left cell neighborhood* of w, if $w \overset{e}{<}_L x$ and $x \overset{e}{<}_L w$ are elementary relations. We denote $-_L$ the reflexive and transitive closure of $\smile_L$, that is, if $x, w \in W$ such that $x -_L w$ there are $x_1, \ldots, x_n \in W$ such that

$$x = x_1 \smile_L x_2 \smile_L \ldots \smile_L x_n = w.$$

Similarly, we define $-_R$ and $-_{LR}$.

We have seen, in Example 1.2, that these equivalences do not coincide with $\sim_L$, $\sim_R$ and $\sim_{LR}$ in general. The first part of Theorem 1.1 is a direct consequence of the following proposition.

Proposition 2.1. *Let W be a finitely generated Coxeter group, then $w \sim_C x$ if and only if $w -_L x$.*

In particular, the equivalence classes of $-_L$ (resp. $-_R$) are the coplactic (resp. plactic) classes of W; and the carpets of W are the equivalence classes of $-_{LR}$.

Remark. In particular, as Knuth-plactic classes are the right cells in symmetric groups (see [1]), the following proposition shows that the above equivalences coincide in symmetric groups, in other words, Knuth elementary relations can be seen with Kazhdan-Lusztig terminology. In Example 1.2, we see that $\sim_C$ and $-_L$ do not coincide in a more general case.

For the proof of $w \sim_C x$ implies $w -_L x$, we follow the symmetric group case one. The proposition is a direct consequence of the the next lemma.

Lemma 2.2. *Let $w, g \in W$ then:*

$$w \smile_C g \iff w \smile_L g.$$

Recall this multiplication rule of the Kazhdan-Lusztig base [12]: Let $s \in S$ and $w \in W$ then

$$(\star) \qquad b_s b_w = \begin{cases} b_{sw} + \displaystyle\sum_{y \prec w,\ sy \leq y} \mu(y,w)\, b_y & \text{if } sw > w \\ (q+q^{-1}) b_w & \text{if } sw < w \end{cases}.$$

Using this result and following [5], we have $y \overset{e}{<}_L w$ if and only if $y \neq w$ and b_y appears with a nonzero coefficient in $b_s b_w$ for some $s \in S$ [5, Lemma 5.3].

Proof. If $w \smile_L g$ then, by definition, $w \overset{e}{<}_L g$ and $g \overset{e}{<}_L w$, and $D(w^{-1}) \not\subset D(g^{-1})$ and $D(g^{-1}) \not\subset D(w^{-1})$. One just has to show the following:

$$(\diamond) \qquad \text{there is } s \in S \text{ such that } w = sg.$$

step 1 As $w \overset{e}{<}_L g$, $g \neq w$ and there is $s \in S$ such that b_w appears with a nonzero coefficient in $b_s b_g$. Thus $g \le sg$. By $(\star)$, either $w = sg$ or $w \prec g$ and $sw \le w$. In the first case, $(\diamond)$ done.

step 2 As $g \overset{e}{<}_L w$, there is $t \in S$ such that $g \neq w$ and b_g appears with a nonzero coefficient in $b_t b_w$. Thus $w \le tw$. By $(\star)$ again, either $g = tw$ or $g \prec w$ and $tg \le g$. In the first case, $(\diamond)$ also done and the second case implies $w = sg$ in the fisrt step.

If $w \smile_C g$, then there is $s \in W$ such that $w = sg$. Therefore $w \le g$ or $g \le w$. By definition of coplactic neighborhood, $D(g^{-1}) \not\subset D(w^{-1})$ and $D(w^{-1}) \not\subset D(g^{-1})$. Assume that $w \le g$, the other case is symmetric. As $\ell(w) = \ell(g) - 1$, $\mu(w, g) = 1 \neq 0$. Hence $w \overset{e}{<}_L g$ and $g \overset{e}{<}_L w$, and then $w \smile_L g$. □

End of the proof of Theorem 1.1 We need the following lemmas. The first one gives a useful caracterisation of plactic neighborhood.

Lemma 2.3. *Let $g, h \in W$. Then the following propositions are equivalent:*

i) $h \smile_P g$ if and only if there are $u \in W$ and $s, t \in S$ such that

$$(\star) \qquad u \le ut \le uts \le utst \quad \text{and} \quad \{g, h\} = \{ut, uts\};$$

and either $g = ut$ and $h = uts$ if $g \le h$, or $g = uts$ and $h = ut$ if $h \le g$.

ii) $h \smile_C g$ if and only if there are $u \in W$ and $s, t \in S$ such that

$$u \le tu \le stu \le tstu \quad \text{and} \quad \{g, h\} = \{tu, stu\};$$

and either $g = tu$ and $h = stu$ if $g \le h$, or $g = stu$ and $h = tu$ if $h \le g$.

Proof. Assume that $g \le h$, the other case is symmetric.

If $h \smile_P g$, there is $s \in S$ such that $h = gs$ and $D(g) \not\subset D(h)$. Hence there is $t \in D(g)$ such that $h \le ht$. Write $u = gt$ then

$$u \le ut = g \le uts = h \le ht = utst.$$

Conversely assume that $(\star)$ holds. Then, as $g \le h$, $g = ut$ and $h = uts$. As $s \in D(h)$ and $g \le gs$ one has $D(h) \not\subset D(g)$. In the same way, with $t \in D(g)$, one shows that $D(g) \not\subset D(h)$, hence $h \smile_P g$. For coplactic classes, one takes inverses. □

Lemma 2.4. *Let W be finite , then*

$$x \smile_P xs \iff w_0x \smile_P w_0xs \iff xw_0 \smile_P xsw_0$$

and

$$x \smile_C sx \iff w_0x \smile_C w_0sx \iff xw_0 \smile_C sxw_0 \ .$$

Proof. Recall that $\ell(w_0w) = \ell(w_0) - \ell(w)$ and that $u \leq w$ if and only if $w_0w \leq w_0u$, for all $u, w \in W$.
One has $x \smile_P xs$ one assumes that $x \leq xs$ (the other case is symmetric). By Lemma 2.3, there is $t \in S$ such that

$$xt \leq x \leq xs \leq xst.$$

Thus

$$(w_0x)st \leq (w_0x)s \leq (w_0x) \leq (w_0x)t.$$

Therefore $w_0x \smile_P w_0xs$ by Lemma 2.3. The equivalence follows from $w_0^2 = e$. On the other hand, as the conjugation by w_0 is a bijection on S, there are $s', t' \in S$ such that $sw_0 = w_0s'$ and $tw_0 = w_0t'$. Thus

$$xstw_0 = (xw_0)s't' \leq xsw_0 = (xw_0)s' \leq (xw_0) \leq xtw_0 = (xw_0)t'.$$

Therefore $xw_0 \smile_P xw_0s' = xsw_0$ by Lemma 2.3 again. The equivalence follows from $w_0^2 = e$. For coplactic neighborhoods, one takes inverses and the lemma is proved. $\square$

We recall here the useful Deodhar's Lemma ([6] or see [8, Lemma 2.1.2]).

Lemma 2.5 (Deodhar's Lemma). *Let $I \subset S$, $x \in X_I$ and $s \in S$ then*

i) if $sx \leq x$, $sx \in X_I$;

ii) if $x \leq sx$, either $sx \in X_I$ or $sx = xr$ with some $r \in I$.

Proof of the second part of Theorem 1.1. (a) follows from definition and Lemma 2.3. (c) is a direct consequence of Lemma 2.4.

For (b): Let $u = u^I u_I \in X_I \cdot C_I$, one just has to show that if $v \smile_C u$ then $v \in X_I \cdot C_I$. As $v \smile_C u$, there is $s \in S$ such that $u = sv$. One uses parabolic components: By Deodhar's Lemma, $sv^I \in X_I$ or $sv^I = v^I r$ for some $r \in I$. In the first case, by uniqueness of the parabolic components, $sv^I = u^I$ and $v_I = u_I$; Therefore $u, v \in X_I.\{e\}$ and $\{e\}$ is a coplactic class of W_I.

In the second case, $u^I = v^I$ and $u_I = rv_I$. It remains to show that $v_I \in C_I$. As $v \leq u$ and $u^I = v^I$, $v_I \leq rv_I = u_I$ in W_I. Hence $D(u_I^{-1}) \not\subset D(v_I^{-1})$. As $u \smile_C v$, there is $t \in S$ such that $tv \leq v$ and $u \leq tu$. If $tu^I = u^I t'$ with $t' \in I$ then $u_I \leq t'u_I$ and $t'v_I \leq v_I$ since $u^I = v^I$. Therefore, $D(v_I^{-1}) \not\subset D(u_I^{-1})$ and $v_I \in C_I$. If $tu^I \in X_I$, then on the one hand, $tv = tv^I v_I \leq v = v^I v_I$ hence $tv^I \leq v^I$. On the other hand, $u^I u_I \leq tu^I u_I$ thus $u = u^I \leq tu^I$ contredicting $u^I = v^I$. $\square$

2.2 Descent Classes

In [2], the author gives a new proof of the Solomon result [25] using descent sets. We extend these results to W to define a equivalence relation related to descent set. This point of view will be useful in Section 3.

Definition Let $g, h \in W$, then g is a *descent neighborhood* of h, denoted $g \smile_D h$, if there is $s \in S$ such that $h = sg$ and there is no $t \in S$ such that $sg = gt$. As easily seen, the relation $\smile_D$ is symmetric.

The smallest equivalence $\sim_D$ on W refining the descent neighborhood is called *the descent equivalence.* That is, if $g \sim_D h$, there are $g_1, \dots, g_n \in W$ such that

$$g = g_1 \smile_D g_2 \smile_D \dots \smile_D g_n = h \ .$$

Let $g \in W$, the equivalence class

$$[g]_D = \{h \in W \,|\, h \sim_D g\}$$

is called the *descent class* of g. Observe that descent classes are left- connected.

The terminology descent is explained in the following result:

Proposition 2.6. *Let $g, h \in W$ then*

$$g \sim_D h \iff D(g) = D(h) \ .$$

Corollary 2.7. *Let $I \subset S$ and $g \in D_{=I}$, then $[g]_D = D_{=I}$.*

Before proving the proposition, we need to recall some well-known results (see [8, Chapter 1, 2] for the two last Lemmas). Let $I \subseteq S$, then the cross section of W/W_I consisting of the unique coset representatives of minimal lengths ([10, Section 5.12]) is given by

$$\begin{aligned} X_I &= \{x \in W \mid \ell(xs) > \ell(x), \ \forall s \in I\} \\ &= \{x \in W \mid x \leq xs, \ \forall s \in I\}. \end{aligned}$$

Let $w \in W$, then there is a unique $(w^I, w_I) \in X_I \times W_I$ such that $w = w^I w_I$. The couple (w^I, w_I) is called the *parabolic components* of w. Moreover, w^I is the unique element of smallest length in the coset wW_I and $\ell(w) = \ell(w^I) + \ell(w_I)$.Observe that X_I is a disjoint union of $D_{=J}$ for $I \subset S \setminus J$.

Lemma 2.8. *Let $I \subset S$ and $w \in W$ such that $D(w) = I$, then W_I is finite. In particular, the longest element $w_{0,I}$ of W_I is well defined.*

Proof. Write $w = w^I w_I$. Thus for all $s \in I$, $w_I s \leq w_I$. Observe that if $e \leq u \leq w_I$, for any $u \in W_I$, then W_I is finite.

One proves that $e \leq u \leq w_I$, for any $u \in W_I$, by induction on $\ell(u)$. Let $u \in W_I$, if $u = e$, the lemma done. Assume that $\ell(u) > 0$, then there is $s \in I$ such that $us \leq u$. By induction, one has $us \leq w$. By [10, Proposition 5.9], one has $u = (us)s \leq w_I$ or $u \leq w_I s$. As $w_I s \leq w_I$ for all $s \in I$, $u \leq w_I$ and $w_I = w_{0,I}$. □

Lemma 2.9. *Let $I \subset S$ and $w \in D_{=I}$. Then there is a unique $x \in X_I$ such that $w = xw_{0,I}$, where $w_{0,I}$ is the longest element of W_I.*

Proof. Let (x, y) be the parabolic components of w. If $y \neq w_{0,I}$, then there is $s \in I$ such that $\ell(ys) > \ell(y)$. Thus

$$\ell(ws) = \ell(xys) = \ell(x) + \ell(ys) > \ell(x) + \ell(y) = \ell(w) \ .$$

Therefore $s \notin D(w) = I$, which is a contradiction. □

Corollary 2.10. *Let $I \subset S$ then $w_{0,I}$ is the unique element of minimal length in $D_{=I}$.*

We recall here the useful *(right) exchange condition* (see [10, 5.8]). Let $w \in W$ and $w = s_1 \dots s_n$ an expression of w, not necessarily reduced, with $s_i \in S$. For all $s \in D(w)$, there is an $1 \leq i \leq n$ such that

$$w = s_1 \dots \hat{s_i} \dots s_n s \ ,$$

where the symbol $\hat{s_i}$ denotes that s_i is omited. If the expression of w is reduced, that is $n = \ell(w)$, then the index i is uniquely determined, and this new expression of w is also reduced. Observe that if $s \in D(w)$, there is a reduced expression of w ending by s. In other words, $w = s_1 \dots s_{n-1} s$ is a reduced expression. In the same way, we have a *(left) exchange conditon.*

Lemma 2.11. *Let $I \subset S$ and $w = s_1 \dots s_n \in X_I$ a reduced expression. Then*

$$s_i \dots s_n \in X_I \qquad \forall 1 \leq i \leq n \ .$$

Proof. Assume there is $1 \leq i \leq n$ such that $g = s_i \dots s_n \notin X_I$. Then there is $s \in I$ such that $\ell(gs) < \ell(g)$. Thus, by exchange condition, there are $i \leq j \leq n$ such that $g = s_i \dots \hat{s_j} \dots s_n s$. Therefore

$$w = s_1 \dots s_{i-1} g = s_1 \dots \hat{s_j} \dots s_n s \ ;$$

and this expression is reduced. Thus $\ell(ws) < \ell(w)$ which is a contradiction, since $w \in X_I$ and $s \in I$. □

Proof of Proposition 2.6. On one hand, if $D(g) = D(h) = I$, there is $x \in X_I$ such that $g = xw_{0,I}$, by Lemma 2.9. Without loss of generality, assume that $h = w_{0,I}$.
Let $x = s_1 \dots s_n$ be a reduced expression. One proves by induction on $n = \ell(x)$ that

$$(\star) \qquad w_{0,I} \smile_D s_n w_{0,I} \smile_D \cdots \smile_D s_2 \dots s_n w_{0,I} \smile_D g.$$

The case $n = 0$ is trivial. If $n > 0$ then $g = xw_{0,I} = s_1 \dots s_n w_{0,I}$. Case $n = 1$: one has to show that $w_{0,I} \smile_D g = s_1 w_{0,I}$. Otherwise, there is $t \in S$ such that $g = s_1 w_{0,I} = w_{0,I} t$ which contradicts $D(g) = I$. Case $n > 1$: Let $x' = s_2 \dots s_n$, then $x' \in X_I$ by Lemma 2.11. Thus $g' = s_1 g = x' w_{0,I}$. By induction one has

$$w_{0,I} \smile_D s_n w_{0,I} \smile_D \cdots \smile_D s_2 \dots s_n w_{0,I} = g',$$

hence $g' \sim_D w_{0,I}$. It remains to show that $g' \smile_D g$. Otherwise, there is $t \in S$ such that $g' = s_1 g = gt$. As $g = g't$ with $\ell(g) = \ell(g') + 1$, one has $t \in D(g) = I$. Thus $g = xw_{0,I} = x'w_{0,I}t$. But $w_{0,I}t \in W_I$. Hence $x = x'$ by uniqueness of the parabolic components, which is a contradiction. Thus $g' \smile_D g$ and $(\star)$ is proved. In particular, $g \sim_D g' \sim_D w_{0,I}$.

On the other hand, if $g \sim_D h$ there are $g_1, \dots g_n \in W$ such that

$$g = g_1 \smile_D g_2 \smile_D \dots \smile_D g_n = h \ .$$

Thus one just has to show that $g \smile_D h$ implies $D(g) = D(h)$. As $g \smile_D h$, there is $s \in S$ such that $g = sh$. If there is $t \in D(g)$ and $t \notin D(h)$, the exchange condition implies that $sht = h$ or $sh = ss_1 \dots \hat{s_i} \dots s_k t$ where $h = s_1 \dots s_k$ is a reduced expression. The first case contradicts $g \smile_D h$. The second case implies that $h = s_1 \dots \hat{s_i} \dots s_k t$ is a reduced expression, which contradicts $t \notin D(h)$. Therefore $D(g) \subset D(h)$. To prove the other inclusion, one proceeds similarly. □

Corollary 2.12 (of the proof). *Let $I \subset S$, $u \in D_{=I}$ and $x \in X_I$ such that $u = xw_{0,I}$. Let $x = s_1 \dots s_n$ be a reduced expression then*

$$w_{0,I} \smile_D s_n w_{0,I} \smile_D \dots \smile_D s_2 \dots s_n w_{0,I} \smile_D u.$$

The next proposition can be obtain as a corollary of Theorem 1.1 and a well-known property linking left cells and descent sets [12]. We give here a direct proof using the descent equivalence relation.

Proposition 2.13. *Let $g, h \in W$ then $g \sim_C h \Longrightarrow D(g) = D(h)$.*

Proof. One just has to show that $g \smile_C h \Longrightarrow g \smile_D h$ to prove (i). Without loss of generality, one assumes that $g \leq h$. By Lemma 2.3, one may assume

$$v \leq sv = g \leq tsv = h \leq stsv \ ,$$

with $s, t \in S$ and $v \in W$. Hence $\ell(stsv) > \ell(g)$. If $g \not\smile_D h$, there is $r \in S$ such that $tsv = svr$. Thus $stsv = vr$ therefore $\ell(stsv) = \ell(vr) \leq \ell(v) + 1 = \ell(g)$ which is a contradiction. □

We end with the following observation, that gives a connection between $\leqq_L$ and $\sim_D$.

Proposition 2.14. *Let $I \subset S$ then for all $u \in D_{=I}$ we have $u \leqq_L w_{0,I}$.*

Proof. Let $x \in X_I$ such that $u = xw_{0,I}$ and $x = s_1 \dots s_n$ be a reduced expression. Denote $u_i = s_i \dots s_n w_{0,I}$. By Lemma 2.12, $u_{i+1} \smile_D u_i$. As $u_{i+1} \leq u_i$ and $\ell(u_i) = \ell(u_{i+1}) - 1$, one has $\mu(u_i, u_{i+1}) = 1$ by definition. Moreover, $D(u_i^{-1}) \not\subset D(u_{i+1})^{-1}$ since $s_i u_i \leq u_i$ and $u_{i+1} \leq s_i u_{i+1}$. Therefore $u_i \leqq_L u_{i+1}$. The proposition follows since $u_n \leqq_L w_{0,I}$. □

3 The case of simply laced Coxeter groups

From now, W is a simply laced Coxeter group, that is, for all $s, t \in S$ we have $1 \leq m_{s,t} \leq 3$.

3.1 Proof of The crochet procedure

The following lemma, with Lemma 2.3, gives a new caracterisation of plactic and coplactic neighborhood in the case of simply laced Coxeter groups.

Lemma 3.1. *Let $v \in W$ and $s, t \in S$. Then*

$$v \leq vs \leq vst \leq vsts \iff v \in X_{\{s,t\}} \text{ and } m_{s,t} = 3$$

and

$$v \leq sv \leq tsv \leq stsv \iff v \in X_{\{s,t\}}^{-1} \text{ and } m_{s,t} = 3 \ .$$

Proof. Observe that the lemma follows for the second equivalence by taking inverses.
If $v \in X_{\{s,t\}}$ and $m_{s,t} = 3$ then $\ell(vy) = \ell(v) + \ell(y)$, for all $y \in W_{\{s,t\}}$. As s, st, sts are reduced expression and $v \in X_{\{s,t\}}$,

$$v \leq vs \leq vst \leq vsts \ .$$

Conversely, observe that $m_{s,t} > 2$. As W is a simply-laced Coxeter group, $m_{s,t} = 3$. By hypothesis, one has $\ell(vs) > \ell(v)$. Assume that $\ell(vt) < \ell(v)$. Then one writes $v = r_1 \ldots r_n t$ (reduced expression). Thus

$$vst = r_1 \ldots r_n tst \ , \qquad \text{reduced expression}$$

and

$$vsts = r_1 \ldots r_n tsts = r_1 \ldots r_n st \ .$$

Therefore $\ell(vsts) \leq n + 2 < n + 3 = \ell(vst)$, which is a contradiction. Thus $v \leq vs$ and $v \leq vt$. Therefore $v \in X_{\{s,t\}}$ and the Lemma is proved. □

The following lemma gives a uniqueness property which strongly depends on the simply laced hypothesis.

Lemma 3.2 (Unicity Lemma). *Let $x, g, h \in W$ such that $x \smile_P g$, $x \smile_P h$ and $g \sim_C h$. Then $g = h$.*

Remark. This result is generally false for Coxeter groups which are not simply laced. As example, Coxeter groups of type $I_2(m)$, where $m \geq 4$.

Proof. By definition, there are $s_1, s_2 \in S$ such that $g = xs_1$ and $h = xs_2$. As $g \sim_C h$, Proposition 2.13 and Proposition 2.6 imply that $D(g) = D(h) = I$. Then there is $x_1, x_2 \in X_I$ such that $g = x_1 w_{0,I}$ and $h = x_2 w_{0,I}$, by Lemma 2.9.
Case 1. If $s_1, s_2 \in I$ then $x = x_1 w_{0,I} s_1 = x_2 w_{0,I} s_2$. As $w_{0,I} s_i \in W_I$, one has

$x_1 = x_2$, by uniqueness of the parabolic components. Therefore $g = h$.
Case 2. If $s_1, s_2 \notin I$ then $s_1, s_2 \in D(x)$. Hence $\ell(h) = \ell(x) - 1$. Let $x = r_1 \dots r_n s_1$ a reduced expression, $r_i \in S$. By exchange condition, with s_2, one has

$$x = \begin{cases} r_1 \dots \hat{r_i} \dots r_n s_1 s_2 \\ \text{or} \\ r_1 \dots r_n s_2 \end{cases} .$$

If $x = r_1 \dots r_n s_2$ then $s_1 = s_2$. Therefore $g = h$. If $x = r_1 \dots \hat{r_i} \dots r_n s_1 s_2$ then $h = r_1 \dots \hat{r_i} \dots r_n s_1$ is a reduced expression, since $\ell(h) = \ell(x) - 1$. Thus $s_2 \in I$, which is a contradiction.
Case 3. If $s_1 \in I$ and $s_2 \notin I$, then $s_2 \in D(x)$ and $\ell(g) = \ell(h) + 2$ (this case is symmetric). As $s_2 \in D(x)$, there is a reduced expression $x = r_1 \dots r_n s_2$. As $h = r_1 \dots r_n = x s_2$ and $s_1 \in I$, one has by exchange condition

$$h = r_1 \dots \hat{r_i} \dots r_n s_1,$$

which is a reduced expression. Therefore

$$g = x s_1 = h s_2 s_1 = r_1 \dots \hat{r_i} \dots r_n s_1 s_2 s_1 .$$

If $s_1 s_2 = s_2 s_1$ then $g = r_1 \dots \hat{r_i} \dots r_n s_2$ which contradicts $\ell(g) > \ell(h)$. As W is simply laced, the only other case is $s_1 s_2 s_1 = s_2 s_1 s_2$. Hence

$$g s_2 = r_1 \dots \hat{r_i} \dots r_n s_2 s_1$$

thus $\ell(g s_2) \leq \ell(h) + 1 < \ell(g)$ which contradicts $s_2 \notin I$ and the lemma is proved. □

In the following, we use a special terminology: we say that $w = u_1 u_2$ is a reduced expression of w if $\ell(w) = \ell(u_1) + \ell(u_2)$. By induction, we say that $w = u_1 \dots u_k$ is a reduced expression of w if $\ell(w) = \ell(u_1) + \dots + \ell(u_k)$, with $u_i \in W$. In particular, if we take a reduced expression for each u_i, we obtain a (usually) reduced expression for w. As example, in our terminology, the parabolic component (w^I, w_I) gives a reduced expression of w.

Lemma 3.3. *Let $u, v \in W$ and $s_1, s_2, t_1, t_2 \in S$ such that*

$$u \leq s_1 u \leq t_1 s_1 u \leq s_1 t_1 s_1 u$$

and

$$v \leq v s_2 \leq v s_2 t_2 \leq v s_2 t_2 s_2 \ .$$

i) If $s_1 u = v s_2$ then either $u = v$ and $t_1 u = v t_2$, or $t_1 v \in X_{\{s_2, t_2\}}$ and $u t_2 \in X^{-1}_{\{s_1, t_1\}}$.

ii) If $s_1 u = v s_2 t_2$ then $t_1 v \in X_{\{s_2, t_2\}}$ and $u t_2 \in X^{-1}_{\{s_1, t_1\}}$.

Proof. **Case** $s_1 u = v s_2$. Observe that $\ell(s_1 u) = \ell(v s_2) = \ell(u) + 1 = \ell(v) + 1$ hence

$$(\star) \qquad \ell(u) = \ell(v) \ .$$

1. First, one proves that $t_1 v \in X_{s_2}$ and $ut_2 \in X_{s_1}^{-1}$.
If $t_1 v \notin X_{s_2}$, $\ell(t_1 v s_2) < \ell(t_1 v)$. By Lemma 3.1, $v \in X_{\{s_2,t_2\}} \subset X_{\{s_2\}}$. As $v \leq t_1 v$ and $t_1 v \notin X_{s_2}$, one has $t_1 v = v s_2$, by Deodhar's Lemma. This implies
$$t_1 s_1 u t_2 s_2 = t_1 v s_2 t_2 s_2 = v t_2 s_2$$
therefore $v = t_1 s_1 u$. By Lemma 3.1, $u \in X_{\{s_1,t_1\}}^{-1}$ thus $\ell(u) < \ell(u) + 2 = \ell(v)$ which contradicts $(\star)$. Hence $\ell(t_1 v s_2) > \ell(t_1 v)$. One proceeds similarly with ut_2.

2. On the other hand, as $s_1 u = v s_2$, $(\star)$ implies
$$\ell(s_1 v s_2) = \ell(u) < \ell(u) + 1 = \ell(v) + 1 = \ell(v s_2) \ .$$
Thus $s_1 v s_2 \leq v s_2$. Let $v = r_1 \dots r_n$ a reduced expression. By left exchange condition, one has
$$s_1 v s_2 = u = \begin{cases} v & \text{or} \\ w s_2 & \end{cases}$$
with $w = r_1 \dots \hat{r_i} \dots r_n$, for some $1 \leq i \leq n$. Observe that the expression of w is reduced. One studies these two cases.

3. **Case 1.** If $u = w s_2$ then $v = s_1 u s_2 = s_1 w$ is also a reduced expression, by $(\star)$. As $v = s_1 w \in X_{\{s_2,t_2\}}$, $w \in X_{\{s_2,t_2\}}$ by Lemma 2.11. Now, observe that $\ell(w) = \ell(u s_2) = \ell(s_1 v) < \ell(v) = \ell(u)$, hence $u = w s_2$ is a reduced expression By Lemma 2.11 again, $w \in X_{\{s_1,t_1\}}^{-1}$. One will show that
$$t_1 v \in X_{\{s_2,t_2\}} \text{ and } u t_2 \in X_{\{s_1,t_1\}}^{-1} \ .$$
By (1), one just has to prove that $\ell(t_1 v t_2) > \ell(t_1 v)$ and $\ell(t_1 u t_2) > \ell(u t_2)$. Assume that $\ell(t_1 v t_2) < \ell(t_1 v)$. As in (1), $t_1 v = v t_2$ by Deodhar's Lemma. This implies that $t_1 s_1 w = s_1 w t_2$. As $w \in X_{\{s_1,t_1\}}^{-1}$, one has
$$\ell(w) + 1 = \ell(w t_2) = \ell(s_1(s_1 w t_2)) = \ell(s_1 t_1 s_1 w) = \ell(w) + 3$$
which is a contradiction. Therefore $\ell(t_1 v t_2) > \ell(t_1 v)$. One proceeds similarly to show that $\ell(t_1 u t_2) > \ell(u t_2)$.

4. **Case 2.** If $u = v$ then $u = v \in X_{\{s_1,t_1\}}^{-1} \cap X_{\{s_2,t_2\}}$, by Lemma 3.1.
If $t_1 u \notin X_{\{s_2,t_2\}}$, as above, $t_1 u = u t_2$, by Deodhar's Lemma.
If $t_1 u \in X_{\{s_2,t_2\}}$ then $u t_2 \in X_{\{s_1,t_1\}}^{-1}$ or Deodhar's Lemma imply $u t_2 = t_1 u \notin X_{\{s_2,t_2\}} \subset X_{s_2}$, since $t_1 u \in X_{s_2}$ by (1).

Case $s_1 u = v s_2 t_2$: Observe that $\ell(s_1 u) = \ell(v s_2 t_2) = \ell(u) + 1 = \ell(v) + 2$, hence
$$(\diamond) \qquad \ell(u) = \ell(v) + 1 \ .$$

1. First, one proves $ut_2 \in X^{-1}_{\{s_1,t_1\}}$.
   Assume that $ut_2 \notin X^{-1}_{\{s_1,t_1\}}$. Thus $s_1u = ut_2$ or $t_1u = ut_2$, by Deodhar's Lemma.

   If $t_1u = ut_2$, one has
   $$ut_2 = t_1u = t_1s_1vs_2t_2$$
   thus $s_1t_1u = vs_2$. As $u \in X^{-1}_{\{s_1,t_1\}}$ and $v \in X_{\{s_2,t_2\}}$, one has $\ell(u)+1 = \ell(v)$ which contradicts $(\diamond)$.

   If $s_1u = ut_2$, one has
   $$vs_2t_2s_2 = s_1us_2 = ut_2s_2 = s_1v.$$
   Therefore one obtains the following contradiction
   $$\ell(vs_2t_2s_2) = \ell(v) + 3 = \ell(s_1v) \leq \ell(v) + 1.$$

2. Now, one proves $t_1v \in X_{\{s_2,t_2\}}$.
   Assume that $t_1v \notin X_{\{s_2,t_2\}}$. By Lemma 3.1, $v \in X_{\{s_2,t_2\}}$. Thus $t_1v = vs_2$ or $t_1v = vt_2$, by Deodhar's Lemma.

   If $t_1v = vs_2$, one has
   $$t_1s_1u = t_1vs_2t_2 = vt_2\ .$$
   As $u \in X^{-1}_{\{s_1,t_1\}}$ and $v \in X_{s_2,t_2}$,
   $$\ell(t_1s_1u) = \ell(u) + 2 = \ell(vt_2) = \ell(v) + 1\ ,$$
   therefore $\ell(u) < \ell(v)$ which contradicts $(\diamond)$.

   If $t_1v = vt_2$ then
   $$s_1t_1s_1u = s_1t_1vs_2t_2 = s_1vt_2s_2t_2 = ut_2s_2t_2s_2t_2 = us_2,$$
   since $s_2t_2s_2 = t_2s_2t_2$ by Lemma 3.1. Therefore one obtains the following contradiction
   $$\ell(s_1t_1s_1u) = \ell(u) + 3 = \ell(us_2) \leq \ell(u) + 1.$$

   The Lemma is proved.

□

Proof of the Crochet Procedure 1.2, (i). The uniqueness is given by Lemma 3.2. As $g \smile_C x$, $D(x) = D(g)$ by Proposition 2.13. By definition $D(x) \not\subset D(h)$. Therefore, if h exists, $D(h) \neq D(g)$ which implies $h \neq g$.
Existence: As $x \smile_P g$ and $y \smile_C g$, by Lemma 2.3, there are $u \in X^{-1}_{\{s_1,t_1\}}$, $v \in X_{\{s_2,t_2\}}$ and $s_1, s_2, t_1, t_2 \in S$ such that
$$u \leq s_1u \leq t_1s_1u \leq s_1t_1s_1u \quad \text{and} \quad \{g,x\} = \{s_1u, t_1s_1u\}$$
and
$$v \leq vs_2 \leq vs_2t_2 \leq vs_2t_2s_2 \quad \text{and} \quad \{g,x\} = \{vs_2, vs_2t_2\}.$$
Therefore, one has to consider four cases:

1. $g = s_1u = vs_2$, $x = vs_2t_2$ and $y = t_1s_1u$,
2. $g = s_1u = vs_2t_2$, $x = vs_2$ and $y = t_1s_1u$,
3. $g = t_1s_1u = vs_2$, $x = vs_2t_2$ and $y = s_1u$,
4. $g = t_1s_1u = vs_2t_2$, $x = vs_2$ and $y = s_1u$.

Observe that the cases (3) and (4) are similar to the cases 2 and 1, by Lemma 2.4.
Case 1. By Lemma 3.3 (i), we have two possibilities.

a) If $t_1u = ut_2$ then $g = s_1u = us_2$, $x = us_2t_2 = s_1t_1u$ and $y = t_1s_1u = ut_2s_2$. Let $h = t_1u = ut_2$, thus

$$u \le h = t_1u \le x = s_1t_1u \le t_1s_1t_1u$$

and

$$u \le h = ut_2 \le y = ut_2s_2 \le ut_2s_2t_2,$$

since $u = v \in X^{-1}_{\{s_1,t_1\}} \cap X_{\{s_2,t_2\}}$. Therefore $x \smile_C h$ and $y \smile_P h$ by Lemma 3.1.

b) If $t_1v \in X_{\{s_2,t_2\}}$ and $ut_2 \in X^{-1}_{\{s_1,t_1\}}$. Let $h = t_1vs_2t_2 = t_1s_1ut_2$, then

$$ut_2 \le x = s_1ut_2 \le h = t_1s_1ut_2 \le s_1t_1s_1ut_2$$

and

$$t_1v \le y = t_1vs_2 \le h = t_1vs_2t_2 \le t_1vs_2t_2s_2\ .$$

As above, $x \smile_C h$ and $y \smile_P h$ and this case is done.

Case 2. Observe that $x = vs_2 = s_1ut_2$ and $y = t_1s_1u = t_1vs_2t_2$.
Let $h = t_1vs_2 = t_1s_1ut_2$ then by Lemma 3.3 (ii)

$$ut_2 \le x = s_1ut_2 \le h = t_1s_1ut_2 \le s_1t_1s_1ut_2$$

and

$$t_1v \le h = t_1vs_2 \le y = t_1vs_2t_2 \le t_1vs_2t_2s_2\ .$$

As above, $x \smile_C h$ and $y \smile_P h$ and the first part of the theorem is proved. □

Now, let g be an involution and $y = x^{-1}$ in the first part of the crochet procedure. As $h \smile_C x$ and $h \smile_P x^{-1}$, one has $h^{-1} \smile_C x$ and $h^{-1} \smile_P x^{-1}$. Hence

$$h^{-1} \smile_P x^{-1} \smile_P h$$

and therefore $h \sim_C h^{-1}$. By Lemma 3.2, one has $h = h^{-1}$. Therefore h is an involution.

The last part of the crochet procedure is a direct consequence of the following lemma.

Lemma 3.4. *Let $w, w' \in W$ be two distinct involutions and $x \in W$ such that $x \smile_P w$ and $x \smile_C w'$. Then we have one of the following cases:*

i) $\ell(w') = \ell(w) + 2$*, and there is* $s \notin D(w)$ *such that* $w' = sws$ *and* $x = ws$*;*

ii) $\ell(w') = \ell(w) - 2$*, and there is* $s \in D(w)$ *such that* $w' = sws$ *and* $x = ws$*;*

iii) $\ell(w') = \ell(w)$*, and there are:*

$$s \notin D(w) \text{ and } r \in D(w) \text{ such that } w' = rsrwrsr \text{ and } x = ws = rw',$$
$$\text{or } s \in D(w) \text{ and } r \notin D(w) \text{ such that } w' = rsrwrsr \text{ and } x = ws = rw'.$$

In particular, w *and* w' *are conjugate.*

Proof. By definitions of plactic and coplactic neighborhoods, there is $s, r \in S$ such that $x = ws = rw'$. Then $x^{-1} = sw = w'r$ and therefore $w = sw'r = rw's$. One begins to show the following properties

$$(\star) \qquad ws \neq sw \text{ and } rw' \neq w'r.$$

If $sw = ws$ then $x = x^{-1}$. As $x \smile_P w$ and $x \smile_C w'$ there is a unique $y \neq x$ such that $y \smile_C w$ and $y \smile_P w'$ (first part of crochet procedure). As $x^{-1} \smile_C w$ and $x^{-1} \smile_P w'$, $y = x^{-1} = x$, by uniqueness of the crochet procedure, which contradicts $y \neq x$. Proceeds similarly with $rw' = w'r$.

Case A: $s \notin D(w)$**, that is,** $\ell(w') \geq \ell(w)$**.**

(A1) One has $s \notin D(sw)$. Otherwise the exchange condition implies $sw = ws$ or $sw = s\hat{w}s$, where $\hat{w}$ means that one has omitted a simple reflection in a reduced expression of w. The case $sw = ws$ contradicts $(\star)$. Therefore $sw = s\hat{w}s$ and $w = \hat{w}s$. If we take a reduced expression of w, then $\hat{w}s$ is a also reduced expression of w. This forces $s \in D(w)$ which contradicts $s \notin D(w)$.

(A2) $(A1)$ implies $\ell(sws) = \ell(w) + 2$. The case $s = r$ implies (i) in the lemma.

(A3) If $s \neq r$ we shall prove that $r \in D(sw)$ and $r \in D(w)$. Suppose that $r \notin D(sw)$, then $\ell(w') = \ell(w) + 2$ since $w' = swr = rws$ and $s \notin D(w)$. Therefore, $s, r \in D(w')$, hence $r \notin D(w)$. By exchange condition with $s \in D(w')$:

$$w' = swr \begin{cases} sws & (a) \\ wrs & (b) \\ s\hat{w}rs & (c) \end{cases}.$$

(a) implies $s = r$ which is a contradiction. (b) implies that $sw = ws$ which contradicts $(\star)$. Finally (c) implies $w = \hat{w}rsr$ (with $\hat{w}$ defined in (A1)). Considering $\ell(w)$ and the reduction condition (see [8, Theorem 1.2.5]), one must delete two generators on the expression $\hat{w}rsr$ to obtain a reduced expression of w. Hence, observe that there is s or r at the right end of a reduced expression of w. Therefore, $r \in D(w)$ or $s \in D(w)$ which contradicts the hypotheses. This contradiction shows that $r \notin D(w)$. Hence $r \in D(sw)$. If $r \notin D(w)$, the exchange condition with $r \in D(sw)$ implies $swr = w$ contredicting $w \neq w' = swr$.

(A4) By $(A3)$, $\ell(w) = \ell(w')$. Moreover, $srs = rsr$. Otherwise $sr = rs$ and

$$w = sw'r = srwsr = srwrs \Rightarrow sws = rwr.$$

In this, (A2) implies:

$$\ell(w) + 2 = \ell(sws) = \ell(rwr) \leq 1 + \ell(wr) = 1 + \ell(w) - 1 = \ell(w),$$

since $r \in D(w)$, by $(A3)$, which is a contradiction. As W is simply laced, the only other case is $srs = rsr$.

(A5) Take $u = srs = rsr = u^{-1}$ then w and w' are conjugate by u as follows:

$$u^{-1}wu = (srs)w(rsr) = sr(swr)sr = s(rw's)r = swr = w'.$$

And the first part of (iii) is proved.

Case B: $s \in D(w)$**, that is,** $\ell(w') \leq \ell(w)$**.**

(B1) One has $s \in D(sw)$. Indeed, otherwise $\ell(sws) = \ell(w)$ and $\ell(ws) = \ell(sw)$. Therefore, $sw = ws$ ([8, Lemma 1.2.6]) which contradicts $(\star)$.

(B2) $(B1)$ implies $\ell(sws) = \ell(w) - 2$. Case (ii) of the lemma follows if $s = r$.

(B3) If $s \neq r$ we shall prove that $r \notin D(sw)$ and $r \notin D(w)$. Suppose that $r \in D(sw)$, then $\ell(w') + 2 = \ell(w)$ since $w' = swr$ and $s \in D(w)$. As $sw' = wr$ and $\ell(w) = \ell(w') + 2$, one has $r, s \in D(w)$. Now, one proceeds as in (A3) interchanching the role of w and w' to obtain a contradiction. Therefore $r \notin D(sw)$ and $\ell(w') = \ell(w)$. If $r \in D(w)$, one has $\ell(rw) = \ell(ws)$ and $\ell(rws) = \ell(w') = \ell(w)$. Therefore, $w = rws = w'$ ([8, Lemma 1.2.6]) which is a contradiction.

(B4) $(B3)$ implies $\ell(w') = \ell(w)$ and $srs = rsr$. Otherwise, if $sr = rs$ then $sws = rwr$ as above. Therefore, (B2) implies:

$$\ell(w) - 2 = \ell(sws) = \ell(rwr) \geq \ell(wr) - 1 = \ell(w) + 1 - 1 = \ell(w),$$

since $r \notin D(w)$, which is a contradiction.

(B5) Take $u = srs = rsr = u^{-1}$ then w and w' are conjugate by u as in $(A5)$. The last part of (iii) is proved.

□

3.2 Proof of Corollary 1.3 and Theorem 1.4

Proof of Corollary 1.3. Let $x \in P(u)$, then there are $g_1, \ldots, g_n \in P(u)$ such that

$$u = g_1 \smile_P \ \ldots \ \smile_P g_n = x \ .$$

By crochet procedure 1.2, one constructs a unique family $h_1, \dots, h_n \in P(v)$ such that $h_i \smile_C g_i$ and

$$v = h_1 \smile_P h_2 \smile_P \dots \smile_P h_n \ .$$

Denotes $y = h_n$. First, one shows that y does not depends of the choice of the g_i. Let $g'_1, \dots, g'_k \in P(u)$ such that

$$u = g'_1 \smile_P \dots \smile_P g'_k = x \ ,$$

and the corresponding family $h'_1, \dots, h'_k \in P(v)$, as above. Denotes $y' = h'_k$, then $y' \smile_C x \smile_C y$ and $y' \sim_P y$. Applying Lemma 3.2 to their inverses, one has $y' = y$.

One defines $\theta_{u,v}$ by $\theta_{u,v}(u) = v$ and $\theta_{u,v}(x) = y$. As the above construction is symmetric in $P(u)$ and $P(v)$, one obtains a unique bijection $\theta_{u,v} \ : P(u) \to P(v)$.

$$\begin{array}{ccccccccc}
u = g_1 & — & g_2 & — & g_3 & — & \dots & — & g_n = x \\
| & & \vdots & & \vdots & & & & \vdots \\
v = h_1 & \dots & h_2 & \dots & h_3 & \dots & \dots & \dots & h_n = y
\end{array}$$

Geometrical illustration of the proof

□

Lemma 3.5. *Let $u \in P_1$. If P_1 and P_2 live in the same carpet $\mathcal{T}$, then there is $v \in P_2$ such that $u \sim_C v$.*

Proof. Let $w \in P_2$. By definition, there is $u_1, \dots, u_n \in \mathcal{T}$ such that $u = u_1$, $w = u_n$, and either $u_i \smile_P u_{i+1}$ or $u_i \smile_C u_{i+1}$, for all $1 \le i \le n-1$. One proceeds by induction on n. The case $n = 1$ is trivial.

Case $n = 2$: if $u \smile_C w$, take $v = w$. If $u \smile_P w$, $P_1 = P_2$ and take $v = u$.

Case $n = 3$: if $u \smile_C u_2 \smile_C w$, take $v = w$. If $u \smile_C u_2 \smile_P w$, take $v = u_2$. Finally, if $u \smile_P u_2 \smile_C w$, there is a bijection f from $P_1 = P(u_2)$ to P_2 given by $u_2 \smile_C w$, by Corollary 1.3. Therefore, one takes $v = f(u)$.

Case $n > 3$: if for all $1 \le i \le n-1$, $u_i \smile_C u_{i+1}$, take $v = w$. Otherwise, there is $1 \le i \le n-1$ such that $u_i \smile_P u_{i+1}$. Denote $P_i = P(u_i) = P(u_{i+1})$ then, by induction, there is $v_i \in P_i$ such that $u \sim_C v_i$. By induction again, there is $v \in P_2$ such that $v_i \sim_C v$. Therefore, $u \sim_C v_i \sim_C v$. □

Proof of Theorem 1.4. *i*) is Lemma 3.5.

ii) Observe that property (c) implies the property (d) in the theorem. Let $v \in P_2$ and $u_1, \dots, u_n$ such that

$$u = u_1 \smile_C u_2 \smile_C \dots \smile_C u_n = v \in P_2.$$

One shows *ii)* by induction on n. If $n = 0$, $u = v$ and there is nothing to prove. If $n = 1$, Corollary 1.3 implies that there is a bijection $\theta_{u,v}$ from P_1 to P_2 verifying (b), (c) and (d) in the theorem. If $n \geq 2$, one defines

$$\theta_{u,v} = \theta_{u_{n-1},v} \circ \cdots \circ \theta_{u,u_2},$$

which verifies the properties by induction.

iii) By *ii)*, $P_1 \cap C_1$ and $P_2 \cap C_1$ are in bijection. Applying *ii)* to coplactic classes (by taking inverses) instead of plactic classes: one has a bijection between $P_i \cap C_1$ and $P_i \cap C_2$, where $i = 1, 2$. Thus there is a bijection from $P_1 \cap C_1$ to $P_2 \cap C_2$. □

Remark. In the proof of *ii)*, $\theta_{u,v}$ depends of the choice of the u_i, that is, of the choosen path from u to v.

Distance between plactic classes. Let P_1, P_2 be two plactic classes. The *distance from P_1 to P_2*, denoted by $d(P_1, P_2)$, is defined as follow: If P_1 and P_2 live in the same carpet, then for all $u \in P_1$, there is $v \in P_2$ such that $u \sim_C v$, by Lemma 3.5. In other words, for all $u \in P_1$, there is $u_1, \ldots, u_n$ such that

$$(\diamond) \qquad u = u_1 \smile_C u_2 \smile_C \ldots \smile_C u_n = v \in P_2.$$

Then $d(P_1, P_2)$ is the minimal number n verifying $(\diamond)$ for all $u \in P_1$ and $v \in P_2$. As example, $d(P_1, P_1) = 0$. If P_1 and P_2 do not live in the same carpet, we take $d(P_1, P_2) = -\infty$.

Corollary 3.6. *Let P_1, P_2 be in the same carpet. For any $u \in P_1$, there is $v \in P_2$ and $u_1, \ldots, u_n$ such that*

$$u = u_1 \smile_C u_2 \smile_C \ldots \smile_C u_n = v \in P_2,$$

and $n = d(P_1, P_2)$.

Proof. Take u', v' such that $(\diamond)$ done. By Theorem 1.4, there is a bijection $\theta_{u',v'} : P_1 \to P_2$. Take $v = \theta_{u',v'}(u)$. One concludes by applying ((ii), (a)) in Theorem 1.4 to $\theta_{u',v'}$ and induction. □

3.3 Carpets and involutions: Proof of Theorem 1.5

Before the proof of the theorem, we introduce a useful object. Let $\mathcal{T}$ be a carpet that contains an involution u. Let $v \in P(u)$, then there are $v_1, \ldots, v_n \in P(u)$ such that:

$$u = v_1 \smile_P v_2 \smile_P \ldots \smile_P v_n = v.$$

Thus

$$u = v_1 \smile_C v_2^{-1} \smile_C \ldots \smile_C v_n^{-1} = v^{-1}.$$

For $i = 1, \ldots, n$ we construct, by induction on i, a family of sets

$$P_i = \{u_{i,1}, \ldots, u_{i,n}\} \subset \mathcal{T}$$

and

$$C_i = \{u_{1,i}, \ldots, u_{n,i}\} \subset \mathcal{T}$$

such that

a) $v_i^{-1} \in P_i$ and $v_j \in C_j$;

b) $|P_i| = |P_k|$, for all $1 \leq i, k \leq n$;

c) $|C_j| = |C_l|$, for all $1 \leq j, l \leq n$;

d) $u_{i,j} \smile_P u_{i,j+1}$ for all $1 \leq i \leq n$ and all $1 \leq j \leq n-1$;

e) $u_{i,j} \smile_C u_{i+1,j}$ for all $1 \leq i \leq n-1$ and all $1 \leq j \leq n$.

[**Initialize**] Set $i = 1$. Denote

$$P_1 = \{u_{1,j} \,|\, j = 1, \ldots, n\},$$

where $u_{1,j} = v_j$, for $j = 1, \ldots, n$.
[**Iteration**] Assume that P_i is constructed and consider the coplactic neighborhood $v_i^{-1} = u_{i,1} \smile_C u_{i+1,1} = v_{i+1}^{-1}$. By Corollary 1.3, there is a bijection f_i from $P(u_{i,1})$ to $P(u_{i+1,1})$. Take $u_{i+1,j} = f_i(u_{i,j})$ and $P_{i+1} = f_i(P_i)$. It is readily seen that P_{i+1} has the desired properties. Set $i \leftarrow i+1$.
[**Loop**] Repeat until $i > n$.

Denote $C_j = \{u_{i,j} \,|\, i = 1, \ldots, n\}$. Observe that $v_j = u_{1,j} \in C_j$ as desired.

Definition The *standard square* of u, v (which depends on the v_i) is the set

$$\mathcal{T}(u,v) = \bigcup_{i=1}^{n} P_i = \bigcup_{j=1}^{n} C_j.$$

We call n the *square order* of $\mathcal{T}(u,v)$. Observe that if the square order $n = d(P(u), P(v^{-1}))$, then $|P_i| = |C_j|$, for any $1 \leq i, j \leq n$.

$$\begin{array}{rcccccccc}
u & = u_{1,1} & — & u_{1,2} & — & \cdots & — & u_{1,n} & = v \\
 & | & & | & & & & | & \\
u_{1,2}^{-1} & = u_{2,1} & — & u_{2,2} & — & \cdots & — & u_{2,n} & \\
 & | & & | & & & & | & \\
 & \vdots & & \vdots & & \ddots & & \vdots & \\
 & | & & | & & & & | & \\
u_{1,n}^{-1} & = u_{n,1} & — & u_{n,2} & — & \cdots & — & u_{n,n} &
\end{array}$$

A Standard square $\mathcal{T}(u,v)$

The following Proposition gives some important results about standard squares.

Proposition 3.7. *Let $\mathcal{T}$ be a carpet, $u \in \mathcal{T}$ an involution, $v \in P(u)$ and $v_1, \ldots, v_n \in P(u)$ such that*

$$u = v_1 \smile_C v_2^{-1} \smile_C \ldots \smile_C v_n^{-1} = v^{-1}.$$

Take $\theta_{u,v^{-1}}$ in Theorem 1.4 which agrees with the v_i^{-1}. Then $\mathcal{T}(u, v)$ has the following properties:

i) $u_{i,j} = u_{j,i}^{-1}$, for all $1 \leq i, j \leq n$;

ii) For all $i = 1, \ldots, n$, $u_{i,i}$ is an involution conjugated to u;

iii) $u_{n,n} = \theta_{u,v^{-1}}(v) \sim_C v$.

Lemma 3.8. *Let $\mathcal{T}(u,v)$ as in the proposition, then the set*

$$\mathcal{T}(u_{2,2}, u_{2,n}) = \{u_{i,j} \,|\, 2 \leq i, j \leq n\}$$

is a standard square contained in $\mathcal{T}(u,v)$. We say that $\mathcal{T}(u_{2,2}, u_{2,n})$ is a standard subsquare of $\mathcal{T}(u,v)$.

Proof. One must prove that the set $\mathcal{T}(u_{2,2}, u_{2,n})$ has the following property:

$$u_{i,2} = u_{2,i}^{-1}, \text{ for all } 2 \leq i \leq n.$$

Let $2 \leq i \leq n$, then by construction of $\mathcal{T}(u, v)$ one has $u_{1,i} \smile_C u_{2,i}$ and $u_{i,1} = u_{1,i}^{-1} \smile_P u_{i,2}$. Thus $u_{2,i}^{-1} \smile_P u_{1,i}^{-1} \smile_P u_{i,2}$, by taking inverses in the above coplactic neighborhood. As $u_{1,1}$ is an involution, the crochet procedure and the construction of P_2 implies that $u_{2,2}$ is an involution, $u_{i,2} \sim_C u_{2,2}$ and $u_{2,i} \sim_P u_{2,2}$. Taking inverses as above, one has $u_{2,i}^{-1} \sim_C u_{2,2}^{-1} = u_{2,2} \sim_C u_{i,2}$. Finally one has $u_{2,i}^{-1} \smile_P u_{1,i}^{-1} \smile_P u_{i,2}$ and $u_{2,i}^{-1} \sim_C u_{i,2}$. Hence $u_{i,2} = u_{2,i}^{-1}$, by Lemma 3.2. □

Proof of Proposition 3.7. One proves (i) - (iii) by induction on n. (i) follows directly by induction on the square order n of $\mathcal{T}(u, v)$, using $\mathcal{T}(u_{2,2}, u_{2,n})$ and Lemma 3.8. To prove (ii), observe that $u_{i,i} = u_{i,i}^{-1}$, by (i). The crochet procedure and the construction of P_2 implies that $u_{2,2}$ is an involution conjugated to $u_{1,1} = u$. The assertion follows again by induction on n using the subsquare $\mathcal{T}(u_{2,2}, u_{2,n})$, since it contains $u_{i,i}$. (iii) follows from $v = u_{1,n} \sim_C u_{n,n}$, the decomposition

$$\theta_{u,v^{-1}} = \theta_{v_{n-1}^{-1}, v^{-1}} \circ \cdots \circ \theta_{u, v_2^{-1}},$$

of Theorem 1.4 ((ii), (a)), and induction as above. □

Proof of Theorem 1.5. *i*): Let $\mathcal{T}$ be a carpet, $u \in \mathcal{T}$ be an involution and C be a coplactic class in $\mathcal{T}$. Denote $P = P(u)$ the plactic class of u, then $P \cap C \neq \emptyset$ by Theorem 1.4. Let $v \in P \cap C$, then $u \sim_P v$ and $u \sim_C v^{-1}$. Construct the standard square $\mathcal{T}(u, v)$. By Proposition 3.7, there is a involution $g \in \mathcal{T}(u, v)$ such that $g \sim_C v$ and g is conjugated to u. For plactic classes, consider inverses.

ii): Let P be a plactic class in $\mathcal{T}$ and $w \in P$ an involution. Then $w \in C(w) = P^{-1} \subset \mathcal{T}$. This proves the first part of (*ii*). Let C be a coplactic class in $\mathcal{T}$, then C and P^{-1} are in bijection, by Theorem 1.4. As P^{-1} and P are in bijection, the second part of *ii*) done.

iii): Choose an involution $u \in P_1 \cap Inv\, W$ and $v \in P_2 \cap C(u)$. By Theorem 1.4, there is a bijection $\theta_{u,v}$ from P_1 to P_2. Consider the standard square $\mathcal{T}(u, v^{-1})$ which agrees with $\theta_{u,v}$. By Proposition 3.7, $\theta_{u,v}(v^{-1}) = g \in P_2 \cap Inv\, W$ is an involution conjugated to u.

Now, let $u_1 \in P_1 \cap Inv\, W$ and $v_1 = \theta_{u,v}(u)$. As above, by Proposition 3.7, $\theta_{u,v}(v_1^{-1}) = g \in P_2 \cap Inv\, W$ is an involution conjugated to u. By this way, one has constructed an injective map Φ from $P_1 \cap Inv\, W$ to $P_2 \cup Inv\, W$. As the above construction is symmetric, the Theorem is proved. □

3.4 Applications: A new proof of a result of Schützenberger

In the symmetric group S_n, it is well-known that all plactic classes contain a unique involution. Hence Theorem 1.5 implies the following well-known result:

Corollary 3.9. *In S_n, all involutions contained in a same carpet are conjugated.*

The above corollary is well-known as a corollary of a result of Schützenberger [23, 4.4]. We give a new proof of this result using Theorem 1.5.

The carpets are indexed by partitions λ of n (see [4]). The set of *fixed points* of a permutation $w \in S_n$ is:

$$\{1 \leq i \leq n \,|\, w(i) = i\},$$

where w is seen acting on the set $\{1, \ldots, n\}$. In [23, 4.4], Schützenberger has shown the following theorem, using Robinson-Schensted correspondence:

Theorem 3.10. *Let λ be a partition of n, $\mathcal{T}^\lambda$ its associated carpet and $w \in \mathcal{T}^\lambda$ be an involution of type, then the number of fixed points of w is equal to the number of odd columns in a standard tableau of shape λ.*

Another proof, using the crochet procedure, has been obtained independently by Blessenohl, Jöllenbeck and Schocker [21]. They use the construction of the unique involution in each plactic class (see [4, Corollary 9.25]).

Recall some notations: let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 1)$ be a partition of n and w_λ be the involution of maximal length in the Young subgroup

$$S_{\lambda_1} \times \cdots \times S_{\lambda_K}.$$

Therefore we have the following expression for w_λ, seen as a word on letters $1, \ldots, n$:

$$w_\lambda = \;\; \lambda_1 \ldots 1 \quad \lambda_1 + \lambda_2 \ldots \lambda_1 + 1 \quad \ldots \quad n \ldots \lambda_1 + \cdots + \lambda_{k-1} + 1 \;\; .$$

Lemma 3.11. *Let λ be a partition, then the number of fixed points of w_λ is equal to the number of odd parts of λ.*

Proof. One proceeds by induction on k. If $k = 1$, $\lambda = (n)$ and $w_\lambda = w_0$. It is well-known that the number of fixed points of w_0 is equal to 1 if n is odd, or 0 if n is even. If $k > 1$, one concludes since the cycle decomposition of w_λ is the product of the cycle decomposition of the w_{λ_i}'s. □

Proof of Theorem 3.10. Let w be an involution in $\mathcal{T}^\lambda$. It is well-known that $\mathcal{T}^\lambda$ is the set of all permutations which are mapped by the Robinson-Schensted correspondence on a pair of tableaux of shape λ. It is also well-known that all plactic classes in S_n contains a unique involution. Therefore, all involutions in $\mathcal{T}^\lambda$ are conjugated to w, by Proposition 1.5. In particular, $w_{\lambda^t} \in \mathcal{T}^\lambda$, where λ^t denote the conjugated partition of λ. One concludes with Lemma 3.11, since the number of fixed points is constant on each conjugacy class and that the number of odd columns in a standard tableau of shape λ is the number of odd parts in λ^t. □

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