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UNIVERSITÉ DU QUÉBEC À MONTRÉAL

CHIRURGIES COSMÉTIQUES SUR UNE 3-VARIÉTÉ

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PAR

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"Raha sendra ka lavitra indray
Ny vohitry ny any aminay,
Ny fo no mandefa vinany
Iampita ny bonga any ho any."

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RÉSUMÉ

Cette thèse étudie les chirurgies cosmétiques exceptionelles le long des noeuds hyperboliques dans une 3-variété orientée. On utilise principalement l'homologie de Heegaard Floer, les invariants de Casson, les résultats sur les chirurgies exceptionelles puis les variétés de caractères dans $PSL_2(\mathbb{C})$. Nous y montrons certaines restrictions sur ces types de chirurgies. Pour les noeuds hyperboliques dans S^3 on montre que la pente d'une telle chirurgie doit être ± 1 et que la variété obtenue est d'un type précis. À partir de cela on montre qu'il ne peut pas y avoir de chirurgie cosmétiques exceptionelles le long de certaines familles de noeuds dans S^3. Pour les noeuds hyperboliques dans une sphère d'homologie entière on donne une liste de résultats possibles. On prouve également que la 3-variété orientée obtenue à partir d'une chirurgie non-trivial sur un noeud non-trivial dans la sphère de Poincaré ne peut pas lui être homéomorphe en préservant l'orientation, comme conséquence on donne une réponse au problème du complément orientée d'un noeud dans $\Sigma(2,3,5)$. Pour le cas plus générale des sphères d'homologie rationelles, on prouve que sous certaines conditions sur la variété des caractères dans $PSL_2(\mathbb{C})$, le nombre de pentes pouvant produire une chirurgie cosmétique dont la variété obtenue est une "petite variété de Seifert" est au plus deux.

Mots-clés: théorie des noeuds, 3-variétés, chirurgies de Dehn, noeuds hyperboliques.

ABSTRACT

This thesis studies cosmetic surgeries along hyperbolic knots in an oriented 3-manifold. Restrictions on the type of surgeries obtained are proven. The tools used are Heegaard Floer theory, the different Casson invariants, and known result about exceptional surgeries and $PSL_2(\mathbb{C})$-character varieties. For hyperbolic knots in S^3 we prove that the slope of exceptional truly cosmetic surgeries must be ± 1 and the manifold obtained is of a certain type. From this we deduce that there are no exceptional truly cosmetic surgeries along certain families of hyperbolic knots in S^3. For the case of integer homology spheres we give a list of possible output of such surgeries. We also prove that the oriented manifold obtained by a non-trivial surgery on a non-trivial knot in the Poincaré homology sphere cannot be orientation preserving homeomorphic to the Poincaré sphere. As a consequence we give an answer to the oriented knot complement problem in $\Sigma(2,3,5)$. For the general case of a rational homology sphere, modulo constraints on its $PSL_2(\mathbb{C})$-character variety, we prove that the number of slopes which could produce small Seifert manifolds as cosmetic surgery is at most two.

Keywords: Dehn surgery, Dehn filling, hyperbolic knots, 3-manifolds.

INTRODUCTION

The complement of a tubular neighbourhood of a knot in a closed oriented 3-manifold is a 3-manifold with torus boundary. Gluing back a solid torus along this boundary gives a new oriented 3-manifold. This procedure, called *Dehn surgery*, named after M. Dehn, is a fundamental tool in constructing 3-manifolds. It had been introduced in 1910 by Dehn (Dehn, 1910) for knots in S^3 and has since been generalized to knots and links in closed oriented 3-manifolds. The surgeries will be parameterized by *slopes*, which are isotopy classes of simple closed curves on the torus boundary of M. In the early 1960's A. Wallace (Wallace, 1960) and W.B.R Lickorish (Lickorish, 1962) have independently shown that every closed oriented 3-manifold can be obtained by surgery on some link in S^3.

The goal of this thesis is to understand under which circumstances two distinct Dehn surgeries on the same knot in a closed oriented 3-manifold give rise to *the same* oriented 3-manifold. Here, by *"the same"* we mean the existence of an (orientation preserving) homeomorphism between them. Such surgeries are called (*truly*) *cosmetic surgeries*. For the trivial knot, it is known that there can be infinitely many distinct surgeries which give the same output. Thus we restrict ourselves to the case of non-trivial knots and the case when the knot complement is not homeomorphic to $D^2 \times S^1$. Gordon and Luecke prove in (Gordon and Luecke, 1989) that on a non-trivial knot in S^3, resp. $S^2 \times S^1$, only the *trivial surgery* can give back S^3, resp. $S^2 \times S^1$. From this result follows the fact that knots in S^3 and $S^2 \times S^1$ are determined by their complements. By contrast Mathieu, in (Mathieu, 1992), gives an infinite family of distinct Dehn surgeries on a trefoil knot in S^3 which give homeomorphic manifolds. Some related results concerning knot complements have also been proved by D. Matignon. In particular he shows in (Matignon,) that the only non-hyperbolic knots in lens spaces (excluding S^3 and $S^2 \times S^1$) which are not determined by their complement are the axes of $L(p,q)$ when $q^2 \equiv \pm 1$ modulo p. In (Matignon, 2010) he also gives an infinite family of pairs (M, K), where M is a lens space and $K \subset M$ is a hyperbolic knot, which produce a manifold homeomorphic to M by a non-trivial Dehn surgery. Rong (Rong, 1993) classified knots in irreducible Seifert fibred 3-manifods, other than lens spaces, whose complements are Seifert fibred and which are not determined by their complements. In (Bleiler et al., 1999) Bleiler, Hodgson and Weeks described an oriented hyperbolic 3-manifold with torus boundary having two distinct *Dehn fillings* which give two oppositely oriented copies of the lens space $L(49, 18)$. Using a Casson-type invariant, Boyer and Lines (Boyer and Lines, 1990) showed that the non-vanishing of the second derivative of Alexander polynomial of the knot evaluated at 1 is an obstruction for having truly cosmetic surgeries. Recently, with help of Heegaard Floer theory and Casson invariant, new criteria for cosmetic surgeries on knots in S^3, and more generally knots in L-space homology spheres, have been established. Zhongtao Wu proved (Wu, 2011c) that if two rational surgeries on a non-trivial knot in an L-space homology sphere give orientation preserving homeomorphic manifolds, then the rational numbers must be of opposite sign. Yi Ni and Zhongtao Wu (Ni and Wu, 2013) gave a refinement of this result for knots in S^3.

Let M be a compact irreducible 3-manifold with boundary a torus. From the work of W. Thurston (Thurston, 1982) we know that M is either hyperbolic (i.e admits a complete finite volume Riemannian metric in its interior), or contains an essential torus, or is atoroidal and Seifert fibred. We are concerned with the case where M is a knot complement in a closed oriented 3-manifold. As discussed above, Matignon, Rong and Mathieu have various results related to cosmetic surgeries on non-hyperbolic knots. As opposed to this approach, our main interest will be on hyperbolic knots in integer homology spheres and, in some cases, rational homology spheres. It is known (Thurston, 1979) that all but finitely many surgeries on such knots give hyperbolic manifolds. The finite exceptions are called *exceptional surgeries.* Due to their more topological nature we will focus on these exceptional cosmetic surgeries. Thus our work will be a study of cosmetic surgeries among exceptional surgeries on hyperbolic knots in integer homology spheres.

Concerning exceptional surgeries, work has been done by various authors: (Culler et al., 1987), (Gordon, 1998), (Gordon and Luecke, 1996), (Lackenby, 1997), (Boyer and Zhang, 1998), (Boyer and Zhang, 2001), (Boyer et al., 2001), (Wu, 1996), (Wu, 2011a), (Wu, 2011b), (Brittenham and Wu, 2001), (Gordon and Wu, 2008), (Ichihara and Masai, 2013). Our work will build on all of these established results on the subject. One of the famous results in this area is the cyclic surgery theorem (Culler et al., 1987): "If M is an irreducible 3-manifold with incompressible torus boundary which is not a Seifert fibred space, then there are at most 3 slopes which can give a 3-manifold with cyclic fundamental group and the distance between them is 1". Here the distance between two slopes is their minimal geometric intersection number. A similar result was proven by Boyer and Zhang for finite surgeries (Boyer and Zhang, 2001): "Under the same conditions, the distance between two slopes which give a 3-manifold with finite fundamental group is at most 3". On the other hand Gordon and Luecke have worked on exceptional surgeries which produce toroidal manifold. In particular pairs of *toroidal slopes* with distance greater or equal to 4 have been completely described by Gordon (Gordon, 1998) and Gordon-Wu (Gordon and Wu, 2008). We will be using all of this to narrow down the possibility of having cosmetic exceptional surgeries.

Together with these classical results and techniques we will be using Heegaard Floer theory. These are homology theories for 3-manifolds and knots introduced by Ozsváth and Szabó (Ozsváth and Szabó, 2004d; Ozsváth and Szabó, 2004c; Ozsváth and Szabó, 2004b; Ozsváth and Szabó, 2006a; Ozsváth and Szabó, 2003a) and, independently, by Rasmussen for knots (Rasmussen, 2003). They are constructed as Lagrangian Floer theory for some special totally real submanifolds in a symplectic manifold which is naturally associated to an oriented 3-manifold or to an oriented knot. We will mainly use the *correction term*, the reduced Heegaard floer homology HF_{red} and the renormalized Euler characteristic for the case of exceptional surgeries on knots. We will then give a very concrete characterisation of truly cosmetic exceptional surgeries on hyperbolic knots in S^3 and a list of families of knots which do not admit such surgeries. For the more general case of knots in integer homology spheres we will give the list of possible types of manifolds obtained after truly cosmetic exceptional surgeries together with some restrictions on the slopes used. Using work of Rasmussen on Heegaard Floer theory we will also settle the oriented knot complement problem for the Poincaré sphere as consequence of a slightly more general result. We also study the case when a surgery on $\Sigma(2,3,5)$ gives $-\Sigma(2,3,5)$.

Finally our last result will be about exceptional cosmetic surgeries on rational homology sphere which yield small Seifert manifolds. This will require a totally different approach since in this particular situation we will be using the theory of $(P)SL_2(\mathbb{C})$-character variety. This theory is about counting representation $\pi_1(M) \to (P)SL_2(\mathbb{C})$ of a 3-manifold group into $(P)SL_2(\mathbb{C})$. It was pioneered by Culler and Shalen (Culler and Shalen, 1983), (Shalen, 2002) and was the source of some breakthroughs in the study of the topology of 3-manifolds. We will study cosmetic surgeries via the Culler-Shalen semi-norm.

Summary of the main results We begin our exploration of cosmetic surgery with the case of hyperbolic knots in S^3. Ni and Wu's result combined with the progress made on exceptional surgeries on S^3 help us to provide a new characterization of cosmetic surgeries on hyperbolic knots.

Theorem A (H. Ravelomanana). *Let K be a hyperbolic knot in S^3, and $r, r' \in \mathbb{Q} \cup \{\infty\}$ two distinct exceptional slopes on $\partial \mathcal{N}(K)$. If $S_K(r)$ is homeomorphic to $S_K(r')$ as oriented manifolds, then the surgery must be toroidal and non-Seifert fibred. Moreover $\{r, r'\} = \{+1, -1\}$.*

As a consequence of this we establish that certain families of hyperbolic knot in S^3 do not admit exceptional cosmetic surgery. This is done with support of the work done by Némethi (Némethi, 2007), Ichihara and Masai (Ichihara and Masai, 2013).

Corollary 5.3.3 (H. Ravelomanana). *There are no exceptional truly cosmetic surgeries on an alternating hyperbolic knot in S^3.*

Corollary 5.3.5 (H. Ravelomanana). *There are no exceptional truly cosmetic surgeries on arborescent knots in S^3.*

Next we go into the slightly more general world of knots in integer homology spheres. In this part we use a combination of results on exceptional surgeries, Heegaard Floer theory, $PSL_2(\mathbb{C})$ character varieties, and some elementary topology to give a description of what could be an exceptional cosmetic surgery.

Proposition 6.1.7 (H. Ravelomanana). *Let Y be a $\mathbb{Z}$-homology sphere, $K \subset Y$ a hyperbolic knot and $M = Y \setminus \mathcal{N}(K)$. Assume we use a preferred basis $\{\mu, \lambda_M\}$ for $\pi_1(\partial M)$. Let $r = p/q$ and $r' = p/q'$ be exceptional slopes such that $0 < p$ and $q < q'$. If $M(r)$ is homeomorphic to $M(r')$ as oriented manifolds, then the surgery gives either* *(a) a reducible manifold in which case* $p = 1$ *and* $q' = q + 1$*,*

(b) a toroidal Seifert fibred manifold in which case $p = 1$ *and* $q' = q + 1$*,*

(c) an atoroidal small Seifert manifold with infinite fundamental group in which case we have the following possibilities

- $p = 1$ *and* $|q - q'| \leq 8$.
- $p = 5$, $q' = q + 1$ *and* $q \equiv 2\,[mod\ 5]$.
- $p = 2$, *and* $q' = q + 2$ *or* $q' = q + 4$.

(d) a toroidal irreducible non-Seifert fibred manifold in which case $p = 1$ *and* $|q' - q| \leq 3$.

As spin off of this, we give a particular attention to the Poincaré homology sphere. We obtain that $\Sigma(2,3,5)$ cannot be obtained, as an oriented manifold, by a non-trivial surgery along a non-trivial knot in $\Sigma(2,3,5)$. Therefore we can answer the oriented knot complement problem for the Poincaré sphere.

Theorem 6.2.1 (H. Ravelomanana). *Let* K *be a non-trivial knot in* $\Sigma(2,3,5)$ *and let* $r \in \mathbb{Q}$*. The result of an* r*-surgery along* K *is never orientation preserving homeomorphic to* $\Sigma(2,3,5)$*.*

Theorem 6.2.3 (H. Ravelomanana). *Non-trivial knots in* $\Sigma(2,3,5)$ *are determined by their oriented complements.*

Theorem 6.2.1 generalises to L-space $\mathbb{Z}$-homology spheres.

Theorem 6.2.2 (H. Ravelomanana). *Let* K *be a non-trivial knot in an oriented L-space* $\mathbb{Z}$*-homology sphere* Y *and let* $r \in \mathbb{Q}$*. The result of an* r*-surgery along* K *is never orientation-preserving homeomorphic to* Y*.*

For the case when a surgery on $\Sigma(2,3,5)$ gives $-\Sigma(2,3,5)$ we have the following result.

Theorem 6.2.7 (H. Ravelomanana). *Let K be a non-trivial knot in $\Sigma(2,3,5)$. If K admits a non-trivial surgery which gives $-\Sigma(2,3,5)$ then the surgery slope is $1/2$ and the result of $+1$-surgery along K is an L-space which admits a tight contact structure.*

Finally we use the theory of Culler-Shalen seminorms to give a bound on the number of small Seifert cosmetic surgeries on hyperbolic knots in a rational homology sphere Y. This bound is obtained modulo some hypothesis on the $PSL_2(\mathbb{C})$-character variety of Y. Here $C(\alpha)$ will denote the set of slopes *cosmetic* to the given slope α.

Theorem B (H. Ravelomanana). *Let Y be a rational homology sphere. Assume that*

$Hom(\pi_1(Y), PSL_2(\mathbb{C}))$ contains only diagonalisable representations, no side of the $PSL_2(\mathbb{C})$-Culler-Shalen ball of M is parallel to λ_M, and α is small-Seifert. Then $C(\alpha) \leq 2$.

Structure of the thesis   This thesis is organized as follows.

Chapter 1. We survey exceptional surgeries, cosmetic surgeries and give the necessary topological background for the subject.

Chapter 2. We briefly review some known results on toroidal surgeries and prove some useful lemmas in the subject.

Chapter 3. We review the theory of the Casson invariant and its variants. We also prove a result of Boyer and Lines that there are no cosmetic surgeries along a knot whose Alexander polynomial has the property that its second derivative evaluated at 1 does not vanish. This is a fact that we will be using later.

Chapter 4. We give a survey on Heegaard Floer theory focusing on the definitions and the principal results we will need later.

Chapter 5. We discuss the case of exceptional cosmetic surgeries on hyperbolic knots in S^3. We review some more specialized material and results from Heegaard Floer theory. We give a proof of Theorem A together with some corollaries. From this we deduce a list of families of knots in S^3 which do not admit exceptional truly cosmetic surgeries.

Chapter 6. We study the more general case of exceptional cosmetic surgeries on integer homology spheres. We establish a proposition which characterizes such surgeries according to the "geometric type" of the output and list all the possible slopes. The last section of the chapter is devoted to the oriented knot complement problem for $\Sigma(2,3,5)$ and to the case when a surgery on $\Sigma(2,3,5)$ gives $-\Sigma(2,3,5)$.

Chapter 7. Finally in Chapter 7 we use the theory of $(P)SL_2(\mathbb{C})$-character varieties to study exceptional cosmetic surgeries on a rational homology sphere which produce small Seifert fibred manifolds.

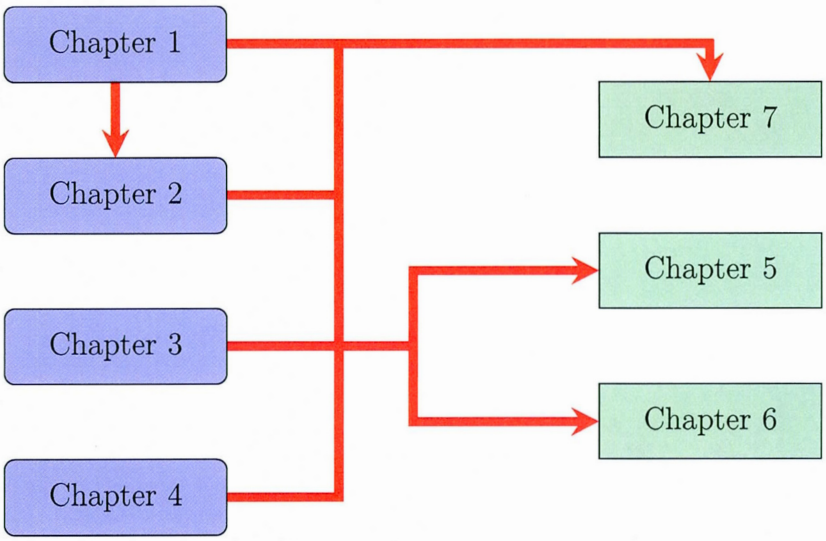


Figure 0.1 Thesis flow chart.

CHAPTER I

BACKGROUND ON COSMETIC SURGERIES

1.1 Dehn surgery

In this chapter we fix some notations and conventions and we briefly outline the necessary background on *Dehn surgery, exceptional surgery* and *cosmetic surgery.* We also give a quick survey on the state of the exceptional surgery problem and the *cosmetic surgery conjecture.* At the end we give some preliminary lemmas.

In what follows, all manifolds will be orientable. We will be precise when a choice of orientation matters. If M is an oriented manifold, we will denote by $-M$ the same topological manifold but with the opposite orientation. Let M_1 and M_2 be two oriented manifolds, we will use the following notations:

$M_1 \cong M_2$ means that the two manifolds are homeomorphic,

$M_1 \cong +M_2$ means that there is an homeomorphism which preserve orientations,

$M_1 \cong -M_2$ means that there is an homeomorphism which reverse orientations.

Let us begin with some definitions.

Definition 1.1.1. *A slope on $S^1 \times S^1$ is a primitive element of $H_1(S^1 \times S^1; \mathbb{Z})/\{\pm 1\}$ representing the isotopy class of a simple closed curve on $S^1 \times S^1$.*

Let M be a compact, connected, oriented 3-manifold and let $T \subset \partial M$ be a torus.

Definition 1.1.2. *A Dehn filling of M along T is a manifold $M(f,T)$ obtained by filling T with a solid torus via a diffeomerphism $f : \partial\,(S^1 \times D^2) \to T$,*

$$i.e \quad M(f,T) := (S^1 \times D^2) \cup_f M.$$

We also refer to the process of constructing $M(f,T)$ as doing a Dehn filling on M along T. The construction of $M(f,T)$ can be done in two steps. The solid torus $S^1 \times D^2$ in $M(f,T)$ is the union of two components, a closed regular neighbourhood A of the disk $\{*\} \times D^2$ and the closure $\overline{S^1 \times D^2 \setminus A}$ of the complement of A. We first attach A to M via f, this process is a 2-handle attachment along a tubular neighbourhood of $f\,(\{*\} \times \partial D^2)$. Such an attachment is uniquely determined by the isotopy class in T of the attaching 1-sphere that is $f\,(\{*\} \times \partial D^2)$. The second stage is to obtain $M(f,T)$ by attaching the 3-ball B to $A \cup M$ along its 2-sphere boundary. Since any homeomorphism of the 2-sphere extends over the 3-ball, the manifold $M(f,T)$ is completely determined by the 2-handle attachment $A \cup M$.

Therefore $M(f,T)$ is uniquely determined by the T isotopy class α of $f\,(\{*\} \times \partial D^2)$, that is, by the slope α on T determined by $f\,(\{*\} \times \partial D^2)$. The curve $S^1 \times \{0\} \subset S^1 \times D^2 \subset M(f,T)$ is called the *core of the Dehn filling.* We will write $M(\alpha,T) := M(f,T)$ and, if the boundary component T is clear from the context, we will simply use the notation $M(\alpha)$ for $M(f,T)$.

Given a set of torus boundary components $T_1, \cdots, T_k \subset \partial M$, and slopes $\alpha_1, \cdots, \alpha_k$ on each component T_i, $i = 1, \cdots, k$; we can do Dehn filling along each T_i to get a new manifold $M(\alpha_1, \cdots, \alpha_k)$.

Let K be a knot in a connected oriented 3-manifold Y. We denote $\mathcal{N}(K)$ a regular neighbourhood of K and $Y_K := \overline{Y \setminus \mathcal{N}(K)}$.

Definition 1.1.3. *A Dehn surgery of slope α on K is the oriented 3-manifold $Y_K(\alpha)$ obtained by doing a Dehn filling of Y_K along $\partial\mathcal{N}(K)$ with slope α.*

The manifold $Y_K(\alpha)$ inherits a preferred orientation from the orientation of Y. Indeed, the orientation on Y induces an orientation on Y_K. Choose the orientation of the filling solid torus $S^1 \times D^2$ so that the orientation on $f(\partial\,(S^1 \times D^2))$ is the opposite of the orientation on ∂Y_K. Then, after gluing, we get an orientation of the whole manifold $Y_K(\alpha) = S^1 \times D^2 \cup_\alpha Y_K$. Therefore a Dehn surgery gives an oriented manifold.

A Dehn surgery along a link $L \subset Y$ is defined in similar fashion.

Meridian and longitude. The knot K determines a distinguished slope μ called the *meridian* of K, up to orientation. It is the class of an essential simple closed curve on $\partial\mathcal{N}(K)$ which bounds a disk in $\mathcal{N}(K)$. A simple closed curve which represents a meridian is called a *meridian curve.* The *trivial Dehn surgery* along K is the Dehn surgery on K with slope μ.

If γ is a slope which can be represented by a simple closed curve which intersects transversally a meridian curve once, then the pair $\{\mu, \gamma\}$ forms a basis of $H_1(\partial\mathcal{N}(K); \mathbb{Z})$ and γ is called a longitude for K. Such choice of basis gives a correspondence:

$$\{\text{Slopes on } \partial M\} \,/\, \{\pm\} \longleftrightarrow \mathbb{Q} \cup \{\infty\}$$
$$\alpha = p\mu + q\gamma \longleftrightarrow \frac{p}{q}$$

In this case we can represent a slope α as an integer point in the $\mathbb{R}^2$ plane or an element of $\mathbb{Q} \cup \{\infty\}$ with the convention that $1/0 = \infty$ represents the meridian. This correspondence is not canonical in general because it depends on the choice of longitude. Since two longitudes differ by an integer multiple of the meridian μ, the choice is infinite. However if the knot K is null-homologous, for instance if Y is an integer homology 3-sphere, then K bounds a Seifert surface F and the curve $F \cap \partial\mathcal{N}(K)$ is an essential simple closed curve which intersect a meridian curve transversally and exactly once. The isotopy class of this curve does not depend on the choice of Seifert surface F. Therefore we have a *canonical longitude*, denoted λ_K, and the correspondence above becomes canonical.

Rational longitude. Let K be a knot in a rational homology 3-sphere Y. In this situation there is also a canonical longitude λ_M called the *rational longitude.* Indeed the knot K has finite order in $H_1(Y, \mathbb{Z})$ so there is an integer n and a surface $\Sigma \subset Y$ such that $nK = \partial\Sigma$. The intersection of Σ with $\partial\mathcal{N}(K)$ is n-parallel copies of a curve λ_M. The isotopy class in $\partial\mathcal{N}(K)$ of this curve does not depend on the choice of the surface Σ. We call the slope λ_M the rational longitude of K. In terms of homology, λ_M is the unique slope on ∂M with the property that the image of λ_M in $H_1(M;\mathbb{Z})$ by the morphism induced by the inclusion $\partial(Y \setminus K) \to Y \setminus K$ is of finite order. For more details on the homological point of view see (Watson, 2009).

Distance between two slopes. The distance, denoted $\Delta(\alpha, \beta)$, between two slopes α and β on T is their minimal geometric intersection number. That is

$$\Delta(\alpha,\beta) = \min\{\,\sharp C_1 \cap C_2 \;:\; C_1, C_2 \text{ simple closed curve representing } \alpha \text{ and } \beta \text{ respectively}\,\}$$

The distance has the following straightforward properties:

- $\Delta(\alpha, \beta) = |\alpha \cdot \beta|$.
- $\Delta(\alpha, \beta) = 0$ iff $\alpha = \beta$.

- $\Delta(\alpha,\beta) = 1$ iff $\{\alpha,\beta\}$ form a basis of $H_1(\partial M;\mathbb{Z})$.
- If we fix a basis $\{\mu,\lambda\}$ of $H_1(T;\mathbb{Z})$, then for $\alpha = p\mu + q\lambda$ and $\beta = p'\mu + q'\lambda$

$$\Delta(\alpha,\beta) = |pq' - qp'|.$$

When M is a compact, connected orientable 3-manifold with torus boundary there is a formula relating the order of the first homology of the filled manifold to the distance of the filling slope from the rational longitude.

Lemma 1.1.4. *(Watson, 2009) Let α be a slope on ∂M. There is a constant c_M such that*

$$|H_1(M(\alpha);\mathbb{Z})| = c_M\ \Delta(\alpha,\lambda_M).$$

If we denote $i : \partial M \to M$ the natural inclusion then the constant c_M is the quantity

$$c_M = |\mathrm{Tor}(H_1(M;\mathbb{Z}))|\ \mathrm{ord}(i_*\lambda_M),$$

where $\mathrm{ord}(i_*\lambda_M)$ is the order of $i_*\lambda_M$ in the homology of M.

Surgery on a link. Assume that Y is an integer homology sphere. Let $L = K_1 \cup \cdots \cup K_m$ be a link in Y. Each component of L has a canonical longitude, therefore every surgery on L can be described by an m-tuple $(p_1/q_1, \cdots, p_m/q_m)$ of elements in $\mathbb{Q} \cup \{\infty\}$. By a *framed link* we mean the data of the link L with such an m-tuple. The m-tuple itself will be called the *framing* of the link. A framed link will be denoted by calligraphic letter, like $\mathcal{L}$. We will write $Y(\mathcal{L})$ for the result of a Dehn surgery on a framed link $\mathcal{L}$. The *framing matrix* of a framed link $\mathcal{L}$ in Y is the matrix $F(\mathcal{L})$ defined by

$$F(\mathcal{L})_{ij} = \begin{cases} q_j\ \mathrm{lk}(K_i,K_j) & \text{if} \quad i \neq j \\ p_i \quad \text{if} \quad i = j \end{cases}$$

where $\mathrm{lk}(,)$ denotes the linking number. The framing matrix gives a presentation for $H_1(Y(\mathcal{L}),\mathbb{Z})$, in particular

$$|\det(F(\mathcal{L}))| = |H_1(Y(\mathcal{L});\mathbb{Z})|\,.$$

For the case of a 2-component link, the framing matrix has the form

$$F(\mathcal{L}) = \begin{pmatrix} p_1 & q_2\,\mathrm{lk}(K_1,K_2) \\ q_1\,\mathrm{lk}(K_2,K_1) & p_2 \end{pmatrix}$$

For more details we refer to (Saveliev, 2002).

Lens spaces. We shall fix here the convention for lens spaces. Let $p \neq 0$ and q be two coprime integers. The *"lens space"* $L(p,q)$ is the oriented 3-manifold obtained by gluing two solid tori V_1 and V_2 via a homeomorphism $f : \partial V_1 \to \partial V_2$ which takes a meridian m on ∂V_1 to a torus knot (p,q) on ∂V_2. In particular $L(p,q)$ is obtained by (p/q)-surgery along the unknot in S^3. A knot in a lens space is said to be an "*axis*", if its complement is homeomorphic to a solid torus. For instance the cores of the solid tori V_1 and V_2 are axes in $L(p,q)$. Indeed, up to isotopy there are at most two axes in $L(p,q)$. Recall that $L(p,q) \cong +L(p,q')$ if and only if $q \equiv q'^{\pm 1}$ [mod p].

Seifert fibred spaces. Let p, q be two coprime integers where $p \geq 1$, and let $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the unit disk in $\mathbb{C}$. Let θ be the rotation $\theta : D^2 \to D^2$ defined by $\theta(z) = e^{2i\pi q/p}\, z$. The "*fibred solid torus of type* (p,q)", denoted by $V_{(p,q)}$, is the quotient space

$$V_{(p,q)} = \frac{D^2 \times [0,1]}{(z,1) \sim (\theta(z),0)}$$

endowed with the (smooth) foliation by circles (called *fibres*) induced from the $[0,1]$-factor in $D^2 \times [0,1]$.

Topologically each $V_{(p,q)}$ is homeomorphic to the solid torus $D^2 \times S^1$, however different pairs of integers may give distinct circle-fibration structures. In general there is a fibre preserving homeomorphism between $V_{(p,q)}$ and $V_{(p',q')}$ if and only if $p = p'$ and $q' \equiv \pm q \pmod p$. The integer p is called the *index* of $V_{(p,q)}$.

Definition 1.1.5.

- *A Seifert structure or Seifert fibration on an oriented 3-manifold M is a smooth foliation of M by circles (called fibres) such that each each fibre ϕ has a closed tubular neighbourhood, consisting entirely of fibres, which is fibre-preserving homeomorphic to some fibred solid torus $V_{(p,q)}$. The index $p \geq 1$ of this fibred solid torus is called the index of ϕ.*

- *A fibre ϕ of index p is called an exceptional fibre if $p > 1$, and a regular fibre otherwise. We say that an exceptional fibre is of type (p, q) if it has a tubular neighbourhood which is fibre-preserving homeomorphic to $V_{(p,q)}$.*

- *A 3-manifold M is called Seifert fibred if there exists a Seifert structure on M.*

- *Two Seifert fibrations on M are said to be equivalent or isomorphic if there exists a fibre preserving diffeomorphism between the two.*

In $V_{(p,q)}$ all fibres are regular except the core fibre $\phi_0 = (\{0\} \times [0,1])/((0,0) = (0,1))$ which has index p. Assume M is compact, it follows that exceptional fibres are isolated and lie in the interior of M. The boundary ∂M is foliated by regular fibres and so it consists of a collection of tori.

The orbit space $\mathcal{B}$ of a Seifert fibred space M is the space of leaves of the given foliation. It can be given the structure of a compact 2-orbifold whose boundary consists of the fibres lying on ∂M. The cone points of the 2-orbifold $\mathcal{B}$ correspond to singular fibres and have index equal to the index of the corresponding fibres in M. The geometry of the 2-orbifold $\mathcal{B}$ is determined by its topological surface type B and the indices $p_1, p_2, \cdots, p_n$ of its cone points. We denote $\mathcal{B} = B(p_1, p_2, \cdots, p_n)$.

The manifold M equipped with a Seifert fibration with base 2-orbifold of genus $\pm g$ (here $-g$ indicates that $\mathcal{B}$ is non-orientable of genus g) and b boundary components, and with n singular fibres of type $(p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)$ will be denoted by

$$M(\pm g, b; (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)).$$

If M is closed we just use the notation $M(\pm g; (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n))$. The collection $(\pm g, b; (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n))$ is called the *unnormalized Seifert invariant.* Reversing the orientation of M has the effect of changing the invariant $(\pm g, b; (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n))$ to $(\pm g, b; (p_1, -q_1), (p_2, -q_2), \cdots, (p_n, -q_n))$.

Theorem 1.1.6. *(Neumann and Raymond, 1978). Let M and M' be two closed Seifert manifolds with associated Seifert invariants $M(g; (p_1, q_1), \cdots, (p_s, q_s))$ and $M(g'; (p'_1, q'_1), \cdots, (p'_t, q'_t))$ respectively. Then M and M' are orientation preserving homeomorphic by a fibre preserving homeomorphism if and only if, after reindexing the Seifert invariants if necessary, there exists an integer k such that*

- $p_i = p'_i$ *for* $i = 1, \cdots, k$ *and* $p_i = p'_j$ *for* $i, j > k$.
- $q_i \equiv q'_i \ [\mathrm{mod}\ p_i]$ *for* $i = 1, \cdots, k$.
- $\sum_{i=1}^{s} q_i/p_i = \sum_{i=1}^{t} q'_i/p'_i$.

According to the theorem, the rational number $e(M) := -\sum_{i=1}^{s} q_i/p_i$ is an invariant of the Seifert structure. It is called the *Euler number* of M. It has the property that

$$e(-M) = -e(M).$$

Using the last theorem, with the appropriate orientation, the Seifert invariant of Seifert fibration has the following unique normal form up to permutation of indices

$$(g; (1, q_0), (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)), \quad 0 < q_i < p_i \text{ for } i = 1, \cdots, n$$

We can also replace $(1, q_0)$ by the Euler number $e(M)$ to get the invariant

$$(g; e(M); (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)).$$

For most Seifert manifolds the Seifert structure is "unique". Indeed, if M is a compact 3-manifold with infinite fundamental group and is distinct from $S^2 \times S^1$, $S^1 \times D^2$, $S^1 \times S^1 \times [0, 1]$, and the twisted interval bundle over the Klein bottle, then the Seifert structure on M is unique up to isotopy, see (Neumann and Raymond, 1978) for details.

In terms of fundamental group, by thinking of a Seifert fibred space M as a circle bundle with base space the 2-orbifold $\mathcal{B}$, we have the following short exact sequence

$$\mathbf{1} \longrightarrow \langle \phi \rangle \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(\mathcal{B}) \longrightarrow \mathbf{1}$$

where $\pi_1^{orb}(\mathcal{B})$ is the orbifold fundamental group of $\mathcal{B}$, $\langle \phi \rangle$ is a cyclic group generated by a regular fibre.

Finally, a particular type of Seifert manifold will be of interest to us. It is the class of *small Seifert manifolds.* They are the Seifert manifolds which are fibred solid tori or which admit the structure of a Seifert fibred space whose base 2-orbifold is the 2-sphere with at most three cone points. They are all irreducible except for $S^1 \times S^2$. If not stated otherwise, we will always assume that a given Seifert manifold is distinct from $S^1 \times S^2$.

When the base 2-orbifold is the 2-sphere with at most 2 cone points, then the manifold is topologically the union of two solid tori, so it is either a lens space or $S^1 \times S^2$. When a closed small Seifert fibred space Y has base 2-orbifold S^2 with exactly three exceptional fibres, then Y contains an *essential surface* (see Definition 1.2.1 below) if and only if $H_1(Y)$ is infinite.

1.2 Exceptional fillings

A compact, connected orientable 3-manifold M will be called *irreducible* if every properly embedded 2-sphere in M bounds a 3-ball. Otherwise M will be called *reducible.* It will be called *boundary irreducible* if every simple closed curve on ∂M which bounds a disk in M bounds a disk in ∂M, and otherwise *boundary reducible.* All embedded surfaces in a 3-manifold we will be considering will be bicollared if not stated otherwise. From now on we will use the following definition.

Definition 1.2.1. *A properly embedded non-empty surface F in a compact, orientable 3-manifold M is said to be essential if it is a 2-sphere which does not bound a 3-ball or if it has the following three properties:*

1. *F has no 2-sphere components,*
2. *the inclusion morphism $\pi_1(F_i) \to \pi_1(M)$ is injective for every component F_i of F,*
3. *no component of F is parallel into ∂M.*[1]

Let $F \subset M$ be a properly embedded surface with boundary and T be a torus component of ∂M. Each component of $\partial F \cap T$ is a simple closed curve on T

[1]A component of F is parallel into M if there is an isotopy of this component onto a boundary component of M.

and they all determine the same slope. A slope r on T is called *boundary slope* if it is the slope of a boundary component of an essential surface in M. If the corresponding surface is a punctured torus then the slope will also be called a *toroidal slope.*

If all the components of ∂M are tori or ∂M is empty, M is said to be *hyperbolic* if its interior admits a complete finite volume Riemannian metric of constant sectional curvature -1. Recall that if M is hyperbolic then it is irreducible, boundary irreducible, contains no essential tori or annuli and is not a Seifert fibred manifold. Thurston's hyperbolization theorem implies that the last statement is an equivalence. A hyperbolic structure on M is unique up to isometry by the Mostow-Prasad rigidity theorem.

Fix M a hyperbolic 3-manifold with ∂M a union of tori. In this section we will discuss the notion of Dehn filling of M. Let T be a component of ∂M. By studying metric completions of incomplete "hyperbolic" 3-manifolds, W. Thurston discovered that, except for a finite number of slopes, all the Dehn fillings of M along T give hyperbolic manifolds.

Theorem 1.2.2. *(Thurston, 1979) Let M be a compact connected oriented 3-manifold with boundary a union of tori. Let T be a component of ∂M. If int(M) admits a complete finite volume hyperbolic structure, for all but finitely many slopes α on T, $M(\alpha)$ is hyperbolic and the core of the Dehn filling is isotopic to the unique shortest geodesic in this manifold.*

Let's consider the set $E(M,T)$ of non-hyperbolic slope on T. A slope in $E(M,T)$ is called an *exceptional slope.* By the above theorem it is a finite set, and one goal of Dehn filling theory is to understand this set of slopes. One of the main "techniques" in this study is to find a bound on the distance $\Delta(r,s)$ between two exceptional slopes r and s.

Theorem 1.2.3. *(Lackenby and Meyerhoff, 2013) Let M be a compact orientable 3-manifold with boundary a torus, and with interior admitting a complete finite-volume hyperbolic structure. If r and s are exceptional slopes on ∂M, then their intersection number $\Delta(r, s)$ is at most 8.*

This bounds is achieved by the figure-8 exterior, indeed

$$E(\text{figure-8}) = \{\infty, 0, \pm 1, \pm 2, \pm 3, \pm 4\}.$$

It has been conjectured by Gordon that the distance of two exceptional slopes is less than 5 for almost all hyperbolic 3-manifold with torus boundary.

Conjecture 1.2.4. *Let M be an hyperbolic 3-manifold with boundary a torus. If α and β are two exceptional slopes on ∂M, then $\Delta(\alpha, \beta) \leq 5$ unless M is one of $W(1), W(2), W(-5/2)$, or $W(-5)$, see Figure 1.1.*

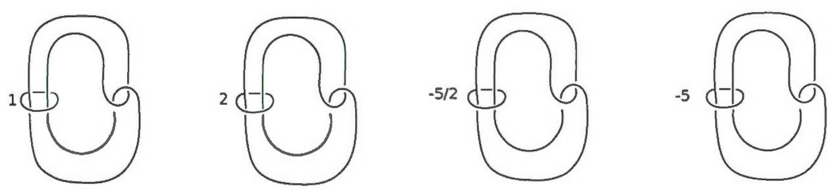


Figure 1.1 $W(1), W(2), W(-5/2), W(-5)$

The conjecture is known to be true if the two slopes are both toroidal. This is the work of Gordon in (Gordon, 1998). We will give a survey on toroidal surgery in chapter 2.

For non-toroidal exceptional surgeries there are three principal results.

Theorem 1.2.5 (Cyclic surgery theorem, (Culler et al., 1987)). *Let M be a compact, oriented, irreducible 3-manifold which is not a Seifert fibred space. Assume* *that ∂M is a torus and let r, s be two slopes on ∂M. If $\pi_1(M(r))$ and $\pi_1(M(s))$ are cyclic, then $\Delta(r, s) \leq 1$.*

Theorem 1.2.6 (Finite surgery theorem, (Boyer and Zhang, 2001)). *Let M be a compact orientable hyperbolic 3-manifold with torus boundary. If r, s are two slopes on ∂M such that $\pi_1(M(r))$ and $\pi_1(M(s))$ are finite, then $\Delta(r, s) \leq 3$.*

Theorem 1.2.7. *(Gordon and Luecke, 1996) Let M be a compact orientable irreducible 3-manifold with torus boundary. If r, s are two slopes on ∂M such that $M(r)$ and $M(s)$ are both reducible, then $\Delta(r, s) \leq 1$.*

We summarize all the results about the bounds on $\Delta(r, s)$ for $r, s \in E(M)$ in table 1.1. We call a slope $r \in E(M)$:

- *reducible*, if $M(r)$ is reducible,
- *toroidal*, if $M(r)$ contains an essential torus,
- *cyclic*, if $\pi_1 M(r)$ is cyclic, *finite*, if $\pi_1 M(r)$ is finite but not cyclic,
- *small Seifert*, if $M(r)$ is a small seifert manifold.

	reducible	cyclic	finite	toroidal	small Seifert
reducible	1	1	1	3	4
cyclic		1	2	8	8
finite			3	8	8
toroidal				8	8
small Seifert					8

Table 1.1 Distance table.

1.3 Cosmetic surgeries

Cosmetic surgery addresses the question: when do two surgeries along the same knot, but with distinct slopes, produce the same manifold? Such a situation does happen, but observations suggest that for generic knots and 3-manifolds this should be very rare.

Definition 1.3.1. *Two Dehn fillings $M(\alpha)$ and $M(\beta)$, where $\alpha \neq \beta$, are called cosmetic if there is a homeomorphism $h : M(\alpha) \to M(\beta)$. They are called truly cosmetic if h can be chosen to be orientation-preserving. We also call two Dehn surgeries cosmetic (resp. truly cosmetic) if the corresponding Dehn fillings are cosmetic (resp. truly cosmetic).*

Example 1.3.2. Here are some examples of cosmetic fillings for two distinct slopes.

- If K is an amphicheiral knot in S^3 and $M = S^3 \setminus \mathcal{N}(K)$, then $M(\alpha)$ is orientation reversing homeomorphic to $M(-\alpha)$.
- It was shown by Mathieu (Mathieu, 1992) that if M is the complement of the trefoil knot in S^3 then we have an infinite family of pairs of distinct slopes which give homeomorphic manifolds. Precisely, for any positive integer k,
$$M\left(\frac{18k+9}{3k+1}\right) \cong -M\left(\frac{18k+9}{3k+2}\right).$$
These Dehn filling manifolds are Seifert fibred with normalized Seifert invariants
$(0; k-3/2; (2,1), (3,1), (3,2))$. Such manifolds do not admit orientation-reversing homeomorphisms. Therefore the fillings are not truly cosmetic.

- If M is the complement of the unknot in S^3, M is a solid torus, then the Dehn filling manifolds are *lens spaces* and

$$M\,(p/q_1) \cong +M\,(p/q_2) \quad \text{iff} \quad q_2 \equiv q_1^{\pm 1}\ [\text{mod}]\ p,$$

for pairs of relatively prime integers (p, q_1) and (p, q_2).

For the first and the third examples one can find a homeomorphism of M which takes one slope to the other. In general we define the following equivalence for slopes.

Definition 1.3.3. *Let M be a compact connected oriented 3-manifold with torus boundary. Two slopes on ∂M are called equivalent if there exists an orientation-preserving homeomorphism of M which takes one to the other.*

The following conjecture is Conjecture (A) in problem 1.81 of the Kirby list of problems in low-dimensional topology (Kirby, 1997).

Conjecture 1.3.4 (Cosmetic surgery conjecture). *Let M be a compact connected oriented irreducible 3-manifold with torus boundary and which is not a solid torus. Let α and β be two inequivalent slopes on ∂M. If $M(\alpha) \cong M(\beta)$, then the homeomorphism is orientation-reversing. Equivalently, two surgeries on inequivalent slopes are never truly cosmetic.*

Gordon and Luecke (Gordon and Luecke, 1989) have proved the first major result toward the resolution of the conjecture.

Theorem 1.3.5. *(Gordon and Luecke, 1989) There is no non-trivial surgery on a non-trivial knot in S^3 or $S^2 \times S^1$ which gives back S^3 or $S^2 \times S^1$.*

For the case where $b_1(Y) > 0$ and the core of the Dehn filling is homotopically trivial in Y the following result was proved by Lackenby.

Theorem 1.3.6. *(Lackenby, 1997) Let* Y *be a compact oriented 3-manifold with* $H_1(Y, \mathbb{Q}) \neq 0$. *Let* K *be a homotopically trivial knot in* Y, *such that* $M = Y \setminus \mathcal{N}(K)$ *is irreducible and atoroidal. Let* $M(p/q)$ *be the Dehn filling along* K *with slope* p/q. *Then there is a natural number* $C(Y, K)$ *which depends only on* Y *and* K *such that, if* $|q| > C(Y, K)$ *then*

$$M(p/q) \quad \textit{is orientation-preserving homeomorphic to } M(p'/q') \text{ iff } p/q = p'/q'.$$

The assumption that K is homotopically trivial can be dropped and replaced by K homologically trivial and Y reducible or K having finite order in $\pi_1(Y)$ (Lackenby, 1997). Taut sutured manifold theory is used to construct the bound $C(Y, K)$.

Relatively recent results has been proven by Zhongtao Wu and Yi Ni, in 2011, for the case of S^3 and L-space $\mathbb{Z}$-homology spheres.

Theorem 1.3.7. *(Ni and Wu, 2013) Suppose* K *is a nontrivial knot in* S^3, $r, r' \in \mathbb{Q} \cup \{\infty\}$ *are two distinct slopes such that* $S_K(r)$ *is homeomorphic to* $S_K(r')$ *as oriented manifolds. Then* r, r' *satisfy*

(a) $r = -r'$;

(b) suppose $r = p/q$, *where* p, q *are coprime integers, then:* $q^2 \equiv -1 \pmod{p}$;

(c) $\tau(K) = 0$, *where* τ *is the concordance invariant defined by Ozsváth-Szabó and Rasmussen.*

Theorem 1.3.8. *(Wu, 2011c) Let* r *and* r' *be two distinct rational numbers with* $rr' > 0$, *let* K *be a non-trivial knot in an* L*-space* $\mathbb{Z}$*-homology sphere* Y *and let* $M = Y \setminus \mathcal{N}(K)$. *Then* $M(r) \not\cong M(r')$.

Yi Ni has also studied cosmetic surgeries for manifolds Y with $b_1(Y) > 0$. For this he used the Thurston norm with Heegaard Floer homology.

Theorem 1.3.9. *(Ni, 2011) Suppose Y is a closed 3–manifold with $b_1(Y) > 0$. Let K be a null-homologous knot in Y, so that the inclusion map $Y - K \to Y$ induces an isomorphism $H_2(Y - K) \cong H_2(Y)$ and we can identify $H_2(Y)$ with $H_2(Y - K)$. Suppose $r \in \mathbb{Q} \cup \{\infty\}$ and let $Y_K(r)$ be the manifold obtained by r–surgery on K. Suppose (Y, K) satisfies that*

$$x_Y(h) < x_{Y-K}(h), \quad \text{for any nonzero element} \quad h \in H_2(Y).$$

where x_M is the Thurston norm in M. The conclusion is: if two rational numbers r, s satisfy that $Y_K(r) \cong \pm Y_K(s)$, then $r = \pm s$.

We can replace the assumption on the Thurston norm with another condition to obtain the following.

Theorem 1.3.10. *(Ni, 2011) Suppose Y is a closed 3–manifold with $b_1(Y) > 0$. Suppose K is a null-homologous knot in Y. Suppose $x_Y \equiv 0$, while the restriction of x_{Y-K} on $H_2(Y)$ is nonzero. Then we have the same conclusion as Theorem 1.3.9. then $r = \pm s$.*

We will be mainly interested in truly cosmetic surgeries along hyperbolic knots K in a rational homology sphere Y. By Theorem 1.2.2, $Y_K(r)$ is hyperbolic for all except a finite number of slopes r on $\partial\mathcal{N}(K)$. Let r and s be such hyperbolic slopes. Assume $Y_K(r)$ is homeomorphic to $Y_K(s)$. Then by Mostow rigidity there is an isometry h between $Y_K(r)$ and $Y_K(s)$. This isometry takes the unique shortest geodesic in $Y_K(r)$ to the unique shortest geodesic in $Y_K(s)$. Apart from a finite number of slopes, the shortest geodesic is isotopic to the core of the Dehn filling, and if this is true for the slopes r and s we can assume that h takes the core of the Dehn filling in $Y_K(r)$ to the core of the Dehn filling $Y_K(s)$. Therefore h takes the meridian r to the meridian s. In particular h restricts to a homeomorphism of Y_K which takes r to s. Moreover a homeomorphism of a one-cusped orientable hyperbolic 3-manifold which changes the slope of some peripheral curve has to be orientation reversing. Therefore the two slopes r and s are not equivalent. One can then deduce the following, see (Bleiler et al., 1999).

Proposition 1.3.11. *(Bleiler et al., 1999) Let M be a compact connected oriented hyperbolic 3-manifold with boundary a torus. Let r and s be distinct slopes on ∂M, such that $M(r)$ (resp. $M(s)$) is hyperbolic and the core of the Dehn filling solid torus is isotopic to the shortest geodesic in $M(r)$ (resp. $M(s)$), which we assume is unique. If $M(r)$ is homeomorphic to $M(s)$, then there is an orientation-reversing homeomorphism of M which takes r to s but no orientation preserving one. In particular, apart from a finite number of slopes, there are no truly cosmetic fillings of M with two inequivalent slopes.*

For cosmetic fillings on a complete finite volume hyperbolic 3-manifold M, the remaining cases are then:

- One of the Dehn filling manifolds has a hyperbolic structure but the core of the Dehn filling is not isotopic to the shortest geodesic.
- The Dehn filling manifold is not hyperbolic.

The second possibility is the case of an exceptional filling. We will focus on this last situation, that is cosmetic surgeries or fillings which are also exceptional.

Using Lemma 1.1.4, we can deduce the following two preliminary lemmas on cosmetic fillings. Let M be a compact, connected, oriented hyperbolic manifold with boundary a torus and assume $b_1(M) = 1$. Fix a canonical basis $\{\mu, \lambda_M\}$ for $H_1(\partial M)$, where λ_M is the rational longitude.

Lemma 1.3.12. *Let p/q and p/q' be exceptional slopes such that $0 < p$ and $q < q'$. If $M(p/q)$ and $M(p/q')$ are homeomorphic then we must be in one of the following cases:*

(a) $p = 1$ and $|q - q'| \leq 8$.

(b) $p \in \{7, 5\}$ and $q' = q + 1$.

(c) $p \in \{4, 3\}$ and $q' \in \{q + 1, q + 2\}$.

(d) $p = 2$ and $q' \in \{q + 2, q + 4\}$.

Proof. We have the bound $\Delta(p/q, p/q') = |pq' - qp| = p|q - q'| \leq 8$, so $p \leq 8$. If $p = 1$ then $|q - q'| \leq 8$. If $p \in \{8, 7, 6, 5\}$ then $|q - q'| \leq 1$ and $q' = q + 1$. On the other hand p and q (resp. p and q') must be relatively prime, thus since one of q and $q+1$ is even and p cannot be 6 or 8. Similarly if $p \in \{4, 3\}$ then $|q - q'| \leq 2$ and $q' \in \{q + 1, q + 2\}$. If $p = 2$ then $|q - q'| \leq 4$ and $q' \in \{q + 1, q + 2, q + 3, q + 4\}$ but we must have $q \equiv q'$ [mod 2] so $q' \in \{q + 2, q + 4\}$. □

For the case of reducible or cyclic fillings we have the following lemma.

Lemma 1.3.13. *Assume the hypotheses of Lemma 1.3.12. If $M(p/q)$ is cyclic or reducible and is homeomorphic to $M(p/q')$ then $p = 1$ and $q' = q + 1$.*

Proof. The distance between two reducible slopes or two cyclic slopes is at most one, so $\Delta(p/q, p/q') = |pq' - qp| = p|q' - q| \leq 1$. It follows that $p = 1$ and $q' = q + 1$. □

CHAPTER II

SURVEY ON TOROIDAL SURGERIES

In this chapter we give a very brief survey on some results about toroidal surgeries on hyperbolic manifolds. Our references are (Gordon and Luecke, 1995), (Gordon, 1998), (Gordon and Luecke, 2004), and (Gordon and Wu, 2008). We begin in section 2.1 by giving some basic background on intersection graphs. In section 2.2 we give a summary of results about toroidal Dehn filling and we prove some lemmas needed for later on.

2.1 Intersection graphs

Litherland was the first to introduce the method of intersection graphs in 1980 as a combinatorial way of studying Dehn surgeries in the solid torus. It was then extensively used by C. McA. Gordon, J. Luecke, M. Scharlemann, and Ying-Qing Wu.

Throughout this chapter, M will be an oriented hyperbolic 3-manifold, with a torus T_0 as a boundary component. We will use a, b to denote the numbers 1 or 2, with the convention that if they both appear in a statement then $\{a, b\} = \{1, 2\}$. We are interested in the transverse intersections of two properly embedded surfaces in M with boundaries on T_0. Let F_1 and F_2 be such surfaces. We assume that there exist two distinct slopes r_1 and r_2 on T_0 such that each component of

∂F_1 (resp. ∂F_2) represents the slope r_1 (resp. r_2). In our particular situation we assume that the two slopes are also toroidal.

We denote by $\widehat{F}_a$ the essential torus in $M(r_a)$ obtained by capping off ∂F_a with meridian disks. Let n_a be the number of boundary components of F_a on T_0. Choose $\widehat{F}_a$ in $M(r_a)$ so that n_a is minimal among all essential tori in $M(r_a)$. Minimizing the number of components of $F_1 \cap F_2$ by an isotopy, we may assume that $F_1 \cap F_2$ consists of arcs and circles which are essential on both F_a. Denote by J_a the attached solid torus in $M(r_a)$, and by u_i $(i = 1, ..., n_a)$ the components of $\widehat{F}_a \cap J_a$, which are all disks, labeled successively when traveling along J_a. Similarly let v_j be the disk components of $\widehat{F}_b \cap J_b$.

Definition of intersection graph We associate to the pair of surfaces $\{F_1, F_2\}$ a pair of graphs $\{\Gamma_1, \Gamma_2\}$ where Γ_a is a graph on F_a defined as follows.

- The vertices of Γ_a are the u_i's. They are drawn like disks. We associate a sign to each vertex of Γ_a as follows: the surface $\widehat{F}_a$ and the curve J_a are oriented and intersect transversally, so if the orientation on $M(r_a)$ is the same as the orientation of the couple $\left(\widehat{F}_a, J_a\right)$ then we say that the vertex is positive, otherwise we say that it is negative.
- The edges of Γ_a are the arc components of $F_1 \cap F_2$.

A *face* of Γ_a is the closure of a connected component of $F_a \setminus \Gamma_a$. A *disk face* of Γ_a is a face which is a disk. The minimality of the number of components in $F_1 \cap F_2$ and the minimality of n_a imply that Γ_a has no trivial loops, and that each disk face of Γ_a in $\widehat{F}_a$ has an interior disjoint from F_b.

Three-manifolds from intersection graphs From the data of F_1 and F_2 we can construct a sub-manifold $X(F_1, F_2)$ of M by taking a regular neighbourhood $\mathcal{N}(F_1 \cup F_2 \cup T_0)$ of $F_1 \cup F_2 \cup T_0$ and capping off its 2-sphere boundary components with 3-balls. Note that $X(F_1, F_2)$ may not be unique.

Now if we start with a pair of graphs $\{\Gamma_1, \Gamma_2\}$. We can still construct a manifold $X(F_1, F_2)$ by building the abstract "2-complex" $F_1 \cup F_2 \cup T_0$ thickening it and capping off the 2-sphere components of its boundary by 3-balls. Lemma 21.1 of (Gordon and Wu, 2008) and Theorem 1.1 of (Gordon, 1998) tell us that if the two slopes r_1, r_2 corresponding to F_1 and F_2 are toroidal then the boundary of $X(F_1, F_2)$ consists of union of tori.

2.2 Toroidal surgeries

In this section we summarize the principal results on toroidal surgeries on hyperbolic 3-manifolds. The list is not exhaustive as we state only what we need for the rest of the thesis. We also prove Lemma 2.2.2, Lemma 2.2.3, and Lemma 2.2.4 which will be useful for Chapter 6.

The following theorem is from Gordon and Ying-Qing Wu.

Theorem 2.2.1. *(Gordon and Wu, 2008) There exist fourteen 3-manifolds M_i, $1 \leq i \leq 14$, such that*

(1) M_i is hyperbolic, $1 \leq i \leq 14$;

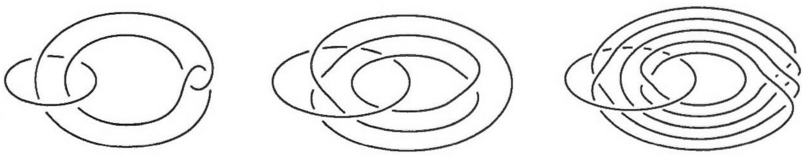
(2) ∂M_i consists of two tori T_0, T_1 if $i \in \{1, 2, 3, 14\}$, and a single torus T_0 otherwise;

(3) there are slopes r_i, s_i on the boundary component T_0 of M_i such that $M(r_i)$ and $M(s_i)$ are toroidal, where $\Delta(r_i, s_i) = 4$ if $i \in \{1, 2, 4, 6, 9, 13, 14\}$, and $\Delta(r_i, s_i) = 5$ if $i \in \{3, 5, 7, 8, 10, 11, 12\}$;

(4) if M is a hyperbolic 3-manifold with toroidal Dehn fillings $M(r), M(s)$ where $\Delta(r,s) = 4$ or 5, then (M, r, s) is equivalent either to (M_i, r_i, s_i) for some $1 \leq i \leq 14$, or to $(M_i(t), r_i, s_i)$ where $i \in \{1, 2, 3, 14\}$ and t is a slope on the boundary component T_1 of M_i.

Here we define two triples (N_1, r_1, s_1) and (N_2, r_2, s_2) to be equivalent if there is a homeomorphism from N_1 to N_2 which sends the boundary slopes (r_1, s_1) to (r_2, s_2) or (s_2, r_2).

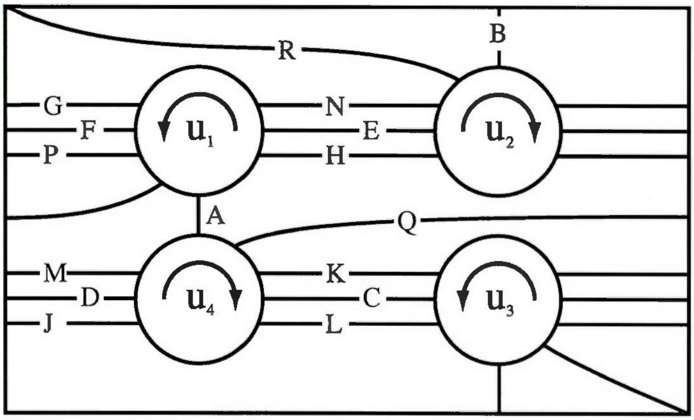
The manifolds M_1, M_2, M_3 are the exteriors of the links L_1, L_2 and L_3 in S^3 which are shown in the following figure



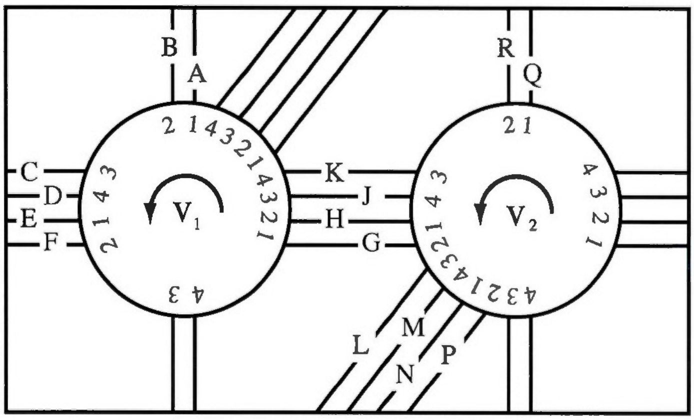
(1) (2) (3)

Figure 2.1 The links L_1, L_2 and L_3.

$M_4, \cdots, M_{14}$ are the manifolds $X(F_1, F_2)$ corresponding to the intersection graphs given in (Gordon and Wu, 2008), and r_i, s_i are the boundary slopes of the corresponding surfaces F_1, F_2. We will reproduce here, in Figure 2.2, Figure 2.3 and Figure 2.4, the intersection graphs for M_4, M_5, and M_{14} as in (Gordon and Wu, 2008). The nine manifolds $M_6, \cdots, M_{14}$ can be constructed as branched covers of a tangle $Q_i = (W_i, K_i)$, for each $i = 6, ..., 14$, where W_i is a 3-ball for $i = 6, ..., 13$, and an $S^2 \times I$ for $i = 14$. More precisely M_i is the double branched cover of W_i with branch set K_i. For more details we refer to (Gordon and Wu, 2008) section 22.



(a)



(b)

Figure 2.2 The intersection graph for the manifold M_4 (Gordon and Wu, 2008).

In Figure 2.2 and Figure 2.3 each graph is on the torus $\widehat{F}_a$ which we draw as a square.

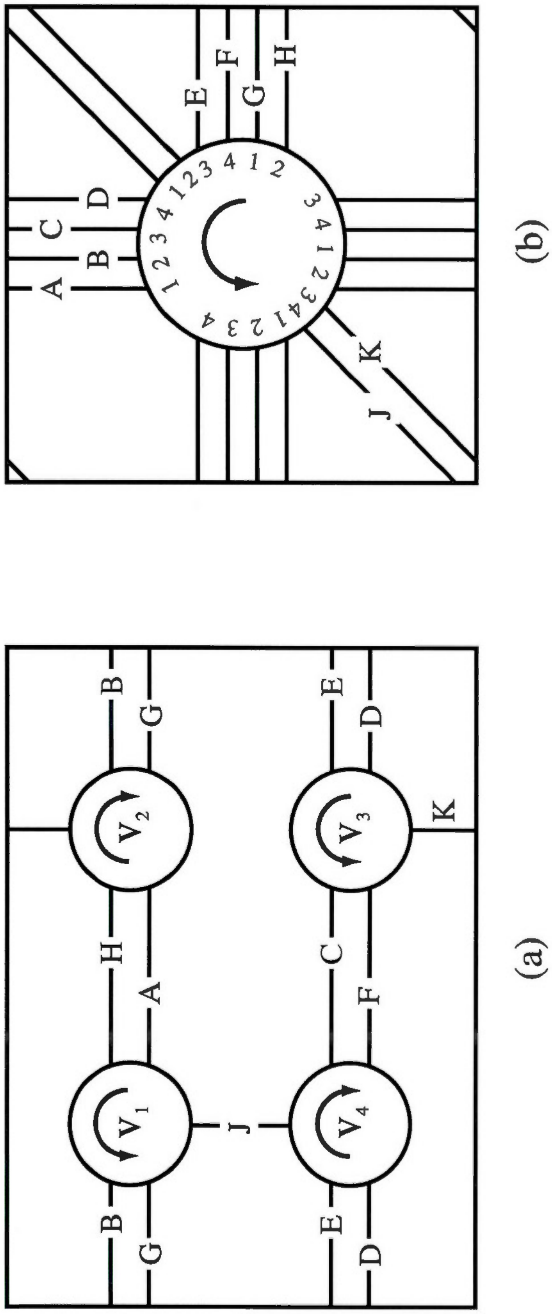


Figure 2.3 The intersection graph for the manifold M_5 (Gordon and Wu, 2008).

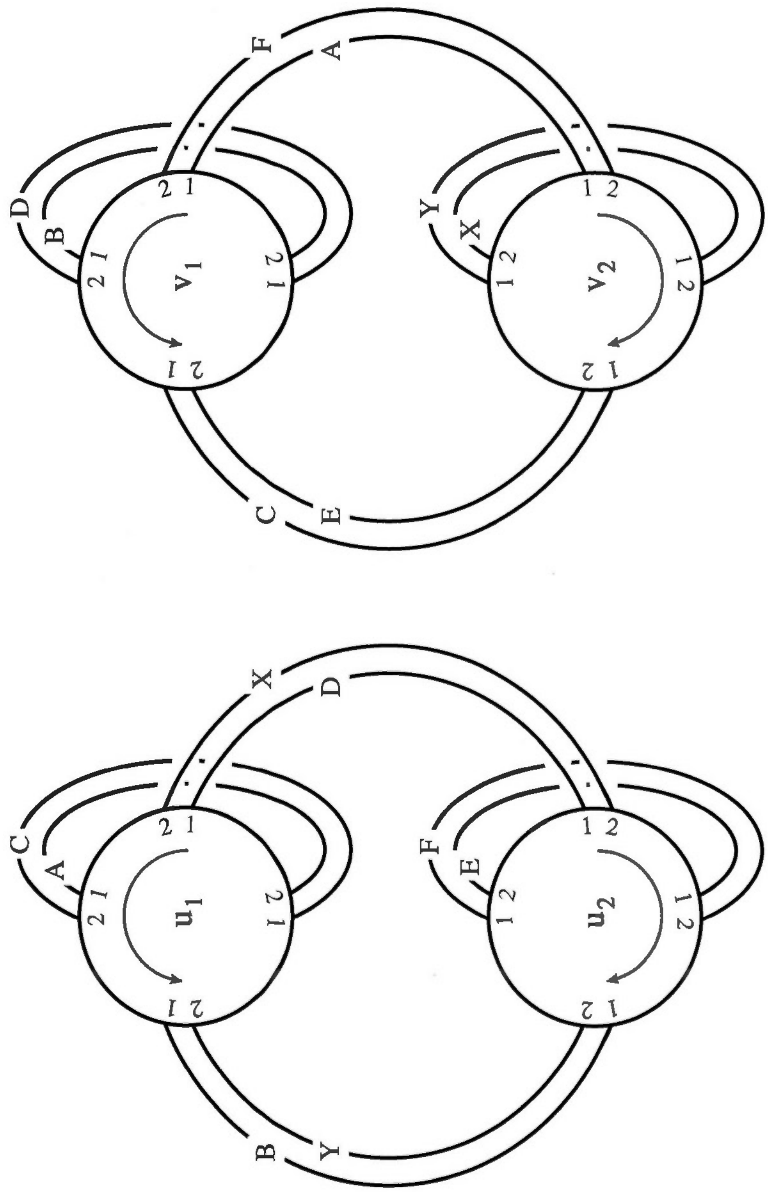


Figure 2.4 The intersection graph for the manifold M_{14} (Gordon and Wu, 2008).

Denote by $T(a_1, a_2)$ a Montesinos tangle which is the sum of two rational tangles of slopes $1/a_1$ and $1/a_2$ respectively, where a_1, a_2 are integers. Denote by $\mathcal{X}(a_1, a_2)$ the collection of Seifert fibre spaces with base 2-orbifold a disk with two cone points c_1, c_2 of index a_1 and a_2, i.e. the cone angle at c_i is $2\pi/a_i$. Note that the double branched cover of $T(a_1, a_2)$ is in $\mathcal{X}(a_1, a_2)$. Denote by $\mathcal{X}(a_1, b_1; a_2, b_2)$ the collection of graph manifolds which are the union of two manifolds X_1, X_2 glued along their boundary, where $X_i \in \mathcal{X}(a_i, b_i)$.

Denote by $K_{p/q}$ the two bridge knot or link associated to the rational number p/q. Denote by $C(p_1, q_1; p_2, q_2)$ the link obtained by replacing each component K_i of a Hopf link by its (p_i, q_i) cable K_i', where q_i is the number of times K_i' winds around K_i. Denote by $Y(p_1, q_1; p_2, q_2)$ the double branched cover of S^3 with branch set $C(p_1, q_1; p_2, q_2)$. Denote by $C(C; p, q)$ the link obtained by replacing one component K_1 of a Hopf link by a Whitehead knot in the solid torus $N(K_1)$, and the other component K_2 by a (p, q) cable of K_2. Let $Y(C; p, q)$ be the double branched cover of S^3 with branch set $C(C; p, q)$.

We can now state the following lemma of Gordon and Ying-Qing Wu.

Lemma 2.2.2. *(Gordon and Wu, 2008) Each M_i $(i = 6, ..., 13)$ admits a lens space surgery $M_i(r_3)$. For each i, let r_1, r_2 be the toroidal slopes r_i, s_i in Theorem 2.2.1. Then the manifolds $M_i(r_1)$, $M_i(r_2)$ and $M_i(r_3)$ are given in the following list.*

$M_6(0) \in X(2,6;2,3)$	$M_6(4) = Y(3,1;5,2)$	$M_6(\infty) = L(9,2)$
$M_7(0) \in \mathcal{X}(2,3;3,3)$	$M_7(-5/2) \in \mathcal{X}(2,3;2,2)$	$M_7(\infty) = L(20,9)$
$M_8(0) \in \mathcal{X}(2,2;2,6)$	$M_8(-5/4) = Y(3,1;2,5)$	$M_8(-1) = L(4,1)$
$M_9(0) \in \mathcal{X}(2,3;2,3)$	$M_9(-4/3) = Y(3,1;2,4)$	$M_9(-1) = L(8,3)$
$M_{10}(0) \in \mathcal{X}(2,3;2,3)$	$M_{10}(-5/2) = Y(C;2,1)$	$M_{10}(\infty) = L(14,3)$
$M_{11}(0) \in \mathcal{X}(2,4;2,4)$	$M_{11}(-5/2) = Y(C;2,1)$	$M_{11}(\infty) = L(24,5)$

$$M_{12}(0) \in \mathcal{X}(2,3;2,3) \qquad M_{12}(5) = Y(3,1;2,3) \qquad M_{12}(\infty) = L(3,1)$$
$$M_{13}(0) \in \mathcal{X}(2,3;2,3) \qquad M_{13}(4) = Z \qquad M_{13}(\infty) = L(4,1)$$

Above Z is a double branched cover of S^3 with branch set some 2-string cable of the trefoil knot.

Proof. See Lemma 22.2 (Gordon and Wu, 2008). □

The next two lemmas can be deduced from the proof of Theorem 22.3 of Gordon and Ying-Qing Wu in (Gordon and Wu, 2008).

Lemma 2.2.3. *(Gordon and Wu, 2008) Let t be a slope on the boundary component T_0 of M_{14}, let K_t be the core of the Dehn filling solid torus in $M_{14}(t)$. Then*

$$H_1\left(M_{14}(t)\right)/H_1(K_t) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Proof. Since $M_{14} = X(F_1, F_2)$ which is determined by the intersection graph in Figure 2.4, we can determine a presentation of the first homology using this picture. Take a regular neighbourhood of $u_1 \cup u_2 \cup D$ on $\hat{F}_a$ as a base point. See Figure 2.4 (a). Then $H_1(M_{14}(r_a))$ is generated by x, y, s_1, s_2, where x is the element of $H_1(\hat{F}_a)$ represented by the edge C on Figure 2.4 (a), oriented from the label 2 endpoint to the label 1 endpoint, y is represented by B, oriented from u_1 to u_2, and s_i by the part of the core of the Dehn filling solid torus running from u_i to u_{i+1} with respect to the orientation of ∂F_b. Then the bigons $B \cup D$, $C \cup E$ and the 4-gon bounded by $C \cup D \cup E \cup Y$ on F_b give relations

$$2s_1 - y = 0, \; 2x = 0, \quad \text{and} \;\; y + 2x = 0.$$

The other faces of Γ_b are parallel to these. Then as an abelian group

$$H_1(M_{14}(r_a)) = \langle x, y, s_1, s_2 \mid 2s_1 - y = 0, \; 2x = 0, \;\; y + 2x = 0 \rangle \,.$$

Now $H_1(K_a)$ is generated by $s_1 + s_2$ thus

$$H_1(M_{14}(r_a))/H_1(K_a) = \langle x, y, s_1, s_2 \mid 2s_1 - y = 0,\ 2x = 0,\ \ y + 2x = 0,\ s_1 + s_2 = 0 \rangle .$$

Therefore $H_1(M_{14}(r_a))/H_1(K_a) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. $\square$

Lemma 2.2.4. *Let r_1, r_2 be two toroidal slopes on M_i, $i \in \{4,5\}$ with $\Delta(r_1, r_2) \in \{4,5\}$. Then for some $b \in \{1,2\}$,*

$$H_1(M_4(r_b)) = \mathbb{Z}, \quad \textit{and} \quad H_1(M_5(r_b)) = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

Proof. Like in Lemma 2.2.3 $M_i = X(F_1, F_2)$, $i \in \{4,5\}$ is determined by the intersection graph in Figure 2.2 and Figure 2.3. From this we are going to get a presentation of the first homology.

- For $i = 4$, choose a regular neighborhood of $v_1 \cup v_2 \cup J$ in Figure 2.2 (b) as a base point. Then $H_1(M_4(r_b))$ is generated by x, y, s_1, s_2, where x, y are represented by the edges L, C in Figure 2.2 (b), oriented from v_1 to v_2, and s_i by the part of the core of the Dehn filling solid torus from v_i to v_{i+1}. The faces bounded by $L \cup C$, $C \cup K$ and $Q \cup K \cup M \cup A$ give the relations

  $$y - s_1 + x + s_2 = 0,\ s_1 - x - s_2 = 0, \quad \text{and} \quad s_2 - s_1 + y = 0.$$

  Then as an abelian group

  $$\begin{aligned} H_1(M_{14}(r_b)) &= \langle x, y, s_1, s_2 \mid y - s_1 + x + s_2 = 0, \\ &\qquad s_1 - x - s_2 = 0,\ \ s_2 - s_1 + y = 0 \rangle \\ &\cong \mathbb{Z}. \end{aligned}$$

- Similarly for $i = 5$, $H_1(M_5(r_b))$ is generated by x, y, s, where x, y are represented by edges E and C on Figure 2.3 (b), oriented from label 3 to label 4, and s is represented by the core of the Dehn filling solid torus. Then the bigon $A \cup H$ and the annulus bounded by $A \cup G \cup C \cup E$ on Figure 2.3 (a) containing J give the relations $x + y = 0$ and $2x - 2y = 0$. Thus

$$H_1(M_{14}(r_a)) = \langle x, y, s \mid x + y = 0,\ 2x - 2y = 0\rangle \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

$\square$

Let W be the exterior of the Whitehead link and let T_0 be a boundary component of W. Choosing a standard meridian-longitude basis μ, λ for $H_1(T_0)$ we can identify slopes T_0 with elements of $\mathbb{Q} \cup \{1/0\}$. The manifolds $W(1)$, $W(2)$, $W(-5)$, $W(-5/2)$ are hyperbolic and they all admits a pair of toroidal slopes r, s with $\Delta(r, s) > 5$. Gordon proved that these examples are the only possibilities for hyperbolic manifolds with pair of toroidal slopes at distance > 5.

Theorem 2.2.5. *(Gordon, 1998) Let M be an irreducible 3-manifold and T a torus component of ∂M. If two slopes r and s on T are toroidal then either*

1. *$\Delta(r, s) \leq 5$; or*
2. *$\Delta(r, s) = 6$ and M is homeomorphic to $W(2)$; or*
3. *$\Delta(r, s) = 7$ and M is homeomorphic to $W(-5/2)$; or*
4. *$\Delta(r, s) = 8$ and M is homeomorphic to $W(1)$ or $W(-5)$.*

For hyperbolic knot in S^3, results about toroidal surgeries are more refined. In particular since there is a canonical *Seifert* longitude we can identify a slope with an element of $\mathbb{Q} \cup \{1/0\}$. One can then obtain bounds on the denominator q of an exceptional slope p/q.

Theorem 2.2.6. *(Gordon and Luecke, 1995) Let K be a hyperbolic knot in S^3 and suppose that $S^3_K(p/q)$ contains an essential torus. Then $|q| \leq 2$.*

Now for the case where the slope is non-integral we have a complete understanding of toroidal surgeries which is given by the following theorem.

Theorem 2.2.7. *(Gordon and Luecke, 2004) Let K be a hyperbolic knot in S^3 that admits a non-integral surgery containing an incompressible torus. Then K is one of the Eudave-Muñoz knots $k(l, m, n, p)$ and the surgery is the corresponding half-integral surgery.*

The list of knots in S^3 which admit pair of toroidal slopes at distance 4 is also known by work of Gordon and Ying-Qing Wu.

Theorem 2.2.8. *(Gordon and Wu, 2008) A knot K in S^3 is hyperbolic and admits two toroidal surgeries $S^3_K(r_1)$, $S^3_K(r_2)$ with $\Delta(r_1, r_2) \geq 4$ if and only if (K, r_1, r_2) is equivalent to one of the following, where n is an integer.*

1. $K = L_1(n)$, $r_1 = 0$, $r_2 = 4$.
2. $K = L_2(n)$, $r_1 = 2 - 9n$, $r_2 = -2 - 9n$.
3. $K = L_3(n)$, $r_1 = -9 - 25n$, $r_2 = -(13/2) - 25n$.
4. *K is the Figure 8 knot,* $r_1 = 4$, $r_2 = -4$.

The knots $L_1(n)$, $L_2(n)$ and $L_3(n)$ are the knots obtained from the right components of the links L_1, L_2, L_3 in Figure 2.1 after $1/n$-surgery on the left components. In the particular case where $\Delta(r_1, r_2) = 4$, then $K = L_1(n)$, $r_1 = 0$, $r_2 = 4$; or $K = L_2(n)$, $r_1 = 2 - 9n$, $r_2 = -2 - 9n$.

CHAPTER III

THE CASSON INVARIANT

In this chapter we give a quick review of two classical 3-manifold invariants: the Casson-Walker invariant and the Casson-Gordon invariant. We also prove a proposition which will be very useful for us. We follow (Walker, 1992) and (Saveliev, 2002) chapter 3 and 4.

3.1 The Casson-Walker invariant

The *Casson invariant* assigns an integer to any oriented integral homology 3-sphere Y. This can be done by counting conjugacy classes of irreducible representations $\pi_1(Y) \to SU(2)$. This invariant was later extended to homology lens spaces by Boyer and Lines (Boyer and Lines, 1990) and then to rational homology 3-spheres by K. Walker (Walker, 1992). Lescop (Lescop, 1996) showed that Walker's invariant also admits a purely combinatorial definition in terms of surgery presentations. We refer to (Lescop, 1996) and (Walker, 1992) for more details. The existence and uniqueness of this invariant together with basic properties are given by the first theorem below taken from [(Saveliev, 2002) section 4.1] and was originally stated in Walker (Walker, 1992) but with a difference of a factor of 2 for the invariant.

Walker give the following definition of the Casson invariant for $\mathbb{Q}$-homology spheres.

Theorem 3.1.1. *(Walker, 1992) There exists a unique invariant λ of oriented rational homology 3-spheres, which satisfies the following properties:*

1. *λ coincides with Casson's invariant on integral homology sphere.*
2. *$\lambda(-Y) = -\lambda(Y)$ where $-Y$ stands for Y with opposite orientation.*
3. *$\lambda(Y_1 \# Y_2) = \lambda(Y_1) + \lambda(Y_2)$ for two rational homology spheres Y_1 and Y_2.*
4. *The number $12 \cdot |H_1(Y;\mathbb{Z})| \cdot \lambda(Y)$ is an integer for any rational homology homology sphere.*
5. *Let K be a knot in an oriented rational homology sphere Y, and let $l \in \partial Y_K$ be a longitude. Then, λ satisfies the surgery formula*

$$\lambda(Y_K(a)) = \lambda(Y_K(b)) + \tau(a, b; l) + \frac{\langle a, b\rangle}{\langle a, l\rangle\langle b, l\rangle} \cdot \frac{1}{2}\Delta''_K(1).$$

for any primitive class $a, b \in H_1(\partial K;\mathbb{Z})$ such that $\langle a, l\rangle \neq 0$ and $\langle b, l\rangle \neq 0$

Here Δ''_k stands for the second order derivative of the Alexander polynomial of K normalized so that $\Delta_K(T) = \Delta_K(T^{-1})$ and $\Delta_K(1) = 1$. The brackets $\langle\, ,\rangle$ denote the intersection pairing $H_1(\partial Y_K;\mathbb{Z}) \otimes H_1(\partial Y_K;\mathbb{Z}) \longrightarrow \mathbb{Z}$. Let us fix a longitude l and choose a basis x, y of $H_1(\partial Y_K;\mathbb{Z})$ such that $\langle x, y\rangle = 1$ and $l = dy$ for some $d \in \mathbb{Z}$. Then

$$\tau(a, b; l) := -s(\langle x, a\rangle, \langle y, a\rangle) + s(\langle x, b\rangle, \langle y, b\rangle) + \frac{d^2 - 1}{12} \cdot \frac{\langle a, b\rangle}{\langle a, l\rangle\langle b, l\rangle}$$

where $s(q, p)$ is the Dedekind sum defined by

$$s(q, p) := \operatorname{sign}(p) \cdot \sum_{k=1}^{|p|-1} ((\frac{k}{p}))((\frac{kq}{p})),$$

with

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

When K is a null-homologous knot the surgery formula simplifies as follows.

Proposition 3.1.2 ((Saveliev, 2002) Corollary 4.5). *Let K be a null-homologous knot in a rational homology three-sphere Y, and let $L(p,q)$ be the lens space obtained by (p/q)-surgery on the unknot in S^3. Then*

$$\lambda(Y_K(p/q)) = \lambda(Y) + \lambda(L(p,q)) + \frac{q}{2p}\Delta_K''(1).$$

Proof. See (Saveliev, 2002) for details. □

Note that by our convention $L(p,q)$ is obtained by (p/q)-surgery on the unknot in S^3, so in the formula we add $+\lambda(L(p,q))$. If we had taken the convention in (Saveliev, 2002) we would have a $-\lambda(L(p,q))$ term instead.

Boyer and Lines have computed the Casson invariant of lens space.

Proposition 3.1.3. *(Boyer and Lines, 1990) For a lens space $L(p,q)$,*

$$\lambda(L(p,q)) = -\frac{1}{2}s(q,p).$$

Proof. See (Boyer and Lines, 1990) or (Saveliev, 2002). □

An interesting example is the Casson invariant of the Poincaré sphere. Recall that the Poincaré sphere, denoted $\Sigma(2,3,5)$, is the oriented manifold obtained by (-1)-surgery on the left handed trefoil in S^3. Since the Alexander polynomial of the trefoil is $T^{-1} - 1 + T$, by the surgery formula in Proposition 3.1.2, $\lambda(\Sigma(2,3,5)) = -1$.

3.2 The Casson-Gordon invariant

We are going to survey the Casson-Gordon version of Casson invariants. We follow their original paper (Casson and Gordon, 1978) section 2. Let Y be a closed oriented 3-manifold and $\phi : H_1(M; \mathbb{Z}) \to \mathbb{Z}_m$ an epimorphism. The invariant associates to the pair (Y, ϕ) a collection of rational numbers $\sigma_r(M, \phi)$, $0 < r < m$. Here are the construction.

By standard topology, the map ϕ induces a canonical m-fold cyclic covering $\tilde{Y} \to Y$. Now by (Casson and Gordon, 1978) Lemma 2.2 we can find $\tilde{X} \to X$ an m-fold cyclic branched covering of 4-manifolds, branched over some surface $F \subset \mathrm{int}(X)$, such that $\partial(\tilde{X} \to X) = (\tilde{Y} \to Y)$. Moreover the rotation through $2\pi/m$ on each fibre of the normal bundle of the branched surface $\tilde{F}$ in $\mathrm{int}(\tilde{X})$ corresponds to the canonical covering translation of $\tilde{X}$.

Recall that there is an intersection form on $H_2(\tilde{X}; \mathbb{Z})$. This extends to a non-singular Hermitian form $\langle , \rangle$ on $H := H_2(\tilde{X}; \mathbb{Z}) \otimes \mathbb{C}$. The covering translation of $\tilde{X}$ which rotates each fibre of the normal bundle of $\tilde{F}$ through $2\pi/m$ induces an automorphism $\theta : H \to H$. The map θ is an isometry of $(H; \langle , \rangle)$ and $\theta^m = \mathrm{id}$. Let $\omega = e^{2\pi i/m}$, and let E_r be the ω^r-eigenspace of θ, $0 \leq r < m$. We have an orthonormal decomposition of $(H; \langle , \rangle)$ as $E_0 \oplus E_1 \oplus \cdots \oplus E_{m-1}$. We denote by $\epsilon_r(\tilde{X})$ the signature of the restriction of $\langle , \rangle$ to E_r, and by $\mathrm{sign}(X)$ the signature of X. Then for $0 < r < m$, we define $\sigma_r(Y, \phi)$ as

$$\sigma_r(Y, \phi) = \mathrm{sign}(X) - \epsilon_r(\tilde{X}) - \frac{2[F]^2 r(m-r)}{m^2}.$$

Using Novikov additivity and the G-signature theorem Casson and Gordon proved that $\sigma_r(Y, \phi)$ depends only on the rational number r and the cyclic cover $\tilde{Y} \to Y$. Therefore when Y has $H_1(Y; \mathbb{Z}) = \mathbb{Z}_m$, we get an invariant of the 3-manifold Y

by taking the sum

$$\sum_{r=1}^{m-1} \sigma_r(Y,\phi) = \sum_{r=1}^{m-1} \operatorname{sign}(X) - \epsilon_r(\tilde{X}) - \frac{2[F]^2 r(m-r)}{m^2}$$
$$= m\operatorname{sign}(X) - \operatorname{sign}(\tilde{X}) - \frac{[F]^2(m^2-1)}{3m}.$$

The Casson-Gordon invariant is then defined as follows.

Definition 3.2.1. *(Casson and Gordon, 1978) The total Casson-Gordon invariant of Y is the rational number*

$$\tau(Y) = m\, sign(X) - sign(\tilde{X}) - \frac{[F]^2(m^2-1)}{3m}.$$

Now assume Y is a $\mathbb{Z}$-homology sphere and K is a knot in Y. Let A be a Seifert matrix for K and let ξ be a complex number with $|\xi| = 1$. We define for each integer $m \neq 0$ the number

$$\sigma(K,m) = \sum_{r=1}^{m-1} \sigma_K(e^{2i\pi r/m}),$$

where $\sigma_K(\xi)$ is the signature of A.

Boyer and Lines found a surgery formula for the Casson-Gordon invariant which involves $\sigma(K,m)$.

Proposition 3.2.2. *(Boyer and Lines, 1990) Let $K \subset Y$ be a knot in a $\mathbb{Z}$-homology sphere Y, then*

$$\tau(Y_{p/q}(K)) = \tau(L(p,q)) - \sigma(K,p).$$

□

They also computed the invariant for the lens space $L(p,q)$.

Proposition 3.2.3. *(Boyer and Lines, 1990) For a lens space $L(p,q)$,*

$$\tau(L(p,q)) = -4p \cdot s(q,p).$$

Proof. See (Saveliev, 2002) or (Boyer and Lines, 1990). □

3.3 A preliminary result on the Casson invariant

The following result was proved by Boyer and Lines (Boyer and Lines, 1990). It requires a non-vanishing condition on the second order derivative of the Alexander polynomial Δ_K of the knot K. We reproduce another proof here for convenience of the reader.

Proposition 3.3.1. *Let K be a non-trivial knot in a 3-manifold Y with trivial first homology and let $M = Y \setminus \mathcal{N}(K)$. If $\Delta''_K(1) \neq 0$, then there is no orientation preserving homeomorphism between $M(r)$ and $M(r')$ if $r \neq r'$.*

Proof. By simple homological reasoning we must have $r = p/q$ and $r' = p/q'$ where q and q' are two integers coprime to p. The surgery formula for Casson-Walker invariant, Proposition 3.1.2, gives

$$\lambda(M(r)) = \lambda(Y) + \lambda(L(p,q)) + \frac{q}{2p}\Delta''_K(1)$$

$$\text{and} \qquad \lambda(M(r')) = \lambda(Y) + \lambda(L(p,q')) + \frac{q'}{2p}\Delta''_K(1)$$

Now since Y has trivial first homology we have a well defined Casson-Gordon invariant τ and a surgery formula from Proposition 3.2.2

$$\tau(M(r)) = \tau(L(p,q)) - \sigma(K,p) \qquad \text{and} \qquad \tau(M(r')) = \tau(L(p,q')) - \sigma(K,p)$$

By Proposition 3.1.3 and Proposition 3.2.3 the two invariant λ and τ for the lens space $L(p,q)$ are related by the following formula

$$\tau(L(p,q)) = -4p \cdot s(q,p) = -2p\lambda(L(p,q)).$$

Now if $M(r) \cong_+ M(r')$, then $\lambda(M(r)) = \lambda(M(r'))$ and $\tau(M(r)) = \tau(M(r'))$. Thus by using the above formula we have the following identities:

$$\tau(L(p,q)) - \sigma(K,p) = \tau(L(p,q')) - \sigma(K,p)$$

$$\tau(L(p,q)) = \tau(L(p,q')) \quad \text{and} \quad \lambda(L(p,q)) = \lambda(L(p,q')) \text{ since } p \neq 0.$$

Therefore we have the equality

$$\lambda(Y) + \lambda(L(p,q)) + \frac{q}{2p}\Delta_K''(1) = \lambda(Y) + \lambda(L(p,q')) + \frac{q'}{2p}\Delta_K''(1)$$

which implies

$$\frac{q}{p}\Delta_K''(1) = \frac{q'}{p}\Delta_K''(1)$$

Since $\Delta_K''(1) \neq 0$ by assumption, we have $q = q'$ and so $r = r'$. $\square$

The proof of the proposition shows that we must have $\lambda(L(p,q)) = \lambda(L(p,q'))$ if p/q and p/q' are two cosmetic slopes. On the other hand for a lens space $L(p,q)$,

$$\lambda(L(p,q)) = -\frac{1}{2}s(q,p).$$

We can then deduce the following lemma which will be useful later.

Lemma 3.3.2. *Let K be a non-trivial knot in a 3-manifold Y with trivial first homology and let $M = Y \setminus \mathcal{N}(K)$. If there is an orientation preserving homeomorphism between $M(p/q)$ and $M(p/q')$ then $s(q,p) = s(q',p)$.* $\square$

CHAPTER IV

HEEGAARD FLOER HOMOLOGY

This chapter provides a brief expository account of Heegaard Floer theory with emphasis on the necessary tools and results needed for the rest of the thesis. Heegaard Floer homology was introduced by Peter Ozsváth and Zoltan Szabó around 2000. The theory had a rapid development and has contributed to progresses on various problems in low dimensional topology. We combine here material from various sources, including the lectures notes (Ozsváth and Szabó, 2006b) and (Ozsváth and Szabó, 2006c), and the original articles: for the three-manifold version we refer to (Ozsváth and Szabó, 2006a), (Ozsváth and Szabó, 2004d), (Ozsváth and Szabó, 2004c), (Ozsváth and Szabó, 2005) and for the knot version we refer to (Ozsváth and Szabó, 2004b). We invite the reader to consult at these papers for further details on the subject. From now on the notation CF° (resp. HF°) will denote collectively the chain complexes $\widehat{CF}, CF^\infty, CF^-, CF^+$ (resp. the homologies $\widehat{HF}, HF^\infty, HF^-, HF^+$). The latter will be defined in this chapter.

Through this chapter Y will denote a closed oriented 3-manifold and K will be a null-homologous knot in Y.

4.1 Three-manifold Heegaard Floer homologies

4.1.1 Heegaard diagrams, Whitney disks and Spin^c structures

Heegaard Diagrams. A *genus g Heegaard splitting* of Y is a decomposition of Y into the union of two oriented genus g handlebodies glued together along their boundaries. It is a known fact that every closed oriented 3-manifold admits a Heegaard splitting. To see this, take a self-indexing Morse function, $f : Y \to \mathbb{R}$ with one index zero and one index three critical points. The level set $\Sigma_g = f^{-1}(3/2)$ is an oriented closed connected surface with genus g equal to the number of index 1 critical points. The decomposition $Y = f^{-1}([0, 3/2]) \cup f^{-1}([3/2, 3])$ then gives a Heegaard splitting of Y. Moreover the intersection of Σ_g with the ascending manifolds of the index 1 critical points is a collection of simple closed curves $\alpha = \{\alpha_1, \cdots, \alpha_g\}$, similarly the intersection of Σ with the descending manifold of the index two critical points is a collection of simple closed curves $\beta = \{\beta_1, \cdots, \beta_g\}$. One can recover Y by attaching 2-handles to the oriented manifold $\Sigma \times [-1, 1]$ along these collections of curves and capping off the resulting manifold with two 3-handles. Thus the data $(\Sigma_g, \alpha, \beta)$ completely determines Y and its orientation and is called a *genus g Heegaard diagram* of Y. Sometimes one needs to add extra data in the form of one or more marked points to get what is called a *pointed Heegaard diagram* $(\Sigma_g, \alpha, \beta, z)$ or a *multi-pointed Heegaard diagram* $(\Sigma_g, \alpha, \beta, z_1, \cdots, z_k)$, where $z, z_1, \cdots, z_k$ are marked points on $\Sigma - \alpha - \beta$. Depending on the circumstance one can also have a Heegaard diagram with more than two set of simple closed curves. If two Heegaard diagrams represent the same 3-manifold then they differ by a finite sequence of the following moves:

- *Isotopy.* This moves the attaching circles in a 1-parameter family which keeps the α's transverse to the β's and such that the α's (resp. β's) remain disjoint among themselves.

- *Handleslide.* Choose two curves γ_1 and γ_2 among the same collection (α or β). Replace γ_1 by a simple closed curve $\tilde{\gamma}$ such that
  - ◇ $\tilde{\gamma}$ is disjoint from $\gamma_1, \gamma_2, \cdots, \gamma_g$
  - ◇ $\tilde{\gamma}$, γ_1 and γ_2 bound an embedded pair of pants (i.e. a disk with two holes) in
    $\Sigma - \gamma_1 - \gamma_2 - \cdots - \gamma_g$
- *Stabilization.* This is a connected sum with a torus and increases the genus of the Heegaard diagram by one. It adds a new α-type curve and a new β-type curve. *Destabilization* is the inverse process of stabilization.

Two Heegaard diagrams are called equivalent if they are related by finite sequences of these modifications. For more details we refer to (Rolfsen, 2003).

If we have a Heegaard diagram with one or more base points, then we require that the supports of the isotopies do not contain the base points and that during handle-slide the base points must be outside the pair of pants region. We then have the notion of *pointed isotopy* and *pointed handle slide.* With stabilization these new moves also define an equivalence relation on the set of pointed Heegaard diagrams.

We have the following fact, first proved by Singer in (Singer, 1933).

Theorem 4.1.1 ((Singer, 1933) Theorem 8). *Two Heegaard diagrams are related by a sequence of isotopies, handleslides, and (de)stabilization if and only if they represent the same manifold up to diffeomorphism.*

We note also that if (Σ, α, β) is a Heegaard diagram for Y then $(-\Sigma, \beta, \alpha)$ is an Heegaard diagram for $-Y$, here the role of α and β has been interchanged.

Whitney disks. Fix a pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y. We associate to it a quadruple denoted by $(\mathrm{Sym}^g(\Sigma), \mathbb{T}_\alpha, \mathbb{T}_\beta, V_z)$ where:

- $\mathrm{Sym}^g(\Sigma)$ is the g-fold symmetric product of Σ (g = genus of Σ), that is the quotient $\Sigma^{\times g}/S_g$ where the permutation group in g letters, S_g , acts on $\Sigma^{\times g}$ by permuting the coordinates. The quotient $\mathrm{Sym}^g(\Sigma)$ is a smooth manifold of real dimension $2g$ and has non-empty sets of symplectic structures and compatible almost-complex structures. In particular a complex structure on Σ induces a complex structure on $\mathrm{Sym}^g(\Sigma) - \Delta$ where $\Delta \subset \mathrm{Sym}^g(\Sigma)$ is the image of the big diagonal[1] in $\Sigma^{\times g}$. This complex structure can be perturbed to give a complex structure on all $\mathrm{Sym}^g(\Sigma)$.
- $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are totally real embedded submanifolds of $\mathrm{Sym}^g(\Sigma)$ (for some choice of almost-complex structure) defined by the attaching circles $\alpha_1, \cdots, \alpha_g$ and $\beta_1, \cdots, \beta_g$:

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g, \qquad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g.$$

  Since the α, resp. β, curves are pairwise disjoint, these submanifolds embed in $\mathrm{Sym}^g(\Sigma)$ via the natural projection. We can also assume that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersect transversally.
- V_z is a codimension 2 submanifold of $\mathrm{Sym}^g(\Sigma)$ defined by $V_z := \{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$. Since $z \in \Sigma - \alpha - \beta$, V_z is disjoint from $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

Let $\mathbb{D}$ be the unit disk in the complex plane. Let S_+, S_- be the arcs in the boundary of $\mathbb{D}$ corresponding to $\mathrm{Im}[z] \geq 0$ and $\mathrm{Im}[z] \leq 0$.

---

[1]The big diagonal is the subset $\{(x_1, \cdots, x_g) | \exists i, j \in \{1, \cdots, g\}, i \neq j \text{ and } x_i = x_j\}$ in $\Sigma^{\times g}$

Definition 4.1.2. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. *A Whitney disk connecting* $\mathbf{x}$ *to* $\mathbf{y}$ *is a continuous map:* $u : \mathbb{D} \longrightarrow \mathrm{Sym}^g\Sigma$ *which sends* $-i$ *to* $\mathbf{x}$, i *to* $\mathbf{y}$, S_+ *inside* $\mathbb{T}_\alpha$ *and* S_- *inside* $\mathbb{T}_\beta$.

When $g \neq 2$ we denote $\pi_2(\mathbf{x}, \mathbf{y})$ the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$. For $g = 2$ we will define this notation later in this subsection. The multiplicity of an element $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ at z is defined to be the integer :

$$n_z(\phi) = \sharp u^{-1}(\mathbb{D} \cap V_z)$$

where u is a smooth representative of ϕ chosen to be transverse to V_z. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and let $a : [0,1] \to \mathbb{T}_\alpha$, resp. $b : [0,1] \to \mathbb{T}_\beta$, be a path from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbb{T}_\alpha$ resp. in $\mathbb{T}_\beta$. The difference $a - b$ gives a loop in $\mathrm{Sym}^g(\Sigma)$. On the other hand we have the following isomorphism (Ozsváth and Szabó, 2004d)

$$\frac{H_1(\mathrm{Sym}^g(\Sigma);\mathbb{Z})}{H_1(\mathbb{T}_\alpha;\mathbb{Z}) \oplus H_1(\mathbb{T}_\beta;\mathbb{Z})} \cong \frac{H_1(\Sigma;\mathbb{Z})}{\langle[\alpha_1], \cdots, [\alpha_g], [\beta_1], \cdots, [\beta_g]\rangle} \cong H_1(Y;\mathbb{Z}).$$

Let $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(Y;\mathbb{Z})$ denote the image of $a - b$ under this isomorphism. The homology class $\epsilon(\mathbf{x}, \mathbf{y})$ is independent of the choice of paths a and b. It follows that if $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0$ then $\pi_2(\mathbf{x}, \mathbf{y})$ is empty. It is also obvious that ϵ is additive: $\epsilon(\mathbf{x}, \mathbf{y}) + \epsilon(\mathbf{y}, \mathbf{z}) = \epsilon(\mathbf{x}, \mathbf{z})$.

For $g = 1$, $\Sigma = \mathrm{Sym}^g\Sigma = S^1 \times S^1$ and it is easy to visualize Whitney disks since they are genuine maps of a disk into the torus. When $g > 1$ we need to introduce the notion of domains.

Let $D_1, \cdots, D_k$ be the closures of the connected components of $\Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$.

Definition 4.1.3. *A domain is a linear combination of the* D_i*'s with integer coefficients.*

Definition 4.1.4. *A domain D is said to be positive if all the coefficients are ≥ 0. We then write $D \geq 0$.*

The set of all domains is then the free Abelian group generated by the set of all D_i's. The boundary of a domain is a linear combination of arcs contained in the α or β curves with integer coefficients.

Definition 4.1.5. *A periodic domain is a domain $\mathcal{P}$ whose boundary is a sum of α and β curves and whose $n_z(\mathcal{P}) = 0$.*

Lemma 4.1.6. *The set of periodic domains is a subgroup isomorphic to $H_2(Y, \mathbb{Z})$.*

Definition 4.1.7. *The domain of a homotopy class ϕ of Whitney disk connecting $\mathbf{x}$ to $\mathbf{y}$ is the formal linear combination*

$$\mathcal{D}(\phi) := \sum_{i=1}^{k} n_{z_i}(\phi)\ D_i$$

where $z_i \in D_i$ are points in the interior of D_i.

We can now define the notation $\pi_2(x, y)$ for genus 2 Heegaard diagram. When Σ has genus 2, $\pi_2(x, y)$ will stand for the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$ modulo the relation $\phi_1 \sim \phi_2$ iff $\mathcal{D}(\phi_1) = \mathcal{D}(\phi_2)$.

Definition 4.1.8. *Let $\mathbf{x} = \{x_1, \cdots, x_g\}$ and $\mathbf{y} = \{y_1, \cdots, y_g\}$ be points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. A domain connecting $\mathbf{x}$ to $\mathbf{y}$ is a domain D such that ∂D consists of α or β curves and $2g$ arcs, g of which connect x_j to $y_{\sigma(j)}$ for some permutation $\sigma \in S_g$, and g of which connect y_j to $x_{\sigma'(j)}$ for some other permutation $\sigma' \in S_g$. We denote $\mathcal{D}(\mathbf{x}, \mathbf{y})$ the set of domains connecting $\mathbf{x}$ to $\mathbf{y}$.*

Proposition 4.1.9. *(Ozsváth and Szabó, 2004d) When $g > 1$, the map $\phi \mapsto \mathcal{D}(\phi)$ gives a bijection between $\pi_2(\mathbf{x}, \mathbf{y})$ and $\mathcal{D}(\mathbf{x}, \mathbf{y})$. For $g = 1$, the map is an injection.*

Counting elements of $\pi_2(\mathbf{x},\mathbf{y})$ is then equivalent to counting domains which do not involve $\mathrm{Sym}^g(\Sigma)$ in their definition.

Let $\mathbf{x},\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $D = \sum n_i D_i \in \mathcal{D}(\mathbf{x},\mathbf{y})$. Since the sum of the D_i's is a generator for $H_2(\Sigma;\mathbb{Z})$, the new domain $\sum(n_i+1)D_i$ is also a domain connecting $\mathbf{x}$ to $\mathbf{y}$ (we have "added Σ" to D). In general, for any $j \in \mathbb{Z}$ the domain $\sum(n_i+j)D_i$ also connects $\mathbf{x}$ to $\mathbf{y}$. For domains with the extra condition $n_z(D) = 0$ we can no longer add a copy of Σ.

When $H_2(Y) = 0$, these domains are the only domains connecting $\mathbf{x}$ to $\mathbf{y}$. Moreover there is at most a unique domain if we require that $n_z(D) = 0$. In general, by Lemma 4.1.6, elements of $H_2(Y)$ correspond to periodic domains. If D is a domain connecting $\mathbf{x}$ and $\mathbf{y}$ and P is a periodic domain, then we can see from the definition that $D+P$ is also a domain connecting $\mathbf{x}$ to $\mathbf{y}$. Since Y is a closed oriented 3-manifold $H_2(Y)$ is trivial or free Abelian. Therefore if $H_2(Y) \neq 0$, we have infinitely many domains connecting $\mathbf{x}$ to $\mathbf{y}$ even with the condition $n_z(D) = 0$. Due to this particularity we will discuss separately the definition of $HF^\circ(Y)$ for the case $b_1(Y) = 0$ and $b_1(Y) > 0$.

For $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ we define $\mathcal{M}(\phi)$ to be the set of (pseudo-)holomorphic representatives of ϕ with respect to some generically chosen almost complex structure.

The group of holomorphic automorphisms of the unit disk is $\mathrm{PSL}(2,\mathbb{R})$, so the subgroup preserving i and $-i$ is isomorphic to $\mathbb{R}$. Therefore $\mathbb{R}$ acts on $\mathcal{M}(\phi)$ by re-parameterization of the unit disk. We denote by $\widehat{\mathcal{M}}(\phi)$ the quotient of $\mathcal{M}(\phi)$ by this $\mathbb{R}$ action. Ozsváth-Szabó specify a set of almost-complex structures that includes those induced by complex structures on Σ. They prove that, for a dense subset of these almost complex structures, $\mathcal{M}(\phi)$ is a smooth manifold whose dimension, denoted $\mu(\phi)$, equals a certain index called the Maslov index of ϕ, which we will not define here. For more details above Maslov index we refer the reader to (Robbin and Salamon, 1993). A result of Gromov says that in every homotopy class ϕ of Maslov index 1 the set $\widehat{\mathcal{M}}(\phi)$ is finite (compact 0-dimensional). We will omit the discussion about the genericity of complex structures and Gromov's result.

Spin^c **structures.** We review here Spin^c structures using (Turaev, 1997), (Ozsváth and Szabó, 2006b) and (Ozsváth and Szabó, 2006c).

Recall that the group $\mathrm{Spin}^{\mathbb{C}}(n)$ is the central $U(1)$ extension of $SO(n)$:

$$\mathbf{1} \longrightarrow U(1) \longrightarrow \mathrm{Spin}^{\mathbb{C}}(n) \longrightarrow SO(n) \longrightarrow \mathbf{1}.$$

Given an oriented manifold X equipped with a Riemannian metric, we have an $SO(n)$ principal bundle over X which is the bundle of oriented orthonormal frame. We can then ask if we can lift this bundle to a principal $\mathrm{Spin}^{\mathbb{C}}$ bundle. This can be done if and only if its second Stiefel-Whitney class w_2 is the mod 2 reduction of an integral cohomology class, see (Milnor, 1963). This is the case if X has dimension 3 or 4, see (Milnor, 1963).

Let g_0 and g_1 be two Riemannian metrics on X which admit two $\mathrm{Spin}^{\mathbb{C}}$-principal bundle ξ_0 and ξ_1 which are lifts of their oriented orthonormal frame bundle. Then the two triple (X, g_0, ξ_0) and (X, g_1, ξ_1) are said to be equivalent if one can find a 1-parameter family of metrics $(g_t)_{0\leq t\leq 1}$ and a continuous 1-parameter family of $\mathrm{Spin}^{\mathbb{C}}$ lift $(\xi_t)_{0\leq t\leq 1}$ of the oriented orthonormal frame bundle of (X, g_t).

Definition 4.1.10. *A* Spin^c *structure on an oriented manifold X is the equivalence class of a* $\mathrm{Spin}^{\mathbb{C}}$ *lift of the oriented orthonormal frame bundle of X with respect to some Riemannian metric.*

Since we are dealing with equivalence classes, a Spin^c structure on X does not depend on any particular metric, they are associated to the manifold itself. For the special case of a closed oriented 3-manifold Y there is a more practical definition of Spin^c structure due to Turaev (Turaev, 1997).

Definition 4.1.11. *Two nowhere-vanishing vector fields v_1, v_2 on Y are homologous if they are homotopic in the complement of a finite number of three-balls in Y.*

Proposition 4.1.12. *(Turaev, 1997) The set of* Spin^c *structures on Y is in one to one correspondence with the set of homology classes of vector fields on Y.*

Definition 4.1.13. *If $\mathfrak{s}$ is a* Spin^c *structure on Y represented by a nowhere-vanishing vector field v, its conjugate* Spin^c *structure $\bar{\mathfrak{s}}$ is the one represented by $-v$.*

The fact that closed oriented 3-manifolds are parallelizable implies that Y always admits a nowhere-vanishing vector field and so a Spin^c structure. Moreover after a choice of trivialization $\varsigma : TY \cong Y \times \mathbb{R}^3$, nowhere-vanishing vector fields on Y correspond to maps $u : Y \to \mathbb{R}^3 \setminus \{0\}$. Therefore homotopy classes of vector fields are in one to one correspondence with homotopy classes of maps $u : Y \to S^2$, since $\mathbb{R}^3 \setminus \{0\}$ has the homotopy type of S^2. The homology class of a vector field in turn is uniquely determined by the induced map $u^* : H^2(S^2; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$. Indeed let $[\omega] \in H^2(Y; \mathbb{Z})$ be a generator, then we have a bijection $\delta^\varsigma : \mathrm{Spin}^c(Y) \to H^2(Y; \mathbb{Z})$, $[v] \to [v^*\omega]$ where we think of v as a map $v : Y \to S^2$.

Proposition 4.1.14. *(Ozsváth and Szabó, 2004d) If $\mathfrak{s}_1, \mathfrak{s}_2 \in$* $\mathrm{Spin}^c(Y)$*, then the difference $\delta^\varsigma(\mathfrak{s}_1) - \delta^\varsigma(\mathfrak{s}_2) \in H^2(Y; \mathbb{Z})$ is independent of the choice of the trivialization ς.*

From now on we will write $\delta^{\varsigma}(\mathfrak{s}_1) - \delta^{\varsigma}(\mathfrak{s}_2)$ as $\mathfrak{s}_1 - s_2$. From this proposition we see that $H^2(Y;\mathbb{Z})$ acts freely and transitively on $\mathrm{Spin}^c(Y)$. Thus $\mathrm{Spin}^c(Y)$ is an affine space over $H^2(Y;\mathbb{Z})$.

Definition 4.1.15. *The first Chern class of* $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ *is the element of* $H^2(Y;\mathbb{Z})$ *defined by* $c_1(\mathfrak{s}) := \mathfrak{s} - \bar{\mathfrak{s}} \in H^2(Y;\mathbb{Z})$.

If $\mathfrak{s}$ is represented by a vector field v, then $c_1(\mathfrak{s})$ is the first Chern class of the orthogonal complement of v, thought of as an oriented real two-plane (hence complex line) bundle over Y. We can see from the definition that $c_1(\mathfrak{s}) = -c_1(\bar{\mathfrak{s}})$.

The choice of base point $z \in \Sigma$ allows us to define a natural map $s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$ as follow. We are going to use Tuarev definition of Spin^c structure for 3-manifold, Proposition 4.1.12. Let f be a Morse function on Y compatible with $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$. Each intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ determines a g-tuple of trajectories for ∇f, connecting the index one critical points to index two critical points. The marked point z also determine a trajectory connecting the index zero critical point to the index three critical point. When we delete tubular neighborhoods of these $g+1$ trajectories, we obtain the complement of disjoint union of balls in Y. The gradient vector field ∇f does not vanish on this complement (we have removed the critical points). Since each trajectory connects critical points of different parities, the gradient vector field has index 0 on all the boundary spheres. It can then be extended as a nowhere vanishing vector field over Y. The homology class of the nowhere vanishing vector field obtained in this manner gives a Spin^c structure. We denote this element by s_z.

Lemma 4.1.16. *(Ozsváth and Szabó, 2004d) Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. *Then we have*

$$s_z(\mathbf{y}) - s_z(\mathbf{x}) = PD[\epsilon(\mathbf{x}, \mathbf{y})] \tag{4.1}$$

In particular $s_z(\mathbf{y}) = s_z(\mathbf{x})$ *if and only if* $\pi_2(\mathbf{x}, \mathbf{y})$ *is non-empty.*

4.1.2 Definition of $\widehat{HF}$, HF^∞, HF^-, HF^+ and HF_{red}

We first give the definition for the case of rational homology spheres.

Definition of $\widehat{HF}$.

Definition 4.1.17. *Let Y be a closed oriented three-manifold with $b_1(Y) = 0$. Choose a pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$ for Y. Choose a complex structure on Σ and a suitable perturbation of the induced complex structure on* $\mathrm{Sym}^g(\Sigma)$, *we define the chain complex $\widehat{CF}(Y; \alpha, \beta)$ to be the free Abelian group freely generated by elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and equipped with following differential: for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$*

$$\widehat{\partial}\mathbf{x} = \sum_{\{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta,\ \phi\in\pi_2(\mathbf{x},\mathbf{y})\ |\ \mu(\phi)=1,\ n_z(\phi)=0\}} \sharp\widehat{\mathcal{M}}(\phi) \cdot \mathbf{y}.$$

The transversal intersection of two compact half-dimensional submanifolds is a finite number of points, so $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is finite. On the other hand the hypothesis $b_1(Y) = 0$ ensures that for any two generators $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ there exists at most one $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with $n_z = 0$. Thus the sum in the definition is finite.

Theorem 4.1.18. *(Ozsváth and Szabó, 2004d) The differential of $\widehat{CF}(Y; \alpha, \beta)$ satisfies $\widehat{\partial}^{\,2} = 0$.*

We define the hat version of the Heegaard Floer homology of Y to be $\widehat{HF}(Y) = \ker \widehat{\partial}/\mathrm{im}\widehat{\partial}$.

By Lemma 4.1.16 if the Spin^c structures $s_z(\mathbf{x})$ and $s_z(\mathbf{y})$ are distinct then the $\mathbf{y}$ component of $\partial\mathbf{x}$ vanishes. Therefore for each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, $\widehat{CF}(Y; \alpha, \beta, \mathfrak{s}) = \mathbb{Z}\{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \ \mid\ s_z(\mathbf{x}) = \mathfrak{s}\}$ is a sub-complex of $\widehat{CF}(Y; \alpha, \beta)$ with homology denoted by $\widehat{HF}(Y, \mathfrak{s})$. Thus we have the following splitting of $\widehat{CF}$ as a direct sum of sub-complexes

$$\widehat{CF}(Y; \alpha, \beta, \mathfrak{s}) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{CF}(Y; \alpha, \beta, \mathfrak{s}).$$

It follows that the homology also splits accordingly,

$$\widehat{HF}(Y, \mathfrak{s}) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s}).$$

Theorem 4.1.19. *(Ozsváth and Szabó, 2004d) The chain homotopy type of $\widehat{CF}(Y, \mathfrak{s})$ is independent of the choices of Heegaard diagram, complex structure, and Riemannian metric.*

As a consequence of this theorem, the homology $\widehat{HF}(Y, \mathfrak{s})$ is a topological invariant of Y for each Spin^c structure $\mathfrak{s}$.

When $b_1(Y) = 0$, we can equip each homology $\widehat{HF}(Y, \mathfrak{s})$ with a relative $\mathbb{Z}$-grading.

Definition 4.1.20. *Assume $b_1(Y) = 0$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\epsilon(\mathbf{x}, \mathbf{y}) = 0$. We define*

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_z(\phi) \tag{4.2}$$

where ϕ is any class in $\pi_2(\mathbf{x}, \mathbf{y})$.

From the discussion after Proposition 4.1.9, $\mathrm{gr}(\mathbf{x}, \mathbf{y})$ is independent of the choice of ϕ since $b_1(Y) = 0$. The differential $\widehat{\partial}$ of $\widehat{CF}$ only count disks ϕ with $\mu(\phi) = 1$ and $n_z(\phi) = 0$, so $\widehat{\partial}$ lower the relative $\mathbb{Z}$-grading by one. This relative grading allows a definition of the Euler characteristic for each group $\widehat{HF}(Y, \mathfrak{s})$ (up to an overall sign). For the case of integer homology spheres the relative $\mathbb{Z}$-grading turn out to be an absolute $\mathbb{Z}$-grading. In particular, for any Heegaard diagram of S^3 the complex $\widehat{CF}(S^3)$ is absolutely $\mathbb{Z}$-graded. The hat version of the Heegaard floer homology of S^3 is isomorphic to $\mathbb{Z}$ and we choose it to be localized in degree 0, that is $\widehat{HF}_0(S^3) \cong \mathbb{Z}$ and $\widehat{HF}_k(S^3) = 0$ for every $k \neq 0$.

The Euler characteristic of $\widehat{HF}(Y)$ determines the order of the first homology of Y as given by the following lemma.

Lemma 4.1.21. *(Ozsváth and Szabó, 2004d) Let Y be a 3-manifold and let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, then*

$$\chi(\widehat{HF}(Y,\mathfrak{s})) = \begin{cases} \pm 1 & \text{if} \quad H_1(Y;\mathbb{Q}) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Using the convention that $|H_1(Y;\mathbb{Z})| = 0$ whenever the manifold has $H_1(Y;\mathbb{Q}) \neq 0$, we get

$$\chi(\widehat{HF}(Y)) = \pm |H_1(Y;\mathbb{Z})|.$$

As a consequence we always have

$$\mathrm{rank}\widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|.$$

Definition 4.1.22. *A rational homology sphere Y is called an L-space if*

$$\mathit{rank}\widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|.$$

L-spaces are Heegaard Floer analogues of lens-spaces. In particular every Lens space $L(p,q)$ with $p \neq 0$ is an L-space. They also includes double branched cover of non-split alternating links.

Definition of HF^∞, HF^- and HF^+. In the definition of $\widehat{HF}$ by requiring $n_z(\phi) = 0$ for each ϕ appearing in the differential, we ensured that this differential is finite. We can also ensure finiteness by introducing new formal generators, this leads to new variants of HF°. We count ϕ with different values of $n_z(\phi)$ as coefficients of different formal generators in the expression for $\partial\mathbf{x}$.

Definition 4.1.23. *With the hypothesis of definition 4.1.17, we define the chain complex $CF^\infty(Y;\alpha,\beta)$ to be the free Abelian group freely generated by formal element of the form $[\mathbf{x},i]$ where $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ and $i\in\mathbb{Z}$, and equipped with the differential defined for each $[\mathbf{x},i]$ by*

$$\partial^\infty[\mathbf{x},i]=\sum_{\{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta,\ \phi\in\pi_2(\mathbf{x},\mathbf{y})\ \mid\ \mu(\phi)=1\}}\sharp\widehat{\mathcal{M}}(\phi)\cdot[\mathbf{y},i-n_z(\phi)].$$

As for $\widehat{HF}$, $\partial^\infty\circ\partial^\infty=0$ and we have a splitting along Spin^c structure, see (Ozsváth and Szabó, 2004d):

$$HF^\infty(Y,\mathfrak{s})=\bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y)}HF^\infty(Y,\mathfrak{s}).$$

Like for $\widehat{HF}$, we can define a relative $\mathbb{Z}$-grading on HF^∞.

Definition 4.1.24. *Assume $b_1(Y)=0$ and let $[\mathbf{x},i]$ and $[\mathbf{y},j]$ be two generators for $CF^\infty(Y;\alpha,\beta)$ with $\epsilon(\mathbf{x},\mathbf{y})=0$. We define*

$$\mathrm{gr}([\mathbf{x},i];[\mathbf{y},j])=\mathrm{gr}(\mathbf{x},\mathbf{y})+2i-2j \tag{4.3}$$

where ϕ is any class in $\pi_2(x,y)$.

The differential ∂^∞ still decreases the degree by one. There is an obvious automorphism of CF^∞, denoted by U , which sends the generator $[\mathbf{x},i]$ to $[\mathbf{x},i-1]$. This automorphism decreases the relative homological grading by 2. Thus, $CF^\infty(Y;\alpha,\beta)$ is naturally a module over $\mathbb{Z}[U,U^{-1}]$ (where here U is a formal variable acting on $CF^\infty(Y;\alpha,\beta)$ via the automorphism U).

Theorem 4.1.25. *(Ozsváth and Szabó, 2004d) For Y with $b_1(Y)=0$ and for any Spin^c structure $\mathfrak{s}$ on Y, $HF^\infty(Y,\mathfrak{s})\cong\mathbb{Z}[U,U^{-1}]$ as $\mathbb{Z}[U,U^{-1}]$-module.*

Thus the homology of CF^∞ does not distinguish Y. However quotients and subcomplexes of $CF^\infty(Y;\alpha,\beta)$ will give interesting homologies.

Definition 4.1.26. *We assume the hypothesis of definition 4.1.17. Let* $\mathfrak{s} \in \mathrm{Spin}^c(Y)$,

- $CF^-(Y;\alpha,\beta,\mathfrak{s})$ *is defined to be the subcomplex of* $CF^\infty(Y;\alpha,\beta,\mathfrak{s})$ *spanned by the generators* $[\mathbf{x},i]$ *with* $i \leq 0$. *It is naturally a module over* $\mathbb{Z}[U]$.
- $CF^+(Y;\alpha,\beta,\mathfrak{s})$ *is defined to be the quotient of* $CF^\infty(Y;\alpha,\beta,\mathfrak{s})$ *by the subcomplex* $CF^-(Y;\alpha,\beta,\mathfrak{s})$. *It is naturally a module over* $\mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U]$.

The corresponding homologies of this complexes will be denoted $HF^-(Y,\mathfrak{s})$ and $HF^+(Y,\mathfrak{s})$ respectively. Summing over all Spin^c structure will give $HF^-(Y)$ and $HF^+(Y)$:

$$HF^-(Y,\mathfrak{s}) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} HF^-(Y,\mathfrak{s}), \qquad \text{and} \qquad HF^+(Y,\mathfrak{s}) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} HF^+(Y,\mathfrak{s}).$$

We use the notation $\mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U]$ to emphasize the fact that it has a natural action of U. However elements of $\mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U]$ are also polynomials in U^{-1} so we can write $\mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U]$ as $\mathbb{Z}[U^{-1}]$.

Let $i : CF^-(Y;\alpha,\beta,\mathfrak{s}) \to CF^\infty(Y;\alpha,\beta,\mathfrak{s})$ denotes the natural inclusion and let $\pi : CF^\infty(Y;\alpha,\beta,\mathfrak{s}) \to CF^+(Y;\alpha,\beta,\mathfrak{s})$ denotes the projection onto the quotient. From the definitions, we have a short exact sequence of chain complexes:

$$0 \longrightarrow CF^-(Y;\alpha,\beta,\mathfrak{s}) \xrightarrow{\ i\ } CF^\infty(Y;\alpha,\beta,\mathfrak{s}) \xrightarrow{\ \pi\ } CF^+(Y;\alpha,\beta,\mathfrak{s}) \longrightarrow 0$$

This will induces a long exact sequence in homology which does not depend on the choice of Heegaard diagram:

Theorem 4.1.27. *(Ozsváth and Szabó, 2004d) There exists a long exact sequence*

$$\cdots \longrightarrow HF^-(Y,\mathfrak{s}) \xrightarrow{i_*} HF^\infty(Y,\mathfrak{s}) \xrightarrow{\pi_*} HF^+(Y,\mathfrak{s}) \longrightarrow \cdots$$

whose isomorphism type depends only on Y and $\mathfrak{s}$.

There is another exact sequence which connects $\widehat{HF}$ and HF^+. First note that there is an embedding of complex $j : \widehat{CF}(Y,\mathfrak{s}) \to CF^+(Y,\mathfrak{s})$ defined by $j(\mathbf{x}) = [\mathbf{x}, 0]$ for each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Together with the automorphism U this will give a short exact sequence of chain complexes:

$$0 \longrightarrow \widehat{CF}(Y;\alpha,\beta,\mathfrak{s}) \xrightarrow{j} CF^+(Y;\alpha,\beta,\mathfrak{s}) \xrightarrow{U} CF^+(Y;\alpha,\beta,\mathfrak{s}) \longrightarrow 0$$

In turn this will induces a second long exact sequence in homology which does not depend on the choice of Heegaard diagram:

Theorem 4.1.28. *(Ozsváth and Szabó, 2004d) There exists a long exact sequence*

$$\cdots \longrightarrow \widehat{HF}(Y,\mathfrak{s}) \xrightarrow{j_*} HF^+(Y,\mathfrak{s}) \xrightarrow{U} HF^+(Y,\mathfrak{s}) \longrightarrow \cdots$$

whose isomorphism type depends only on Y and $\mathfrak{s}$.

Because there are no absolute $\mathbb{Z}$-gradings on the complexes, these two long exact sequences are actually exact triangles. In other words, the map on the far right of the sequence is the same as the map on the far left, and the sequence keeps repeating in this manner.

From this long exact sequence we have a vanishing criterion for HF^+ in term of $\widehat{HF}$.

Proposition 4.1.29. *(Ozsváth and Szabó, 2004d) Let $(Y, \mathfrak{s})$ be a closed oriented 3-manifold equipped with a* Spin^c *structure. Then $HF^+(Y, \mathfrak{s}) = 0$ if and only if $\widehat{HF}(Y, \mathfrak{s}) = 0$.*

Proof. From the long exact sequence in Theorem 4.1.28, if $HF^+(Y, \mathfrak{s}) = 0$ then clearly $\widehat{HF}(Y, \mathfrak{s})$ vanishes. Conversely assume $\widehat{HF}(Y, \mathfrak{s}) = 0$ then we have an isomorphism $U : HF^+(Y, \mathfrak{s}) \to HF^+(Y, \mathfrak{s})$. Let $[x, i] \in HF^+(Y, \mathfrak{s})$, then $U^{i+1}[x, i] = 0$ by definition of U. But since U is an isomorphism on homology, we must have $[x, i] = 0$. Thus $HF^+(Y, \mathfrak{s}) = 0$. □

Definition of HF_{red}. The homology HF_{red} is a finitely generated $\mathbb{Z}$-module variant of the Heegaard Floer homology. It is extracted from the homology HF^+ which is infinitely generated over $\mathbb{Z}$. For clarity let us denote the induced actions of U on the subcomplex CF^- by U^- and the induced actions of U on the quotient complex CF^+ by U^+.

Lemma 4.1.30. *(Ozsváth and Szabó, 2004d) For k sufficiently large,*

$$\mathrm{im}(U^+)^k = \mathrm{im}(\pi_*), \qquad \textit{and} \qquad \ker(U^{-1})^k = \ker i_*.$$

Definition 4.1.31. *For k sufficiently large, let*

$$HF^+_{\text{red}}(Y, \mathfrak{s}) = HF^+(Y, s)/\mathrm{im}(U^+)^k, \qquad \textit{and} \qquad HF^-_{\text{red}}(Y, \mathfrak{s}) = \ker(U^-)^k \subset HF^-(Y, \mathfrak{s}).$$

Proposition 4.1.32. *(Ozsváth and Szabó, 2004d) The boundary homomorphism of the long exact sequence induces a U-equivariant isomorphism*

$$HF^+_{\text{red}}(Y, \mathfrak{s}) \cong HF^-_{\text{red}}(Y, \mathfrak{s}).$$

Moreover, both are finitely generated $\mathbb{Z}$-modules.

We will denote $HF_{\mathrm{red}}(Y,\mathfrak{s})$ both $HF^+_{\mathrm{red}}(Y,\mathfrak{s})$ and $HF^-_{\mathrm{red}}(Y,\mathfrak{s})$. Again summing other all Spin^c structure will give another invariant $HF_{\mathrm{red}}(Y)$ which is called the *reduced Heegaard Floer homology* of Y:

$$HF_{\mathrm{red}}(Y) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} HF_{\mathrm{red}}(Y,\mathfrak{s}).$$

To close the discussion about HF_{red}, we will state the fact that its vanishing characterise L-spaces.

Proposition 4.1.33. *Let Y be an oriented rational homology sphere. Then Y is an L-space if and only if $HF_{\mathrm{red}}(Y) = 0$.*

Proof. Assume Y is an L-space let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Then $\widehat{HF}(Y,\mathfrak{s}) \cong \mathbb{Z}$ and the exact sequence in Theorem 4.1.28 gives a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j_*} HF^+(Y,\mathfrak{s}) \xrightarrow{U^+} HF^+(Y,\mathfrak{s}) \longrightarrow 0$$

hence U^+ is surjective on homology and for k sufficiently large $\mathrm{im}(U^+)^k = HF^+(Y,\mathfrak{s})$ and it follows that $HF^+_{\mathrm{red}}(Y,\mathfrak{s}) = HF^+(Y,s)/\mathrm{im}(U^+)^k = 0$. Conversely suppose that $HF_{\mathrm{red}}(Y,\mathfrak{s}) = 0$, then from Lemma 4.1.30 we have $\ker i_* = 0$ and $\mathrm{im}\pi_* = HF^+(Y,\mathfrak{s})$. Next, the long exact sequence in Theorem 4.1.27 becomes

$$0 \longrightarrow HF^-(Y,\mathfrak{s}) \xrightarrow{i_*} HF^\infty(Y,\mathfrak{s}) \xrightarrow{\pi_*} HF^+(Y,\mathfrak{s}) \longrightarrow 0$$

On the other hand from Theorem 4.1.25 we know that $HF^\infty(Y,\mathfrak{s}) \cong \mathbb{Z}[U,U^{-1}]$. Consequently we have a splitting of $\mathbb{Z}[U]$ module $HF^-(Y,\mathfrak{s}) \oplus HF^+(Y,\mathfrak{s}) \cong \mathbb{Z}[U,U^{-1}]$. From this we can deduce that $HF^-(Y,\mathfrak{s}) \cong U\mathbb{Z}[U]$ and $HF^+(Y,\mathfrak{s}) \cong \mathbb{Z}[U^{-1}]$. It follows that the exact sequence in Theorem 4.1.28 simplifies as

$$0 \longrightarrow \widehat{HF}(Y,\mathfrak{s}) \xrightarrow{j_*} \mathbb{Z}[U^{-1}] \xrightarrow{U^+} \mathbb{Z}[U^{-1}] \longrightarrow 0$$

Thus $\widehat{HF}(Y,\mathfrak{s}) \cong \mathbb{Z}$, and since $|\mathrm{Spin}^c(Y)| = |H_1(Y;\mathbb{Z})|$ we have that $\mathrm{rank}\widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. $\square$

4.1.3 Three-manifolds Y with $b_1(Y) > 0$

For a 3-manifold Y with $b_1(Y) > 0$, the definition of CF° does not change, however to ensure finiteness of the sum in the definition of the boundary map we need to restrict to special classes of Heegaard diagrams which are called *weakly admissible* and *strongly admissible* Heegaard diagrams. Indeed for $\mathbf{x},\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\epsilon(\mathbf{x},\mathbf{y}) = 0$, there may be infinitely many $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ with $\mu(\phi) = 1$ and $n_z = 0$.

From Proposition 4.1.9 counting elements of $\pi_2(\mathbf{x},\mathbf{y})$ is the same as counting elements of $\mathcal{D}(\mathbf{x},\mathbf{y})$. The discussion after this proposition tells us that the requirement $n_z(\phi) = 0$ determines at most one element of $\mathcal{D}(\mathbf{x},\mathbf{y})$ only if $H_2(Y;\mathbb{Z}) = 0$. In case $b_1(Y) > 0$, there is an $H_2(Y;\mathbb{Z})$-degree of freedom in ϕ coming from the addition of periodic domains. It may be the case that infinitely many of these possible ϕ have $\mu(\phi) = 1$. To avoid this problem, we use the special type of Heegaard diagrams defined as follows.

Definition 4.1.34. *Let (Σ,α,β) be a Heegaard diagram for Y.*

- *(Σ,α,β) is weakly admissible if all (nonzero) periodic domains have at least one positive and at least one negative coefficient.*
- *Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, (Σ,α,β) is strongly admissible for $\mathfrak{s}$ if, for any periodic domain D such that $\langle c_1(\mathfrak{s}),[D]\rangle = 2n$, some coefficient of D is greater than n. Here $[D]$ denotes the element of $H_2(Y;\mathbb{Z})$ corresponding to D under*

the one to one correspondence between $H_2(Y;\mathbb{Z})$ and the set of periodic domains.

Weak admissibility implies that the differential of $\widehat{HF}$ is finite. Indeed, suppose $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ has a holomorphic representative. Then imϕ is a complex submanifold of $\mathrm{Sym}^g(\Sigma)$ which intersect the complex submanifolds V_w for any w. Since the coefficients in the domain $D(\phi)$ were defined to be intersection numbers with V_w for various w, they must all be positive. But if all periodic domains P have both positive and negative coefficients, then only finitely many domains of the form $D(\phi)+P$ have all positive coefficients. Therefore among the elements $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ with $n_z(\phi) = 0$ only a finite number can have holomorphic representatives.

For $\widehat{HF}(Y)$ we only need weak admissibility but for $HF^{\infty}(Y,\mathfrak{s})$ and $HF^{-}(Y,\mathfrak{s})$ strong admissibility for $\mathfrak{s}$ is required.

In this context of $b_1(Y) > 0$, we only have a relative $\mathbb{Z}/2\mathbb{Z}$-grading. Give the α and β curves some fix orientations, inducing orientations on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. From this choice of orientations one can define an absolute $\mathbb{Z}/2\mathbb{Z}$-grading on elements of the intersection $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The resulting relative grading will not depend on the choices made. Let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

- $\mathrm{gr}(\mathbf{x}) = 1$ if, at $\mathbf{x}$, the oriented basis of $T_{\mathbf{x}}\mathrm{Sym}^g(\Sigma)$ resulting from the juxtaposition of that of $T_{\mathbf{x}}\mathbb{T}_\alpha$ and $T_{\mathbf{x}}\mathbb{T}_\beta$ match with the orientation of $T_{\mathbf{x}}\mathrm{Sym}^g(\Sigma)$ induced by the orientation on Σ.
- $\mathrm{gr}(\mathbf{x}) = -1$ otherwise.

Changes of orientations on the α and β curves affect the grading of each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, in the same way. The relative $\mathbb{Z}/2\mathbb{Z}$-grading is therefore independent of these choices. Fixing orientation on the α and β curves, we may find $\mathrm{gr}(\mathbf{x})$

for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ without using the symmetric product $\mathrm{Sym}^g(\Sigma)$. Indeed, $\mathbf{x}$ corresponds to g points $x_1, \cdots, x_g \in \Sigma$ where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation $\sigma \in S_g$. Let $\epsilon_i(\mathbf{x})$ be $+1$ if α_i intersects $\beta_{\sigma(i)}$ positively at x_i , and let it be -1 otherwise. Therefore we get the following formula for the grading

$$\mathrm{gr}(\mathbf{x}) = \mathrm{sign}(\sigma) \sum_{i=1}^{g} \epsilon_i(x)$$

This relative $\mathbb{Z}/2\mathbb{Z}$-grading is useful since it allows us to still define the Euler characteristic of $\widehat{HF}(Y, \mathfrak{s})$ for each Spin^c structure $\mathfrak{s}$ even if the manifold has $b_1 = 1$.

4.1.4 Examples of Heegaard Floer homology

Lens space $L(p, q)$. The lens space $L(p, q)$ admits a genus one Heegaard diagram so we can take $\Sigma = S^1 \times S^1$, thus $\mathrm{Sym}^g\Sigma = S^1 \times S^1$ and $\mathbb{T}_\alpha = \alpha$, $\mathbb{T}_\beta = \beta$ are genuine simple closed curves on Σ. Let $\{\lambda, \mu\}$ be the classical longitude meridian basis of $H_1(S^1 \times S^1; \mathbb{Z})$, then α and β are the curves which represent the homology classes $[\alpha] = y$, and $[\beta] = p\lambda + q\mu$. It follows that $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \alpha \cap \beta$ is a collection of p distinct points. Let $\mathbf{x}, \mathbf{y} \in \alpha \cap \beta$, then the curve $\epsilon(\mathbf{x}, \mathbf{y})$ go around the longitude at least once, see Figure 4.1 for the case of $L(2, 1)$, so $[\epsilon(\mathbf{x}, \mathbf{y})] = a\lambda + b\mu$ for some $a \neq 0$.

In the following figure, the red curve represents α, the green curves represents β and the blue curve represents the class of $\epsilon(\mathbf{x}, \mathbf{y})$.

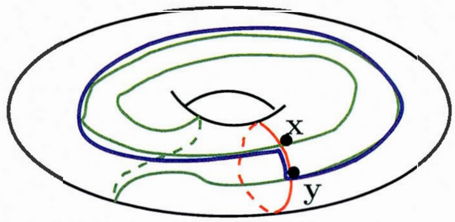


Figure 4.1 Heegaard diagram for $L(2,1)$.

Therefore if we choose a marked point z on $\Sigma - \alpha - \beta$,

$$s_z(\mathbf{y}) - \mathfrak{s}_z(\mathbf{x}) = PD[\epsilon(\mathbf{x},\mathbf{y})] \neq 0,$$

in particular each intersection point corresponds to a different Spin^c structure. Also, by Lemma 4.1.16 $\pi_2(\mathbf{x},\mathbf{y})$ is empty if $\mathbf{x} \neq \mathbf{y}$. It follows that for each Spin^c structure $\mathfrak{s}$ on $L(p,q)$,

$$\widehat{CF}(L(p,q),\alpha,\beta,\mathfrak{s}) = \mathbb{Z} < \mathbf{x} >, \qquad \text{for some } \ \mathbf{x} \in \alpha \cap \beta,$$

and the boundary map becomes

$$\widehat{\partial}\mathbf{x} = \sum_{\emptyset} \widehat{\mathcal{M}}(\phi) \ \cdot \ \mathbf{y} = 0.$$

Therefore $\widehat{HF}(L(p,q),\mathfrak{s}) = \widehat{CF}(L(p,q),\alpha,\beta,\mathfrak{s}) = \mathbb{Z} < \mathbf{x} >$. By the same arguments we can deduce that for each Spin^c structure $\mathfrak{s}$,

$$HF^-(L(p,q),\mathfrak{s}) = U\mathbb{Z}[U] < \mathbf{x} >,$$

$$HF^+(L(p,q),\mathfrak{s}) \cong \mathbb{Z}[U,U^{-1}]/U\mathbb{Z}[U].$$

In particular we can see that $L(p,q)$ is an example of L-space since

$$\widehat{HF}(L(p,q)) = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(L(p,q))} \widehat{HF}(L(p,q),\mathfrak{s}) \cong \mathbb{Z}^p.$$

The manifold $S^1 \times S^2$. In the case of $S^1 \times S^2$ the first Betti number is 1 so in order to compute $\widehat{HF}(S^1 \times S^2)$ we must use a weakly admissible Heegaard diagram for each Spin^c structure. It turns out that there is one genus one Heegaard diagram which is weakly admissible for all Spin^c structures. This is the one shown in Figure 4.2.

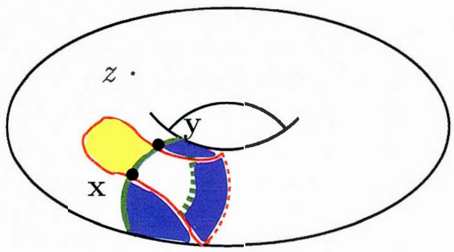


Figure 4.2 Heegaard diagram for $S^1 \times S^2$, α curve (red), β curve (green).

Therefore there is only one pair of intersection points $\{\mathbf{x}, \mathbf{y}\}$ with a pair of non-homotopic holomorphic disks, both with Maslov index one connecting $\mathbf{x}$ to $\mathbf{y}$, we refer to (Ozsváth and Szabó, 2004c) for more details. These disks are shown in yellow and blue in Figure 3.2. From the figure we can see that $\epsilon(\mathbf{x}.\mathbf{y})$ bounds a disk so $s_z(\mathbf{y}) - \mathfrak{s}_z(\mathbf{x}) = PD[\epsilon(\mathbf{x}, \mathbf{y})] = 0$ and $\mathbf{x}, \mathbf{y}$ belong to the same Spin^c structure $\mathfrak{s}_0$. It turns out that $\mathfrak{s}_0$ is the unique torsion Spin^c structure of $S^1 \times S^2$, the one which is identified to 0 under the correspondence Spin$^c(S^1 \times S^2) \cong H^2(S^1 \times S^2) \cong \mathbb{Z}$ (see (Ozsváth and Szabó, 2004c) again for details). We can then deduce that

$$\widehat{CF}(S^1 \times S^2, \alpha, \beta, \mathfrak{s}_0) = \mathbb{Z} < \mathbf{x} > \ \oplus \mathbb{Z} < \mathbf{y} >,$$

and under the choice of coherent orientation, the differential is defined by

$$\widehat{\partial}\mathbf{y} = 0, \quad \text{since there are no disk connecting } y \text{ to } x$$

$$\widehat{\partial}\mathbf{x} = \mathbf{y} - \mathbf{y} = 0,$$

since there is a pair of non-homotopic disk connecting $\mathbf{x}$ to $\mathbf{y}$ and which are counted with opposite sign. It follows that $\widehat{\partial}$ is trivial and

$$\widehat{HF}(S^1 \times S^2, \mathfrak{s}_0) = \widehat{CF}(S^1 \times S^2, \alpha, \beta, \mathfrak{s}_0) = \mathbb{Z} < \mathbf{x} > \ \oplus \mathbb{Z} < \mathbf{y} >,$$

For the non torsion Spin^c-structures it is immediate from the Heegaard diagram, which is always weakly admissible, that

$$\widehat{HF}(S^1 \times S^2, \mathfrak{s}) = 0, \qquad \text{if } \mathfrak{s} \neq \mathfrak{s}_0.$$

By the same arguments we can deduce that

$$HF^-(S^1 \times S^2, \mathfrak{s}_0) = (\mathbb{Z} \oplus \mathbb{Z}) \otimes (U\mathbb{Z}[U]),$$

$$HF^+(S^1 \times S^2, \mathfrak{s}_0) \cong (\mathbb{Z} \oplus \mathbb{Z}) \otimes (\mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]),$$

$$HF^\pm(S^1 \times S^2, \mathfrak{s}) = 0, \qquad \text{if } \mathfrak{s} \neq \mathfrak{s}_0.$$

The Poincaré sphere $\Sigma(2,3,5)$. Instead of computing Heegaard Floer homologies via Heegaard diagrams one can use the surgery presentation of $\Sigma(2,3,5)$ as (-1)-surgery along the left handed trefoil to determine HF°. This is done in (Ozsváth and Szabó, 2003a) and gives the following answer.

Proposition 4.1.35. *(Ozsváth and Szabó, 2003a) Seen as a $\mathbb{Z}$-module,*

$$HF_k^+(\Sigma(2,3,5)) = \begin{cases} \mathbb{Z} & \textit{if } k \textit{ is even and } k \geq 2 \\ 0 & \textit{otherwise} \end{cases}$$

Moreover,

$$U : HF_{k+2}^+(\Sigma(2,3,5)) \longrightarrow HF_k^+(\Sigma(2,3,5))$$

is an isomorphism for $k \geq 2$.

From this proposition we can deduce that $HF^+(\Sigma(2,3,5)) \cong \mathbb{Z}[U, U^-1]/U\mathbb{Z}[U]$ as $\mathbb{Z}[U]$-module and that $HF_{\mathrm{red}}(\Sigma(2,3,5)) = 0$. Therefore $\Sigma(2,3,5)$ is also an L-space and in particular $\widehat{HF}(\Sigma(2,3,5)) \cong \widehat{HF}(S^3) \cong \mathbb{Z}$. However the two manifolds can be distinguished by a numerical invariant which comes from a "refinement" of HF°.

A connected sum formula. Connected sum is a natural operation in 3-manifold theory so it is natural to ask if there is a connected sum formula for HF°. The answer was given by P. Ozsváth and Z. Szabó in (Ozsváth and Szabó, 2004c).

Proposition 4.1.36. *(Ozsváth and Szabó, 2004c) Let Y_1 and Y_2 be a pair of oriented three-manifolds, and fix $\mathfrak{s}_1 \in \mathrm{Spin}^c(Y_1)$ and $\mathfrak{s}_2 \in \mathrm{Spin}^c(Y_2)$. Let $\widehat{CF}(Y_1, \mathfrak{s}_1)$ and $\widehat{CF}(Y_2, \mathfrak{s}_2)$ denote the corresponding chain complexes for calculating $\widehat{HF}$. Then,*

$$\widehat{CF}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \cong \widehat{CF}(Y_1, \mathfrak{s}_1) \otimes_{\mathbb{Z}} \widehat{CF}((Y_2, \mathfrak{s}_2), \qquad \textit{and}$$

$$\widehat{HF}_k(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \cong \left(\bigoplus_{i+j=k} \widehat{HF}_i(Y_1, \mathfrak{s}_1) \otimes_{\mathbb{Z}} \widehat{HF}_j((Y_2, \mathfrak{s}_2) \right)$$
$$\oplus \left(\bigoplus_{i+j=k-1} \mathrm{Tor}(\widehat{HF}_i(Y_1, \mathfrak{s}_1), \widehat{HF}_j((Y_2, \mathfrak{s}_2)) \right).$$

4.1.5 Cobordisms and absolute $\mathbb{Q}$-grading

In this subsection we will discuss briefly the fact that the homologies HF° are functorial with respect to cobordisms. We will then define the absolute $\mathbb{Q}$-grading which is a lift of the relative $\mathbb{Z}$-grading. We follow (Ozsváth and Szabó, 2006a), (Ozsváth and Szabó, 2004c) and (Ozsváth and Szabó, 2003a).

A Heegaard diagram (Σ, α, β) specifies a unique oriented 3-manifold. If we add a third set of attaching circles $\gamma = \{\gamma_1, \cdots, \gamma_g\}$ then we get a triple $(\Sigma, \alpha, \beta, \gamma)$

called *triple Heegaard diagram.* Out of this triple, we can form three Heegaard diagrams: (Σ, α, β), (Σ, β, γ) and (Σ, α, γ), which in turn determine three oriented 3-manifolds $Y_{\alpha,\beta}$, $Y_{\beta,\gamma}$ and $Y_{\alpha,\gamma}$. Moreover we can naturally associate a 4-manifold $X_{\alpha,\beta,\gamma}$ to it as follow. Let Δ denote the two-simplex, with vertices $v_\alpha, v_\beta, v_\gamma$ labelled clockwise, and let e_i denote the edge from v_j to v_k, where $\{i, j, k\} = \{\alpha, \beta, \gamma\}$. We form the identification space

$$X_{\alpha,\beta,\gamma} = \frac{(\Delta \times \Sigma) \coprod (e_\alpha \times U_\alpha) \coprod (e_\beta \times U_\beta) \coprod (e_\gamma \times U_\gamma)}{(e_\alpha \times \Sigma) \sim (e_\alpha \times \partial U_\alpha), (e_\beta \times \Sigma) \sim (e_\beta \times \partial U_\beta), (e_\gamma \times \Sigma) \sim (e_\gamma \times \partial U_\gamma)}.$$

Over the vertices of Δ, this space has corners but they can be smoothed to obtain a smooth oriented cobordism between the three-manifolds $Y_{\alpha,\beta}$, $Y_{\beta,\gamma}$, and $Y_{\alpha,\gamma}$. In fact, if we consider the implicit orientation conventions in the above description,

$$\partial X_{\alpha,\beta,\gamma} = -Y_{\alpha,\beta} \sqcup -Y_{\beta,\gamma} \sqcup Y_{\alpha,\gamma}$$

Given a Heegaard triple we have three embedded g-dimensional tori in $\mathrm{Sym}^g(\Sigma)$:

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g, \qquad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g, \qquad \text{and} \qquad \mathbb{T}_\gamma = \gamma_1 \times \cdots \times \gamma_g$$

Definition 4.1.37. *Let* $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. *A Whitney triangle connecting* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{w}$ *is a continuous map* $u \colon \Delta \longrightarrow \mathrm{Sym}^g(\Sigma)$, *such that* $u(v_\gamma) = \mathbf{x}$, $u(v_\alpha) = \mathbf{y}$, *and* $u(v_\beta) = \mathbf{w}$, *and* $u(e_\alpha) \subset \mathbb{T}_\alpha$, $u(e_\beta) \subset \mathbb{T}_\beta$, $u(e_\gamma) \subset \mathbb{T}_\gamma$.

There is a naturally defined map

$$\epsilon \colon (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times (\mathbb{T}_\beta \cap \mathbb{T}_\gamma) \times (\mathbb{T}_\alpha \cap \mathbb{T}_\gamma) \longrightarrow H_1(X; \mathbb{Z})$$

such that $\epsilon(\mathbf{x}, \mathbf{y}, \mathbf{w}) = 0$ if and only if there is a Whitney triangle connecting $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{w}$. We call Two Whitney triangles *homotopic* if the corresponding maps are homotopic through maps which are all Whitney triangles. For fixed

$\mathbf{x}$, $\mathbf{y}$, and $\mathbf{w}$, we denote $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ the space of homotopy classes of Whitney triangles connecting $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{w}$. To any two elements of $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ there is a naturally associated difference, which is a periodic domain (after we subtract off a sufficient multiple of Σ). When $g > 1$, this gives an affine isomorphism, whenever $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ is non-empty,

$$\pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \cong \mathbb{Z} \oplus H_2(X_{\alpha,\beta,\gamma};\mathbb{Z}),$$

above the first factor is given by the local multiplicity at the marked point z. When $g = 1$, and $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \neq \emptyset$ we also have an isomorphism

$$\pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \cong H_2(X_{\alpha,\beta,\gamma};\mathbb{Z}).$$

We can use a Whitney triangle to build a singular two-plane field in X whose underlying Spin^c structure is independent of the choices made in its construction, see Section 6 of (Ozsváth and Szabó, 2004c). This gives rise to a map

$$\mathrm{Spin}^c_z \colon \pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \longrightarrow \mathrm{Spin}^c(X),$$

where $\mathrm{Spin}^c_z(\psi)$ restricts to $\mathrm{Spin}^c_z(\mathbf{x})$, $\mathrm{Spin}^c_z(\mathbf{y})$, and $\mathrm{Spin}^c_z(\mathbf{w})$ at the three boundary components.

Let $(\Sigma, \alpha, \beta, \gamma, z)$ be a pointed Heegaard triple, and fix a Spin^c structure $\mathfrak{s}$ over $X_{\alpha,\beta,\gamma}$. Under suitable admissibility hypotheses (Ozsváth and Szabó, 2006a), there are chain maps:

$$f^\circ(\cdot;\mathfrak{s})\colon CF^\circ(\alpha,\beta;\mathfrak{s}_{\alpha,\beta}) \otimes CF^{\leq 0}(\beta,\gamma;\mathfrak{s}_{\beta,\gamma}) \longrightarrow CF^\circ(\alpha,\gamma;\mathfrak{s}_{\alpha,\gamma}),$$

where $\mathfrak{s}_{\xi,\eta}$ is the restriction of $\mathfrak{s}$ to $Y_{\xi,\eta}$ and where $CF^{\leq 0} \cong CF^-$ is the chain complex generated by pairs $[\mathbf{x}, j]$ with $j \leq 0$, given by the formula:

$$f^\circ([\mathbf{x},i]\otimes[\mathbf{y},j];\mathfrak{s}) = \sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma} \sum_{\{\psi\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \ \mid \ \mathrm{Spin}^c_z(\psi)=\mathfrak{s},\mu(\psi)=0\}} (\#\mathcal{M}(\psi))\cdot[\mathbf{w}, i+j-n_z(\psi)],$$

above $\mathcal{M}(\psi)$ denotes the moduli space of pseudo-holomorphic triangles in the homotopy class ψ, and $\mu(\psi)$ denotes its expected dimension.

For the admissibility hypothesis we need the notion of a new type of domain. A *triply-periodic domain* is a 2-chain $\mathcal{D}$ on Σ whose boundary are sum of α, β and γ curves or arcs and such that $n_z(\mathcal{D}) = 0$. The admissibility then consists of requiring that for each non-trivial triply-periodic domain which can be written as a sum of doubly-periodic domains (i.e periodic domains as we have defined for Heegaard diagram of 3-manifolds)

$$\mathcal{D} = \mathcal{D}_{\alpha,\beta} + \mathcal{D}_{\beta,\gamma} + \mathcal{D}_{\alpha,\gamma}$$

with the property that

$$\langle c_1(\mathfrak{s}_{\alpha,\beta}), H(\mathcal{D}_{\alpha,\beta})\rangle + \langle c_1(\mathfrak{s}_{\beta,\gamma}), H(\mathcal{D}_{\beta,\gamma})\rangle + \langle c_1(\mathfrak{s}_{\alpha,\gamma}), H(\mathcal{D}_{\alpha,\gamma})\rangle = 2n \geq 0,$$

we have some local multiplicity of $\mathcal{D}$ which is strictly greater than n.

Theorem 4.1.38. *(Ozsváth and Szabó, 2004d) Suppose that $(\Sigma, \alpha, \beta, \gamma, z)$ is an admissible Heegaard triple for the* Spin^c *structure $\mathfrak{s}$. Then, the induced maps on homology*

$$F^\circ(\cdot \otimes \cdot\,, \mathfrak{s})\colon HF^\circ(\alpha, \beta; \mathfrak{s}_{\alpha,\beta}) \otimes HF^{\leq 0}(\beta, \gamma; \mathfrak{s}_{\beta,\gamma}) \longrightarrow HF^\circ(\alpha, \gamma; \mathfrak{s}_{\alpha,\gamma}),$$

is $\mathbb{Z}[U]$-equivariant, is independent of the analytic choices made and is invariant under the special moves which define the equivalence between Heegaard triples.

See (Ozsváth and Szabó, 2004d), and (Ozsváth and Szabó, 2006a) for further details and proof.

The pairing introduced in Theorem 4.1.38 above can be used to associate maps to cobordisms. Every cobordism between two connected 3-manifolds Y and Y' can be decomposed into 1-handles, 2-handles and 3-handles (cf. Proposition 4.2.13 in (Gompf and Stipsicz, 1999)). However we are only interested in cobordisms obtained via surgery on knots or links. Such cobordisms corresponds to 2-handle attachment to $Y \times I$. Thus we will not discuss 1-handles and 3-handles.

Let K be a null-homologous knot in Y. Fix a framing λ for K which we can assume to be a longitude for simplicity (there is no loss of generality if we are doing integer surgery), also fix an admissible Heegaard diagram subordinate to K. We can choose the diagram such that $\beta_1 = \mu$ is a meridian of the first torus component of Σ. The framing of K is given, by pushing K off itself onto the Heegaard surface. The resulting knot on Σ is determined by $\lambda + n \cdot \mu$, for a suitable $n \in \mathbb{Z}$. With this done, we can represent the surgery by the Heegaard triple diagram $(\Sigma, \alpha, \beta, \gamma)$ where γ_i, $i \geq 2$, are isotopic push-offs of the β_i, perturbed, such that γ_i intersects β_i in a pair of cancelling intersection points. The curve γ_1 equals $\lambda + n \cdot \mu$. We now have a Heegaard triple $(\Sigma, \alpha, \beta, \gamma)$ such that $Y_{\alpha,\beta} = Y$, $Y_{\beta,\gamma} = \sharp^{\,g-1}(S^2 \times S^1)$ and $Y_{\alpha,\gamma} = Y_K(\lambda)$.

Proposition 4.1.39. *(Ozsváth and Szabó, 2003a) The cobordism $X_{\alpha\beta\gamma} \cup_\partial (\#^{g-1} D^3 \times S^1)$ is diffeomorphic to the cobordism W_K given by the framed surgery along K.*

We are now going to describe the construction of the map induced by a cobordism for the case of $\widehat{HF}$. The HF^+, HF^-, HF^∞ cases are similar. First of all we recall from subsection 4.1.4 that $\widehat{HF}(S^2 \times S^1; \mathfrak{s}_0) \cong \mathbb{Z} \oplus \mathbb{Z}$, thus from Proposition 4.1.36

$$\widehat{HF}\left(\sharp^{\,g-1}(S^2 \times S^1)\right) \cong \mathbb{Z}^{2g-2}.$$

There is a top dimensional generator $\widehat{\Theta}_{\beta\gamma}$ for this homology. For more details we refer to (Ozsváth and Szabó, 2004d). By evaluating the pairing $F^\circ(\cdot \otimes \cdot\,, \mathfrak{s})$ on this generator we get contracted map

$$\widehat{F}(\cdot \otimes \widehat{\Theta}_{\beta\gamma}; \mathfrak{s})\colon \widehat{HF}(\alpha, \beta; \mathfrak{s}_{\alpha,\beta}) \longrightarrow \widehat{HF}(\alpha, \gamma; \mathfrak{s}_{\alpha,\gamma}),$$

We then define the map $\widehat{F}_{W_K,\mathfrak{s}}\colon \widehat{HF}(Y;\mathfrak{s}_{\alpha,\beta}) \longrightarrow \widehat{HF}(Y_K(\lambda);\mathfrak{s}_{\alpha,\gamma})$ induced by the cobordism W_K to be

$$\widehat{F}_{W_K,\,\mathfrak{s}} := \widehat{F}(\,\cdot\, \otimes \widehat{\Theta}_{\beta\gamma}\,, \mathfrak{s}).$$

Theorem 4.1.40. *(Ozsváth and Szabó, 2006a) The map $\widehat{F}_{W_K}$ does not depend on the choices made in its definition.*

Given a framed link $L = K_1 \sqcup \cdots \sqcup K_m$, we can also define a map

$$\widehat{F}_L\colon\ \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y_L),$$

where Y_L is the obtained by surgery along L in Y, it is similar to what we did for a single attachment. The map $\widehat{F}_L$ associated to multiple attachments is then a composition

$$\widehat{F}_L = \widehat{F}_{K_m} \circ \cdots \circ \widehat{F}_{K_1}$$

There are similar maps for HF^∞, HF^+ and HF^-.

We can now introduce the absolute $\mathbb{Q}$-grading on HF°. Let Y be an oriented three-manifold, equipped with a torsion Spin^c structure (i.e. one for which $c_1(\mathfrak{t})$ is torsion). Let $\mathfrak{A}$ be the homogeneous generating set of CF°. We have seen that $HF^\circ(Y,\mathfrak{t})$ is a relatively $\mathbb{Z}$-graded Abelian group with relative grading function

$$\mathrm{gr}\ :\ \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{Z}.$$

Theorem 4.1.41. *(Ozsváth and Szabó, 2003a) Let $\mathfrak{t}$ be a torsion* Spin^c *structure of Y. Then, the homology groups $HF^\circ(Y,\mathfrak{t})$ can be endowed with an absolute grading*

$$\widetilde{\mathrm{gr}}\ :\ \mathfrak{A} \longrightarrow \mathbb{Q}$$

satisfying the following properties:

- *The homogeneous elements of least grading in $HF^+(S^3;\mathfrak{s}_0)$ have absolute grading zero.*

- *The absolute grading lifts the relative $\mathbb{Z}$-grading, in the sense that if $\xi, \eta \in \mathfrak{A}$, then*
$$\mathrm{gr}(\xi,\eta) = \widetilde{\mathrm{gr}}(\xi) - \widetilde{\mathrm{gr}}(\eta).$$

- *The natural maps ι and π in the long exact sequence (Theorem 4.1.27) preserve the absolute grading, while the coboundary map decreases absolute degree by one, and the U action decreases it by two.*

- *If W is a cobordism from Y_1 to Y_2 endowed with a* Spin^c *structure whose restriction $\mathfrak{t}_i$ to Y_i is torsion for $i = 1, 2$, then*
$$\widetilde{\mathrm{gr}}(F_{W,\mathfrak{s}}(\xi)) - \widetilde{\mathrm{gr}}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}, \tag{4.4}$$
where $\mathfrak{t}_i = \mathfrak{s}|Y_i$ for $i = 1, 2$.

To define $\widetilde{\mathrm{gr}}$, we present Y as a surgery on a link $L \subset S^3$, so that $\mathfrak{t}$ is the restriction of a Spin^c structure $\mathfrak{s}$ over the induced cobordism $W(S^3, L)$ from S^3 to Y. Let $(\Sigma, \alpha, \beta, \gamma, z)$ be a Heegaard triple subordinate to some bouquet for the link L, so that $Y_{\alpha,\beta} \cong S^3$, $Y_{\beta,\gamma} \cong \#^n(S^1 \times S^2)$, and $Y_{\alpha,\gamma} \cong Y$. Fix intersection points $\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{x}_1 \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ so that $\mathbf{x}_0$, resp. $\mathbf{x}_1$, are in the same degree as the highest non-zero generators of $\widehat{HF}(S^3, \mathfrak{t}_0)$ and $\widehat{HF}(\#^n(S^2 \times S^1), \mathfrak{t}_0)$ respectively. (We say that the intersection points $\mathbf{x}_0$ and $\mathbf{x}_1$ lie *in the canonical degree*.) Therefore, by fixing the absolute $\mathbb{Q}$-grading on S^3 equation 4.4 determines the grading for Y using the cobordism $W(S^3, L)$.

4.1.6 The correction term

From the absolute $\mathbb{Q}$-grading we can derive a new numerical invariant for rational homology three-spheres equipped with Spin^c structures.

Definition 4.1.42. *(Ozsváth and Szabó, 2003a) Let $(Y, \mathfrak{s})$ be a rational homology three-sphere equipped with a* Spin^c *structure. The correction term $d(Y, \mathfrak{s})$ is the minimal $\mathbb{Q}$-grading of any non-torsion element in the image of $HF^\infty(Y, \mathfrak{s})$ in $HF^+(Y, \mathfrak{s})$, i.e*

$$d(Y, \mathfrak{s}) = \min\{\widetilde{\mathrm{gr}}\,(\pi_*(x)) \mid x \in HF^\infty(Y; \mathfrak{s})\}$$

where $\pi_* : HF^\infty(Y; \mathfrak{s}) \to HF^+(Y; \mathfrak{s})$ is the map in the exact sequence of Theorem 4.1.27.

There is another interpretation of $d(Y, \mathfrak{s})$ using the reduced homology $HF_{\mathrm{red}}(Y; \mathfrak{s})$. By definition of $HF_{\mathrm{red}}(Y; \mathfrak{s})$, we have the isomorphism:

$$HF^+(Y; \mathfrak{s}) \cong \frac{\mathbb{Z}[U, U^{-1}]}{U\mathbb{Z}[U]} \oplus HF_{\mathrm{red}}(Y; \mathfrak{s}),$$

then $d(Y, \mathfrak{s})$ is the $\mathbb{Q}$-grading of the lowest degree generator of $\mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$.

For the 3-sphere, the homologies $HF^\circ(S^3)$ are all supported in degree zero so $d(S^3) = 0$.

The correction term satisfies the following properties.

Proposition 4.1.43. *(Ozsváth and Szabó, 2003a) Let $(Y, \mathfrak{s})$ and $(Y', \mathfrak{t})$ be rational homology three-spheres equipped with* Spin^c *structures. Then, we have that*

$$d(Y, \mathfrak{s}) = d(Y, \overline{\mathfrak{s}}) \tag{4.5}$$

$$d(Y, \mathfrak{s}) = -d(-Y, \mathfrak{s}) \tag{4.6}$$

$$d(Y \# Y', \mathfrak{s} \# \mathfrak{t}) = d(Y, \mathfrak{s}) + d(Y', \mathfrak{t}) \tag{4.7}$$

Proof. The proof can be seen in (Ozsváth and Szabó, 2003a) section 4. □

For the case of a 3-manifold Y_0 with $H_1(Y_0;\mathbb{Z}) \cong \mathbb{Z}$, there is a unique Spin^c structure $\mathfrak{s}_0$ suth that $c_1(\mathfrak{s}_0) = 0$. We can then define two correction terms as follow.

Definition 4.1.44. *(Ozsváth and Szabó, 2003a) We define the correction terms $d_{+1/2}(Y_0)$, resp. $d_{-1/2}(Y_0)$, to be the minimal $\mathbb{Q}$-grading of any non-torsion element in the image of $HF^\infty(Y_0,\mathfrak{s}_0)$ in $HF^+(Y_0,\mathfrak{s}_0)$ with grading* $+1/2$ resp. $-1/2$ modulo 2.

Proposition 4.1.45. *(Ozsváth and Szabó, 2003a) Let $H_1(Y_0;\mathbb{Z}) \cong \mathbb{Z}$. Then,*

$$d_{1/2}(Y_0) - 1 \leq d_{-1/2}(Y_0). \tag{4.8}$$

$$d_{\pm 1/2}(Y_0,\mathfrak{s}) = d_{\pm 1/2}(Y_0,\bar{\mathfrak{s}}), \tag{4.9}$$

$$d_{\pm 1/2}(Y_0,\mathfrak{s}) = -d_{\mp 1/2}(-Y_0,\mathfrak{s}). \tag{4.10}$$

Proof. See (Ozsváth and Szabó, 2003a), section 4.2. □

For integral homology spheres there is only one Spin^c structure so there is a unique correction term. The following proposition relates correction terms for integral homology spheres and correction terms for zero-surgeries on knots inside them.

Proposition 4.1.46. *(Ozsváth and Szabó, 2003a) Let $K \subset Y$ be a knot in an integral homology three-sphere and let Y_1 be the manifold obtained by $+1$ surgery on K. Then,*

$$d(Y) - \frac{1}{2} \leq d_{-1/2}(Y_0), \qquad \textit{and} \qquad d_{+1/2}(Y_0) - \frac{1}{2} \leq d(Y_1).$$

Proof. See (Ozsváth and Szabó, 2003a) section 4.2. □

In light of Proposition 4.1.46, for three-manifold with the homology of $S^1 \times S^2$, we can use the correction terms to have obstructions for a that manifold to be the result of a zero-surgery on a knot in S^3. More precisely, since $d(S^3) = 0$, we see that if Y_0 is obtained as such surgery, then

$$-\frac{1}{2} \leq d_{-1/2}(Y_0).$$

Moreover, by reflecting the knot and using Proposition 4.1.43, we also obtain the bound

$$d_{1/2}(Y_0) \leq \frac{1}{2}.$$

Proposition 4.1.47. *(Ozsváth and Szabó, 2003a) Let $K \subset S^3$ be an oriented knot in the three-sphere. Then,*

$$\begin{aligned} d_{1/2}(S^3_K(0)) - \frac{1}{2} &= d(S^3_K(1)) \\ d(S^3_K(-1)) - \frac{1}{2} &= d_{-1/2}(S^3_K(0)) \end{aligned}$$

Proof. This is a direct consequence of the surgery long exact sequence, in view of the structure of $HF^+(S^3)$. □

Ozsváth and Z. Szabó also proved the following result concerning the correction term for $1/n$-surgery.

Proposition 4.1.48. *(Ozsváth and Szabó, 2003a) Let $K \subset Y$ be a knot in a $\mathbb{Z}$-homology 3-sphere. Then, we have the following inequalities (where here n is any positive integer):*

$$d_{1/2}(Y_K(0)) - \frac{1}{2} \leq d(Y_K(1/(n+1))) \leq d(Y_K(1/n)) \leq d(Y)$$

$$d(Y) \leq d(Y_K(-1/n)) \leq d(Y_K(-1/(n+1))) \leq d_{-1/2}(Y_K(0)) + \frac{1}{2}.$$

Proof. See (Ozsváth and Szabó, 2003a) Corollary 9.14. □

Let us now give some examples of correction terms for some classical 3-manifolds.

- From subsection 4.1.4, $HF^+(\Sigma(2,3,5)) \cong \mathbb{Z}[U, U^-1]/U\mathbb{Z}[U]$ as $\mathbb{Z}[U]$-module and
$$HF^+_k(\Sigma(2,3,5)) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even and } k \geq 2 \\ 0 & \text{otherwise} \end{cases}$$
Therefore since the absolute $\mathbb{Q}$-grading is a lift of the $\mathbb{Z}$-grading, which is also absolute for integer homology sphere, the $\mathbb{Q}$-grading of the lowest degree generator of $\mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$ in $HF^+(\Sigma(2,3,5))$ is 2. Thus
$$d(\Sigma(2,3,5)) = 2.$$
In particular since $d(S^3) = 0$, $\Sigma(2,3,5)$ and S^3 are distinct manifolds.

- For the lens space $L(p,q)$, P. Ozsváth and Z. Szabó gave an inductive formula for the correction terms. Let p and q be a pair of relatively prime, positive integers. First we fix and identification of $\text{Spin}^c(L(p,q))$ with $\mathbb{Z}/p\mathbb{Z} \cong H_1(L(p,q))$. We consider the genus one Heegaard diagram $(\Sigma, \alpha, \beta, z)$ of $-L(p,q)$ (opposite orientation) such that $\Sigma = S^1 \times S^1$ with a prescribed orientation, $\alpha = S^1 \times \{0\}$, β is a simple closed curve homologous to $-q\alpha + p\lambda$ where $\lambda = \{0\} \times S^1$. Finally the base point z is placed so that all the coefficients of the triply-periodic domains connecting α, λ, and β are negative, and we order the intersection points of α with γ circularly (about α), so that the $(p-1)^{st}$ one modulo p is the one adjacent to the base point. We can now state the following proposition.

Proposition 4.1.49. *(Ozsváth and Szabó, 2003a) Fix positive, relatively prime integers $p > q$, and also choose an integer with $0 \leq i < p + q$. Then,* *with respect to the above ordering of the* Spin^c *structures over* $-L(p,q)$*, we have the following inductive formula:*

$$d(-L(p,q),i) = \left(\frac{pq-(2i+1-p-q)^2}{4pq}\right) - d(-L(q,r),j), \tag{4.11}$$

where r *and* j *are the reductions modulo* q *of* p *and* i *respectively.*

We end this section with some results relating the torsion of the Alexander polynomial of a knot in an integer homology sphere to +1-surgery. In what follows, Y will be an oriented integer homology sphere. We define

$$N(Y) := \mathrm{rank} HF_{\mathrm{red}}(Y)$$

Definition 4.1.50. *Let* K *be a knot in a rational homology sphere* Y *and let its normalized Alexander polynomial be*

$$\Delta_K(T) = a_0 + \sum_{i=1}^{d} a_i\ (T^i + T^{-i}).$$

We define the i*-th torsion invariant of the Alexander polynomial to be*

$$t_i = \sum_{j=1}^{d} j a_{|i|+j}.$$

Theorem 4.1.51. *(Ozsváth and Szabó, 2003a) Let* Y *be an integral homology three-sphere and* $K \subset Y$ *be a knot. Then there is a bound:*

$$|t_0(K)| + 2\sum_{i=1}^{d} |t_i(K)| \leq \mathrm{rank} HF_{\mathrm{red}}(Y) + \frac{d(Y)}{2} + \mathrm{rank} HF_{\mathrm{red}}(Y_K(1)) - \frac{d(Y_K(1))}{2},$$

Proof. See (Ozsváth and Szabó, 2003a) Theorem 6.1. □

4.2 Knot Floer homology

Knot Heegaard Floer homology or Knot Floer homology is the version of Heegaard Floer homology for homologically trivial knots in an oriented closed 3-manifold. It was independently developed by Ozsváth and Szabó in (Ozsváth and Szabó, 2004b) and by Rasmussen in (Rasmussen, 2003). We will use both references in this section.

4.2.1 Knots and Heegaard diagrams

We begin with the description of a knot in a closed oriented 3-manifold in the sense of P. Ozsváth and Z. Szabó.

Definition 4.2.1. *(Ozsváth and Szabó, 2004b) In this section, a* knot *will consist of a pair (Y, K), where Y is an oriented three-manifold, and $K \subset Y$ is an embedded, oriented, null-homologous circle.*

The requirement that the knot should be homologically trivial is a technical assumption which is needed for a refined invariant.

A knot (Y, K) has a Heegaard diagram $(\Sigma, \alpha, \beta^0, \mu)$, where here α is an unordered g-tuple of pairwise disjoint attaching circles $\alpha = \{\alpha_1, ..., \alpha_g\}$, β^0 is a $(g-1)$-tuple of pairwise disjoint attaching circles $\{\beta_2, ..., \beta_g\}$, μ is an embedded, oriented circle in Σ which is disjoint from the β^0, and, g is the genus of Σ. This data is chosen so that $(\Sigma, \alpha, \beta^0)$ specifies the knot-complement $Y \setminus \mathcal{N}(K)$, i.e. if we attach disks along the α and β^0, and then add a three-ball, we obtain the knot-complement. Moreover, μ represents the "meridian" for the knot in Y; thus, $(\Sigma, \alpha, \{\mu\} \cup \beta^0)$ is a Heegaard diagram for Y.

Definition 4.2.2. *(Ozsváth and Szabó, 2004b) A* marked Heegaard diagram for a knot *(Y, K) is a quintuple $(\Sigma, \alpha, \beta^0, \mu, m)$, where here $m \in \mu \cap (\Sigma - \alpha_1 - ... - \alpha_g)$.*

Note that here the marked point is "on the meridian" μ as opposed to the marked point for a pointed Heegaard diagram for 3-manifold. A marked Heegaard diagram for a knot K can also be used to construct a doubly-pointed Heegaard diagram for Y:

Definition 4.2.3. *(Ozsváth and Szabó, 2004b) A* doubly-pointed Heegaard diagram for a three-manifold Y *is a tuple* $(\Sigma, \alpha, \beta, w, z)$*, where* (Σ, α, β) *is a Heegaard diagram for* Y*, and* w *and* z *are a pair of distinct basepoints in* Σ *which do not lie on any of the* α *or* β*.*

We associate a doubly-pointed Heegaard diagram to Y from a Heegaard diagram for (Y, K) as follows. Let $(\Sigma, \alpha, \beta^0, \mu, m)$ be a marked Heegaard diagram for (Y, K). We write $\beta = \beta^0 \cup \mu$. Fix an arc δ which meets μ transversely in a single intersection point, which is the basepoint m, and which is disjoint from all the α and β^0. Then, let w be the initial point of δ, and z be its final point. The orientation of K specifies the ordering of the two points w and z. Take a longitude λ for K, which we think of as a curve in the Heegaard surface. The orientation on K induces an orientation for λ. Now choose z so that if δ is oriented as a path from w to z, then we have an equality of algebraic intersection numbers: $\#(\delta \cap \mu) = \#(\lambda \cap \mu)$ (for either orientation of μ).

4.2.2 Definition and properties of the knot Floer homology

The Heegaard Floer homology for knots comes as the homology of a filtered chain complex, so we will begin with a quick remainder about the subject. Fix a partially ordered set S. An S-filtered group is a free Abelian group C generated freely by a distinguished set of generators $\mathfrak{S}$ which admits a map $\mathcal{F}\colon \mathfrak{S} \longrightarrow S$. We write elements of C as sums

$$\sum_{\sigma \in \mathfrak{S}} a_\sigma \cdot \sigma, \qquad \text{where} \qquad a_\sigma \in \mathbb{Z}.$$

If

$$a = \sum_{\sigma \in \mathfrak{S}} a_\sigma \cdot \sigma \quad \text{and} \quad b = \sum_{\sigma \in \mathfrak{S}'} b_\sigma \cdot \sigma$$

are elements of S-filtered groups $(C, \mathcal{F}, \mathfrak{S})$ and $(C', \mathcal{F}', \mathfrak{S}')$ respectively, then we write $a \leq b$ if

$$\max_{\{\sigma \in \mathfrak{S} \,|\, a_\sigma \neq 0\}} \mathcal{F}(\sigma) \leq \min_{\{\sigma \in \mathfrak{S}' \,|\, b_\sigma \neq 0\}} \mathcal{F}'(\sigma).$$

A morphism of S-filtered groups

$$\phi\colon (C, \mathcal{F}, \mathfrak{S}) \longrightarrow (C', \mathcal{F}', \mathfrak{S}')$$

is a group homomorphism with the property that

$$\phi(a) \leq a, \qquad \text{for all} \quad a \in C.$$

An *S-filtered chain complex* is an S-filtered group equipped with a differential which is an S-filtered morphism; a morphism of S-filtered chain complexes is a chain map which is also an S-filtered morphism. Let $T \subset S$ be a subset of S with the property that if $b \in T$, then all elements $a \in S$ such that $a \leq b$ are contained in T. If $(C_*, \partial, \mathcal{F})$ is an S-filtered complex, then T gives rise to a subcomplex of C_*, which transforms naturally under morphisms.

We are now ready to define the Heegaard Floer knot homology. We equip $\mathbb{Z} \oplus \mathbb{Z}$ with the partial order defined by $(i, j) \leq (i', j')$ when $i \leq i'$ and $j \leq j'$. Let $(\Sigma, \alpha, \beta, w, z)$ be a doubly-pointed Heegaard diagram. We can associate to it a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex

$$CFK^\infty(\Sigma, \alpha, \beta, w, z) = \mathbb{Z}\{[\mathbf{x}; i, j] \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ i, j \in \mathbb{Z}\}$$

equipped with the differential

$$\partial^\infty[\mathbf{x}; i, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \,|\, \mu(\phi) = 1\}} \# \left(\widehat{\mathcal{M}}(\phi)\right) [\mathbf{y}; i - n_w(\phi), j - n_z(\phi)],$$

where $\widehat{\mathcal{M}(\phi)}$ stands for the quotient of the moduli space of pseudo-holomorphic disks which represent the homotopy type of ϕ by the natural action of $\mathbb{R}$ on this space, and $\mu(\phi)$ denotes the formal dimension of $\mathcal{M}(\phi)$. The chain complex has a filtration given by

$$\mathcal{F}\colon (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times (\mathbb{Z} \oplus \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}, \qquad \mathcal{F}[\mathbf{x}; i, j] = (i, j).$$

We then equip the chain complex with the structure of a $\mathbb{Z}[U]$-module, by defining:

$$U \cdot [\mathbf{x}; i, j] = [\mathbf{x}; i-1, j-1].$$

The complex CFK^∞ naturally splits into a sum of subcomplexes. More precisely, two generators $[\mathbf{x}; i, j]$ and $[\mathbf{y}, l, m]$ are in the same summand when there is a homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with

$$n_w(\phi) = i - \ell \quad \text{and} \quad n_z(\phi) = j - m.$$

In our case the doubly-pointed Heegaard diagram comes from the marked Heegaard diagram of an oriented, null-homologous knot K. In this situation, we can interpret the splitting in terms of Spin^c structures over $Y_K(0)$. Precisely, fix a Spin^c structure $\mathfrak{s}$ over Y and let $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y, K) = \mathrm{Spin}^c(Y_0(K))$ be a Spin^c structure which extends it. The complex $CFK^\infty(\Sigma, \alpha, \beta, w, z)$ has the following filtered subcomplex

$$\begin{aligned} CFK^\infty(\Sigma, \alpha, \beta, w, z; \underline{\mathfrak{t}}) = \mathbb{Z}\{\, [\mathbf{x}; i, j] \;\mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ i, j \in \mathbb{Z},\ \mathfrak{s}_{w(\mathbf{x})} = \mathfrak{s}, \\ \underline{\mathfrak{s}}_m(\mathbf{x}) + (i-j)PD[\beta_1] = \underline{\mathfrak{t}}\,\} \end{aligned}$$

here $\beta_1 = \mu$ and $[\mu] \in H_1(Y_K(0))$ is the homology class obtained by thinking of μ as a closed curve in $Y_K(0)$ and $\underline{\mathfrak{s}}_m$ is the map

$$\underline{\mathfrak{s}}_m\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c(Y_0(K))$$

defined in similarly as in section 4.1.16.

We have a splitting of CFK^{∞} as direct sum of these subcomplexes:

$$CFK^{\infty}(\Sigma, \alpha, \beta, w, z) = \bigoplus_{\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y,K)} CFK^{\infty}(\Sigma, \alpha, \beta, w, z; \underline{\mathfrak{t}}).$$

As for 3-manifolds, there are variants CFK^{-}, CFK^{+} which are subcomplexes, quotient complexes. For our purposes we will be interested in the complex

$$\widehat{CFK}(\Sigma, \alpha, \beta, w, z, \underline{\mathfrak{t}}) = \{[\mathbf{x}; 0, j] \in CFK^{\infty}(\Sigma, \alpha, \beta, w, z; \underline{\mathfrak{t}})\}.$$

and the sum over all relative Spin^c structure

$$\widehat{CFK}(\Sigma, \alpha, \beta, w, z) = \bigoplus_{\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y,K)} \widehat{CFK}(\Sigma, \alpha, \beta, w, z; \underline{\mathfrak{t}}).$$

The differential is defined by

$$\widehat{\partial}[\mathbf{x}; 0, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \,|\, \mu(\phi)=1,\ n_w(\phi)=n_z(\phi)=0\}} \#\left(\widehat{\mathcal{M}}(\phi)\right)[\mathbf{y}; 0, j],$$

Theorem 4.2.4 ((Ozsváth and Szabó, 2004b) and (Rasmussen, 2003)). *The maps $\widehat{\partial}$ and ∂^{∞} satisfy $\widehat{\partial} \circ \widehat{\partial} = 0$ and $\partial^{\infty} \circ \partial^{\infty} = 0$.*

We define the knot Floer homology to be

$$\widehat{HFK}(Y, K, \underline{\mathfrak{t}}) = H_*\left(\widehat{CFK}(\Sigma, \alpha, \beta, w, z; \underline{\mathfrak{t}}), \widehat{\partial}\right)$$

$$\text{and} \qquad \widehat{HFK}(Y, K) = \bigoplus_{\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y,K)} \widehat{HFK}(Y, K, \underline{\mathfrak{t}}).$$

Theorem 4.2.5. *(Ozsváth and Szabó, 2004b) Let (Y, K) be an oriented knot, and let $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y, K)$. Then the filtered chain homotopy type of the chain complexes $CFK^{\infty}(Y, K, \underline{\mathfrak{t}})$ is a topological invariant of the oriented knot K and the* Spin^c *structure $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}}^c(Y, K)$; in particular it is independent of the choice of admissible, marked Heegaard diagram $(\Sigma, \alpha, \beta^0, \mu, m)$ used in its definition and all other analytic choices.*

An immediate consequences of the theorem is that knot Floer homology is a topological invariant.

Corollary 4.2.6 ((Ozsváth and Szabó, 2004b) and (Rasmussen, 2003)). *The "knot Floer homology groups" $\widehat{HFK}(Y, K, \underline{\mathfrak{t}})$ are topological invariants of the knot $K \subset Y$ and $\underline{\mathfrak{t}} \in \underline{\mathrm{Spin}^c}(Y, K)$.*

Since K is homologically trivial we can find a Seifert surface $F \subset Y$ for K. Let $\widehat{F}$ be the surface in $Y_K(0)$ obtained by capping off F with the core of the attaching 0-framed 2-handle. We have the following one to one correspondence, (Ozsváth and Szabó, 2004b),

$$\mathrm{Spin}^c(Y_K(0)) \cong \mathrm{Spin}^c(Y) \times \mathbb{Z}, \qquad \mathfrak{t} \mapsto (\mathfrak{s}, j)$$

where $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ is uniquely determined by the restriction of $\mathfrak{t}$ to $Y \setminus \mathcal{N}(K)$ and

$$j = \frac{1}{2} \left\langle c_1(\mathfrak{t}), [\widehat{F}] \right\rangle .$$

Therefore we can define

$$\widehat{HFK}(Y, K, j) := \widehat{HFK}(Y, K, \underline{\mathfrak{t}}).$$

Definition 4.2.7. *The grading j in $\widehat{HFK}(Y, K, j)$ is called the Alexander grading. We denote the Alexander filtration level of $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ by $A(\mathbf{x})$.*

The Alexander filtration is uniquely determined by the following condition, (Ozsváth and Szabó, 2004b): for $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$$\mathbb{A}(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi), \qquad \text{for} \quad \phi \in \pi_2(\mathbf{x}, \mathbf{y}).$$

Like for 3-manifold, the knot Floer homology HFK^∞ and $\widehat{HFK}$ also admit relative $\mathbb{Z}$-grading which lifts to an absolute $\mathbb{Q}$-grading for torsion $\mathfrak{s}$. We have the following conjugation symmetry for $\widehat{HFK}$

Proposition 4.2.8. *(Ozsváth and Szabó, 2004b) Let $(Y, \mathfrak{s})$ be a 3-manifold equipped a torsion* Spin^c *structure. Then*

$$\widehat{HFK}_d(Y, K, m) = \widehat{HFK}_{d-2m}(Y, K, -m)$$

where the subscript denote the absolute grading.

Proof. See (Ozsváth and Szabó, 2004b) section 3. $\square$

We are now going to give some properties of knot Floer homology. The first one is the fact that it categorifies the Alexander polynomial of a knot.

Theorem 4.2.9. *(Rasmussen, 2003) Let K be a knot in an oriented rational homology sphere Y. Then*

$$\Delta_K(T) = \sum_{i,j} (-1)^i \operatorname{rank} \widehat{HFK}_i(K, j)\, T^j,$$

where $\Delta_K(T)$ is the Alexander polynomial of K normalized so that $\Delta_K(T) = \Delta_K(T^{-1})$ and $\Delta_K(1) = 1$.

The second property is that it detect the genus of a knot in S^3.

Theorem 4.2.10. *(Ozsváth and Szabó, 2004a) Let K be a knot in S^3, then the genus of K is given by*

$$g(K) = \max\{j \mid \widehat{HFK}(S^3, K, j) \neq 0\}.$$

Theorem 4.2.11. *(Ni, 2007) Let K be a null-homologous knot in a closed oriented 3-manifold Y. Assume $Y \setminus K$ is irreducible, then K is a fibred knot if and only if $\widehat{HFK}(Y, K, g(K)) \cong \mathbb{Z}$.*

CHAPTER V

EXCEPTIONAL COSMETIC SURGERIES ON S^3

The purpose of this chapter is to prove the following theorem.

Theorem A. *Let K be a hyperbolic knot in S^3, and $r, r' \in \mathbb{Q} \cup \{\infty\}$ two distinct exceptional slopes on $\partial\mathcal{N}(K)$. If $S_K(r)$ is homeomorphic to $S_K(r')$ as oriented manifolds, then the surgery must be toroidal and non-Seifert fibred, moreover $\{r, r'\} = \{+1, -1\}$.*

The theorem tells us that the only slopes which can yield exceptional truly cosmetic surgeries on S^3 are ± 1. Moreover, there are no truly cosmetic surgeries on hyperbolic knot in S^3 if the slope is cyclic or finite or reducible or Seifert fibred.

Theorem A has the following immediate corollaries about the nature of some Heegaard Floer invariants.

Corollary 5.2.3. *If a 3-manifold Y is the result of an exceptional truly cosmetic surgery on a hyperbolic knot K in S^3 then:*

$$|t_0(K)| + 2\sum_{i=1}^{n} |t_i(K)| \leq rank HF_{\text{red}}(Y),$$

where the number $t_i(K)$ for $i \in \mathbb{Z}$ is the torsion invariant of the Alexander polynomial $\Delta_K(T)$ of K and n is the degree of $\Delta_K(T)$.

Corollary 5.2.4. *If a hyperbolic knot $K \subset S^3$ admits an exceptional truly cosmetic surgery then the Heegaard Floer correction term of any $1/n$ ($n \in \mathbb{Z}$) surgery on K vanishes: $d(S^3_K(1/n)) = 0$.*

The chapter is structured as follow. In Section 5.1 we review the mapping cone construction in Heegaard Floer homology and then state the rank formula for $\widehat{HF}$. We also prove a corollary of the rank formula and other important results. In Section 5.2 we give the proof of Theorem A and its two corollaries. Finally in Section 5.3 we give examples of families of knots in S^3 which do not admit truly exceptional cosmetic surgery.

5.1 Results from Heegaard Floer theory

Recall from chapter 4 section 4.2 that knot Floer homology associates to a null-homologous knot K a $\mathbb{Z}\oplus\mathbb{Z}$-filtered $\mathbb{Z}[U]$-complex $CFK^\infty(Y,K)$, generated over $\mathbb{Z}$ by $(\mathbb{T}_\alpha \cap \mathbb{T}_\beta)\times(\mathbb{Z}\oplus\mathbb{Z})$ equipped with a function $\mathcal{F} : (\mathbb{T}_\alpha\cap\mathbb{T}_\beta)\times(\mathbb{Z}\oplus\mathbb{Z}) \longrightarrow \mathbb{Z}\oplus\mathbb{Z}$ with the property that $\mathcal{F}(U\cdot[\mathbf{x};i,j]) = (i-1,j-1)$ and $\mathcal{F}([\mathbf{y};i',j']) \leq \mathcal{F}([\mathbf{x};i,j])$ for all $\mathbf{y}$ having nonzero coefficient in $\partial\mathbf{x}$.

For a region S in the plane with the property that $(i,j) \in S$ implies $(i+1,j)$, or $(i,j+1) \in S$, let $C_K\{S\}$ be the natural quotient complex of $CFK^\infty(Y,K)$ generated by $[\mathbf{x};i,j]$ with $(i,j)\in S$. For an integer s, we define

$$\widehat{A}_s(K) := C_K\{\max(i, j-s) = 0\} \quad \text{and} \quad \widehat{B}(K) := C_K\{i=0\}.$$

There are two canonical chain maps $\widehat{v}_s : \widehat{A}_s \longrightarrow \widehat{B}$ and $\widehat{h}_s : \widehat{A} \longrightarrow \widehat{B}$. The map $\widehat{v}_s$ is projection onto $C\{i=0\}$, while $\widehat{h}_s$ is projection onto $C\{j=s\}$, followed by the identification with $C\{j=0\}$, followed by the chain homotopy equivalence from $C\{j=0\}$ to $C\{i=0\}$.

Now fix two coprime integers p, q with $q > 0$, and consider the two chain complexes

$$\widehat{\mathbb{A}} = \bigoplus_{t\in\mathbb{Z}} (t, \widehat{A}_{\lfloor \frac{t}{q} \rfloor}), \quad \widehat{\mathbb{B}} = \bigoplus_{t\in\mathbb{Z}} (t, \widehat{B}),$$

where $\lfloor x \rfloor$ is the largest integer not greater than x. An element of $\widehat{\mathbb{A}}$ could be written as $\{(t, a_t)\}_{t\in\mathbb{Z}}$ with $a_t \in \widehat{A}_{\lfloor \frac{t}{q} \rfloor}$. Define a chain map $\widehat{D}_{p/q} : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ by

$$\widehat{D}_{p/q}\{(t, a_t)\} = \{(t, b_t)\},$$

where

$$b_t = \widehat{v}(a_t) + \widehat{h}(a_{t-p}).$$

Let $\widehat{\mathbb{X}}_{p/q}(K)$ be the mapping cone of $\widehat{D}_{p/q}$.

The new complex $\widehat{\mathbb{X}}_{p/q}$ naturally splits into the direction sum of p subcomplexes

$$\widehat{\mathbb{X}}_{p/q} = \bigoplus_{i=0}^{p-1} \widehat{\mathbb{X}}_{i,p/q},$$

where $\widehat{\mathbb{X}}_{i,p/q}$ is the subcomplex of $\widehat{\mathbb{X}}_{p/q}$ containing all $\widehat{A}_t$ and $\widehat{B}_t$ with $t \equiv i \pmod p$.

The Heegaard Floer homology $\widehat{HF}(Y_K(p/q))$ is determined by the mapping cone $\widehat{\mathbb{X}}_{p/q}$:

Theorem 5.1.1. *(Ozsváth and Szabó, 2011) Let $K \subset Y$ be a nullhomologous knot and p, q a pair of coprime integers. Then, for each $i \in \mathbb{Z}/p\mathbb{Z}$, there is a relatively graded isomorphism of groups*

$$\widehat{HF}(Y_K(p/q), \mathfrak{s}_i) \cong H_*(\widehat{\mathbb{X}}_{i,p/q}),$$

where $\mathfrak{s}_i$ is the Spin^c *structure corresponding to $i \in \mathbb{Z}/p\mathbb{Z}$.*

Ozsváth and Szabó (Ozsváth and Szabó, 2011) used this rational surgery formula for the case of S^3 and, later, Zhongtao Wu (Wu, 2011c) did the same thing for integral homology L-spaces to find a rank formula for $\widehat{HF}(Y_K(p/q)$. We are now going to describe this rank formula. Let Y be an integral homology L-space and let K be a knot in Y.

Definition 5.1.2. *Define $\nu_Y(K)$ by*

$\nu_Y(K) = \min\{s \in \mathbb{Z} | \widehat{v}_s : \widehat{A}_s \longrightarrow \widehat{CF}(Y)$ *induces a non-trivial map in homology*$\}$.

Let $m(K) \subset -Y$ be the *mirror image* of the knot $K \subset Y$; that is $m(K)$ is topologically the same as K with its original orientation but now the ambient manifold is given the opposite orientation. When $Y = S^3$, $m(K)$ is the mirror image in the usual sense if we apply the reflection in S^3 which is orientation reversing. Either $\nu_Y(K)$ or $\nu_{-Y}(m(K))$ is non-negative, so since the rank of $\widehat{HF}$ does not depend on orientation and $Y_K(p/q) \cong -Y_{m(K)}(-p/q)$, we can assume without loss of generality that $\nu_Y(K) \geq 0$.

We can now state the rank formula for the Heegaard Floer homology of a rational surgery.

Proposition 5.1.3. *(Ozsváth and Szabó, 2011) Let K be a knot in an integral homology L-space Y, and fix a pair of relatively prime integers p and q with $p \neq 0$ and $q > 0$; and suppose that $\nu_Y(K) \geq \nu_{-Y}(m(K))$. Then, if $\nu_Y(K) > 0$ or $p > 0$,*

$$\mathrm{rank}(\widehat{HF}(Y_{p/q}(K)) = p + 2\max(0, (2\nu_Y(K) - 1)q - p) + q(\sum_s(\mathrm{rank}H_*(\hat{A}_s) - 1));$$

if $\nu_Y(K) = 0$, we have that

$$\mathrm{rank}(\widehat{HF}(Y_{p/q}(K)) = |p| + q(\sum_s(\mathrm{rank}H_*(\hat{A}_s) - 1)).$$

Notice that the term $\hat{A}_s$ is the complex in the mapping cone formula for rational surgery and satisfies $\mathrm{rank}H_*(\hat{A}_s) \geq 1$ for each s so that

$$\sum_s(\mathrm{rank}H_*(\hat{A}_s) - 1) \geq 0.$$

This proposition has the following immediate corollary about L-space surgeries on Y.

Corollary 5.1.4. *Let K be a knot in an L-space integral homology sphere Y and fix a pair of relatively prime integers p and q with $p \neq 0$ and $q > 0$. If $Y_K(p/q)$ is an L-space then $Y_K(p)$ is also an L-space.*

Proof. Without loss of generality we can assume $\nu_Y(K) \geq 0$.

If $\nu_Y(K) = 0$, and $Y_K(p/q)$ is an L-space then

$$\mathrm{rank}(\widehat{HF}(Y_K(p/q)) = |p| + q(\sum_s (\mathrm{rank} H_*(\hat{A}_s) - 1)) = |p|.$$

Since $q > 0$ and $\sum_s (\mathrm{rank} H_*(\hat{A}_s) - 1) \geq 0$, we obtain that

$$\sum_s (\mathrm{rank} H_*(\hat{A}_s) - 1) = 0.$$

Now for $Y_K(p)$ we have the rank formula

$$\mathrm{rank}(\widehat{HF}(Y_K(p)) = |p| + \sum_s (\mathrm{rank} H_*(\hat{A}_s) - 1) = |p|.$$

Assume $\nu_Y(K) > 0$, then $\nu_Y(K) \geq 1$ since it is an integer. Similarly $q \geq 1$, so

$$1 \leq (2\nu_Y(K) - 1) \leq (2\nu_Y(K) - 1)q.$$

Now assume that $Y_K(p/q)$ is an L-space and let us consider the case $p > 0$, we obtain

$$\mathrm{rank}(\widehat{HF}(Y_K(p/q)) = p + 2\max\{0, (2\nu_Y(K) - 1)q - p\} + q(\sum_s (\mathrm{rank} H_*(\hat{A}_s) - 1)) = p > 0.$$

Since both p, $\max\{0, (2\nu_Y(K) - 1)q - p\}$ and $(\sum_s (\mathrm{rank} H_*(\hat{A}_s) - 1)$ are non-negative we have

$$\max\{0, (2\nu_Y(K) - 1)q - p\} = \sum_s (\mathrm{rank} H_*(\hat{A}_s) - 1) = 0.$$

Hence $(2\nu_Y(K)-1) \leq (2\nu_Y(K)-1)q \leq p$, and $(2\nu_Y(K)-1)-p \leq 0$. It follows that $\max\{0,(2\nu_Y(K)-1)-p\}=0$ and the rank formula for $Y_K(p)$ gives

$$\mathrm{rank}(\widehat{HF}(Y_K(p)) = p + 2\max\{0,(2\nu_Y(K)-1)-p\} + (\sum_s(\mathrm{rank}H_*(\hat{A}_s)-1)) = p.$$

If $p<0$ and $Y_K(p/q)$ is an L-space, we can write the rank formula as follow:

$$\mathrm{rank}(\widehat{HF}(Y_K(p/q)) = -|p| + 2\max\{0,(2\nu_Y(K)-1)q+|p|\} + q(\sum_s(\mathrm{rank}H_*(\hat{A}_s)-1)) = |p|.$$

Using the fact that $0 \leq 1 \leq (2\nu_Y(K)-1)q$ we have

$$\begin{aligned}\mathrm{rank}(\widehat{HF}(Y_K(p/q)) &= -|p| + 2(2\nu_Y(K)-1)q + 2|p| + q(\sum_s(\mathrm{rank}H_*(\hat{A}_s)-1)) \\ &= |p| + 2(2\nu_Y(K)-1)q + q(\sum_s(\mathrm{rank}H_*(\hat{A}_s)-1)) = |p|\end{aligned}.$$

Since both summands in the last line are non-negative we have

$$(2\nu_Y(K)-1) = \sum_s(\mathrm{rank}H_*(\hat{A}_s)-1) = 0.$$

From this we can deduce the rank of $\widehat{HF}(Y_K(p)$

$$\mathrm{rank}(\widehat{HF}(Y_K(p)) = |p| + 2(2\nu_Y(K)-1) + (\sum_s(\mathrm{rank}H_*(\hat{A}_s)-1)) = |p|.$$

Therefore in all cases $\mathrm{rank}(\widehat{HF}(Y_K(p)) = |p|$ and $Y_K(p)$ is an L-space. □

Using the same mapping cone construction as in the rational surgery formula, Ni and Zhongtao Wu proved an inequality between the correction terms of the lens space $L(p,q)$ and of the manifold obtained after surgery.

Theorem 5.1.5. *(Ni and Wu, 2013) Suppose $p,q>0$ are coprime integers and $r=p/q$. Then*

$$d(S^3_K(p/q),i) \leq d(L(p,q),i)$$

for all $i \in \mathbb{Z}/p\mathbb{Z}$, where the Spin^c *structures on $S^3_K(r)$ has been identified to $\mathbb{Z}/p\mathbb{Z}$ according to the construction of the mapping cone.*

When K is a knot in an integral homology L-space Y admitting an L-space surgery, the following characterization of $\widehat{HFK}(Y,K)$ will be useful. It was proved in [(Ozsváth and Szabó, 2005) theorem 1.2] for the case of S^3 and stated in [(Wu, 2011c) Proposition 3.7] for the more general case of L-space homology spheres.

Proposition 5.1.6. *Suppose $K \subset Y$ is a knot in an integral homology L-space. If there is a rational number r for which $Y_r(K)$ is an L-space, then there is an increasing sequence of integers $n_{-k} < \ldots < n_k$ with the property that $n_i = -n_{-i}$, and $\widehat{HFK}(K,j) = 0$ unless $j = n_i$ for some i, in which case $\widehat{HFK}(K,j) \cong \mathbb{Z}$.*

An immediate corollary [(Wu, 2011c) Corollary 3.8] is a simplified expression for the Alexander polynomials of such knots.

Corollary 5.1.7. *Let K be a knot that admits an L-space surgery. Then the Alexander polynomial of K has the form*

$$\Delta_K(T) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(T^{n_j} + T^{-n_j}),$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < \ldots < n_k$.

Next we state a result by Rustamov which gives a relation between Casson-Walker invariant and the "renormalized Euler characteristic" of the Heegaard Floer homology $\widehat{HF}$.

Theorem 5.1.8. *(Rustamov, 2004) For any rational homology sphere M we have*

$$|H_1(M)|\ \lambda(M) = \sum_{\mathfrak{s} \in Spin^c(M)} \left(\chi(HF_{\mathrm{red}}(M,\mathfrak{s})) - \frac{1}{2} d(M,\mathfrak{s}) \right)$$

where $\lambda(M)$ is the Casson-Walker invariant of M and d stands for the correction term in Heegaard Floer homology.

We can now prove the following proposition, a part of a result by Zhongtao Wu and Yi Ni. We reproduce a proof here for convenience of the reader. This is one of the main ingredients for the proof of Theorem A. More precisely, it will imply that the cosmetic surgery cannot be Seifert fibred.

Proposition 5.1.9. *(Ni and Wu, 2013) Let $p, q > 0$ be two coprime integers. If there is an orientation preserving homeomorphism between $S^3_K(p/q)$ and $S^3_K(-p/q)$ then*

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(p/q))} \chi(HF_{\mathrm{red}}(S^3_K(p/q),\mathfrak{s})) = 0.$$

Proof. Letting $r = p/q$, from Theorem 5.1.8 we have

$$|H_1(S^3_K(r))|\ \lambda(S^3_K(r)) = \sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(r))} \left(\chi(HF_{\mathrm{red}}(S^3_K(r),\mathfrak{s}) - \frac{1}{2}d(S^3_K(r),\mathfrak{s})\right).$$

On the other hand we have the following surgery formula for Casson-Wlaker invariant from Proposition 3.1.2,

$$\lambda(S^3_K(p/q)) = \lambda(S^3) + \lambda(L(p,q)) + \frac{q}{2p}\Delta''_K(1).$$

However $\lambda(S^3) = 0$ and we must have $\Delta''_K(1) = 0$ by Proposition 3.3.1. Hence

$$\lambda(S^3_K(p/q)) = \lambda(L(p,q)).$$

Using again Theorem 5.1.8, the correction terms for lens space $L(p,q)$ satisfies

$$p\ \lambda(L(p,q)) \ = \sum_{\mathfrak{s}\in \mathrm{Spin}^c(L(p,q))} -\frac{1}{2}d(L(p,q),\mathfrak{s}).$$

Therefore

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(r))} \left(\chi(HF_{\mathrm{red}}(S^3_K(r),\mathfrak{s}) - \frac{1}{2}d(S^3_K(r),\mathfrak{s})\right) \ = \sum_{\mathfrak{s}\in \mathrm{Spin}^c(L(p,q))} -\frac{1}{2}d(L(p,q),\mathfrak{s}),$$

From Theorem 5.1.5 we have that for each Spin^c structure $\mathfrak{s}$

$$d(S^3_K(r),\mathfrak{s}) \leq d(L(p,q),\mathfrak{s}).$$

Thus,

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(r))} \chi(HF_{\mathrm{red}}(S^3_K(r),\mathfrak{s})) \leq 0.$$

On the other hand, we can do similar argument for $S^3_K(r') = S^3_K(-r)$ to get

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(-r))} (\chi(HF_{\mathrm{red}}(S^3_K(-r),\mathfrak{s}) - \frac{1}{2}d(S^3_K(-r),\mathfrak{s}))$$
$$= \sum_{\mathfrak{s}\in \mathrm{Spin}^c(L(p,-q))} -\frac{1}{2}d(L(p,-q),\mathfrak{s}).$$

Since we have an orientation preserving homeomorphism between $S^3_K(r)$ and $S^3_K(-r)$ and that the total rank of HF_{red} does not depend on orientation, we have

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(r))} (\chi(HF_{\mathrm{red}}(S^3_K(r),\mathfrak{s}) - \frac{1}{2}d(S^3_K(r),\mathfrak{s})) = \sum_{\mathfrak{s}\in \mathrm{Spin}^c(L(p,-q))} -\frac{1}{2}d(L(p,-q),\mathfrak{s}).$$

Recall that for a knot K in S^3 we have $S^3_K(r) = -S^3_{m(K)}(-r)$ where $m(K)$ is the mirror image of K. Then if $S^3_K(r) = S^3_K(-r)$, using the properties of d, we get for all $\mathfrak{s}$

$$d(S^3_K(r),\mathfrak{s}) = d(S^3_K(-r),\mathfrak{s}) = -d(S^3_{m(K)}(r),\mathfrak{s}).$$

On the other hand, by exchanging the role of $m(K)$ and K, we also have for all $\mathfrak{s}$

$$d(S^3_{m(K)}(r),\mathfrak{s}) \ \leq d(L(p,q),\mathfrak{s}).$$

From this it follows that

$$-d(S^3_{m(K)}(r),\mathfrak{s}) \ \geq d(L(p,-q),\mathfrak{s}) \quad \text{i.e} \quad d(S^3_K(r),\mathfrak{s}) \ \geq d(L(p,-q),\mathfrak{s}).$$

Thus,

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(Z)} \chi(HF_{\mathrm{red}}(S^3_K(r),\mathfrak{s})) \geq 0.$$

This implies

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(r))} \chi(HF_{\mathrm{red}}(S^3_K(r),\mathfrak{s}) = 0.$$

□

The next proposition, due to Ozsváth and Szabó, will be essential for excluding the possibility of a rational homology 3-sphere Seifert fibred cosmetic surgery.

Proposition 5.1.10. *(Ozsváth and Szabó, 2003b) Let Y be a rational homology 3-sphere Seifert fibred space. Then for one of the orientations of Y, $HF_{\mathrm{red}}(Y)$ is supported in even degree.*

Corollary 5.1.11. *(Ni and Wu, 2013) There are no truly cosmetic surgeries on a non-trivial knot in S^3 which yield a rational homology sphere Seifert fibred space.*

Proof. Let K be a non-trivial knot in S^3. Let us suppose that there is an orientation preserving homeomorphism between $S^3_K(r)$ and $S^3_K(-r)$, by Proposition 5.1.9

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(r))} \chi(HF_{\mathrm{red}}(S^3_K(r),\mathfrak{s})) = 0.$$

On the other hand by Proposition 5.3.1, we can assume $HF_{\mathrm{red}}(S^3_K(r))$ is supported in even degree so

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(S^3_K(r))} \chi(HF_{\mathrm{red}}(S^3_K(r),\mathfrak{s})) = \pm \mathrm{rank}\, HF_{\mathrm{red}}(S^3_K(r)).$$

Therefore we must have $HF_{\text{red}}(S^3_K(r)) = 0$ in which case $S^3_K(r)$ is an L-space. In particular the knot K admits an L-space surgery. By Corollary 5.1.4 of the rank formula, K admits an integral L-space surgery and by Corollary 5.1.7 the knot Floer homology satisfies: there is an increasing sequence of integers $n_{-k} < ... < n_k$ with the property that $n_i = -n_{-i}$, and $\widehat{HFK}(K,j) = 0$ unless $j = n_i$ for some i, in which case $\widehat{HFK}(K,j) \cong \mathbb{Z}$. This implies that the Alexander polynomial of K has the form

$$\Delta_K(T) = (-1)^k + \sum_{j=1}^{k}(-1)^{k-j}(T^{n_j} + T^{-n_j}),$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < ... < n_k$.

If $\Delta_K(T) = 1$, then $\widehat{HFK}(K,0) = \mathbb{Z}$, and $\widehat{HFK}(K,j) = 0$ for any other j. Hence $g(K) = 0$ and K is the unknot, which we have excluded.

Thus $\Delta_K(T) \neq 1$ and by a straightforward computation

$$\Delta''_K(1) = 2\sum_{j=1}^{k}(-1)^{k-j}n_j^2.$$

Then the fact that $0 < n_1 < n_2 < ... < n_k$ implies $\Delta''_K(1) \neq 0$. Using Proposition 3.3.1, K does not admit a truly cosmetic surgery.

□

5.2 Proof of Theorem A

Lemma 5.2.1. *Let K be a hyperbolic knot in S^3, and $r, r' \in \mathbb{Q}\cup\{\infty\}$ two distinct exceptional slopes on $\partial\mathcal{N}(K)$. If $S_K(r)$ is homeomorphic to $S_K(r')$ as oriented manifolds, then r and r' are in the following table*

r	2	1	1/2	1/3	1/4
r'	-2	-1	$-1/2$	$-1/3$	$-1/4$

Table 5.1 Slopes of exceptional truly cosmetic surgeries on S^3.

Proof. Write $r = p/q$ and $r' = p/q'$. By Ni and Zhongtao Wu (Ni and Wu, 2013) $r = -r'$ so $q = -q'$, then $\Delta(r, r') = p|q - q'| = 2p|q| \leq 8$ i.e $p|q| \leq 4$. Therefore $p \in \{1, 2, 3, 4\}$. If $p = 1$ then $|q| \leq 4$ and we have the case $r \in \{1, 1/2, 1/3, 1/4\}$. If $p = 2$ then $|q| \leq 2$, since q is odd we have $|q| = 1$ so $r = 2$. Now we need to exclude the case $p \in \{3, 4\}$.

An orientation preserving homeomorphism $f : M(p/q) \to M(p/q')$ induces an isomorphism $f_* : H_1(M(p/q)) \to H_1(M(p/q'))$. Since $H_1(M(p/q)) = \mathbb{Z}/p\mathbb{Z}$ is generated by the class $[\mu]_q$ of the meridian,

$$f_* [\mu]_q = [\mu]_{q'} u, \qquad \text{for some unit } u \in (\mathbb{Z}/p\mathbb{Z})^*.$$

Let's recall that the linking pairing of $M(p/q)$ is a non-degenerate bilinear form

$$lk_q : \mathrm{Tor}(H_1(M(p/q))) \otimes \mathrm{Tor}(H_1(M(p/q))) \to \mathbb{Q}/\mathbb{Z},$$

which is defined via some intersection count. One can check that

$$lk_q([\mu]_q, [\mu]_q) = -q/p.$$

To see this, let D be a meridian disk for the surgery torus such that $p\mu + q\lambda = \partial D$, in $M(p/q)$. Since Y is a $\mathbb{Z}$-homology sphere $\lambda_M = \partial\Sigma$ for some surface Σ. Then $p\mu = \partial D - q\partial\Sigma = \partial (D - q\Sigma)$ and by definition

$$lk_q([\mu]_q, [\mu]_q) = \frac{(D - q\Sigma) \cdot \mu}{p} \quad [\mathrm{mod}\ \mathbb{Z}]$$

where the dot $\bullet$ denotes the intersection number. Now μ can be pushed off of D so $D \bullet \mu = 0$, and $\partial\Sigma = \lambda_M \Sigma$ so $\Sigma \bullet \mu = 1$. Therefore

$$lk_q([\mu]_q, [\mu]_q) = -\frac{q}{p} \quad [\text{mod } \mathbb{Z}]\,.$$

The map f induces an isomorphism between the linking pairing of $M(p/q)$ and $M(p/q')$ since it preserves oriented intersection numbers. Therefore

$$\begin{aligned} lk_q([\mu]_q, [\mu]_q) &= lk_q(f_*[\mu]_q, f_*[\mu]_q) \quad [\text{mod } \mathbb{Z}] \\ &= lk_{q'}([\mu]_{q'} u, [\mu]_{q'} u) \quad [\text{mod } \mathbb{Z}] \\ &= lk_q([\mu]_{q'}, [\mu]_{q'}) u^2 \quad [\text{mod } \mathbb{Z}]\,. \end{aligned}$$

Thus

$$-\frac{q}{p} \equiv -\frac{q'}{p} u^2 \quad [\text{mod } \mathbb{Z}]\,, \quad \text{i.e} \quad q \equiv q' u^2 \quad [\text{mod } p]\,.$$

We apply this congruence to the case $p \in \{4, 3\}$. For $p = 4$ (resp. $p = 3$), $u \in \{1, 3\}$ (resp. $u \in \{1, 2\}$). Therefore $u^2 = 1$ and $q \equiv q' \quad [\text{mod } 4]$, but $q' \in \{q + 1, q + 2\}$ by Lemma 1.3.12 which is not possible. Thus $p \notin \{3, 4\}$.

□

In these two lemmas it is essential that $\text{int}(M)$ has a complete finite volume hyperbolic structure since the bound is on the diameter of $E(M)$. Thus the examples given in (Mathieu, 1992) do not fall into this category.

For a hyperbolic knot K in S^3, if we take into account the type of manifold obtained after surgery we have the following lemma.

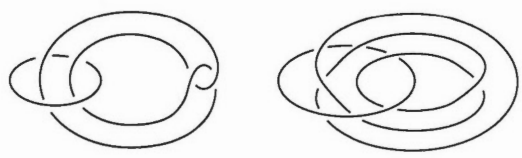
Lemma 5.2.2. *There are no truly cosmetic surgeries on hyperbolic knot in S^3 which yield a reducible manifold.*

Proof. If $K \subset S^3$ is a hyperbolic knot and r, r' are two reducible slopes on $\mathcal{N}(K)$ then $\Delta(r, r') \leq 1$, see table 1.1. However by Lemma 5.2.1, the distance between two cosmetic slopes must be at least two. This is not possible. □

Proof of Theorem A. Let K be a hyperbolic knot in S^3, and $r, r' \in \mathbb{Q} \cup \{\infty\}$ two distinct exceptional slopes on $\partial\mathcal{N}(K)$. Assume $S_K(r)$ is homeomorphic to $S_K(r')$ as oriented manifolds. By Lemma 5.2.2 and Corollary 5.1.11 $S_K(r)$ is irreducible and non-Seifert fibred. Therefore since the manifold is not hyperbolic it contains an essential torus. On the other hand, by Theorem 2.2.6, if $S_K(r)$ is toroidal and $r = p/q$ then $|q| \leq 2$. Therefore using the distance table 1.1 and table of Lemma 5.2.1 we can deduce that: $r = 2$ and $r' = -2$, or $r = 1$ and $r' = -1$, or $r = 1/2$ and $r' = -1/2$.

By Theorem 2.2.7, if there is a non-integral slope on $\partial(S^3 \setminus \mathcal{N}(K))$ which gives a toroidal manifold then K is one of the Eudave-Muñoz knots $k(l, m, n, p)$ and the surgery is the corresponding half-integral surgery. Therefore there is at most one slope which can give an essential torus. Thus there is no non-integral cosmetic slope which gives a toroidal manifold. This excludes the case $r = 1/2$ and $r' = -1/2$.

Now we have either $r = 1$ or $r = 2$. If $r = 2$ then $\Delta(2, -2) = 4$ and Theorem 2.2.8 gives the complete list of all hyperbolic knots in S^3 with two toroidal slopes r_1 and r_2 at distance 4. Precisely, there is an integer n and an homeomorphism of S^3 which send the triple (K, r_1, r_2) to $(L_1(n); 0, 4)$, $\;n \neq 0, 1$ or $(L_2(n); 2 - 9n, -2 - 9n)$, $\;n \neq 0, \pm 1$. Where $L_i(n), i = 1, 2$ denotes the knot obtained from the right component of the link L_i, $\;i = 1, 2$ (see figure below) after $1/n$ surgery on the left component



The links L_1 and L_2 from left to right

The manifold obtained after surgery is then $S^3(L_2(n); 2-9n)$ or $S^3(L_2(n); -2-9n)$ or $S^3(L_1(n); 0)$ or $S^3(L_2(n); 4)$. Therefore we can check that

$$|2-9n| = |H_1\left(S^3(L_2(n); 2-9n)\right)| \neq |H_1\left(S^3(L_2(n); -2-9n)\right)| = |2+9n|$$

$$0 = |H_1\left(S^3(L_1(n); 0)\right)| \neq |H_1\left(S^3(L_2(n); 4)\right)| = 4.$$

Since $n \neq 0$, this completes the proof of Theorem A. □

Some corollaries. Theorem A also induces the following results about the reduced Heegaard Floer Homology of Y correction terms.

Corollary 5.2.3. *If a 3-manifold Y is the result of an exceptional truly cosmetic surgery on a hyperbolic knot K in S^3 then:*

$$|t_0(K)| + 2\sum_{i=1}^{n} |t_i(K)| \leq rankHF_{\text{red}}(Y),$$

where the number $t_i(K)$ for $i \in \mathbb{Z}$ is the torsion invariant of the Alexander polynomial $\Delta_K(T)$ of K and n is the degree of $\Delta_K(T)$.

This lower bound is strictly positive if $\Delta_K(T) \neq 1$ since in this situation not all the torsion $t_i(K)$ are zero.

Proof. Let $K \subset S^3$ be a hyperbolic knot such that there is an orientation preserving homeomorphism between $S^3_K(r)$ and $S^3_K(r')$ for two distinct rational numbers

r and r'. Let $Y = S^3_K(r)$, by Theorem A we can assume $r = +1$. By Theorem 4.1.51 we have the inequality:

$$|t_0(K)| + 2\sum_{i=1}^{n} |t_i(K)| \leq \mathrm{rank} HF_{\mathrm{red}}(S^3) + \frac{d(S^3)}{2} + \mathrm{rank} HF_{\mathrm{red}}(Y) - \frac{d(Y)}{2},$$

where the number $t_i(K)$ for $i \in \mathbb{Z}$ is the torsion invariant of the Alexander polynomial $\Delta_K(T)$ of K and n is the degree of $\Delta_K(T)$. On the other hand we also have the identity:

$$\lambda(Y) = \chi(HF_{\mathrm{red}}(Y)) - \frac{d(Y)}{2}$$

from Theorem 5.1.8. We also know that $\mathrm{rank} HF_{\mathrm{red}}(S^3) = d(S^3) = 0$. Now by the surgery formula for Casson invariant

$$\lambda(Y) = \lambda(S^3) + \lambda(L(1,1)) + \Delta''_K(1) = \Delta''_K(1)$$

and by Proposition 3.3.1 $\Delta''_K(1) = 0$, thus $\lambda(Y) = 0$. By Proposition 5.1.9 $\chi(HF_{\mathrm{red}}(Y)) = 0$, hence $d(Y) = 0$. This proves the desired result. □

Corollary 5.2.4. *If a hyperbolic knot $K \subset S^3$ admits exceptional truly cosmetic surgeries then the Heegaard Floer correction term of any $1/n$ ($n \in \mathbb{Z}$) surgery on K satisfies*

$$d(S^3_K(1/n)) = 0.$$

Proof. Let K be as in the proof of Corollary 5.2.3. Let $d_{1/2}(S^3_K(0))$ and $d_{-1/2}(S^3_K(0))$ be the two correction terms for the 0-surgery along K as defined in Chapter 4. Let n be a positive integer, by Proposition 4.1.48,

$$d_{1/2}(S^3_K(0)) - \frac{1}{2} \leq d(S^3_K(1/(n+1))) \leq d(S^3_K(1/n)) \leq d(S^3) = 0$$

By Proposition 4.1.47 we have

$$d_{1/2}(S^3_K(0)) - \frac{1}{2} = d(S^3_K(+1)).$$

By the proof of Corollary 5.2.3 $d(S^3_K(+1)) = 0$, this completes the proof. □

5.3 Cosmetic surgeries on some special classes of knots

As consequences of Theorem A, let us give some results about cosmetic surgeries along algebraic knots, alternating knots and arborescent knots in S^3.

5.3.1 Algebraic knots.

An *algebraic knot* $K \subset S^3$ is the link of an *irreducible complex plane curve singularity* that is, K is the transversal intersection

$$K = \{f = 0\} \cap S^3_\epsilon$$

where $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is an irreducible polynomial, and $S^3_\epsilon = \{z \in \mathbb{C}^2 : ||z|| = \epsilon\}$ for $\epsilon > 0$ sufficiently small. The natural orientations of S^3 and of the regular part of $\{f = 0\}$ induces a natural orientation on K. When $\{f = 0\}$ is not smooth at the origin, K is not the unknot. The Heegaard Floer homology of the result of a surgery on an algebraic knot has been computed by Némethi in (Némethi, 2007).

Proposition 5.3.1. *(Némethi, 2007) Let $K \subset S^3$ be an algebraic knot, $p, q > 0$ two coprime integers and $Y = -S^3_K(-p/q)$. Then $HF_{\mathrm{red}}(Y)$ is supported in even degree.*

This leads to the following corollary.

Corollary 5.3.2. *There are no truly cosmetic surgeries on non-trivial algebraic knot in S^3.*

Proof. The proof is similar to the proof of Corollary 5.1.11 since the HF_{red} of the resulting manifold is supported in even degree by Proposition 5.3.1. $\square$

5.3.2 Alternating knots.

Combining Theorem A with work of Ichihara and Masai, we have the following result for alternating knot in S^3.

Corollary 5.3.3. *There are no exceptional truly cosmetic surgeries on an alternating hyperbolic knot in S^3.*

Proof. The corollary is a consequence of the classification of exceptional surgeries on alternating knots (Ichihara and Masai, 2013) combined with Theorem A. By their classification if an alternating hyperbolic knot in S^3 admits an exceptional surgery with slope r, then either:

- K is a twist knot $K[2n, \pm 2]$ for $n \neq 0$ (which includes the figure-8),
- K is a two bridge knot $K_{[a,b]}$ with $|a|, |b| > 2$ and $r = 0$ if both a, b are even and $r = 2b$ if a is odd and b is even,
- K is a pretzel knot $P(a, b, c)$ with $a, b, c \neq 0, \pm 1$ and $r = 0$ if a, b, c are all odd and $r = 2(b + c)$ if a is even and b, c are odd. Moreover $S^3_K(r)$ is toroidal but not Seifert fibred.

The Alexander polynomial of a twist knot $K[2n, \pm 2]$ is $\Delta_K(t) = 2n+1-n(t+t^{-1})$, so

$\Delta''_K(1) = -2n \neq 0$. Therefore by Proposition 3.3.1 we cannot have truly cosmetic surgery for the first case. For the last two cases $r \neq \pm 1$, so these exceptional slopes cannot be truly cosmetic slopes by Theorem A. $\square$

5.3.3 Arborescent knots.

Another class of knots in S^3 is the class of *arborescent knots.* Let (B, t) be the pair consisting of the 3-ball B^3 and a pair t of two properly embedded arcs in B. The pair (B, t) is called a *2-tangle.* It is called a *rational tangle* if there exists an orientation preserving homeomorphism of pairs:

$$f : (B, t) \longrightarrow (D^2 \times [0, 1], \{x, y\} \times [0, 1])$$

where $x, y \in \operatorname{int}(D^2)$. Two examples of rational tangles are the tangle [0] and $[\infty]$ shown in Figure 5.1. The four endpoints of a 2-tangle lie on the boundary S^2 of

Figure 5.1 [0] and $[\infty]$ tangles.

B^3. Any rational tangle can be obtain from [0] or $[\infty]$ by doing a finite number of twists of the neighbourhood of the endpoints in S^2.

Now a *Montesinos tangle* is a *sum* of several rational tangles. Here the sum operation of two 2-tangles consists of putting them next to each other and connecting them with two horizontal arcs as shown in Figure 5.2.

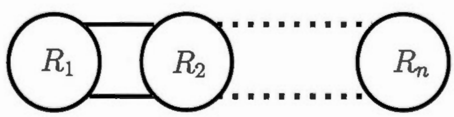


Figure 5.2 A Montesinos tangle.

In Figure 5.2 we sum n rational tangles $R_1, \cdots, R_n$ to get a Montesinos tangle.

Now let **T** and **S** be two 2-tangles. We can perform three operations with **T** and **S**: sum, product and inversion. These operations are shown in Figure 5.3.

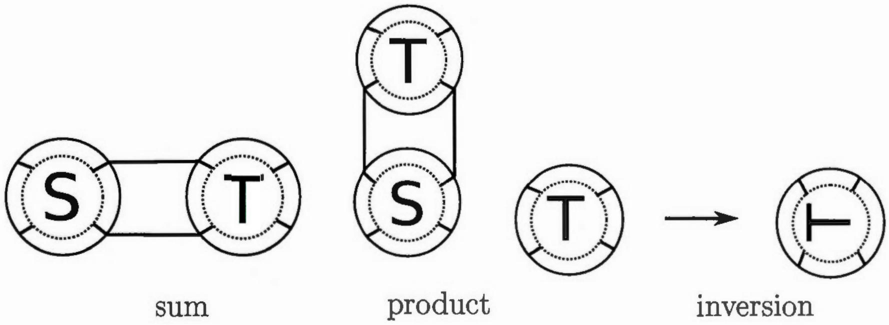


Figure 5.3 Sum, product and inversion of tangles.

We can also close a 2-tangle and get a link or a knot. There are two ways of doing this: the *numerator closure* and the *denominator closure.* The first is done by connecting the four endpoints by two horizontal arcs and the last is done by joining them with two vertical arcs. These processes are shown in Figure 5.4.

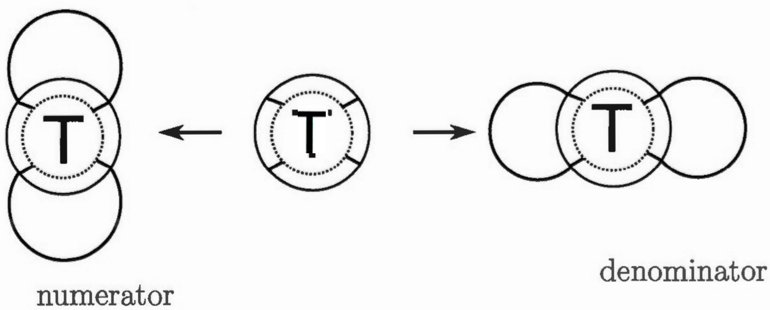


Figure 5.4 Numerator and denominator closure of a 2-tangle.

Finally an *arborescent tangle* is a 2-tangle obtained by performing a finite number of sum, product and inversion operation in any order with Montesinos tangles. We can now define what is an *arborescent* link or knot.

Definition 5.3.4. *An arborescent link is a link in S^3 obtained by taking the numerator or denominator closure of an arborescent tangle. An arborescent knot is a one component arborescent link.*

Now Theorem A implies the following result about cosmetic surgeries along arborescent knots.

Corollary 5.3.5. *There are no exceptional truly cosmetic surgeries on arborescent knots in S^3.*

Proof. In light of Theorem A we only have to check that ± 1 surgery on an arborescent knot do not yield two homeomorphic toroidal manifold. Following Ying-Qing Yu, there are three types of arborescent knots: type I, type II and type III. By Theorem 3.6 of (Wu, 1996), every non-trivial surgery on a type III arborescent knot gives a hyperbolic manifold, therefore we shall focus on type II and type I arborescent knots. For type I, they are Montesinos knots with length at most 3, which again split as 2-bridge knots and Montesinos knots of length 3. The 2-bridge knots which admit toroidal surgeries are given in Theorem 1.1 of (Brittenham and Wu, 2001) and they are among the knots in Corollary 5.3.3, hence they do not admit truly cosmetic surgeries. The case of Montesinos knots of length 3 is dealt with in (Wu, 2011a), precisely if K is a Montesinos knot of length 3 and δ is a slope on $\partial\mathcal{N}(K)$, then $S^3_K(\delta)$ is toroidal if and only if, following notation in (Wu, 2011a), (K, δ) is equivalent to one of

- $K = K(1/q_1,\ 1/q_2,\ 1/q_3)$, q_i odd, $|q_i| > 1$, $\delta = 0$.
- $K = K(1/q_1,\ 1/q_2,\ 1/q_3)$, q_1 even, q_2, q_3 odd, $|q_i| > 1$, $\delta = 2(q_2 + q_3)$.
- $K = K(-1/2,\ 1/3,\ 1/(6 + 1/n))$, $n \neq 0, -1$, $\delta = 16$ if n is odd, and 0 if n is even.

- $K = K(-1/3,\ -1/(3+1/n),\ 2/3)$, $n \neq 0, -1$, $\delta = -12$ when n is odd, and $\delta = 4$ when n is even.
- $K = K(-1/2,\ 1/5,\ 1/(3+1/n))$, n even, and $n \neq 0$, $\delta = 5 - 2n$.
- $K = K(-1/2,\ 1/3,\ 1/(5+1/n))$, n even, and $n \neq 0$, $\delta = 1 - 2n$.
- $K = K(-1/(2+1/n),\ 1/3,\ 1/3)$, n odd, $n \neq -1$, $\delta = 2n$.
- $K = K(-1/2,\ 1/3,\ 1/(3+1/n))$, n even, $n \neq 0$, $\delta = 2 - 2n$.
- $K = K(-1/2,\ 2/5,\ 1/9)$, $\delta = 15$.
- $K = K(-1/2,\ 2/5,\ 1/7)$, $\delta = 12$.
- $K = K(-1/2,\ 1/3,\ 1/7)$, $\delta = 37/2$.
- $K = K(-2/3,\ 1/3,\ 1/4)$, $\delta = 13$.
- $K = K(-1/3,\ 1/3,\ 1/7)$, $\delta = 1$.

After checking this list for knots which are listed more than once we notice that there are at most three toroidal slopes. A knot $K(t_1, t_2, t_3)$ admits exactly two toroidal surgeries if and only if it is equivalent to one of the following 5 knots:

$$\begin{aligned}
&K(-1/2,\ 1/3,\ 2/11), \quad \delta = 0 \text{ and } -3;\\
&K(-1/3,\ 1/3,\ 1/3), \quad \delta = 0 \text{ and } 2;\\
&K(-1/3,\ 1/3,\ 1/7), \quad \delta = 0 \text{ and } 1;\\
&K(-2/3,\ 1/3,\ 1/4), \quad \delta = 12 \text{ and } 13;\\
&K(-1/3,\ -2/5,\ 2/3), \quad \delta = 4 \text{ and } 6.
\end{aligned}$$

and it admits exactly three toroidal surgeries if and only if it is the figure-8 or the knot $K(-1/2,\ 1/3,\ 1/7)$. No pair of toroidal slopes corresponding to these knots admitting more that one toroidal slope are ± 1 in the standard Seifert framing so this excludes the possibility of truly cosmetic surgery.

The remaining case is that of type II arborescent knots. There are knots that have a Conway sphere cutting it into two Montesinos tangles of type $T(r_i, 1/2),\ \ i = 1, 2$ where $r_i \in \mathbb{Q} \cup \{\infty\}$. According to Theorem 1.1 of (Wu, 2011b), there are three distinct knots K_1, K_2, K_3 such that an arborescent knot $K \in S^3$ admits an exceptional slope δ if and only if (K, δ) is isotopic to $(K_1, 3),\ (K_2, 0),\ (K_3, -3)$, in which case the slope is toroidal. Therefore since there is exactly one slope for each knot, there are no truly cosmetic surgeries.

$\square$

CHAPTER VI

EXCEPTIONAL COSMETIC SURGERIES ON $\mathbb{Z}$-HOMOLOGY SPHERES

In this chapter we deal with the case of exceptional cosmetic surgeries along knots in an integer homology sphere. This is a generalisation of the case of S^3. In what follows μ will denote a meridian for a knot and λ_M its preferred longitude. The main result for this slightly more general case is:

Proposition 6.1.7. *Let Y be a $\mathbb{Z}$-homology sphere, $K \subset Y$ a hyperbolic knot and $M = Y \setminus \mathcal{N}(K)$. Assume we use a preferred basis $\{\mu, \lambda_M\}$ for $\pi_1(\partial M)$. Let $r = p/q$ and $r' = p/q'$ be exceptional slopes such that $0 < p$ and $q < q'$. If $M(r)$ is homeomorphic to $M(r')$ as oriented manifolds, then the surgery gives either*

(a) a reducible manifold in which case $p = 1$ and $q' = q + 1$,

(b) a toroidal Seifert fibred manifold in which case $p = 1$ and $q' = q + 1$,

(c) an atoroidal small Seifert manifold with infinite fundamental group in which case we have the following possibilities

- *$p = 1$ and $|q - q'| \leq 8$.*
- *$p = 5$, $q' = q + 1$ and $q \equiv 2\,[mod\ 5]$.*
- *$p = 2$, and $q' = q + 2$ or $q' = q + 4$.*

(d) a toroidal irreducible non-Seifert fibred manifold in which case $p = 1$ *and* $|q' - q| \leq 3$.

We give a proof of Proposition 6.1.7 in Section 6.1. In Section 6.2 we deal with questions related to the knot complement problem for $\Sigma(2, 3, 5)$.

6.1 Cosmetic surgeries on hyperbolic knots in integer homology spheres

We will prove Proposition 6.1.7 by using a series of lemmas and preliminary results.

Lemma 6.1.1. *Let* K *be a knot in a* $\mathbb{Z}$*-homology sphere* Y, $M = Y \setminus \mathcal{N}(K)$ *and let* λ_M *be the preferred longitude. Consider the basis* $\{\mu, \lambda_M\}$ *for* $\pi_1(\partial M)$. *Let* p/q *and* p/q' *be exceptional slopes such that* $0 < p$ *and* $q < q'$. *If* $M(p/q) \cong M(p/q')$ *then one of the following holds:*

(a) $p = 1$ *and* $|q - q'| \leq 8$.

(b) $p = 5$, $q' = q + 1$ *and* $q \equiv 2\,[mod\; 5]$.

(c) $p = 2$, *and* $q' = q + 2$ *or* $q' = q + 4$.

Proof. Since Y is a $\mathbb{Z}$-homology sphere, $H_1(M(p/q)) = \mathbb{Z}/p\mathbb{Z}$ is generated by the class $[\mu]_q$ of the meridian μ. As argued in the proof of Lemma 5.2.1, an orientation preserving homeomorphism $f : M(p/q) \to M(p/q')$ will induce an isomorphism $f_* : H_1(M(p/q)) \to H_1(M(p/q'))$ such that $f_*[\mu]_q = u\,[\mu]_{q'}$ for some unit u in $\mathbb{Z}/p\mathbb{Z}$. Moreover, it gives an isomorphism between the linking pairing of $M(p/q)$ and $M(p/q')$, and we must have

$$-\frac{q}{p} \equiv -\frac{q'}{p}u^2 \quad [\text{mod}\;\mathbb{Z}]\,, \quad \text{i.e} \quad q \equiv q'u^2 \quad [\text{mod}\;p]\,.$$

From Lemma 1.3.12 we have $p \in \{7, 5, 4, 3, 2, 1\}$. We then apply the above congruence relation to these cases to obtain the result. For $p = 1$ it is Lemma 1.3.12 (a).

- Case $p = 7$. By Lemma 1.3.12 $q' = q + 1$. The squares modulo 7 are 1, 2 and 4, they are all units so

  $$q \equiv q+1 \quad [\text{mod } 7] \quad \text{or} \quad q \equiv 2(q+1) \quad [\text{mod } 7] \quad \text{or} \quad q \equiv 4(q+1) \quad [\text{mod } 7]\,.$$

  The first equation is impossible and the last two are equivalent to

  $$q \equiv 5 \quad [\text{mod } 7] \quad \text{or} \quad q \equiv 1 \quad [\text{mod } 7]\,.$$

  By a straightforward computation

  $$s(5,7) = \frac{-1}{14}, \quad s(6,7) = \frac{-5}{14}, \quad s(1,7) = \frac{5}{14}, \quad s(2,7) = \frac{1}{14}.$$

  Using the fact that $s(a,p) = s(b,p)$ if $a \equiv b \quad [\text{mod } p]$, we get

  If $q \equiv 5 \quad [\text{mod } 7]$,

  $$s(q,7) = s(5,7) = \frac{-1}{14} \neq \frac{-5}{14} = s(6,7) = s(q+1,7)$$

  If $q \equiv 1 \quad [\text{mod } 7]$,

  $$s(q,7) = s(1,7) = \frac{5}{14} \neq \frac{1}{14} = s(2,7) = s(q+1,7)$$

  This contradicts Lemma 3.3.2 which says that we must have $s(q,p) = s(q',p)$.

- Case $p = 5$. By Lemma 1.3.12 $q' = q + 1$. The squares modulo 5 are 1 and 4, the only unit among them is 1, therefore

  $$q \equiv q+1 \quad [\text{mod } 5] \quad \text{or} \quad q \equiv 4(q+1) \quad [\text{mod } 5]\,.$$

  the first equation has no solution and the second is equivalent to

  $$q \equiv 2 \quad [\text{mod } 5]\,.$$

- Case $p = 4$. By Lemma 1.3.12 $q' \in \{q+1, q+2\}$. The only square modulo 4 is 1 therefore

$$q \equiv q+1 \quad [\text{mod } 4] \qquad \text{or} \qquad q \equiv q+2 \quad [\text{mod } 4]\,.$$

these equations have no solutions so the case $p = 4$ is not possible.

- Case $p = 3$. By Lemma 1.3.12 $q' \in \{q+1, q+2\}$. The only square modulo 3 is 1, therefore this case is also impossible.

- Case $p = 2$. By Lemma 1.3.12 $q' \in \{q+2, q+4\}$.

□

By using result of Gordon on toroidal exceptional slopes at large distance, we can get more restrictions on the slopes which gives cosmetic toroidal fillings.

Lemma 6.1.2. *Let M be a hyperbolic knot manifold in an integral homology sphere and let r, s be two slopes on ∂M. If $M(r)$ and $M(s)$ are toroidal and if there is an orientation preserving homeomorphism between them, then $\Delta(r, s) \leq 3$.*

Proof. We will distinguish the cases $\Delta(r, s) > 5$ and $\Delta(r, s) = 5$, or 4.

Let W be the Whitehead link exterior. By Theorem 2.2.5 if $\Delta(r, s) > 5$ then either

- $\Delta(r, s) = 6$ and M is homeomorphic to $W(2)$
- $\Delta(r, s) = 7$ and M is homeomorphic to $W(-5/2)$
- $\Delta(r, s) = 8$ and M is homeomorphic to $W(1)$ or $W(-5)$

Then manifold $M(r)$ is then obtained by surgery on the Whitehead link with coefficients $\{2, a_1/b_1\}$ or $\{-5/2, a_2/b_2\}$ or $\{1, a_3/b_3\}$ or $\{-5, a_4/b_4\}$. We can then compute the order of the first homology using this coefficient,

$$|H_1(M(r))| = \begin{vmatrix} 2 & b_1 \operatorname{lk}(K_1, K_2) \\ \operatorname{lk}(K_2, K_1) & a_1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & a_1 \end{vmatrix} = 2a_1$$

where K_1, K_2 denotes the two components of the Whitehead link and $\operatorname{lk}(K_1, K_2)$ their linking number. Similarly we get for the other possibilities

$$|H_1(M(r))| = \begin{vmatrix} -5 & 0 \\ 2 & a_2 \end{vmatrix} = -5a_2, \quad \text{or} \quad |H_1(M(r))| = \begin{vmatrix} -5 & 0 \\ 0 & a_4 \end{vmatrix} = -5a_4.$$

On the other hand if $\Delta(r, s) > 5$ then $M(r)$ must be a homology sphere by Lemma 6.1.1. Therefore these three possibilities cannot occur. The remaining case is $M(r) = W(1)$ which is the figure-8 complement and there are no truly cosmetic surgery along the figure-8 knot, as we can check for instance that $\Delta''_{\text{figure-8}}(1) \neq 0$ and use Proposition 3.3.1.

Now we can assume $\Delta(r, s) \in \{4, 5\}$. Let's recall Theorem 2.2.1

There exist fourteen 3-manifolds M_i, $1 \leq i \leq 14$, such that

(1) M_i is hyperbolic, $1 \leq i \leq 14$;

(2) ∂M_i consists of two tori T_0, T_1 if $i \in \{1, 2, 3, 14\}$, and a single torus T_0 otherwise;

(3) there are slopes r_i, s_i on the boundary component T_0 of M_i such that $M(r_i)$ and $M(s_i)$ are toroidal, where $\Delta(r_i, s_i) = 4$ if $i \in \{1, 2, 4, 6, 9, 13, 14\}$, and $\Delta(r_i, s_i) = 5$ if $i \in \{3, 5, 7, 8, 10, 11, 12\}$;

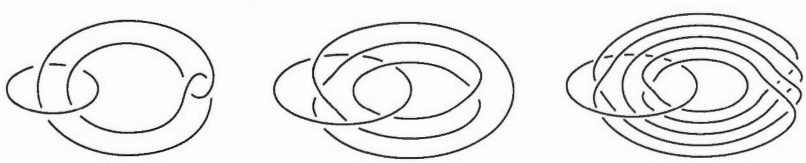
(4) if M is a hyperbolic 3-manifold with toroidal Dehn fillings $M(r), M(s)$ where $\Delta(r, s) = 4$ or 5, then (M, r, s) is equivalent either to (M_i, r_i, s_i) for some $1 \leq i \leq$

14, or to $(M_i(t), r_i, s_i)$ where $i \in \{1, 2, 3, 14\}$ and t is a slope on the boundary component T_1 of M_i.

Here we define two triples (N_1, r_1, s_1) and (N_2, r_2, s_2) to be *equivalent* if there is a homeomorphism from N_1 to N_2 which sends the boundary slopes (r_1, s_1) to (r_2, s_2) or (s_2, r_2).

- Case M is one of M_1, M_2, M_3.

  The manifolds M_1, M_2, M_3 are the exterior of the following links



(1) (2) (3)

  For $i \in \{1, 2, 3\}$ we will denote K_i' and K_i'' the leftmost and rightmost components of the above links. Let $t = a/b$ be a slope on $\mathcal{N}(K_i')$ written in its Seifert framing. If $a \neq 1$ then $H_1(M_i(t)) = \mathbb{Z} \oplus \mathbb{Z}/a\mathbb{Z} \neq \mathbb{Z} = H_1(M)$. Therefore we must have $a = 1$. But in this case we have a knot complement in S^3 and we know from Theorem A that the surgery must be ± 1 surgery with respect to the Seifert framing, and therefore $\Delta(r, s) = 2$ which is not the case here.

- Case $M \cong M_{14}$. Let t be a slope on the boundary component T_0 of M_{14}, and let K_t be the core of the Dehn filling solid torus in $M_{14}(t)$. By Lemma 2.2.3

$$H_1(M_{14}(t)) / H_1(K_t) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

  Therefore $H_1(M_{14}(t)) \neq \mathbb{Z} = H_1(M)$.

- Case M is one of M_4 and M_5. By Lemma 2.2.4 we have

$$H_1(M_4(r)) = \mathbb{Z}, \quad \text{and} \quad H_1(M_5(r)) = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

These situations are not possible since $M(r)$ is a rational homology sphere.

- Case M is one of $M_6, M_7, M_{10}, M_{11}, M_{12}, M_{13}$. By Lemma 2.2.2 M admits a Lens space surgery. With respect to the framing in (Gordon and Wu, 2008) these lens space surgeries are

$$\begin{array}{lll} M_6(\infty) = L(9,2) & M_7(\infty) = L(20,9) & M_{10}(\infty) = L(14,3) \\ M_{11}(\infty) = L(24,5) & M_{12}(\infty) = L(3,1) & M_{13}(\infty) = L(4,1) \end{array}$$

From this we can deduce that $|\mathrm{Tor}\,(H_1(M))\,| \neq 1$ which is not possible since $H_1(M) = \mathbb{Z}$.

- Case M is one of M_8 and M_9. From Lemma 2.2.2 the manifolds M_8 and M_9 has two toroidal surgeries and one lens space surgery listed as follows with respect to the framing used in (Gordon and Wu, 2008)

$$\begin{array}{lll} M_8(0), & M_8(-5/4), & M_8(-1) = L(4,1) \\ M_9(0), & M_9(-4/3), & M_9(-1) = L(8,3) \end{array}$$

For $i \in \{8,9\}$ let $a = |\mathrm{Tor}\,(H_1(M_i))\,|$ and l be the order of the preferred rational longitude λ_{M_i}. We are going to express the framing used in (Gordon and Wu, 2008) according to our standard basis $\{\mu, \lambda_{M_i}\}$. Let λ be the framing used in (Gordon and Wu, 2008). Then the -1 slope in this framing can be written $-\mu + (p\mu + \lambda_{M_i}) = (p-1)\mu + \lambda_{M_i}$. Using the fact that $|H_1\,(L(4,1))\,| = 4$ and $|H_1\,(L(8,3))\,| = 4$, with Lemma 1.1.4 we get

$$|H_1\,(M_8(-1))\,| = 4 = \Delta\left((p-1)\mu + \lambda_{M_8}\ ;\ \lambda_{M_8}\right)\ la = |p-1|la,$$

$$|H_1\,(M_9(-1))\,| = 8 = \Delta\left((p-1)\mu + \lambda_{M_9}\ ;\ \lambda_{M_9}\right)\ la = |p-1|la.$$

Since M is a knot complement in an integer homology sphere, if M is one of M_8 or M_9 then we must have $l = a = 1$. Therefore $p \in \{-3, 5\}$ for M_8 and $p \in \{9, -7\}$ for M_9. We can then deduce

$$H_1(M_8(0)) = \mathbb{Z}/5\mathbb{Z} \text{ or } \mathbb{Z}/3\mathbb{Z}, \qquad H_1(M_8(-5/4)) = \mathbb{Z}/15\mathbb{Z} \text{ or } \mathbb{Z}/17\mathbb{Z},$$
$$H_1(M_9(0)) = \mathbb{Z}/9\mathbb{Z} \text{ or } \mathbb{Z}/7\mathbb{Z}, \qquad H_1(M_9(-4/3)) = \mathbb{Z}/23\mathbb{Z} \text{ or } \mathbb{Z}/25\mathbb{Z},$$

Therefore $M_8(0)$ and $M_8(-5/4)$ are not homeomorphic and the same is true for $M_9(0)$ and $M_8(-4/3)$. We can conclude that M cannot be one of M_8 or M_9.

□

The last lemma implies in particular that toroidal truly cosmetic surgeries on integer homology sphere must be integer homology spheres.

The next preliminary result addresses the case of Seifert fibred toroidal surgeries. Before going into it we need a bit of $PSL_2(\mathbb{C})$-character variety theory. We refer to (Boyer and Zhang, 1998) and Chapter 7 for more details on the subject.

Let $X(G)$ denote the $PSL_2(\mathbb{C})$-character variety of a finitely generated group G. When $G = \pi_1(Z)$ where Z is a path-connected space, we shall sometimes write $X(Z)$ for $X(\pi_1(Z))$. Recall that $X(G)$ is a complex algebraic variety and a surjective homomorphism $G \twoheadrightarrow H$ induces an injective morphism $X(H) \hookrightarrow X(G)$ by precomposition. A curve $X_0 \subset X(G)$ is called non-trivial if it contains the character of an irreducible representation. Each $\gamma \in X(G)$ determines an element f_γ of the coordinate ring $\mathbb{C}[X(G)]$ where if $\rho : G \to PSL_2(\mathbb{C})$ is a representation and χ_ρ the associated point in $X(G)$, then $f_\gamma(\chi_\rho) = \text{trace}(\rho(\gamma))^2 - 4$. When $G = \pi_1(M)$, any slope r on ∂M determines an element of $\pi_1(M)$, well-defined up to conjugation and taking inverse. Hence it induces a well-defined element $f_r \in \mathbb{C}[X(M)]$.

Lemma 6.1.3. *Let Y be a $\mathbb{Z}$-homology sphere, $K \subset Y$ a hyperbolic knot and $M = Y \setminus \mathcal{N}(K)$. Assume we use a basis $\{\mu, \lambda_M\}$ for $\pi_1(\partial M)$. Let $r = p/q$ and $r' = p/q'$ be exceptional slopes such that $0 < p$ and $q < q'$. If $M(r)$ is homeomorphic to $M(r')$ as oriented manifolds and is Seifert fibred and toroidal, then $p = 1$ and $q' = q + 1$.*

Proof. Let $\mathcal{B}$ be the base orbifold for $M(r)$. Since $M(r)$ is toroidal with finite first homology $\mathcal{B}$ cannot be spherical. Moreover it cannot be a sphere with strictly less than 4 cone points. Thus $\mathcal{B}$ must be either hyperbolic or one among: $S^2(2,2,2,2)$, $\mathbb{T}^2$, $\mathbb{RP}^2(2,2)$ or the Klein bottle. Since we assume that $M(r)$ and $M(r')$ are toroidal, by lemma 6.1.2 $\Delta(r,r') \leq 3$, so $p \leq 3$. Lemma 6.1.1 then implies that $p = 1$ or $p = 2$. If $p = 2$ then $q' = q + 2$ or $q' = q + 4$ and it follows that $\Delta(r,r') = 4$ or 8, this contradict the fact that $\Delta(r,r') \leq 3$. Therefore we must have $p = 1$. Furthermore using the fact that $M(r)$ is a Seifert fibred manifold, we have the following surjection in first homology: $H_1(M(r)) \twoheadrightarrow H_1(\mathcal{B})$, thus $|H_1(M(r))| = p = 1 \geq |H_1(\mathcal{B})|$. However we know that $|H_1(S^2(2,2,2,2))| = |H_1(\mathbb{RP}^2(2,2))| = 8$, $H_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_1(\text{Klein bottle}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Thus $\mathcal{B}$ must be hyperbolic.

By the same argument as above $\mathcal{B}$ cannot be $\mathbb{RP}^2(a,b)$ since $|H_1(\mathbb{RP}^2(a,b))| = 2ab > 1$.

By work of Thurston (Thurston, 1979), since $\mathcal{B} \neq \mathbb{RP}^2(a,b)$ the real dimension of the Teichmüller space $\mathcal{T}(\mathcal{B})$ of $\mathcal{B}$ is at least 2. Moreover $\mathcal{T} \subset X(\pi_1^{orb}(\mathcal{B}))$ where $\pi_1^{orb}(\mathcal{B})$ is the orbifold fundamental group of $\mathcal{B}$. On the other hand we have

$$\pi_1(M) \twoheadrightarrow \pi_1(M(r)) \twoheadrightarrow \pi_1^{orb}(\mathcal{B})$$

which induce a sequence of inclusions

$$X(M) \supset X(M(r)) \supset X(\pi_1^{orb}(\mathcal{B})) \supset \mathcal{T}(\mathcal{B}).$$

Therefore the complex dimension of $X(M(r))$ is at least 1. We want to prove that it contains a subvariety of complex dimension at least 2. Assume in the contrary that all components of $X(M(r))$ have complex dimension 1. In this case $\mathcal{T}(\mathcal{B})$ would be an open set in a non-trivial curve $X_0 \subset X(M(r))$. When $\chi_\rho \in \mathcal{T}(\mathcal{B})$, ρ is the holonomy of a hyperbolic structure on $\mathcal{B}$ and it is well known that if $\gamma \in \pi_1^{orb}(\mathcal{B})$ has infinite order, then $f_\gamma(\chi_\rho)$ is a real number. Deforming χ_ρ in $\mathcal{T}(\mathcal{B})$ shows that $f_\gamma|_{X_0}$ is non-constant and must take some non-real values. This contradicts the fact that it is real-valued on the open subset $\mathcal{T}(\mathcal{B}) \subset X_0$. Thus $X(M)$ has a subvariety of complex dimension 2 or larger on which f_r is constant and which contains the character of an irreducible representation. Hence if $r' \neq r$ is any other slope, we can then construct a non-trivial curve $X_0 \subset X(M)$ on which both f_r and $f_{r'}$ are constant. Indeed let X be this two dimensional subvariety, if $f_{r'}|_X$ is constant then we are done, otherwise we can take a regular value $z_0 \in \mathbb{C}$ of $f_{r'}|_X$, the preimage $f_{r'}|_X^{-1}(z_0)$ is a codimension one subvariety of X and we can take $X_0 = f_{r'}|_X^{-1}(z_0)$. It follows that $f_r|_{X_0}$ is constant for each slope. In particular for each ideal point $\tilde{x}$ of X_0 and slope $s \in \partial M$, $\tilde{f}_s(\tilde{x}) \in \mathbb{C}$. Now Proposition 4.10 and Claim (pg. 786) of (Boyer and Zhang, 1998) imply that there is a closed essential surface $S \subset M$ which compresses in $M(r)$ but stays incompressible in $M(s)$ if $\Delta(s, r) > 1$.

Suppose we have $\Delta(r, r') \geq 2$, then S must be incompressible in $M(r')$. Since M is hyperbolic it has no incompressible torus. Therefore S must have genus at least 2 and is a horizontal surface.

On the other hand $M \subset Y$ and Y is a $\mathbb{Z}$-homology sphere so S must separate M and also $M(r')$. Indeed $H_2(Y) = 0$ so $[S] = 0$ and S separates. Let M_1 and M_2 be the two components of $M(r') \setminus S$. They are both interval semi-bundles with base $\mathcal{B}$. It follows that if Σ_i is the core surface of M_i, then $\pi_1(\Sigma_i) \cong \pi_1(M_i)$

for $i = 1, 2$. On the other hand since $\partial M_i = S$ is connected, we have a 2 to 1 connected cover $\partial M_i \to \Sigma_i$. Then $\pi_1(S)$ is an index two subgroup of $\pi_1(\Sigma_i)$, in particular it is normal. Using Van-Kampen theorem we have

$$\pi_1(M(r')) = \pi_1(\Sigma_1) *_{\pi_1(S)} \pi_1(\Sigma_2)$$

and $\pi_1(S)$ is normal in $\pi_1(M(r'))$ since it is normal in both component of the amalgam. Hence

$$\frac{\pi_1(M(r'))}{\pi_1(S)} = \left(\frac{\pi_1(\Sigma_1)}{\pi_1(S)}\right) * \left(\frac{\pi_1(\Sigma_2)}{\pi_1(S)}\right) \cong \mathbb{Z}/2Z * \mathbb{Z}/2\mathbb{Z}$$

and we have a surjection $\pi_1(M(r')) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. This induces a surjection in first homology $H_1(M(r')) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, which contradicts the fact that $H_1(M(r'))$ is cyclic. Therefore $\Delta(r, r') = p|q - q'| \leq 1$ which implies $p = 1$ and $q' = q + 1$. $\square$

Before continuing with the next lemmas, we need the following results of Rasmussen, (Rasmussen, 2007) and of Ni (Ni, 2009).

Theorem 6.1.4. *(Rasmussen, 2007) If Z is an L-space with $H_1(Z) = \mathbb{Z}/p\mathbb{Z}$, and $K \subset Z$ is a primitive knot (i.e K generates $H_1(Z)$) with a homology sphere non-trivial surgery X. Then X is an L-space if and only if one of the following condition holds:*

1. *$\widehat{HFK}(K) \cong \mathbb{Z}^p$ and width $\widehat{HFK}(K) < 2p$.*
2. *$\widehat{HFK}(K) \cong \mathbb{Z}^{p+2}$ and width $\widehat{HFK}(K) = 2p$.*

Here width $\widehat{HFK}(K)$ is the difference $Max - Min$, where Max is the maximum value of j for which $\widehat{HFK}(K, j)$ is nontrivial and Min is the minimum value.

On the other hand, Ni gives a formula to compute the genus of the knot K in the situation of Theorem 6.1.4.

Theorem 6.1.5. *(Ni, 2009) Suppose K is a primitive knot in a rational homology sphere Z, and that $H_1(Z;\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Then*

$$g(K) = \frac{(\textit{width } \widehat{HFK}(K) - p + 1)}{2}.$$

The following lemma will be useful for the study of L-space homology sphere surgeries in L-space $\mathbb{Z}$-homology spheres and truly cosmetic surgeries in such manifolds as the Poincarés sphere.

Lemma 6.1.6. *Let Z be an L-space $\mathbb{Z}$-homology sphere and let K be any non-trivial knot in Z. If K admits an L-space $\mathbb{Z}$-homology sphere non-trivial surgery, then $\Delta_K(T) = T^{-1} - 1 + T$, $\Delta''_K(1) = 2$ and K has genus 1.*

Proof. By Theorem 6.1.4 either

1. $\widehat{HFK}(K) \cong \mathbb{Z}^p$ and width $\widehat{HFK}(K) < 2p$.
2. $\widehat{HFK}(K) \cong \mathbb{Z}^{p+2}$, width $\widehat{HFK}(K) = 2p$.

where width $\widehat{HFK}(K)$ is the difference $Max - Min$, where Max is the maximum value of j for which $\widehat{HFK}(K, j)$ is nontrivial and Min is the minimum value.

By Theorem 6.1.5 $g(K) = (\text{width } \widehat{HFK}(K) - p + 1)/2$.

In our case $p = 1$ so all the hypothesis are satisfied and we have either

1. $\widehat{HFK}(K) \cong \mathbb{Z}$ and width $\widehat{HFK}(K) < 2$.
2. $\widehat{HFK}(K) \cong \mathbb{Z}^3$, width $\widehat{HFK}(K) = 2$.

The first case implies that width $\widehat{HFK}(K) = 0$, so $g(K) = 0$. Therefore we are left with the second case. We can then compute the Euler characteristic of $\widehat{HFK}(K)$ to obtain the symmetrized Alexander polynomial of K using the formula

$$\Delta_K(T) = \sum_{i,j} (-1)^i \operatorname{rank}\widehat{HFK}_i(K,j)\, T^j.$$

We obtain

$$\Delta_K(T) = a_{j_0}\, T^{j_0} + a_{j_0+1}\, T^{j_0+1} + a_{j_0+2}\, T^{j_0+2},$$

for some j_0. Using the fact that $\Delta_K(T) = \Delta_K(T^{-1})$ we get $j_0 = -1$. Since width $\widehat{HFK}(K) = 2$ and $\widehat{HFK}(K) \cong \mathbb{Z}$ we have $a_{-1} = a_1 = \pm 1$, so

$$\Delta_K(T) = \pm T^{-1} + a_0 \pm T.$$

On the other hand, from Corollary 5.1.7

$$\Delta_K(T) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(T^{n_j} + T^{-n_j}),$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < ... < n_k$. Therefore $j = 1, k = 1,\ n_j = 1$ and

$$\Delta_K(T) = T^{-1} - 1 + T.$$

Now computing the second derivative gives $\Delta''_K(1) = 2$. Finally since width $\widehat{HFK}(K) = 2$ and $g(K) = \max\{k \mid \widehat{HFK}_*(K,k) \neq 0\}$, we must have $g(K) = 1$. $\square$

We can now state and prove the main proposition of this section.

Proposition 6.1.7. *Let Y be a $\mathbb{Z}$-homology sphere, $K \subset Y$ a hyperbolic knot and $M = Y \setminus \mathcal{N}(K)$. Assume we use a preferred basis $\{\mu, \lambda_M\}$ for $\pi_1(\partial M)$. Let $r = p/q$ and $r' = p/q'$ be exceptional slopes such that $0 < p$ and $q < q'$. If $M(r)$ is homeomorphic to $M(r')$ as oriented manifolds, then the surgery gives either* *(a) a reducible manifold in which case* $p = 1$ *and* $q' = q + 1$,

(b) a toroidal Seifert fibred manifold in which case $p = 1$ *and* $q' = q + 1$,

(c) an atoroidal small Seifert manifold with infinite fundamental group in which case we have the following possibilities

- $p = 1$ *and* $|q - q'| \leq 8$.
- $p = 5$, $q' = q + 1$ *and* $q \equiv 2\,[mod\ 5]$.
- $p = 2$, *and* $q' = q + 2$ *or* $q' = q + 4$.

(d) a toroidal irreducible non-Seifert fibred manifold in which case $p = 1$ *and* $|q' - q| \leq 3$.

Proof. The manifold $M(r)$ is either reducible, Seifert fibred or toroidal. If it is reducible then we have (a) which is given by Lemma 1.3.13. If it is toroidal and Seifert fibred then we have (b) which is given by Lemma 6.1.3. The remaining possibilities are then (c), (d) and the case $\pi_1(M(r))$ is finite. The proofs of (c) and (d) follow from Lemma 6.1.1. We are now left with the last possibility. Assume that $\pi_1(M(r))$ is finite. The distance between two finite slopes is at most 3, so $\Delta(p/q, p/q') = p|q' - q| \leq 3$. In particular $p \in \{1, 2, 3\}$, but by Lemma 6.1.1, $p \in \{1, 2, 5\}$ thus $p = 1$ or $p = 2$. If $p = 2$ then $|q' - q| \geq 2$ by Lemma 6.1.1 and $\Delta(p/q, p/q') = 4 > 3$. Therefore $p = 1$. Then $M(r)$ is a homology sphere with finite fundamental group which implies $M(r) = \Sigma(2, 3, 5)$ or $M(r) = S^3$. If $M(r) = \Sigma(2, 3, 5)$ or S^3 then $M \subset \Sigma(2, 3, 5)$ or S^3. Let Z denote either $\Sigma(2, 3, 5)$ or S^3. Then $M = Z \setminus \mathcal{N}(K)$ where K is a non-trivial knot in Z (since M is hyperbolic) for which there is a non trivial slope which gives $\Sigma(2, 3, 5)$. We notice that both $\Sigma(2, 3, 5)$ and S^3 are L-space homology spheres, so by Lemma 6.1.6 $\Delta''_K(1) = 2 \neq 0$. Therefore by Proposition 3.3.1 there is no orientation preserving homeomorphism between $M(r)$ and $M(r')$ since $r \neq r'$. Thus there cannot be two distinct slopes which give the same L-space homology sphere up to orientation preserving homeomorphism. $\square$

6.2 The knot complement problem for $\Sigma(2,3,5)$.

The Poincaré sphere $\Sigma(2,3,5)$ is the oriented 3-manifold obtained by -1-surgery on the left handed trefoil in S^3. From the surgery formula for the Casson invariant we can deduce that $\lambda(\Sigma(2,3,5)) = -1$. In this section we do not assume that the knot is hyperbolic. A knot K in a 3-manifold Y is *determined by its complement* if the existence of a homeomorphism between $Y \setminus K$ and $Y \setminus K'$ for some other knot K', implies the existence of homeomorphism between the pair (Y, K) and (Y, K'). Here we do not expect these homeomorphisms to be orientation preserving. Gordon and Luecke (Gordon and Luecke, 1989) have proved that all non-trivial knots in S^3 and $S^2 \times S^1$ are determined by their complements. D. Matignon has proved in (Matignon, 2010) that, if one only considers orientation preserving homeomorphisms, then non-trivial non-hyperbolic knots are determined by their complements in closed, atoroidal and irreducible Seifert fibred 3-manifolds; except the axes in $L(p,q)$ when $q^2 \equiv \pm 1$ [mod p]. In general there are some knots in 3-manifolds which are not determined by their complement. In (Rong, 1993) Y. Rong classified all Seifert fibered knots which are not determined by their complements in closed 3-manifolds other than lens spaces. In (Matignon,) D. Matignon gave examples of hyperbolic knots in lens spaces which are not determined by their oriented complements.

This section will be concerned with the question: Can a non-trivial surgery on a non-trivial knot in $\Sigma(2,3,5)$, oriented in this time, yield a manifold homeomorphic (orientation preserving or reversing) to $\Sigma(2,3,5)$? The answer when all the homeomorphisms are orientation preserving is given by Theorem 6.2.1. As a related question, we answer the question of whether non-trivial knots in $\Sigma(2,3,5)$ are determined by their oriented complements or not.

Theorem 6.2.1. *Let K be a non-trivial knot in $\Sigma(2,3,5)$ and let $r \in \mathbb{Q}$. The result of an r-surgery along K is never orientation-preserving homeomorphic to $\Sigma(2,3,5)$.*

Proof. Since $\Sigma(2,3,5)$ is an L-space $\mathbb{Z}$-homology sphere, Lemma 6.1.6 implies that

$$\Delta_K''(1) = 2 \neq 0.$$

Therefore by Proposition 3.3.1 the homeomorphism must be orientation reversing. □

This theorem can be stated for the general case of L-space $\mathbb{Z}$-homology spheres.

Theorem 6.2.2. *Let K be a non-trivial knot in an oriented L-space $\mathbb{Z}$-homology sphere Y and let $r \in \mathbb{Q}$. The result of an r-surgery along K is never orientation-preserving homeomorphic to Y.*

Proof. Same as for Theorem 6.2.1 □

Theorem 6.2.1 has a direct consequence which is the answer to the "oriented knot complement problem" in $\Sigma(2,3,5)$.

Theorem 6.2.3. *Non-trivial knots in $\Sigma(2,3,5)$ are determined by their oriented complements.*

Proof. Let K and K' be two non-trivial knots in $\Sigma(2,3,5)$, let us denote V and V' their complements with the induced orientations. Suppose there is an orientation-preserving homeomorphism $f : V \to V'$. Let μ_K, respectively $\mu_{K'}$, be the meridional slope of K, respectively K', and let $r = f(\mu_K)$. The oriented manifold $V'(r)$ is orientation preserving homeomorphic to $\Sigma(2,3,5)$ and therefore by Theorem 6.2.1 $r = \pm\mu_{K'}$. It follows that we can extend f to an orientation preserving homeomorphism between $(\Sigma(2,3,5), K)$ and $(\Sigma(2,3,5), K')$. □

We can ask if there is still a non-trivial surgery along a knot K in $\Sigma(2,3,5)$ which gives $-\Sigma(2,3,5)$ (the orientation is reversed). The purpose of the rest of the section is to study this possibility. In particular we prove the existence of tight contact structure on the $+1$-surgery along K using a result of Motoo Tange.

Lemma 6.2.4. *Let $K \subset \Sigma(2,3,5)$ be a non-trivial knot and let $r \in \mathbb{Q}$. If the result of an r-surgery along K is homeomorphic to $-\Sigma(2,3,5)$ then $r = 1/2$.*

Proof. Let $Y = \Sigma(2,3,5)$. Since $H_1(Y,\mathbb{Z}) = 0$ we must have $r = 1/q$. Now by the Casson invariant surgery formula

$$\lambda(-Y) = \lambda(Y) + \lambda(L(1,q)) + \frac{q}{2}\Delta''_K(1).$$

Since $\lambda(-Y) = -\lambda(Y)$ and $\lambda(L(1,q)) = \lambda(S^3) = 0$ we have

$$\frac{q}{2}\Delta''_K(1) = -2\lambda(Y) = 2.$$

Now Lemma 6.1.6 implies that $\Delta''_K(1) = 2$, thus $q = 2$. □

Lemma 6.2.5. *Let $K \subset \Sigma(2,3,5)$ be a non-trivial knot and let $r \in \mathbb{Q}$. If the result of an r-surgery along K is homeomorphic to $-\Sigma(2,3,5)$ then the result of $+1$-surgery along K is an L-space.*

Proof. From Lemma 6.2.4 $r = 1/2$ and from Corollary 5.1.4 the the result of $+1$-surgery along K is also an L-space. □

Before going into the existence of a tight contact structure let us discuss the notion of a "reducible" knot. We say that a knot K in a 3-manifold Y is *irreducible* if its complement $Y \setminus K$ is irreducible. In case Y is the Poincaré sphere with either orientation, a non-trivial knot K is irreducible if and only if it does not lie in a ball. Indeed if $Y \setminus K$ was reducible then there are two 3-manifolds M and N such that M is distinct from S^3, $K \subset N$ and $Y = M \sharp (N \setminus K)$. However meridional surgery on K yields Y again so $Y_K(\infty) = Y$. Thus

$$Y_K(\infty) = Y = M \sharp N_K(\infty).$$

But Y is irreducible and M is distinct from S^3. Therefore $N_K(\infty) \cong S^3$ and it follows that $N \cong S^3$. Therefore K lies in a ball. Conversely if a non-trivial knot K in Y lies in ball, then its the complement is obviously reducible.

The knot complement problem for a non-trivial knot which lies in a ball is equivalent to the knot complement problem for knots in S^3. Since it is known that non-trivial knots in S^3 are determined by their complements, we can assume that our knot K does not lie in a ball. That is $Y \setminus K$ is irreducible.

Theorem 6.2.6. *(Tange, 2011) Let Y be an L-space homology sphere, K a knot in Y and p a positive integer. If $Y \setminus K$ is irreducible and $Y_K(p)$ is an L-space, then Y admits positive tight contact structures.*

Theorem 6.2.7. *Let K be a non-trivial knot in $\Sigma(2,3,5)$. If K admits a non-trivial surgery which gives $-\Sigma(2,3,5)$ then the surgery slope is $1/2$ and the result of $+1$-surgery along K is an L-space which admits a tight contact structure.*

Proof. Since we can assume that $Y \setminus K$ is irreducible, the theorem follows from Lemma 6.2.4 Theorem 6.2.6 and Lemma 6.2.5. ☐

CHAPTER VII

COSMETIC SURGERIES AND CHARACTER VARIETIES

From what we have seen so far, we can say that cosmetic surgeries are very rare. In particular for the case of a hyperbolic knot complement in S^3 there are at most two exceptional cosmetic slopes that is ± 1. For hyperbolic knot complements in $\mathbb{Z}$-homology spheres, according to chapter 6 there are limited possibilities for exceptional cosmetic surgeries. In this chapter we will focus on small Seifert cosmetic surgeries along hyperbolic knots in rational homology spheres.

Let Y be a rational homology sphere and K be a hyperbolic knot in Y. We denote $M = Y_K$ the corresponding knot exterior. If α is a slope on ∂M, we define the set $C(\alpha)$ to be

$$C(\alpha) = \{\text{slope } \ \beta |\ M(\alpha) \cong M(\beta)\}.$$

The main theorem of this chapter is the following.

Theorem B. *Let Y be a rational homology sphere. Assume that Hom $(\pi_1(Y), PSL_2(\mathbb{C}))$ contains only diagonalisable representations, no side of the $PSL_2(\mathbb{C})$-Culler-Shalen ball of M is parallel to λ_M, and α is small-Seifert. Then $\#C(\alpha) \leq 2$.*

The bound in the theorem is sharp. In (Bleiler et al., 1999), there is a construction of a one-cusped hyperbolic 3-manifold with a pair of distinct slopes which gives oppositely oriented copies of the lens space $L(49/18)$. An infinite family of hyperbolic manifolds which admit pairs (α, β) of reducible filling slopes, of which some pairs yield homeomorphic manifolds are presented in (Matignon and Hoffman, 2003). The result of Zhongtao Wu in Theorem 1.3.8 implies this result for the case of integral homology L-spaces.

The Dehn filling manifold $M(\alpha)$ we are considering is small Seifert. For the proof of Theorem B we will distinguish the cases $b_1(M(\alpha)) = 1$ and $b_1(M(\alpha)) = 0$. In section 7.1 and section 7.2 we give some background on character varieties. In next section 7.3 we give the necessary material which are needed for the proof of Theorem B. This proof will be given in section 7.4.

7.1 Preliminaries

Let Γ be a finitely generated group, for instance $\Gamma = \pi_1(M)$ where M is a compact 3-manifold. The $PSL_2(\mathbb{C})$*-representation variety* of Γ is the set of homomorphism

$$\overline{R}(\Gamma) := \hom(\Gamma, PSL_2(\mathbb{C}))$$

equipped with the compact-open topology. Given a set of generators $g_1, \cdots, g_n$ of Γ, the space $\overline{R}(\Gamma)$ can be embedded in $\mathbb{C}^{4n}$ via

$$\begin{aligned} \overline{R}(\Gamma) &\longrightarrow \mathbb{C}^{4n} \\ \rho &\longmapsto (\rho(g_1), \cdots, \rho(g_n)) \end{aligned}$$

where each $\rho(g_i)$ is a 2 by 2 complex matrix. This will give $\overline{R}(\Gamma)$ the structure of an affine complex algebraic set whose defining polynomials are obtained by requiring that successive 4-tuples have determinant ± 1 and that the relators equal to the identity matrix.

Proposition 7.1.1. *(Culler and Shalen, 1983) The affine complex algebraic variety structure of the $PSL_2(\mathbb{C})$-representation variety $\overline{R}(\Gamma)$ is independent of the choices of generators and relators.*

Note that this algebraic set may have many irreducible components. However we have the following property from (Culler and Shalen, 1983).

Proposition 7.1.2. *(Culler and Shalen, 1983) Let V be an irreducible component of $\overline{R}(\Gamma)$. Then any representation equivalent to a representation in V must itself belong to V.*

We call a representation $\rho : \Gamma \to PSL_2(\mathbb{C})$ *reducible* if there exists a proper non-trivial vector sub-space of $\mathbb{C}^2$ which is invariant under ρ. We call ρ *irreducible* otherwise.

If a representation $\rho : \Gamma \to PSL_2(\mathbb{C})$ is *reducible* then there exists a basis of $\mathbb{C}^2$ on which ρ has the following form

$$\rho(g) = \begin{pmatrix} a_g & b_g \\ 0 & c_g \end{pmatrix}, \qquad g \in \Gamma.$$

The group $PSL_2(\mathbb{C})$ acts algebraically on $\overline{R}(\Gamma)$ by conjugation. Two representations are called equivalent if they are conjugate to each other. The set of equivalence classes of representations corresponds to the quotient $\overline{R}(\Gamma)//PSL_2(\mathbb{C})$, where the quotient is taken in the algebraic geometric category. In order to understand this set, Culler and Shalen introduced the $PSL_2(\mathbb{C})$-*character variety* of Γ using the *trace function.* For each representation $\rho \in \overline{R}(\Gamma)$, the $(PSL_2(\mathbb{C}))$*character* of ρ is the map χ_ρ defined by

$$\chi_\rho : \Gamma \to \mathbb{C}, \quad \chi_{\rho(g)} = \operatorname{trace}\left(\rho(g)\right)^2.$$

The set of all characters $\overline{X}(\Gamma) = \left\{\chi_\rho \mid \rho \in \overline{R}(\Gamma)\right\}$ is also a complex algebraic set in a natural way such that the following map is regular, in the sense of algebraic geometry,

$$\bar{t} : \overline{R}(\Gamma) \longrightarrow \overline{X}(\Gamma), \quad \bar{t}(\rho) = \chi_\rho.$$

Moreover its corresponds to the set of equivalence classes of representations under the action of $PSL_2(\mathbb{C})$ by inner automorphisms. Like $\overline{R}(\Gamma)$, $\overline{X}(\Gamma)$ may have many irreducible components.

Let $\overline{R}^{irr}(\Gamma)$ be the subset of irreducible representations and let $\overline{X}^{irr}(\Gamma) = \bar{t}\left(\overline{R}^{irr}(\Gamma)\right)$. Culler and Shalen have shown that $\overline{R}^{irr}(\Gamma)$ and $\overline{X}^{irr}(\Gamma)$ are Zariski open subsets of $\overline{R}^{irr}(\Gamma)$ and $\overline{X}^{irr}(\Gamma)$ respectively. Moreover $\bar{t}^{-1}\left(\overline{X}^{irr}(\Gamma)\right) = \overline{R}^{irr}(\Gamma)$.

Recall that a complex analytic variety is a set locally defined by zeros of holomorphic functions, which may have singularities. The map $\bar{t}$ has the following nice property.

Proposition 7.1.3. *(Culler and Shalen, 1983) The induced morphism $\bar{t} : \overline{R}^{irr}(\Gamma) \to \overline{X}^{irr}(\Gamma)$ is a principal analytic fibration with structure group $PSL_2(\mathbb{C})$.*

7.2 Twisted cohomology and tangent spaces

Twisted cohomology is useful for understanding the space $\overline{X}(\Gamma)$ locally and infinitesimally near a character χ_ρ. Recall that if we have a Γ-module $\mathfrak{M}$, then we can defined the group cohomology $H^*(\Gamma, \mathfrak{M})$ as follow. For each integer $n \geq 0$ define the cochain complex $C^n(\Gamma; \mathfrak{M})$ to be

$$C^n(\Gamma; \mathfrak{M}) := \{\text{map } \ \phi : \underbrace{\Gamma \times \cdots \times \Gamma}_{n \text{ times}} \to \mathfrak{M}\}$$

equipped with the differential $d_n : C^n(\Gamma; \mathfrak{M}) \to C^{n+1}(\Gamma; \mathfrak{M})$ defined by

$$\begin{aligned} d_n\phi \ (g_1, g_2, \cdots, g_{n+1}) &= g_1 \cdot \phi\,(g_2, \cdots, g_{n+1}) + (-1)^{n+1}\phi\,(g_1, g_2, \cdots, g_n) \\ &\quad + \sum_{i=1}^{n}(-1)^i \ \phi\,(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_{n+1}) \end{aligned}$$

Let $B^0(\Gamma, \mathfrak{M}) = \mathbf{0}$, $B^k(\Gamma, \mathfrak{M}) = \mathrm{Im}(d_{k-1})$ for $k > 0$ and $Z^n(\Gamma, \mathfrak{M}) = \ker(d_n)$. We define the n-th cohomology group of Γ associated to $\mathfrak{M}$ to be

$$H^n(\Gamma, \mathfrak{M}) := Z^n(\Gamma, \mathfrak{M})/B^n(\Gamma, \mathfrak{M}).$$

The module $C^0(\Gamma; \mathfrak{M})$ can be identified with the set of constants $\mathfrak{M}$. A 0-cochain m can be thought of as an element of $\mathfrak{M}$, so for every $g \in \Gamma$

$$d_0 m(g) = g \cdot m - m.$$

It follows that

$$Z^0(\Gamma, \mathfrak{M}) = \{m \in \mathfrak{M} \mid g \cdot m = m, \ \forall g \in \Gamma\}.$$

Thus $H^0(\Gamma; \mathfrak{M}) = Z^0(\Gamma; \mathfrak{M})$ is the set of Γ-invariant elements of $\mathfrak{M}$. On the other hand $B^1(\Gamma; \mathfrak{M})$ is the set of maps $\phi : \Gamma \to \mathfrak{M}$ defined by $\phi(g) = g \cdot m - m$ for some $m \in \mathfrak{M}$.

The module $C^1(\Gamma; \mathfrak{M})$ is the set of maps $\phi : \Gamma \to \mathfrak{M}$. The differential d_1 then gives

$$d_1\phi(g_1, g_2) = g_1 \cdot \phi(g_2) + \phi(g_1) - \phi(g_1 g_2).$$

Thus $Z^1(\Gamma; \mathfrak{M})$ is the set of maps ϕ which satisfy

$$\phi(g_1 g_2) = \phi(g_1) + g_1 \cdot \phi(g_2).$$

Going back to character varieties, a representation $\overline{\rho} \in \overline{R}(\Gamma)$ defines a morphism $Ad \circ \overline{\rho} : \pi_1(M) \longrightarrow \mathrm{Aut}(sl_2(C))$ by post-composing with the adjoint representation $Ad : PSL_2(\mathbb{C}) \longrightarrow \mathrm{Aut}(sl_2(C))$. The Lie-algebra $sl_2(\mathbb{C})$ then becomes a Γ-module. The cohomology $H^*(\Gamma; Ad_{\overline{\rho}})$ is defined to be the cohomology of Γ associated to this module.

Let $\overline{\rho} \in \overline{R}(\Gamma)$ and consider a path ρ_t in $\overline{R}(\Gamma)$, differentiable with respect to the parameter t. Up to first order we can expand ρ_t as

$$\rho_t(g) = \exp(tu(g) + o(t^2)) \ \rho(g)$$

for $g \in \Gamma$, t in some interval centred at 0 which depends on g, and $u : \Gamma \to sl_2(C)$. When we differentiate the homomorphism condition

$$\rho_t(gh) = \rho_t(g)\rho_t(h)$$

at $t = 0$, we get

$$\frac{\mathrm{d}[\rho_t(g)\rho_t(h)]}{\mathrm{d}t} = u(g)\rho(g)\rho(h) + \rho(g)u(h)\rho(h),$$
$$\frac{\mathrm{d}[\rho_t(gh)]}{\mathrm{d}t} = u(gh)\rho(gh) = u(gh)\rho(g)\rho(h).$$

Therefore

$$u(gh)\rho(g)\rho(h) = u(g)\rho(g)\rho(h) + \rho(g)u(h)\rho(h).$$

Multiplying both side by $\rho(h)^{-1}\rho(g)^{-1}$ on the right gives

$$u(gh) = u(g) + \rho(g)u(h)\rho(g)^{-1} = u(g) + Ad_{\rho(g)} \cdot u(h).$$

Thus $u \in Z^1(\Gamma; Ad_{\overline{\rho}})$, and it follows that the Zariski tangent space $T_{\overline{\rho}}^{Zar}\overline{R}(\Gamma)$ injects into $Z^1(\Gamma; Ad_{\overline{\rho}})$.

Definition 7.2.1. *A representation $\rho \in \overline{R}(\Gamma)$ is called scheme reduced if the inclusion of $T_{\overline{\rho}}^{Zar}\overline{R}(\Gamma)$ into $Z^1(\Gamma; Ad_{\overline{\rho}})$ is an isomorphism.*

For the character variety $\overline{R}(\Gamma)$ we can also show that $T_{\overline{\rho}}^{Zar}\overline{X}(\Gamma)$ injects into $H^1(\Gamma; Ad_{\overline{\rho}})$ when ρ is irreducible. To see this we can think of a tangent vector of $\overline{X}(\Gamma)$ at $\overline{\rho}$ as an element of $T_{\overline{\rho}}^{Zar}\overline{R}(\Gamma)$ modulo the tangent space of the orbit $\Gamma \cdot \rho$. To compute the tangent space of the orbit consider a deformation ρ_t induced by a differentiable path $g_t \in PSL_2(\mathbb{C})$:

$$\rho_t(x) = g_t^{-1}\rho(x)g_t, \quad x \in \Gamma.$$

Then if we expand g_t as $g_t = \exp(tu_0 + o(t^2))$ where $u_0 \in sl_2(\mathbb{C})$ and differentiate ρ_t at $t = 0$ we get

$$\frac{\mathrm{d}\rho_t(g)}{\mathrm{d}t}_{|t=0} = \frac{\mathrm{d}g_t^{-1}}{\mathrm{d}t}_{|t=0} \rho(x)g_0 + g_0^{-1}\rho(x) \frac{\mathrm{d}g_t}{\mathrm{d}t}_{|t=0} = -u_0\, \rho(x) + \rho(x)\, u_0.$$

Therefore a tangent vector u of the orbit can be written

$$u(x) = \frac{\mathrm{d}\rho_t(g)}{\mathrm{d}t}_{|t=0}\rho^{-1}(x) = \rho(x)\ u_0\rho(x)^{-1} - u_0 = Ad_{\rho(x)} \cdot u_0 - u_0.$$

It follows that u is the coboundary $du_0 \in B^1(\Gamma; Ad_{\overline{\rho}})$. This gives an inclusion of $T^{Zar}_{\overline{\rho}}\overline{X}(\Gamma)$ into $H^1(\Gamma; Ad_{\overline{\rho}}) = Z^1(\Gamma; Ad_{\overline{\rho}})/B^1(\Gamma; Ad_{\overline{\rho}})$.

Definition 7.2.2. *A character $\chi_\rho \in \overline{X}(\Gamma)$ is called scheme reduced if the inclusion of $T^{Zar}_{\overline{\rho}}\overline{X}(\Gamma)$ into $H^1(\Gamma; Ad_{\overline{\rho}})$ is an isomorphism.*

7.3 Character varieties and Culler-Shalen norm

By fixing a base point on ∂M we have a morphism $\pi_1(\partial M) \longrightarrow \pi_1(M)$. Using the Hurewicz isomorphism we get an identification $\pi_1(\partial M) \cong H_1(\partial M)$ and therefore a morphism $H_1(\partial M) \longrightarrow \pi_1(M)$. Since $\mathrm{int}(M)$ is assumed to be a one-cusped complete finite volume hyperbolic 3-manifold, this morphism is injective. We therefore think of a slope as an element of $\pi_1(\partial M)$, $\pi_1(M)$ or $H_1(\partial M)$ interchangeably. The main references for this section are (Ben Abdelghani and Boyer, 2001), (Shalen, 2002) and (Boyer and Zhang, 1998).

For simplicity we will denote by $\overline{R}(M)$ (resp. by $\overline{X}(M)$) the $PSL_2(\mathbb{C})$-representation variety $\overline{R}(\pi_1(M))$ (resp. the $PSL_2(\mathbb{C})$-character variety $\overline{X}(\pi_1(M))$). We also denote $H^*(M; Ad_{\overline{\rho}})$ the corresponding twisted cohomology.

For each $\gamma \in \pi_1(M)$ we have the following function

$$f_\gamma : \overline{X}(M) \longrightarrow \mathbb{C}, \quad f_\gamma(\chi) = \mathrm{trace}(\rho(\gamma))^2 - 4 = \chi(\gamma) - 4.$$

The function f_γ is a regular function and the zeros of f_γ are the characters of representations $\overline{\rho}$ for which $\overline{\rho}(\gamma)$ is parabolic or $\overline{\rho}(\gamma) = [\pm\mathrm{Id}]$. We will use the same notation f_γ for the restriction of f_γ to a curve $X_0 \subset \overline{X}(M)$. For our purposes, we need f_γ to be a non-constant function. This is provided by the following lemma which is a variation of Corollary 4.5.2 of (Shalen, 2002).

Lemma 7.3.1. *Let N be a connected orientable one-cusped hyperbolic 3-manifold of finite volume. Then there is a 1-dimensional irreducible component X_0 of $\overline{X}(N)$, containing the character of a representation associated to the hyperbolic structure of N, such that if γ is any non-trivial element of $\pi_1(N)$ carried in the boundary of the compact core of N, the function f_γ is non-constant on X_0.*

Proof. The proof was done in (Shalen, 2002) for the case of $SL_2(\mathbb{C})$ representations. Now if $X(N)$ is the $SL_2(\mathbb{C})$ character variety then we have a finite-to-one "algebraic" morphism $\pi : X(N) \to \overline{X}(N)$ which induces an inclusion $X(N)//H^1(N, \mathbb{Z}/2\mathbb{Z}) \cong \overline{X}(N)$. Therefore π sends one-dimensional components of $X(N)$ to one dimensional components of $\overline{X}(N)$. Since the expression for f_γ is the same for $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ character varieties, the result still holds for $PSL_2(\mathbb{C})$. □

Here by the core of N one means a compact manifold M with boundary such that $\text{int}(M) \cong N$. Let $X_0 \subset \overline{X}(M)$ be a *non-trivial* irreducible component. Here *non-trivial* means that it contains the character of an irreducible representation. Let $\widehat{X}_0$ be the normalized projective completion of X_0. There is an isomorphism between function fields

$$\mathbb{C}(X_0) \cong \mathbb{C}(\widehat{X}_0), \quad f \longmapsto \widehat{f}.$$

We can then define the degree of f to be the degree of $\widehat{f}$. For $x \in \widehat{X}$ we denote by $Z_x(\widehat{f}_\gamma)$ the multiplicity of x as a zero of $\widehat{f}_\gamma$. By convention $Z_x(\widehat{f}_\gamma) = \infty$ if $\widehat{f}_\gamma \equiv 0$. Now we denote $\Lambda = \pi_1(\partial M)$ seen as a subgroup of $\pi_1(M)$. We can also think of Λ as a lattice in $H_1(\partial M, \mathbb{R})$. An element $\gamma \in \Lambda$ satisfies, see (Shalen, 2002),

$$\deg(\widehat{f}_\gamma) = \sum_{x \in \widehat{X}_0} Z_x(\widehat{f}_\gamma).$$

The degree is finite if $\widehat{f}_\gamma$ is non-constant on $\widehat{X}_0$, for instance if X_0 is as in Lemma 7.3.1. The key property of $\deg(\widehat{f}_\gamma)$ is that for each curve $X_0 \subset X(M)$ it defines a semi-norm $||.||_{X_0}$ on $H_1(\partial M, \mathbb{R})$ which for each $\gamma \in \Lambda$ satisfies

$$||\gamma||_{X_0} = \begin{cases} 0 \text{ if } & f_\gamma | X_0 \text{ is constant} \\ \deg(\widehat{f}_\gamma) \text{ if } & f_\gamma | X_0 \neq 0. \end{cases}$$

See (Boyer and Zhang, 1998) for instance. This semi-norm is called the *Culler-Shalen semi-norm* associated to the curve X_0. There is at least one curve for which $||.||_{X_0}$ is a norm. This is given in the following proposition which is a summary of Property 8.1.10, Property 8.1.11 and Equation (9.2.2) of (Shalen, 2002).

Proposition 7.3.2. *Let X_0 be as in Lemma 7.3.1. Then the semi-norm $||.||_{X_0}$ is a norm. Moreover the unit ball is a compact convex set with boundary a finite sided balanced polygon.*

Note that if B_r is the ball of radius r centred at the origin, then B_r can be view as the unit ball for the norm $\frac{1}{r}||.||_{X_0}$ therefore B_r has the same properties as the unit ball.

Let $X_1, \cdots, X_k$ be all the non-trivial irreducible curve components in $\overline{X}(M)$. We can define an "absolute" semi-norm $||.||$ on $H_1(\partial M, \mathbb{R})$ by

$$||.|| = ||.||_{X_1} + \cdots + ||.||_{X_k}.$$

Proposition 7.3.2 then implies that this is a norm. We will call this norm the *absolute Culler-Shalen norm* or the absolute norm if there is no risk of confusion.

Let $X_0 \subset \overline{X}(M)$ be a curve. There is a unique 4-dimensional subvariety $R_0 \subset \overline{R}(M)$ for which $t(R_0) = X_0$, see Lemma 4.1 of (Boyer and Zhang, 1998). If

$\overline{\rho}(\alpha) = I$ for some slope α, then we have an induced representation $\overline{\rho}' : \pi_1(M(\alpha)) \longrightarrow PSL_2(C)$ and a cohomology group $H^1(M(\alpha); Ad_{\overline{\rho}'})$.

Let X_0 be as in Lemma 7.3.1 and $\nu : X_0^\nu := \widehat{X}_0 \setminus \{\text{ideal points}\} \longrightarrow X_0$ be the map which corresponds to the affine normalization of X_0. There is an affine normalization $R_0^\nu \longrightarrow R_0$, which we still denote by ν, such that the following diagram commutes.

$$\begin{tikzcd} R_0^\nu \arrow[r, "\nu"] \arrow[d, "\overline{t}^\nu"'] & R_0 \arrow[d, "\overline{t}"] \\ X_0^\nu \arrow[r, "\nu"'] & X_0 \end{tikzcd}$$

The map t^ν and ν are all surjective, see (Culler et al., 1987).

Let $N \subset PSL_2(\mathbb{C})$ denote the subgroup

$$\left\{ \pm \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \pm \begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix} \;\middle|\; z, w \in \mathbb{C}^* \right\}$$

For each $\gamma \in \pi_1(M)$ we are going to consider the following subset of $\overline{X}(M)$:

$$A(\gamma) = \{\chi_{\overline{\rho}} \in \overline{X}(M) \mid \overline{\rho}(\gamma) = \pm I; \quad \overline{\rho} \text{ is non-abelian and conjugates into } N\}.$$

$$B(\gamma) = \{\chi_{\overline{\rho}} \in \overline{X}(M) \mid \overline{\rho}(\gamma) = \pm I; \quad \overline{\rho} \text{ is non-abelian and does not conjugates into } N\}.$$

Note that elements of $A(\gamma)$ must be irreducible but not necessarily those of $B(\gamma)$.

We now state the next theorem needed for the proof of Theorem B in the case of small-Seifert filling. It is from (Ben Abdelghani and Boyer, 2001) Theorem 2.1.

Theorem 7.3.3. *Fix a slope α on ∂M and consider a non-trivial, irreducible curve $X_0 \subset \overline{X}(M)$. Suppose that $x \in \widehat{X}_0$ is not an ideal point and corresponds to a character χ_ρ for some representation $\overline{\rho} \in R_0$ with non-abelian image and which satisfies $\rho(\alpha) \in \{\pm I\}$. Assume that $H^1(M(\alpha); Ad_\rho) = 0$ and $\rho(\pi_1(\partial M)) \not\subseteq \{\pm I\}$.*

1. *If $\beta \in \pi_1(\partial M)$ and $\rho(\beta) \neq \pm I$, then*

$$Z_x(\widehat{f}_\alpha) \geq \begin{cases} Z_x(\widehat{f}_\beta) + 1 & \text{if } \overline{\rho} \text{ conjugates into } N, \\ Z_x(\widehat{f}_\beta) + 2 & \text{otherwise.} \end{cases}$$

2. *If $\beta \in \pi_1(\partial M)$ and $Z_x(\widehat{f}_\alpha) > Z_x(\widehat{f}_\beta)$, then $\widehat{f}_\alpha | X_0 \neq 0$, $\rho(\beta) \neq \pm I$ and*

$$Z_x(\widehat{f}_\alpha) = \begin{cases} Z_x(\widehat{f}_\beta) + 1 & \text{if } \overline{\rho} \text{ conjugates into } N, \\ Z_x(\widehat{f}_\beta) + 2 & \text{otherwise.} \end{cases}$$

The condition $\rho(\pi_1(\partial M)) \not\subseteq \{\pm I\}$ may not be satisfied in general. We will need the auxiliary assumption that the manifold Y has only diagonalisable $PSL_2(\mathbb{C})$ representations.

The following proposition is Proposition 1.5.4 of (Culler et al., 1987).

Proposition 7.3.4. *Let α and β be non-zero elements of Λ. Suppose that x is a point of X_0^ν such that $Z_x(f_\alpha) > Z_x(f_\beta)$. Then for every $\tilde{\rho} \in R_0^\nu$ with $t^\nu(\tilde{\rho}) = x$, the representation $\rho = \nu(\tilde{\rho})$ satisfies $\rho(\alpha) = \pm I$.*

We end this section by a result about zeros at ideal points. Recall that a slope α on ∂M is a "*boundary slope*" if it is the slope of the boundary of an essential surface. We say that α is a "*strict boundary slope*" if it is the slope of the boundary of an essential surface which is not a (semi) fibre in any (semi) fibration of M over S^1. A slope α not being a strict boundary slope means that if it is the slope of the boundary of an essential surface, then that surface is a fibre in a fibration of M over S^1. The following proposition comes from Proposition 1.6.1 of (Culler et al., 1987).

Proposition 7.3.5. *Let x be an ideal point of $\widehat{X}_0$. Let α and β be non-zero elements of Λ. Suppose that α is primitive and is not a boundary class, and that*

$$Z_x(f_\alpha) > Z_x(f_\beta).$$

Then there is a closed essential surface in M which is incompressible in $M(\alpha)$.

7.4 Proof of theorem B

Before doing the proof we give two preliminary lemmas.

Lemma 7.4.1. *Assume that* $\mathrm{rank}_{\mathbb{Z}}(H_1(M)) = 1$. *Then for each ordinary point $x \in \widehat{X}_0$ there is a representation $\overline{\rho} \in R_0$, with non-abelian image, such that $\chi_{\overline{\rho}} = \nu(x)$.*

Proof. Let $Z_0 \subset X(M)$ ($SL_2(\mathbb{C})$-character variety) be an irreducible curve component of $\pi^{-1}(X_0)$, and S_0 a component of $t^{-1}(Z_0)$.

Let $X(\Gamma)$ be the $SL_2(\mathbb{C})$-character variety of a finitely generated group Γ. In (Boyer, 2002) Proposition 2.8 it is shown that if x is a reducible trivial character in a non-trivial curve inside $X(\Gamma)$ then $b_1(\Gamma) \geq 2$. Here since $\mathrm{int}(M)$ is a one-cusped complete finite volume hyperbolic manifold, $b_1(\pi_1(M)) = \mathrm{rank}_{\mathbb{Z}}(H_1(M))$. Since $\mathrm{rank}_{\mathbb{Z}}(H_1(M)) = 1$ by assumption, any character in a non-trivial curve inside $X(M)$ is non-trivial, in particular any element of Z_0 is non-trivial. The same Proposition 2.8 of (Boyer, 2002) applied to $\pi_1(M)$ implies that if a character $z \in Z_0$ is non-trivial then there is a representation $\rho \in S_0 \cap t^{-1}(z)$ with non-abelian image. Since for each $x \in \widehat{X}_0$, $\nu(x) \in X_0$ we can take $z \in \pi^{-1}(\nu(x))$ to get such a representation ρ and then take the corresponding $PSL_2(\mathbb{C})$ representation $\overline{\rho}$. $\square$

Recall that we have chosen M to be the complement of regular neighbourhood of a knot in a rational homology 3-sphere. Therefore M satisfies $\text{rank}_{\mathbb{Z}}\left(H_1(M)\right) = 1$.

Lemma 7.4.2. *If $\alpha \in \Lambda$ is not a boundary slope then $||\alpha||_{X_0} \neq 0$ for each curve $X_0 \subset X(M)$ for which $||.||_{X_0} \not\equiv 0$.*

Proof. See the proof of Proposition 5.4 and 5.5 in (Boyer and Zhang, 1998).

□

Lemma 7.4.3. *Let α and β be two slopes on ∂M such that $\pi_1(M(\alpha)) \cong \pi_1(M(\beta))$. Then there is a one to one correspondence between $A(\alpha)$ and $A(\beta)$, and between $B(\alpha)$ and $B(\beta)$,*

Proof. Let $\Psi : \pi_1(M(\beta)) \to \pi_1(M(\alpha))$ be an isomorphism, $p_\alpha : \pi_1(M) \to \pi_1(M(\alpha))$, and $p_\beta : \pi_1(M) \to \pi_1(M(\beta))$ be the obvious projections. Let $\chi_{\overline{\rho}} \in A(\alpha)$, we have a representation $\Phi_\alpha(\overline{\rho}) : \pi_1(M(\alpha)) \to PSL_2(\mathbb{C})$ obtained via the following factorisation of $\overline{\rho}$

$$\begin{tikzcd}
\pi_1(M) \arrow[d, "p_\alpha"'] \arrow[dr, "\overline{\rho}"] & \\
\pi_1(M(\alpha)) \arrow[r, dashed, "\Phi_\alpha(\overline{\rho})"'] & PSL_2(\mathbb{C})
\end{tikzcd}$$

We also have an equivalent representation $\Phi_\beta(\overline{\rho}) : M(\beta) \to PSL_2(\mathbb{C})$ for the β-case. Let $\overline{\rho}'$ be the composition $\overline{\rho}' := \Phi(\overline{\rho}) \circ \Psi \circ p_\beta$

$$\begin{tikzcd}
\pi_1(M) \arrow[r, dashed, "\overline{\rho}'"] \arrow[d, "p_\beta"'] & PSL_2(\mathbb{C}) \\
\pi_1(M(\beta)) \arrow[r, "\Psi"'] & \pi_1(M(\alpha)) \arrow[u, "\Phi(\overline{\rho})"']
\end{tikzcd}$$

The maps p_β, p_α and Ψ are all surjective so im$\overline{\rho}$ = im$\overline{\rho}'$. In particular if $\overline{\rho}$ does not conjugate into N then neither does $\overline{\rho}'$, and if $\overline{\rho}$ is irreducible then so is $\overline{\rho}'$. The representation $\overline{\rho}'$ satisfies $\overline{\rho}'(\beta) = \pm I$ by construction. Next we need to check that if $\chi_{\overline{\rho_1}} = \chi_{\overline{\rho}_2}$ then $\chi_{\overline{\rho}'_1} = \chi_{\overline{\rho}'_2}$.

We first assume that $\chi_{\overline{\rho}_1}$ is an irreducible character. Therefore $\overline{\rho}_2 = \overline{g}\,\overline{\rho}_1\,\overline{g}^{-1}$ for some

$g \in SL_2(\mathbb{C})$. Then we deduce that

$$\begin{aligned}
\overline{\rho}'_2 &= \Phi_\alpha\left(\overline{\rho}_2\right) \circ \Psi \circ p_\beta \\
&= \Phi_\alpha\left(\overline{g}\,\overline{\rho}_1\,\overline{g}^{-1}\right) \circ \Psi \circ p_\beta \\
&= \left(\overline{g}\;\Phi_\alpha(\overline{\rho}_1)\;\overline{g}^{-1}\right) \circ \Psi \circ p_\beta \\
&= \overline{g}\;\left(\Phi_\alpha(\overline{\rho}_1) \circ \Psi \circ p_\beta\right)\;\overline{g}^{-1} \\
&= \overline{g}\;\overline{\rho}'_1 \overline{g}^{-1}
\end{aligned}$$

which implies $\overline{\rho}'_2 = \overline{g}\;\overline{\rho}'_1\;\overline{g}^{-1}$. Therefore $\chi_{\overline{\rho}'_1} = \chi_{\overline{\rho}'_2}$. If $\gamma \in \pi_1(\partial M)$ we denote $X^{irr}(\gamma)$ and $X^{red}(\gamma)$ the sets

$$X^{irr}(\gamma) = \left\{\chi_{\overline{\rho}} \in \overline{X}(M) \mid \overline{\rho}(\gamma) = \pm I, \quad \text{and } \overline{\rho} \text{ is irreducible}\right\}$$

$$X^{red}(\gamma) = \left\{\chi_{\overline{\rho}} \in \overline{X}(M) \mid \overline{\rho}(\gamma) = \pm I, \quad \text{and } \overline{\rho} \text{ is reducible}\right\}.$$

We then have a well defined map

$$F : X^{irr}(\alpha) \longrightarrow X^{irr}(\beta), \quad \overline{\rho} \longmapsto \overline{\rho}'.$$

The map F sends $A(\alpha)$ to $A(\beta)$, and $B(\alpha)$ to $B(\beta)$. The next step is to extend F to the reducible representations.

Now assume $x = \chi_{\overline{\rho}_1} = \chi_{\overline{\rho}_2}$ is a reducible character. By analogy with (Boyer, 2002) there exists a representation $a : \pi_1(M) \to \mathbb{C}^*$ such $t^{-1}(x) = R^a_x \cup R^{a^{-1}}_x$ where

$$\begin{aligned} R^a_x &= \left\{A\overline{\rho}A^{-1} | A \in PSL_2(\mathbb{C}),\ \overline{\rho} \in U^a_x\right\}, \\ R^{a^{-1}}_x &= \left\{A\overline{\rho}A^{-1} | A \in PSL_2(\mathbb{C}),\ \overline{\rho} \in U^{a^{-1}}_x\right\} \\ U^a_x &= \left\{ \text{representation } \ \overline{\rho} = \pm \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}, \\ U^{a^{-1}}_x &= \left\{ \text{representation } \ \overline{\rho} = \pm \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} \right\} \end{aligned}$$

If $\overline{\rho}_1$ and $\overline{\rho}_2$ are conjugate, by the same argument as for irreducible characters $\chi_{\overline{\rho}'_1} = \chi_{\overline{\rho}'_2}$. Assume that $\overline{\rho}_1$ and $\overline{\rho}_2$ are not conjugate. Without loss of generality we can suppose that

$$\overline{\rho}_1 = \pm \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \quad \text{and } \overline{\rho}_2 = \pm A \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} A^{-1}, \quad \text{for some } \ A \in PSL_2(\mathbb{C}).$$

Since $\Phi(\overline{\rho}_i)$ and $\overline{\rho}_i$, $\ i = 1, 2$ have the same image we can use the same matrices to represent $\Phi(\overline{\rho}_i)$, $\ i = 1, 2$. Hence

$$\overline{\rho}'_1 = \Phi(\overline{\rho}_1) \circ \Psi \circ p_\beta = \pm \begin{pmatrix} a \circ \Psi \circ p_\beta & b \circ \Psi \circ p_\beta \\ 0 & a^{-1} \circ \Psi \circ p_\beta \end{pmatrix}$$

$$\text{and } \ \overline{\rho}'_2 = \Phi(\overline{\rho}_2) \circ \Psi \circ p_\beta = \pm A \begin{pmatrix} a^{-1} \circ \Psi \circ p_\beta & b \circ \Psi \circ p_\beta \\ 0 & a \circ \Psi \circ p_\beta \end{pmatrix} A^{-1}.$$

Therefore

$$\text{trace}(\rho_1') = \pm\left[a \circ \Psi \circ p_\beta + a^{-1} \circ \Psi \circ p_\beta\right]$$

$$\text{trace}(\rho_2') = \pm\left[a^{-1} \circ \Psi \circ p_\beta + a \circ \Psi \circ p_\beta\right]$$

It follows that $\text{trace}(\rho_1')^2 = \text{trace}(\rho_2')^2$, that is $\chi_{\rho_1'} = \chi_{\rho_2'}$. Thus F is well defined on the set of reducible characters.

Finally, we show that F is bijective. If $\overline{\rho}' \in X^{irr}(\beta)$ (resp. $X^{red}(\beta)$) , we get $\overline{\rho} \in X^{irr}(\alpha)$ (resp. $X^{red}(\alpha)$) as follow: we first define $\Phi_\alpha(\overline{\rho})$ to be $\Phi_\alpha(\overline{\rho}) = \Phi_\beta(\overline{\rho}') \circ \Psi^{-1}$ then $\overline{\rho}' = \Phi_\alpha(\overline{\rho}) \circ p_\alpha$. This uniquely determine $\overline{\rho}$, therefore the map F is bijective. $\square$

It is a known from (Cooper et al., 1994) 2.4 that $X(M)$ has no 0-dimensional component. Let $X_1, \cdots, X_k$ be the curve components of $X(M)$. If $\alpha \in \Lambda$ is not a boundary slope, then Lemma 7.4.2 allows us to write the X_i-norm of α in terms of the zeros of $f_\alpha|_{X_i}$ for each $i \in \{1, \cdots, k\}$

$$||\alpha||_{X_i} = \sum_{x \in \widehat{X_i}} Z_x(\widehat{f_\alpha}|_{\widehat{X_i}}).$$

Let $\widehat{X}$ be the abstract disjoint union of all the $\widehat{X_i}$, $i \in \{1, \cdots, k\}$, then we have the following formula for the absolute norm

$$||\alpha|| = \sum_{x \in \widehat{X}} Z_x(\widehat{f_\alpha})$$

where $\widehat{f_\alpha}$ is understood to be the restriction to the appropriate component. Let $x \in \widehat{X}$, we define the number m_x and m_0 to be

$$m_x = \min\left\{Z_x(\widehat{f_\gamma}) \mid \gamma \in \Lambda \setminus \{0\}\right\}, \quad \text{and} \quad m_0 = \sum_{x \in \widehat{X}} m_x.$$

We can then deduce

$$||\alpha|| = m_0 - m_0 + \sum_{x \in \widehat{X}} Z_x(\widehat{f}_\alpha) = m_0 + \sum_{x \in \widehat{X}} \left(Z_x(\widehat{f}_\alpha) - m_x \right).$$

Let us suppose that x is an ideal point of $\widehat{X}_0 \subset \widehat{X}$. If $Z_x(\widehat{f}_\alpha) - m_x > 0$ then $Z_x(\widehat{f}_\alpha) > Z_x(\widehat{f}_\gamma)$ for some $\gamma \in \Lambda \setminus \{0\}$. Since α is primitive and is not a boundary class, Lemma 7.3.5 implies that there is a closed surface in M which is incompressible in $M(\alpha)$. This situation does not occur if we assume $M(\alpha)$ is small-Seifert with $b_1(M(\alpha)) = 0$. Therefore we always have $Z_x(\widehat{f}_\alpha) - m_x = 0$ at an ideal point.

Let $x \in \widehat{X}$ be an ordinary point, x is contained in some $\widehat{X}_0$ and by Lemma 7.4.1 there is a representation $\overline{\rho} \in R_0$ with non-abelian image, such that $\chi_{\overline{\rho}} = \nu(x)$. Let $\tilde{\rho} = \nu^{-1}(\rho)$, we have the following equality

$$\nu\left(t^\nu(\tilde{\rho})\right) = t\left(\rho\right) = \nu(x).$$

The normalization map $\nu : X_0^\nu \to X_0$ is an "isomorphism" outside singular points, so if x is a smooth point then $t^\nu(\tilde{\rho}) = x$. This smoothness is provided by Theorem A of (Boyer, 2002). A direct consequence of this is that for an ordinary point x, $\nu(x)$ is contained in only one irreducible component. Therefore if we consider instead of $\widehat{X}$, the "natural" union $\widehat{X_1} \cup \cdots \cup \widehat{X_k}$, we can write the absolute norm of α as

$$||\alpha|| = \sum_{x \in \widehat{X_1} \cup \cdots \cup \widehat{X_k}} Z_x(\widehat{f}_\alpha).$$

Now if we assume that $Z_x(\widehat{f}_\alpha) > m_x$ then by Lemma 7.3.4 the representation $\rho = \nu\left(\tilde{\rho}\right)$ satisfies $\rho(\alpha) = \pm I$. If we add the extra condition that Y have only Abelian $PSL_2(\mathbb{C})$-representations then $\rho(\Lambda) \nsubseteq \{\pm I\}$ and all the hypothesis of

Theorem 7.3.3 are satisfied. In particular if $m_x = Z_x(\widehat{f_\beta})$ for some $\beta \in \Lambda \setminus \{0\}$ then $\widehat{f_\alpha}|X_0 \neq 0$, $\rho(\beta) \neq \pm I$ and

$$\begin{cases} Z_x(\widehat{f_\alpha}) - m_x = Z_x(\widehat{f_\alpha}) - Z_x(\widehat{f_\beta}) = 1 & \text{if } \overline{\rho} \text{ conjugates into } D, \\ Z_x(\widehat{f_\alpha}) - m_x = Z_x(\widehat{f_\alpha}) - Z_x(\widehat{f_\beta}) = 2 & \text{otherwise.} \end{cases}$$

Therefore

$$||\alpha|| = \overline{m}_0 + A(\alpha) + 2B(\alpha).$$

Where

$$\overline{m}_0 = \sum_{x \in \widehat{X_1} \cup \cdots \cup \widehat{X_k}} m_x.$$

The next lemma is a straightforward consequence of this formula.

Lemma 7.4.4. *Let α be a slope on ∂M which is not a boundary slope. Assume that $M(\alpha)$ is small-Seifert, Y has only diagonalisable $PSL_2(\mathbb{C})$-representations and $b_1(M(\alpha)) = 0$. If β is a slope such that $M(\beta) \cong M(\alpha)$, then $||\beta|| = ||\alpha||$.*

Proof. This follow from Lemma 7.4.3, $\sharp A(\alpha) = \sharp A(\beta)$ and $\sharp B(\alpha) = \sharp B(\beta)$. □

The last step before going into the proof of Theorem B is the following result of Culler, Gordon, Luecke and Shalen.

Theorem 7.4.5. *(Culler et al., 1987) Suppose that $H_1(M;\mathbb{Q})$ is one dimensional. If α is a boundary slope, then either*

(i) $M(\alpha)$ contains a closed essential surface of strictly positive genus.

(ii) $M(\alpha)$ is the connected sum of two lens spaces.

(iii) There is a closed essential surface $S \subset M$ which compresses in $M(\alpha)$ but which remains incompressible in $M(\delta)$ as long as $\Delta(\alpha, \delta) > 1$.

(iv) $M(\alpha) \cong S^1 \times S^2$.

Proof of Theorem B. Let α be an exceptional slope on ∂M and $\beta \in C(\alpha)$.

First we deal with the case $b_1(M(\alpha)) \neq 0$. Since M is the complement of regular neighbourhood of a knot in a rational homology 3-sphere, $b_1(M) = \mathrm{rank}_{\mathbb{Z}}(H_1(M)) = 1$ and so is $b_1(M(\alpha))$. Now let λ be the preferred longitude of ∂M. Since $b_1(M(\alpha)) = 1$, we must have $\alpha = q\lambda$ and $\beta = q'\lambda$ for some integer q and q'. Then because α and β are primitive elements, $\beta = \pm\alpha$.

We can now assume that $b_1(M(\alpha)) = 0$. Let us suppose that α is a boundary slope. By Theorem 7.4.5 we have the following possibilities:

(i) $M(\alpha)$ contains a closed essential surface of strictly positive genus.

(ii) $M(\alpha)$ is the connected sum of two lens spaces.

(iii) There is a closed essential surface $S \subset M$ which compresses in $M(\alpha)$ but which remains incompressible in $M(\delta)$ as long as $\Delta(\alpha, \delta) > 1$.

(iv) $M(\alpha) \cong S^1 \times S^2$.

Since $M(\alpha)$ is small-Seifert with $b_1(M(\alpha)) = 0$, only (iii) can occur. Then the fact that $M(\alpha) \cong M(\beta)$ implies that S also compresses in $M(\beta)$ so $\Delta(\alpha, \beta) \leq 1$. The condition $\Delta(\alpha, \beta) \leq 1$ implies that there are at most three of such slopes. Assume $\beta \in C(\alpha)$ is distinct from α, then $\Delta(\alpha, \beta) = 1$ and we have either

$$C(\alpha) = \{\alpha, \beta, \alpha + \beta\}, \quad \text{or} \quad C(\alpha) = \{\alpha, \beta\}.$$

Now recall from Lemma 1.1.4 that there is a constant c_M independent of α such that

$$|H_1(M(\alpha);\mathbb{Z})| = c_M \Delta(\alpha, \lambda_M).$$

Therefore since $H_1(M(\alpha);\mathbb{Z}) = H_1(M(\beta);\mathbb{Z}) = H_1(M(\alpha+\beta);\mathbb{Z})$, we must have

$$\Delta(\alpha, \lambda_M) = \Delta(\beta, \lambda_M) = \Delta(\alpha+\beta, \lambda_M).$$

Thus all three elements $\alpha, \beta, \alpha+\beta$ lie on the same line l in $\mathbb{R}^2$ which is at fixed distance from λ_M. This is impossible since α and β are linearly independent. It follows that the first case cannot occur and we have:

$$C(\alpha) = \{\alpha, \beta\}.$$

Now if α is not a boundary slope we can apply Lemma 7.4.4 to obtain

$$||\alpha|| = ||\beta||.$$

Let $r = ||\alpha||$, then every element of $C(\alpha)$ lies on the boundary of the ball $B(0,r)$ of the absolute norm. On the other hand they must all have the same μ component since they give homeomorphic manifolds after Dehn filling. Hence they also lie on a line l parallel to λ. Since $B(0,r)$ is convex, $\partial B(0,r) \cap l$ has at most two points unless l contains a side of $\partial B(0,r)$. $\square$

CONCLUSION

This study has offered new perspectives on cosmetic surgeries. For a hyperbolic knot K in S^3, the result that exceptional truly cosmetic surgeries must have slope ± 1 is a new advance on the topic, together with the finding that the manifold obtained is toroidal but not Seifert fibred. One consequence of the latter is the existence of an essential separating torus in both $S^3_K(-1)$ and $S^3_K(1)$. To rule out the latter possibility appears to require refinements in Heegaard Floer theory. The other ingredient, graph intersection theory, also has its limitations in this context since $\Delta(+1,-1) = 2$. In fact the main results on the existence of pairs $\{r, s\}$ of toroidal surgery slopes using this technique require $\Delta(r, s) \geq 4$. Indeed the smaller the distance between the slopes the more complicated the graphs become. To push the study further we then need to get around these problems or to devise a new approach to the subject.

For the case of exceptional truly cosmetic surgeries on integer homology spheres, we have a precise list of possibilities which suggest in which directions we should look for a more elaborate investigation of the cosmetic surgery conjecture. The fact that in most cases the result of the surgery has to be an integer homology sphere is notable as well. In particular the slope must be of the form $1/q$, except, possibly, for atoroidal small Seifert surgeries. Therefore investigating cosmetic $1/q$ surgeries for some integer q will give more insight into the problem. Some tools from Heegaard Floer theory are available in this situation. Nonetheless we have the same challenges as for S^3 in terms of the techniques used.

The examination of small Seifert cosmetic surgeries on rational homology sphere done in Chapter 7 is quite particular since it uses character varieties, a totally different method. Contrary to Heegaard Floer homology, with this tool we do not need the knot to be null-homologous when we work in $\mathbb{Q}$-homology spheres. Graph intersection methods were also excluded here since we do not look for a pair of slopes each giving an essential surface. However some of the assumptions in Theorem B seem too technical. In order to get around this issue we may need to go deeper into the theory of the theory of $PSL_2(\mathbb{C})$-character varieties of 3-manifolds. Theorem B is a starting point for the study of cosmetic surgery on manifold Y with $b_1(Y) = 0$ but $H_1(Y) \neq 0$. We can think of it as a rational homology analogue of Theorem 1.3.8 of Zhongtao Wu or part (a) of Theorem 1.3.7 of Zhongtao Wu and Yi Ni, modulo suitable hypotheses.

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