Minimal and maximal elements in two-sided cells of S_n and Robinson-Schensted correspondence

Christophe Hohlweg

Institut de Recherche Mathématique Avancée,
Université Louis Pasteur et CNRS,
7,rue René Descartes
67084 Strasbourg, France
hohlweg@math.u-strasbg.fr

February 1, 2008

Abstract

In symmetric groups, a two-sided cell is the set of all permutations which are mapped by the Robinson-Schensted correspondence on a pair of tableaux of the same shape. In this article, we show that the set of permutations in a two-sided cell which have a minimal number of inversions is the set of permutations which have a maximal number of inversions in conjugated Young subgroups. We also give an interpretation of these sets with particular tableaux, called reading tableaux. As corollary, we give the set of elements in a two-sided cell which have a maximal number of inversions.

1 Introduction

In this article, we consider the symmetric group S_n. For $w \in S_n$, *the length of w*, denoted $\ell(w)$, is the number of inversions of w.

The Robinson-Schensted correspondence [19] is the well-known bijection $\pi : w \mapsto (P(w), Q(w))$ between S_n and pairs of standard tableaux of the same shape (a partition of n). For each partition λ of n, we denote by $\mathcal{T}^\lambda$ the set of all permutations which are mapped by π on a pair of tableaux of shape λ. In the Kazhdan-Lusztig theory, which we use in this article, the sets $\mathcal{T}^\lambda$ are called *two-sided cells* (see [11, 22, 1]).

Our goal is to describe, for any partition λ of n, the set $\mathcal{T}^\lambda_{min}$ of elements of minimal length in $\mathcal{T}^\lambda$ and the set $\mathcal{T}^\lambda_{max}$ of elements of maximal length in $\mathcal{T}^\lambda$.

For each composition $c = (n_1, \ldots, n_k)$ of n (with $n_i \geq 1$), the *Young subgroup* $S_c = S_{n_1} \times \cdots \times S_{n_k}$ contains a unique permutation σ_c of maximal length. It is well-known that σ_c is an involution, called *the longest element* of S_c. Denote $\lambda(c)$ the decreasing reordering of c. It is well-known that two Young subgroups S_{c_1} and S_{c_2} are conjugated in S_n if and only if $\lambda(c_1) = \lambda(c_2)$.

Schützenberger [21] has shown that the map $T \mapsto w_T = \pi^{-1}(T, T)$ is a bijection between the standard tableaux of shape a partition of n and the involutions of S_n (see also [4] where another interesting description of this bijection is given).

A standard tableau T is a *reading tableau* if it has the following property: for any $1 \leq p \leq n$, either p is in the first line of T, or if p is in the ith line of T ($i > 1$) then $p - 1$ is in the $(i-1)^{\text{th}}$ line of T.

As example, the *column superstandard tableaux* (which are tableaux numbered from the bottom to the top of each column, from left to right) are reading tableaux. These particular tableaux are the transposed of those defined by Garsia and Remmel in [8]. Our main result is the following:

Theorem 1.1. *Let λ be a partition of n and $\mathcal{T}^\lambda$ be its associated two-sided cell, then*

$$
\begin{aligned}
\mathcal{T}^\lambda_{min} &= \{\sigma_c \mid \lambda(c) = \lambda^t\} \\
&= \{w_T \mid T \text{ is a reading tableau of shape } \lambda\},
\end{aligned}
$$

where λ^t denotes the conjugate partition of λ.

Example. Consider the partition $\lambda = (3, 2, 1, 1)$ of 7; $\lambda^t = (4, 2, 1)$. Then the reading tableaux of shape λ are

$$T_5 = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \equiv \begin{array}{c} \boxed{2} \\ \boxed{1} \\ \quad\boxed{3} \\ \quad\quad\boxed{7} \\ \quad\quad\boxed{6} \\ \quad\quad\boxed{5} \\ \quad\quad\boxed{4} \end{array} \quad ; \quad T_6 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \equiv \begin{array}{c} \boxed{1} \\ \quad\boxed{3} \\ \quad\boxed{2} \\ \quad\quad\boxed{7} \\ \quad\quad\boxed{6} \\ \quad\quad\boxed{5} \\ \quad\quad\boxed{4} \end{array}$$

The right skew tableau is taken in the plactic class of the corresponding tableau viewed in the plactic monoid [13] (see also [7]). The corresponding involutions are then

$$w_{T_1} = 4\,3\,2\,1\,6\,5\,7 = \sigma_{(4,2,1)} \quad ; \quad w_{T_2} = 2\,1\,6\,5\,4\,3\,7 = \sigma_{(2,4,1)}$$

$$w_{T_3} = 1\,5\,4\,3\,2\,7\,6 = \sigma_{(1,4,2)} \quad ; \quad w_{T_4} = 4\,3\,2\,1\,5\,7\,6 = \sigma_{(4,1,2)}$$

$$w_{T_5} = 2\,1\,3\,7\,6\,5\,4 = \sigma_{(2,1,4)} \quad ; \quad w_{T_6} = 1\,3\,2\,7\,6\,5\,4 = \sigma_{(1,2,4)}$$

and $\mathcal{T}_{min}^{(3,2,1,1)} = \{\sigma_{(4,2,1)}, \sigma_{(2,4,1)}, \sigma_{(1,4,2)}, \sigma_{(4,1,2)}, \sigma_{(2,1,4)}, \sigma_{(1,2,4)}\}$.

Let λ be a partition of n and T be a standard tableau of shape λ. The *Schützenberger evacuation* of T, denoted by $ev(T)$, is a tableau of shape λ [20] (see also [18, p.128-130]). The evacuation illustrates the conjugation and the left (and right) multiplication by the longest element $\sigma_{(n)}$ in S_n. In particular $Q(w\sigma_{(n)}) = ev(Q(w)^t)$, for any $w \in S_n$, and $\mathcal{T}^\lambda \sigma_{(n)} = \sigma_{(n)}\mathcal{T}^\lambda = \mathcal{T}^{\lambda^t}$. Denote $d_c = \sigma_{(n)}\sigma_c$. As $\ell(\sigma_{(n)}w) = \ell(\sigma_{(n)}) - \ell(w)$, we obtain the following corollary:

Corollary 1.2. *Let λ be a partition of n and $\mathcal{T}^\lambda$ be its associated two-sided cell, then*

$$\begin{aligned} \mathcal{T}_{max}^\lambda &= \{d_c \mid \lambda(c) = \lambda\} \\ &= \{w \mid ev(Q(w)^t) = P(w\sigma_{(n)}) \text{ is a reading tableau of shape } \lambda^t\}. \end{aligned}$$

In the theory of Coxeter groups, the element d_c is well-known as the unique element of maximal length in the set of minimal right coset representatives of S_c [9, Chapter 2].

As a by-product of our proof, we obtain, in Section 3, that if w is an involution, the Kazhdan-Lusztig polynomial $P_{e,w} = 1$ if and only if w is the longest element of a Young subgroup, where e denotes the identity of S_n. More precisely, we show that an involution w avoids the pattern 3412 and 4231 if and only if w is the longest element of a Young subgroup of S_n.

To our knowledge, our results are the first results relating the Robinson-Schensted transformation and the length function (number of inversions) of a permutation. There is no evident link between both. Our proof is non combinatorial and uses heavily the $a-$function of Lusztig [15, 14] (questions about the leading term of Kazhdan-Lusztig polynomials and the $a-$function are heavily

studied, see for instance [23, 24]). Trying to find a combinatorial proof (which is a challenge) leads first to the following difficulty: for any permutations w, x in a two sided-cell $\mathcal{T}^\lambda$, there are permutations $w_1, \ldots, w_k \in \mathcal{T}^\lambda$ such that $w_1 = w$, $w_k = x$ and w_{i+1} is obtained from w_i by a Knuth or a dual-Knuth relation (see [18, 7]). But a permutation w may be 'locally minimal', that is whenever w (or w^{-1}) admits a Knuth or a dual-Knuth elementary relation, this relation increases the length. The involution $\sigma = 632541 \in S_6$ is locally minimal, but not of minimal length, in its two-sided cell.

It would be interesting to find a purely combinatorial proof of the main result. It is apparently an open problem to read the length of a permutation w directly on the pair of tableaux $\pi(w)$ (however, see [17], where the author gives a way to read the signature on the pair of tableaux). Fortunately, the Lusztig a-function gives us a way to avoid this problem.

2 Consequences of the main result

We denote a partition of n by $\lambda = (\lambda_1, \ldots, \lambda_k)$, with $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. Our reference for the general theory of the symmetric group is [18].

For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, we define, for $i > 0$,

$$m_i(\lambda) = |\{j \mid \lambda_j = i\}|.$$

The number $m_i(\lambda)$ is called the *multiplicity* of i in λ (see [16]). Observe that $m_i(\lambda) = 0$ for all $i > n$, since $\sum_i \lambda_i = n$. It is well-known that the multinomial coefficient

$$\left(\begin{array}{c} m_1(\lambda) + m_2(\lambda) + \cdots + m_n(\lambda) \\ m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda) \end{array} \right)$$

is the number of compositions associated to λ. Hence, we obtain the following corollary.

Corollary 2.1. *Let λ be a partition of n, then*

$$|\mathcal{T}_{min}^\lambda| = \left(\begin{array}{c} m_1(\lambda^t) + m_2(\lambda^t) + \cdots + m_n(\lambda^t) \\ m_1(\lambda^t), m_2(\lambda^t), \ldots, m_n(\lambda^t) \end{array} \right),$$

which is the number of compositions c of n such that $\lambda(c) = \lambda^t$.

The minimal elements in two-sided cells are linked to another important number in combinatorics

$$n(\lambda) = \sum_{i=1}^{k} \binom{\lambda_i^t}{2},$$

see [16, p.2-3].

Corollary 2.2. *Let λ be a partition of n and write $\lambda^t = (\lambda_1^t, \ldots, \lambda_k^t)$. Then $\ell(w) = n(\lambda)$ for all $w \in \mathcal{T}_{min}^\lambda$.*

Proof. Let c be a composition of n such that $\lambda(c) = \lambda^t$. Then $\ell(\sigma_c) = \ell(\sigma_{\lambda^t}$. Let w_i be the longest element of the Young subgroup $S_{\lambda_i^t}$, then $\ell(w_i) = \binom{\lambda_i^t}{2}$. Therefore

$$\ell(\sigma_{\lambda^t}) = \sum_{i=1}^{k} \binom{\lambda_i^t}{2}$$

since $\sigma_{\lambda^t} = w_1 \ldots w_k$ (seen as a word on the letters $1, \ldots, n$) and that the letters in w_{i+1} are greater than the letters in w_i. The corollary follows from Theorem 1.1. $\qquad\square$

As in the case of minimal elements, we have the following corollaries:

Corollary 2.3. *Let λ be a partition of n, then*

$$| \mathcal{T}_{max}^\lambda | = \begin{pmatrix} m_1(\lambda) + m_2(\lambda) + \cdots + m_n(\lambda) \\ m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda) \end{pmatrix},$$

which is the number of compositions c of n such that $\lambda(c) = \lambda$.

Corollary 2.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n, then*

$$\ell(w) = \binom{n}{2} - \sum_{i=1}^{k} \binom{\lambda_i}{2}$$

for all $w \in \mathcal{T}_{max}^\lambda$.

Proof. As $\ell(\sigma_{(n)}) = \binom{n}{2}$ and $\ell(\sigma_{(n)} w) = \ell(\sigma_{(n)}) - \ell(w)$, for any $w \in S_n$, the corollary follows from same argument than in the proof of Corollary 2.2. $\qquad\square$

3 Proof of Theorem 1.1

The following proposition implies that

$$\{\sigma_c \,|\, \lambda(c) = \lambda^t\} = \{w_T \,|\, T \text{ is a reading tableau of shape } \lambda\} \subset \mathcal{T}^\lambda.$$

Proposition 3.1. *Let λ be a partition of n; then the following conditions are equivalent:*

i) T is a reading tableau of shape λ;

ii) $w_T = \sigma_c$, where c is a composition of n such that $\lambda(c) = \lambda^t$.

Proof. Recall that the longest element of a Young subgroup is an involution, since it is unique.

Assume (i). As T is a reading tableau, if n is in the row T_i, one has $1 \le p \le n-1$ such that $p+1$ is in the first row of T, $p+i = n$ and $p+j$ is at the end

5

of the row T_j, for all $1 \leq j \leq i$. One applies the i first steps of the inverse of Robinson-Schensted correspondence, hence

$$w_T = w_{T'} \; n \; \ldots \; p+1.$$

where T' is the standard Young tableau obtained by deleting $p+1,\ldots,n$ in T. Thus $w_{T'}$ is a permutation on the set $\{1,\ldots,p\}$. Observe that T' is also a reading tableau. The shape of T' is denoted by λ'. By induction on n, $w_{T'}$ is the longest element of the Young subgroup $S_{c'}$, where $\lambda(c') = \lambda'$. Then w_T is the longest element of the Young subgroup $S_{c'} \times S_i$. Let $c = (c', i)$; it is now easy to see that $\lambda(c) = \lambda^t$.

Conversely, let $c = (n_1, \ldots n_k)$ and use induction and similar arguments with direct Robinson-Schensted correspondence on the permutation

$$w_T = n_1 \; \ldots \; 1 \; w',$$

where $n_1 \; \ldots \; 1$ is the longest element of the Young subgroup S_{n_1} and w' is the longest element of the Young subgroup $S_{n_2} \times \cdots \times S_{n_k}$. $\qquad\square$

Now, it remains to prove that $\{\sigma_c \,|\, \lambda(c) = \lambda^t\} = \mathcal{T}_{min}^{\lambda}$, to end the proof of Theorem 1.1.

The Lusztig a-function: We consider the symmetric group S_n as a Coxeter system (W, S) of type A_{n-1} with $W = S_n$ and generating set S consisting of the $n-1$ simple transpositions $\tau_i = (i, \; i+1)$, where $i = 1, \ldots, n-1$. Then $\ell(w)$ is also the length of w as a reduced word in the elements of S. A classical bijection between subsets of S and compositions of n is obtained as follow: Let $I \subset S$ and $S \setminus I = \{\tau_{i_1}, \ldots, \tau_{i_k}\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$. Set $n_1 = i_1$, $n_2 = i_2 - i_1 + 1$, $\ldots$, $n_k = n - i_k$, then n_i are non-negative integers. By this way, we have obtained a unique composition $c_I = (n_1, \ldots, n_k)$ of n associated to I. Moreover,

$$W_I = S_{n_1} \times \cdots \times S_{n_k}.$$

Therefore, as is well-known the Young subgroups of S_n are precisely the *parabolic subgroups* of S_n (see [9, Proposition 2.3.8]).

Our basic references for the work of Kazhdan and Lusztig are [11], [15] (see also [5]). We denote by $\leq$ the Bruhat order on S_n.

Let $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ where $q^{1/2}$ is an indeterminate. Let $\mathcal{H}$ be the Hecke algebra over $\mathcal{A}$ corresponding to S_n. Let $(T_w)_{w \in S_n}$ be the standard basis of $\mathcal{H}$ and $(\widetilde{T}_w)_{w \in S_n}$ the basis defined as follows:

$$\widetilde{T}_w = q^{-\ell(w)/2} T_w.$$

In [11, Theorem 1.1], Kazhdan and Lusztig have shown that there is a basis $(b_w)_{w \in S_n}$ of $\mathcal{H}$, called the *Kazhdan-Lusztig basis*, such that

$$b_w = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} q^{(\ell(w)-\ell(y))/2} P_{y,w}(q^{-1}) \widetilde{T}_y,$$

where $P_{y,w} \in \mathcal{A}$ are the Kazhdan-Lusztig polynomials. Moreover, they have defined three equivalence relations on S_n, with equivalence classes that are called *left cells*, *right cells* and *two-sided cells*. In our case, the following result of Vogan and Jantzen result on S_n [10, 22] gives the link with the Robinson-Schensted correspondence (see also [1]): the set $\mathcal{T}^\lambda$ is a two-sided cell for all partitions λ of n; and any two-sided cell of S_n arises by this way.

Following Lusztig [14, 15], let $h_{x,y,w}$ be the structure constants of the Kazhdan-Lusztig base $(b_w)_{w \in W}$, that is

$$b_x b_y = \sum_{w \in W} h_{x,y,w} \, b_w.$$

Denote $\delta(w)$ the degree of the Kazhdan-Lusztig polynomial $P_{e,w}$ as a polynomial in q. Write $u = q^{1/2}$. Let $a(w)$ be the smallest integer such that for any $x, y \in S_n$, $u^{a(w)} h_{x,y,w} \in \mathcal{A}^+$, where $\mathcal{A}^+ = \mathbb{Z}[u]$ (this is well defined for any Weyl group). In [14, 15], Lusztig has shown the following properties about the $a-$function:

a) $a(w) \leq \ell(w) - 2\delta(w)$ ([15, Section 1.3]);

b) The $a-$function is constant on two-sided cells ([14, Theorem 5.4]).

c) For any $I \subset S$, $a(\sigma_{c_I}) = \ell(\sigma_{c_I})$ ([15, Corollary 1.9 (d) and Theorem 1.10]). In other words, for any composition c of n, $a(\sigma_c) = \ell(\sigma_c)$.

d) Let $\mathcal{D} = \{w \in W \mid a(w) = \ell(w) - 2\delta(w)\}$, then each element in $\mathcal{D}$ is an involution, called a *Duflo involution* ([15, Proposition 1.4]). In symmetric groups, all involutions are Duflo involutions. Indeed, each left cell contains a unique Duflo involution [15]; left cells are precisely coplactic classes (the sets of permutations having the same right tableau under π, see for instance [1]), and each coplactic class contains a unique involution.

Let λ be a partition of n and $\mathcal{T}^\lambda$ be its associated two-sided cells. Properties (b) and (c) imply that $a_\lambda := a(\sigma_{\lambda^t}) = a(w)$, for all $w \in \mathcal{T}^\lambda$. Therefore, by (a),

$$\ell(\sigma_c) = a_\lambda = a(w) \leq \ell(w),$$

for any $w \in \mathcal{T}^\lambda$. Thus

$$\{\sigma_c \mid \lambda(c) = \lambda^t\} \subset \mathcal{T}^\lambda_{min}.$$

Now, let $w \in \mathcal{T}^\lambda_{min}$, then $a(w) = a_\lambda = \ell(\sigma_{\lambda^t}) = \ell(w)$, since $\sigma_{\lambda^t} \in \mathcal{T}^\lambda_{min}$. Property (d) implies that w is a Duflo involution and $\delta(w) = 0$.

By Proposition 3.1, $\sigma_c \in \mathcal{T}^\lambda$ implies $\lambda(c) = \lambda^t$ Therefore, Theorem 1.1 is a direct consequence of the following result, which gives a surprising criterion about the degree $\delta(w)$ of the Kazhdan-Lusztig polynomial $P_{e,w}$, for $w \in S_n$ an involution.

Proposition 3.2. *Let $w \in S_n$, then the following conditions are equivalent:*

i) *w is an involution and $\delta(w) = 0$;*

ii) *$w = \sigma_c$, for some composition c of n.*

KL Polynomials and smoothness of Schubert Varieties: We say that a permutation $w \in S_n$, seen as a word $w = x_1 \ldots x_n$, *avoids the pattern* 4231 (resp. *avoids the pattern* 3412) if there is no $1 \leq i < j < k < l \leq n$ such that $x_l < x_j < x_k < x_i$ (resp. $x_k < x_l < x_i < x_j$). In other words, there is no subword of w with the same relative order as the word 4231 (resp. 3412).

Here, we link these definitions with Kazhdan-Lusztig polynomials by the way of the following well-known criterion: Let $w \in S_n$, then

$$(\diamond) \qquad P_{e,w} = 1 \iff w \text{ avoids the patterns } 4231 \text{ and } 3412.$$

Indeed, on one hand, Lakshmibai and Sandhya have shown that a Schubert variety $X(w)$, $w \in S_n$, is smooth if and only if w avoids the pattern 3412 and 4231 ([12] or see [2, Theorem 8.1.1]).

On the other hand, Deodhar [6] has shown a useful characterization of the smoothness by the way of Kazhdan-Lusztig polynomials: Let $w \in S_n$ then $P_{e,w} = 1$ if and only if $X(w)$ is smooth.

Proof of Proposition 3.2: By the above discussion, Proposition 3.2 is a direct consequence of the following lemma.

Lemma 3.3. *Let $w \in S_n$ an involution, then the following statements are equivalent*

i) w avoids the patterns 4231 and 3412;

ii) there is a composition c of n such that $w = \sigma_c$;

Proof. $(ii) \Rightarrow (i)$: write $c = (c_1, \ldots, c_k)$. If $k = 1$ then $\sigma_c = \sigma_{(n)}$ avoids the patterns 4231 and 3412. If $k > 1$ then σ_c is the image of $(\sigma_{(c_1)}, \ldots, \sigma_{(c_k)})$ under the canonical isomorphism between $S_{c_1} \times \cdots \times S_{c_k}$ and S_c. Conclude by induction on n.

$(i) \Rightarrow (ii)$: one sees $w = x_1 \ldots x_n$ as a word on the letters $1, \ldots, n$.

One proceeds by induction on n. Therefore, one may suppose that $(i) \Rightarrow (ii)$ for all proper Young subgroups of S_n. If $n \leq 4$, it is readily seen. Suppose $n > 4$.

If $x_1 = 1$, then $w \in S_1 \times S_{n-1}$, and the lemma follows by induction.

If $n > x_1 = p > 1$, then $x_p = 1$ and $1 \leq x_i \leq p$, for all $1 \leq i \leq p$. Otherwise, there is $1 < i < p$ such that $x_i > p$. In other words, there is $1 < i < p < x_i$ such that $x_p = 1 < x_{x_i} = i < x_1 = p < x_i$, that is, w has the pattern 3412 which is a contradiction.

Hence $w \in S_p \times S_{n-p}$ and the lemma follows by induction.

If $x_1 = n$, then $x_n = 1$ one just has to show that $w = w_0$. Otherwise, there is $1 < i < n-1$ such that $x_i < x_{i+1}$ (since if $i = 1$, $x_1 = n < x_2$ and if $i+1 = n$, $x_{n-1} < x_n = 1$ which are contradictions). Thus there is $1 < i < i+1 < n$ such that $x_n < x_i < x_{i+1} < x_1$, that is, w has the pattern 4231 which is a contradiction. $\square$

Acknowledgements

I am indebted to C. Chauve and C. Reutenauer for useful discussions about Robinson-Schensted correspondence and plactic classes and to P. Polo for his useful remarks. I would thank P. Baumann, R. Bédard and F. Chapoton for pointing out some references.

References

[1] S. Ariki, *Robinson-Schensted correspondence and left cells*, Combinatorial methods in representation theory (Kyoto, 1998), Adv. Stud. Pure Math. **28** (2000), 1–20.

[2] S. Billey and V. Lakshmibai, *Singular loci of Schubert Varieties*, Progress in Math, Birkhäuser, **182** (2000).

[3] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 4-6, Hermann (1968).

[4] W. H. Burge, *Four correspondences betweengraphs and generalized Young tableaux*, J. of Combin. Theory Ser. A **17** (1974), 193–242.

[5] C. W. Curtis, *Representations of Hecke algebras*, Soc. Math. de France Astérisque **168** (1988), 13–60.

[6] V. V. Deodhar, *Local Poincaré duality and non-singularity of Schubert varieties*, Comm. Algebra **13** (1985), 1379–1388.

[7] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts **35** (1997).

[8] A. M. Garsia and J. Remmel, *Shuffles of permutations and the Kronecker product*, Graphs Combin. 1 **3** (1985), 217–263.

[9] M. Geck and G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*, London Math. Soc. Mon. New Series **21** (2000).

[10] J. C. Jantzen, *Einhüllende Algebren halbeinfacher Lie-Algebren*, Springer-Verlag, Berlin (1983).

[11] D. Kazhdan and G. Lusztig, *Representation of Coxter groups and Hecke Algebras*, Inventiones math **53** (1979), 165–184.

[12] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $SL(n)/B$*, Proc. Indiann Acad. Sci. (Math. Sci.), **100** (1990), 42–52.

[13] A. Lascoux and M. P. Schützenberger, *Le monoïde plaxique*, Noncommutative structures in algebra and geometric combinatorics (Naples, 1978), Quad. "Ricerca Sci." **109**, CNR, Rome, (1981), 129–156.

[14] G. Lusztig, *Cells in affine Weyl groups, I* , in "Algebraic groups and Related Topics", Advanced Studies in Pure Math. Vol. 6 (1986), 255–285.

[15] G. Lusztig, *Cells in affine Weyl groups, II*, Journal of Algebra **109** (1987), 536–548.

[16] I. G. Macdonald, *Symmetric functions and Hall Polynomials*, Oxford Math. Mono. (1979).

[17] A. Reifegerste, *Permutation sign under the Robinson-Schensted-Knuth correspondence*, preprint (2003).

[18] B. Sagan, *The symmetric group*, Wadsworth and Brooks/Cole math series (1991).

[19] C. Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math **13** (1961), 179–191.

[20] M.-P. Schützenberger, *Quelques remarques sur une construction de Schensted*, Math. Scand. **12** (1963), 117–128.

[21] M.-P. Schützenberger, *La correspondance de Robinson*, In Combinatoire et représentation du groupe symétrique, Lecture Notes in Math. **579** (1977), Springer, 59–113.

[22] D. A. Vogan, *A Generalized τ-Invariant for the primitive spectrum of a Semisimple Lie Algebra*, Math. Ann. **242** (1979), 209–224.

[23] N. Xi, *The based ring of two-sided cells of affine Weyl groups of type $\tilde{A}_{n-1}$*, Mem. of AMS No. 749 vol. 157 (2002).

[24] N. Xi, *The Leading Coefficient of Certain Kazhdan-Lusztig Polynomials of the Permutation Group $\mathfrak{S}_n$*, preprint (2004).