POSETS RELATED TO THE CONNECTIVITY SET OF COXETER GROUPS

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Abstract. We define the notion of connectivity set for elements of any finitely generated Coxeter group. Then we define an order related to this new statistic and show that the poset is graded and each interval is a shellable lattice. This implies that any interval is Cohen-Macauley. We also give a Galois connection between intervals in this poset and a boolean poset. This allows us to compute the Möbius function for any interval.

Introduction

Let $\mathfrak{S}_n$ denote the symmetric group, the group of permutations on the set $[n] = \{1, 2, \dots, n\}$. The length $\ell(w)$ of a permutation $w \in \mathfrak{S}_n$ is its number of inversions i.e. the number of pair (i, j) with $1 \leq i < j \leq n$ and $w(i) > w(j)$. The *connectivity set* (also known as *global ascent set*) of a permutation $w = w(1)w(2)\cdots w(n) \in \mathfrak{S}_n$ is equal to

(1) $$C(w) = \{i \in [n-1] : w(j) < w(k), \forall 1 \leq j \leq i < k \leq n\}.$$

This set has recently been the subject of an article by R. Stanley [10] which is related to the number of connected components of a permutation (see [5, 6]). It turns out that the connectivity set appears also to be linked with the study of two combinatorial Hopf algebras: the Malvenuto-Reutenauer Hopf algebra and the quasisymmetric functions in non-commuting variables [1, 2, 7].

In their analysis of the Malvenuto-Reutenauer Hopf algebra of permutations [1], Aguiar and Sottile consider the set $\mathrm{GDes}(w)$ of global descents of a permutation $w \in \mathfrak{S}_n$ which is related to the connectivity set by the formula $\mathrm{GDes}(w) = w_0(C(ww_0))$ where $w_0 = n\, n-1 \cdots 2\, 1$ is the unique permutation of maximal length in $\mathfrak{S}_n$. Similar notions are also used in [7].

In the work of Bergeron and Zabrocki [2] a natural order on set compositions arose out of the Hopf algebra structure of the quasisymmetric functions in non-commuting variables. This order came out of some simple conditions placed on the comultiplicative structure of the Hopf algebra. Restricting attention to set compositions with one element in each part, the partial order can be viewed as a partial order $\leq$ on permutations: let $u, v \in \mathfrak{S}_n$, then

(2) $$u \leq v \text{ if and only if } C(v) \subset C(u) \text{ and } \mathrm{std}_{C(u)}(v) = \mathrm{std}_{C(u)}(u).$$

Here $\mathrm{std}_I(w)$ denotes the standardization of $w \in \mathfrak{S}_n$ along the blocks of $I \subset [n-1]$. For instance $\mathrm{std}_{\{1,4\}}(4.623.51) = 1.423.65$. A few of the Hasse diagrams for these posets appear in Figures 1 and 2.

The study of the poset $(\mathfrak{S}_n, \leq)$ led us to consider a generalization of these concepts in the language of Coxeter groups. Not only is the definition of this poset strongly simplified, but the connectivity set turns out to be closely linked with a classical result in the theory of Coxeter groups:

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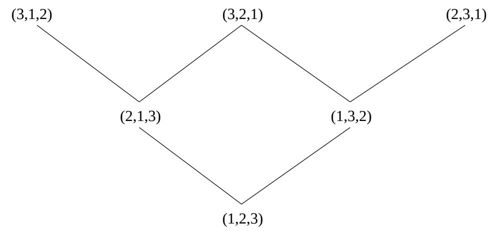

FIGURE 1. Hasse diagram for posets of $(\mathfrak{S}_3, \leq)$

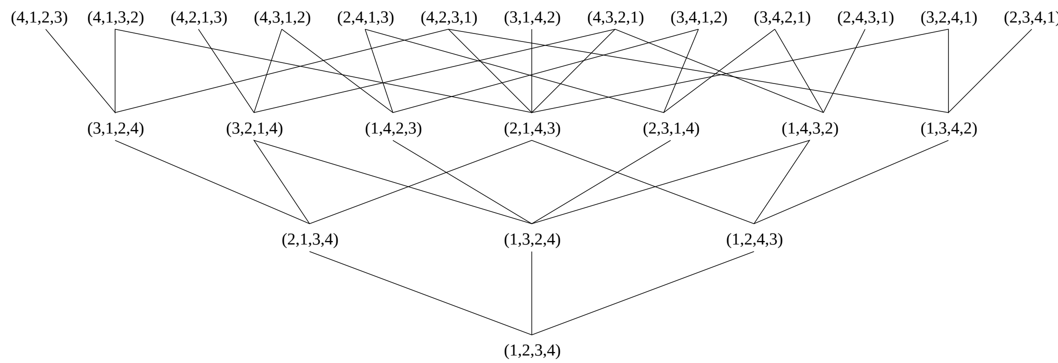

FIGURE 2. Hasse diagram for posets of $(\mathfrak{S}_4, \leq)$

the word property, due to Tits [15] and independently to Matsumoto [13] (see for instance [4, Theorem 3.3.1]).

Let (W, S) be a finitely generated Coxeter system whose length function is denoted by $\ell : W \to \mathbb{N}$. Denote by e the identity of W. We denote by W_I the *parabolic subgroup* of W generated by $I \subset S$. Let $w \in W$, the word property says that any pair of reduced expressions for w can be linked by a sequence of braid relation transformations. In particular, the set

$$S(w) = \{s_i \in S \,:\, w = s_1 \ldots s_{\ell(w)} \text{ reduced}\} = \bigcap_{\substack{I \subset S \\ w \in W_I}} I \tag{3}$$

is independent of the choice of a reduced expression for w. It is clear that $w \in W_{S(w)}$.

The *descent set* of $w \in W$ is the set

$$\mathrm{Des}(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

Let $I \subset S$, it is well-known that the set

$$X_I = \{u \in W : \ell(us) > \ell(u), \forall s \in I\} = \{u \in W : \mathrm{Des}(u) \subseteq S \backslash I\}$$

is a set of minimal length coset representatives of W/W_I. Each element $w \in W$ has a unique decomposition $w = w^I w_I$ where $w^I \in X_I$ and $w_I \in W_I$ and moreover $\ell(w) = \ell(w^I) + \ell(w_I)$. The pair (w^I, w_I) is generally referred to as the *parabolic components of w along I* (see [4, Proposition 2.4.4] or [8, 5.12]).

Definition 1.

(1) The *connectivity set of* $w \in W$ is the set $C(w) = S \setminus S(w)$.
(2) Let $u, v \in W$, $u \le v$ if and only if the parabolic component $v_{S(u)} = u$.

In other words, $C(w)$ is the set of simple reflections which do not appear in a reduced expression for w. We will show in §1.2 that in the case where $W = \mathfrak{S}_n$ and S is the set of simple transpositions $\tau_i = (i, i+1)$, with $i \in [n-1]$, the definitions coincide with those in equations (1) and (2).

Let P be a poset and $u \in P$, a *cover* of U is an element $v > u$ in P such that the interval $[u, v] = \{u, v\}$. Recall that a lattice L is called *upper semimodular* if for $w, g, h \in L$ such that g and h cover w, there is a x which covers both g and h.

Our main results is the following theorem.

Theorem 2. *Let (W, S) be a finitely generated Coxeter system.*

(1) *The poset $(W, \le)$ is graded. The rank function is $w \mapsto |S(w)|$.*
(2) *The interval $[u, v]$ is an upper semimodular lattice, for any $u \le v$ in W.*

We obtain immediately the following corollary. We refer the reader to [3, 11, 12] and [4, Appendix A2] for more information about the concepts of shellability and Cohen-Macauliness.

Corollary 3. *The order complex of $[u, v]$ is shellable (hence Cohen-Macaulay), for any $u \le v$ in W.*

In §1, we construct a *Galois connection* between any interval $[u, v]$ and a boolean poset. This will allow us to show our second main result.

Theorem 4. *For any $u \le v$ in W, the Möbius function is given by*

$$\mu(u,v) = \begin{cases} (-1)^{|S(v)|-|S(u)|} & \text{if } S(u) = S(v)\backslash \mathrm{Des}(v^{S(u)}) \text{ or } u = v \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\mu(e,v) = \begin{cases} (-1)^{|S(v)|} & \text{if } v = w_0(S(v)) \\ 0 & \text{otherwise,} \end{cases}$$

where $w_0(I)$ denote the element of maximal length in W_I.

It is interesting to remark that this Möbius function in type A correspond to the coefficients of the primitives of the Malvenuto-Reutenauer Hopf algebra. This was done by Duchamp, Hivert and Thibon [7, §3.4]. They defined operations $p_I(w)$ for a permutation w that correspond to our $\mathrm{std}_I(w)$, but they do not consider the order it induces.

Finally in §2, we will give a characterization of the connectivity set when the Coxeter group is finite of type B and D and when they are considered as subgroups of the symmetric group acting on the set $(-[n]) \cup [n]$. We also give formulas for generating functions for the numbers of elements of a Coxeter group of type A, B and D with exactly k elements in the connectivity set.

Acknowledgment

The authors would like to thank Vic Reiner for helpful suggestions on this research. He has outlined a proof that the poset $W - \{e\}$ is contractible and remarked that the order complex of $(W, \le)$ was shellable for $W = \mathfrak{S}_n$ and small values of n. It is still open to show this in general, since $(W, \le)$ is not a single interval. We also thank Hugh Thomas for valuable discussions.

1. Main Results

For additional information about finitely generated Coxeter groups we refer the reader to [8, 4]. In this section, (W, S) is an arbitrary finitely generated Coxeter system (except at the end of §1.2 where we will discuss the case of symmetric groups).

1.1. Preliminaries.

We start by recalling a well-known lemma whose proof follows immediately from definitions.

Lemma 5. *For $u, v \in W$ such that $\ell(uv) = \ell(u) + \ell(v)$, we have $C(uv) = C(u) \cap C(v)$ (or alternatively, $S(uv) = S(u) \cup S(v)$).*

The *(left) weak order* $\leq_{\mathcal{L}}$ on W may be defined as follows: $u \leq_{\mathcal{L}} v$ if and only if there is $v' \in W$ such that $v = v'u$ and $\ell(v) = \ell(v') + \ell(u)$.

Proposition 6. *Let $u, v \in W$.*

1. *If $u \leq v$ in W, then $C(v) \subset C(u)$ (or alternatively, $S(u) \subset S(v)$).*
2. *For $u \leq v$ in W, $[u, v] = \{v_I : S(u) \subset I \subset S(v)\}$.*
3. *if $u \leq v$, then $u \leq_{\mathcal{L}} v$.*
4. *If $W_{S(u)}$ is of finite index k in W, then $|\{w \in W : w \geq u\}| = k$.*

Proof. (1) As $v = v^{S(u)}u$ with $\ell(v) = \ell(v^{S(u)}) + \ell(u)$, then $C(v) = C(v^{S(u)}) \cap C(u) \subseteq C(u)$ by Lemma 5.

(2) and (3) follow from definitions.

(4) There is a bijection between the set $X_{S(u)}$ and $\{w \in W | w \geq u\}$. $w \geq u$ if and only if $w = w^{S(u)}u$, hence the map

$$\{w \in W | w \geq u\} \longrightarrow X_{S(u)}$$

which sends $w \mapsto w^{S(u)}$ has as inverse $x \mapsto xu$ and so is a bijection. □

Remark 7. In fact, we could define the order on the set of coset W/W_I, for any $I \subset S$. Indeed, we can take an element w such that $C(w) = I$. Then by Proposition 6 (4) the poset $(W/W_I, \leq)$ is obtained by taking the set $\{gW_I : g \geq w\}$ and the induced order. For the symmetric group $(W = \mathfrak{S}_n)$, this gives the order on set compositions considered in [2].

Now we recall some useful facts about minimal coset representatives. If $I \subseteq J \subseteq S$, then $X_I^J = X_I \cap W_J$ is the set of minimal coset representatives of W_J/W_I and $X_J X_I^J = X_I$. The following lemma is well-known.

Lemma 8. *Let $I \subset J \subset S$ and $w \in W$, then $w_I = (w_J)_I$. In particular, $w_I \leq w_J$.*

Proof. Write $w = w^I w_I = w^J w_J$ and $w_J = (w_J)^I (w_J)_I$ and conclude by uniqueness of the parabolic components. □

Let $K \subseteq S$. If W_K is finite, then W_K contains a unique element $w_0(K)$ of maximal length. It can be characterized as the unique element w in W_K such that $\mathrm{Des}(w) = K$. In fact, for any $w \in W$, $W_{\mathrm{Des}(w)}$ is finite (see for instance [4, Proposition 2.3.1]).

Lemma 9. *Let $I \subset J \subset S$ such that W_J is finite, then $(w_0(J))_I = w_0(I)$.*

Proof. Assume there is an $s \in I$ such that $\ell((w_0(J))_I s) > \ell((w_0(J))_I)$. As $((w_0(J))^I, (w_0(J))_I s)$ are the parabolic components of $w_0(J)s$ along I we obtain

$$\ell(w_0(J)s) = \ell((w_0(J))^I) + \ell((w_0(J))_I s) > \ell(w_0(J)).$$

Hence $s \notin Des(w_0(J)) = J$ which contradicts $s \in I \subset J$. Therefore $Des((w_0(J))_I) = I$. □

1.2. **The symmetric group.** We end this preliminary discussion by proving the equivalence of definitions on the case of $W = \mathfrak{S}_n$ and S is the set of simple transpositions $\tau_i = (i, i+1)$, with $i \in [n-1]$.

The *standardization* of a word $w = a_1 a_2 \cdots a_n$ of length n in an totally ordered alphabet, denoted by $\mathrm{std}(w)$, is the unique permutation $\sigma \in \mathfrak{S}_n$ such that for all $i < j$ we have $\sigma(i) > \sigma(j)$ if and only if $a_i > a_j$. For instance, for $w = cbbcaa$ in the alphabet $\{a < b < c\}$ we have $\mathrm{std}(w) = 534612$. A *composition* of n is a sequence $\mathbf{c} = (c_1, \dots, c_k)$ of positive integers whose sum is n. There is a well-known bijection between compositions of n and subsets of $[n-1]$ defined by

$$I = \{i_1, i_2, \dots, i_k\} \quad \mapsto \quad \mathbf{c}_I = (i_1, i_2 - i_1, \dots, i_k - i_{k-1}, n - i_k).$$

Let $I \subset S$ and $\mathbf{c}_{S\setminus I} = (c_1, c_2, \dots, c_k)$. Set $t_i = c_1 + c_2 + \cdots + c_i$ for all i. Given a k-tuple $(\sigma_1, \sigma_2, \dots, \sigma_k) \in \mathfrak{S}_{c_1} \times \mathfrak{S}_{c_2} \times \cdots \times \mathfrak{S}_{c_k}$ of permutations, we define $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_k \in \mathfrak{S}_n$ as the permutation that maps an element a belonging to the interval $[t_{i-1}+1, t_i]$ onto $t_{i-1} + \sigma_i(a - t_{i-1})$. This assignment defines an isomorphism $\mathfrak{S}_{c_1} \times \mathfrak{S}_{c_2} \times \cdots \times \mathfrak{S}_{c_k} \simeq W_I$. We have also the well-known characterization

$$X_I = \left\{\sigma \in \mathfrak{S}_n \;\middle|\; \forall i,\ \sigma \text{ is increasing on the interval } [t_{i-1}+1, t_i]\right\}.$$

Specifically, write $\sigma \in \mathfrak{S}_n$ as the concatenation $\sigma_1 \cdot \ldots \cdot \sigma_k$ of words in the alphabet $\mathbb{N}$ such that the length of the word σ_i is c_i. It is then easy to check that the parabolic components of σ along I are

$$\sigma_I = \mathrm{std}_{S\setminus I}(\sigma) := \mathrm{std}(\sigma_1) \times \mathrm{std}(\sigma_2) \times \cdots \times \mathrm{std}(\sigma_k) \in \mathfrak{S}_{\mathbf{c}} \quad \text{and} \quad \sigma^I = \sigma\sigma_{S\setminus I}^{-1} \in X_I. \tag{4}$$

Proposition 10. *For $u \in \mathfrak{S}_n$,*

$$C(u) = \{\tau_i \in S : u(j) < u(k), \forall\ 1 \le j \le i < k < n\}.$$

Proof. For $u \in \mathfrak{S}_n$, write $C(u) = \{\tau_{i_1}, \tau_{i_2}, \dots, \tau_{i_k}\}$ with $1 \le i_1 < i_2 < \cdots < i_k \le n-1$. Then we have that

$$u \in W_{S(u)} \simeq \mathfrak{S}_{c_1} \times \mathfrak{S}_{c_2} \times \cdots \times \mathfrak{S}_{c_k}.$$

If $\tau_i \in C(u)$, then for each $j < i$, $u(j) \le i$ and for each $k > i$, $u(k) > i$. Hence $C(u) \subseteq \{\tau_i : u(j) < u(k), \forall j \le i < k\}$. Conversely, if $\tau_k \in \{\tau_i : u(j) < u(k), \forall j \le i < k\}$, then $u \in \mathfrak{S}_k \times \mathfrak{S}_{n-k}$. Hence $\tau_k \notin S(u)$ implying $\tau_k \in C(u)$. □

As $u_{S(u)} = u$ and by (4) and Proposition 6 (1), we obtain immediately the following corollary.

Corollary 11. *Let $u, v \in \mathfrak{S}_n$, the following propositions are equivalent:*

1. *$u \le v$;*
2. *$\mathrm{std}_{C(u)}(v) = u$;*
3. *$C(v) \subset C(u)$ and $\mathrm{std}_{C(u)}(v) = \mathrm{std}_{C(u)}(u)$.*

1.3. **Proof of Theorem 2.** From the examples given in the previous section, we see that this poset does not form a lattice, but each interval $[u, v]$ in this poset does.

Lemma 12. *Let $u, v \in W$ such that $u < v$. For any $w, g \in [u, v]$ we have*

(a) *$v_{S(w)\cup S(g)}$ is the unique least upper bound of g and w in $[u, v]$ (i.e. $w \vee g := \min_{\le}\{x : v \ge x \ge w, v \ge x \ge g\} = v_{S(w)\cup S(g)}$). Moreover, $S(w \vee g) = S(w) \cup S(g)$.*

(b) *$v_{S(w)\cap S(g)}$ is the unique greatest lower bound of w and g in $[u, v]$ (i.e. $w \wedge g := \max_{\le}\{x : u \le x \le w, u \le x \le g\} = v_{S(w)\cap S(g)}$). Moreover, $S(w \wedge g) \subseteq S(w) \cap S(g)$.*

Proof. $S(w) \subseteq S(w) \cup S(g)$, so by Lemma 8, $w = v_{S(w)} \le v_{S(w)\cup S(g)}$ and hence $S(w) \subseteq S(v_{S(w)\cup S(g)})$. Similarly, $S(g) \subseteq S(v_{S(w)\cup S(g)})$. From this, we have $S(w) \cup S(g) \subseteq S(v_{S(w)\cup S(g)}) \subseteq S(w) \cup S(g)$ and therefore $S(v_{S(w)\cup S(g)}) = S(w) \cup S(g)$. Conversly, if $h \ge w, g$, then $S(h) \supseteq S(w) \cup S(g)$ and so $h = v_{S(h)} \ge v_{S(w)\cup S(g)}$.

For part (b), observe that $S(v_{S(w)\cap S(g)}) \subseteq S(w) \cap S(g)$ since $v_{S(w)\cap S(g)} \in W_{S(w)\cap S(g)}$. Now $S(w) \cap S(g) \subseteq S(w), S(g)$ gives $v_{S(w)\cap S(g)} \le v_{S(w)} = w, v_{S(g)} = g$. Moreover, if $h \le g, w$, then $S(h) \subseteq S(w) \cap S(g)$ and hence $h = v_{S(h)} \le v_{S(w)\cap S(g)}$. □

Remark 13. We do not have $S(w \wedge g) = S(w) \cap S(g)$ in general. For instance, in $\mathfrak{S}_4$, the two elements s_2s_1 and s_2s_3. $s_2s_1 \wedge s_2s_3 = e$ and $S(s_2s_1) \cap S(s_2s_3) = \{s_2\}$.

Lemma 14. *Let $u, v \in W$ be such that $u < v$. For any $s \in \mathrm{Des}(v^{S(u)})$ we have*

$$u < (v_{S(u)\cup\{s\}})^{S(u)} u = v_{S(u)\cup\{s\}} < v.$$

Moreover, $C((v_{S(u)\cup\{s\}})^{S(u)} u) = C(u)\backslash\{s\}$.

Proof. Let $x = (v_{S(u)\cup\{s\}})^{S(u)}$ so that $u = v_{S(u)} \le v_{S(u)\cup\{s\}} = xu$. Choose $s \in \mathrm{Des}(v^{S(u)})$, then $s \notin S(u)$ since $v^{S(u)} \in X_{S(u)} = \{w \in W : \mathrm{Des}(w) \subseteq S\backslash S(w)\}$. As $x \in X_{S(u)}$, any reduced expression of x must end in a simple reflection $r \in S \setminus S(u)$ (see for instance [4, Lemma 2.4.3]). But $x \in W_{S(u)\cup\{s\}}$, therefore $r \in S(u) \cup \{s\} \setminus S(u)$. In other words, any reduced expression of x must end in s. Hence $s \in S(x)$. By Lemma 5, $S(xu) = S(u) \cup S(x) = S(u) \cup \{s\}$.

This implies that $C(xu) = C(u)\backslash\{s\}$. Moreover, $v = v^{S(u)\cup\{s\}} xu$ and $xu \in W_{S(u)\cup\{s\}}$. By the uniqueness of the parabolic components, we have that $v_{S(u)\cup\{s\}} = xu$ and so can conclude that $xu \le v$. □

The next proposition identifies exactly which elements cover u.

Proposition 15. *Let $u < v$. If u' is a cover of u in $[u, v]$, then $u' = (v^{S(u)})_{S(u)\cup\{s\}} u$ for some $s \in \mathrm{Des}(v^{S(u)})$. Moreover, $C(u') = C(u)\backslash\{s\}$.*

Proof. Let $u < u'$ be a cover. By Lemma 14, there is an $s \in \mathrm{Des}((u')^{S(u)})$ and an $x = ((u')_{S(u)\cup\{s\}})^{S(u)}$ such that $x \ne id$ and $u < xu < u'$. Since u' is a cover of u, then $xu = u'$. Since $v = v^{S(u)\cup\{s\}} xu$ with $\ell(v) = \ell(v^{S(u)\cup\{s\}}) + \ell(x) + \ell(u)$, therefore $v^{S(u)} = v^{S(u)\cup\{s\}} x$ and $s \in \mathrm{Des}(v^{S(u)})$. Hence $x = (v^{S(u)})_{S(u)\cup\{s\}}$ as expected. □

Corollary 16.

$$|\{u' : u' \text{ covers } u \text{ in } [u, v]\}| = |\mathrm{Des}(v^{S(u)})|.$$

Remark 17. The poset $(W, \le)$ does not not form a lattice, even after adding a maximal element, since $u \vee v$ does not exist in general (see for example the Hasse diagram for $(\mathfrak{S}_4, \le)$, where the two elements $(1,4,2,3)$ and $(2,3,1,4)$ do not have a unique least upper bound). In addition, the intervals are not in general distributive lattices (see for example the interval $[(1,2,3,4),(4,2,3,1)]$ in the poset $(\mathfrak{S}_4, \le)$). Moreover an interval is not modular since is not lower semimodular. Take in $\mathfrak{S}_4$ the interval $[1234, 4231]$. Then 4231 covers both 3124 and 1342 but $3124 \wedge 1342 = 1234$.

Proof of Theorem 2. Choose $u, v \in W$ such that $u < v$.

(1) By Lemma 14 we can construct a path $u_0 = u < u_1 < u_2 < \cdots < u_k < v = u_{k+1}$ such that $|S(u_{i+1})| = |S(u_i)| + 1$. Since if u' covers u, then it also holds that $|S(u')| = |S(u)| + 1$, hence the poset is graded.

(2) The fact that $([u,v],\wedge,\vee)$ is a lattice follows from Lemma 12. Now take $w \in [u,v]$ and $g,h \in [u,v]$ covering w. By Proposition 15 we know that there are distinct $s,r \in \mathrm{Des}(v^{S(w)})$ such that $S(g) = S(w)\cup\{s\}$ and $S(h) = S(w)\cup\{r\}$. By Lemma 12 we know that $S(g\vee h) = S(w)\cup\{r,s\}$. In other words $g \vee h$ covers both h and g. Hence $[u,v]$ is upper semimodular. □

Proof of Corollary 3. By [3, Theorem 3.1], a upper semilattice is shellable, hence Cohen-Macaulay. □

1.4. **Galois connection and proof of Theorem 4.**

For $I \subseteq J \subseteq S$, we will denote the poset

$$\mathcal{P}_I(J) = \{K \subseteq S | I \subseteq K \subseteq J\}$$

which is ordered by inclusion of subsets.

Choose $u,v \in W$ such that $u < v$ and set $I = S(u)$ and $J = S(v)$. We define two maps:

$$G_v : \mathcal{P}_I(J) \longrightarrow [u,v]$$

where $G_v(K) = v_K$ and

$$F : [u,v] \longrightarrow \mathcal{P}_I(J)$$

with $F(w) = S(w)$.

Theorem 18. *Let $u,v \in W$ such that $u \leq v$, and set $I = S(u)$ and $J = S(v)$.*

(*a*) $G_v \circ F = id_{[u,v]}$ *and* $F \circ G_v(K) \subseteq K$ *for all* $K \in \mathcal{P}_I(J)$.

(*b*) *The map*

$$F : ([u,v],\leq) \longrightarrow (\mathcal{P}_I(J),\subseteq)$$

is a poset monomorphism. Moreover, for $w,g \in [u,v]$, $F(w \vee g) = F(w) \cup F(g)$.

(*c*) *The map*

$$G_v : (\mathcal{P}_I(J),\subseteq,\cup,\cap) \longrightarrow ([u,v],\leq,\vee,\wedge)$$

is a poset epimorphism. Moreover, $G_v(K \cap L) = G_v(K) \wedge G_v(L)$ for $K,L \in \mathcal{P}_I(J)$.

(*d*) *For any $w \in [u,v]$, $G_v^{-1}(w)$ is a sublattice of $(\mathcal{P}_I(J),\subseteq,\cup,\cap)$. Moreover,*

$$\min G_v^{-1}(w) = S(w) \in \mathcal{P}_I(J)$$

and

$$\max G_v^{-1}(w) = J\backslash \mathrm{Des}(v^{S(w)}) \in \mathcal{P}_I(J)$$

Let $P = (P,\leq_P)$ and $Q = (Q,\leq_Q)$ be posets and $f : P \to Q$ and $g : Q \to P$ be order preserving maps. Recall that the pair (f,g) is called a Galois connection if for any $x \in Q$ and $y \in P$,

$$f(x) \leq_Q y \Leftrightarrow x \leq_P g(y).$$

Corollary 19. *The pair (F,G_v) is a Galois connection from $(\mathcal{P}_I(J),\subseteq)$ to $([u,v],\leq)$.*

Proof. By Theorem 18 G_v and F are order preserving. In addition, for $w \in [u,v]$ and $K \in \mathcal{P}_I(J)$, if $w \leq G_v(K)$, then by Theorem 18 part (d), $F(w) \subseteq F(G_v(K)) \subseteq K$ and if $F(w) \subseteq K$, then $w = G_v(F(w)) \leq G_v(K)$. □

Proof of Theorem 18. (part (a)) For a $w \in [u,v]$, we have that $G_v(F(w)) = G_v(S(w)) = v_{S(w)} = w$ by definition of $w \leq v$. For $K \in \mathcal{P}_I(J)$, then $F(G_v(K)) = S(v_K) \subseteq K$.

(part (b)) F is injective since $G_v \circ F = id_{[u,v]}$. It is order preserving by Lemma 5. Lemma 12 part (a) implies that $F(w \vee g) = F(w) \cup F(g)$.

(part (c)) G_v is surjective since $G_v \circ F = id_{[u,v]}$. Take K,L such that $I \subseteq K \subseteq L \subseteq J$, then by Lemma 8, $v_K \leq v_L$ and hence is order preserving.

(part (d)) Let $K, L \in \mathcal{P}_I(J)$ be such that $G_v(K) = G_v(L) = w \in [u, v]$. We have to show that $K \cap L$ and $K \cup L$ are in $G_v^{-1}(w)$. Note that $v_K = v_L = w$, therefore $w \in W_K \cap W_L = W_{K\cap L}$. Since $X_K, X_L \subseteq X_{K\cap L}$, then $v_{K\cap L} = w$ because of the uniqueness of the parabolic components. Hence $G_v(K \cap L) = w$.

From part (a), $S(w) = F \circ G_v(K) \subseteq K$ which implies that $\min G_v^{-1}(w) = S(w)$. Moreover, since $v = v^K v_K = v^L v_L$ and $v_K = v_L = w$, we have that $v^K = v^L \in X_K \cup X_L \subseteq X_{K\cup L}$. Also $w \in W_K \cup W_L \subseteq W_{K\cup L}$. Therefore $v_{K\cup L} = w$ and $K \cup L \in G_v^{-1}(w)$. Let $A := J\backslash \mathrm{Des}(v^{S(w)})$. By definition of $v^{S(w)}$, we have $S(w) \cap \mathrm{Des}(v^{S(w)}) = \emptyset$. Therefore $S(w) \subseteq A$, and Lemma 8 gives $w \leq v_A$. Given that $I \subset S(w)$ we have $I \cap \mathrm{Des}(v^{S(w)}) = \emptyset$ and this gives that $A \subset \mathcal{P}_I(J)$. Since $v^{S(w)} \in X_A$ and $v_{S(w)} \in W_{S(w)} \subseteq W_A$, we have $v^{S(w)} = v^A$ and $w = v_A$, therefore $G_v(A) = w$. □

The value of the Möbius function in Theorem 4 follows from the following classical result due to Rota.

Proposition 20. *([9] Theorem 1) If (f, g) is a Galois connection between P and Q, then for $a \in P$ and $b \in Q$,*

$$\sum_{\substack{x\in P\\ f(x)=b}} \mu_P(a, x) = \sum_{\substack{y\in Q\\ g(y)=a}} \mu_Q(y, b).$$

Moreover, both sums are equal to 0 unless $g(f(a)) = a$ and $f(g(b)) = b$, in which case they are both equal to $\mu_Q(f(a), b) = \mu_P(a, g(b))$.

Proof of Theorem 4. Corollary 19 implies that Proposition 20 applies for $P = [u, v]$ and $Q = \mathcal{P}_I(J)$ where $b = J = S(v)$, $I = S(u)$, and $a = u$. Since $F : [u, v] \to \mathcal{P}_I(J)$ is injective, we have

$$\mu(u, v) = \sum_{\substack{w\in[u,v]\\ F(w)=J}} \mu_{[u,v]}(u, w) = \sum_{\substack{L\in\mathcal{P}_I(J)\\ G_v(L)=u}} \mu_{\mathcal{P}_I(J)}(L, J). \tag{5}$$

Now Proposition 20 says that $\mu(u, v) = 0$ unless $G_v^{-1}(u)$ is a single element. Theorem 18 part (d) identifies that $G_v^{-1}(u)$ will contain exactly one element if and only if $S(u) = J\backslash\mathrm{Des}(u)$. In this case $\mu(u, v) = \mu(I, J) = (-1)^{|S(v)|-|S(u)|}$.

Now take $u = e$. Then the nonzero condition is equivalent to $\mathrm{Des}(v) = S(v) = J$, since $\mathrm{Des}(e) = S(e) = \emptyset$. This condition is a characterization of $w_0(J)$. □

Corollary 21.

$$\sum_{x\leq v} \mu(e, x) t^{|S(x)|} = (1 - t)^{|\mathrm{Des}(v)|}$$

Proof.

$$\begin{aligned}
\sum_{x\leq v} \mu(e, x) t^{|S(x)|} &= \sum_{\substack{x\leq v\\ x=w_0(S(x))}} (-1)^{|S(x)|} t^{|S(x)|} \\
&= \sum_{K\subseteq \mathrm{Des}(v)} (-1)^{|K|} t^{|K|} \\
&= (1 - t)^{|\mathrm{Des}(v)|}
\end{aligned}$$

□

2. Examples: Coxeter groups of type A, B and D

2.1. $W = \mathfrak{S}_n$**.** Let $\mathfrak{S}_n^{(k)}$ denote the set of permutations $\{w \in \mathfrak{S}_n : |C(w)| = k\}$. From the previous section, $(\mathfrak{S}_n, \leq)$ forms a graded poset with the elements of rank k are $\mathfrak{S}_n^{(n-k)}$. It is well known that $f_A(x) = 1 - 1/\sum_{n\geq 0} n!x^n = x + x^2 + 3x^3 + 13x^4 + 71x^5 + \cdots$ is a generating function for the number of elements in $\mathfrak{S}_n^{(0)}$ (see [6] or [14, A003319] and references therein). Now any permutation can be identified with an ordered sequence of permutations with empty connectivity set by breaking the permutation at the positions of the elements in the connectivity set and standardizing the segments. More precisely, if $C(u) = \{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_k}\}$, write $w = w_1 \ldots w_{k+1}$ as in Equation (4), then

$$\mathrm{std}_{C(u)}(w) = \mathrm{std}(w_1) \times \cdots \times \mathrm{std}(w_{k+1}) \in \mathfrak{S}_{i_1}^{(0)} \times \mathfrak{S}_{i_2-i_1}^{(0)} \times \cdots \times \mathfrak{S}_{n-i_k}^{(0)},$$

and this is clearly a bijection between $\mathfrak{S}_n$ and $\biguplus_\alpha \mathfrak{S}_{\alpha_1}^{(0)} \times \mathfrak{S}_{\alpha_2}^{(0)} \times \cdots \times \mathfrak{S}_{\alpha_{\ell(\alpha)}}^{(0)}$ where the index α runs over all compositions $(\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ of n. This implies that the generating function for $|\mathfrak{S}_n^{(k)}|$ with $n \geq 1$ is $f_A(x)^{k+1}$ and the proposition below follows from this remark (see also [1] Corollary 6.4).

Proposition 22. *The coefficient of $x^n t^k$ in the generating function $\frac{f_A(x)}{1-tf_A(x)}$ is equal to the number of permutations in the set $\mathfrak{S}_n^{(k)}$.*

$$\frac{f_A(x)}{1 - t f_A(x)} = 1 + x + x^2\,(t+1) + x^3\left(t^2 + 2\,t + 3\right) + x^4\left(t^3 + 3\,t^2 + 7\,t + 13\right) + x^5\left(t^4 + 4\,t^3 + 12\,t^2 + 32\,t + 71\right) + \cdots$$

2.2. $W = \mathfrak{B}_n$**.** Denote by $\mathfrak{S}_{[\pm n]}$ the symmetric group acting on the set $(-[n]) \cup [n]$. When W is equal to the hyperoctahedral group $\mathfrak{B}_n$ of order $2^n n!$ we can characterize the elements of this partial order more combinatorially on on signed permutations. Let $\mathfrak{B}_n$ represent the signed permutations: it is the subgroup of $\mathfrak{S}_{[\pm n]}$ consisisting of the element w such that $w(-i) = -w(i)$ for all $i \in [n]$. As a Coxeter group, it is generated by the elements $\{\tau_0, \tau_1, \ldots \tau_{n-1}\}$ where τ_0 is the transposition $(-1, 1)$ and τ_i is the product of the transpositions $(i, i+1)$ and $(-i, -i-1)$.

Proposition 23. *For $u \in \mathfrak{B}_n$ with $S = \{\tau_0, \tau_1, \ldots \tau_{n-1}\}$,*

$$C(u) = \{\tau_i : |u(j)| < u(k), \forall\ 0 \leq j \leq i < k < n\}. \tag{6}$$

with the convention that $u(0) = 0$.

Proof. Let $\tau_i \in C(u)$, then for $k > i$ and $j < i$, $\tau_k\tau_j = \tau_j\tau_k$ and hence $u \in W_{\{\tau_0,\tau_1,\ldots,\tau_{i-1}\}} \times W_{\{\tau_{i+1},\ldots,\tau_{n-1}\}} \cong \mathfrak{B}_i \times \mathfrak{S}_{n-i}$. For this reason $|u(j)| \leq i$ and $u(k) > i$ and hence $\tau_i \in \{\tau_i : |u(j)| < u(k), \forall 0 \leq j \leq i < k \leq n\}$.

Now assume that $\tau_i \in \{\tau_i : |u(j)| < u(k), \forall 0 \leq j \leq i < k \leq n\}$, then for $k > i$, $u(k) > 0$ and since $u(k) > \max\{|u(1)|, |u(2)|, \ldots, |u(i)|\}$, by the pigeon hole principle we know that $u(k) > i$ (since it is larger than i different positive values). For $j \leq i$, we know that $|u(j)| < \min\{u(i+1), \ldots, u(n)\} \leq i + 1$ and hence $u \in \mathfrak{B}_i \times \mathfrak{S}_{n-i}$ and hence $\tau_i \in C(u)$. □

Denote $\mathfrak{B}_n^{(k)} = \{w \in \mathfrak{B}_n : |C(w)| = k\}$. This characterization of the connectivity set in type B gives a method for calculating the number elements of $\mathfrak{B}_n^{(k)}$.

Proposition 24.

$$f_B(x) = \frac{\sum_{n\geq 0} 2^n n! x^n}{\sum_{n\geq 0} n! x^n} = 1 + x + 5\,x^2 + 35\,x^3 + 309\,x^4 + 3287\,x^5 + 41005\,x^6 + \cdots$$

is a generating function for the number of elements of $\mathfrak{B}_n$ with empty connectivity set. Moreover, the coefficient of $x^n t^k$ in the generating function $f_B(x)/(1 - tf_A(x))$ is equal to the number of elements in the set $\mathfrak{B}_n^{(k)}$.

$$\frac{f_B(x)}{1 - t f_A(x)} = 1 + x\,(t+1) + x^2\left(t^2 + 2\,t + 5\right) + \ x^3\left(t^3 + 3\,t^2 + 9\,t + 35\right) + x^4\left(t^4 + 4\,t^3 + 14\,t^2 + 56\,t + 309\right) + \cdots$$

Proof. Let $r = \min\{i : \tau_i \in C(u)\}$, then $u(1)u(2)\cdots u(r)$ is a word of a signed permutation representing an element with empty connectivity set and $std(u(r+1)u(r+2)\cdots u(n))$ is a word of a permutation of size $n-r$. It is not hard to see that this operation defines a bijection between $\mathfrak{B}_n$ and the set $\biguplus_{r=0}^{n} \mathfrak{B}_r^{(0)} \times \mathfrak{S}_{n-r}$. Therefore if $f_B(x) = \sum_{n\geq 0} |\mathfrak{B}_n^{(0)}| x^n$, then the generating function will satisfy $\sum_{n\geq 0} 2^n n! x^n = f_B(x) \sum_{n\geq 0} n! x^n$. Moreover, the proof for Proposition 22 shows that $f_B(x) f_A(x)^k$ is a generating function for the number of elements of $\mathfrak{B}_n^{(k)}$. $\square$

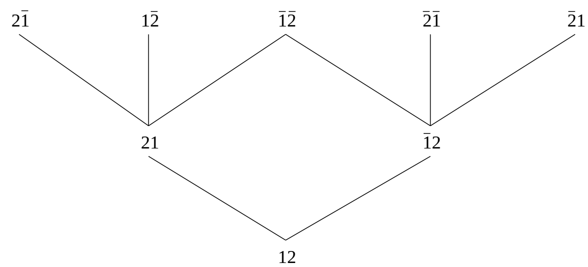

FIGURE 3. Hasse diagram for posets of $(\mathfrak{B}_2, \leq)$

2.3. $W = \mathfrak{D}_n$**.** For $n \geq 2$, let $\mathfrak{D}_n$ denote the Coxeter group of type D of order $2^{n-1}n!$. $\mathfrak{D}_n$ can be realized as a subgroup of $\mathfrak{B}_n$ of signed permutations and is generated by $\{t_0, t_1, \ldots, t_{n-1}\}$ where for $i \geq 1$, $t_i = \tau_i$ and $t_0 = \tau_0 t_1 \tau_0$. In other words, $t_0(1) = -2$ and $t_0(2) = -1$.

Denote $\mathfrak{D}_n^{(k)} = \{w \in \mathfrak{D}_n : |C(w)| = k\}$.

Proposition 25. *For $u \in \mathfrak{D}_n$, if ($u(1) = 1$ and $u(k) > 0$ for $1 \leq k \leq n$) or ($u(1) < -1$ and $u^{-1}(1) < -1$ and $|\{1 \leq i \leq n : u(i) < 0\}| = 2$), then*

$$C(u) = \{t_1\} \cup \{t_i : i \neq 1 \text{ and } |u(j)| < u(k), \forall\ 0 \leq j \leq i < k < n\} \tag{7}$$

otherwise

$$C(u) = \{t_i : i \neq 1 \text{ and } |u(j)| < u(k), \forall\ 0 \leq j \leq i < k < n\}. \tag{8}$$

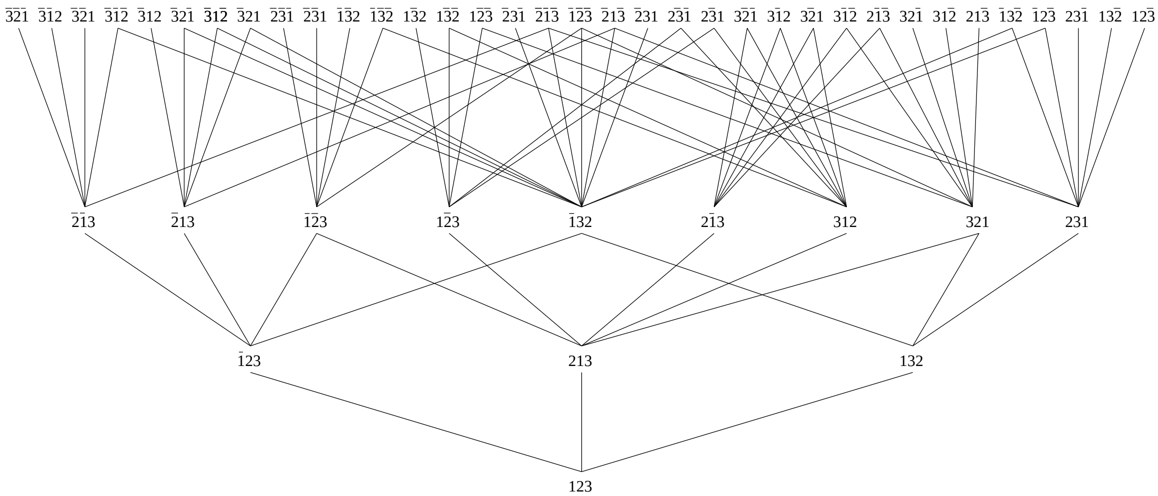

FIGURE 4. Hasse diagram for posets of $(\mathfrak{B}_3, \leq)$

Proof. For $u \in \mathfrak{D}_n \subset \mathfrak{B}_n$, we denote $C_B(u)$ the connectivity set of u viewed as an element of $\mathfrak{B}_n$. Then observe that $C_B(u) \setminus \{t_1\} = C(u) \setminus \{t_1\}$.

We are left to consider when t_1 is in $C(u)$. Clearly $\{t_0, t_1\} \subseteq C(w)$ if and only if $u(i) > 0$ for all $i > 0$ and $u(1) = 1$. Consider now the case where $u \in W_{\{t_0,t_2,\ldots,t_{n-1}\}}$. If $u \in W_{\{t_2\ldots t_{n-1}\}}$, then $u(1) = 1$ and $u(k) = 2$ for some $k \geq 2$. Hence $t_0u(1) = -2$ and $t_0u(k) = -1$ and $|\{1 \leq i \leq n : u(i) < 0\}| = 2$. Assume by induction that $u \in W_{\{t_0,t_2,t_3\ldots t_{n-1}\}}$ and that the only two negative values of $u(i)$ are $u(1) = -\ell$ and $u(k) = -1$ for some $k \geq 2$. If $\ell = 2$, then t_0u has no negative signs and $t_0u(1) = 1$ and hence $t_0u \in W_{\{t_2\ldots t_{n-1}\}}$. If $\ell \neq 2$, then $u(d) = 2$ for some d and $t_0u(d) = -1$ and $t_0u(k) = 2$ and $t_0u(1) = -\ell$. Hence by induction, if $u \in W_{\{t_0,t_2,t_3,\ldots t_{n-1}\}}$, then $u(1) = 1$ and $u(k) > 0$ for $1 \leq k \leq n$, or $u(1) < -1$ and $u^{-1}(1) < -1$ and $|\{1 \leq i \leq n : u(i) < 0\}| = 2$.

For the converse, assume that $u(1) < -1$ and $u^{-1}(1) < -1$ and $|\{1 \leq i \leq n : u(i) < 0\}| = 2$. Then let $k = -u(1)$ and set $w = t_0t_2t_3\cdots t_{k-1}u$. It is easy to check that $w \in \mathfrak{S}_n$ and $w(1) = 1$. Therefore $t_1 \in C(w)$ and so t_1 does not appear in a reduced expression for w. Therefore if $t_{i_1}\cdots t_{i_{\ell(w)}}$ is a reduced word for w, then $t_{k-1}t_{k-2}\cdots t_2t_0t_{i_1}\cdots t_{i_{\ell(w)}}$ is a word for u which does not contain a t_1 and so $t_1 \in C(u)$. □

Proposition 26.

$$f_D(x) = \frac{3 + \sum_{n\geq 0} 2^n n! x^n}{2\sum_{n\geq 0} n! x^n} + x - 2 = x^2 + 13\,x^3 + 135\,x^4 + 1537\,x^5 + 19811\,x^6 + \cdots \tag{9}$$

is a generating function for the number of elements of $\mathfrak{D}_n$ with empty connectivity set. Moreover, the coefficient of x^nt^k in the generating function $\frac{2tf_A(x)+t^2xf_A(x)+f_D(x)}{(1-tf_A(x))}$ is equal to the number of elements in the set $\mathfrak{D}_n^{(k)}$.

$$\frac{2tf_A(x) - 2tx + t^2xf_A(x) + f_D(x)}{(1 - tf_A(x))} = x^2\,(t^2 + 2t + 1) + x^3\,\left(t^3 + 3\,t^2 + 7\,t + 13\right) + x^4\,(t^4 + 4\,t^3 + 12\,t^2 + 40\,t + 135) + \cdots$$

Proof. Let $f_D(x) = \sum_{n\geq 2} |\mathfrak{D}_n^{(0)}|x^n$ be the generating function for the number of elements of $\mathfrak{D}_n$ with empty connectivity set and $f_A(x)$ be the generating function for the number of elements of $\mathfrak{S}_n$ with empty connectivity set from Proposition 22.

Now take w to be an element of $\mathfrak{D}_n^{(k)}$ such that neither t_0 nor t_1 are elements in $C(w) = \{t_{i_1}, \ldots, t_{i_k}\}$ then $i_1 > 1$ and $w \in W_{\{t_0,t_1,\ldots,t_{i_1-1}\}} \times W_{\{t_{i_1+1},\ldots,t_{i_2-1}\}} \times \cdots \times W_{\{i_k+1,\ldots n-1\}} \cong \mathfrak{D}_{i_1} \times \mathfrak{S}_{i_2-i_1} \times \cdots \times \mathfrak{S}_{n-i_k}$. Since each component must have empty connectivity set, the generating function for these elements is $f_D(x)f_A(x)^k$.

Take w to be an element of $\mathfrak{D}_n^{(k)}$ such that both of t_0 or t_1 are elements in $C(w)$, then $w \in W_{\{t_2,\ldots,t_{n-1}\}}$ and has exactly $k-2$ other elements in the connectivity set. By Proposition 22 the generating function for these elements will be $xf_A(x)^{k-1}$.

Now for an element $w \in \mathfrak{D}_n^{(k)}$ with exactly one of $t_0, t_1 \in C(w)$ (take w.l.o.g. t_0), then $w \in W_{\{t_1,t_2,\ldots,t_{n-1}\}} \cong \mathfrak{S}_n$ and has exactly $k-1$ other elements in the connectivity set, but is not in $W_{\{t_2,\ldots,t_{n-1}\}}$. Therefore the generating function for these elements is given by $f_A(x)^k - xf_A(x)^{k-1}$.

Since $\mathfrak{D}_n = \biguplus_{k\geq 0} \mathfrak{D}_n^{(k)}$, we have the following generating function equation

$$\sum_{n\geq 2} 2^{n-1}n!x^n = f_D(x) + \sum_{k\geq 1}(f_D(x)f_A(x)^k + xf_A(x)^{k-1} + 2(f_A(x)^k - xf_A(x)^{k-1})) - x.$$

A bit of algebraic manipulation yields the equation (9) and

$$\sum_{n\geq 2} |\mathfrak{D}_n^{(k)}(x)|x^nt^k = f_D(x) + \sum_{k\geq 1} t^k(f_D(x)f_A(x)^k + xf_A(x)^{k-1} + 2(f_A(x)^k - xf_A(x)^{k-1})) - tx$$

is an expression for the bigraded generating function formula in the proposition. □

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