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# Generalized Descent Algebras

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*Abstract.* If $A$ is a subset of the set of reflections of a finite Coxeter group $W$, we define a sub-$\mathbb{Z}$-module $\mathcal{D}_A(W)$ of the group algebra $\mathbb{Z}W$. We discuss cases where this submodule is a subalgebra. This family of subalgebras includes strictly the Solomon descent algebra, the group algebra and, if $W$ is of type $B$, the Mantaci–Reutenauer algebra.

## Introduction

Let $(W, S)$ be a finite Coxeter system whose length function is denoted by $\ell\colon W \to \mathbb{N} = \{0, 1, 2, \dots\}$. In 1976, Solomon introduced a remarkable subalgebra $\Sigma W$ of the group algebra $\mathbb{Z}W$, called the *Solomon descent algebra* [11]. Let us recall its definition. If $I \subset S$, let $W_I$ denote the *standard parabolic subgroup* generated by $I$. Then

$$X_I = \{w \in W \mid \forall s \in I,\ \ell(ws) > \ell(w)\}$$

is a set of *minimal length coset representatives* of $W/W_I$. Let $x_I = \sum_{w \in X_I} w \in \mathbb{Z}W$. Then $\Sigma W$ is defined as the sub-$\mathbb{Z}$-module of $\mathbb{Z}W$ spanned by $(x_I)_{I \subset S}$. The study of this algebra is strongly related to the study of many problems in symmetric groups and Coxeter groups, see for instance [2, 5, 10, 12].

In [3], the authors constructed a subalgebra $\Sigma'(W_n)$ of the group algebra $\mathbb{Z}W_n$ of the Coxeter group $W_n$ of type $B_n$: it turns out that this subalgebra is defined from "generalized descent sets" relative to a larger set of reflections than $S$ and that it contains $\Sigma W_n$. In fact $\Sigma'(W_n)$ is the Mantaci–Reutenauer algebra [8].

It is natural to ask whether this kind of construction can be generalized to other groups. Let us explain now what kind of subalgebras we are looking for.

Let $T = \{wsw^{-1} \mid w \in W \text{ and } s \in S\}$ be the set of reflections in $W$. Let $A$ be a fixed subset of $T$. If $w \in W$, let $D_A(w) = \{s \in A \mid \ell(ws) < \ell(w)\} \subset A$ be the $A$-descent set of $w$. A subset $I$ of $A$ is said to be $A$-*admissible* if there exists $w \in W$ such that $D_A(w) = I$. Let $\mathcal{P}_{\mathrm{ad}}(A)$ denote the set of $A$-admissible subsets of $A$. If $I \in \mathcal{P}_{\mathrm{ad}}(A)$, we set $D_I^A = \{w \in W \mid D_A(w) = I\}$ and $d_I^A = \sum_{w \in D_I^A} w \in \mathbb{Z}W$. Now, let

$$\mathcal{D}_A(W) = \bigoplus_{I \in \mathcal{P}_{\mathrm{ad}}(A)} \mathbb{Z} d_I^A.$$

As an example, $\mathcal{D}_S(W) = \Sigma_S(W) = \Sigma W$. In fact, they are precisely the $\Sigma W$-modules defined in [9] (see Remark 1.12). The main theorem of this paper is the following (here, $C(w)$ denotes the conjugacy class of $w$ in $W$).

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***Theorem A*** *If there exist two subsets $S_1$ and $S_2$ of $S$ such that $A = S_1 \cup (\bigcup_{s\in S_2} C(s))$, then $\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$.*

This theorem is a generalization of Atkinson's proof [1] of Solomon's result (take $A = S$). It provides a generalization of Atkinson's proof to the case of the Mantaci–Reutenauer algebra (take $A = \{s_1, \dots, s_{n-1}\} \cup C(t)$, where $S = \{t, s_1, \dots, s_{n-1}\}$ satisfies $C(t) \cap \{s_1, \dots, s_{n-1}\} = \varnothing$). But for instance the theorem also gives another algebra in type $B_n$ (take $A = \{t\} \cup C(s_1)$) and a new algebra in type $F_4$ of $\mathbb{Z}$-rank 300 (take $A = \{s_1, s_2\} \cup C(s_3)$, where $S = \{s_1, s_2, s_3, s_4\}$ satisfies $\{s_1, s_2\} \cap C(s_3) = \varnothing$). Moreover, if $A = T$, we get that $\mathcal{D}_A(W) = \mathbb{Z}W$ (see Example 1.7). In the case of dihedral groups, we get another family of algebras.

***Theorem B*** *If $W$ is a dihedral group of order $4m$ ($m \geq 1$), $S = \{s, t\}$ and $A = \{s, t, sts\}$ or $A = \{t, sts\}$, then $\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$.*

It must be noted that the algebras constructed in Theorems A and B are not necessarily unitary. More precisely, $1 \in \mathcal{D}_A(W)$ if and only if $S \subset A$. Moreover, if $S \subset A$, then $\Sigma W \subset \mathcal{D}_A(W)$. Some computations with GAP suggest that the following question has a positive answer. If $\mathcal{D}_A(W)$ is a unitary subalgebra of $\mathbb{Z}W$, is it true that $A$ is one of the subsets mentioned in Theorems A and B?

This paper is organized as follows. Section 1 is essentially devoted to the proofs of Theorems A and B. In Section 2, we discuss more precisely the case of dihedral groups.

## 1 Descent Sets

Let $(W, S)$ be a finitely generated Coxeter system (not necessary finite). If $s, s' \in S$, we denote by $m(s, s')$ the order of $ss' \in W$. If $W$ is finite, we denote by $w_0$ its longest element.

### 1.1 Root System

Let $V$ be an $\mathbb{R}$-vector space endowed with a basis indexed by $S$ denoted by $\Delta = \{\alpha_s \mid s \in S\}$. Let $B\colon V \times V \to \mathbb{R}$ be the symmetric bilinear form such that

$$B(\alpha_s, \alpha_{s'}) = -\cos\Big(\frac{\pi}{m(s, s')}\Big)$$

for all $s, s' \in S$. If $s \in S$ and $v \in V$, we set $s(v) = v - 2B(\alpha_s, v)\alpha_s$. Thus $s$ acts as the reflection in the hyperplane orthogonal to $\alpha_s$ (for the bilinear form $B$). This extends to an action of $W$ on $V$ as a group generated by reflections. It stabilizes $B$.

We recall some basic terminology on root systems (see for instance [4, 6]). The *root system* of $(W, S)$ is the set $\Phi = \{w(\alpha_s) \mid w \in W, s \in S\}$ and the elements of $\Delta$ are the *simple roots*. The roots contained in

$$\Phi^+ = \Big(\sum_{\alpha\in\Delta} \mathbb{R}^+\alpha\Big) \cap \Phi$$

are said to be *positive*, while those contained in $\Phi^- = -\Phi^+$ are said to be *negative*. Moreover, $\Phi$ is the disjoint union of $\Phi^+$ and $\Phi^-$. If $w \in W$, $\ell(w) = |N(w)|$, where

$$N(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}.$$

Let $\alpha = w(\alpha_s) \in \Phi$; then $s_\alpha = wsw^{-1}$ acts as the reflection in the hyperplane orthogonal to $\alpha$ and $s_\alpha = s_{-\alpha}$. Therefore, the *set of reflections of W*

$$T = \bigcup_{w \in W} wSw^{-1}$$

is in bijection with $\Phi^+$ (and thus $\Phi^-$).

Let us recall the following well-known result.

***Lemma 1.1*** *Let $w \in W$.*

(i) *If $\alpha \in \Phi^+$, then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) \in \Phi^+$.*

(ii) *If $s \in S$, then*

$$N(sw) = \begin{cases} N(w) \coprod \{w^{-1}(\alpha_s)\} & \text{if } \ell(sw) > \ell(w), \\ N(w) \setminus \{-w^{-1}(\alpha_s)\} & \text{otherwise.} \end{cases}$$

Henceforth, we fix a subset $A$ of $T$. We start with easy observations.

As a consequence of Lemma 1.1(i), we get that

$$D_A(w) = \{s_\alpha \in A \mid \alpha \in \Phi^+ \text{ and } w(\alpha) \in \Phi^-\}.$$

We also set

$$N_A(w) = \{\alpha \in \Phi^+ \mid s_\alpha \in A \text{ and } w(\alpha) \in \Phi^-\}.$$

The map $N_A(w) \to D_A(w)$, $\alpha \mapsto s_\alpha$ is then a bijection.

## 1.2 Properties of the Map $D_A$

First, using Lemma 1.1(ii), we get the following.

***Corollary 1.2*** *If $s \in S$ and if $w \in W$ is such that $w^{-1}sw = s_{w^{-1}(\alpha_s)} \notin A$, then $N_A(w) = N_A(sw)$ (and $D_A(w) = D_A(sw)$).*

***Remark 1.3*** If $A_1 \subset A_2 \subset T$, then $D_{A_1}(w) = D_{A_2}(w) \cap A_1$ for all $w \in W$. Therefore if $W$ is finite, $\mathcal{D}_{A_1}(W) \subset \mathcal{D}_{A_2}(W)$.

***Proposition 1.4*** *We have*

(i) *$\varnothing$ is A-admissible;*

(ii) *$D_\varnothing^A = \{1\}$ if and only if $S \subset A$.*

**Proof** We have $D_A(1) = \varnothing$ so (i) follows. If $s \in S \setminus A$, then $D_A(s) = \varnothing$. This shows (ii). ■

The notion of $A$-descent set is obviously compatible with direct products:

***Proposition 1.5*** *Assume that $W = W_1 \times W_2$ where $W_1$ and $W_2$ are standard parabolic subgroups of $W$. Then for all $I \in \mathcal{P}_{\mathrm{ad}}(A)$, we have*

$$D_I^A = D_{I\cap W_1}^{A\cap W_1} \times D_{I\cap W_2}^{A\cap W_2}.$$

***Corollary 1.6*** *Assume that $W$ is finite and that $W = W_1 \times W_2$ where $W_1$ and $W_2$ are standard parabolic subgroups of $W$. Then $\mathcal{D}_A(W) = \mathcal{D}_{A\cap W_1}(W_1) \otimes_{\mathbb{Z}} \mathcal{D}_{A\cap W_2}(W_2)$.*

***Example 1.7*** Consider the case where $A = T$ (then $N_A(w) = N(w)$). It is well known [4, Chapter VI, Exercise 16] that the map $w \mapsto N(w)$ from $W$ onto the set of subsets of $\Phi^+$ is injective (observe that if $\alpha \in N(w_1 w_2^{-1})$, then $\pm w_2^{-1}(\alpha)$ lives in the union, but not in the intersection, of $N(w_1)$ and $N(w_2)$). Therefore, the map $W \to \mathcal{P}_{\mathrm{ad}}(T)$, $w \mapsto D_T(w)$ is injective. In particular, if $W$ is finite, then $\mathcal{D}_T(W) = \mathbb{Z}W$.

In the case of finite Coxeter groups, the multiplication on the left by the longest element has the following easy property.

***Proposition 1.8*** *If $W$ is finite and if $w \in W$, then $D_A(w_0 w) = A \setminus D_A(w)$.*

***Corollary 1.9*** *If $W$ is finite, then*

(i) *$A$ is $A$-admissible;*
(ii) *$I \in \mathcal{P}_{\mathrm{ad}}(A)$ if and only if $A \setminus I \in \mathcal{P}_{\mathrm{ad}}(A)$;*
(iii) *$D_A^A = \{w_0\}$ if and only if $S \subset A$.*

**Proof** $D_A(w_0) = A$, so (i) follows. (ii) follows from Proposition 1.8. (iii) follows from Proposition 1.8 and Proposition 1.4(ii). ■

## 1.3 Left-Connectedness

Atkinson gave a new proof of Solomon's result by using an equivalence relation to describe descent sets [1]. We extend his result to $A$-descent sets. It shows in particular that the subsets $D_I^A$ are left-connected (recall that a subset $E$ of $W$ is said to be *left-connected* if, for all $w, w' \in E$, there exists a sequence $w = w_1, w_2, \ldots, w_r = w'$ of elements of $E$ such that $w_{i+1} w_i^{-1} \in S$ for every $i \in \{1, 2, \ldots, r-1\}$).

Let $w$ and $w'$ be two elements of $W$. We say that $w$ is an *$A$-descent neighborhood* of $w'$, and write $w \smile_A w'$, if $w' w^{-1} \in S$ and $w^{-1} w' \notin A$. It is easily seen that $\smile_A$ is a symmetric relation. The reflexive and transitive closure of the $A$-descent neighborhood relation is called *the $A$-descent equivalence*, and is denoted by $\sim_A$. The next proposition characterizes this equivalence relation in terms of $A$-descent sets.

***Proposition 1.10*** *Let $w, w' \in W$. Then*

$$w \sim_A w' \Leftrightarrow D_A(w) = D_A(w') \Leftrightarrow N_A(w) = N_A(w').$$

**Proof** The second equivalence is clear. If $w \smile_A w'$, then it follows from Corollary 1.2 that $N_A(w) = N_A(w')$. It remains to show that if $N_A(w) = N_A(w')$, then $w \sim_A w'$.

So, assume that $N_A(w) = N_A(w')$. Write $x = w'w^{-1}$ and let $m = \ell(x)$. If $\ell(x) = 0$, then $w = w'$ and we are done. Assume that $m \geq 1$, and write $x = s_1 s_2 \cdots s_m$ with $s_i \in S$. We now want to prove by induction on $m$ that

$$(*) \qquad w \smile_A s_m w \smile_A s_{m-1}s_m w \smile_A \cdots \smile_A s_2 \cdots s_m w \smile_A s_1 s_2 \cdots s_m w = w'.$$

First, assume that $w \not\smile_A s_m w$. In other words, $w^{-1}s_m w \in A$. For simplification, let $\alpha_i = \alpha_{s_i}$. By Lemma 1.1(ii), we have

$$N_A(s_m w) = N_A(w) \coprod \{w^{-1}(\alpha_m)\} \quad \text{or} \quad N_A(s_m w) = N_A(w) \setminus \{-w^{-1}(\alpha_m)\}.$$

In the first case, as $N_A(w) = N_A(w')$, and by applying Lemma 1.1(ii) again, there exists a step $i \in \{1, 2, \ldots, m-1\}$ between $N_A(w)$ to $N_A(w')$ where $w^{-1}(\alpha_m)$ is removed from $N_A(s_i \cdots s_m w)$, that is, $(s_{i+1} \cdots s_m w)^{-1}(\alpha_i) = -w^{-1}(\alpha_m)$. In the same way, we get the same result in the second case. In other words, we have proved that there exists $i \in \{1, 2, \ldots, m-1\}$ such that $s_m \cdots s_{i+1}(\alpha_i) = -\alpha_m$, so, by Lemma 1.1(i), we have $\ell(s_m \cdots s_{i+1} s_i) < \ell(s_m \cdots s_{i+1})$. This contradicts the fact that $m = \ell(x)$. So $w \smile_A s_m w$, and then $N_A(w) = N_A(s_m w)$. Hence $N_A(s_m w) = N_A(w')$ and $w'(s_m w)^{-1} = s_1 s_2 \cdots s_{m-1}$. We then get by induction that

$$s_m w \smile_A s_{m-1} s_m w \smile_A \cdots \smile_A s_1 s_2 \cdots s_m w = w',$$

which shows $(*)$. ■

***Corollary 1.11*** *If $I \in \mathcal{P}_{\mathrm{ad}}(A)$, then $D_I^A$ is left-connected.*

***Remark 1.12*** The above corollary was first stated by Tits [13, Theorem 2.19] as follows: Let $s \in T$; we denote by $X_s$ the set of $w \in W$ such that $\ell(ws) > \ell(w)$. Then for all $J \subset A \subset T$, the set $Y_J^A = \bigcap_{s \in J} X_s \cap \bigcap_{s \in A \setminus J} (W \setminus X_s)$ is left-connected. An easy computation shows that $Y_J^A = D_{A \setminus J}^A$ (possibly empty). Proposition 1.10 provides a new (shorter) proof of Tits' theorem.

We mention that, using this result and the terminology of Tits, Moszkowski [9] has shown that for all $A \subset T$, $\mathcal{D}_A(W)$ is a $\Sigma W$-module for the left multiplication.

## 1.4 Nice Subsets of $T$

We say that $A$ is *nice* if for every $s \in A$ and $w \in W$ such that $w^{-1}sw \notin A$, we have $D_A(sw) = D_A(w)$. Notice that every subset of $S$ is nice, by Corollary 1.2.

If $w \in W$ and $I, J \in \mathcal{P}_{\mathrm{ad}}(A)$, we set

$$D_A(I, J, w) = \{(u, v) \in D_I^A \times D_J^A \mid uv = w\}.$$

The next lemma gives a characterization of the fact that $\mathcal{D}_A(W)$ is an algebra in terms of these sets.

***Lemma 1.13*** *Assume that $W$ is finite. Then the following are equivalent.*

(i) *$\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$.*
(ii) *For all $I, J \in \mathcal{P}_{\mathrm{ad}}(A)$ and for all $w, w' \in W$ such that $D_A(w) = D_A(w')$, we have $|D_A(I, J, w)| = |D_A(I, J, w')|$.*

*If these conditions are fulfilled, we choose for any $I \in \mathcal{P}_{\mathrm{ad}}(A)$ an element $z_I$ in $D_I^A$. Then*

$$d_I^A d_J^A = \sum_{K \in \mathcal{P}_{\mathrm{ad}}(A)} |D_A(I, J, z_K)| d_K^A.$$

**Proof** That (ii) implies (i) is obvious.

If (i) is true, then, on the one hand,

$$d_I^A d_J^A = \sum_{K \in \mathcal{P}_{\mathrm{ad}}(A)} c_K d_K^A$$

for some $c_K \in \mathbb{Z}$. On the other hand,

$$d_I^A d_J^A = \sum_{D_A(u)=I} \sum_{D_A(v)=J} uv = \sum_{K \in \mathcal{P}_{\mathrm{ad}}(A)} \sum_{D_A(w)=K} |D_A(I, J, w)| w.$$

(ii) follows by identification. ∎

Let us now fix $s \in S$ and let $(u, v) \in W \times W$. If $u \smile_A su$, we set $\psi_s^A(u, v) = (su, v)$. If $u \not\smile_A su$, then we set $\psi_s^A(u, v) = (u, u^{-1}suv)$. Note that in the last case, $u^{-1}su \in A$. We have $(\psi_s^A)^2 = \mathrm{Id}_{W \times W}$. In particular, $\psi_s^A$ is a bijection. Using $\psi_s^A$, one can relate the notion of nice subsets to the property (ii) stated in Lemma 1.13.

***Proposition 1.14*** *Assume that $A$ is nice. Let $I, J \in \mathcal{P}_{\mathrm{ad}}(W)$, let $w \in W$, and let $s \in S$ be such that $w \smile_A sw$. Then $\psi_s^A(D_A(I, J, w)) = D_A(I, J, sw)$.*

**Proof** Let $(u, v)$ be an element of $D_A(I, J, w)$. By symmetry, we only need to prove that $\psi_s^A(u, v) \in D_A(I, J, sw)$. If $u \smile_A su$, then $D_A(su) = D_A(u) = I$ by Proposition 1.10, so $\psi_s^A(u, v) = (su, v) \in D_A(I, J, sw)$. So, we may assume that $u \not\smile_A su$. Let $s' = u^{-1}su \in A$. Note that $w^{-1}sw = v^{-1}s'v \notin A$. Then $\psi_s^A(u, v) = (u, s'v)$ and $us'v = sw$. So we only need to prove that $D_A(s'v) = D_A(v)$. But this just follows from the definition of nice subset of $T$. ∎

***Corollary 1.15*** *If $W$ is finite and if $A$ is nice, then $\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$. It is unitary if and only if $S \subset A$.*

**Proof** This follows from Lemma 1.13 and from Propositions 1.10 and 1.14. ∎

## 1.5 Proof of Theorems A and B

Using Corollary 1.15, we see that Theorems A and B are direct consequences of the following theorem (which holds also for infinite Coxeter groups).

***Theorem 1.16*** *Assume that one of the following holds.*

(i) *There exist two subsets $S_1$ and $S_2$ of $S$ such that $A = S_1 \cup \left(\bigcup_{s \in S_2} C(s)\right)$.*
(ii) *$S = \{s, t\}$, $m(s, t)$ is even or $\infty$, and $A = \{s, t, sts\}$ or $A = \{t, sts\}$.*

*Then $A$ is nice.*

**Proof** Assume that (i) or (ii) holds. Let $r \in A$ and let $w \in W$ be such that $w^{-1}rw \notin A$. We want to prove that $D_A(rw) = D_A(w)$. By symmetry, we only need to show that $D_A(rw) \subset D_A(w)$. If $r \in S$, then this follows from Corollary 1.2. So we may assume that $r \notin S$.

Assume that (i) holds. Write $A' = \bigcup_{s \in S_2} C(s)$ and $S' = A \setminus A'$. Then $S' \subset S$, $A = A' \coprod S'$ and $A'$ is stable under conjugacy. Then $r \in A'$ and $w^{-1}rw \in A' \subset A$, which contradicts our hypothesis.

Assume that (ii) holds. If $m(s, t) = 2$, then $A$ is contained in $S$ and therefore is nice by Corollary 1.2. So we may assume that $m(s, t) \geq 4$. Since $r \notin S$, we have $r = sts$. Assume that $D_A(rw) \not\subset D_A(w)$. Let

$$\Phi_A = \{\alpha \in \Phi^+ \mid s_\alpha \in A\} \subset \{\alpha_s, \alpha_t, s(\alpha_t)\}.$$

There exists $\alpha \in \Phi_A$ such that $rw(\alpha) \in \Phi^-$ and $w(\alpha) \in \Phi^+$. So $w(\alpha) \in N(r) = \{\alpha_s, s(\alpha_t), st(\alpha_s)\}$. Since $w^{-1}rw \notin A$, we have that $ws_\alpha w^{-1} \neq r$, so $w(\alpha) \neq s(\alpha_t)$. So $w(\alpha) = \alpha_s$ or $st(\alpha_s)$. But the roots $\alpha_s$ and $\alpha_t$ lie in different $W$-orbits, so $\alpha = \alpha_s$ and $w(\alpha_s) \in \{\alpha_s, st(\alpha_s)\}$. If $W$ is infinite, this gives that $w \in \{1, st\}$, which contradicts the fact that $w^{-1}rw \notin A$. If $W$ is finite, this gives that $w \in \{1, st, w_0 s, stsw_0\}$. But again, this contradicts the fact that $w^{-1}rw \notin A$. ■

## 1.6 A Remark Concerning the Solomon Homomorphism

Let $\mathbb{Z}\operatorname{Irr} W$ denote the characters algebra of $W$, $\Sigma W$ is endowed with a $\mathbb{Z}$-linear map $\theta\colon \Sigma W \to \mathbb{Z}\operatorname{Irr} W$ satisfying $\theta(x_I) = \operatorname{Ind}_{W_I}^W 1_{W_I}$. This is an algebra homomorphism [11].

If $W$ is the symmetric group $\mathfrak{S}_n$, then $\theta$ becomes an epimorphism and the pair $(\Sigma\mathfrak{S}_n, \theta)$ provides a nice construction [7] of $\operatorname{Irr}(\mathfrak{S}_n)$, which is the first ingredient of several recent works, see for instance [12]. However, the morphism $\theta$ is surjective if and only if $W$ is a product of symmetric groups.

In [3], the authors have shown that the Mantaci–Reutenauer algebra is also endowed with an algebra homomorphism $\theta'\colon \Sigma'(W_n) \to \mathbb{Z}\operatorname{Irr} W_n$ extending $\theta$. Moreover, $\theta'$ is surjective and $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta'$ is the radical of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_n)$. This leads to a construction of the irreducible characters of $W_n$ following Jőllenbeck's strategy.

However, the situation seems to be much more complicated in the other types. Let us define another sub-$\mathbb{Z}$-module of $\mathbb{Z}W$. If $I \subset A$, we still denote by $W_I$ the subgroup of $W$ generated by $I$ and we still set

$$X_I = \{w \in W \mid \forall\, x \in W_I,\ \ell(wx) \geq \ell(w)\}.$$

Then $X_I$ is again a set of representatives for $W/W_I$. Now, let $x_I = \sum_{w \in X_I} w \in \mathbb{Z}W$. Then $\Sigma_A(W) = \sum_{I \subset A} \mathbb{Z}x_I$ is a sub-$\mathbb{Z}$-module of $\mathbb{Z}W$. However, it is not in general a subalgebra of $\mathbb{Z}W$.

The bad point in the above construction is that in many cases one has $\Sigma_A(W) \neq \mathcal{D}_A(W)$. Also, we are not able to construct in general a morphism of algebras $\theta_A \colon \mathcal{D}_A(W) \to \mathbb{Z}\operatorname{Irr} W$ extending $\theta$ if $S \subset A$ (see Remark 2.2 and Proposition 2.3 below).

***Example 1.17*** Assume here that $W$ is of type $F_4$, that $S = \{s_1, s_2, s_3, s_4\}$ and that $A = C(s_1) \cup S$. Then, using GAP, one can see that $\operatorname{rank}_{\mathbb{Z}} \mathcal{D}_A(W) = 300$ and $\operatorname{rank}_{\mathbb{Z}} \Sigma_A(W) = 149$. Moreover, $\Sigma_A(W)$ is not a subalgebra of $\mathcal{D}_A(W)$.

# 2 Example: The Dihedral Groups

The aim of this section is to study the unitary subalgebras $\mathcal{D}_A(W)$ constructed in Theorems A and B whenever $W$ is finite and dihedral. Henceforth, we assume that $S = \{s, t\}$ with $s \neq t$ and that $m(s, t) = 2m$, with $2 \leq m < \infty$.

Note that $w_0 = (st)^m$ is central. In what follows, we will need some facts on the character table of $W$. Let us recall here the construction of $\operatorname{Irr} W$. First, let $H$ be the subgroup of $W$ generated by $st$. It is normal in $W$, of order $2m$ (in other words, of index 2). We choose the primitive $(2m)$-th root of unity $\zeta \in \mathbb{C}$ of argument $\pi/m$. If $i \in \mathbb{Z}$, we denote by $\xi_i \colon H \to \mathbb{C}^\times$ the unique linear character such that $\xi_i(st) = \zeta^i$. Then $\operatorname{Irr} H = \{\xi_i \mid 0 \leq i \leq 2m - 1\}$. Now let $\chi_i = \operatorname{Ind}_H^W \xi_i$. Then $\chi_i = \chi_{2m-i}$ and, if $1 \leq i \leq m - 1$, $\chi_i \in \operatorname{Irr} W$. Also, $\chi_i$ has values in $\mathbb{R}$. More precisely, for $1 \leq i \leq m - 1$ and $j \in \mathbb{Z}$

$$\chi_i((ts)^j) = \zeta^j + \zeta^{-j} = 2\cos\Big(\frac{ij\pi}{m}\Big) \quad \text{and} \quad \chi_i(s(ts)^j) = 0.$$

Let 1 denote the trivial character of $W$, let $\varepsilon$ denote the sign character and let $\gamma \colon W \to \{1, -1\}$ be the unique linear character such that $\gamma(s) = -\gamma(t) = 1$. Then

$$\operatorname{Irr} W = \{1, \varepsilon, \gamma, \varepsilon\gamma\} \cup \{\chi_i \mid 1 \leq i \leq m - 1\}. \tag{1}$$

In particular, $|\operatorname{Irr} W| = m + 3$.

## 2.1 The Subset $A = \{s, t, sts\}$

From now on, we assume that $A = \{s, t, sts\}$. We set $\bar{s} = A \setminus \{s\} = \{t, sts\}$ and $\bar{t} = A \setminus \{t\} = \{s, sts\}$. It is easy to see that $\mathcal{P}_{\mathrm{ad}}(A) = \{\varnothing, \{s\}, \{t\}, \bar{s}, \bar{t}, A\}$. For simplification, we will denote by $d_I$ the element $d_I^A$ of $\mathbb{Z}W$ (for $I \in \mathcal{P}_{\mathrm{ad}}(W)$) and we set $d_s = d_{\{s\}}$ and $d_t = d_{\{t\}}$. We have

$$d_\varnothing = 1, \quad d_{\bar{s}} = w_0 s, \quad d_s = s, \quad d_A = w_0,$$
$$d_t = \sum_{i=1}^{m-1} \big( (st)^i + (ts)^{i-1} t \big), \quad d_{\bar{t}} = \sum_{i=1}^{m-1} \big( (st)^i s + (ts)^i \big).$$

The multiplication table of $\mathcal{D}_A(W)$ is given by

| | $1$ | $d_s$ | $d_{\bar{s}}$ | $d_A$ | $d_t$ | $d_{\bar{t}}$ |
|---|---|---|---|---|---|---|
| $1$ | $1$ | $d_s$ | $d_{\bar{s}}$ | $d_A$ | $d_t$ | $d_{\bar{t}}$ |
| $d_s$ | $d_s$ | $1$ | $d_A$ | $d_{\bar{s}}$ | $d_t$ | $d_{\bar{t}}$ |
| $d_{\bar{s}}$ | $d_{\bar{s}}$ | $d_A$ | $1$ | $d_s$ | $d_{\bar{t}}$ | $d_t$ |
| $d_A$ | $d_A$ | $d_{\bar{s}}$ | $d_s$ | $1$ | $d_{\bar{t}}$ | $d_t$ |
| $d_t$ | $d_t$ | $d_{\bar{t}}$ | $d_t$ | $d_{\bar{t}}$ | $z_A$ | $z_A$ |
| $d_{\bar{t}}$ | $d_{\bar{t}}$ | $d_t$ | $d_{\bar{t}}$ | $d_t$ | $z_A$ | $z_A$ |

where $z_A = (m-1)\big(1 + d_A + d_s + d_{\bar{s}}\big) + (m-2)\big(d_t + d_{\bar{t}}\big)$. We now study the sub-$\mathbb{Z}$-module $\Sigma_A(W)$; we will show that it coincides with $\mathcal{D}_A(W)$. First, it is easily seen that $\mathcal{P}_0(A) = \{\varnothing,\ \{s\},\ \{t\},\ \{sts\},\ \bar{s},\ A\}$ is the set of subsets $I$ of $A$ such that $W_I \cap A = I$ and that

$$\begin{aligned}
x_A &= 1 \\
x_{\bar{s}} &= 1 + d_s \\
x_{sts} &= 1 + d_s + d_t \\
x_t &= 1 + d_s + d_{\bar{t}} \\
x_s &= 1 + d_t + d_{\bar{s}} \\
x_\varnothing &= 1 + d_s + d_t + d_{\bar{t}} + d_{\bar{s}} + d_A
\end{aligned}$$

Therefore, $\Sigma_A(W) = \mathcal{D}_A(W) = \bigoplus_{I \in \mathcal{P}_0(A)} \mathbb{Z} x_I$. So we can define a map

$$\theta_A \colon \Sigma_A(W) \to \mathbb{Z} \operatorname{Irr} W$$

by $\theta_A(x_I) = \operatorname{Ind}_{W_I}^W 1_{W_I}$.

***Proposition 2.1*** *Assume that $S = \{s, t\}$ with $s \neq t$, that $m(s,t) = 2m$ with $m \geq 2$ and that $A = \{s, t, sts\}$. Then*

(i) $\Sigma_A(W) = \mathcal{D}_A(W)$ *is a subalgebra of* $\mathbb{Z}W$ *of* $\mathbb{Z}$*-rank* 6.
(ii) $\theta_A$ *is a morphism of algebras.*
(iii) $\operatorname{Ker} \theta_A = \mathbb{Z}(x_t - x_{sts})$.
(iv) $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Ker} \theta_A$ *is the radical of* $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma_A(W)$.
(v) $\theta_A$ *is surjective if and only if* $m = 2$ *that is, if and only if* $W$ *is of type* $B_2$.

**Proof** (i) has already been proven. For proving the other assertions, we need to compute explicitly the map $\theta_A$. It is given by the following table:

| $d_I$ | $1$ | $d_s$ | $d_{\tilde{s}}$ | $d_A$ | $d_t$ | $d_{\tilde{t}}$ |
|---|---|---|---|---|---|---|
| $\theta_A(d_I)$ | $1$ | $\varepsilon\gamma$ | $\gamma$ | $\varepsilon$ | $\sum_{i=1}^{m-1}\chi_i$ | $\sum_{i=1}^{m-1}\chi_i$ |

(iii) This shows that $\operatorname{Ker}\theta_A = \mathbb{Z}(d_{\tilde{t}} - d_t) = \mathbb{Z}(x_t - x_{sts})$, so (iii) holds.

(ii) To prove that $\theta_A$ is a morphism of algebras, the only difficult point is to prove that $\theta_A(d_t^2) = \theta_A(d_t)^2$. Let $\rho$ denote the regular character of $W$. Then

$$\theta_A(d_t) = \frac{1}{2}(\rho - 1 - \gamma - \varepsilon\gamma - \varepsilon).$$

Therefore, $\theta_A(d_t)^2 = (m-2)\rho+1+\gamma+\varepsilon\gamma+\varepsilon$. But, $d_t^2 = z_A = (m-2)x_\varnothing+1+d_s+d_{\tilde{s}}+d_A$. This shows that $\theta_A(d_t^2) = \theta_A(d_t)^2$.

(iv) Let $R$ denote the radical of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma_A(W)$. We only need to prove that $\mathbb{C} \otimes_{\mathbb{Z}} R = \mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A$. Since $\mathbb{C}\operatorname{Irr} W$ is a split semisimple commutative algebra, every subalgebra of $\mathbb{C}\operatorname{Irr} W$ is semisimple. So $(\mathbb{C} \otimes_{\mathbb{Z}} \Sigma_A(W))/(\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A)$ is a semisimple algebra. This shows that $R$ is contained in $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A$. Moreover, since $(x_t - s_{sts})^2 = (d_t - d_{\tilde{t}})^2 = 0$, $\operatorname{Ker}\theta_A$ is a nilpotent two-sided ideal of $\Sigma_A(W)$. So $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A$ is contained in $\mathbb{C} \otimes_{\mathbb{Z}} R$. This shows (iv).

(v) If $m = 2$, then $\operatorname{Irr} W = \{1, \gamma, \varepsilon\gamma, \varepsilon, \chi_1\} = \theta_A(\{1, d_s, d_{\tilde{s}}, d_A, d_t\})$ so $\theta_A$ is surjective. Conversely, if $\theta_A$ is surjective, then $|\operatorname{Irr} W| = 5$ (by (i) and (iii)). Since $|\operatorname{Irr} W| = m + 3$, this gives $m = 2$. ■

We close this subsection by giving a complete set of orthogonal primitive idempotents for $\Sigma_A(W)$, extending to our case those given in [2]:

$$E_\varnothing = \frac{1}{4m}x_\varnothing, \quad E_s = \frac{1}{2}\left(x_s - \frac{1}{2}x_\varnothing\right), \quad E_t = \frac{1}{2}\left(x_t - \frac{1}{2}x_\varnothing\right),$$
$$E_{\tilde{s}} = \frac{1}{2}\left(x_{\tilde{s}} - \frac{1}{2}x_t - \frac{1}{2}x_{sts} + \frac{m-1}{2m}x_\varnothing\right),$$
$$E_A = 1 - \frac{1}{2}x_s - \frac{1}{4}x_t + \frac{1}{4}x_{sts} - \frac{1}{2}x_{\tilde{s}} + \frac{1}{4}x_\varnothing.$$

## 2.2 The Subset $B = \{s\} \cup C(t)$

Let $B = \{s\} \cup C(t)$ (so that $|B| = m + 1$). It is easy to see that $\mathcal{P}_{\mathrm{ad}}(B)$ consists of the sets

$$\varnothing = D_B(1), \quad B = D_B(w_0), \quad \{s\} = D_B(s), \quad C(t) = D_B(w_0 s),$$
$$D_B((ts)^i) = D_B(s(ts)^i), \quad 1 \le i \le m-1,$$
$$D_B((st)^j) = D_B((ts)^{j-1}t), \quad 1 \le j \le m-1.$$

Therefore $\mathcal{D}_B(W)$ is a subalgebra of $\mathbb{Z}W$ of $\mathbb{Z}$-rank $(2m+2)$.

Using GAP, we can see that, in general, $\Sigma_B(W) \neq \mathcal{D}_B(W)$. First examples are given in the following table

| $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{Z}$-rank of $\mathcal{D}_B(W)$ | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| $\mathbb{Z}$-rank of $\Sigma_B(W)$ | 6 | 8 | 10 | 10 | 14 | 12 | 18 | 18 | 22 | 16 |

***Remark 2.2*** The linear map $\theta_B\colon \Sigma_B(W) \to \mathbb{Z}\operatorname{Irr} W$, $x_I \mapsto \operatorname{Ind}_{W_I}^W 1_{W_I}$, is well-defined and surjective if and only if $m \in \{2,3\}$ (recall that $m \geq 2$). Indeed, the image of $\theta_B$ cannot contain a non-rational character. But all characters of $W$ are rational if and only if $W$ is a Weyl group, that is, if and only if $2m \in \{2,3,4,6\}$.

Moreover, if $m=2$, then $A=B$ and $\theta_B=\theta_A$ is surjective by Proposition 2.1. If $m=3$, then this follows from Proposition 2.3 below.

## 2.3 The Algebra $\mathcal{D}_B(G_2)$

From now on $m=3$, that is, $W$ is of type $I_2(6)=G_2$. For convenience, we keep the same notation as in §2.1. We have

$$\begin{aligned} d_\varnothing &= 1, & d_1 &= d^B_{\{t\}} = t + st,\\ d_s &= s, & d_2 &= d^B_{\{s,sts\}} = ts + sts,\\ d_{\bar{s}} &= d^B_{C(t)} = w_0 s, & d_3 &= d^B_{\{s,sts,tstst\}} = tsts + ststs,\\ d_A &= d^B_B = w_0, & d_4 &= d^B_{\{t,tstst\}} = tst + stst. \end{aligned}$$

Let us now show that $\Sigma_B(W) = \mathcal{D}_B(W)$. First, it is easily seen that

$$\mathcal{P}_0(B) = \{\varnothing,\ \{s\},\ \{t\},\ \{sts\},\ \bar{s},\ \{tstst\}, \{s,tstst\}, B\}$$

is the set of subsets $I$ of $B$ such that $W_I \cap B = I$ and that

$$\begin{aligned} x_A = x_B &= 1\\ x_{\bar{s}} &= 1 + d_s\\ x_{\{s,tstst\}} &= 1 + d_1\\ x_{tstst} &= 1 + d_s + d_1 + d_2\\ x_t &= 1 + d_s + d_2 + d_3\\ x_{sts} &= 1 + d_s + d_1 + d_4\\ x_s &= 1 + d_1 + d_4 + d_{\bar{s}}\\ x_\varnothing &= 1 + d_s + d_1 + d_2 + d_3 + d_4 + d_{\bar{s}} + d_A \end{aligned}$$

Therefore, $\Sigma_B(W) = \mathcal{D}_B(W) = \bigoplus_{I\in\mathcal{P}_0(B)} \mathbb{Z}x_I$. So we can define a map

$$\theta_B\colon \Sigma_B(W) \to \mathbb{Z}\operatorname{Irr} W$$

by $\theta_B(x_I) = \operatorname{Ind}_{W_I}^W 1_{W_I}$.

***Proposition 2.3*** *Assume that $S = \{s, t\}$ with $s \neq t$, that $m(s, t) = 6$ and that $B = \{s, t, sts, tstst\}$. Then*

(i) $\Sigma_B(W) = \mathcal{D}_B(W)$ *is a subalgebra of* $\mathbb{Z}W$ *of* $\mathbb{Z}$*-rank* 8.

(ii) $\theta_B$ *is a surjective linear map (and not a morphism of algebras).*

(iii) $\operatorname{Ker}\theta_B = \mathbb{Z}(x_{tstst} - x_t) \oplus \mathbb{Z}(x_{tstst} - x_{sts})$.

(iv) $\operatorname{Irr} W = \theta_B\left(\{1, d_s, d_{\tilde{s}}, d_A, d_1, d_2\}\right)$.

**Proof** With computations similar to §2.1, we obtain the following table for the map $\theta_B$.

| $d_I^B$ | 1 | $d_s$ | $d_{\tilde{s}}$ | $d_A$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|---|---|---|---|
| $\theta_B(d_I^B)$ | 1 | $\varepsilon\gamma$ | $\gamma$ | $\varepsilon$ | $\chi_2$ | $\chi_1$ | $\chi_2$ | $\chi_1$ |

This shows (ii) and (iv). As $d_1 - d_3 = x_{\{tstst\}} - x_t$ and $d_2 - d_4 = x_{tstst} - x_{sts}$, (iii) is proved. Finally, the fact that $\theta_B$ is not a morphism of algebras follows from $\theta_B(d_1^2)(w_0) = (1 + \varepsilon\gamma + \chi_1)(w_0) = -2 \neq \theta_B(d_1)^2(w_0) = \chi_2(w_0)^2 = 4$. ■

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