Sarmanov Family of Bivariate Distributions for Multivariate Loss Reserving Analysis

Anas Abdallah [*], Jean-Philippe Boucher[†], Hélène Cossette[‡], and Julien Trufin [§]

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Abstract

The correlation among multiple lines of business plays a critical role in aggregating claims and thus determining loss reserves for an insurance portfolio. We show that the Sarmanov family of bivariate distributions is a convenient choice to capture the dependencies introduced by various sources, including the common calendar year, accident year and development period effects. The density of the bivariate Sarmanov distributions with different marginals can be expressed as a linear combination of products of independent marginal densities. This pseudo-conjugate property greatly reduces the complexity of posterior computations. In a case study, we analyze an insurance portfolio of personal and commercial auto lines from a major US property-casualty insurer.

Keywords: Runoff triangles, Sarmanov, Random effects, Maximum Likelihood Estimation, Bootstrap.

[*]École d'Actuariat, Université Laval
[†]Département de mathématiques, UQAM
[‡]École d'Actuariat, Université Laval
[§]Département de mathématiques, Université Libre de Bruxelles

1 Introduction

Provisions generally represent most of the liabilities of a property/casualty insurance company. It is therefore crucial for a company to estimate its provisions well. With the advent of the new regulatory standards (e.g. Solvency II in Europe and the upcoming ORSA[1] guidelines in North America), it is now necessary for an insurer to be more accurate and rigorous to settle the amount of provisions for the entire portfolio. This involves taking into account the correlation between the lines of business.

To incorporate dependencies among multiple runoff triangles, the literature can be separated into two different schools of thought.

The first strand of research examines distribution-free methods, where the (conditional) mean squared prediction error can be derived to measure prediction uncertainty. For example, Braun (2004) takes into account the correlations between the segments by introducing a correlation between development factors, while Schmidt (2006) adopts a multivariate approach, by performing a simultaneous study of all segments of the portfolio.

The other approach relies on parametric methods based on distributional families, allowing predictive distribution of unpaid losses, which is believed to be more informative to actuaries in setting a reasonable reserve range than a single mean squared prediction error. We will focus on the parametric approach.

Parametric reserving methods mainly involve copulas to model dependence between lines of business. For example, Brehm (2002) uses a Gaussian copula to model the joint distribution of unpaid losses, while De Jong (2012) models dependence between lines of business with a Gaussian copula correlation matrix. Shi et al. (2012) and Wüthrich et al. (2013) also use multivariate Gaussian copula, to accommodate correlation due to accounting years within and across runoff triangles. Bootstrapping is another popular parametric approach used to forecast the predictive distribution of unpaid losses for correlated lines of business. Kirschner et al. (2008) use a synchronized bootstrap and Taylor and McGuire (2007) extend this result to a generalized linear model context. More recently, Abdallah et al. (2015) use Hierarchical Archimedean copulas to accommodate correlation within and between runoff triangles.

We use random effects to accommodate correlation due to calendar year, accident year and development period effects within and across runoff triangles. Bayesian methods are not new to the loss reserving literature (see Shi et al. (2012) for an excellent review). In this paper, to capture dependence between the lines of business (through random effects), we introduce the Sarmanov Family of bivariate distributions to the reserving literature. This family of bivariate distributions was first presented in Sarmanov (1966) and appeared in more detail in Lee (1996). The Sarmanov family includes Farlie–Gumbel–Morgenstern (FGM) distributions as special cases.

The applicability of Sarmanov's distribution results from its versatile structure that offers us flexibility in the choice of marginals and allows a closed form for the joint density. We aim to show the potential of this family of distributions in a loss reserving context.

In Section 2, we review the modeling of runoff triangles, where notations are set and

[1]ORSA: Own Risk and Solvency Assessment

random effects defined. In Section 3, we present the Sarmanov Family of Bivariate Distributions and introduce them to the loss reserving context in Section 4. We apply the model to a casualty insurance portfolio from a U.S. insurer and demonstrates the flexibility of the proposed approach in Section 5. Section 6 concludes the paper.

2 Modeling

2.1 General notations

In this paper, a dependence model within and between lines of business through calendar year, accident year and development period effects is presented. To simplify the notations, we will consider the calendar year case. The notations could be easily generalized to accident year and development period cases.

Let us consider an insurance portfolio with ℓ lines of business ($\ell \in \{1, ..., L\}$). We define by $X_{i,j}^{(\ell)}$, the incremental payments of the i^{th} accident year ($i \in \{1, ..., n\}$), and the j^{th} development period ($j \in \{1, ..., n\}$). To take into account the volume of each line of business, we work with standardized data which we denote by $Y_{i,j}^{(\ell)} = X_{i,j}^{(\ell)}/p_i^{(\ell)}$, where $p_i^{(\ell)}$ represents the exposure variable in the i^{th} accident year for the ℓ^{th} line of business. The exposure variable can be the number of policies, the number of open claims, or the earned premiums. The latter option is the one chosen in this paper. We suppose that the accident year effect is independent of the development period effect. Hence, a regression model with two independent explanatory variables, accident year and development period, is used. Assume that $\alpha_i^{(\ell)}$ ($i \in \{1, 2, ..., n\}$) and $\beta_j^{(\ell)}$ ($j \in \{1, 2, ..., n\}$) characterize the accident year effect and the development period effect respectively. In such a context, a systematic component for the ℓ^{th} line of business can be written as

$$\eta_{i.j}^{(\ell)} = \zeta^{(\ell)} + \alpha_i^{(\ell)} + \beta_j^{(\ell)},\ \ell = 1, ..., L, \tag{1}$$

where $\zeta^{(\ell)}$ is the intercept, and for parameter identification, the constraint $\alpha_1^{(\ell)} = \beta_1^{(\ell)} = 0$ is supposed. In our empirical illustration, and in the following, we work with two runoff triangles ($L = 2$) of cumulative paid losses exhibited in Tables 1 and 2 of Shi and Frees (2011). They correspond to paid losses of Schedule P of the National Association of Insurance Commissioners (NAIC) database. These are 1997 data for personal auto and commercial auto lines of business, and each triangle contains losses for accident years 1988-1997 and at most ten development years. Shi and Frees (2011) show that a lognormal distribution and a gamma distribution provide a good fit for the Personal Auto and the Commercial Auto line data respectively. To demonstrate the reasonable model fits for the two triangles, the authors exhibit the qq-plots of marginals for personal and commercial auto lines (see Figure 3 in Shi and Frees (2011)). We work with their conclusion and continue with the same continuous distributions for each line of business. More specifically, we consider the form $\mu_{i.j}^{(1)} = \eta_{i.j}^{(1)}$ for a lognormal distribution with location (log-scale) parameter $\mu_{i.j}^{(1)}$ and shape parameter σ. However, for the gamma distribution, as noted by Abdallah et al. (2015) we use the exponential link instead of the canonical inverse link to ensure positive means, with $\mu_{ij}^{(2)} = \dfrac{\exp(\eta_{ij}^{(2)})}{\phi}$, where $\mu_{ij}^{(2)}$ and ϕ are the scale (location) and the shape parameters respectively.

2.2 Random effects

The models with random effects can be interpreted as models where hidden characteristics are captured by this additional random term. Here, we want to detect the effects characterizing the loss of a given calendar year (accident year or development period) through a random variable. The latter will capture correlations within the runoff triangles for the L lines of business.

As mentioned earlier, we keep the same assumptions of Shi and Frees (2011) for the marginals, i.e. a lognormal distribution for the first line of business and a gamma distribution for the second line of business. Hence, as an associated conjugate prior, we take normal and gamma distributions, for the first and second runoff triangle respectively.

2.2.1 Prior distributions

Let the random variable $\Theta_t^{(\ell)}$ characterize the losses of the business line ℓ ($\ell = 1, 2$) for a given calendar year t with probability density function (pdf) denoted by $u^{(\ell)}$.

Let $\mathbf{Y_t}^{(\ell)} = \left(Y_{t,1}^{(\ell)}, ..., Y_{1,t}^{(\ell)}\right)$ be the vector of losses for the t^{th} calendar year of the business line ℓ. This vector can also be written as $\mathbf{Y_t}^{(\ell)} = \left(Y_1^{(\ell)}, ..., Y_j^{(\ell)}, ..., Y_t^{(\ell)}\right)$ where j indicates the j^{th} development period. Also, let $\mu_j^{(\ell)} = \mu_{t-j+1,j}^{(\ell)}$.

Let us assume that, given $\Theta_t^{(\ell)}$, the random variables $Y_1^{(\ell)}, ..., Y_t^{(\ell)}$ are conditionally independent. For $\ell = 1$, we suppose that

$$\left[Y_{i,j}^{(1)} \mid \Theta_t^{(1)} = \theta^{(1)}\right] \sim \text{Logn.}\left(\mu_{i,j}^{(1)}\theta^{(1)}, \sigma^2\right) ,$$

and

$$f_{Y_j^{(1)}|\Theta_t^{(1)}}\left(y_j^{(1)}; \theta^{(1)}\mu_j^{(1)}, \sigma^2\right) = \left(\frac{1}{y_j^{(1)}\sqrt{2\pi}\sigma}\right)\exp\left(\frac{-(\log y_j^{(1)} - \mu_j^{(1)}\theta^{(1)})^2}{2\sigma^2}\right),$$

with $E[Y_{i,j}^{(1)} \mid \Theta_t^{(1)} = \theta_t^{(1)}] = e^{\mu_{i,j}^{(1)}\theta_t^{(1)} + \sigma^2/2}$ and $\text{Var}[Y_{i,j}^{(1)} \mid \Theta_t^{(1)} = \theta_t^{(1)}] = \left(e^{\sigma^2} - 1\right)\left(e^{2\mu_{i,j}^{(1)}\theta_t^{(1)} + \sigma^2}\right)$.

Also, let

$$\Theta_t^{(1)} \sim \text{Normal}\left(a, b^2\right),$$

with

$$u^{(1)}\left(\theta^{(1)}; a, b^2\right) = \frac{1}{b\sqrt{2\pi}}\exp\left(\frac{-\left(\theta^{(1)} - a\right)^2}{2b^2}\right).$$

Given these assumptions, the law of total probability leads to the following joint density function for $\mathbf{Y_t}^{(1)}$, denoted by $f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)}; a, b^2\right)$

$$\begin{aligned}
& f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)}; a, b^2\right) \\
= & \int_0^{\infty} \prod_{j=1}^{t} f_{Y_j^{(1)} | \Theta_t^{(1)}}\left(y_j^{(1)} \mid \Theta_t^{(1)} = \theta^{(1)}\right) u^{(1)}\left(\theta^{(1)}; a, b^2\right) d\theta^{(1)} \\
= & \prod_{j=1}^{t}\left(\frac{1}{y_j^{(1)}\sqrt{2\pi}\sigma}\right) \frac{\sigma}{\sqrt{\sum_{j=1}^{t} \mu_j^2 b^2 + \sigma^2}} \\
& \times \exp\left(-\frac{\left(\frac{1}{\sigma^2}\sum_{j=1}^{t}\log(y_j^{(1)})^2(b^2\sum_{j=1}^{t}\mu_j^2 + \sigma^2) + \frac{1}{b^2}a^2(b^2\sum_{j=1}^{t}\mu_j^2 + \sigma^2) - \frac{(\sum_{j=1}^{t}\log(y_j^{(1)})\mu_j b^2 + a\sigma^2)^2}{b^2\sigma^2}\right)}{2(b^2\sum_{j=1}^{t}\mu_j^2 + \sigma^2)}\right).
\end{aligned} \tag{2}$$

For the second line of business $\ell = 2$, we assume that

$$\left[Y_{i,j}^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}\right] \sim \text{Gamma}\left(\phi, \frac{\mu_{i,j}^{(2)}}{\theta^{(2)}}\right),$$

and

$$f_{Y_j^{(2)} | \Theta_t^{(2)}}\left(y_j^{(2)}; \phi, \frac{\mu_j^{(2)}}{\theta^{(2)}}\right) = \frac{y_j^{(2)\phi - 1}}{\Gamma(\phi)\left(\frac{\mu_j^{(2)}}{\theta^{(2)}}\right)^{\phi}} \exp\left(-\frac{y_j^{(2)}}{\frac{\mu_j^{(2)}}{\theta^{(2)}}}\right),$$

with $E[Y_{i,j}^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}] = \phi\mu_{i,j}^{(2)}\frac{1}{\theta^{(2)}}$ and $\text{Var}[Y_{i,j}^{(2)} \mid \Theta_t^{(2)} = \theta^{(2)}] = \phi\mu_{i,j}^{(2)2}\frac{1}{\theta^{(2)2}}$. For the random effect $\Theta_t^{(2)}$, we suppose

$$\Theta_t^{(2)} \sim \text{Gamma}(\alpha, \tau),$$

with

$$u^{(2)}\left(\theta^{(2)}; \alpha, \tau\right) = \frac{\theta^{(2)\alpha - 1}}{\Gamma(\alpha)(\tau)^{\alpha}} \exp\left(-\frac{\theta^{(2)}}{\tau}\right).$$

The joint density function of $\mathbf{Y}_{\mathbf{t}}^{(2)}$ denoted by $f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)}; \alpha, \tau\right)$ is hence given by

$$
\begin{aligned}
f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\tau\right) &= \int_0^\infty \prod_{k=1}^{t} f_{Y_k^{(2)}|\Theta_t^{(2)}}\left(y_k^{(2)} \mid \Theta_t^{(2)}=\theta^{(2)}\right) u^{(2)}\left(\theta^{(2)};\tau,\alpha\right) d\theta^{(2)} \\
&= \left(\prod_{k=1}^{t} \frac{y_k^{(2)\phi-1}}{\Gamma(\phi)(\mu_k^{(2)})^{\phi}}\right) \frac{\Gamma(t\phi+\alpha)}{\Gamma(\alpha)(\tau)^{\alpha}} \frac{1}{\left(\sum_{k=1}^{t} \frac{y_k^{(2)}}{\mu_k^{(2)}} + \frac{1}{\tau}\right)^{t\phi+\alpha}}. \qquad (3)
\end{aligned}
$$

For parameter identification, we suppose that $a = 1$ and $\tau = \frac{1}{\alpha-1}$ in our empirical illustration.

2.2.2 Posterior distribution

Using Bayes theorem, the posterior distributions for $[\Theta_t^{(1)} = \theta^{(1)} \mid \mathbf{Y}_{\mathbf{t}}^{(1)}]$ and $[\Theta_t^{(2)} = \theta^{(2)} \mid \mathbf{Y}_{\mathbf{t}}^{(2)}]$ are given by

$$
\begin{aligned}
u^{(1)}\left(\theta^{(1)} \mid \mathbf{Y}_{\mathbf{t}}^{(1)}\right) \quad &\propto f_{\mathbf{Y}_{\mathbf{t}}^{(1)}|\Theta_t^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)} \mid \Theta_t^{(1)}=\theta^{(1)}\right) u^{(1)}\left(\theta^{(1)};a,b^2\right) \\
&\propto u^{(1)}\left(\theta^{(1)};a_{post},b^2_{post}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
u^{(2)}\left(\theta^{(2)} \mid \mathbf{Y}_{\mathbf{t}}^{(2)}\right) \quad &\propto f_{\mathbf{Y}_{\mathbf{t}}^{(2)}|\Theta_t^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)} \mid \Theta_t^{(2)}=\theta^{(2)}\right) u^{(2)}\left(\theta^{(2)};\alpha,\tau\right) \\
&\propto u^{(2)}\left(\theta^{(2)};\alpha_{post},\tau_{post}\right).
\end{aligned}
$$

This shows that the posterior distributions for $[\Theta_t^{(1)} = \theta^{(1)} \mid \mathbf{Y}_{\mathbf{t}}^{(1)}]$ and $[\Theta_t^{(2)} = \theta^{(2)} \mid \mathbf{Y}_{\mathbf{t}}^{(2)}]$, are again Normal and gamma distributions with updated parameters

$$
\begin{cases}
a_{post} = \frac{\sum_{k=1}^{t}\log(y_k^{(1)})\mu_k^{(1)}b^2+a\sigma^2}{\sum_{k=1}^{t}\mu_k^{(1)2}b^2+\sigma^2} \text{ and } b^2_{post} = \frac{b^2\sigma^2}{\sum_{k=1}^{t}\mu_k^{(1)2}b^2+\sigma^2}; \\
\alpha_{post} = \alpha + t\phi \text{ and } \tau_{post} = \left(\frac{1}{\tau} + \sum_{k=1}^{t}\frac{y_k^{(2)}}{\mu_k^{(2)}}\right)^{-1}.
\end{cases}
$$

These results of posterior distributions will be very helpful in the calculation of the joint Sarmanov distribution, and for the moments calculation of the total as well.

3 Sarmanov Family of Bivariate Distributions

Sarmanov's bivariate distribution was introduced in the literature by Sarmanov (1966), and was also proposed in physics by Cohen (1984) under a more general form. Lee (1996) suggests a multivariate version and discusses several applications in medicine. Recently, due to its flexible structure, Sarmanov's bivariate distribution gained interest in different applied studies. For example, Schweidel et al. (2008) use a bivariate Sarmanov model to capture the relationship between a prospective customer's time until acquisition of a particular service and the subsequent duration for which the service is retained. Miravete (2009) presents two models based on Sarmanov distribution and uses them to compare the number of tariff plans offered by two competing cellular telephone companies. Danaher and Smith (2011) discuss applications to marketing (see also the references therein). In the insurance field, Hernández-Bastida et al. (2009) and Hernández-Bastida and Fernández-Sánchez (2012) use the bivariate Sarmanov distribution for premium evaluation. Here, we want to highlight and show its usefulness in loss reserving modeling.

We suppose a dependence between the calendar years (accident years or development periods) of the two runoff triangles, i.e the elements of a given calendar year of a line of business are assumed to be correlated with the corresponding elements of the other line of business through common random effects. This will create dependence between $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$. For this purpose, we propose to use the Sarmanov Family of bivariate distributions to model the joint distribution of the random effect $\Theta_t^{(\ell)}$ with $\ell \in \{1,2\}$.

3.1 Definitions

Let $\psi^{(\ell)}(\theta^{(\ell)})$, $\ell = 1,2$ be two bounded non-constant functions such that $\int_{-\infty}^{\infty} \psi^{(\ell)}(t)\, u^{(\ell)}(t)\, dt = 0$. Let $\left(\Theta^{(1)}, \Theta^{(2)}\right)$ have a bivariate Sarmanov distribution, the joint distribution can then be expressed as

$$u^S\left(\theta^{(1)}, \theta^{(2)}\right) = u^{(1)}\left(\theta^{(1)}; a, b^2\right) u^{(2)}\left(\theta^{(2)}; \alpha, \tau\right)\left(1 + \omega\psi^{(1)}\left(\theta^{(1)}\right)\psi^{(2)}\left(\theta^{(2)}\right)\right), \tag{4}$$

provided that ω is a real number that satisfies the condition

$$1 + \omega\psi^{(1)}\left(\theta^{(1)}\right)\psi^{(2)}\left(\theta^{(2)}\right) \geq 0 \text{ for all } \theta^{(\ell)}\ ,\ \ell \in \{1,2\}.$$

One of the main interesting properties of the Sarmanov is that the bivariate distribution can support a wide range of marginals, such as in this case, the normal and the gamma distributions. Different methods are proposed in Lee (1996) to construct mixing functions $\psi^{(\ell)}$ for different types of marginals. As mentioned in Lee (1996), different types of mixing functions can be used to yield different multivariate distributions with the same set of marginals. Based on Corollary 2 in Lee (1996), a mixing function can be defined as $\psi^{(\ell)}\left(\theta^{(\ell)}\right) = \exp\left(-\theta^{(\ell)}\right) - L_{(\ell)}(1)$, where $L_{(\ell)}$ is the Laplace transform of $u^{(\ell)}$, evaluated at 1. Hence, given our choice of distribution for $\Theta^{(\ell)}$, $\ell = 1,2$, we have

$$\psi^{(1)}(\theta^{(1)}) = \exp\left(-\theta^{(1)}\right) - \exp\left(-a + \frac{b^2}{2}\right)$$
$$\psi^{(2)}(\theta^{(2)}) = \exp\left(-\theta^{(2)}\right) - (1+\tau)^{-\alpha}.$$

As for the dependence parameter ω of the Sarmanov bivariate distribution, in the case of normal and gamma marginals, it is bounded as follows

$$-\frac{1}{b\exp(-a+\frac{b^2}{2})\sqrt{\alpha}\tau\left(1+\tau\right)^{-\alpha-1}} \leq \omega \leq \frac{1}{b\exp(-a+\frac{b^2}{2})\sqrt{\alpha}\tau\left(1+\tau\right)^{-\alpha-1}}.$$

The proof of this result is a direct consequence of Lee's (1996) Theorem 2.

3.2 Joint distribution

A critical problem when modeling dependence between runoff triangles is to obtain a joint distribution of unpaid losses. The Sarmanov distribution will be a good ally to circumvent to this problem. With normal and gamma marginals for $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$ respectively, the prior joint pdf of $\left(\Theta_t^{(1)},\Theta_t^{(2)}\right)$ is given by

$$\begin{aligned} u^S\left(\theta^{(1)},\theta^{(2)}\right) &= u^{(1)}\left(\theta^{(1)};a,b^2\right)u^{(2)}\left(\theta^{(2)};\alpha,\tau\right)\left(1+\omega\exp\left(-a+\frac{b^2}{2}\right)\left(1+\tau\right)^{-\alpha}\right) \\ &\quad +u^{(1)}\left(\theta^{(1)};a-b^2,b^2\right)u^{(2)}\left(\theta^{(2)};\alpha,\frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)\left(1+\tau\right)^{-\alpha} \\ &\quad -u^{(1)}\left(\theta^{(1)};a-b^2,b^2\right)u^{(2)}\left(\theta^{(2)};\alpha,\tau\right)\omega\exp\left(-a+\frac{b^2}{2}\right)\left(1+\tau\right)^{-\alpha} \\ &\quad -u^{(1)}\left(\theta^{(1)};a,b^2\right)u^{(2)}\left(\theta^{(2)};\alpha,\frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)\left(1+\tau\right)^{-\alpha}, \end{aligned} \tag{5}$$

which corresponds to a linear combination of the product of univariate pdfs. This last expression highlights an attractive feature of the Sarmanov family of distributions. Its simplicity and form greatly facilitate many calculations.

The joint distribution $f_{\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)}\right)$ in the case of the Sarmanov family of bivariate distributions with normal and gamma marginals is expressed by

$$\begin{aligned} & f_{\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)}\right) \\ &= \int_0^{+\infty}\int_{-\infty}^{\infty}\prod_{k=1}^{t} f_{Y_k^{(1)}|\Theta_t^{(1)}}\left(y_k^{(1)}\mid\Theta_t^{(1)}=\theta^{(1)}\right)f_{Y_k^{(2)}|\Theta_t^{(2)}}\left(y_k^{(2)}\mid\Theta_t^{(2)}=\theta^{(2)}\right)u^S\left(\theta^{(1)},\theta^{(2)}\right)d\theta^{(1)}d\theta^{(2)}. \end{aligned}$$

Following (2), (3) and (5), we obtain a closed-form expression for the density function of $\left(\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}\right)$, namely

$$
\begin{aligned}
f_{\mathbf{Y}_{\mathbf{t}}^{(1)},\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)},\mathbf{y}_{\mathbf{t}}^{(2)}\right) &= f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)};a,b^2\right) f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\tau\right)\left(1+\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right) \\
&\quad + f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)};a-b^2,b^2\right) f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
&\quad - f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)};a-b^2,b^2\right) f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\tau\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
&\quad - f_{\mathbf{Y}_{\mathbf{t}}^{(1)}}\left(\mathbf{y}_{\mathbf{t}}^{(1)};a,b^2\right) f_{\mathbf{Y}_{\mathbf{t}}^{(2)}}\left(\mathbf{y}_{\mathbf{t}}^{(2)};\alpha,\frac{\tau}{1+\tau}\right)\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}. \quad (6)
\end{aligned}
$$

3.3 Posterior Sarmanov distribution

The posterior distribution can be used for the calculation of the moments of the total reserve. The posterior bivariate joint density function of the couple $\left(\Theta_t^{(1)},\Theta_t^{(2)}\right)$ conditioned on $\left(\mathbf{Y}_{\mathbf{t}}^{(1)},\mathbf{Y}_{\mathbf{t}}^{(2)}\right)$ is given by

$$
\begin{aligned}
& u^S(\theta^{(1)},\theta^{(2)} \mid \mathbf{y}_{\mathbf{t}}^{(1)},\mathbf{y}_{\mathbf{t}}^{(2)}) \\
&= \frac{f(\mathbf{y}_{\mathbf{t}}^{(1)},\mathbf{y}_{\mathbf{t}}^{(2)} \mid \theta_t^{(1)},\theta_t^{(2)})u^S(\theta_t^{(1)},\theta_t^{(2)})}{f_{\mathbf{Y}_{\mathbf{t}}^{(1)},\mathbf{Y}_{\mathbf{t}}^{(2)}}(y_1^{(1)},...,y_t^{(1)},y_1^{(2)},...,y_t^{(2)})} \\
&= C_1 u^{(1)}\left(\theta_t^{(1)};a_{post},b_{post}^2\right)u^{(2)}\left(\theta_t^{(2)};\alpha_{post},\tau_{post}\right) + C_2 u^{(1)}\left(\theta_t^{(1)};a_{post}^{'},b_{post}^2\right)u^{(2)}\left(\theta_t^{(2)};\alpha_{post},\tau_{post}^{'}\right) \\
& -C_3 u^{(1)}\left(\theta_t^{(1)};a_{post}^{'},b_{post}^2\right)u^{(2)}\left(\theta_t^{(2)};\alpha_{post},\tau_{post}\right) - C_4 u^{(1)}\left(\theta_t^{(1)};a_{post},b_{post}^2\right)u^{(2)}\left(\theta_t^{(2)};\alpha_{post},\tau_{post}^{'}\right)
\end{aligned} \quad (7)
$$

where

$$
\begin{aligned}
a_{post} &= \frac{\sum_{k=1}^{t}\log(y_k^{(1)})\mu_k^{(1)}b^2 + a\sigma^2}{\sum_{k=1}^{t}\mu_k^{(1)2}b^2+\sigma^2} \\
a_{post}^{'} &= \frac{\sum_{k=1}^{t}\log(y_k^{(1)})\mu_k^{(1)}b^2 + \left(a-b^2\right)\sigma^2}{\sum_{k=1}^{t}\mu_k^{(1)2}b^2+\sigma^2} \\
b_{post}^2 &= \left(\frac{\sum_{k=1}^{t}\mu_k^{(1)2}}{\sigma^2}+\frac{1}{b^2}\right)^{-1}
\end{aligned}
$$

and

$$\begin{aligned} \alpha_{post} &= t\phi + \alpha \\ \tau_{post} &= \left(\sum_{k=1}^{t} \frac{y_k^{(2)}}{\mu_k^{(2)}} + \frac{1}{\tau} \right)^{-1} \\ \tau^{,}_{post} &= \left(\sum_{k=1}^{t} \frac{y_k^{(2)}}{\mu_k^{(2)}} + \frac{1}{\tau} + 1 \right)^{-1}, \end{aligned}$$

with

$$\begin{aligned} C_1 &= \frac{1}{f_{\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)}\right)} f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)}; a, b^2\right) f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)}; \alpha, \tau\right) \left(1 + \omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right) \\ C_2 &= \frac{1}{f_{\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)}\right)} f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)}; a - b^2, b^2\right) f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)}; \alpha, \frac{\tau}{1+\tau}\right) \omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\ C_3 &= \frac{1}{f_{\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)}\right)} f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)}; a - b^2, b^2\right) f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)}; \alpha, \tau\right) \omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\ C_4 &= \frac{1}{f_{\mathbf{Y_t}^{(1)},\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(1)},\mathbf{y_t}^{(2)}\right)} f_{\mathbf{Y_t}^{(1)}}\left(\mathbf{y_t}^{(1)}; a, b^2\right) f_{\mathbf{Y_t}^{(2)}}\left(\mathbf{y_t}^{(2)}; \alpha, \frac{\tau}{1+\tau}\right) \omega \exp\left(-a + \frac{b^2}{2}\right)(1+\tau)^{-\alpha}. \end{aligned}$$

This last expression shows that the posterior bivariate density function of $\left(\Theta^{(1)}, \Theta^{(2)}\right)$, is again a linear combination of the product of univariate normal and gamma pdfs. The posterior density is hence a pseudo-conjugate to the prior density in the sense that the posterior density is a linear combination of products of densities from the univariate natural exponential family of distributions (normal and gamma in our case). It would be interesting to investigate the link between the posterior Sarmanov distribution and the linear credibility theory, where the Bayesian premium is considered linear.

4 Claims reserving

4.1 Calendar year dependence

To accommodate correlation, most multivariate loss reserving methods focus on a pair-wise association between corresponding cells in multiple runoff triangles. Recently, Shi and Frees (2011) successfully incorporated dependence between two lines of business with a pair-wise association. However, such a practice usually relies on an independence assumption across accident years and ignores the calendar year effects that could affect all open claims simultaneously and induce dependencies among loss triangles. In fact, most dependencies among loss triangles could arguably be driven by certain calendar year effects and exogenous common factors such as inflation, interest rates, jurisprudence or strategic decisions such as the acceleration of the payments for the entire portfolio can have simultaneous impacts on all lines of business of a given sector, which could be the case here for the two lines of business considered in the present paper.

Such a calendar year effect has already been analyzed, for example by Barnett and Zehnwirth (1998) who add a covariate to capture the calendar year effect. De Jong (2006) models the growth rates in cumulative payments in a calendar year, and Wüthrich (2010) examines the accounting year effect for a single line of business. Wüthrich and Salzmann (2012) use a multivariate Bayes Chain-Ladder model that allows modeling of dependence along accounting years within runoff triangles. The authors derive closed form solutions for the posterior distribution, claims reserves and corresponding prediction uncertainty. Kuang et al. (2008) also consider a canonical parametrization with three factors for a single line of business.

In our proposed model, instead of adding an explanatory variable for the calendar year effect, the dependence relation between the paid claims of a diagonal will be based on a random effect. More specifically, the same random variable $\Theta_t^{(\ell)}$ is assumed for each diagonal of a runoff triangle. The likelihood function of this model can be easily derived from (2) and (3).

4.2 Line of Business dependence

4.2.1 Motivations

In the same view of Abdallah et al. (2015), we propose a model that allows a dependence relation between all the observations that belong to the same calendar year for each line of business using random effects instead of multivariate Archimedean copulas. Additionally, we use another dependence structure that links the losses of calendar years of different lines of business with a Sarmanov family of bivariate distributions instead of hierarchical copula. With this second level of dependence, we capture the dependence between two different runoff triangles in a pairwise manner between corresponding diagonals, instead of between cells. Hence, instead of pairing cells with a copula as in Shi and Frees (2011), we will pair diagonals through random effects using the Sarmanov family of bivariate distributions.

The calendar year effect has rarely been studied with more than one line of business. Two recent examples are De Jong (2012), where the calendar year effect was introduced through the correlation matrix and Shi et al. (2012), who used random effects to accommodate the correlation due to accounting year effects within and across runoff triangles. Shi et al. (2012) work with a Bayesian perspective, using a multivariate lognormal distribution, along with a multivariate Gaussian correlation matrix. The predictive distributions of outstanding payments are generated through Monte Carlo simulations. The calendar year effect is taken into account through an explanatory variable. Again with a Bayesian framework, Wüthrich et al. (2013) used a multivariate lognormal Chain-Ladder model and derived predictors and confidence bounds in closed form. Their analytical solutions are such that they allow for any correlation structure. Their models permit dependence between and within runoff triangles, along with any correlation structure. It has also been shown in this paper that the pair-wise dependence form is rather weak compared with calendar year dependence. More recently, Shi (2014) captures the dependencies introduced by various sources, including the common calendar year effects via the family of elliptical copulas, and uses parametric bootstrapping to quantify the associated reserving variability.

In this paper, to model the complex dependence structure between two runoff triangles, we introduce models based on the Sarmanov family of bivariate distributions. The idea is to use random effects to capture dependence within lines of business, and then join the two random effects through a Sarmanov distribution to capture dependence between lines of business. Empirical results are shown in the next section. Finally, the log-likelihood function of this model can be obtained from (6).

4.2.2 Mean and Variance

To compute the resulting reserve for this model, the estimated total unpaid losses for $i+j > n+1$, can be expressed as follows

$$E\left[R_{tot}\right] = E\left[R^{(1)} + R^{(2)}\right] = E\left[\sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} Y_{i,j}^{(\ell)}\right] = \sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} E[Y_{i,j}^{(\ell)}],$$

where

$$E[Y_{i,j}^{(1)}] = E[E[Y_{i,j}^{(1)} \mid \Theta_t^{(1)}]] = E[e^{\mu_{i,j}^{(1)}\Theta_t^{(1)}+\sigma^2/2}] = e^{a\mu_{i,j}^{(1)}+\frac{1}{2}b^2\mu_{i,j}^{(1)2}+\sigma^2/2}$$

and

$$E[Y_{i,j}^{(2)}] = E[E[Y_{i,j}^{(2)} \mid \Theta_t^{(2)}]] = E\left[\phi\mu_{i,j}^{(2)}\frac{1}{\Theta_t^{(2)}}\right] = \phi\mu_{i,j}^{(2)}\frac{1}{\tau(\alpha-1)},$$

with $t = i + j - 1$.

Consequently, the total unpaid losses can be written as

$$E\left[R_{tot}\right] = \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} e^{a\mu_{i,j}^{(1)}+\frac{1}{2}b^2\mu_{i,j}^{(1)2}+\sigma^2/2} + \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} \frac{p_i^{(2)}\phi\mu_{i,j}^{(2)}}{\tau(\alpha-1)}. \quad (8)$$

When we model dependence between loss triangles, the global variance can be very informative. Knowing that the two runoff triangles are correlated, it is interesting to observe how the two random effects $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$ change together, i.e whether the two variables tend to show similar (positive dependence) or opposite behavior (negative dependence). Note that when $\Theta_t^{(1)}$ and $\Theta_t^{(2)}$ are assumed unrelated (independent case), we will have $\mathrm{Cov}(R^{(1)}, R^{(2)}) = 0$.

The total claims reserve variance can be written as

$$\begin{aligned}\mathrm{Var}(R_{tot}) &= \mathrm{Var}\left(R^{(1)} + R^{(2)}\right) = \mathrm{Var}(R^{(1)}) + \mathrm{Var}(R^{(2)}) + 2\mathrm{Cov}(R^{(1)}, R^{(2)})\\ &= \sum_{\ell=1}^{2}\mathrm{Var}(R^{(\ell)}) + 2\mathrm{Cov}(R^{(1)}, R^{(2)}).\end{aligned}$$

Using the conditional independence of $Y_{i,j}^{(\ell)}$ given $\Theta_t^{(\ell)} = \theta^{(\ell)}$ $(t = i + j - 1))$, we have

$$\begin{aligned}
\mathrm{Var}(R^{(1)}) &= E\left[\mathrm{Var}\left(\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} Y_{i.j}^{(1)} \mid \Theta_t^{(1)}\right)\right] + \mathrm{Var}\left[E\left(\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} Y_{i.j}^{(1)} \mid \Theta_t^{(1)}\right)\right] \\
&= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)2}\left(E\left[\mathrm{Var}\left(Y_{i.j}^{(1)} \mid \Theta_t^{(1)}\right)\right] + \mathrm{Var}\left(E\left[Y_{i.j}^{(1)} \mid \Theta_t^{(1)}\right]\right)\right) \\
&= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)2}\left(e^{2a\mu_{i,j}^{(1)}+2b^2\mu_{i,j}^{(1)2}+2\sigma^2} - e^{2a\mu_{i,j}^{(1)}+b^2\mu_{i,j}^{(1)2}+\sigma^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(R^{(2)}) &= E\left[\mathrm{Var}\left(\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(2)} Y_{i.j}^{(2)} \mid \Theta_t^{(2)}\right)\right] + \mathrm{Var}\left(E\left[\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(2)} Y_{i.j}^{(2)} \mid \Theta_t^{(2)}\right]\right) \\
&= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(2)2}\left(E\left[\mathrm{Var}\left(Y_{i.j}^{(2)} \mid \Theta_t^{(2)}\right)\right] + \mathrm{Var}\left(E\left[Y_{i.j}^{(2)} \mid \Theta_t^{(2)}\right]\right)\right) \\
&= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} \phi(p_i^{(2)}\mu_{i,j}^{(2)})^2\left(\frac{\alpha+\phi-1}{\tau^2(\alpha-1)^2(\alpha-2)}\right).
\end{aligned}$$

For the covariance calculation, we have

$$\mathrm{Cov}(R^{(1)}, R^{(2)}) = \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} p_i^{(2)}\left(E\left[Y_{i,j}^{(1)} Y_{i,j}^{(2)}\right] - E\left[Y_{i,j}^{(1)}\right] E\left[Y_{i,j}^{(2)}\right]\right).$$

From (8) we have

$$E\left[Y_{i,j}^{(1)}\right] E\left[Y_{i,j}^{(2)}\right] = \left(e^{a\mu_{i,j}^{(1)}+\frac{1}{2}b^2\mu_{i,j}^{(1)2}+\sigma^2/2}\right)\left(\frac{\phi\mu_{i,j}^{(2)}}{\tau(\alpha-1)}\right),$$

and given (5), we obtain

$$\begin{aligned}
E[Y_{i,j}^{(1)} Y_{i,j}^{(2)}] &= E\left[E\left(Y_{i.j}^{(1)} Y_{i.j}^{(2)} \mid \Theta_t^{(1)}, \Theta_t^{(2)}\right)\right] \\
&= e^{\sigma^2/2}\phi\mu_{i,j}^{(2)} E[e^{\mu_{i,j}^{(1)}\Theta_t^{(1)}}\frac{1}{\Theta_t^{(2)}}],
\end{aligned}$$

with

$$E[e^{\mu_{i,j}^{(1)}\Theta_t^{(1)}}\frac{1}{\Theta_t^{(2)}}] = \int_0^{\infty}\int_{-\infty}^{\infty} e^{\mu_{i,j}^{(1)}\theta^{(1)}}\frac{1}{\theta^{(2)}} u^S(\theta^{(1)}, \theta^{(2)}) d\theta^{(1)} d\theta^{(2)}.$$

Consequently, the total variance of unpaid losses is expressed as follows

$$
\begin{aligned}
\mathrm{Var}(R_{tot}) \;=\; & \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)2}\left(e^{2a\mu_{i,j}^{(1)}+2b^2\mu_{i,j}^{(1)2}+2\sigma^2} - e^{2a\mu_{i,j}^{(1)}+b^2\mu_{i,j}^{(1)2}+\sigma^2}\right) \\
+ \; & \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} \phi(p_i^{(2)}\mu_{i,j}^{(2)})^2\left(\frac{\alpha+\phi-1}{\tau^2(\alpha-1)^2(\alpha-2)}\right) \\
+ \; & 2\sum_{i=2}^{n}\sum_{j=n-i+2}^{n}\Bigg\{ p_i^{(1)}p_i^{(2)}e^{\sigma^2/2}\phi\mu_{i,j}^{(2)} \times \\
& \Bigg[e^{a\mu_{i,j}^{(1)}+\frac{b^2}{2}\mu_{i,j}^{2(1)}}\frac{1}{\tau(\alpha-1)}\left(1+\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha}\right) \\
& + e^{\left(a-b^2\right)\mu_{i,j}^{(1)}+\frac{b^2}{2}\mu_{i,j}^{2(1)}}\frac{1+\tau}{\tau(\alpha-1)}\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
& - e^{\left(a-b^2\right)\mu_{i,j}^{(1)}+\frac{b^2}{2}\mu_{i,j}^{2(1)}}\frac{1}{\tau(\alpha-1)}\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \\
& - e^{a\mu_{i,j}^{(1)}+\frac{b^2}{2}\mu_{i,j}^{2(1)}}\frac{1+\tau}{\tau(\alpha-1)}\omega\exp\left(-a+\frac{b^2}{2}\right)(1+\tau)^{-\alpha} \Bigg] \\
& - \left(p_i^{(1)}e^{a\mu_{i,j}^{(1)}+\frac{1}{2}b^2\mu_{i,j}^{(1)2}+\sigma^2/2}\right)\left(\frac{p_i^{(2)}\phi\mu_{i,j}^{(2)}}{\tau(\alpha-1)}\right)\Bigg\}. \qquad (9)
\end{aligned}
$$

4.3 Accident year and development period dependence

We consider here a dependence structure captured through accident year and development period effects. In fact, some exogenous factors could result in an accident year trend. Change in reserving practices for example, in the way case reserves are settled at the opening of the claim, for current accident year claims. Further, a court judgment, a change in legislation affecting future losses, major events and disasters can all result in an accident year trend as well. The development period trend could result from the same exogenous factors cited for the calendar year case, but also from management decisions. For example, a revision of inactive claims or a changing pace of payments (internal or external initiative) are widespread practices in the industry that might affect several lines of business simultaneously.

4.3.1 Credibility loss reserving

As discussed earlier in this paper, the flexibility of the Sarmanov family of bivariate distribution allows us to easily change the dependence structure. Hence, as in extension and in addition to the calendar year approach, we will consider here two other approaches in which the random effect characterizes the loss of a given accident year or development period. Such modeling is well illustrated in Figure 1. In fact, we can see that a given accident year or development period effect will also impact the observations in the lower triangle belonging to the same accident year or development period. This is a great advantage when working with random effects rather than copulas, where the predictive power for the lower triangle might be limited.

Henceforth, we consider a situation where an insurer has access to claims experience and has the potential to improve prediction of outstanding liabilities by incorporating past information. The link here with linear credibility is pretty straightforward.

The accident year and development period effect has rarely been studied in the literature. It is interesting to note that depending on the dependence structure we use, we could get different conclusions from the analysis of dependence between the two business lines. This was also well illustrated in Figure 4 of Shi et al. (2012). This will be discussed in greater detail in the next section, where an empirical illustration is presented.

The idea here is that future payments will be updated through past experience. In fact, the random effect characterizing the loss of a given accident year or development period affects payments in the lower triangle as well. More importantly, it would be interesting here to see how these random effects impact the two runoff triangles simultaneously.

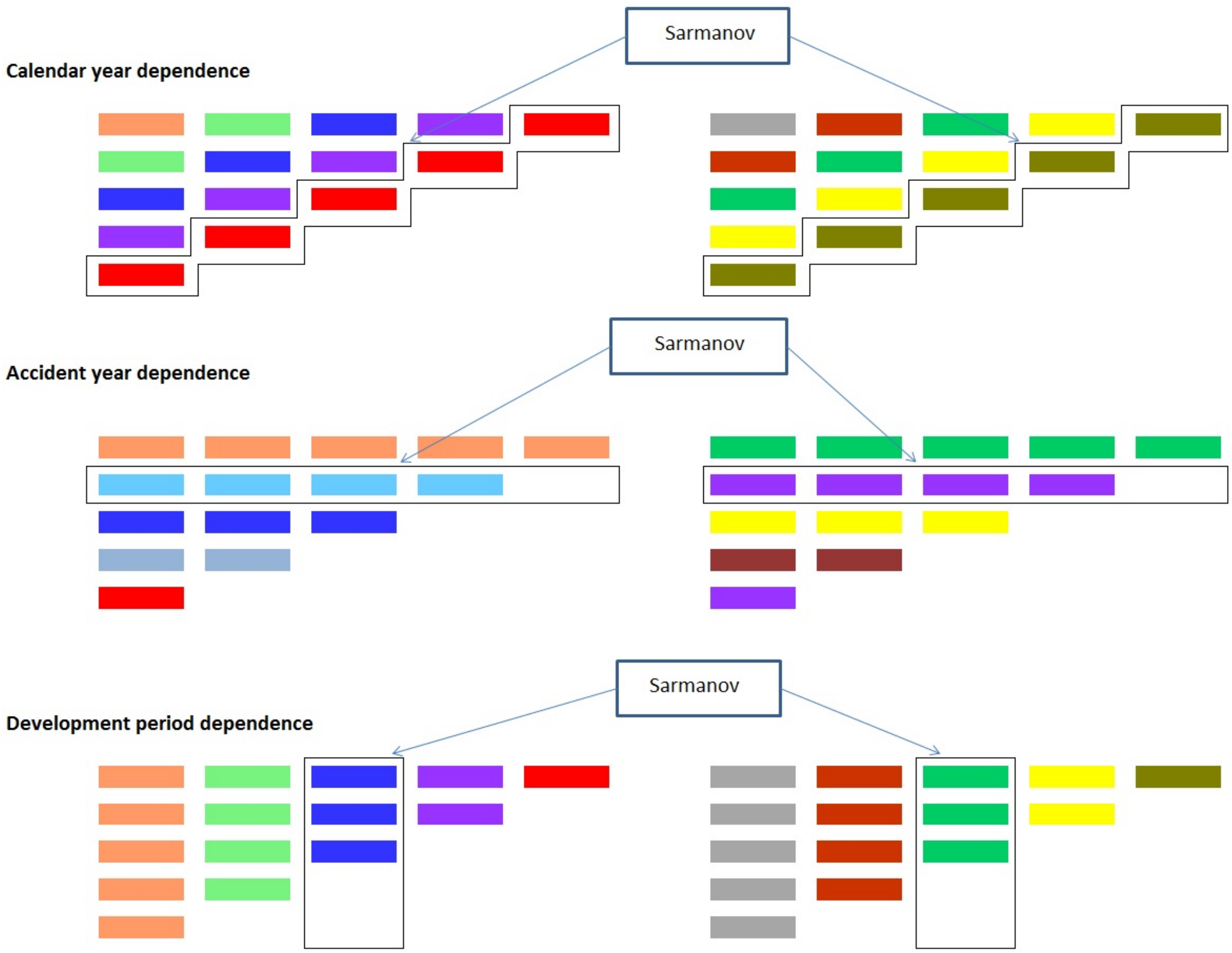

Figure 1: Modeling dependence with a Sarmanov bivariate distribution

4.3.2 Expected claim reserve

For the accident year or development period approach, unlike the calendar year case, the projections in the lower part of the triangle will be now impacted by the values of the upper part, because they are, henceforth, linked by the random effect $\Theta_t^{(\ell)}$.

Let $\Theta_t = \left(\Theta_t^{(1)}, \Theta_t^{(2)}\right)$ and $\Im_t = \left(\mathbf{Y_t^{(1)}}, \mathbf{Y_t^{(2)}}\right)$ where $\Theta_t^{(\ell)}$, $\ell = 1, 2$, characterizes the loss of a given accident year ($t = i$) or development period ($t = j$).

Given the conditional independence of $Y_{i,j}^{(\ell)}$ given $\Theta_t^{(\ell)} = \theta^{(\ell)}$, the total estimated projected paid loss ratio is given by

$$\begin{aligned} E\left[R_{tot} \mid \Im_t\right] &= E\left[R^{(1)} + R^{(2)} \mid \Im_t\right] \\ &= E\left[\sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} Y_{i.j}^{(\ell)} \mid \Im_t\right] \\ &= \sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} E\left[Y_{i.j}^{(\ell)} \mid \Im_t\right], \end{aligned}$$

where $E[Y_{i,j}^{(\ell)} \mid \Im_t] = E[E[Y_{i,j}^{(\ell)} \mid \Theta_t, \Im_t] \mid \Im_t] = E[E[Y_{i,j}^{(\ell)} \mid \Theta_t^{(\ell)}] \mid \Im_t]$ with $u^{(1)}\left(\theta^{(1)} \mid \Im_t\right) = \int_0^{+\infty} u^S(\theta^{(1)}, \theta^{(2)} \mid \Im_t) d\theta^{(2)}$ and $u^{(2)}\left(\theta^{(2)} \mid \Im_t\right) = \int_{-\infty}^{+\infty} u^S(\theta^{(1)}, \theta^{(2)} \mid \Im_t) d\theta^{(1)}$.

Hence, from (7), we have

$$u^{(1)}\left(\theta^{(1)} \mid \Im_t\right) = \left(C_1 - C_4\right) u^{(1)}\left(\theta^{(1)}; a_{post}, b_{post}^2\right) + \left(C_2 - C_3\right) u^{(1)}\left(\theta^{(1)}; a'_{post}, b_{post}^2\right),$$

which leads to

$$\begin{aligned} E[Y_{i,j}^{(1)} \mid \Im_t] &= e^{\sigma^2/2} \int_{-\infty}^{+\infty} e^{\mu_{i,j}^{(1)}\theta^{(1)}} u^{(1)}\left(\theta^{(1)} \mid \Im_t\right) d\theta^{(1)} \\ &= \left(C_1 - C_4\right) e^{a_{post}\mu_{i,j}^{(1)} + \frac{b_{post}^2}{2}\mu_{i,j}^{2(1)} + \sigma^2/2} + \left(C_2 - C_3\right) e^{a'_{post}\mu_{i,j}^{(1)} + \frac{b_{post}^2}{2}\mu_{i,j}^{2(1)} + \sigma^2/2}. \end{aligned}$$

Similarly, we obtain for the second line of business

$$u^{(2)}\left(\theta^{(2)} \mid \Im_t\right) = \left(C_1 - C_3\right) u^{(2)}\left(\theta^{(2)}; \alpha_{post}, \tau_{post}\right) + \left(C_2 - C_4\right) u^{(2)}\left(\theta^{(2)}; \alpha_{post}, \tau'_{post}\right),$$

and hence

$$\begin{aligned} E[Y_{i,j}^{(2)} \mid \Im_t] &= \phi\mu_{i,j}^{(2)} E[\frac{1}{\Theta_t^{(2)}} \mid \Im_t] \\ &= \phi\mu_{i,j}^{(2)} \int_0^{+\infty} \frac{1}{\theta^{(2)}} u^{(2)}\left(\theta^{(2)} \mid \Im_t\right) d\theta^{(2)} \\ &= \phi\mu_{i,j}^{(2)} \left(\frac{C_1 - C_3}{\tau_{post}(\alpha_{post}-1)} + \frac{C_2 - C_4}{\tau_{post}^{,}(\alpha_{post}-1)} \right). \end{aligned}$$

Consequently, the total unpaid losses in this case can be written as

$$\begin{aligned} E\left[R_{tot} \mid \Im_t\right] &= \sum_{\ell=1}^{2}\sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(\ell)} E[Y_{i.j}^{(\ell)} \mid \Im_t] \\ &= \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(1)} \left((C_1 - C_4)\, e^{a_{post}\mu_{i,j}^{(1)} + \frac{b_{post}^2}{2}\mu_{i,j}^{2(1)} + \sigma^2/2} + (C_2 - C_3)\, e^{a_{post}^{,}\mu_{i,j}^{(1)} + \frac{b_{post}^2}{2}\mu_{i,j}^{2(1)} + \sigma^2/2} \right) \\ &+ \sum_{i=2}^{n}\sum_{j=n-i+2}^{n} p_i^{(2)} \phi\mu_{i,j}^{(2)} \left(\frac{C_1 - C_3}{\tau_{post}(\alpha_{post}-1)} + \frac{C_2 - C_4}{\tau_{post}^{,}(\alpha_{post}-1)} \right), \end{aligned} \quad (10)$$

with parameters a_{post}, $a_{post}^{,}$, b_{post}^2, α_{post}, τ_{post}, $\tau_{post}^{,}$ and $C_i, i \in \{1, 2, 3, 4\}$ as given in (7). The expression of the claims reserve variance is more cumbersome for this approach but can be handily derived given that the posterior density of the Sarmanov family is a pseudo-conjugate prior.

5 Empirical illustration

5.1 Model calibration

We implement the three models proposed in the previous sections with the runoff triangles described in section 2.1. We want to compare the fit of our models with that obtained in Shi and Frees (2011), where pairwise dependence (PWD) between cells is supposed through a copula. The Gaussian copula was selected for this model based on Akaike's Information Criterion (AIC). In our empirical study, we first use a model that supposes independence between lines of business, with dependence within runoff triangles captured through random effects. This model is described in section 4.1. Fit statistics are shown in Table 1. In terms of the AIC, we observe that the three models offer a better fit than the PWD model, which is a promising result for what follows. Now, we suppose pairwise dependence between random effects that affect a given calendar year, accident year or development period. This dependence between runoff triangles is captured with the Sarmanov family of bivariate distributions. The fit statistics and the reserves obtained for this model are shown in Tables 2 and 3 respectively.

The reserve estimations, for the calendar year approach are based on (8), with the systematic component described in (1). As for the accident year and development period approach, the calculation is performed following (10). However, the accident year (development period) parameter is missing in the mean specification for accident year (development period) approach. Hence, we borrow the information from the calendar year trend to complete the projection of the lower triangle. We note that a gamma curve, also known as a Hoerl's curve, could also have been investigated for this case.

	Dependence			
Fit Statistics	**PWD**	**Dev. period**	**Calendar year**	**Accident year**
Log-Likelihood	350.5	376.4	396.4	402.3
AIC	-618.9	-669.0	-708.9	-720.8
BIC	-508.3	-656.2	-696.2	-708.1

Table 1: Fit Statistics of PWD model vs Independent lines of business with random effects

	Dependence		
Fit Statistics	**Dev. period**	**Calendar year**	**Accident year**
Dependence parameter	628.76 (194.20)	-387.10 (746.77)	12083 (22300)
Log-Likelihood	381.1	396.6	403.1
AIC	-676.2	-707.2	-718.8
BIC	-663.1	-694.2	-705.8

Table 2: Fit Statistics of Sarmanov model

We observe that the model with accident year dependence offers the best fit of all the models. Indeed, according to the fit statistics, the data seem to favour the model emphasising accident year effects. However, the model with development period dependence seems to favor dependence between lines of business. Given that the three models nest the independence case as a special case, we can perform a likelihood ratio test to examine the model fit. Compared with the independent case, the accident year model gives a χ^2 statistics of 0.2, the calendar year model gives a χ^2 statistics of 0.4, whereas the development period model gives a χ^2 statistics of 9.4. Henceforth, the dependence is rejected over the independence model for the calendar year and accident year cases, because ω is not statistically significant, meanwhile a dependence model is preferred for the development period case. A Wald test (see Boucher et al. (2007) for a detailed discussion on one-sided statistic tests)

		Dependence		
Reserves estimation	PWD	**Dev. period**	**Calendar year**	**Accident year**
Personal	6,423,180	6,547,988	6,476,093	6,616,171
Commercial	495,989	504,928	551,478	438,716
Total	6,919,169	7,052,916	7,027,571	7,054,888

Table 3: Reserve estimation with different models

based on the estimated values of ω and its standard errors (see Table 2) leads to the same conclusions drawn from the likelihood ratio test. Interestingly, the model has a better fit when incorporating dependence between the two lines of business only for the development period approach. This is also confirmed by the results of the AIC.

5.2 Predictive distribution

In practice, actuaries are interested in knowing the uncertainty of the reserve. A modern parametric technique, the bootstrap, not only gives such information but most importantly provides the entire predictive distribution of aggregated reserves for the portfolio. The predictive distribution notably allows assessment of risk capital for an insurance portfolio. Bootstrapping is also ideal from a practical point of view, because it avoids complex theoretical calculations and can easily be implemented. Moreover, it tackles potential model overfitting, typically encountered in loss reserving problems due to the small sample size. Henceforth, we implement a parametric bootstrap analysis to quantify predictive uncertainty.

The bootstrap technique is increasingly popular in loss reserving, and allows a wide range of applications. It was first introduced in a loss reserving context with a distribution-free approach by Lowe (1994). For a multivariate loss reserving analysis, Kirschner et al. (2008) used a synchronized parametric bootstrap to model dependence between correlated lines of business, and Taylor and McGuire (2007) extended this result to a generalized linear model context. Shi and Frees (2011), and more recently Shi (2014), have also performed a parametric bootstrap to quantify the uncertainty in parameter estimates, while modeling dependence between loss triangles using copulas.

5.2.1 Sarmanov simulation

The parametric bootstrap allows us to obtain the whole distribution of the reserves. We follow the same bootstrap algorithm as Taylor and McGuire (2007), also summarized in Shi and Frees (2011).

The first step of the parametric bootstrap is to generate pseudo-responses of normalized incremental paid losses $y_{ij}^{*(\ell)}$, for i, j such that $i + j - 1 \leq n$ and $\ell = 1, 2$.

For the first line of business, we generate a realization $y_{ij}^{*(1)}$ of a lognormal distribution with location (log-scale) parameter $\widehat{\mu}_{ij}^{(1)}\Theta^{(1)}$ and shape parameter $\widehat{\sigma}$. As for the second line of business, $y_{ij}^{*(2)}$ is a generated realization of a gamma distribution with location (scale) parameter $\frac{\widehat{\mu}_{ij}^{(2)}}{\Theta^{(2)}}$ and shape parameter $\widehat{\phi}$.

Therefore, a technique to generate realizations of the couple $\left(\theta^{(1)}, \theta^{(2)}\right)$ from a Sarmanov family of bivariate distributions should be used.

Given that the calendar year dependence is the most widely used for its intuitive and practical purposes, we focus solely on this approach.

To generate a bivariate Sarmanov distribution we follow the method based on the conditional simulation. Thus, for a given calendar year t, the algorithm for a Sarmanov bivariate distribution between the lines of business is as follows

	Estimated reserve	Bootstrap reserve	Estimation error	Process error
CY Sarmanov	7,027,571	7,047,931	312,331	153,413

Table 4: Bootstrap results for the calendar year Sarmanov model

1. Generate a realization $\theta^{(1)}$, from the random variable $\Theta_t^{(1)} \sim Normal\left(\widehat{a}, \widehat{b^2}\right)$.
2. Generate a realization from the conditional cumulative distribution of the random variable $\left(\Theta_t^{(2)} \mid \Theta_t^{(1)} = \theta^{(1)}\right)$.
3. Get a realization $\theta^{(2)}$ from the previous stage.

Consequently, we have obtained realizations of the couple $\left(\theta^{(1)}, \theta^{(2)}\right)$ from a Sarmanov family of bivariate distributions.

5.2.2 MSEP

A common statistic to measure the total variance uncertainty of the portfolio R_{tot}, is the mean squared error of prediction (MSEP).

The MSEP is a combination of process error and estimation error. Estimation error is linked to past observations and process error is due to the variation of future observations. The definition can be expressed as follows

$$
\begin{aligned}
MSEP[\widehat{R_{tot}}] &= E[(R_{tot} - \widehat{R_{tot}})^2] \\
&= E[((R_{tot} - E[R_{tot}]) - (\widehat{R_{tot}} - E[R_{tot}]))^2].
\end{aligned}
$$

Assuming $E[(R_{tot} - E[R_{tot}])(\widehat{R_{tot}} - E[\widehat{R_{tot}}])] = 0$, i.e. future observations are independent of past observations, we get

$$
\begin{aligned}
MSEP[\widehat{R_{tot}}] &\approx E[(R_{tot} - E[R_{tot}])^2] + E[(\widehat{R_{tot}} - E[\widehat{R_{tot}}])^2] \\
&= \underbrace{Var[R_{tot}]}_{\text{Process error}^2} + \underbrace{Var[\widehat{R_{tot}}]}_{\text{Estimation error}^2} .
\end{aligned}
$$

The main advantage of using the Sarmanov family of bivariate distributions lies in the fact that we are able to derive a closed-form expression for the process error of the whole portfolio (see (9)), which is not straightforward to obtain analytically with a copula model. We quantify the estimation error with the parametric bootstrap. In our empirical illustration, the obtained bootstrap results are exhibited in Table 4.

Also, because we can obtain the estimation error and process error for a Sarmanov model, it would be interesting to compare them with their analytic equivalent from Mack's model, which has long been considered as a benchmark model. This comparison is shown in Table 5. We note that the two methods provide results in the same order of magnitude.

Model	Reserve	$\sqrt{MSEP}$
Sarmanov	7,027,571	347,947
Mack	6.925.951	334,929

Table 5: Comparison between Sarmanov model and Mack model

5.2.3 Risk capital analysis

In addition to the bootstrap results for the calendar year dependence model with a Sarmanov family of bivariate distributions exhibited in Table 4, we provide a histogram of the reserve distribution, with the corresponding percentiles in Figure 2. The latter information is important and useful for actuaries when they want to select a reserve at a desired level of conservatism. We also superimposed kernel density estimates on the histogram of Figure 2 in Figure 3 with several choices for the bandwidth parameter to determine the smoothness and closeness of the fit. Smoothing the data distribution with a kernel density estimate can be more effective than using a histogram to identify features that might be obscured by the choice of histogram bins.

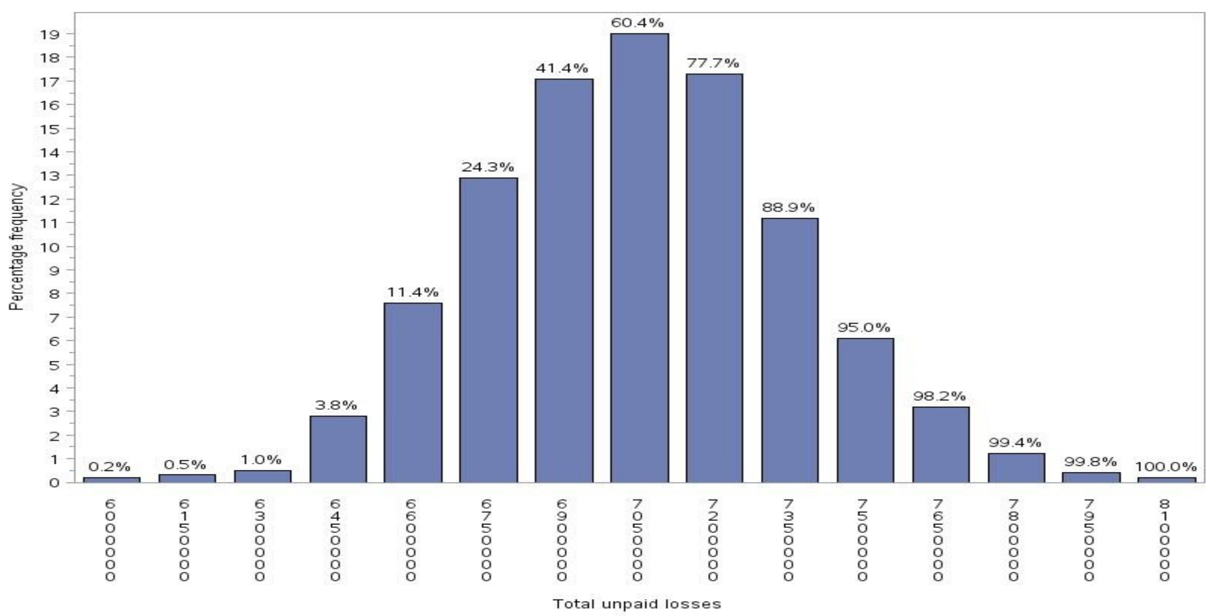

Figure 2: Percentiles of total unpaid losses (in millions) - Sarmanov calendar year model

The predictive distribution of unpaid losses is very helpful to obtain reserve ranges, but it is also useful from a risk capital standpoint. Risk capital is the amount that property/casualty insurers set aside as a buffer against potential losses from extreme and adverse events.

We want to show here the impact of assuming a dependence structure based on the

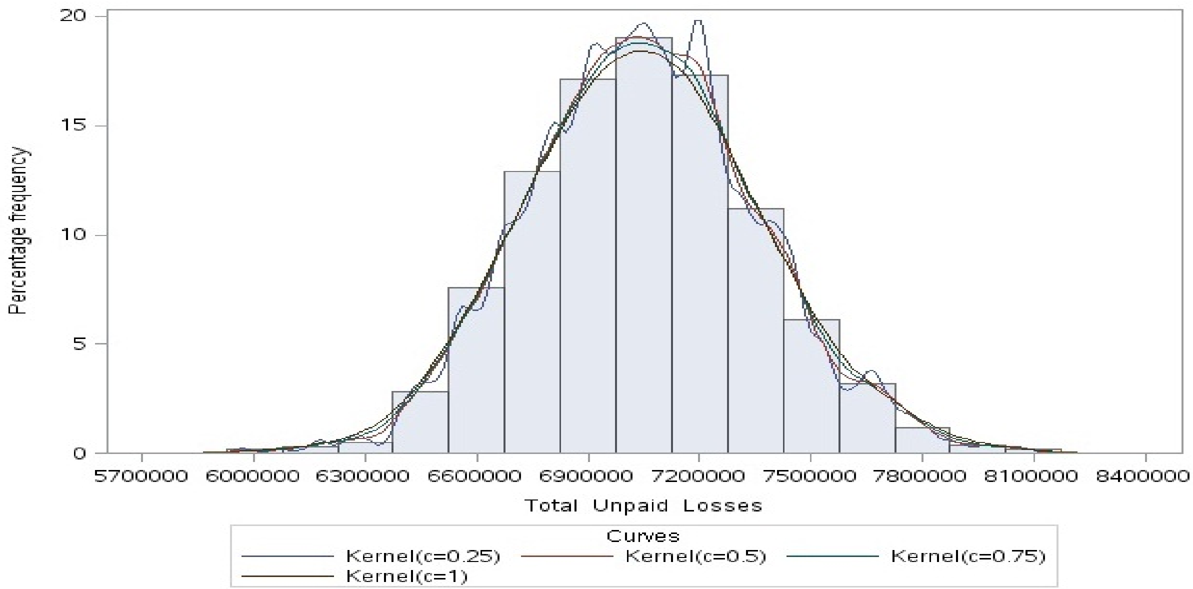

Figure 3: Total unpaid losses distribution with density Kernel estimation (in millions) - Sarmanov calendar model

Sarmanov family of bivariate distribution on the risk capital calculation instead of summing up the risk capital for each subportoflio. In fact, the most common approach in practice, called the "Silo" method, is to divide the portfolio into several subportfolios and to evaluate the risk capital for each silo and then add them up for the portfolio. The main criticism to this method is that it implicitly assumes a perfect positive linear relationship among subportfolios, which does not allow any form of diversification. We aim to show, following the parametric bootstrap, that one can take advantage of this diversification between the two lines of business, allowing risk capital analysts to be less conservative.

Mathematically, the risk capital is the difference between the risk measure and the expected unpaid losses of the portfolio, which are $7,047,931$. For the risk measure, we consider the tail value-at-risk (TVaR) that has been widely used by actuaries. This measure is more informative than the value at risk (VaR) in the distribution tail, and the subadditivity of VaR is not guaranteed in general.

To examine the role of dependencies we calculate the risk measure for each sub-portfolio (i.e. the personal auto line and the commercial auto line), and then use the simple sum as the risk measure for the entire portfolio. This is the result reported under the silo method. The silo method gives the largest estimates of risk measures because it does not account for any diversification effect in the portfolio. We provide the results for the case where no random effects are considered within lines of business (Silo - independent), and the case where random effects within lines of business are assumed (Silo - random effects). Both cases assume independence between lines of business and are compared with the case that treats the two lines of business as related through the Sarmanov bivariate distribution. We show in Table 6 that the gain in terms of risk capital is important when we capture the

Risk measure	TVaR (80%)	TVaR (85%)	TVaR (90%)	TVaR (95%)	TVaR (99%)
Silo - random effects	7,671,066	7,755,618	7,862,446	8,041,361	8,441,168
Silo - independent	7,582,963	7,656,635	7,760,671	7,922,635	8,259,798
Sarmanov	7,491,092	7,542,301	7,609,383	7,720,910	7,910,013
Risk capital					
Silo - random effects	623,135	707,686	814,515	993,429	1,393,237
Silo - independent	535,032	608,703	712,739	874,704	1,211,866
Sarmanov	443,160	494,369	561,451	672,979	862,082
Gain					
vs independent	17.17%	18.78%	21.23%	23.06%	28.86%
vs random effect	28.88%	30.14%	31.07%	32.26%	38.12%

Table 6: Risk capital estimation with different scenarios

association between the two triangles, and this difference is even greater in the distribution tail where most adversed situations are encountered for the two lines of business. This result indicates that the silo method leads to more conservative risk capital, while the Sarmanov model leads to more aggressive risk capital.

6 Conclusion

In this paper, we have studied different approaches to model dependence between loss triangles. If losses in different lines of business are correlated, aggregate reserves must reflect this dependence. To allow a flexible dependence relation, we propose the use of the Sarmanov family of bivariate distributions. To illustrate the model, an empirical illustration was performed using the same data as that used by Shi and Frees (2011). Based on the AIC and on the BIC, we show that our models provide a better fit than the PWD model does.

With the proposed model, we can derive analytically the expression of total the reserve and the total process variance with a calendar year, accident year and development period dependence model, thanks to the pseudo-conjugate properties. Also, we use a parametric bootstrap to derive a predictive distribution and incorporate parameter uncertainty in our analysis.

By coupling various sources of dependencies with a Sarmanov bivariate distribution through random effects, we propose a new approach to model dependence structures between runoff triangles. This model is a promising tool to better take into account dependencies within and between business lines. Indeed, this approach can easily be generalized to more than two lines of business because it is possible to extend the Sarmanov's family of distributions to the multivariate case. As an extension, one can also consider these random effects dynamic or evolutionary, i.e. that they evolve over time and are updated through past experience. We leave the detailed discussion of this complicated case to the future study.

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