Time Series of Correlated Count Data using Multifractal Process

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Abstract

This paper generalizes Poisson-Multifractal for correlated time series of count data. We show that the model has useful properties; it captures long-term time dependence and flexible dependence between types of count. Based on real data, the correlated multifractal model is used to model the number of claims of two separate coverages in automobile insurance. Smoothed values of the underlying process can be estimated, and a specific property of the model allows us to split the unobserved process into separate elements. These elements can be considered as climatic, economic or social factors affecting the frequency of claims, which can be associated with exogeneous informations. Even if the model proposed in this paper implies dependence between count variables, we think that it can be easily generalized in many directions: to model dependence between claim cost and frequency, or between the claims frequency of different insurance products.

Key Words: Time Series, Count Data, Multivariate Analysis, Multifractal process, Maximum Likelihood, Poisson, Dependence.

1. INTRODUCTION

Models of time series of counts can be classified in many ways. Cox (1981) proposed to separate the models into two families: observation driven models and parameter driven models. More recently, a detailed overview of time series models for counts has been done by Jung and Tremayne (2011), who identified several categories of models, such as static regression models, autoregressive conditional mean models, integer autoregressive models, and generalized linear autoregressive models (GLARMA). In their classification, a Poisson process with an autoregressive intensity is a parameter-driven model called the Poisson Stochastic Autoregressive Mean (Poisson-SAM).

The word "fractal' emerged on the scientific scene with the work of Mandelbrot(1982) in the 1960s and 1970s. Subsequently, multifractal processes became popular means of modeling financial time series. We refer the interested reader to the numerous publications of Mandelbrot (e.g. (1997) and (2001)), for applications of these processes to finance. In a recent paper, Boucher and Hainaut (2013) generalized the approach for count data, and proposed a Poisson-Multifractal model that can be advantageously compared to the Poisson-SAM model, even if the underlying unobserved process is not continuous. The authors also showed that this new Poisson model can capture particular unobserved time dependence structure, not present in the other model.

There are many practical reasons to construct a model where a dependence between two time series of counts is supposed. To our knowledge, no dependent time series model of counts using correlated autoregressive mean models has been developed. Recently, Jung, Liesenfeld and Richard (2011) use the efficient importance sampling (EIS) techniques to estimate the parameters of correlated times series. They apply independent underlying processes to each time series of count and add a common autoregressive process to introduce dependence. The model that we propose in this paper directly supposes flexible dependence with multifractal processes.

In Section 2, the univariate Poisson-Multifractal model is reviewed. Generalizations of the Poisson-Multifractal model for bivariate time series of count is presented in Section 3. In Section 4, an applied example using insurance data is used to illustrate the model, where we use another conditional distribution with the Multifractal process, namely the Negative Binomial distribution. The last section concludes the paper.

2. THE POISSON MULTIFRACTAL MODEL

We suppose that the number of events observed on the time interval $[t, t+1)$ has the following probability function:

$$\Pr\left(N_t = n \middle| F_t\right) = \frac{\lambda(t)^n e^{-\lambda(t)}}{n!}. \tag{2.1}$$

where we suppose that the parameter $\lambda(t)$ for the period of time $[t, t+1)$ is modeled as:

$$\lambda(t) = d_t \exp(\beta^\top x_t) F_t = \tau(t) F_t. \tag{2.2}$$

The vector x_t is used to include covariates, d_t is the risk exposure, and F_t is a multifractal process. The random process F_t is used here to introduce overdispersion and time dependence. This approach was inspired by the binomial multifractal process used to model volatility in a Gaussian time series, as studied by Calvet and Fisher (2008).

The process F_t has interesting interpretations because it is the product of m random factors, which may be seen as climatic, economic or social, for example. In insurance, we can then compare the underlying processes with economic statistics, such as unemployment and the cost of oil. Those m factors cannot be observable directly, and are then modeled by a Markov state vector, $\overline{M}_t$, of m components:

$$\overline{M}_t = (M_{1,t}\,, M_{2,t}\, \ldots M_{m,t}) \in \mathbb{R}_+^m.$$

Formally, the process F_t is the product of those factors:

$$F_t = \prod_{j=1}^{m} M_{j,t} \tag{2.3}$$

The $M_{j,t}$, $j = \{1, \ldots, m\}$, are built in a recursive manner. Let us assume that the vector of $M_{j,t-1}$ exists up to period $t-1$. For each $j = \{1, \ldots, m\}$, the next period $M_{j,t}$ is drawn from a fixed distribution $M(j)$ with probability γ_j, and is otherwise equal to its previous value $M_{j,t} = M_{j,t-1}$. It can be expressed as:

$$M_{j,t} = \begin{cases} M_{j,t-1} & \text{with probability } 1-\gamma_j \\ M(j) & \text{with probability } \gamma_j \end{cases}. \tag{2.4}$$

The random variable $M(j)$ is a simple binomial variable that is worth $m_{0,j}$ with probability p_0 and $2 - m_{0,j}$ with probability $1 - p_0$. Formally, we have:

$$M(j) = \begin{cases} m_{0,j} & p_0 = \frac{1}{2} \\ 2 - m_{0,j} \equiv m_{1,j} & 1 - p_0 = \frac{1}{2} \end{cases}. \tag{2.5}$$

The $m_{0,j=1\ldots m} \in (0,1)$ are parameters to be estimated and p_0 is set to $\frac{1}{2}$. If the underlying Markov process $M_{j,t}$ equals $2 - m_{0,j}$, it will increase the intensity rate. Conversely, if $M_{j,t} = m_{0,j}$, the intensity of the process will be reduced.

By construction, the parameter γ_j represents the probability that the factor $M_{j,t}$ changes its value. If γ_j is inversely proportional to j (and we will impose this relation in the inference), the last factor $M_{m,t}$ changes its value less frequently than the first factor $M_{1,t}$. This approach allows us to capture low-valued regime shifts and long volatility cycles of the counting process.

To reduce the number of parameters to be estimated, a specific parametrization of $m_{0,j=1\ldots m}$ and $\gamma_{i=1\ldots m}$ have been proposed. In particular, we assume that the probability parameters γ_j and the possible values of $M_{j,t}$, $j = 1, \ldots, m$, $t = 1, \ldots, T$ are given by the functions:

$$\gamma_j \equiv \gamma_1^{b^{j-1}} \quad j = 1, \ldots, m. \tag{2.6}$$

$$m_{0,j} \equiv (m_0)^{j^c}, \tag{2.7}$$

where m_0 and c are such that $m_{0,j} \in (0,1)$. The $2m$ parameters $m_{0,1}, ..., m_{0,m}, \gamma_1, ..., \gamma_m$ are then replaced by four parameters γ_1, b, m_0, c.

Basic properties of the Poisson-Multifractal model, as well as the method to estimate the parameters (using Hamilton's filter) can be found in Boucher and Hainaut (2013).

3. BIVARIATE MULTIFRACTAL PROCESS

The dependence between two time series of counts can also be managed with multifractal processes. As indicated in the introduction, to our knowledge, no dependent time series model of counts using correlated autoregressive mean models has been developed. Jung, Liesenfeld and Richard (2011) used two Poisson-SAM models with independent underlying processes to each time series of count and add a common autoregressive process to introduce dependence. The model proposed in this paper directly supposes flexible dependence between each underlying fractal process.

3.1 Model

We consider in this section, two time series of counts $N^{(A)}(t)$ and $N^{(B)}(t)$. The time series $N^{(A)}(t)$ and $N^{(B)}(t)$ depend respectively on multifractal processes $F_t^{(A)}$ and $F_t^{(B)}$, which will be defined later. Conditionally on multifractal processes, both series of counts are Poisson random variables with an autoregressive mean. More precisely, we have

$$\Pr(N^{(A)}(t) = a_t | F_t^{(A)}) = \frac{(\lambda_t^{(A)})^{a_t} e^{-\lambda_t^{(A)}}}{a_t!}, \tag{3.1}$$

$$\Pr(N^{(B)}(t) = b_t | F_t^{(B)}) = \frac{(\lambda_t^{(B)})^{b_t} e^{-\lambda_t^{(B)}}}{b_t!}, \tag{3.2}$$

with intensities defined as:

$$\lambda_t^{(A)} = d_t^{(A)} \exp(x_t^{(A)}\beta^{(A)})F_t^{(A)} = \tau(t)F_t^{(A)} \tag{3.3}$$

$$\lambda_t^{(B)} = d_t^{(B)} \exp(x_t^{(B)}\beta^{(B)})F_t^{(B)} = \kappa(t)F_t^{(B)}, \tag{3.4}$$

where $d_t^{(A)}$, $d_t^{(B)}$ are the exposures and vectors $x_t^{(A)}, x_t^{(B)}, \beta^{(A)}, \beta^{(B)}$ are respectively covariates and their coefficients. As for the one-dimensional multifractal model, we define processes $M_{j,t}^{(A)}$ and $M_{j,t}^{(B)}$, for $j = 1$ to m. The marginal distribution of $M_{j,t}^{(A)}$ is such that

$$M_{j,t}^{(A)} = \begin{cases} M_{j,t-1}^{(A)} & \text{with probability } 1 - \gamma_{A,j} \\ M^{(A)}(j) & \text{with probability } \gamma_{A,j} \end{cases}. \tag{3.5}$$

where $M^{(A)}(j)$ is again a simple binomial random variable equal to $m_{0,j}^{(A)}$ with probability $\frac{1}{2}$ and $2 - m_{0,j}^{(A)} = m_{1,j}^{(A)}$ with probability $\frac{1}{2}$. $M_{j,t}^{(B)}$ is defined in the same way as $M_{j,t}^{(A)}$. Multifractal processes $F_t^{(A)}$ and $F_t^{(B)}$ are the same as in the one-dimensional model defined as the product of fractal components:

$$F_t^{(A)} = \prod_{j=1}^{m} M_{j,t}^{(A)} \qquad F_t^{(B)} = \prod_{j=1}^{m} M_{j,t}^{(B)}.$$

Following Fisher and Calvet (2008, chapter 4), we introduce specific dependence for the two-dimensional process $\{F_t^{(A)}, F_t^{(B)}\}$ through each pair of fractal components $M_{j,t}^{(A)}$ and $M_{j,t}^{(B)}$. The level of dependence is controlled by a vector of parameters $\theta_j \in [0, 1]$ for , $j = 1, ..., m$.

3.2 Dependence Structure

We note $A_{j,t} = 1$ (resp. $A_{j,t} = 0$), the fact that the process $M_{j,t}^{(A)}$ is (resp. not) drawn from the distribution $M^{(A)}(j)$. We define in a similar way a process $B_{j,t}$ related to $M_{j,t}^{(B)}$. We have the following marginal distributions for $A_{j,t}$ and $B_{j,t}$:

$$\begin{aligned} \Pr(A_{j,t}=1) &= \gamma_{j,A} \qquad (3.6)\\ \Pr(B_{j,t}=1) &= \gamma_{j,B}, \qquad (3.7) \end{aligned}$$

that are used with the following dependence structure:

$$\begin{aligned} \Pr(B_{j,t}=1|A_{j,t}=1) &= (1-\theta_j)\gamma_{j,B}+\theta_j \\ &= \gamma_{j,B}+\theta_j(1-\gamma_{j,B}) \qquad (3.8)\\ \Pr(B_{j,t}=0|A_{j,t}=1) &= (1-\theta_j)(1-\gamma_{j,B}) \\ &= (1-\gamma_{j,B})-\theta_j(1-\gamma_{j,B}). \qquad (3.9) \end{aligned}$$

With this structure, the processes $M_{j,t}^{(A)}$ and $M_{j,t}^{(B)}$ are pairwise dependent. We can infer the following properties for the two-dimensional multifractal process:

Proposition 3.1. *Using the dependence structure defined in equations (3.6) to (3.9), we have the following joint probabilities:*

$$\begin{aligned} \Pr(A_{j,t}=1,B_{j,t}=1) &= \gamma_{j,A}\gamma_{j,B}+\phi_j \\ \Pr(A_{j,t}=0,B_{j,t}=1) &= (1-\gamma_{j,A})\gamma_{j,B}+\phi_j \\ \Pr(A_{j,t}=1,B_{j,t}=0) &= \gamma_{j,A}(1-\gamma_{j,B})+\phi_j \\ \Pr(A_{j,t}=0,B_{j,t}=0) &= (1-\gamma_{j,A})(1-\gamma_{j,B})+\phi_j, \end{aligned}$$

where $\phi_j=\theta_j\gamma_{j,A}(1-\gamma_{j,B})$, for $j=1,...,m$.

Proof. The proof can be found in the appendix. □

Let us denote $M_t=(M_{1,t}^{(A)}\dots M_{m,t}^{(A)}, M_{1,t}^{(B)}\dots M_{m,t}^{(B)})$ the m vector of volatility components. M_t can take $d=2^{2m}$ possible values, $s_1,\dots,s_d\in\mathbb{R}_+^{2m}$. This two-dimensional multifractal model is then defined by parameters $m_{0,j}^{(A)}, m_{0,j}^{(B)}, \gamma_{A,j}, \gamma_{B,j}$ for $j = 1, ...m$, with extra parameters $\theta_j, j = 1, ...m$ that introduce a dependence between processes. As for the one-dimensional process, we define $m_{0,j}^{(A)}$ and $m_{0,j}^{(B)}$ as function of j:

$$m_{0,j}^{(A)} = (m_{A,0})^{j^{Ac}} \qquad m_{0,j}^{(B)} = (m_{B,0})^{j^{Bc}}, \tag{3.10}$$

that have 4 parameters $m_{A,0}$, $m_{B,0}$, Ac, Bc . Similarly, the frequencies at which $A_{j,t}$ and $B_{j,t}$ switches are an increasing function of j defined as follows:

$$\gamma_{A,j} \equiv 1 - (1 - \gamma_{A,1})^{b_A^{j-1}} \quad j = 1, \ldots, m \tag{3.11}$$

$$\gamma_{B,j} \equiv 1 - (1 - \gamma_{B,1})^{b_B^{j-1}} \quad j = 1, \ldots, m, \tag{3.12}$$

where $\gamma_{A,1}$, $\gamma_{B,1}$ and b_A, b_B are constant parameters. Calvet and Fisher (2008) introduced this type of dependence in a two-dimensional autoregressive Gaussian process but they assume that all θ_j, $j = 1, ...m$ are identical. In our approach, a better fit is obtained when we differentiate the θ_j, $j = 1, ...m$. In certain cases, a parametrization for $\theta_j, j = 1, ...m$ can be used so as to reduce the number of parameters to assess. For example, the use of a specific parametrization such as $\theta_j = \theta^{j^\eta}$ can lead, for positive values of η, to stronger dependence for long-term cycle processes, and stronger dependence for short-term cycle processes if η is negative. Other parametrizations of θ_j allows us to construct several kinds of dependence structures.

3.3 Inference

We can see that the vector

$$M_t = \{M_{1,t}^{(A)}, \ldots, M_{m,t}^{(A)}, M_{1,t}^{(B)}, \ldots, M_{m,t}^{(B)}\}$$

takes $d = 4^m$ possible values, $s_1, \ldots, s_d \in \mathbb{R}_+^{2m}$. Each value corresponds to a certain state of M_t. We denote in the remainder of this section $M_t^{(A)} = \{M_{1,t}^{(A)}, \ldots, M_{m,t}^{(A)}\}$, $M_t^{(B)} = \{M_{1,t}^{(B)}, \ldots, M_{m,t}^{(B)}\}$. The first (resp. the last) m elements of $s_{k=1\ldots d}$ are noted $s_k^{(A)}$ (resp $s_k^{(B)}$). The vector M_t is a Markov chain with a transition matrix $A = (a_{x,y})_{1\leq x,y\leq d}$:

$$
\begin{aligned}
a_{x,y} &= Pr(M_{t+1} = s_y \,|\, M_t = s_x\,) \\
&= Pr\left(M_{t+1}^{(A)} = s_y^{(A)},\, M_{t+1}^{(B)} = s_y^{(B)} \,|\, M_t^{(A)} = s_x^{(A)},\, M_t^{(B)} = s_x^{(B)}\right) \\
&= \prod_{j=1}^{m} Pr\left(M_{j,t+1}^{(A)} = s_y^{(A)}(j),\, M_{j,t+1}^{(B)} = s_y^{(B)}(j) \,|\, M_{j,t}^{(A)} = s_x^{(A)}(j),\, M_{j,t}^{(B)} = s_x^{(B)}(j)\right).
\end{aligned}
$$

The probability in this last equation can be split as follows

$$
\begin{aligned}
Pr\left(M_{j,t+1}^{(A)} = s_y^{(A)}(j),\, M_{j,t+1}^{(B)} = s_y^{(B)}(j) \,|\, M_{j,t}^{(A)} = s_x^{(A)}(j),\, M_{j,t}^{(B)} = s_x^{(B)}(j)\right) = \\
I_{00}^{(j)} p_{00}(j,t) + I_{01}^{(j)} p_{01}(j,t) + I_{10}^{(j)} p_{10}(j,t) + I_{11}^{(j)} p_{11}(j,t),
\end{aligned}
$$

where $I_{00}^{(j)}$, $I_{00}^{(j)}$, $I_{00}^{(j)}$, $I_{00}^{(j)}$ are indicator variables

$$
\begin{aligned}
I_{00}^{(j)} &= I_{(s_y^{(A)}(j)=s_x^{(A)}(j),s_y^{(B)}(j)=s_x^{(B)}(j))} \\
I_{01}^{(j)} &= I_{(s_y^{(A)}(j)=s_x^{(A)}(j),s_y^{(B)}(j)\neq s_x^{(B)}(j))} \\
I_{10}^{(j)} &= I_{(s_y^{(A)}(j)\neq s_x^{(A)}(j),s_y^{(B)}(j)=s_x^{(B)}(j))} \\
I_{11}^{(j)} &= I_{(s_y^{(A)}(j)\neq s_x^{(A)}(j),s_y^{(B)}(j)\neq s_x^{(B)}(j))},
\end{aligned}
$$

and where the associated probabilities are

$$\begin{aligned}
p_{00}(j,t) &= (1-\gamma_{A,j})\left[\Pr(B_{j,t}=0|A_{j,t}=0)+\frac{1}{2}\Pr(B_{j,t}=1|A_{j,t}=0)\right] \\
&\quad +\frac{1}{2}\gamma_{A,j}\left[\Pr(B_{j,t}=0|A_{j,t}=1)+\frac{1}{2}\Pr(B_{j,t}=1|A_{j,t}=1)\right] \\
p_{10}(j,t) &= \frac{1}{2}\gamma_{A,j}\left[\Pr(B_{j,t}=0|A_{j,t}=1)+\frac{1}{2}\Pr(B_{j,t}=1|A_{j,t}=1)\right] \\
p_{01}(j,t) &= (1-\gamma_{A,j})\left[\Pr(B_{j,t}=0|A_{j,t}=0)+\frac{1}{2}\Pr(B_{j,t}=1|A_{j,t}=0)\right] \\
p_{11}(j,t) &= \frac{1}{2}\gamma_{A,j}\left[\frac{1}{2}\Pr(B_{j,t}=1|A_{j,t}=1)\right].
\end{aligned}$$

Let us denote $o_{t=1,\ldots,T}=(n_t^{(A)},n_t^{(B)})_{t=1,\ldots,T}$ the observed numbers of $N_t^{(A)}$ and $N_t^{(B)}$, on T periods of time. The probabilities of the presence in a certain state $j=1...d$ of the Markov chain M_t is noted as in the one-dimensional model:

$$\Pi_t^{(j)} = P\left(M_t=s_j \,|\, o_1,\ldots,o_t,x_t^{(A)},x_t^{(B)},d_t^{(A)},d_t^{(B)}\right).$$

The 4^m vector $\Pi_t=\left(\Pi_t^j\right)_{j=1,\ldots,d}$ can be calculated recursively by Hamilton's filter:

$$\Pi_t=\frac{p(t,o_t,x_t^{(A)},x_t^{(B)},d_t^{(A)},d_t^{(B)})*(\Pi_{t-1}A)}{\left\langle p(t,o_t,x_t^{(A)},x_t^{(B)},d_t^{(A)},d_t^{(B)})*(\Pi_{t-1}A)\ ,\ \mathbf{1}\right\rangle}, \tag{3.13}$$

where $p(t,o_t,x_t^{(A)},x_t^{(B)},d_t^{(A)},d_t^{(B)})$ is the likelihood vector of Poisson functions, computed for each state of M_t.

If the set of parameters is noted $\Upsilon=\{m_{A,0},m_{B,0},Ac,Bc,\gamma_{A,1},\gamma_{A,2},b_A,b_B,\theta_{j=1...m}\}$, the loglikelihood is:

$$\ln L(o_1\ldots o_T\,|\,\Upsilon)=\sum_{t=0}^{T}\ln\left\langle p(t,x_t^{(A)},x_t^{(B)},d_t^{(A)},d_t^{(B)}),(\Pi_{t-1}A)\right\rangle. \tag{3.14}$$

The parameters are obtained numerically by maximization of this loglikelihood.

4. EMPIRICAL ILLUSTRATION USING INSURANCE DATA

We apply the two-dimensional Poisson-Multifractal to an insurance database that contains information about the weekly number of car accidents reported to an insurance company, over a period of 4 years. Analyses of the number of claims for an insurer is an important research topic in statistics and in actuarial sciences (see e.g. Denuit et al.(2007) or Frees and Valdez (2008) for reviews). Modeling these data can provide insight into the claim process and is practical for solvency purposes.

We work with a sample of the automobile portfolio of a major company operating in Canada. Only private use cars have been considered in this sample of $1,393,401$ insured vehicles, observed over 206 weeks, on the time period 2004 to 2007. We split the number of claims into two categories: at fault and non-at fault accidents.

The insurance portfolio exhibits clear seasonality for at-fault claims, as shown in Figure 4.1. Indeed, each winter, the frequency for both type of claims is much higher than during the rest of the year. The claim frequencies decrease until mid-spring, and then increase again in the summer to attain a peak in July. The claim frequencies are at their lowest in mid-fall, and rise again for the winter. This seasonality is explained by big differences between road conditions in winter, spring, summer, and fall in Canada. Snowstorms and very low temperatures creating ice on roads represent driving hazards. In summer, clement weather conditions, greater car use because of annual vacations create another kind of situation in which insurers observe higher claim frequency. Consequently, we consider the following covariates to model seasonality:

$$x_t = \left(1, \frac{t}{1000}, \cos\left(\frac{2\pi}{12}t\right), \sin\left(\frac{2\pi}{12}t\right), \cos\left(\frac{2\pi}{6}t\right), \sin\left(\frac{2\pi}{6}t\right)\right), \tag{4.1}$$

One purpose of the research is to model claim frequency, with a presumption of the existence of time dependence between the claim frequency of consecutive weeks. Intuitively,

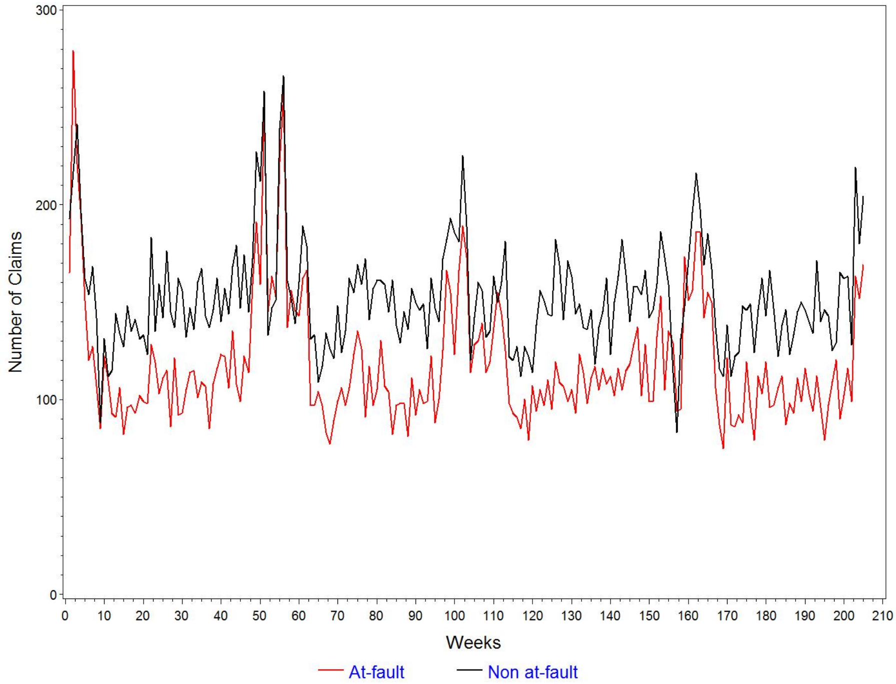

Figure 4.1: Claim frequencies per week between 2004 and 2008

we can think that a strong dependence exists between these types of claims. Indeed, if in a specific week we observe a high number of at-fault accidents, we can suppose that a high number of non-at fault accidents also arise because at-fault accidents generate non-at-fault claims. However, a single insurance company does not insure the whole population, but only some specific profiles, meaning that a single accident does not necessarily produce an at-fault claim and a non-at-fault claim with the same company. We think that the dependence can be explained by the fact that similar weather conditions can be observed during several consecutive weeks. Other social conditions, such as road, economic or environmental conditions, can also influence claim experience for long periods of time. By using a bivariate count distribution, we look for underlying dependence between the number of each types of claim.

4.1 Independent Poisson-Multifractal Models

First, we fit two independent one-dimensional Poisson-Multifractal models, with $m = 8$, to the two categories of claims. To compare the results of the independent Poisson-Multifractal model, we also used a popular generalization of the Poisson distribution, where an autoregressive lognormal process has been added in the mean function. The model, denoted Poisson-AR(1) or noted Poisson-SAM (stochastic autoregressive mean) (Jung and al., 2011), is described as:

$$N_t|\theta_t \sim Poisson(\lambda(t)\theta_t), \; \theta_t = \exp(W_t), \tag{4.2}$$

with:

$$W_t = \delta_1 W_{t-1} + \nu^2 \epsilon_t, , \tag{4.3}$$

where $\epsilon_t \sim N(0,1)$. To account for trend and seasonality, the same covariate vector as the Poisson-Multifractal models is used.

We use the EIS method of Jung and Liesenfeld (2001) to estimate the parameters of the model, that is a simulation-based method. Contrary to the Poisson-Multifractal, the Poisson-AR(1) is fitted by simulations. Procedures based on simulations are more time consuming and involve loss of precision due to variations in the estimates. The Poisson-Multifractal model is easy to estimate because it takes countable values only, but at the same time, it is flexible because the underlying process takes a very large number of possible values (for $m = 8$, $2^8 = 512$ values are possible for F_t).

Results of the Poisson-AR(1) are in Table 4.2. Based on loglikelihoods, the fits are quite similar for both types of claims, despite the fact that the Multifractal model uses more parameters than the Poisson-AR(1). The parameters $\widehat{\boldsymbol{\beta}}$ of the Poisson-AR(1) are different from those of the Multifractal model. This can be explained by the differences between models. The underlying unobserved process (that follows a multifractal process or a lognormal process) captures different kinds of time dependence.

To highlight differences between models, we can illustrate smoothed values of the processes. As expressed in Boucher and Hainaut (2013), given all T observed counts $\boldsymbol{N_T} = n_1, ..., n_T$, smoothed values $E[F_t|\boldsymbol{N_T}]$ and $E[N_t|\boldsymbol{N_T}]$ can easily be computed. Indeed, the conditional distribution of F_t is noted $g(F_t|\boldsymbol{N_T})$ and is provided by the following relation:

$$g(F_t|\boldsymbol{N_T}) = \frac{\Pr(n_{t+1}, ..., n_T|F_t)\Pr(\boldsymbol{N_t})}{\Pr(\boldsymbol{N_T})} g(F_t|\boldsymbol{N_t}). \tag{4.4}$$

All elements of this ratio can be retrieved in the estimation procedure. Figures 4.2, 4.3 and 4.4 present the smoothed values of the random effects (F_t and e^{W_t}), the means ($d_t e^{\beta x_t}$) and smoothed values of means ($d_t e^{\beta x_t} F_t$ and $d_t e^{\beta x_t} e^{W_t}$), respectively. Even if the smoothed values of means are almost similar for both models, by looking at Figures 4.2 and 4.3, we can see that the mean of each model is constructed differently. This is even more apparent when we look at the non at-fault claims. The Poisson-AR(1) model proposes $\boldsymbol{\beta}$ parameters that are close to the observed frequency, while the Poisson-Multifractal model seems to overestimate their values. To compensate, the underlying multifractal process is much lower than the lognormal process.

To better understand the time dependence of the underlying multifractal process of the Poisson distribution, we will use a specific property of the model highlighted by Boucher and Hainaut (2013). The multifractal component of the Poisson distribution, F_t can be split into m elements, because F_t is equal to the product of $M_{j,t},\ j = 1..., m$. We can then compute the smoothed value of each $M_{j,t},\ j = 1..., m$, using a similar development of (4.4):

$$g(M_{j,t}|\boldsymbol{N_T}) = \frac{\Pr(n_{t+1}, ..., n_T|M_{j,t})\Pr(\boldsymbol{N_t})}{\Pr(\boldsymbol{N_T})} g(M_{j,t}|\boldsymbol{N_t}) \quad j = 1, ..., m, \tag{4.5}$$

or using the decomposition of $g(F_t|\boldsymbol{N_T}) = \prod_{j=1}^{m} g(M_{j,t}|\boldsymbol{N_T})$. Figure 4.5 illustrates the underlying random effects in detail. The first component of the multifractal process, for at-fault and non at-fault claims, exhibits long-term cycles. These two long-term cycles are quite

	At-Fault		Non-at-Fault	
Parameters	Estimate	(std.err.)	Estimate	(std.err.)
γ_1	0.067	(0.039)	0.081	(0.030)
b	12.332	(10.905)	4.5920	(2.298)
m_0	0.839	(0.021)	0.813	(0.018)
β_0	-5.349	(0.026)	-5.022	(0.030)
β_1	0.135	(0.021)	0.030	(0.023)
β_2	-0.026	(0.020)	-0.073	(0.021)
β_3	0.109	(0.020)	0.041	(0.020)
β_4	0.005	(0.020)	-0.015	(0.019)
c	-1.084	(0.184)	-1.359	(0.137)
Loglikelihood	-886.388		-925.4908	

Table 4.1: Parameter estimates for the Poisson-Multifractal model

	At-Fault	Non-at-Fault
Parameters	Estimate	Estimate
β_0	-5.492	-5.221
β_1	0.204	0.090
β_2	0.010	-0.043
β_3	0.138	0.094
β_4	0.050	-0.003
ν	0.129	0.119
δ_1	0.490	0.436
Loglikelihood	-888.468	-922.3601

Table 4.2: Parameter estimates for the Poisson-AR(1)

similar for both type of claims and seems to validate our assumption that some dependence exists between them. The second processes M_2 present mid-term cycles for both type of claims, and shows some form of dependence because peaks of both process happen in the same time periods. It is more difficult to conclude something about the other processes. We can even considered them as random noise.

Such decomposition of the cycles can be useful to analyze, and to compare with climatic data or economic factors, for example. In our paper, however, we limit our analysis to the study of potential dependence between the random effects of each type of claim.

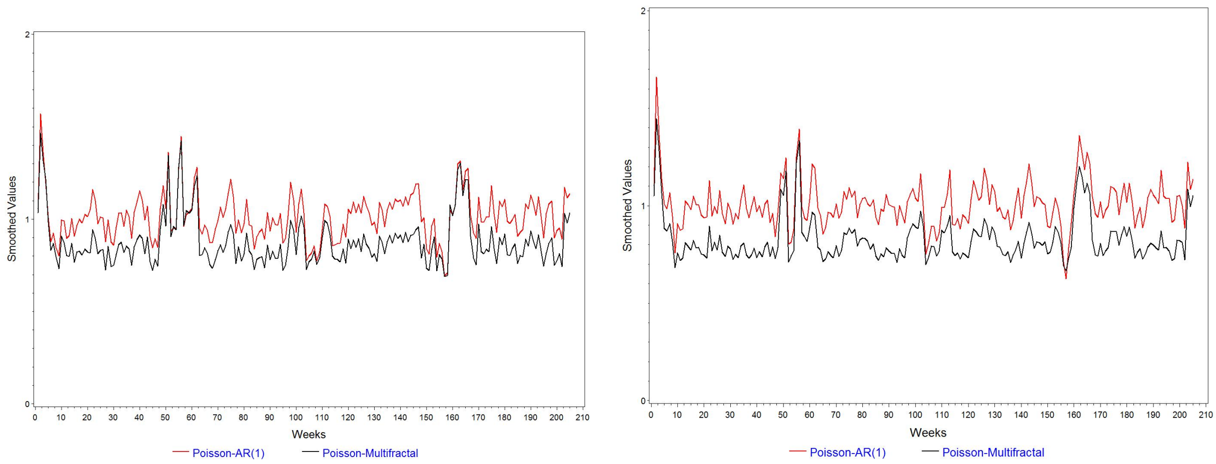

Figure 4.2: Smoothed values of the random effects (F_t and e^{W_t}) for the Poisson-Multifractal and the Poisson-AR(1) models, for the number of at-fault (left) and non at-fault (right) claims

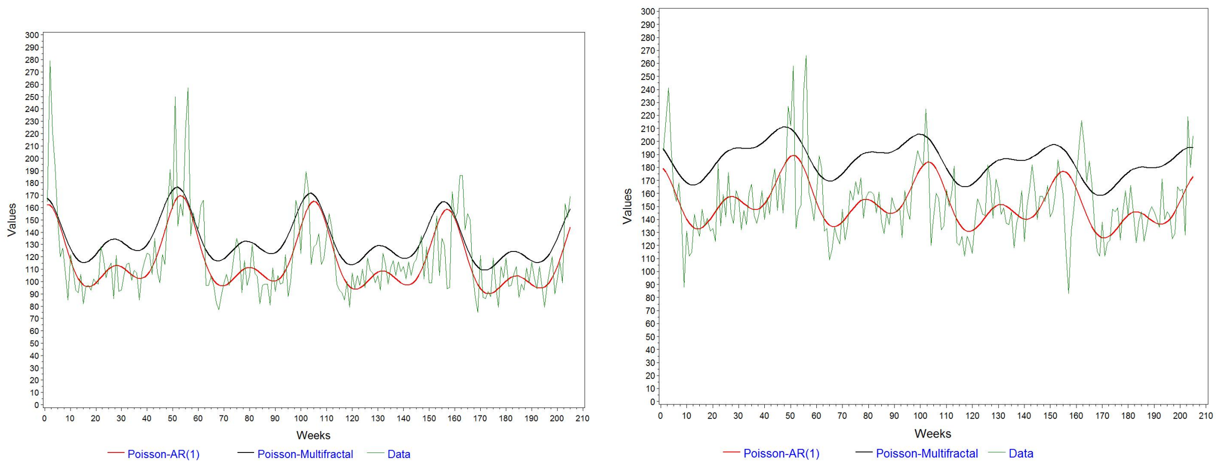

Figure 4.3: Mean frequencies ($d_t e^{\beta x_t}$) of Poisson-Multifractal and Poisson-AR(1) models, for the number of at-fault (left) and non at-fault (right) claims

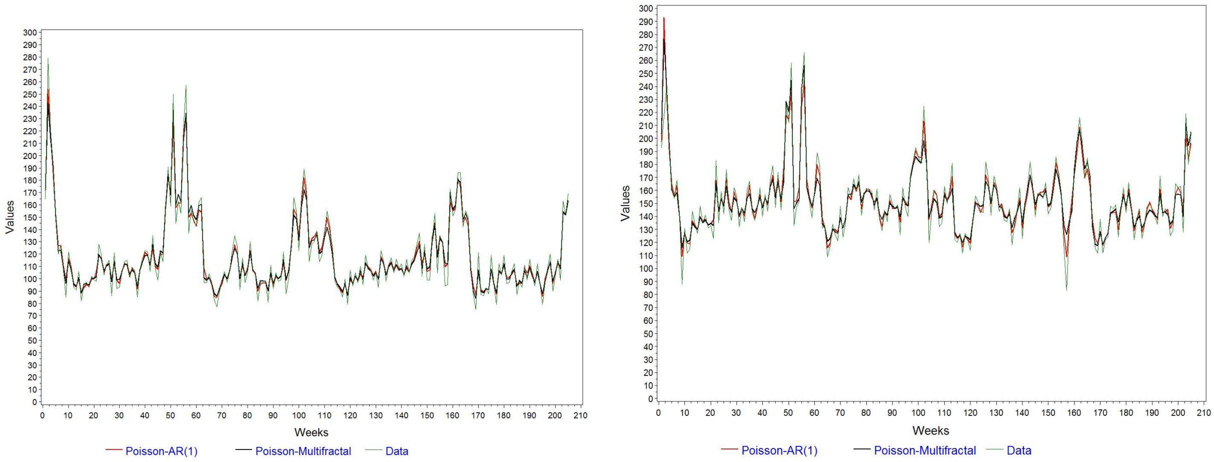

Figure 4.4: Smoothed frequencies ($d_t e^{\beta x_t} F_t$ and $d_t e^{\beta x_t} e^{W_t}$) of Poisson-Multifractal and Poisson-AR(1) models, for the number of at-fault (left) and non at-fault (right) claims

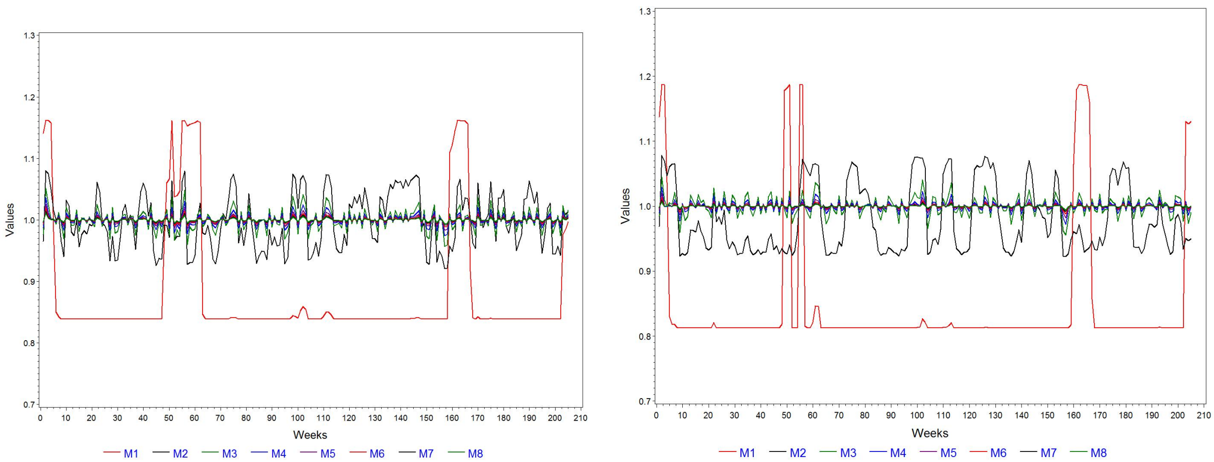

Figure 4.5: Smoothed values of the 5 fractal processes of the Poisson-Multifractal model, for the number of at-fault (left) and non at-fault (right) claims

4.2 Dependent Count Distribution

A bivariate version of the Poisson Multifractal model is proposed to fit the frequency of claims for both types of coverage. This model will allow us to test for dependence between frequencies of both type of claims. We choose to work with five fractal components per type of claims ($m = 5$). The dimension of the transition matrix is $1024(= 2^{2\times5}) \times 1024$. A higher number of fractal components requires more computer memory than what we possess.

In the previous subsection, we illustrate the count distribution with a Multifractal process by using a conditional Poisson distribution. Other conditional count distributions can be used easily. To model possible overdispersion that does not come from the time dependence structure of the data, we tested the negative binomial (NB) distribution for both types of claims. The NB distribution has the following probability distribution:

$$\Pr(N_t = n|F_t) = \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(a)}\Big(\frac{\lambda(t)}{a+\lambda(t)}\Big)^n\Big(\frac{a}{a+\lambda(t)}\Big)^a.$$

where we still suppose that the parameter $\lambda(t)$ for the period of time $[t, t+1)$ is modeled as:

$$\lambda(t) = d_t \exp(\beta^\top x_t)F_t = \tau(t)F_t. \tag{4.6}$$

Empirical analyses and classic statistical tests show us that the Poisson distribution is rejected against the NB distribution for the number of non at-fault claims, but is not rejected for the number of at-fault claims. Consequently, we used a bivariate count distribution with conditional Poisson distribution for at-fault claims, and with a conditional NB for non at-fault claims.

Results of the bivariate count distribution are shown in Table 4.3. Notations (A) and (B) refer to claims respectively related to at-fault and non-at-fault claims. A better model is obtained with $\gamma_{A,1} = \gamma_{B,1} = \gamma_1$ and $b_A = b_B = b$, because statistical tests show that these

Parameters	Estimate	(std. err.)
$\beta_0^{(A)}$	-5.318	(0.022)
$\beta_1^{(A)}$	0.145	(0.0252)
$\beta_2^{(A)}$	-0.010	(0.019)
$\beta_3^{(A)}$	0.093	(0.026)
$\beta_4^{(A)}$	0.020	(0.018)
$\beta_0^{(B)}$	-5.054	(0.022)
$\beta_1^{(B)}$	0.027	(0.017)
$\beta_2^{(B)}$	-0.064	(0.0159)
$\beta_3^{(B)}$	0.060	(0.018)
$\beta_4^{(B)}$	-0.025	(0.015)
$a^{(B)}$	0.004	(0.002)
γ_1	0.071	(0.022)
b	6.678	(3.625)
$m_{A,0}$	0.817	(0.0168)
Ac	-1.256	(0.137)
$m_{B,0}$	0.821	(0.016)
Bc	-1.820	(0.329)
θ_1	1.000	(0.001)
θ_2	1.000	(0.002)
θ_3	1.000	(0.024)
Loglikelihood	-1787.13	

Table 4.3: Parameter estimates for the two-dimensional Poisson-multivariate model

parameters were not statistically different.

For the dependence structure, only the dependence parameters for the first three processes (θ_1, θ_2 and θ_3) were statistically significant. Consequenty, we can suppose independence between the fourth and the fifth processes ($\theta_4 = \theta_5 = 0$). These results tend to validate the existence of some dependence the frequencies of at-fault and non-at-fault claims.

As done with the independent models, smoothed values of the two-dimensional multifractal process can be found. Figure 4.6 illustrates the smoothed mean value (and the observed number of claims), for at-fault and non-at-fault claims.

Analysis of the smoothed factors for $m = 1, ..., 5$ also allows us to undertand the model in more details. The dependence parameter θ_1 of the first processes $M_{1,t}^{(A)}$ and $M_{1,t}^{(B)}$, is positive and can be seen as a measure of correlation. As illustrated in in Figure 4.7, which presents the

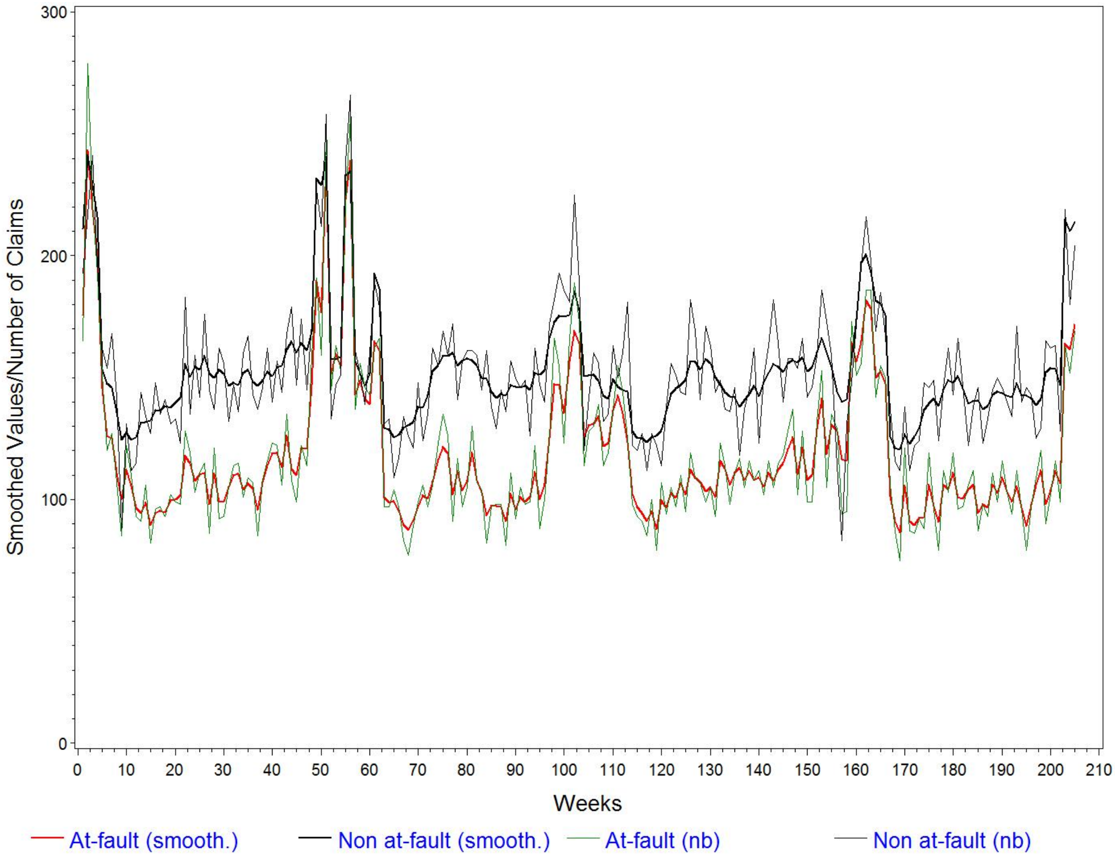

Figure 4.6: Smoothed frequencies in the two-dimensional Poisson-Multifractal model, for the number of at-fault and non-at-fault claims

smoothed values of $M_{1,t}^{(A)}$ and $M_{1,t}^{(B)}$, this dependence is clearly visible. As mentioned earlier, this dependence can come from common endegenous factors, but can also be considered as situations where the number of expected claims is clearly different from what we expect. For some period of time in the year, the count distributions with covariates (4.1) seems to have difficulties modelling the number of claims correctly. With a bivariate multifractal model, we saw that this *error* of prediction often affects at-fault and non at-fault claims simultaneously. For solvency purposes, it is important to consider these possibilities of contagion.

Finally, the analysis of independent processes $M_{4,t}$ and $M_{5,t}$ for both type of counts, allows us to consider these elements as random noise. Because we used a negative binomial distribution to model the number of non-at-fault claims, we can see that the magnitude of the $M_{4,t}^{(B)}$ and $M_{5,t}^{(B)}$ is smaller than the magnitude of $M_{4,t}^{(A)}$ and $M_{5,t}^{(A)}$.

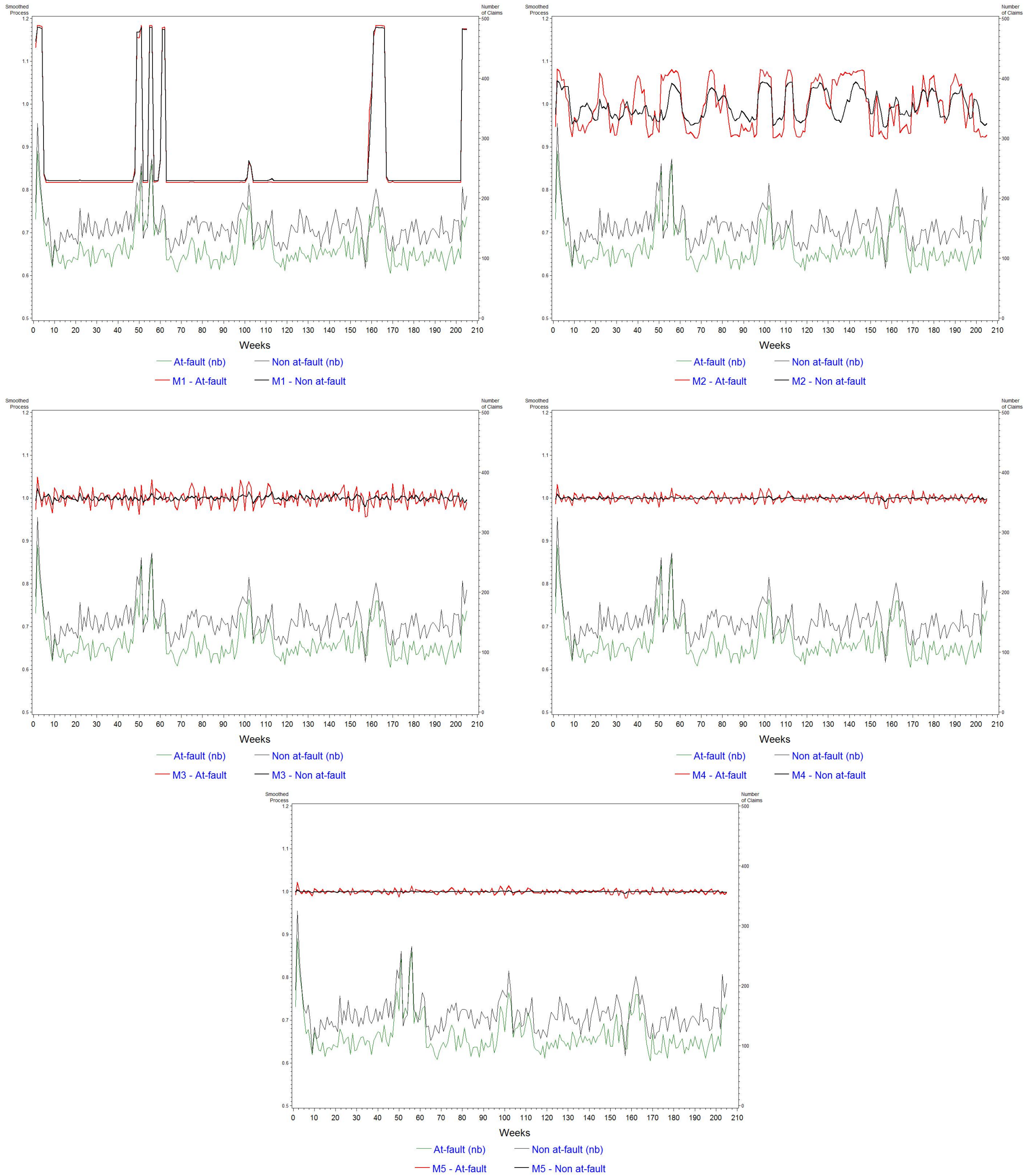

Figure 4.7: Smoothed values of the m processes, for the number of at-fault and non-at-fault claims

In this numerical example, only complete dependence ($\theta = 1$) or independence ($\theta = 0$) is observed. As explained in the construction of the bivariate model in Section 3.2, the model is flexible enough to use many values smaller than 1 for all θs. Indeed, other empirical analyses of dependence between other insurance coverages generates such values. In the modeling of dependence between the number of at-fault and non-at-fault claims, we observed that dependence between the two time series of count is not perfect. The dependence obtained between the two time series consists, however, of a strong dependence for the first processes, and independence for the last ones.

5. CONCLUSION

This work introduces the parametric count distribution with Multifractal process to the actuarial science literature. This is a flexible approach for time series of count, particularly when data exhibits overdispersion and periodic time dependence. The model has been shown to be have useful properties, particularly the fact that it does not need complex simulation methods to estimate the parameters.

We also generalize the model for bivariate count data, and show that the dependence structure that can be used in the multivariate fractal model is flexible and easily interpretable. Indeed, the bivariate model has been applied to at-fault and non-at-fault claims, where we dependence between the random variables has been shown to be significant. Because we can decompose each underlying multifractal process into m independent processes, we can understand the dependence between these two types of claims more precisely.

We think that the models introduced in this paper are first step toward constructing general models where dependence between two time series is possible. Indeed, the model can be easily generalized in many directions. For example, other count distributions can be used, as we did in using not only a conditional Poisson distribution but also a negative binomial distribution. Moreover, a model where a dependence between the costs and the frequency of claims can be constructed using the same structure as the one presented in this paper.

APPENDIX

Proof of Proposition 3.1. We have (subscripts have been removed for the sake of clarity):

$$\begin{aligned}
\Pr(B=1|A=1)\Pr(A=1)+\Pr(B=1|A=0)\Pr(A=0) &= \Pr(B=1) \\
((1-\theta)\gamma_B+\theta)\gamma_A+\Pr(B=1|A=0)(1-\gamma_A) &= \gamma_B,
\end{aligned}$$

from which we find:

$$\begin{aligned}
\Pr(B=1|A=0) &= \gamma_B-\theta\gamma_A\frac{1-\gamma_B}{1-\gamma_A} \\
&= \gamma_B-\theta(1-\gamma_B)\frac{\gamma_A}{1-\gamma_A}.
\end{aligned}$$

Similarly, we then have:

$$\begin{aligned}
\Pr(B=0|A=0) &= (1-\gamma_B)+\theta\gamma_A\frac{1-\gamma_B}{1-\gamma_A} \\
&= (1-\gamma_B)+\theta(1-\gamma_B)\frac{\gamma_A}{1-\gamma_A} \\
&= 1-\Pr(B=1|A=0).
\end{aligned}$$

We then define the joint probabilities as:

$$
\begin{aligned}
\Pr(A=1,B=1) &= \Pr(B=1|A=1)\Pr(A=1) \\
&= \left[\gamma_{j,B} + \theta(1-\gamma_{j,B})\right]\gamma_A \\
&= \gamma_A\gamma_B + \theta\gamma_A(1-\gamma_B) \\
\Pr(A=0,B=1) &= \Pr(B=1|A=0)\Pr(A=0) \\
&= \left[\gamma_B - \theta\gamma_A\frac{1-\gamma_B}{1-\gamma_A}\right](1-\gamma_A) \\
&= (1-\gamma_A)\gamma_B + \theta\gamma_A(1-\gamma_B) \\
\Pr(A=1,B=0) &= \Pr(B=0|A=1)\Pr(A=1) \\
&= \left[(1-\gamma_{j,B}) - \theta(1-\gamma_{j,B})\right]\gamma_A \\
&= \gamma_A(1-\gamma_B) + \theta\gamma_A(1-\gamma_B) \\
\Pr(A=0,B=0) &= \Pr(B=0|A=0)\Pr(A=0) \\
&= \left[(1-\gamma_B) + \theta(1-\gamma_B)\frac{\gamma_A}{1-\gamma_A}\right](1-\gamma_A) \\
&= (1-\gamma_A)(1-\gamma_B) + \theta\gamma_A(1-\gamma_B).
\end{aligned}
$$

Remember that $0 \leq \Pr(A,B) \leq 1$, leading to some constraints about the θs.

$\square$

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