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SL_K-TILINGS AND LAURENT POLYNOMIALS

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THÈSE

PRÉSENTÉE

COMME EXIGENCE PARTIELLE

DU DOCTORAT EN MATHÉMATIQUES

PAR

ADOLFO RODRÍGUEZ

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I would like to start by thanking my parents, Professors Clara Rondón and Adolfo Rodríguez, and my siblings Justo and Clara Rodríguez, for raising me in what I can only describe as the best possible environment for a child to develop personally and intellectually. In many years of being surrounded with some of the smartest people in academia, I have never found myself within a setting where critical thinking and creativity were valued as much as in my childhood home.

I would not have discovered the beautiful world of difficult elementary mathematics if it were not for the efforts of Professors Rafael Sánchez, Henry Martínez, and everyone involved in mathematical olympiads in Venezuela. They have set an amazing example of dedication and altruism that I hope one day to be able to follow. This gratitude is extended to the each of the people who helped me develop as a mathematician, especially my undergraduate advisor, Professor Miguel Méndez.

My alma mater, UCV, welcomed me into the world of non-elementary mathematics, and taught me many of the academic and social values that I still hold close to my heart. It also taught me that those two types of values are often non-exclusive. It is within its *shadow defeating* walls that I met my wife, Mariolys, to whom I owe constant support and motivation during the best and most important moments of my life, and the virtue of being able to smile and dance throughout the rest.

Finally, but not least, I would like to thank my supervisor for his invaluable encouragement and assistance during my doctoral studies, the members of my thesis commitee for their useful suggestions, and all of the people that I have met at UQAM, especially those in LaCIM and the Department of Mathematics, for repeatedly proving that Montréal is one of the best places in the world for doing research and living life.

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RÉSUMÉ

Un SL_k-pavage est une fonction, sur l'ensemble $\mathbb{Z} \times \mathbb{Z}$, à valeur dans un certain corps de caractéristique zéro. On exige aussi que chaque mineur connexe, de format $k \times k$, soit égal à l'unité multiplicative du corps. Ces pavages sont intimement liés à la résolution d'une récurrence dite “octaèdrale”, dont les solutions apparaissent naturellement comme des systèmes dynamiques dans le contexte de la mécanique statistique, de même que dans la théorie des algèbres amassés. Plus précisément, via l'identité de Desnanot-Jacobi, certains SL_k-pavage (ceux satisfaisant une condition supplémentaire de positivité) sont équivalents à une solution de la récurrence d'octaèdrale dans la grille à trois dimensions discrète. Les entrées de cette solution font intervenir toutes les entrées, et tous les mineurs connexes dudit pavage. D'autres SL_k-pavages importants sont spécifiés par des conditions de bord, souvent prenant une forme d'escalier irrégulier. Dans ce cas, il a été conjecturé que les entrées et les mineurs sont des polynômes de Laurent à coefficients non négatifs. Plusieurs preuves de cas particuliers de cette conjecture ont été proposées dans la littérature. Le but principal de cette thèse est de présenter des propositions couvrant de nouveaux cas, et de développer des modèles combinatoires permettant d'élaborer une preuve complète de cette conjecture. Un de nos résultats principaux est la preuve de la conjecture pour le cas $k = 2$. Cette preuve est basée sur la combinatoire des chemins discrets pour décrire les entrées du pavage. Ce modèle fournit des formules en terme de polynômes de Laurent à coefficient positif dont les variables sont les entrées apparaissant sur le bord spécifié. La structure combinatoire du modèle introduit est basée sur la notion d'intersection de chemins dans un graphe, et d'une extension du lemme de Lindström-Gessel-Viennot qui est cohérente avec cette nouvelle notion d'intersection de chemins.

Au cours de ce travail, nous avons été amenés à introduire des généralisations naturelles des lemmes de Lindström-Gessel-Viennot et de Stembridge, permettant de compter des ensembles de n-uples de chemins sans intersections. En particulier, ces généralisations permettent d'énumérer des familles de tableaux de Young semi-standard de forme gauche (skew tableaux), ainsi que de tableaux décalés (shifted tableaux). Notre contribution dans ce contexte concerne non seulement l'obtention de nouvelles preuves de résultats connus dans l'énumération des tableaux, mais aussi des résultats nouveaux fournissant des formules énumératives pour de plus larges familles de tableaux. D'autre part, nous avons développé une toute nouvelle approche permettant d'établir des identités de déterminants, par l'énumération de chemins. Celle-ci fournit des preuves plus courtes et plus élémentaires d'identités classiques, ainsi que de nouveaux résultats algébriques généraux reliés aux déterminants. Nous concluons cette partie de notre travail avec un théorème qui contient une vaste famille d'identités déterminantales originales, et qui permet d'exprimer le déterminant d'une matrice de mineurs d'une matrice

générique. Nous utilisons ensuite ces identités déterminantales pour obtenir une preuve récursive de la non-négativité de Laurent pour chaque entrée (et certains mineurs) de SL_k-pavages déterminés par des conditions de bords générales. Ceci produit un nouvel algorithme efficace de calcul d'entrées et de mineurs, qui évite en particulier la division par des polynômes autres que des monômes.

Plusieurs des résultats de notre travail relèvent de contextes comme la théorie des algèbres amassées, ou les algèbres de Lie, dans lesquels les calculs nécessitent l'utilisation de techniques algébriques complexes. Nous réussissons cependant à les aborder avec des outils purement combinatoires, beaucoup plus simples. En particulier, nos méthodes combinatoires se sont avérées beaucoup plus efficaces que celles utilisées auparavant, pour démontrer la non-négativité de Laurent dans plusieurs cas particuliers d'algèbres amassées.

ABSTRACT

SL_k-tilings are functions on the set $\mathbb{Z} \times \mathbb{Z}$, taking values in certain field of characteristic zero, with the additional condition that every $k \times k$ connected minor is equal to the multiplicative unit of the field. These tilings are closely related to the *octahedron recurrence*, whose solutions appear naturally as dynamical systems in contexts of statistical mechanics, as well as in the theory of cluster algebras. More precisely, via the Desnanot-Jacobi identity, certain SL_k-tilings are equivalent to a solution to the octahedron recurrence on a discrete three dimensional grid. The entries of this solution include all of the entries and all of the connected minors of the tiling. Other important SL_k tilings are specified by boundary conditions in the shape of an irregular staircase. In this case, it has been conjectured that the entries and the minors of the tiling are nonnegative Laurent polynomials. Several partial proofs of this conjecture have been proposed in the literature. The principal purpose of this thesis is to exhibit some propositions covering new cases, and to present combinatorial models that could help us develop a full proof of this conjecture. One of our most important results is the proof of the conjecture for the case $k = 2$. This proof is based on the combinatorics of discrete paths describing the entries of the tiling. Our model provides formulas for all entries of the tiling, which are nonnegative Laurent polynomials whose variables are the entries appearing in the specified boundary. The combinatorial structure of the introduced model is based on a generalized notion of intersection of paths in a graph, and in an extension of the the Lindström-Gessel-Viennot lemma which is consistent with this notion of path intersection.

Within our work we present natural generalizations of the Lindström-Gessel-Viennot and Stembridge's lemmas, allowing us to count sets of non-intersecting tuples of paths in certain graphs. In particular these extensions can be used for counting families of semi-standard Young tableaux of skew and shifted shapes. Our contribution in this context does not only include new proofs of some well known results on tableau enumeration, but also some original results providing enumerative formulas for broad families of tableaux. Additionally, we develop a simple approach allowing us to establish determinantal identities using path enumeration. We thus obtain short elementary proofs of classic identities and some general algebraic results involving determinants. We conclude this section of our work with a theorem which holds a large family of original determinantal identities yielding a formula for the determinant of a matrix of minors of a generic matrix. We use some of these determinantal identities to obtain a recursive proof of Laurent nonnegativity for every entry (and some of the minors) of SL_k-tilings under general boundary conditions. This produces a new efficient algorithm for calculating these entries and minors, by avoiding polynomial division other than by monomials.

Many of the results stated in this work appear in contexts such as cluster algebras and Lie algebras, where proofs often require the use of complex algebraic techniques. However, we have decided to approach them with purely combinatorial tools. Our proposed combinatorial methods appear to be more effective than those used in the past for the proof of Laurent nonnegativity in particular cluster algebras.

INTRODUCTION

The purpose of this thesis is to provide a combinatorial survey and several new results on the theory of SL_k-tilings (Bergeron and Reutenauer, 2010) and related subjects. An SL_k-tiling ($k \geq 2$) is a tiling of the integer plane $\mathbb{Z} \times \mathbb{Z}$ with elements of a zero-characteristic field so that every $k \times k$ connected sub-matrix has determinant equal to 1. Figure 0.1 below shows part of an SL_2-tiling with entries in $\mathbb{Z}$.

The study of SL_k-tilings is in part motivated by Fomin and Zelevinsky's (2002b) theory of *cluster algebras*, which enclose a very general family of dynamical systems, whose elements (*cluster variables*) result from applying so called *mutations* to a set of generators. An important and non trivial property of these systems is that the cluster variables turn out to be Laurent polynomials in the generators, i.e., polynomials in these generators and their reciprocals. Another apparent (yet still unproven in general) property is that the coefficients of these Laurent polynomials are all non negative integers. A large amount of particular cluster algebras have been investigated in great detail, and combinatorial arguments play a very important role in most of the relevant literature (See for example Musiker, Schiffler, and Williams, 2011).

The precise relation between SL_k-tilings and cluster algebras is formalized by Di Francesco and Kedem (2009) and Di Francesco (2010). These authors review certain discrete integrable systems called *type A T-systems*, relevant to areas of statistical mechanics, and showed that they yield interesting examples of cluster algebras under certain boundary conditions and mutations on such conditions. The type A_{k-1} T-systems turn out to be in natural correspondence with certain subsets of the *positive zero-free* SL_k*-tilings* of Bergeron and Reutenauer (2010). We explain this correspondence at the beginning of chapter 2, page 75.

$$\begin{array}{cccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 7 & 3 & 2 & 1 & 1 & 1 & \cdots \\ \cdots & 2 & 1 & 1 & 1 & 2 & 3 & \cdots \\ \cdots & 1 & 1 & 2 & 3 & 7 & 11 & \cdots \\ \cdots & 1 & 2 & 3 & 8 & 19 & 30 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Figure 0.1 Part of an SL_2-tiling.

Di Francesco (2010) describes most of the solutions of a type A_r T-system ($r \geq 1$) under general boundary conditions. Bergeron and Reutenauer (2010) focus on a smaller set of possible boundary conditions, although their tilings are more general since they do not restrict every minor of order $< k$ to be non-zero. The latter also provide several interesting results describing linear algebraic properties of SL_k-tilings.

By picturing the integer plane $\mathbb{Z} \times \mathbb{Z}$ in matrix form (with the first coordinate increasing downwards, and the second one increasing to the right), one can see an SL_k-tiling as a matrix $P := [P_{ij}]_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$, which is infinite in every direction. The condition that P is an SL_k-tiling may be written as follows;

$$\forall (a,b) \in \mathbb{Z} \times \mathbb{Z}; \quad \det(P_{[a,a+k)[b,b+k)}) = 1, \tag{1}$$

where $[m,n)$ $(m, n \in \mathbb{Z},\ m < n)$ denotes the closed-open integer integral delimited by m, n;

$$[m,n) := \{m, m+1, \ldots, n-1\}, \tag{2}$$

and P_{IJ} $(I, J \subseteq \mathbb{Z})$ denotes the sub-matrix of P with entries P_{ij} for $i \in I$, $j \in J$. The interval notation (2) is often used throughout this work, along with the similarly defined intervals $[m,n]$, $(m,n]$, (m,n).

It is sometimes possible to determine all entries of an SL_k-tiling P from a collection of *boundary conditions* of the form;

$$\det(P_{[a,a+m)[b,b+m)}) = x, \tag{3}$$

for $a, b \in \mathbb{Z}$, $m \in [1, k)$, $x \in R$. One could identify such collection with a family of quadruplets (a, b, m, x) from $\mathbb{Z} \times \mathbb{Z} \times [1, k) \times R$. We are interested in very general families of boundary conditions which we describe in Section 2.1. For the moment let us refer to these still to be described conditions as *proper families of boundary conditions* or simply *proper boundary conditions.* Also we refer to the elements $x \in R$ from these conditions as *boundary variables.*

Di-Francesco's work implies that, under proper boundary conditions and the additional property that each minor of order $< k$ is non-zero, every minor of P which does not completely enclose any of the sub-matrices $P_{[a,a+m)[b,b+m)}$ appearing in the boundary conditions, is a Laurent polynomial with nonnegative integer coefficients in the boundary variables. One of the main motivations of this work is to provide some steps towards the extension of this result to every possible minor of P. We successfully achieve this for $k = 2$ and conjecture a combinatorial model providing a partial argument for all k. This model is inspired by well-known results of Lindström (1973), Gessel and Viennot (1989), and Stembridge (1990), recalled in Section 1.2, though our approach diverges somewhat from these simple yet powerful results.

Although the paths from our model are substantially shorter than the ones appearing in Di Francesco's paper, it appears that a fairly simple and natural bijection may be constructed between them. We do not pursue this bijection, but we underline that the main innovation of our work is the much broader notion of *intersection* of paths, which ultimately leads to a complete Laurent positivity result for $k = 2$ and a partial proof of this result for $k \geq 3$.

An interesting specialization of SL_k-tilings appears upon letting $k \to \infty$. This limit is equivalent to ignoring the SL_k property (1), and having instead a proper family of boundary conditions of the form (3) for arbitrarily large values of m. These SL_∞*-tilings* were reviewed by Speyer (2007) in the form of solutions to the so called *octahedron recurrence*. His work implies the Laurent non negativity of *connected minors* (minors obtained from sub-matrices with adjacent columns and rows) of an SL_∞-tiling P under proper boundary conditions.

It is important to mention that our combinatorial model provides a first step towards the proof of a conjecture by Fomin and Zelevinsky (2000) (Conjecture 19), since their *chamber minors* are closely related to the minors appearing in certain boundary conditions.

Hereafter in this introduction we provide a brief framework of this thesis, followed by an outline of the original results from our work.

In Chapter 1 we furnish the reader with the preliminary tools and definitions of this work. Section 1.1 consists mostly of notation and definitions, serving to introduce the language and basic combinatorial objects needed to understand most of this thesis. The Lindström-Gessel-Viennot and Stembridge's Lemmas (Theorems 1 and 2), along with some natural extensions, are introduced in Section 1.2. We use them extensively in Section 1.3 to derive identities involving determinants, some of which are occasionally employed in Chapter 2. Sections 1.3 and 1.4 may be viewed as standalone examples of applications of the results from Section 1.2. In fact, they may be entirely skipped by a reader interested in SL_k-tilings, since only a few already well-known results from Section 1.3 are referenced elsewhere. Chapter 2 comprises the main results of this work, providing theorems and conjectures on the theory of SL_k-tilings.

Our main original results are listed below, in the order that they appear in this thesis;

- Theorems 1 and 2 (pages 16, 22) provide natural extensions of the Lindström-Gessel-Viennot and Stembridge's Lemmas (Corollaries 1 and 2 on pages 17, 24).
- Theorem 3 (page 37) yields a very general determinantal identity. Bernard Leclerc has said in private communication, that he believes this identity to be original. He has also provided a Lie-theoretical argument for it. Our proof is entirely

elementary.

- Theorem 6 provides a general formula for the Kostka numbers of shifted shapes.
- Theorem 7 provides the Laurent nonnegativity property for the minors of an SL_2-tiling under proper boundary conditions.

The proofs presented in this work of the following well-known results were devised independently by the author and could be valued for their simplicity. For some of them we have found a recent proof in the revised literature using combinatorial techniques but missing the simple path-based arguments.

- Propositions 1, 2 and 3 (pages 33-35) are well-known results for which we provide independent elementary proofs by utilizing the very useful matrix $\widehat{A}$ (defined in page 31). See the work by Fulmek (2012) for another recent example of path-based arguments to deriving well-known determinantal identities.
- Theorem 5 generalizes Lederer's (2006) formula for Kostka numbers of skew shapes. This formula may be obtained from arguments similar to those of Lederer or, as mentioned by that author, from Schur symmetric function identities. Our proof is based on our extension (Theorem 1) of the Lindström-Gessel-Viennot Lemma (Corollary 1).
- In Section 2.3 (page 126) we provide an inductive approach to the proof of the Laurent nonnegativity property of the entries of an SL_k-tiling under proper boundary conditions and the restriction that every minor of order $< k$ is non-zero. Although this Laurent nonnegativity property also results from Di Francesco's (2010) work, our proof has the advantage of providing a recursive formula for each of these entries. This recursive formula involves no polynomial division other than by monomials, which in turn provides a fast recursive algorithm for their calculation.

CHAPTER I

NON-INTERSECTING PATHS AND DETERMINANTS

In this chapter we review some basic graph theoretical notions, along with two well-known results involving non-intersecting paths in directed graphs, and a few important applications of these results. In Section 1.1 we introduce the concepts of directed graphs, paths and tuples of paths, which constitute the elementary theoretical basis of our research. We then use these combinatorial objects in the subsequent sections as counting (or "weighted" counting) tools. In Section 1.2 we recall two classic results on enumeration of tuples of non-intersecting paths, namely the Lindstrom-Gessel-Viennot and Stembridge's Lemmas. We also provide new natural extensions of these results. Sections 1.3 and 1.4 are applications of these lemmas and our extensions, and may be skipped by a reader interested in our main results involving SL_k-tilings, since only a couple of well-known identities from Section 1.3 are used in the subsequent chapters.

1.1 Digraphs and Paths

In this section we review the basic notions of directed graphs and paths, along with all the language necessary to understand the rest of our work.

Definition 1. A *directed graph* or *digraph* is a pair $G = (V, E)$ where V is any finite set and $E \subseteq V \times V$. The elements of V are called the *vertices* (or *points*) of G and the elements of E are called the *edges* of G. A digraph can be pictured as a collection of points in the plane, labeled by the elements of V, along with arrows $v \to w$ for all $v, w \in V$ satisfying $(v, w) \in E$ (see Figure 1.1).

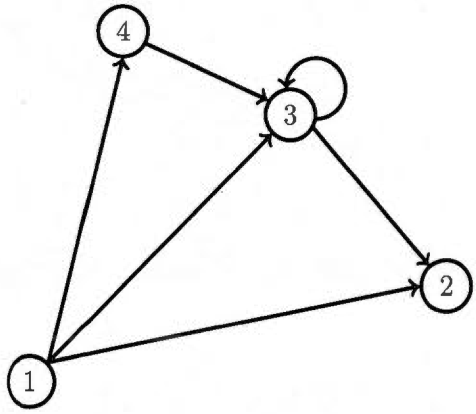


Figure 1.1 The digraph $G = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4),(3,2),(3,3),(4,3)\})$.

Definition 2. A *directed path* or simply *path* of a digraph $G = (V, E)$ is an ordered sequence $v_1, v_2, \ldots, v_m \in V$ of (not necessarily distinct) vertices such that $(v_i, v_{i+1}) \in E$ for $i = 1, \ldots, m-1$ (see Figure 1.2). We denote this path by $p = v_1 \to v_2 \to \cdots \to v_m$. We respectively call v_1 and v_m the *starting vertex* (or *starting point*) and *ending vertex* (or *ending point*) of p, respectively. The terms *source* and *sink* are common in the literature and we may occasionally use them here. We say that p is a path *from* v_1 *to* v_m, or that p *starts at* v_1 and *ends at* v_m. Also, we say that each of the vertices $v_1, \ldots, v_m$ and edges $(v_1, v_2), \ldots, (v_{m-1}, v_m)$ *are visited by* (or simply *are in the path*) p, and write $v_i \in p$ $(1 \leq i \leq m)$ and $(v_i, v_{i+1}) \in p$ $(1 \leq i \leq m-1)$. The edges (v_i, v_{i+1}) are called the *steps* of p, and the number $m-1$ of steps of p is called the *length* of p. The path with no vertices is called the *empty path.* It has length 0. Another trivial example is the path consisting of a single vertex vertex $v \in V$. This path has length 0 as well.

As stated above, a path does not necessarily consist of different vertices. For example $3 \to 3 \to 2$ and $4 \to 3 \to 3 \to 3 \to 2$ are paths in the graph of Figure 1.2. However, in the rest of our work, we focus only on acyclic digraphs (defined below), which satisfy the property no path may visit any vertex more than once.

Definition 3. A *cycle* of a digraph G is a path which starts and ends at the same

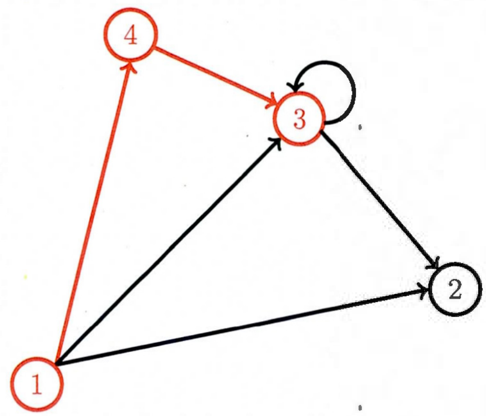


Figure 1.2 The path $1 \to 4 \to 3$ highlighted on a digraph.

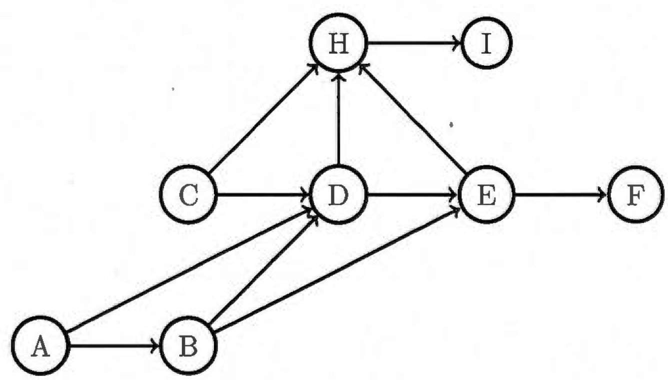


Figure 1.3 An acyclic digraph.

vertex, having at least one step. A *loop* is a cycle with one single step. For example $3 \to 3$ is a loop in the graph of Figure 1.2. A graph is said to be *acyclic* if it contains no cycles (see Figure 1.3).

At this point it is convenient to introduce some notation on *concatenation* and *truncation* of paths. We define these notions for acyclic digraphs.

Right truncation. If p is a path of an acyclic digraph and $v \in p$, then we denote by $p(\to v)$ the path whose vertices are those of p up to the vertex v. For example, if $p = v_1 \to v_2 \to v_3 \to v_4 \to v_5$, then $p(\to v_3) = v_1 \to v_2 \to v_3$ and $p(\to v_5) = p$.

Left truncation. If p is a path of an acyclic digraph and $v \in p$, then we denote by $p(v \rightarrow)$ the path whose vertices are those of p starting from the vertex v. For example, if $p = v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_5$, then $p(v_4 \rightarrow) = v_4 \rightarrow v_5$ and $p(v_1 \rightarrow) = p$.

Concatenation at a common vertex. If $p_1 = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m$ and $p_2 = v_m \rightarrow v_{m+1} \rightarrow \cdots \rightarrow v_{m+n}$ (p_1's ending vertex is equal to p_2's starting vertex), then we denote

$$p_1p_2 := v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{m+n}.$$

For example, if p is a path with $v \in p$, then $p(\rightarrow v)p(v \rightarrow) = p$.

Acyclic digraphs may often be drawn in such a way that all the edges are oriented in the same general direction (see Figure 1.3). A more formal statement of this property is provided by the following lemma and remark, as a simple result of their definition.

Lemma 1. *If the vertices $v \neq w$ of an acyclic digraph are such that there is a path from v to w, then there is no path from w to v.*

Proof. Let $p_1 = v \rightarrow v_1 \rightarrow \cdots \rightarrow v_s \rightarrow w$ be a path between v and w, and suppose, in order to obtain a contradiction, that there is a path $p_2 = w \rightarrow w_1 \rightarrow \cdots \rightarrow w_t \rightarrow v$ between w and v. This would mean that there is a path $p_1p_2 = v \rightarrow v_1 \rightarrow \cdots \rightarrow v_s \rightarrow w_1 \rightarrow \cdots \rightarrow w_t \rightarrow v$ starting and ending at v, which is not possible since the digraph is acyclic. $\square$

Remark. The lemma above implies that for any acyclic digraph G, the relation

$$v \leq_G w \quad \Leftrightarrow \quad \text{There is a path from } v \text{ to } w, \tag{1.1}$$

is antisymmetric. It is also clearly transitive and reflexive. Thus any acyclic digraph defines a partial order on its vertices. We refer to this partial order as the order *induced by G*, and denote it by $\leq$ when there is no ambiguity concerning the graph G.

Next we define the notion of intersection of paths:

Definition 4. Two paths p_1, p_2 of a digraph G are said to *intersect* (or to be *intersecting*) if they share at least one vertex. Otherwise they are said to be *non-intersecting*.

More generally, the paths $p_1, \ldots, p_k$ are said to be non-intersecting if any pair of paths p_i, p_j $(1 \leq i < j \leq k)$ is non-intersecting. We introduce the notation $p_1 \cap p_2$ for the set of vertices which are visited both by p_1 and p_2. Thus these two paths intersect at a vertex v if $v \in p_1 \cap p_2$ and they are non-intersecting if $p_1 \cap p_2 = \emptyset$.

The following construction is essential to our work.

Switching two paths at a common vertex. When two paths p_1, p_2 of an acyclic digraph G satisfy $v \in p_1 \cap p_2$, we can construct two new paths p_1', p_2' defined by $p_1' := p_1(\rightarrow v)p_2(v \rightarrow)$ and $p_2' := p_2(\rightarrow v)p_1(v \rightarrow)$, which together contain exactly the same vertices and edges as p_1, p_2. More generally, suppose that the paths p_i, p_j from the tuple $(p_1, \ldots, p_k)$ intersect at a vertex v. We introduce the following notation:

$$\mathsf{X}_{ij}^{v}(p_1, \ldots, p_k) := (p_1', \ldots, p_k'), \text{ where for } l = 1, \ldots, k;$$
$$p_l' := \begin{cases} p_l & \text{if } l \neq i, j, \\ p_i(\rightarrow v)p_j(v \rightarrow) & \text{if } l = i, \\ p_j(\rightarrow v)p_i(v \rightarrow) & \text{if } l = j, \end{cases} \tag{1.2}$$

In simple words, X_{ij}^{v} *switches* the paths p_i, p_j at their common vertex v, while leaving every other path intact. Notice that X_{ij}^{v} is an involution, i.e., for every tuple $(p_1, \ldots, p_k)$ of paths such that $v \in p_i \cap p_j$, it is also true that $v \in p_i', \cap p_j'$, and;

$$\mathsf{X}_{ij}^{v}(\mathsf{X}_{ij}^{v}(p_1, \ldots, p_k)) = (p_1, \ldots, p_k).$$

Section 1.2 is concerned with the enumeration of tuples $(p_1, \ldots, p_k)$ of non-intersecting paths satisfying certain conditions. It is often convenient that the starting and ending points of these paths are fixed or restricted to certain sets of vertices, and that these satisfy one of the two Definitions 5 or 6 below.

Definition 5. Let $G = (V, E)$ be an acyclic digraph. Let $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_m)$ $(k \leq m)$ be two finite sequences of distinct vertices from V. We say that these sequences are *G-compatible* or simply *compatible* if for any function $g : \{1, \ldots, k\} \rightarrow \{1, \ldots, m\}$, and any tuple $(p_1, \ldots, p_k)$ of paths such that p_i starts at u_i and ends at $v_{g(i)}$, the following condition holds: If the paths $p_1, \ldots, p_k$ are non-intersecting, then g is strictly increasing.

The definition above is especially important when $k = m$. In this case, the compatibility of the sequences $(u_1, \ldots, u_k)$, $(v_1, \ldots, v_k)$, means that any non-intersecting tuple $(p_1, \ldots, p_k)$ of paths from the vertices $u_1, \ldots, u_k$ to the vertices $v_{g(1)}, \ldots, v_{g(k)}$ respectively, must necessarily satisfy that g is the identity function on the set $\{1, \ldots, k\}$.

In particular when $k = m = 2$, the notion of compatibility above means that there are no pairs (p_1, p_2) of non-intersecting paths such that p_1 starts at u_1 and ends at v_2, and p_2 starts at u_2 and ends at v_1. For example, the sequences (A, B) and (I, F) of vertices from the graph in Figure 1.3, are compatible. An intimately related notion is that of pairwise compatibility:

Definition 6. Let $G = (V, E)$ be an acyclic digraph. Let $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_m)$ $(k \leq m)$ be two finite sequences of distinct vertices from V. We say that these sequences are *pairwise G-compatible* or simply *pairwise compatible* if for any $1 \leq i_1 < i_2 \leq k$, $1 \leq j_1 < j_2 \leq k$, the sequences (u_{i_1}, u_{i_2}) and (v_{j_1}, v_{j_2}) are compatible.

Definitions 5 and 6 are not equivalent. For example, in Figure 1.3, the sequences (C, A, B) and (I, D, F) are compatible, but they are not pairwise compatible. However, pairwise compatibility does imply compatibility:

Lemma 2. *Let $G = (V, E)$ be an acyclic digraph. Suppose that $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_m)$ $(k \leq m)$ are pairwise compatible sequences of distinct vertices from V. Then they are compatible.*

Proof. Let $(u_1, \ldots, u_k)$, $(v_1, \ldots, v_m)$ $(k \leq m)$ be pairwise compatible sequences. Let g be a function from $\{1, \ldots, k\}$ to $\{1, \ldots, m\}$ and suppose that there are non-intersecting paths $p_1, \ldots, p_k$ such that p_i starts at u_i and ends at $v_{g(i)}$ for $i = 1, \ldots, k$. Clearly g must be injective, since otherwise two of these paths would intersect at their ending vertex. Suppose, in order to obtain a contradiction, that g is not increasing. Thus there

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & (-2,1) \arrow[r] \arrow[u] & (-1,1) \arrow[r] \arrow[u] & (0,1) \arrow[r] \arrow[u] & (1,1) \arrow[r] \arrow[u] & (2,1) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & (-2,0) \arrow[r] \arrow[u] & (-1,0) \arrow[r] \arrow[u] & (0,0) \arrow[r] \arrow[u] & (1,0) \arrow[r] \arrow[u] & (2,0) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & (-2,-1) \arrow[r] \arrow[u] & (-1,-1) \arrow[r] \arrow[u] & (0,-1) \arrow[r] \arrow[u] & (1,-1) \arrow[r] \arrow[u] & (2,-1) \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

Figure 1.4 The north-east lattice graph.

exist i_1, i_2 such that $1 \leq i_1 < i_2 \leq k$ and $1 \leq g(i_2) < g(i_1) \leq m$. Hence (u_{i_1}, u_{i_2}) and $(v_{g(i_2)}, v_{g(i_1)})$ are not compatible (since there exist non-intersecting paths p_{i_1} from u_{i_1} to $v_{g(i_1)}$ and p_{i_2} from u_{i_2} to $v_{g(i_2)}$), which contradicts the pairwise compatibility of $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_m)$. $\square$

We end this section by discussing some properties of the north-east lattice graph, which we use extensively in the following sections.

Definition 7. The *north-east lattice graph*, which we denote by $\mathcal{L}_{NE}$ is the digraph whose vertex set is the integer lattice $\mathbb{Z} \times \mathbb{Z}$ and whose edges are those of the form $(x, y) \to (x + 1, y)$ and $(x, y) \to (x, y + 1)$. This graph is visualized in Figure 1.4, as an infinite grid with vertical edges pointing north and horizontal edges pointing east. The paths of $\mathcal{L}_{NE}$ are called *north-east lattice paths* or simply *lattice paths* when there is no ambiguity. North-east lattice paths may be viewed as paths in a cartesian coordinate system (see Figure 1.5).

Let us review some of $\mathcal{L}_{NE}$'s properties:

1. $\mathcal{L}_{NE}$'s edge set is invariant under integral vertical and horizontal translations. More formally if for fixed values of $a, b \in \mathbb{Z}$ we replace every edge $(x, y) \to (x', y')$

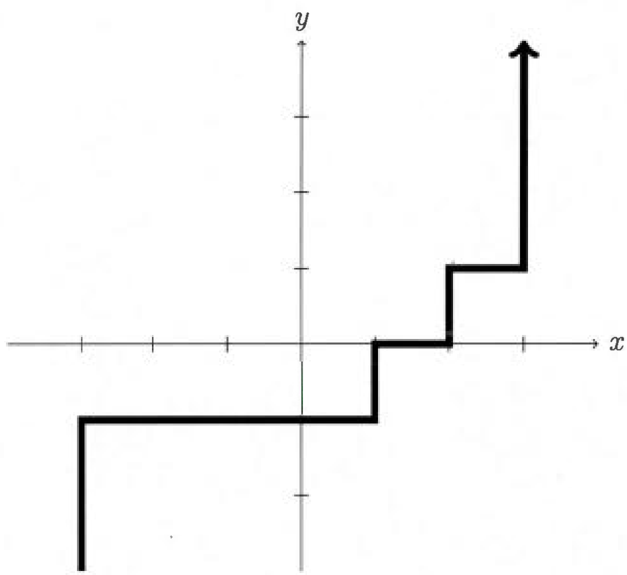


Figure 1.5 A north-east lattice path from $(-3,-3)$ to $(3,4)$.

of L_{NE} with $(x+a, y+b) \to (x'+a, y'+b)$, then the new edge set obtained is identical to the original one. This digraph is also invariant under reflections through diagonal lines of the form $y - x = c$ (c being a constant). We refer to these and similar properties as the *symmetry* of $\mathcal{L}_{NE}$.

2. $\mathcal{L}_{NE}$ is acyclic.

3. The partial order $\leq$ induced by this graph is given by

$$(x_1, y_1) \leq (x_2, y_2) \quad \Leftrightarrow \quad x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

4. For $(x_1, y_1) \leq (x_2, y_2)$, the number of paths from (x_1, y_1) to (x_2, y_2) is given by the binomial coefficient

$$\binom{x_2 + y_2 - x_1 - y_1}{x_2 - x_1} = \binom{x_2 + y_2 - x_1 - y_1}{y_2 - y_1}$$

1.2 Lindstrom-Gessel-Viennot and Stembridge's Lemmas

In this section we state some natural extensions of well-known theorems which provide enumeration formulae for tuples of non-intersecting paths in an acyclic digraph. We apply these tools extensively in Sections 1.3 and 1.4 to derive determinantal identities and tableau enumeration formulae respectively. The results and proofs listed in this section are also the inspiration for the combinatorial model of Section 2.1.

Lindström (1973), and independently Gessel and Viennot (1989) discovered a remarkable result counting the number of k-tuples $(p_1, \dots, p_k)$ of non-intersecting paths in an acyclic digraph between compatible k-tuples of vertices $(u_1, \dots, u_k)$ and $(v_1, \dots, v_k)$. Recall that this compatibility relation implies that the path starting at u_i will necessarily end at v_i for $i = 1, \dots, k$. The Lindstrom-Gessel-Viennot lemma is Corollary 1 below. We remark that some sources state this result under a pairwise compatibility hypothesis. However, compatibility turns out to be sufficient.

Stembridge (1990) later extended this result by providing a formula for the number of tuples $(p_1, \dots, p_k)$ of non-intersecting paths between pairwise compatible tuples $(u_1, \dots, u_k)$ and $(v_1, \dots, v_m)$ with $k \leq m$. Stembridge's result is Corollary 2.

Theorems 1 and 2 below are very natural generalizations of these two results. In fact their proofs are essentially the same as those of the well-known theorems. All of these results provide expressions for the number of elements in some collection of objects. However, in order to preserve more information, it is convenient to state these results in terms of weighted sums, rather than cardinalities. The weights we are interested in are given by the following definition:

Definition 8. A *weight* on an acyclic digraph $G = (V, E)$ is a function $w : V \cup E \to R$, where R is a ring. It is extended to paths by defining the weight of a path p to be the product of the weights of all its vertices and edges. More precisely:

$$w(v_1 \to \cdots \to v_m) := \left(\prod_{i=1}^{m} w(v_i) \right) \left(\prod_{i=1}^{m-1} w(v_i \to v_{i+1}) \right)$$

A weight is also extended multiplicatively to tuples of paths by the rule:

$$w(p_1, \dots, p_k) := w(p_1) \cdots w(p_k)$$

For any set S of vertices, edges, paths or tuples of paths of G, we define the *weighted cardinality* of S, denoted $|S|_w$, as the sum of the weights of all elements of S. We may also refer to this sum as the *weighted sum* of the elements of S. Clearly the cardinality $|S|$ is the weighted cardinality of S for the constant weight $w = 1$.

Remark. Observe that the involution X_{ij}^{v} ($v \in V$) defined by (1.2) is weight-preserving for any weight w on G. This is evident by definition, since p_i', p_j' visit together exactly the same vertices and edges as p_i, p_j.

Definition 9. Let G be an acyclic digraph and let T be any finite set of k-tuples of paths in G ($k \geq 2$). Denote by $\mathfrak{S}_k$ the group of all permutations of $\{1, \ldots, k\}$. We say that a function $F : T \to \mathfrak{S}_k$ is a *k-arrangement* of T if it satisfies the following two properties:

A1. If $(p_1, \ldots, p_k) \in T$ consists of k non-intersecting paths, then $F(p_1, \ldots, p_k) = \mathrm{id}$ (the identity permutation).

A2. If $(p_1, \ldots, p_k) \in T$, and $v \in p_i \cap p_j$ (for some $i \neq j$), then $\mathsf{X}_{ij}^{v}(p_1, \ldots, p_k) \in T$ and $F(\mathsf{X}_{ij}^{v}(p_1, \ldots, p_k)) = F(p_1, \ldots, p_k) \circ (i\,j)$, where $(i\,j)$ denotes the transposition permutation interchanging i and j.

Remark. Not every set of k-tuples of paths admits a k-arrangement. In particular T has to be stable under X_{ij}^{v} for all i, j, v.

Theorem 1. *Let $G = (V, E)$ be an acyclic digraph and let $w : V \cup E \to R$ be any weight on G. Let T be any set of k-tuples of paths in G and suppose that T admits a k-arrangement $F : T \to \mathfrak{S}_k$. Then the weighted sum of all non-intersecting tuples in T is equal to*

$$\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma) |F^{-1}(\sigma)|_w \tag{1.3}$$

Proof. Denote $T_\sigma := F^{-1}(\sigma)$ and

$$T_0 := \{(p_1, \ldots, p_k) \in T : p_1, \ldots, p_k \text{ are non-intersecting}\}$$

Since F is a k-arrangement, $T_0 \subseteq T_{\mathrm{id}}$. Define $\mathrm{sign}(t) \in \{1, -1\}$ for all $t \in T$ by:

$$\mathrm{sign}(t) := \mathrm{sign}(F(t)) \quad \text{(the sign of the permutation } F(t))$$

In particular $\text{sign}(t) = 1$ for all $t \in T_0$. Expression (1.3) is then equal to:

$$\sum_{t\in T} \text{sign}(t)w(t) = \sum_{t\in T_0} w(t) + \sum_{t\in T\setminus T_0} \text{sign}(t)w(t) = |T_0|_w + \sum_{t\in T\setminus T_0} \text{sign}(t)w(t)$$

It is then enough to prove that the second sum in the right hand side cancels out, for which it suffices to find a function:

$$f : (T \setminus T_0) \to (T \setminus T_0)$$

such that for all $t \in T \setminus T_0$:

1. $w(f(t)) = w(t)$,
2. $f(f(t)) = t$, and
3. $\text{sign}(f(t)) = -\text{sign}(t)$.

Start by fixing an arbitrary total order $\preceq$ on V. The tuple $f(t)$ is defined constructively from t as follows: Suppose that $t = (p_1, \ldots, p_k) \in T_\sigma$. Since $t \notin T_0$, there is at least one intersection vertex $v \in p_i \cap p_j$ $(1 \le i < j \le k)$. Choose v_t to be the smallest such intersection vertex, according to the total order $\preceq$, then choose i_t to be the smallest possible value of i given $v = v_t$, and finally choose $j_t > i_t$ to be the smallest possible value of j given $v = v_t$ and $i = i_t$. Define $f(t) := \mathsf{X}^{v_t}_{i_t j_t}(t) \in T_{\sigma\circ(i_t j_t)}$. Thus

$$\text{sign}(f(t)) = \text{sign}(\sigma)\text{sign}(i_t\, j_t) = -\text{sign}(\sigma) = -\text{sign}(t).$$

On the other hand, since the paths of $f(t)$ have the same intersection vertices as those of t, we know that $v_{f(t)} = v_t$. We also know that $i_{f(t)} = i_t$ and $j_{f(t)} = j_t$, since every path other than p_{i_t} or p_{j_t} remains unchanged when constructing $f(t)$ from t. Hence $f(f(t)) = t$. Finally the equality $w(f(t)) = w(t)$ results from the fact that X^{v}_{ij} is weight-preserving for all i, j, v. Therefore the function f exists as wanted, which proves Equation (1.3). $\square$

Corollary 1 (Lindstrom-Gessel-Viennot lemma). *Let $G = (V, E)$ be an acyclic digraph and let $w : V \cup E \to R$ be a weight on G. Let $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ be compatible* *sequences of vertices from G and let S_{ij} $(1 \leq i, j \leq k)$ denote the set of all paths from u_i to v_j. Then the weighted sum of all non-intersecting tuples in $S_{11} \times S_{22} \times \cdots \times S_{kk}$ is equal to the determinant of the square matrix $[|S_{ij}|_w]_{1 \leq i,j \leq k}$.*

Proof. We may assume that the u_i's are all different. Otherwise there would be no non-intersecting tuples, and two rows from the matrix above would be equal, which cancels out the determinant. Similarly we assume that the v_i's are all different. For each permutation $\sigma \in S_k$, define

$$T_\sigma := S_{1\sigma(1)} \times S_{2\sigma(2)} \times \cdots \times S_{k\sigma(k)},$$

and set

$$T := \bigcup_{\sigma \in \mathfrak{S}_n} T_\sigma.$$

Since the starting points $u_1, \ldots, u_k$ are all different, and so are the ending points $v_1, \ldots, v_k$, we have that the T_σ's are disjoint. Thus the function $F : T \to \mathfrak{S}_k$ given by $F(t) := \sigma$ for $t \in T_\sigma$ is well defined.

We claim that F is a k-arrangement of T. Indeed if $(p_1, \ldots, p_k) \in T_\sigma$ consists of non-intersecting paths, then by the compatibility of $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ we have that σ is increasing, and so it must be the identity permutation. On the other hand, if the path p_i from u_i to $v_{\sigma(i)}$, intersects the path p_j from u_j to $v_{\sigma(j)}$, at the vertex v, then $p_i(\to v)p_j(v \to)$ ends at $v_{\sigma(j)} = v_{\sigma\circ(i\,j)(i)}$ and $p_j(\to v)p_i(v \to)$ ends at $v_{\sigma(i)} = v_{\sigma\circ(i\,j)(j)}$. Thus F is a k-arrangement as claimed.

Therefore, by Theorem 1, the sum of the weights of all non-intersecting tuples in T is equal to:

$$\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma)|T_\sigma|_w = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) \prod_{i=1}^{k} |S_{i\sigma(i)}|_w = \det\left(|S_{ij}|_w\right)_{1 \leq i,j \leq k}$$

as wanted. □

Rather than determinants, Stembridge's lemma (Corollary 2 below) involves closely

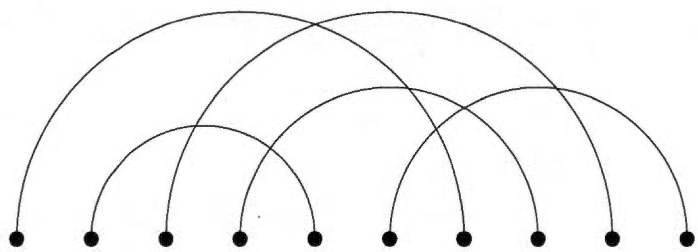


Figure 1.6 The matching $\{\{1,7\},\{2,5\},\{3,9\},\{4,8\},\{6,10\}\}$ of 10.

related polynomials called Pfaffians. Before presenting our extension to Stembridge's lemma (Theorem 2), some theoretical background is needed.

Definition 10. Let k be an even positive integer. A *matching* of k is a partition of the set $\{1,2,\ldots,k\}$ into 2-element subsets.

For example $\{\{1,6\},\{2,3\},\{4,5\}\}$ is a matching of 6. We denote the set of all matchings of k by M_k. An *inversion* of a matching $\pi \in M_k$ is a 4-element set $\{a,b,c,d\} \subseteq \{1,2,\ldots,k\}$ with $a<b<c<d$ such that $\{a,c\},\{b,d\} \in \pi$. If $\{i,j\} \in \pi$ we say that i and j are *adjacent* in π. A matching π can be represented as a diagram composed of k points labeled by the numbers $1,2,\ldots,k$ in that order on a horizontal segment, and upper semicircles joining i,j for all $\{i,j\} \in \pi$ (see Figure 1.6). Inversions are then the crossings between these semicircles. The number of inversions of π is denoted $\text{inv}(\pi)$ and the *sign* of π is given by $\text{sign}(\pi) := (-1)^{\text{inv}(\pi)}$. For example, if π is the matching from Figure 1.6, then $\text{inv}(\pi) = 7$ and $\text{sign}(\pi) = -1$.

Definition 11. Let k be an even positive integer and let $A = [a_{i,j}]_{1\leq i<j\leq k}$ be a strictly upper triangular array. Define the *Pfaffian* of A by:

$$\text{pfaff}(A) := \sum_{\pi \in M_k} \text{sign}(\pi) \prod_{\{i,j\}\in\pi,\, i<j} a_{i,j}$$

An interesting result relating Pfaffians to determinants is that for any matrix $A = (a_{i,j})_{1\leq i,j\leq k}$ satisfying $a_{i,j} = -a_{j,i}$ for all $i,j = 1,\ldots,k$ (in particular $a_{i,i} = 0$ for $i = 1,\ldots,k$. This is called a *skew-symmetric* matrix), the following equality holds:

$$\det(A) = (\text{pfaff}[a_{i,j}]_{1\leq i<j\leq k})^2$$

A combinatorial proof of this result can be found in Stembridge's (1990) paper. The following two lemmas found in the same article are essential for the rest of this section

Lemma 3. *Let $\pi \in M_k$ and assume that $\{i_1, i_2\} \notin \pi$ for some i_1, i_2 with $1 \leq i_1 < i_2 \leq k$. Let π' be the matching obtained by exchanging the values of i_1 and i_2 in π. If $\{i_1, i\}, \{i, i_2\} \notin \pi$ for all i between i_1 and i_2, then* $\text{sign}(\pi) = -\text{sign}(\pi')$.

Proof. Let j_1 and j_2 be the numbers adjacent to i_1 and i_2 respectively in π. By hypothesis we know that j_1 and j_2 are not between i_1 and i_2. Observe that:

- Any inversion $\{a, b, c, d\}$ of π (respectively π') with $\{a, b, c, d\} \cap \{i_1, i_2, j_1, j_2\} = \emptyset$ is also an inversion of π' (respectively π).
- Any inversion $\{a, b, c, d\}$ of π (respectively π') with $|\{a, b, c, d\} \cap \{i_1, i_2, j_1, j_2\}| = 2$ is not an inversion of π' (respectively π).
- $\{i_1, i_2, j_1, j_2\}$ is an inversion of either π or π', and only one of them.

Thus

$$\begin{aligned}
\text{inv}(\pi) - \text{inv}(\pi') = \pm 1 &+ |\{\{a, b\} : \{a, b, i_1, j_1\} \text{ is an inversion of } \pi\}| \\
&- |\{\{a, b\} : \{a, b, i_2, j_1\} \text{ is an inversion of } \pi'\}| \\
&+ |\{\{a, b\} : \{a, b, i_2, j_2\} \text{ is an inversion of } \pi\}| \\
&- |\{\{a, b\} : \{a, b, i_1, j_2\} \text{ is an inversion of } \pi'\}|
\end{aligned}$$

Let us find the parity of the difference

$$|\{\{a, b\} : \{a, b, i_1, j_1\} \text{ is inv of } \pi\}| - |\{\{a, b\} : \{a, b, i_2, j_1\} \text{ is inv of } \pi'\}|, \tag{1.4}$$

We say that two sets $\{a, b\}, \{c, d\}$ are *crossing* if for any matching γ with $\{a, b\}, \{c, d\} \in \gamma$, the set $\{a, b, c, d\}$ is an inversion of γ. The parity of (1.4) is then equal to the parity of

$$|\{\{a, b\} \in \pi \cap \pi' : \{a, b\} \text{ is crossing with exactly one of } \{i_1, j_1\} \text{ or } \{i_2, j_1\}\}|$$

Since j_1 is either less than or greater than both i_1 and i_2, the number above is equal to:

$$|\{\{a,b\} \in \pi \cap \pi' : a \in \{i_1, i_1+1, \dots, i_2\},\, b \notin \{i_1, i_1+1, \dots, i_2\}\}|$$

Similarly, this is also the parity of the difference

$$|\{\{a,b\} : \{a,b,i_2,j_2\} \text{ is inv of } \pi\}| - |\{\{a,b\} : \{a,b,i_1,j_2\} \text{ is inv of } \pi'\}|$$

Hence $\operatorname{inv}(\pi) - \operatorname{inv}(\pi')$ is odd, and so $\operatorname{sign}(\pi) = -\operatorname{sign}(\pi')$. □

Lemma 4. *The Pfaffian of* $[1]_{1\le i<j\le k}$ *(k even) is equal to 1. Equivalently*

$$\sum_{\pi \in M_k} \operatorname{sign}(\pi) = 1$$

Proof. Let $\pi \in M_k$ be a matching of k. Say that the pair (i_1, i_2) is *π-admissible* if $i_1 < i_2$, $\{i_1, i_2\} \notin \pi$, and $\{i_1, i\}, \{i, i_2\} \notin \pi$ for all i between i_1 and i_2. Let (i_1, i_2) be the lexicographically smallest π-admissible pair and let π' be the matching resulting from exchanging i_1 and i_2 in π. By Lemma 3, $\operatorname{sign}(\pi) = -\operatorname{sign}(\pi')$. To prove that everything but a 1 in the sum above cancels out, it suffices to show that $\pi'' = \pi$ for all $\pi \in M_k$ (where π'' is obtained from π' by interchanging the lexicographically smallest π'-admissible pair), and that the only matching π with no π-admissible pairs is the matching $\{\{1,k\},\{2,k-1\},\dots\}$, which has sign 1. For the first claim notice that (i_1, i_2) and every π-admissible pair (i_1', i_2') with $\{i_1', i_2'\} \cap \{i_1, i_2\} = \emptyset$ are also π'-admissible pairs. Thus if there were a lexicographically smaller π'-admissible pair, it would have to be of either of the forms

$$(i, i_1)\,(i < i_1), \quad (i, i_2)\,(i < i_1) \quad \text{or} \quad (i_1, i)\,(i_1 < i < i_2).$$

For the first two cases, this would mean that (i, i_2) or (i, i_1) (respectively) is a π-admissible pair, contradicting the minimality of (i_1, i_2). Less evidently, for the third case, this would mean that (i_1, i) is a π-admissible pair, contradicting again the minimality of (i_1, i_2). Hence (i_1, i_2) is also the lexicographically minimal π'-admissible pair, and so $\pi'' = \pi$. It remains to find the matchings π with no π-admissible pairs. Take $\{a,b\}, \{c,d\} \in \pi$ with $\{a,b\} \cap \{c,d\} = \emptyset$. The only way that no two elements of these sets form a π-admissible pair is that one of these sets is completely contained in the interval bounded by the other one. Since this must happen for every two sets in π, this relation defines a total order in the elements of π. The greatest of them is necessarily $\{1, k\}$ and the others are found to be $\{i, k-i\}$ $(i = 2, 3, \ldots)$ inductively. $\square$

Definition 12. Let G be an acyclic digraph, let $I = (v_1, \ldots, v_m)$ be a tuple of distinct vertices of G, and let T be a set of k-tuples of paths in G, for some integer $k \leq m$. We say that T is *I-stable* if these three properties are satisfied:

S1. All of the paths from the tuples in T end at vertices in I.

S2. If $(p_1, \ldots, p_k) \in T$ and the paths $v \in p_i \cap p_j$ $(1 \leq i < j \leq k)$, then $X^v_{ij}(p_1, \ldots, p_k) \in T$.

S3. If $(p_1, \ldots, p_k) \in T$ and the paths p_i, p_j $(1 \leq i < j \leq k)$, ending at v_a, v_b respectively, are non-intersecting, then $a < b$.

Theorem 2. *Let $G = (V, E)$ be an acyclic digraph and let $w : V \cup E \to R$ be any weight on G. Let $I = (v_1, \ldots, v_m)$ be a tuple of vertices of G, and let T be an I-stable sets of k-tuples (k even, $k \leq m$) of paths in G. Then the weighted sum of all non-intersecting tuples in T is equal to:*

$$\sum_{\pi \in M_k} \operatorname{sign}(\pi) \left|\{(p_1, \ldots, p_k) \in T : p_i \cap p_j = \emptyset \ \forall \ \{i, j\} \in \pi\}\right|_w \quad . \tag{1.5}$$

Proof. Denote by T_π the set of all tuples $(p_1, \ldots, p_k)$ in T such that p_i, p_j are non-intersecting for all $\{i, j\} \in \pi$, and let T_0 denote the set of non-intersecting tuples in T. Notice that $T_0 \subseteq T_\pi$ for all $\pi \in M_k$, so (1.5) becomes:

$$\begin{aligned}
\sum_{\pi \in M_k} \operatorname{sign}(\pi) |T_\pi|_w &= \sum_{\pi \in M_k} \operatorname{sign}(\pi) |T_0|_w + \sum_{\pi \in M_k} \operatorname{sign}(\pi) |T_\pi \setminus T_0|_w \\
&= |T_0|_w + \sum_{\pi \in M_k} \operatorname{sign}(\pi) |T_\pi \setminus T_0|_w
\end{aligned}$$

where the last equality results from Lemma 4. Thus we just need to show that the last sum is equal to 0. It is convenient to write this sum as;

$$\sum_{\pi \in M_k} \sum_{t \in T_\pi \setminus T_0} W(\pi, t),$$

where $W(\pi, t) = \text{sign}(\pi) w(t)$. To prove that this sum cancels out it suffices to find a function

$$f : \bigcup_{\pi \in M_k} \{\pi\} \times (T_\pi \setminus T_0) \to \bigcup_{\pi \in M_k} \{\pi\} \times (T_\pi \setminus T_0)$$

such that $f(f(\pi, t)) = (\pi, t)$ and $W(f(\pi, t)) = -W(\pi, t)$ for all (π, t) in the set union above. To define f start by considering a total order $\preceq$ on V. In particular we choose one that is consistent with the order $\leq_G$ induced by the graph G ($u \leq_G v$ if there is a path from u to v), i.e., one that satisfies $u \leq_G v \Rightarrow u \preceq v$ for all $u, v \in V$. We construct $(\pi', (p'_1, \ldots, p'_k)) = f(\pi, (p_1, \ldots, p_k))$ from $t = (p_1, \ldots, p_k) \in T_\pi \setminus T_0$ as follows. Let v be the minimal (in the order $\preceq$) vertex of intersection between paths of t. Let $i_1 < i_2$ be the smallest indexes such that p_{i_1} and p_{i_2} visit v. Define π' by exchanging the values of i_1 and i_2 in π, and define $t' = (p'_1, \ldots, p'_k) := \chi^v_{i_1 i_2}(t)$. Observe the following facts:

1. $t' \in T$. This is true because T is I-stable.

2. $\{i_1, i_2\} \notin \pi, \pi'$. Indeed, the paths p_{i_1} and p_{i_2} intersect at v, and so do the paths $p'_{i_1} = p_{i_1}(\to v) p_{i_2}(v \to)$ and $p'_{i_2} = p_{i_2}(\to v) p_{i_1}(v \to)$.

3. $t' \in T_{\pi'}$ (This clearly implies that $t' \in T_{\pi'} \setminus T_0$). Indeed, suppose that for some $\{i, j\} \in \pi'$, the paths p'_i and p'_j intersect. If $i, j \neq i_1, i_2$, then $\{i, j\} \in \pi$, but the paths $p_i = p'_i$, $p_j = p'_j$ intersect, which contradicts that $t \in T_\pi$. Otherwise, if $i = i_1$ and $j \neq i_1, i_2$, then the paths $p'_{i_1} = p_{i_1}(\to v) p_{i_2}(v \to)$ and $p'_j = p_j$ intersect, which means that p_j intersects either $p_{i_1}(\to v)$ or $p_{i_2}(v \to)$. The first option would contradict the minimality of v, while the second one would contradict the non-intersection of p_j and p_{i_2}, resulting from the fact that $\{i_2, j\} \in \pi$. The case $i = i_2$, $j \neq i_1, i_2$ generates a similar contradiction.

4. $f(f(\pi', t')) = (\pi, t)$. Indeed, since t' shares the same vertices and edges of t, they have the same minimal (in the order $\preceq$) intersection vertex v. Also since every

index i other than i_1 or i_2 satisfies $p_i = p'_i$, we know that i_1, i_2 are the smallest indexes such that both p'_{i_1} and p'_{i_2} visit v. Hence the same construction above yields t from t' and π from π'.

5. $w(t') = w(t)$. This true because t and t' share the same vertices and edges with the same corresponding multiplicities.

6. $\{i_1, i\}, \{i, i_2\} \notin \pi$ for all i between i_1 and i_2. Indeed we claim that for all such i, the path p_i intersects both p_{i_1} and p_{i_2}. Assume without loss of generality that $i_1 < i_2$, so that $i_1 < i < i_2$. To show that p_i intersects p_{i_1}, consider their ending points v_a, v_b. If $a \leq b$, then by the I-stability of T, we know that p_i must necessarily intersect p_{i_1}. Otherwise, if $b < a$, then by I-stability, $p_i = p'_i$ must intersect $p'_{i_2} = p_{i_2}(\to v)p_{i_1}(v \to)$. It must then necessarily intersect $p_{i_1}(v \to)$, to avoid contradicting the minimality of v. Thus p_i intersects p_{i_1}. By a similar argument it intersects p_{i_2}.

7. $\text{sign}(\pi') = -\text{sign}(\pi)$. This is a result of facts 2 and 6 above, along with Lemma 3.

Therefore f exists as wanted. □

Corollary 2 (Stembridge's Lemma). *Let $G = (V, E)$ be an acyclic digraph and let $w : V \cup E \to R$ be any weight on G. Let $(u_1, \dots, u_k)$, $(v_1, \dots, v_m)$ (k even, $k \leq m$) be pairwise compatible tuples of vertices of G, and let S_i denote the set of all paths from u_i to $\{v_1, \dots, v_m\}$ for $i = 1, \dots, k$. Then the weighted sum of all non-intersecting tuples in $S_1 \times \dots \times S_k$ is equal to the Pfaffian of the array $[|N_{ij}|_w]_{1 \leq i < j \leq k}$, where N_{ij} is the set of all non-intersecting tuples in $S_i \times S_j$.*

Proof. Set $T := S_1 \times \dots \times S_k$, and $I := (v_1, \dots, v_m)$. Clearly T is I-stable (properties **S1** and **S2** are true by the definition of T, and property **S3** is a result of the compatibility of the two sequences). Thus by Theorem 2, the weighted sum of all non-intersecting tuples in T is equal to;

$$
\begin{aligned}
& \sum_{\pi \in M_k} \operatorname{sign}(\pi)\, \left|\{(p_1,\ldots,p_k) \in T : (p_i,p_j) \in N_{ij} \text{ for all } \{i,j\} \in \pi\}\right|_w \\
= & \sum_{\pi \in M_k} \operatorname{sign}(\pi) \prod_{\{i,j\}\in\pi,\, i<j} |N_{ij}|_w \\
= & \ \operatorname{pfaff}[|N_{ij}|_w]_{1\le i<j\le k},
\end{aligned}
$$

as wanted. □

The next two sections are applications of Theorems 1, 2 and Corollaries 1, 2. Section 1.3 employs Corollary 1 to derive alternate proofs of determinantal identities, at least one of which (Theorem 3) appears to be new. Section 1.4 lists some results on Young tableau enumeration which arise from Theorems 1 and 2.

1.3 Determinantal identities

A common application of the Lindstrom-Gessel-Viennot lemma (Corollary 1) is the derivation and proof of identities involving determinants (*determinantal identities*) in a combinatorial manner. There are several methods to carry out these proofs, and they usually involve an interpretation of each entry of a matrix as the weighted sum of all paths between two vertices of a certain graph. Here we focus on a digraph isomorphic to the bottom-right quadrant of the north-east lattice path, and from it we define a matrix whose entries are weighted cardinalities of sets of paths in this digraph. We then argue that any determinantal identity which holds for this matrix, must also hold in general for any matrix. We thus derive some well-known identities and a very general new one (Theorem 3), using combinatorial arguments along with some basic algebraic ones. Some elementary concepts must be reviewed prior to defining our graph and its weight.

Let A be an $m \times n$ matrix:

$$A := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Denote $[n] := [1, n] = \{1, 2, \dots, n\}$ for all $n \geq 1$. For each pair of subsets $I = \{i_1, \dots, i_p\} \subseteq [m]$, $J = \{j_1, \dots, j_q\} \subseteq [n]$ with $1 \leq i_1 < \cdots < i_p \leq m$ and $1 \leq j_1 < \cdots < j_p \leq n$, define:

$$A_{IJ} := [a_{i_b j_c}]_{\substack{1 \leq b \leq p \\ 1 \leq c \leq q}}$$

We call this a *sub-matrix* of A. When $|I| = |J| = k$, the matrix A_{IJ} is a $k \times k$ square matrix, and we refer to its determinant $\det(A_{IJ})$ as a $k \times k$ *minor* of A. For instance the entry a_{ij} may be regarded as the 1×1 minor $\det(A_{\{i\}\{j\}})$, more conveniently denoted $\det(A_{ij})$ or simply A_{ij}.

A *determinantal polynomial* f is a function which assigns to each $m \times n$ matrix A, an expression of the form:

$$f(A) = \sum_{k\geq 0} \sum_{\boldsymbol{I},\boldsymbol{J}} C_{\boldsymbol{I},\boldsymbol{J}} \prod_{i=1}^{k} \det(A_{I_i J_i}), \tag{1.6}$$

where the second sum is over all pairs of k-tuples $\boldsymbol{I} := (I_1, \ldots, I_k), \boldsymbol{J} := (J_1, \ldots, J_k)$ with

$$I_i \subseteq [m],\ J_i \subseteq [n],\ |I_i| = |J_i|, \quad i = 1, \ldots, k,$$

and the coefficients $C_{\boldsymbol{I},\boldsymbol{J}}$ are integer constants (not depending on A), only a finite number of which are non-zero. In other words, a determinantal polynomial is a polynomial of integer coefficients on the minors of a matrix. The following is an example of a determinantal polynomial evaluated in a generic matrix A:

$$\begin{aligned} f(A) \quad := \quad & \det(A_{\{1,2,3\}\{1,2,3\}}) \det(A_{\{1,2,4\}\{1,2,4\}}) \\ & - \det(A_{\{1,2,3\}\{1,2,4\}}) \det(A_{\{1,2,4\}\{1,2,3\}}) \\ & - \det(A_{\{1,2\}\{1,2\}}) \det(A_{\{1,2,3,4\}\{1,2,3,4\}}) \end{aligned} \tag{1.7}$$

A *determinantal identity* is an equation of the form "$f = 0$" (or an equivalent expression), where f is a determinantal polynomial, and such that the equality $f(A) = 0$ holds true for every $m \times n$ matrix A. For example "$f = 0$," where f is given by equation 1.7 above, is a determinantal identity, since the equality holds true for every 4×4 matrix A (see Proposition 1).

Adding new rows to the right, or columns at the bottom of a matrix does not change the expression of a determinantal polynomial or identity. Thus it is often convenient to write determinantal polynomials and identities in terms of an $\mathbb{N}^+ \times \mathbb{N}^+$ matrix;

$$A := \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.8}$$

One of the best known determinantal identities is the Leibniz formula for the determinant of an $n \times n$ matrix:

$$\det(A_{\{1,\ldots,n\}\{1,\ldots,n\}}) - \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sign}(\sigma)\prod_{i=1}^{n} a_{i\sigma(i)} = 0$$

This is actually a list of determinantal identities, one for each value of $n \geq 1$. We conveniently regard this formula as the definition of the determinant. The literature on determinants is very rich in identities which are far from trivial. See for example Muir's (2003) survey. The following lemma allows us to simplify determinantal identities by conveniently reordering rows and columns.

Lemma 5. *Let σ be any permutation of $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$. For $I \subseteq \mathbb{N}^+$, denote by $\operatorname{inv}_I(\sigma)$ the number of pairs $(i,j) \in I^2$ such that $i < j$ and $\sigma(i) > \sigma(j)$. Set $\operatorname{sign}_I(\sigma) := (-1)^{\operatorname{inv}_I(\sigma)}$. Let δ be any other permutation of $\mathbb{N}^+$ (possibly equal to σ). Let f be any determinantal polynomial and let $f^{\sigma\delta}$ be the determinantal polynomial such that $f^{\sigma\delta}(A)$ results from $f(A)$ by replacing every minor $\det(A_{IJ})$ with $\operatorname{sign}_I(\sigma)\operatorname{sign}_J(\delta)\det(A_{\sigma(I)\delta(J)})$ (where $\sigma(I) := \{\sigma(i) : i \in I\}$ and $\delta(J) := \{\delta(j) : j \in J\}$). Then $f = 0$ is a determinantal identity if and only if $f^{\sigma\delta} = 0$ is a determinantal identity.*

Proof. Let $A = [a_{ij}]_{i,j\geq 1}$ be any $\mathbb{Z}\times\mathbb{Z}$ matrix. Define

$$A^{\sigma\delta} := [a_{\sigma(i)\delta(j)}]_{i,j\geq 1}.$$

Let $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$ be subsets of $\mathbb{N}^+$. Assume without loss of generality that $i_1 < \cdots < i_k$ and $j_1 < \ldots < j_k$. Let α, β be the unique permutations of the set $\{1, \ldots, k\}$ such that $\sigma(i_{\alpha(1)}) < \cdots < \sigma(i_{\alpha(k)})$ and $\delta(j_{\beta(1)}) < \cdots < \delta(j_{\beta(k)})$. One can easily verify that:

$$A_{\sigma(I)\delta(J)} = [a_{\sigma(i_{\alpha(a)})\delta(j_{\beta(b)})}]_{1\leq a,b\leq k}$$

An inversion of α is a pair $(a,b) \in \{1,\ldots,k\}^2$ such that $a < b$ and $\alpha(a) > \alpha(b)$. Since the first inequality is equivalent to $\sigma(i_{\alpha(a)}) < \sigma(i_{\alpha(b)})$, and the second one is equivalent to $i_{\alpha(a)} > i_{\alpha(b)}$, the inversion (a,b) of α is in natural correspondence with the inversion $(i_{\alpha(b)}, i_{\alpha(a)})$ of σ. Thus $\operatorname{sign}(\alpha) = \operatorname{sign}_I(\sigma)$. Similarly $\operatorname{sign}(\beta) = \operatorname{sign}_J(\delta)$. By properties of determinants:

$$\begin{aligned}\det(A^{\sigma\delta}_{IJ}) &= \det[a_{\sigma(i_a)\delta(j_b)}]_{1\le a,b\le k}\\ &= \operatorname{sign}(\alpha)\operatorname{sign}(\beta)\det[a_{\sigma(i_{\alpha(a)})\delta(j_{\beta(b)})}]_{1\le a,b\le k}\\ &= \operatorname{sign}_I(\sigma)\operatorname{sign}_J(\delta)\det(A_{\sigma(I)\delta(J)})\end{aligned}$$

Therefore $f(A^{\sigma\delta}) = f^{\sigma\delta}(A)$ and the lemma follows. □

Notice that $f = 0$ is a determinantal identity if and only if it vanishes when expanded as a polynomial on the entries of a generic matrix. It follows that:

Lemma 6. *Let A be an $\mathbb{N}^+ \times \mathbb{N}^+$ matrix whose entries are algebraically independent over the rationals. Then $f = 0$ is a determinantal identity if and only if $f(A) = 0$.*

We now proceed to introduce the combinatorial concepts which allow us to derive determinantal identities. For every matrix A we define a new matrix $\widehat{A}$ which greatly simplifies the expansions of several interesting determinantal polynomials. It will turn out (see Lemma 7) that the the entries of $\widehat{A}$ are algebraically independent if those of A are so. Start by defining a planar digraph G_A whose vertexes are the entries a_{ij} of A, and whose edges are those of the form $a_{ij} \to a_{(i+1)j}$ and $a_{ij} \to a_{i(j-1)}$. This is perhaps made clearer in a visual format:

$$G_A = \begin{array}{ccccccc} a_{11} & \to & a_{12} & \to & a_{13} & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ a_{21} & \to & a_{22} & \to & a_{23} & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ a_{31} & \to & a_{32} & \to & a_{33} & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ \vdots & & \vdots & & \vdots & & \ddots \end{array}$$

For $i, j \geq 1$, let $G_A(i,j)$ denote the set of all paths starting at a_{i1} and ending at a_{1j}.

For example:

$$\begin{aligned} G_A(3,1) &= \{a_{31} \to a_{21} \to a_{11}\} \\ G_A(2,3) &= \{a_{21} \to a_{11} \to a_{12} \to a_{13},\ a_{21} \to a_{22} \to a_{12} \to a_{13}, \\ &\quad a_{21} \to a_{22} \to a_{23} \to a_{13}\} \end{aligned}$$

We extend this by defining $G_A^{(r)}(i,j)$ to be the set of all paths in $G_A(i,j)$ which do not visit a_{pq} for $1 \le p, q \le r$ (in particular $G_A^{(0)}(i,j) = G_A(i,j)$). For example:

$$G_A^{(1)}(2,3) = \{a_{21} \to a_{22} \to a_{12} \to a_{13}, a_{21} \to a_{22} \to a_{23} \to a_{13}\}$$

We refer to each element of $\bigcup_{i,j \ge 1} G_A(i,j)$ as an *A-path*. For each A-path ρ, define the *weight* of ρ, denoted by $\omega(\rho)$, to be product of all entries of A visited by ρ (including a_{i1} and a_{1j}). For example $\omega(a_{31} \to a_{21} \to a_{11}) = a_{31}a_{21}a_{11}$. As usual we extend the definition of the weight multiplicatively by setting

$$\omega(\rho_1, \dots, \rho_r) := \omega(\rho_1) \cdots \omega(\rho_r)$$

for every r-tuple $(\rho_1, \dots, \rho_r)$ of A-paths.

For $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\}$ with $1 \le i_1 < \cdots < i_k$ and $1 \le j_1 < \cdots < j_k$, denote by $G_A(I,J)$ the set of all k-tuples $(\rho_1, \dots, \rho_k)$ of non-intersecting A-paths such that $\rho_l \in G_A(i_l, j_l)$ for $l = 1, \dots, k$. In other words, $G_A(I,J)$ is the set of all non-intersecting tuples in the cartesian product $G_A(i_1,j_1) \times \cdots \times G_A(i_k,j_k)$. Similarly, denote by $G_A^{(r)}(I,J)$ the set of all non-intersecting tuples in the cartesian product $G_A^{(r)}(i_1,j_1) \times \cdots \times G_A^{(r)}(i_k,j_k)$. In particular $G_A^{(0)}(I,J) = G_A(I,J)$.

Finally, denote:

$$\begin{aligned} G_A(i,*) &:= \bigcup_{j \ge 1} G_A(i,j), \\ G_A(*,j) &:= \bigcup_{i \ge 1} G_A(i,j), \\ G_A(I,*) &:= \bigcup_{|J|=|I|} G_A(I,J), \\ G_A(*,J) &:= \bigcup_{|I|=|J|} G_A(I,J), \end{aligned}$$

and similarly define $G_A^{(r)}(i,*)$, $G_A^{(r)}(*,j)$, $G_A^{(r)}(I,*)$, and $G_A^{(r)}(*,J)$.

Let $\widehat{A}$ denote the matrix:

$$\widehat{A} := \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad \text{where} \quad p_{ij} := |G_A(i,j)|_w = \sum_{\rho \in G_A(i,j)} \omega(\rho).$$

The first few entries of $\widehat{A}$ are given by:

$$\widehat{A} = \begin{bmatrix} a_{11} & a_{11}a_{12} & \cdots \\ a_{21}a_{11} & a_{21}a_{11}a_{12} + a_{21}a_{22}a_{12} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Below we see how several minors of the matrix $\widehat{A}$ have simple expressions in terms of the entries of A, for example;

$$\det \begin{bmatrix} a_{11} & a_{11}a_{12} \\ a_{21}a_{11} & a_{21}a_{11}a_{12} + a_{21}a_{22}a_{12} \end{bmatrix} = a_{11}a_{12}a_{21}a_{22}$$

The following lemma is essential to the development of this chapter.

Lemma 7. *Let A be a matrix of algebraically independent entries a_{ij} $(i,j \geq 1)$, over the rationals. Then*

(a) *The entries p_{ij} $(i,j \geq 1)$ of $\widehat{A}$ are also algebraically independent over the rationals.*

(b) *$f = 0$ is a determinantal identity if and only if $f(\widehat{A}) = 0$.*

Proof. To prove (a), consider integers $r, i_1, \ldots, i_r, j_1, \ldots, j_r \geq 1$ and let f be a non-vanishing polynomial of integer coefficients on r variables. It will be sufficient to show that $f(p_{i_1 j_1}, p_{i_2 j_2}, \ldots, p_{i_r, j_r})$ is a non-vanshing polynomial in the a_{ij}'s. We proceed by induction on r. If $r = 1$, then f is a non-vanishing constant. Now assume that $r \geq 1$. Suppose without loss of generality that the pairs $(i_1, j_1), \ldots, (i_r, j_r)$ are ordered lexicographically, and that the degree of $p_{i_r j_r}$ in f is $d > 0$. Thus:

$$f(p_{i1j1}, \ldots, p_{irjr}) = \sum_{k=0}^{d} g_k(p_{i1j1}, \ldots, p_{i_{r-1}j_{r-1}}) p_{i_r j_r}^k,$$

where the g_k's are polynomials and g_d is non-vanishing. By the inductive hypothesis, $g_d(p_{i_1j_1},\dots,p_{i_{r-1}j_{r-1}})$ is a non-vanishing polynomial in the a_{ij}'s. From the lexicographical order of the pairs (i_m, j_m), we know that $p_{i_rj_r}$ is the only one of the $p_{i_mj_m}$'s whose expansion involves the variable $a_{i_rj_r}$. Also we know that the degree of $a_{i_rj_r}$ in $p_{i_rj_r}$ is 1. Hence the degree of $a_{i_rj_r}$ in $f(p_{i_1j_1},p_{i_2j_2},\dots,p_{i_r,j_r})$ is k, and the coefficient of $a^k_{i_rj_r}$ in $f(p_{i_1j_1},p_{i_2j_2},\dots,p_{i_r,j_r})$ is $g_d(p_{i_1j_1},\dots,p_{i_{r-1}j_{r-1}}) \neq 0$. Therefore $f(p_{i_1j_1},p_{i_2j_2},\dots,p_{i_r,j_r})$ does not vanish as a polynomial in the a_{ij}'s, as wanted.

Part (b) is a result of part (a) and Lemma 6. $\square$

Next we illustrate how, despite the complicated nature of the entries of $\widehat{A}$, proving determinantal identities in the matrix $\widehat{A}$ turns out to be relatively easy. First recall that by the Lindstrom-Gessel-Viennot lemma, if $I, J \subseteq \mathbb{N}^+$ with $|I| = |J| = m$, then

$$\det(\widehat{A}_{\{i_1,\dots,i_m\}\{j_1,\dots,j_m\}}) = |G_A(I,J)|_w. \tag{1.9}$$

In particular, if $I = J = \{1,\dots,m\}$, we know that the set $G_A(I,J)$ has exactly one element $(\rho_1,\dots,\rho_m)$, which is given by

$$\rho_t = a_{t1} \to a_{t2} \to \cdots \to a_{tt} \to a_{(t-1)t} \to \cdots \to a_{1t} \tag{1.10}$$

for $1 \leq t \leq m$, and so

$$\det(\widehat{A}_{[m][m]}) = \prod_{1\leq i,j\leq m} a_{ij} \tag{1.11}$$

Similarly, if $m < m'$,

$$\det(\widehat{A}_{[m](m'-m,m']}) = \prod_{\substack{1\leq i\leq m\\ 1\leq j\leq m'}} a_{ij}, \tag{1.12}$$

and

$$\det(\widehat{A}_{(m'-m,m'][m]}) = \prod_{\substack{1\leq i\leq m'\\ 1\leq j\leq m}} a_{ij}. \tag{1.13}$$

We refer to paths of the form

$$a_{i1} \to a_{i2} \to \cdots \to a_{ij} \to a_{(i-1)j} \to \cdots \to a_{1j}$$

as *rectangular A-paths.* Observe that for each $i, j \geq 1$, there is exactly one rectangular A-path in $G_A(i,j)$.

Equations (1.11)-(1.13) provide a taste of the advantage of working with minors of $\widehat{A}$, rather than minors of the generic matrix A. To better illustrate this advantage, we now present a short proof of the Desnanot-Jacobi identity:

Proposition 1 (The Desnanot-Jacobi Identity). *Let P be any $n \times n$ matrix ($n \geq 3$). Then P satisfies the equation:*

$$\begin{aligned}\det(P_{I\cup\{1\}I\cup\{1\}})\det(P_{I\cup\{n\}I\cup\{n\}}) - \det(P_{I\cup\{1\}I\cup\{n\}})\det(P_{I\cup\{n\}I\cup\{1\}})\\ = \det(P_{II})\det(P_{I\cup\{1,n\}I\cup\{1,n\}}),\end{aligned} \tag{1.14}$$

where $I = \{2, \ldots, n-1\}$.

Proof. From Lemma 5 we know that the equality holds in 1.14 for all P if and only if the equality:

$$\begin{aligned}&\det(P_{J\cup\{n-1\}J\cup\{n-1\}})\det(P_{J\cup\{n\}J\cup\{n\}})\\ &-\det(P_{J\cup\{n-1\}J\cup\{n\}})\det(P_{J\cup\{n\}J\cup\{n-1\}})\\ &=\det(P_{JJ})\det(P_{J\cup\{n-1,n\}J\cup\{n-1,n\}}),\end{aligned} \tag{1.15}$$

holds for all P, where $J = \{1, \ldots, n-2\}$. We thus proceed to prove equation 1.15. By Lemma 7(b), we only need to show this equality for $P = \widehat{A}$, where $A = [a_{ij}]_{i,j\geq 1}$ is an infinite matrix whose entries are algebraically independent over the rationals. From equation 1.9, if $\{b, c\} \subseteq \{n-1, n\}$ (b, c may be equal), then

$$\det(P_{J\cup\{b\}J\cup\{c\}}) = |G_A(J \cup \{b\}, J \cup \{c\})|_w.$$

Since $\rho_1, \ldots, \rho_{n-2}$ do not intersect in the sum above, ρ_i must necessarily be the rectangular path from a_{i1} to a_{1i}, for $i = 1, \ldots, n-2$. Thus

$$\det(P_{J\cup\{b\}J\cup\{c\}}) = \left(\prod_{1\leq i,j\leq n-2} a_{ij}\right) \left|G_A^{(n-2)}(b,c)\right|_{\omega},$$

The left hand side of 1.14 is then equal to:

$$\left(\prod_{1\leq i,j\leq n-2} a_{ij}\right)^2 \det B, \tag{1.16}$$

where B is the 2×2 matrix with entries;

$$B_{i,j} := \left| G_A^{(n-2)}(n-2+i, n-2+j) \right|_\omega .$$

Thus by the Linstrom-Gessel-Viennot lemma (Corollary 1), the expression (1.16) becomes

$$\left(\prod_{1 \le i,j \le n-2} a_{ij} \right)^2 \left| G_A^{(n-2)}(\{n-1,n\},\{n-1,n\}) \right|_\omega$$

The set $G_A^{(n-2)}(\{n-1,n\},\{n-1,n\})$ has only one element, consisting of two rectangular paths, so the expression above simplifies to:

$$\left(\prod_{1 \le i,j \le n-2} a_{ij} \right)^2 (a_{(n-1)1} \cdots a_{(n-1)(n-1)} \cdots a_{1(n-1)})(a_{n1} \cdots a_{nn} \cdots a_{1n}),$$

$$= \left(\prod_{1 \le i,j \le n-2} a_{ij} \right) \left(\prod_{1 \le i,j \le n} a_{ij} \right) = \det(P_{[n-2][n-2]}) \det(P)$$

(where the last equality holds by equation 1.11), as wanted. □

Setting the $(n-1)$-th column of P to be equal to $((-1)^{n-1}, 0, \ldots, 0)$ in equation 1.15, we get;

$$\begin{aligned} &\det(P_{[2,n-1][n-2]}) \det(P_{\{1,\ldots,n-2,n\}\{1,\ldots,n-2,n\}}) \\ &- \det(P_{[n-1]\{1,\ldots,n-2,n\}}) \det(P_{\{2,\ldots,n-2,n\}[n-2]}) \\ &= \det(P_{[n-2][n-2]}) \det(P_{[2,n]\{1,\ldots,n-2,n\}}), \end{aligned} \tag{1.17}$$

Writing the same identity for an $(n+1) \times (n+1)$ matrix and then permuting columns n and $n+1$, the identity above becomes;

$$\begin{aligned} &\det(P_{[2,n][n-1]}) \det(P_{\{1,\ldots,n-1,n+1\}[n]}) \\ &- \det(P_{[n][n]}) \det(P_{\{2,\ldots,n-1,n+1\}[n-1]}) \\ &= \det(P_{[n-1][n-1]}) \det(P_{[2,n+1][n]}), \end{aligned} \tag{1.18}$$

Interchanging columns 1 and n we obtain;

$$\begin{aligned} &\det(P_{[2,n][2,n]}) \det(P_{\{1,\ldots,n-1,n+1\}[n]}) \\ &- \det(P_{[n][n]}) \det(P_{\{2,\ldots,n-1,n+1\}[2,n]}) \\ &= \det(P_{[n-1][2,n]}) \det(P_{[2,n+1][n]}), \end{aligned} \tag{1.19}$$

These identities are useful later in this work. The same idea from the proof of proposition 1 is used to show the more general identity:

Proposition 2. *If P is an $n \times n$ matrix and $1 \leq \ell < n$, then*

$$\det\left(\left[\det\left(P_{J\cup\{b\}J\cup\{c\}}\right)\right]_{\ell+1\leq b,c\leq n}\right) = \det\left(P_{JJ}\right)^{n-\ell-1}\det\left(P\right), \tag{1.20}$$

where $J = \{1, \ldots, \ell\}$.

Proof. We proceed as in the proof of Proposition 1, by showing the equality above for $P = \widehat{A}$, where $A = [a_{ij}]_{i,j\geq 1}$ is an infinite matrix whose entries are algebraically independent over the rationals. By equation 1.9,

$$\det\left(P_{J\cup\{b\}J\cup\{c\}}\right) = \left|G_A(J\cup\{b\}, J\cup\{c\})\right|_\omega$$

The non-intersecting property of a tuple $(\rho_1, \ldots, \rho_{\ell+1}) \in G_A(J\cup\{b\}, J\cup\{c\})$ determines $\rho_1, \ldots, \rho_\ell$ uniquely as rectangular paths. Thus the left hand side of equation 1.20 is equal to

$$\det\left[\left(\prod_{1\leq i,j\leq \ell} a_{ij}\right)\left|G_A^{(\ell)}(b,c)\right|_\omega\right]_{\ell+1\leq b,c\leq n}$$

$$= \left(\prod_{1\leq i,j\leq \ell} a_{ij}\right)^{n-\ell} \det\left(\left[\left|G_A^{(\ell)}(b,c)\right|_\omega\right]_{\ell+1\leq b,c\leq n}\right)$$

$$= \left(\prod_{1\leq i,j\leq \ell} a_{ij}\right)^{n-\ell} \left|G_A^{(\ell)}([\ell+1,n],[\ell+1,n])\right|_\omega$$

Any tuple $(\rho_{\ell+1}, \ldots, \rho_n) \in G_A^{(\ell)}([\ell+1,n],[\ell+1,n])$ is again determined uniquely as a tuple of rectangular paths by the non-intersecting property. The expression above thus simplifies to:

$$\left(\prod_{1\leq i,j\leq \ell} a_{ij}\right)^{n-\ell-1}\left(\prod_{1\leq i,j\leq n} a_{ij}\right) = \det\left(P_{[1,\ell][1,\ell]}\right)^{n-\ell-1}\det\left(P_{[1,n][1,n]}\right),$$

(where the last equality holds by equation 1.11), as wanted. $\square$

Propositions 1 and 2 are part of a much larger family of well-known determinantal identities on which the determinant of a matrix of minors is equated to a product of minors. The proposition below is another example of this type of identity.

Proposition 3. *Let P be an $n \times n$ matrix. Then*

$$\det\left(\left[\det\left(P_{\{1,b\}\{c-1,c\}}\right)\right]_{2\leq b,c\leq n}\right) = \left(\prod_{j=2}^{n-1} P_{1j}\right)\det(P) \tag{1.21}$$

Proof. As before we just need to prove equation 1.21 for $P = \widehat{A}$ where $A = [a_{ij}]_{i,j\geq 1}$ is a matrix whose entries are algebraically independent over the rationals. Let us simplify the minor;

$$\det(P_{\{1,b\}\{c-1,c\}}) = |G_A(\{1,b\},\{c-1,c\})|_w = \sum_{(\rho',\rho)\in G_A(\{1,b\},\{c-1,c\})} \omega(\rho',\rho).$$

Since $G_A(1,c-1)$ has exactly one element $\rho' = a_{11} \to \cdots \to a_{1(c-1)}$, the sum above is over all paths $\rho \in G_A(b,c)$ which avoid edges of the form $a_{1i} \to a_{1(i+1)}$ $(i \geq 1)$. Let $\overline{G_A(b,c)}$ denote the set of such paths. The minor above becomes:

$$\det(P_{\{1,b\}\{c-1,c\}}) = a_{11}\cdots a_{1(c-1)} \sum_{\rho\in\overline{G_A(b,c)}} \omega(\rho),$$

Thus the left hand side of 1.21 is equal to:

$$\begin{aligned} & a_{11}^{n-1}a_{12}^{n-2}\cdots a_{1(n-1)}^{1}\det\left(\left[\sum_{\rho\in\overline{G_A(b,c)}}\omega(\rho)\right]_{2\leq b,c\leq n}\right) \\ &= a_{11}^{n-1}a_{12}^{n-2}\cdots a_{1(n-1)}^{1}\sum_{\rho_2,\ldots,\rho_n}\omega(\rho_2,\ldots,\rho_n), \end{aligned}$$

where the last sum is over all $(n-1)$-tuples $(p_2,\ldots,p_n)$ of A-paths such that $p_k \in \overline{G_A(k,k)}$ for $k = 2,\ldots,n$. The non-intersecting property determines these paths uniquely as rectangular paths. Therefore the left hand side of 1.21 may be further simplified as follows:

$$\begin{aligned} a_{11}^{n-1}a_{12}^{n-2}\cdots a_{1(n-1)}^{1}\left(\frac{1}{a_{11}}\prod_{1\leq i,j\leq n} a_{ij}\right) &= \left(\prod_{j=2}^{n-1} a_{11}\cdots a_{1j}\right)\left(\prod_{1\leq i,j\leq n} a_{ij}\right) \\ &= \left(\prod_{j=2}^{n-1} P_{1j}\right)\det(P), \end{aligned}$$

(where the last equality results from equations 1.12 and 1.11), as wanted. $\square$

Our main goal for the rest of this section is to provide a very wide generalization of Proposition 2 (and thus of Proposition 1). We are interested in identities of the form:

$$\det\left([\det(P_{I_b I_c})]_{1\leq b,c\leq m}\right) = \prod_{i=1}^{l} \det(P_{J_i J_i}), \tag{1.22}$$

where $I_1, \ldots, I_m, J_1, \ldots, J_l \subseteq \mathbb{N}^+$ and $|I_1| = |I_2| = \cdots = |I_m|$. The following is the main theorem of this section:

Theorem 3. *Let $d \geq 1$ be an integer. Let $I_1, \ldots, I_m$ be distinct d-element subsets of $\mathbb{N}^+$ such that the following property is satisfied for all $q \geq 2$, $j \in \{1, \ldots, m\}$:*

$$\textit{If } q \in I_j \textit{ and } q-1 \notin I_j, \textit{ then } (I_j - \{q\}) \cup \{q-1\} \in \{I_1, \ldots, I_m\}. \tag{1.23}$$

Then for every matrix P;

$$\det\left([\det\left(P_{I_b I_c}\right)]_{1\leq b,c\leq m}\right) = \prod_{r\geq 1} \det\left(P_{[r][r]}\right)^{\ell_r - \ell_{r+1}}, \tag{1.24}$$

where ℓ_r denotes the number of indices $k \in [m]$ such that $r \in I_k$.

Before proving this theorem, we illustrate its scope with a few examples:

Example 1. Let n, k be integers with $1 \leq k < n$. Take $d = k+1$, $m = n-k$ and $I_j = \{1, \ldots, k, k+j\}$ for $j = 1, \ldots, n-k$, to obtain Proposition 2.

Example 2. Let n, d be integers with $1 \leq d \leq n$. Set $m = \binom{n}{d}$ and let $I_1, \ldots, I_m$ be all the d-element subsets of $\{1, \ldots, n\}$. In this case $n_\ell = \binom{n-1}{d-1}$ for $\ell = 1, \ldots, n$ and $n_\ell = 0$ for every other value of ℓ. Thus Sylvester's identity below holds for every $n \times n$ matrix P:

$$\det\left([\det\left(P_{I_b I_c}\right)]_{1\leq b,c\leq \binom{n}{d}}\right) = \det(P)^{\binom{n-1}{d-1}} \tag{1.25}$$

Example 3. Take $d = 2$, $m = 4$, $I_1 = \{1,2\}$, $I_2 = \{1,3\}$, $I_3 = \{1,4\}$, $I_4 = \{2,3\}$, to obtain:

$$\det\begin{pmatrix} \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} & \begin{vmatrix} p_{11} & p_{13} \\ p_{21} & p_{23} \end{vmatrix} & \begin{vmatrix} p_{11} & p_{14} \\ p_{21} & p_{24} \end{vmatrix} & \begin{vmatrix} p_{12} & p_{13} \\ p_{22} & p_{23} \end{vmatrix} \\ \begin{vmatrix} p_{11} & p_{12} \\ p_{31} & p_{32} \end{vmatrix} & \begin{vmatrix} p_{11} & p_{13} \\ p_{31} & p_{33} \end{vmatrix} & \begin{vmatrix} p_{11} & p_{14} \\ p_{31} & p_{34} \end{vmatrix} & \begin{vmatrix} p_{12} & p_{13} \\ p_{32} & p_{33} \end{vmatrix} \\ \begin{vmatrix} p_{11} & p_{12} \\ p_{41} & p_{42} \end{vmatrix} & \begin{vmatrix} p_{11} & p_{13} \\ p_{41} & p_{43} \end{vmatrix} & \begin{vmatrix} p_{11} & p_{14} \\ p_{41} & p_{44} \end{vmatrix} & \begin{vmatrix} p_{12} & p_{13} \\ p_{42} & p_{43} \end{vmatrix} \\ \begin{vmatrix} p_{21} & p_{22} \\ p_{31} & p_{32} \end{vmatrix} & \begin{vmatrix} p_{21} & p_{23} \\ p_{31} & p_{33} \end{vmatrix} & \begin{vmatrix} p_{21} & p_{24} \\ p_{31} & p_{34} \end{vmatrix} & \begin{vmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{vmatrix} \end{pmatrix} \tag{1.26}$$

$$= p_{11}\det\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}\det\begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix},$$

While Examples 1 and 2 above are widely known identities, we have not found any theorem in the revised literature which is wide enough to imply identities such as the one in Example 3. Before presenting our proof of Theorem 3, we need to state five lemmas, the last three of which are well-known results in the theory of determinants. For the sake of completeness we provide combinatorial proofs for all of them. We start by introducing three maps (namely **first**$(\cdot)$, **last**$(\cdot)$ and **both**$(\cdot)$) on some particular families of A-path tuples ($A = [a_{ij}]_{i,j\geq 1}$ being a generic matrix). Let $k \geq 1$ be an integer, and let I be a k-element subset of $\mathbb{N}^+$. Define

$$\mathbf{first} : \bigcup_{i\in I} G_A(i,*) \times G_A(I-\{i\},*) \longrightarrow \bigcup_{i\in I} G_A(i,*) \times G_A(I-\{i\},*) \tag{1.27}$$

by setting $\mathbf{first}(\gamma, \rho_1, \ldots, \rho_{k-1}) := (\gamma', \rho'_1, \ldots, \rho'_{k-1})$, where the paths $\gamma', \rho'_1, \ldots, \rho'_{k-1}$ are constructed as follows: If γ does not intersect any of the paths $\rho_1, \ldots, \rho_{k-1}$, set $(\gamma', \rho'_1, \ldots, \rho'_{k-1}) := (\gamma, \rho_1, \ldots, \rho_{k-1})$. Otherwise let a_{pq} be the *first* intersection (i.e., the one with the smallest value of $q - p$) between γ and any of the paths $\rho_1, \ldots, \rho_{k-1}$, let ρ_r be the path which meets γ at a_{pq}, and set:

$$\begin{aligned}\gamma' &:= \rho_r(\to a_{pq})\gamma(a_{pq} \to),\\ \rho'_r &:= \gamma(\to a_{pq})\rho_r(a_{pq} \to),\\ \rho'_l &:= \rho_l \text{ for } l \neq r.\end{aligned} \tag{1.28}$$

Similarly define

$$\mathbf{last} : \bigcup_{i\in I} G_A(*,i) \times G_A(*,I-\{i\}) \longrightarrow \bigcup_{i\in I} G_A(*,i) \times G_A(*,I-\{i\}) \tag{1.29}$$

by setting $\mathbf{last}(\gamma, \rho_1, \ldots, \rho_{k-1}) := (\gamma', \rho'_1, \ldots, \rho'_{k-1})$, where the paths $\gamma', \rho'_1, \ldots, \rho'_{k-1}$ are constructed as follows: If γ does not intersect any of the paths $\rho_1, \ldots, \rho_{k-1}$, set $(\gamma', \rho'_1, \ldots, \rho'_{k-1}) := (\gamma, \rho_1, \ldots, \rho_{k-1})$. Otherwise let a_{pq} be the *last* intersection (i.e., the one with the greatest value of $q - p$) between γ and any of the paths $\rho_1, \ldots, \rho_{k-1}$, let ρ_r be the path which meets γ at a_{pq}, and set:

$$\begin{aligned}\gamma' &:= \gamma(\to a_{pq})\rho_r(a_{pq} \to),\\ \rho'_r &:= \rho_r(\to a_{pq})\gamma(a_{pq} \to),\\ \rho'_l &:= \rho_l \text{ for } l \neq r.\end{aligned} \tag{1.30}$$

Finally define the map $\mathbf{both}(\cdot)$ from the set

$$\bigcup_{i,j\in I} G_A(*,i) \times G_A(j,*) \times G_A(I-\{j\}, I-\{i\})$$

into itself by setting $\mathbf{both}(\gamma, \zeta, \rho_1, \ldots, \rho_{k-1}) := (\gamma', \zeta', \rho'_1, \ldots, \rho'_{k-1})$, where the paths $\gamma', \zeta', \rho'_1, \ldots, \rho'_{k-1}$ are constructed as follows: If γ intersects any of the paths $\rho_1, \ldots, \rho_{k-1}$ above the main diagonal (i.e., at an entry a_{pq} with $q \geq p$), set $\zeta' := \zeta$ and set $(\gamma', \rho'_1, \ldots, \rho'_{k-1}) := \mathbf{last}(\gamma, \rho_1, \ldots, \rho_{k-1})$. Otherwise, if ζ intersects any of the paths $\rho_1, \ldots, \rho_{k-1}$ below the main diagonal (i.e., at an entry a_{pq} with $q \leq p$), set $\gamma' := \gamma$ and set $(\zeta', \rho'_1, \ldots, \rho'_{k-1}) := \mathbf{first}(\zeta, \rho_1, \ldots, \rho_{k-1})$. Else set $(\gamma', \zeta', \rho'_1, \ldots, \rho'_{k-1}) := (\gamma, \zeta, \rho_1, \ldots, \rho_{k-1})$.

Lemma 8. *The maps defined above satisfy the following properties:*

(a) $\mathbf{first}(\cdot)$, $\mathbf{last}(\cdot)$ *and* $\mathbf{both}(\cdot)$ *are involutions which preserve the weight of tuples and the position of each of their intersections.*

(b) *For* $I = \{u+1, \dots, u+k\}$;

(b1) $\mathbf{first}(\cdot)$ *maps the set* $U_i := G_A(i,*) \times G_A(I - \{i\}, *)$ *into* $U_{i-1} \cup U_i \cup U_{i+1}$, *where the only elements whose image falls in* U_i *are fixed points.*

(b2) $\mathbf{last}(\cdot)$ *maps the set* $V_i := G_A(*,i) \times G_A(*, I - \{i\})$ *into* $V_{i-1} \cup V_i \cup V_{i+1}$, *where the only elements whose image falls in* V_i *are fixed points.*

(b3) $\mathbf{both}(\cdot)$ *maps the set* $W_{ij} := G_A(*,i) \times G_A(j,*) \times G_A(I - \{j\}, I - \{i\})$ *into* $W_{(i-1)j} \cup W_{(i+1)j} \cup W_{ij} \cup W_{i(j-1)} \cup W_{i(j+1)}$, *where the only elements whose image falls in* W_{ij} *are fixed points.*

Proof. **(a)** The fact that $\mathbf{first}(\cdot)$ preserves intersections and weight is evident by 1.28, since the vertices of (γ', ρ_r') are the same (with multiplicities) as those of (γ, ρ_r). The same argument holds for $\mathbf{last}(\cdot)$, and the result for $\mathbf{both}(\cdot)$ follows immediately by definition. This also shows that they are all involutions, since the choice of a_{pq} in each definition depends uniquely on the position of intersections.

(b) We only show **(b1)**, as the argument is similar for **(b2)** and **(b3)**. Consider any tuple $\boldsymbol{\rho} = (\gamma, \rho_1, \dots, \rho_{k-1}) \in U_i$. If $\boldsymbol{\rho}$ has no intersections, then $\mathbf{first}(\boldsymbol{\rho}) = \boldsymbol{\rho} \in U_i$. Otherwise, using the notation from 1.28, we claim that ρ_r starts at one of the entries $a_{(i-1)1}$ or $a_{(i+1)1}$. Suppose, in order to obtain a contradiction, that ρ_r starts at a_{l1} for some l with $|l - i| > 1$. Then there is at least one path among $\rho_1, \dots, \rho_{k-1}$ which starts at an entry between a_{i1} and a_{l1}. This path must intersect either $\gamma(\to a_{pq})$ or $\rho_r(\to a_{pq})$. The first case would contradict the minimality of $q-p$, while the second one would contradict the fact that the paths $\rho_1, \dots, \rho_{k-1}$ are non-intersecting. From this contradiction we conclude that ρ_r starts at $a_{(i\pm1)1}$, and so does $\gamma' = \rho_r(\to a_{pq})\gamma(a_{pq} \to)$. Therefore $\mathbf{first}(\boldsymbol{\rho}) \in U_{i-1} \cup U_{i+1}$ as wanted. $\square$

Lemma 9. *Let* $u \geq 0$; $k, b, c \geq 1$ *be integers, and for* $I := \{u+1, \dots, u+k\}$, *define;*

$$\begin{aligned} U_{lij} &:= G_A^{(u)}(l,i) \times G_A^{(u)}(I - \{l\}, I - \{j\}) \\ V_{lij} &:= G_A^{(u)}(i,l) \times G_A^{(u)}(I - \{j\}, I - \{l\}) \\ W_{ij} &:= G_A^{(u)}(b,i) \times G_A^{(u)}(j,c) \times G_A^{(u)}(I - \{j\}, I - \{i\}). \end{aligned}$$

Then if we denote $X := \dfrac{\det\left(\widehat{A}_{[u+k][u+k]}\right)}{\det\left(\widehat{A}_{[u][u]}\right)}$, *the following equalities hold;*

$$\sum_{l\in I}\sum_{\rho\in U_{lij}}(-1)^l\omega(\rho) = \delta_{i-j}(-1)^i X \tag{1.31}$$

$$\sum_{l\in I}\sum_{\rho\in V_{lij}}(-1)^l\omega(\rho) = \delta_{i-j}(-1)^i X \tag{1.32}$$

$$\sum_{i,j\in I}\sum_{\rho\in W_{ij}}(-1)^{i+j}\omega(\rho) = X\sum_{i\in I}\left|\left\{\eta\in G_A^{(u)}(b,c)\,:\,a_{ii}\in\eta\right\}\right|_\omega \tag{1.33}$$

Proof. To prove 1.31, observe that by Lemma 8, the involution **first**$(\cdot)$ sends all but the fixed points of U_{lij} to $U_{(l-1)ij}\cup U_{(l+1)ij}$. Thus the only surviving terms in the sum are those for which $\boldsymbol{\rho} = (\gamma, \rho_1, \ldots, \rho_{k-1})$ consists of k non-intersecting paths. This is only possible when $i = j$ (otherwise one of the paths $\rho_1, \ldots, \rho_{k-1}$ would meet γ at its ending point). The tuple $\boldsymbol{\rho}$ must then be some permutation of an element of $G_A^{(u)}(I, I)$, but since this set has exactly one element (the one consisting entirely of rectangular paths), it is necessary that $l = i = j$, and

$$\omega(\rho) = \prod_{t\in I} a_{t1}a_{t2}\cdots a_{tt}a_{(t-1)t}\cdots a_{1t} = \frac{\prod_{1\le t,s\le u+k} a_{ts}}{\prod_{1\le t,s\le u} a_{ts}},$$

which equals the desired expression by 1.11. Equation 1.32 follows from a similar argument.

To prove 1.33, observe that by Lemma 8, the involution **both**$(\cdot)$ sends all but the fixed points of W_{ij} to $W_{(i-1)j}\cup W_{(i+1)j}\cup W_{i(j-1)}\cup W_{i(j+1)}$. Thus the only surviving terms in the sum correspond to the tuples $\rho = (\gamma, \zeta, \rho_1, \ldots, \rho_{k-1})$ such that γ does not intersect $\rho_1, \ldots, \rho_{k-1}$ above the main diagonal, and ζ does not intersect those paths below the main diagonal. Let us describe more precisely one of these tuples. Suppose without loss of generality that $i \le j$. Then

$$(\rho_1, \rho_2, \ldots, \rho_{i-1-u}) \in G_A^{(u)}(\{u+1, u+2, \ldots, i-1\}, \{u+1, u+2, \ldots, i-1\})$$

However, the set above has only one element, which consists of rectangular paths. In particular;

$$\rho_{i-1-u} = a_{(i-1)1} \rightarrow a_{(i-1)2} \rightarrow \cdots \rightarrow a_{(i-1)(i-1)} \rightarrow a_{(i-2)(i-1)} \rightarrow \cdots \rightarrow a_{1(i-1)}.$$

Since γ ends at a_{1i} and does not intersect $\rho_{i-1-u}(a_{(i-1)(i-1)} \rightarrow)$, it must necessarily be vertical above the main diagonal. More precisely; γ visits a_{ii} and;

$$\gamma(a_{ii} \rightarrow) = a_{ii} \rightarrow a_{(i-1)i} \rightarrow \cdots \rightarrow a_{1i}.$$

If j is strictly greater than i, then ρ_{i-u} starts at a_{i1}, and so it must either intersect ρ_{i-1-u} somewhere below the main diagonal, or meet γ at the diagonal entry a_{ii}. Both of these options produce a contradiction. Thus these tuples occur only when $j = i$. Since ζ starts at a_{i1} and does not intersect $\rho_{i-1-u}(\rightarrow a_{(i-1)(i-1)})$, it must be horizontal below the main diagonal. More precisely; ζ visits a_{ii} and;

$$\zeta(\rightarrow a_{ii}) = a_{i1} \rightarrow a_{i2} \rightarrow \cdots \rightarrow a_{ii}.$$

Set:

$$\rho := \zeta(\rightarrow a_{ii})\gamma(a_{ii} \rightarrow) = a_{i1} \rightarrow \cdots \rightarrow a_{ii} \rightarrow \cdots \rightarrow a_{1i}.$$

The paths $\rho, \rho_1, \ldots, \rho_{k-1}$ must be non-intersecting, which determines them uniquely as being the components of the only element of the set $G_A^{(u)}(I, I)$, satisfying;

$$\omega(\rho, \rho_1, \ldots, \rho_{k-1}) = \frac{\prod\limits_{1 \leq t,s \leq u+k} a_{ts}}{\prod\limits_{1 \leq t,s \leq u} a_{ts}} = \frac{\det\left(\widehat{A}_{[u+k][u+k]}\right)}{\det\left(\widehat{A}_{[u][u]}\right)}$$

Setting $\eta := \gamma(\rightarrow a_{ii})\zeta(a_{ii} \rightarrow)$, the sum on the left hand side of 1.33 becomes;

$$\frac{\det\left(\widehat{A}_{[u+k][u+k]}\right)}{\det\left(\widehat{A}_{[u][u]}\right)} \sum_{i \in I} \sum_{\substack{\eta G_A^{(u)}(b,c) \\ a_{ii} \in \eta}} \omega(\eta),$$

as wanted. $\square$

Note that by setting $u := 0$, the lemma above yields;

$$\sum_{l\in I}\sum_{\rho\in U_{lij}}(-1)^{l}\omega(\rho) \;=\; \delta_{i-j}(-1)^{i}\det\left(\widehat{A}_{\{1,\ldots,k\}\{1,\ldots,k\}}\right) \tag{1.34}$$

$$\sum_{l\in I}\sum_{\rho\in V_{lij}}(-1)^{l}\omega(\rho) \;=\; \delta_{i-j}(-1)^{i}\det\left(\widehat{A}_{\{1,\ldots,k\}\{1,\ldots,k\}}\right) \tag{1.35}$$

$$\sum_{i,j\in I}\sum_{\rho\in W_{ij}}(-1)^{i+j}\omega(\rho) \;=\; \det\left(\widehat{A}_{\{1,\ldots,k\}\{1,\ldots,k\}}\right)\sum_{i=1}^{k}\sum_{\substack{\eta\in G_A(b,c)\\ a_{ii}\in\eta}}\omega(\eta) \tag{1.36}$$

The following classical results affords a nice combinatorial proof.

Lemma 10 (Classical). *For any $k \times k$ matrix P;*

$$P\,\mathrm{adj}(P) = \det(P)\mathrm{Id}_k, \tag{1.37}$$

where Id_k *denotes the* $k\times k$ *identity matrix and* $\mathrm{adj}(P)$ *is the adjoint matrix of* P*, whose entries are given by:*

$$\mathrm{adj}(P)_{ij} = (-1)^{i+j}\det\left(P_{\{1,\ldots,k\}-\{j\}\;\{1,\ldots,k\}-\{i\}}\right). \tag{1.38}$$

Proof. This is not a determinantal identity by definition. However all the entries of the matrices appearing in 1.37 are integer polynomials in the entries of P, so it is enough to show the equality for a matrix P whose entries are algebraically independent over the rationals. In particular it is sufficient to show it for $P = \widehat{A}$, where A is a matrix whose entries are algebraically independent. In that case P_{il} is the weighted sum of the elements of $G_A(i,l)$, and from 1.38, $(\mathrm{adj}(P))_{lj}$ is the weighted and signed sum of the elements of $G_A(I-\{j\}, I-\{l\})$, where $I = \{1,\ldots,k\}$. Thus;

$$\begin{aligned}(P\,\mathrm{adj}(P))_{ij} &= \sum_{l=1}^{k}\sum_{\rho\in V_{lij}}(-1)^{l+j}\omega(\rho)\\ &= (-1)^{j}\delta_{i-j}(-1)^{i}\det(P)\\ &= \delta_{i-j}\det(P),\end{aligned} \tag{1.39}$$

where $V_{lij} = G_A(i,l)\times G_A(I-\{j\}, I-\{l\})$, and the second equality holds by 1.35. Therefore $P\,\mathrm{adj}(P) = \det(P)\mathrm{Id}_k$, as wanted. □

Lemma 11 (Classical). *Let P be an $n \times n$ matrix partitioned in blocks B, C, D, E of orders $k \times k$, $k \times (n-k)$, $(n-k) \times k$ and $(n-k) \times (n-k)$ respectively, as follows:*

$$P = \begin{bmatrix} B & C \\ D & E \end{bmatrix},$$

and assume B is invertible. Then

$$\det(P) = \det(B)\det(E - DB^{-1}C). \tag{1.40}$$

Proof. Again this is not exactly a determinantal identity, but it would still suffice to show it for a matrix of algebraically independent entries. Indeed, both sides of equation 1.40 are rational functions on the entries of P. We thus just need to show 1.40 for $P = \widehat{A}$, where A is a matrix of algebraically independent entries. From Lemma 10;

$$\begin{aligned} B^{-1} &= \frac{1}{\det(B)}\operatorname{adj}(B) \\ &= \frac{1}{\det(B)}\left[(-1)^{i+j} \sum_{\rho \in G_A(I-\{j\}, I-\{i\})} \omega(\rho)\right]_{1 \leq i,j \leq k}, \end{aligned}$$

where $I = \{1, \ldots, k\}$. We index the entries of B, C, D, E as their corresponding entries in P (for instance, we denote the upper-left entry of D by $D_{(k+1)1}$). To be consistent with this notation, we index the entries of $DB^{-1}C$ with pairs (b, c) such that $k+1 \leq b, c \leq n$. The usual matrix multiplication yields;

$$\begin{aligned} (DB^{-1}C)_{bc} &= \frac{1}{\det(B)} \sum_{1 \leq i,j \leq k} \sum_{\rho \in W_{ij}} (-1)^{i+j}\omega(\rho) \\ &= \sum_{i=1}^{k} \sum_{\substack{\eta \in G_A(b,c) \\ a_{ii} \in \eta}} \omega(\eta) \end{aligned} \tag{1.41}$$

where $W_{ij} = G_A(b,i) \times G_A(j,c) \times G_A(I - \{j\}, I - \{i\})$, and the last equality holds by 1.36. Since every path from $G_A(b,c)$ has exactly one diagonal element;

$$E_{bc} = \sum_{i \geq 1} \sum_{\substack{\eta \in G_A(b,c) \\ a_{ii} \in \eta}} \omega(\eta) \tag{1.42}$$

Subtracting 1.41 from 1.42, we get;

$$(E - DB^{-1}C)_{bc} = \sum_{i \geq k} \sum_{\substack{\eta \in G_A(b,c) \\ a_{ii} \in \eta}} \omega(\eta) = \sum_{\eta \in G_A^{(k)}(b,c)} \omega(\eta), \tag{1.43}$$

and so;

$$\det(E - DB^{-1}C) = \sum_{\eta \in G_A^{(k)}(J,J)} \omega(\eta), \tag{1.44}$$

where $J = \{k+1, \ldots, n\}$. The set $G_A^{(k)}(J,J)$, has exactly one element, consisting of rectangular path, which yields;

$$\det(E - DB^{-1}C) = \prod_{i=k+1}^{n} a_{i1} \cdots a_{ii} \cdots a_{1i} = \frac{\prod\limits_{1 \leq i,j \leq n} a_{ij}}{\prod\limits_{1 \leq i,j \leq k} a_{ij}} = \frac{\det(P)}{\det(B)}, \tag{1.45}$$

(where the last equality holds by 1.11), as wanted. □

Lemma 12 (Muir's (2003) law of extensible determinants). *Any homogeneous determinantal identity can be extended by adding a new set of indices to the rows and columns of each minor. More formally, if for some $m, k \geq 1$ and some integer coefficients $C_{\boldsymbol{I},\boldsymbol{J}}$ given for all $\boldsymbol{I} = (I_1, \ldots, I_k)$, $\boldsymbol{J} = (J_1, \ldots, J_k)$ with $I_i, J_i \in [m]$ and $|I_i| = |J_i|$ $(i = 1, \ldots, k)$, the equation*

$$\sum_{\boldsymbol{I},\boldsymbol{J}} C_{\boldsymbol{I},\boldsymbol{J}} \prod_{i=1}^{k} \det(P_{I_i J_i}) = 0, \tag{1.46}$$

is a determinantal identity, then so is the equation

$$\sum_{\boldsymbol{I},\boldsymbol{J}} C_{\boldsymbol{I},\boldsymbol{J}} \prod_{i=1}^{k} \det(P_{I_i \cup (m,m+n]\ J_i \cup (m,m+n]}) = 0, \tag{1.47}$$

Proof. It will be sufficient to show 1.47 for $P = \widehat{A}$, where A is an $(m+n) \times (m+n)$ matrix whose entries are algebraically independent over the rationals. Define the permutation $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ by:

$$\sigma(x) := \begin{cases} x + n & \text{if } 1 \leq x \leq m \\ x - m & \text{if } m + 1 \leq x \leq m + n \\ x & \text{if } m + n \leq x \end{cases}$$

Notice that for all $I \subseteq [m]$;

$$\sigma(I \cup (m, m+n]) = [n] \cup (I+n),$$

where $I + n := \{x + n : x \in I\}$. Also;

$$\mathrm{sign}_{I \cup (m,m+n]}(\sigma) = (-1)^{n|I|}$$

Thus by Lemma 5, equation 1.47 is a determinantal identity if and only if the equation

$$\sum_{I,J} C_{I,J} \prod_{i=1}^{k} \det(P_{I'_i J'_i}) = 0, \tag{1.48}$$

is a determinantal identity, where we denote $I' := [n] \cup (I+n)$ for $I \subseteq [m]$. Recall that

$$\det(P_{I'_i J'_i}) = \left|G_A(I'_i, J'_i)\right|_{\omega},$$

Since $\{1, \dots, n\} \subseteq I'_i, J'_i$, then for any tuple $(\rho_1, \dots, \rho_\ell) \in G_A(I'_i, J'_i)$, the sub-tuple $(\rho_1, \dots, \rho_n)$ is the unique element of the set $G_A([n],[n])$, which consists entirely of rectangular paths. Hence;

$$\begin{aligned} \det(P_{I'_i J'_i}) &= \det(P_{[n][n]}) \left|G_A^{(n)}(I_i + n, J_i + n)\right|_{\omega} \\ &= \det(P_{\{1,\dots,n\}\{1,\dots,n\}}) \det(Q_{I_i J_j}), \end{aligned}$$

where Q is the $m \times m$ matrix whose entry (b, c) is the weighted sum of the elements of $G_A^{(n)}(b+n, c+n)$. Therefore the left hand side of 1.48 is equal to:

$$\det(P_{[n][n]})^k \sum_{I,J} C_{I,J} \prod_{i=1}^{k} \det(Q_{I_i J_i}),$$

which vanishes because 1.46 is a determinantal identity. □

We are now ready to introduce the notation and terminology which is necessary for the proof of Theorem 3. Let $\mathcal{N}_d$ denote the family of all d-element subsets of $\mathbb{N}^+$. We define a partial ordering $\preceq$ on $\mathcal{N}_d$ as follows: If $I = \{i_1, \dots, i_d\}, J = \{j_1, \dots, j_d\} \in \mathcal{N}_d$ with $i_1 < \dots < i_d$ and $j_1 < \dots < j_d$, then;

$$I \preceq J \Leftrightarrow i_t \leq j_t \text{ for } t = 1, \dots, d. \tag{1.49}$$

This allows us to simplify condition 1.23 of Theorem 3, by simply requiring that $\{I_1, \ldots, I_m\}$ is an *initial segment* of $(\mathcal{N}_d, \preceq)$, that is, a subset $M \subseteq \mathcal{N}_d$ such that if $J \in M$ and $I \preceq J$, then $I \in M$. The *projection map* $\boldsymbol{\pi} : \mathcal{N}_d \to \mathcal{N}_{d-1}$ is defined by:

$$\boldsymbol{\pi}(I) := I - \{\max(I)\}. \tag{1.50}$$

For example $\boldsymbol{\pi}(\{1,2,4,8,9\}) = \{1,2,4,8\}$. The inverse of the projection map is given by:

$$\boldsymbol{\pi}^{-1}(J) := \{I \in \mathcal{N}_d : \boldsymbol{\pi}(I) = J\} = \{J \cup \{j\} : j > \max(J)\},$$

for $J \in \mathcal{N}_{d-1}$. Clearly if M is an initial segment of $(\mathcal{N}_d, \preceq)$, then $\boldsymbol{\pi}(M)$ is an initial segment of $(\mathcal{N}_{d-1}, \preceq)$, and if M is an initial segment of $(\mathcal{N}_{d-1}, \preceq)$, then $\boldsymbol{\pi}^{-1}(M)$ is an (infinite) initial segment of $\mathcal{N}_d$.

Let M be an initial segment of $(\mathcal{N}_d, \preceq)$ and let $N \subseteq \boldsymbol{\pi}(M)$ be an initial segment of $(\mathcal{N}_{d-1}, \preceq)$. Denote

$$M/N := M \cap \boldsymbol{\pi}^{-1}(\boldsymbol{\pi}(M) - N). \tag{1.51}$$

This may be described as the set of elements of M whose projection is not in N. Equivalently;

$$M/N = M - (M \cap \boldsymbol{\pi}^{-1}(N)), \tag{1.52}$$

For example, for

$$M = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \{1,4,5\}\},$$

we have that

$$\boldsymbol{\pi}(M) = \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}\}.$$

Take $N = \{\{1,2\}, \{1,3\}\} \subseteq \boldsymbol{\pi}(M)$. Thus

$$M/N = \{\{2,3,4\}, \{2,3,5\}, \{1,4,5\}\}$$

From the definition we have;

$$\boldsymbol{\pi}(M/N) = \boldsymbol{\pi}(M) - N. \tag{1.53}$$

Getting back to the subject of A-paths, for $(\rho_1, \ldots, \rho_d) \in G_A(I, J)$ $(I, J \in \mathcal{N}_d)$, define

$$\operatorname{Diag}(\rho_1, \ldots, \rho_d) := \{i : a_{ii} \in \rho_j \text{ for some } j \in \{1, \ldots, d\}\}.$$

This set is an element of $\mathcal{N}_d$, since every path visits exactly one diagonal entry. Observe that

$$\begin{aligned} \operatorname{Diag}(\rho_1, \ldots, \rho_d) &\preceq I \\ \operatorname{Diag}(\rho_1, \ldots, \rho_d) &\preceq J \end{aligned} \tag{1.54}$$

Proof of Theorem 3. As in the previous proofs, we show the theorem for $P = \widehat{A}$, where A is a matrix of algebraically independent entries. We actually prove a more general statement involving A-paths. Let M be a finite initial segment of $(\mathcal{N}_d, \preceq)$ and let $N \subseteq \pi(M)$ be an initial segment of $(\mathcal{N}_{d-1}, \preceq)$. Suppose that $M/N = \{I_1, \ldots, I_m\}$. Let Q be the $m \times m$ matrix whose entries are given by:

$$Q_{ij} := \sum_{\substack{\rho \in G_A(I_i, I_j) \\ \operatorname{Diag}(\rho) \notin N}} \omega(\rho). \tag{1.55}$$

We claim that

$$\det(Q) = \prod_{I \in M/N} \prod_{r \in I} a_{r1} \cdots a_{rr} \cdots a_{1r}, \tag{1.56}$$

We prove this claim by induction on m. We skip the induction base case $m = 1$, as it follows exactly the same idea as the inductive step below (alas with $k = m = 1$).

For the inductive step, observe that by being finite and non-empty, the projection set $\pi(M/N) = \pi(M) - N$ admits at least one minimal element with respect to $\preceq$. From its minimal elements, choose J for which the number $\max\{\max(I) : I \in M \cap \pi^{-1}(J)\}$ takes the greatest value (this is only used at the end of the proof). Set $u := \max(J)$, and recall that:

$$\pi^{-1}(J) = \{J \cup \{u+1\}, J \cup \{u+2\}, J \cup \{u+3\}, \ldots\},$$

Thus since M is an initial segment, there is some $k \geq 1$ for which

$$M \cap \pi^{-1}(J) = \{J \cup \{u+1\}, \ldots, J \cup \{u+k\}\}$$

Applying the same permutation to the columns and to the rows of Q leaves its determinant unchanged, so we may assume that $I_i = J \cup \{u+i\}$ for $i = 1, \ldots, k$. From Lemma 11;

$$\det(Q) = \det(B)\det(E - DB^{-1}C),$$

where $B = [Q_{ij}]_{1\leq i,j\leq k}$, $C = [Q_{ij}]_{\substack{1\leq i\leq k \\ k+1\leq j\leq m}}$, $D = [Q_{ij}]_{\substack{k+1\leq i\leq m \\ 1\leq j\leq k}}$, $E = [Q_{ij}]_{k+1\leq i,j\leq m}$. By 1.55;

$$B_{ij} := \sum_{\substack{\rho\in G_A(J\cup\{u+i\},J\cup\{u+j\}) \\ \mathrm{Diag}(\rho)\notin N}} \omega(\rho). \tag{1.57}$$

From 1.54, any tuple $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_d)$ from the sum above satisfies

$$\mathrm{Diag}(\rho_1, \ldots, \rho_{d-1}) \preceq J.$$

However since J is a minimal element of $\boldsymbol{\pi}(M)-N$, then N contains every $(d-1)$-element set which strictly precedes J in the order $\preceq$. Hence $\mathrm{Diag}(\rho_1, \ldots, \rho_{d-1})$ must necessarily be equal to J. This means that each one of the paths $\rho_1, \ldots, \rho_{d-1}$ is rectangular, so that

$$\omega(\rho_1, \ldots, \rho_{d-1}) = \prod_{x\in J} a_{x1}\cdots a_{xx}\cdots a_{1x}.$$

In particular $\rho_{d-1} = a_{u1} \to \cdots \to a_{uu} \to \cdots \to a_{1u}$. Equation 1.57 then becomes:

$$B_{ij} := \left(\prod_{x\in J} a_{x1}\cdots a_{xx}\cdots a_{1x}\right) \sum_{\rho\in G_A^{(u)}(u+i,u+j)} \omega(\rho), \tag{1.58}$$

Thus;

$$\det(B) := \left(\prod_{x\in J} a_{x1}\cdots a_{xx}\cdots a_{1x}\right)^k \sum_{\rho\in G_A^{(u)}(I,I)} \omega(\rho_1, \ldots, \rho_k), \tag{1.59}$$

where $I = \{u+1, \ldots, u+k\}$. There is only one element in the set $G_A^{(u)}(I, I)$, consisting entirely of diagonal paths, which yields;

$$\begin{aligned} \det(B) &= \left(\prod_{x\in J} a_{x1}\cdots a_{xx}\cdots a_{1x}\right)^k \left(\prod_{i=1}^{k} a_{(u+i)1}\cdots a_{(u+i)(u+i)}\cdots a_{1(u+i)}\right), \\ &= \prod_{I\in M\cap\boldsymbol{\pi}^{-1}(J)} \prod_{i\in I} a_{r1}\cdots a_{rr}\cdots a_{1r}. \end{aligned} \tag{1.60}$$

If $k = m$, then $\det(Q) = \det(B)$, and we are done, since J is the only element of $\pi(M/N)$, and so $M \cup \pi^{-1}(J) = M$ (the particular case $k = m = 1$ is the induction base case which we skipped at the beginning). Otherwise, let us now compute $DB^{-1}C$. By Lemma 10 and equation 1.58;

$$B^{-1}_{ij} = (-1)^{i+j} \frac{\left(\prod_{x\in J} a_{x1}\cdots a_{xx}\cdots a_{1x}\right)^{k-1}}{\det(B)} \sum_{\rho\in G_A^{(u)}(I-\{u+j\},I-\{u+i\})} \omega(\rho),$$

where $S = \{u+1,\ldots,u+k\}$. Indexing entries of C and D as the corresponding entries in Q (denoting, for instance, the upper-left entry of D by $D_{(k+1)1}$), and indexing entries of $DB^{-1}C$ with pairs (b,c) such that $k+1 \leq b, c \leq n$, we obtain from the usual matrix multiplication;

$$(DB^{-1}C)_{bc} = \frac{\left(\prod_{x\in J} a_{x1}\cdots a_{xx}\cdots a_{1x}\right)^{k-1}}{\det(B)} \sum_{1\leq i,j\leq k} \sum_{\substack{\gamma_1,\ldots,\gamma_d,\\ \zeta_1,\ldots,\zeta_d,\\ \rho_1,\ldots,\rho_{k-1}}} (-1)^{i+j}\omega(\gamma_1,\ldots,\rho_{k-1}),$$

where the paths in the sum are such that

$$\begin{aligned}
(\gamma_1,\ldots,\gamma_d) &\in G_A(I_b, J\cup\{u+i\}), \quad \mathrm{Diag}(\gamma_1,\ldots,\gamma_{d-1}) \notin N,\\
(\zeta_1,\ldots,\zeta_d) &\in G_A(J\cup\{u+j\}, I_c), \quad \mathrm{Diag}(\eta_1,\ldots,\eta_{d-1}) \notin N,\\
(\rho_1,\ldots,\rho_{k-1}) &\in G_A^{(u)}(I-\{u+j\}, I-\{u+i\})
\end{aligned} \tag{1.61}$$

Since $(\gamma_1,\ldots,\gamma_{d-1}) \in G_A(I_b - \max(I_b), J)$, we know that $\mathrm{Diag}(\gamma_1,\ldots,\gamma_{d-1}) \preceq J$, and since all elements of $\mathcal{N}_{d-1}$ which strictly precede J are in N, then $\mathrm{Diag}(\gamma_1,\ldots,\gamma_{d-1})$ must necessarily be equal to J, and so the paths $\gamma_1,\ldots,\gamma_{d-1}$ are vertical above the main diagonal. Similarly $\mathrm{Diag}(\zeta_1,\ldots,\zeta_{d-1}) = J$ and the paths $\zeta_1,\ldots,\zeta_{d-1}$ are horizontal below the main diagonal. Thus if $j_1 < \cdots < j_{d-1} = u$ are the elements of J, then γ_t meets η_t at $a_{j_tj_t}$ for $t = 1,\ldots,d-1$. Setting

$$\kappa_t := \zeta(\to a_{j_tj_t})\gamma(a_{j_tj_t}\to) = a_{j_t1} \to \cdots \to a_{j_tj_t} \to \cdots \to a_{1j_t},$$

$$\eta_t := \gamma(\to a_{j_tj_t})\zeta(a_{j_tj_t}\to),$$

for $t = 1, \dots, d-1$, it is immediate that

$$\omega(\kappa_1, \dots, \kappa_t) = \prod_{x \in J} a_{x1} \cdots a_{xx} \cdots a_{1x},$$

and so;

$$\begin{aligned}
(DB^{-1}C)_{bc} &= \frac{\left(\prod_{x\in J} a_{x1} \cdots a_{xx} \cdots a_{1x}\right)^k}{\det(B)} \sum_{1\le i,j\le k} \sum_{\substack{\gamma_d,\zeta_d,\\ \eta_1,\dots,\eta_{d-1},\\ \rho_1,\dots,\rho_{k-1}}} (-1)^{i+j}\omega(\eta_1,\dots,\rho_{k-1}) \\
&= \frac{\left(\prod_{1\le i,j\le u} a_{ij}\right)}{\left(\prod_{1\le i,j\le u+k} a_{ij}\right)} \sum_{1\le i,j\le k} \sum_{\substack{\gamma_d,\zeta_d,\\ \eta_1,\dots,\eta_{d-1},\\ \rho_1,\dots,\rho_{k-1}}} (-1)^{i+j}\omega(\eta_1,\dots,\rho_{k-1}),
\end{aligned}$$

where the conditions 1.61 may be translated as:

$$(\eta_1, \dots, \eta_{d-1}) \in G_A(I_b - \{\max(I_b)\}, I_c - \{\max(I_b)\}), \quad \mathrm{Diag}(\eta_1, \dots, \eta_{d-1}) = J,$$

$$\gamma_d \in G_A^{(u)}(\max(I_b), u+i) \text{ does not meet } \eta_1, \dots, \eta_{d-1} \text{ below the main diagonal,}$$

$$\zeta_d \in G_A^{(u)}(u+j, \max(I_c)) \text{ does not meet } \eta_1, \dots, \eta_{d-1} \text{ above the main diagonal,}$$

$$(\rho_1, \dots, \rho_{k-1}) \in G_A^{(u)}(I - \{u+j\}, I - \{u+i\})$$

For fixed paths $\eta_1, \dots, \eta_{d-1}$, the set of tuples $(\gamma_d, \zeta_d, \rho_1, \dots, \rho_{k-1})$ as above is stable under the map $\mathbf{both}(\cdot)$ (see the text preceding Lemma 8). Hence an argument similar to the proof of equation 1.33 in Lemma 8 yields;

$$\begin{aligned}
(DB^{-1}C)_{bc} &= \sum_{i=u+1}^{u+k} \sum_{\substack{(\eta_1,\dots,\eta_d)\in G_A(I_b,I_c)\\ \mathrm{Diag}(\eta_1,\dots,\eta_{d-1})=J\\ a_{ii}\in\eta_d}} \omega(\eta_1, \dots, \eta_d) \\
&= \sum_{\substack{(\eta_1,\dots,\eta_d)\in G_A(I_b,I_c)\\ \mathrm{Diag}(\eta_1,\dots,\eta_{d-1})=J}} \omega(\eta_1, \dots, \eta_d),
\end{aligned} \tag{1.62}$$

This last equality is not evident, as it is not immediate that the path η_d must necessarily visit a_{ii} for some $i \in \{u+1, \dots, u+k\}$. It is clear, since η_{d-1} visits a_{uu}, that the diagonal element which η_d visits must be some a_{ii} for $i \ge u+1$. To prove that $i \le u+k$, observe that $\mathrm{Diag}(\eta_1, \ldots, \eta_d) \preceq I_b$, so it is sufficient to show that $\max(I_b) \leq u + k$. We argue this as follows. Notice that $\pi(I_b) \in \pi(M) - N$ (since $I_b \in M/N$). Let J' be a minimal element of $\pi(M) - N$ such that $J' \preceq \pi(I_b)$. Clearly $J' \cup \{\max(I_b)\} \preceq I_b$, so $J' \cup \{\max(I_b)\} \in M$. More precisely $J' \cup \{\max(I_b)\} \in M \cap \pi^{-1}(J')$. Hence:

$$\begin{aligned}
\max(I_b) &= \max(J' \cup \{\max(I_b)\}) \\
&\leq \max\{\max(I) : I \in M \cap \pi^{-1}(J')\} \\
&\leq \max\{\max(I) : I \in M \cap \pi^{-1}(J)\} \\
&= u + k
\end{aligned}$$

(where the second inequality holds by our choice of J).

Subtracting 1.62 to the definition of E_{bc}, we obtain;

$$(E - DB^{-1}C)_{bc} = \sum_{\substack{\eta \in G_A(I_b, I_c) \\ \mathrm{Diag}(\eta) \notin N \cup \{J\}}} \omega(\kappa_1, \ldots, \kappa_{d-1}, \kappa).$$

By the inductive hypothesis;

$$\det(E - DB^{-1}C) = \prod_{I \in M/(N \cup \{J\})} \prod_{r \in I} a_{r1} \cdots a_{rr} \cdots a_{1r},$$

and since $M/N = (M \cap \pi^{-1}(J)) \cup (M/(N \cup \{J\}))$;

$$\det(Q) = \det(B) \det(E - DB^{-1}C) = \prod_{I \in M/N} \prod_{r \in I} a_{r1} \cdots a_{rr} \cdots a_{1r},$$

concluding our proof of equation 1.56. Theorem 3 is derived by assuming N to be empty. In this case $M/N = M$ and $Q_{bc} = \det(P_{I_b I_c})$. Defining the numbers ℓ_r as in the statement of the theorem;

$$\begin{aligned}
\det(Q) &= \prod_{t=1}^{m} \prod_{r \in I_t} a_{r1} \cdots a_{rr} \cdots a_{1r} \\
&= \prod_{t=1}^{m} \prod_{r \in I_t} \frac{\det\left(P_{[r][r]}\right)}{\det\left(P_{[r-1][r-1]}\right)} \\
&= \prod_{r \geq 1} \frac{\det\left(P_{[r][r]}\right)^{\ell_r}}{\det\left(P_{[r-1][r-1]}\right)^{\ell_r}} \\
&= \prod_{r \geq 1} \det\left(P_{[r][r]}\right)^{\ell_r - \ell_{r+1}},
\end{aligned}$$

as wanted. □

To keep this work as elementary and self-contained as possible, we have provided combinatorial proofs of Theorem 3 and of all the lemmas that precede it. However it is important to point out that, through private correspondence with the author, Bernard Leclerc has provided a simple Lie-theoretic argument effectively proving Theorem 3. In fact, both sides of 1.24 are the maximum-weight element of certain irreducible bimodule of the action of the Lie algebra of all $n \times n$ matrices on the ring of complex polynomials in the entries p_{ij}.

1.4 Enumeration of Tableaux

Combinatorial proofs of enumeration formulae for standard and semistandard Young tableaux (defined below) are popular in the algebraic combinatorics literature, valued for their simplicity and inherent beauty. Refer to Stanley (1997), Chapter 7 for a comprehensive survey of some of the results and definitions presented here. The best known application of the Lindstrom-Gessel-Viennot lemma to weighted tableau enumeration is the Jacobi-Trudi formula for Schur symmetric functions in terms of the complete homogeneous symmetric functions. This formula itself provides several purely enumerative results when combined with certain algebraic arguments involving coefficient extraction. Our goal for this section is to provide a direct combinatorial method to derive those enumerative results, bypassing the use of symmetric functions or any type of algebraic argument involving generating functions. We achieve this through our slight extensions of Lindstrom-Gessel-Viennot and Stembridge's lemmas (Corollaries 1 and 2), namely Theorems 1 and 2, which allow us to enforce restrictions on some tuples of lattice paths accounting for the entries of certain corresponding tableaux.

1.4.1 Partitions, Compositions and Diagrams

Definition 13. Let n be a nonnegative integer. A *partition* λ of n (written $\lambda \vdash n$) is a tuple $(\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \cdots \geq \lambda_k > 0$ and $\lambda_1 + \cdots + \lambda_k = n$. The λ_i's $(1 \leq i \leq k)$ are called the *parts* of λ, and the number k of parts is called the *length* of λ, denoted $\ell(\lambda)$. The number n is called the *size* of λ and is denoted $|\lambda|$. The only partition of 0 is the *empty partition*, denoted $\emptyset$.

Remark. Sometimes it might be convenient to add zeroes at the end of a partition λ, without changing its value. For example $(4,3,1,0,0,0) = (4,3,1,0) = (4,3,1)$ and $\emptyset = (0) = (0,0)$. With this notation the length $\ell(\lambda)$ is defined to be the number of positive parts of λ. This notation is useful, for example, when writing the set of all partitions of n with at most k parts: $\{(\lambda_1, \ldots, \lambda_k) : \lambda_1 + \cdots + \lambda_k = n, \lambda_1 \geq \cdots \geq \lambda_k \geq 0\}$.

In order to give a pictorial representation of partitions, we must first define *cells* and *diagrams*:

Definition 14. A *cell* is a square of side 1 whose four vertices have nonnegative integer coordinates. Cells are denoted by (i, j) where $j-1$ and $i-1$ are respectively the x-coordinate and the y-coordinate of its lower-left corner in the cartesian plane (see Figure 1.7). Notice that the number i increases upwards and j increases from left to right. Any set D of cells is called a *diagram*. The number of elements of D, denoted $|D|$, is called the *size of D*. We define the *i-th row* (or *row i*) of D to be the intersection $D \cap \{(i,1), (i,2), (i,3), \ldots\}$ and the *j-th column* (or *column j*) of D as the intersection

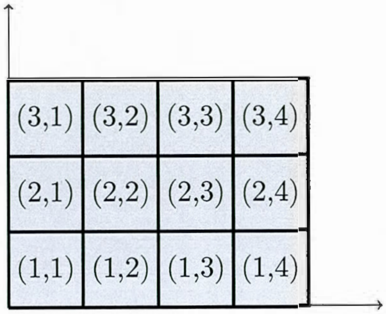


Figure 1.7 Some cells and their names.

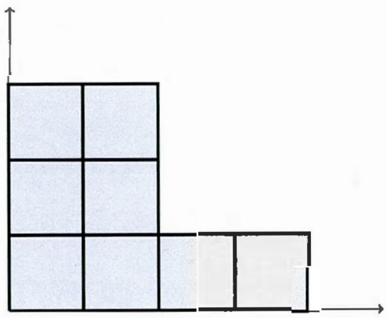


Figure 1.8 The Young diagram $D(\lambda)$ of the partition $\lambda = (4, 2, 2)$.

$D \cap \{(1, j), (2, j), (3, j), \ldots\}$.

Definition 15. The *Young diagram* (or simply *diagram*) *of a partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is the set of all the cells (i, j) satisfying $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$ (see Figure 1.8). We denote this set by $D(\lambda)$.

Remark. Notice that the diagram $D(\lambda)$ of the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ has exactly λ_i cells in its i-th row $(1 \leq i \leq k)$. Thus it is easy to obtain λ from the diagram $D(\lambda)$. In the rest of this work we refer indistinctively to a partition λ and its diagram $D(\lambda)$, writing for example $(i, j) \in \lambda$ whenever $(i, j) \in D(\lambda)$. We may also refer to the the number $k = \ell(\lambda)$ as the "number of rows" of λ.

A natural way to generalize the concept of partitions is by lifting the weakly decreasing condition, as in the following definition.

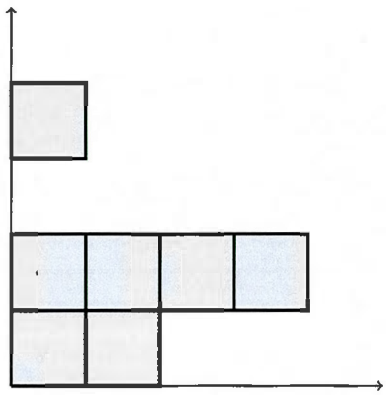


Figure 1.9 The diagram $D(\alpha)$ of the composition $\alpha = (2, 4, 0, 1)$.

Definition 16. A *composition* of the nonnegative integer n is a tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_k = n$. The number n is called the *size* of α and is denoted $|\alpha|$. Any partition is also a composition. As with partitions, any zeroes at the right of α leave the composition unchanged. For example $(1, 3, 0, 4, 1, 0, 0) = (1, 3, 0, 4, 1)$ and $(1, 0, 0) = (1)$. The *diagram of the composition* α, denoted $D(\alpha)$, is the set of cells (i, j) satisfying $1 \leq i \leq k$ and $1 \leq j \leq \alpha_i$ (see Figure 1.9). In what follows, we make no distinction between α and its diagram $D(\alpha)$, making the notation "$D(\alpha)$" unnecessary in most cases.

The above visual representations of partitions and compositions (Figures 1.8 and 1.9) suggest more simple generalizations of these objects. Definitions 18 and 19 below are the ones more relevant to our work.

Definition 17. Given two compositions $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_k)$, we say that β *contains* γ (or that γ *is contained in* β) if $\gamma_i \leq \beta_i$ for $1 \leq i \leq k$. This relation is denoted $\gamma \subseteq \beta$ or $\beta \supseteq \gamma$.

Definition 18. Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ such that $\mu \subseteq \lambda$ (see Definition 17 above), define the *skew partition* λ/μ as the diagram containing all cells (i, j) which satisfy $(i, j) \in \lambda$ and $(i, j) \notin \mu$ (see Figure 1.10). In other words

$$\lambda/\mu := D(\lambda) \setminus D(\mu),$$

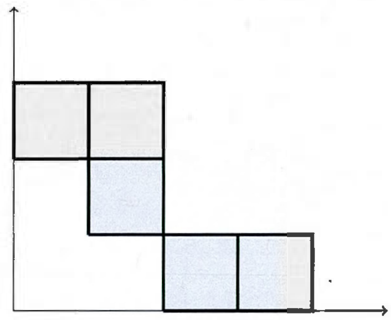


Figure 1.10 The skew partition $(4,2,2)/(2,1)$.

where the right hand side denotes the set-theoretic difference between $D(\lambda)$ and $D(\mu)$. A more precise equivalent definition is

$$\lambda/\mu := \{(i,j) : 1 \leq i \leq k,\ \mu_i + 1 \leq j \leq \lambda_i\}.$$

Partitions are special cases of skew partitions, since $\lambda = \lambda/\emptyset$ for every partition λ.

The following generalizes all of the definitions above.

Definition 19. Given two compositions $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_k)$ such that $\gamma \subseteq \beta$, define the *row-convex diagram* β/γ as follows (see Figure 1.11):

$$\beta/\gamma := \{(i,j) : 1 \leq i \leq k,\ \gamma_i + 1 \leq j \leq \beta_i\} = D(\beta) - D(\gamma).$$

Skew partitions are special cases of row convex diagrams.

Remark. The reason for the name *row-convex* is that each of the rows of these diagrams are topologically convex in the following sense: If (i, j_1) and (i, j_2) are in the i-th row of some row-convex diagram D, then so is (i, j) for any integer j between j_1 and j_2. Similarly one could define *column-convex diagrams* to be such that if $(i_1, j), (i_2, j) \in D$ for some $1 \leq i_1 < i_2$, $j \geq 1$, then $(i, j) \in D$ for all i with $i_1 \leq i \leq i_2$. A diagram that is both row-convex and column-convex is referred to simply as a *convex* diagram. We are only interested in diagrams that are convex.

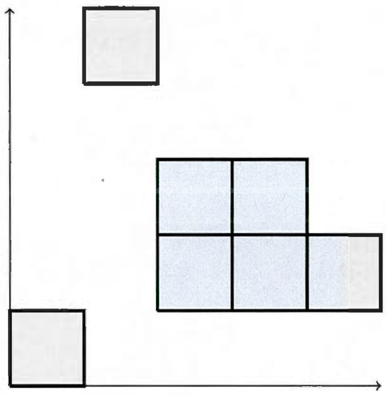


Figure 1.11 The row-convex diagram $(1,5,4,0,2)/(0,2,2,0,1)$.

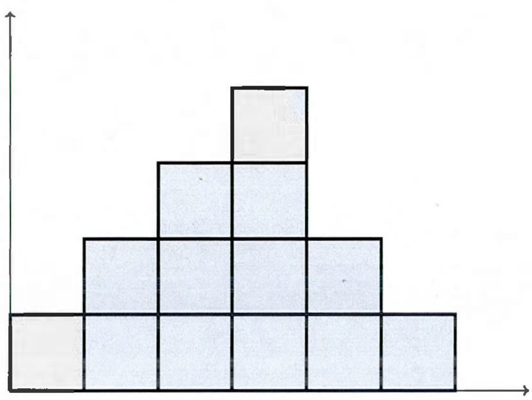


Figure 1.12 The shifted diagram η^* for $\eta = (6,4,2,1)$.

Besides partitions and skew partitions, another interesting class of diagrams is that of *shifted diagrams.* If $\eta = (\eta_1, \ldots, \eta_k)$ is a strictly decreasing partition (i.e. $\eta_1 > \cdots > \eta_k$), then the shifted diagram η^* is the one given by (see Figure 1.12)

$$\begin{aligned}\eta^* &:= \{(i,j) : 1 \leq i \leq k \text{ and } i \leq j \leq i + \eta_i - 1\} \\ &= (\eta_1 + 0, \eta_2 + 1, \ldots, \eta_k + k - 1)/(0, 1, \ldots, k-1)\end{aligned}$$

A *staircase* is a shifted diagram η^* where η is of the form $(s+k, s+k-1, \ldots, s+1)$ for some $s \geq 0$, $k \geq 1$.

1.4.2 Young Tableaux

The enumeration of positive integer fillings of diagrams provides important results in a wide array of areas ranging from graph theory (Adin, King, and Roichman, 2011) to statistical mechanics, positioning them among the most important objects in algebraic combinatorics. In this section we review some basic definitions regarding these objects.

Definition 20. A *filling* of a diagram D is a function $\varphi : D \to \mathbb{N}^+$. This can be seen as one of the possible ways of placing an integer into each of the cells of D (see Figure 1.13). Each of these numbers are called the *entries* of D. We say that D is the *shape* of φ and write $\text{shape}(\varphi) = D$. The *content* of the filling φ, denoted $\text{content}(\varphi)$, is the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ where α_j is the number of times the entry j appears in φ $(j \geq 1)$.

Definition 21. Let D be a convex diagram. A *semistandard Young tableau* of shape D is a filling $\tau : D \to \mathbb{N}^+$ which is weakly increasing on every row from left to right and strictly increasing on every column upwards (see Figure 1.14). In other words, the following two conditions are satisfied for all $i, j, i_1, i_2, j_1, j_2 \geq 1$:

$$(i, j_1), (i, j_2) \in D \quad \text{and} \quad j_1 < j_2 \quad \Rightarrow \quad \tau(i, j_1) \leq \tau(i, j_2)$$

$$(i_1, j), (i_2, j) \in D \quad \text{and} \quad i_1 < i_2 \quad \Rightarrow \quad \tau(i_1, j) < \tau(i_2, j)$$

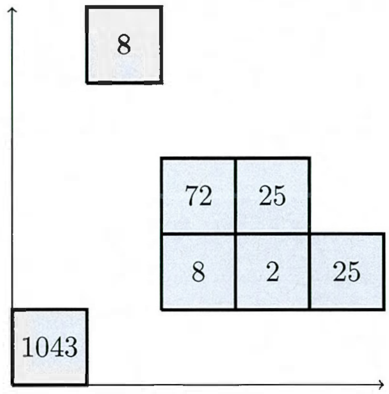


Figure 1.13 A filling of the diagram $D = \{(1,1),(2,3),(2,4),(2,5),(3,3),(3,4),(5,2)\}$.

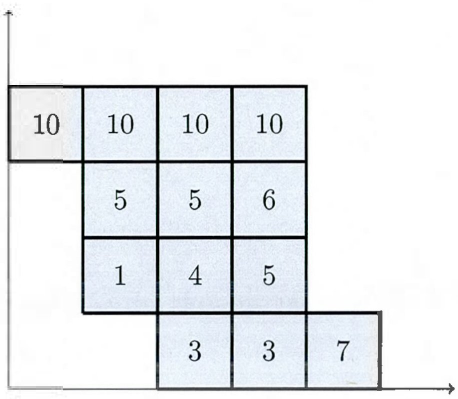


Figure 1.14 A semistandard Young tableau of shape $(5,4,4,4)/(2,1,1)$.

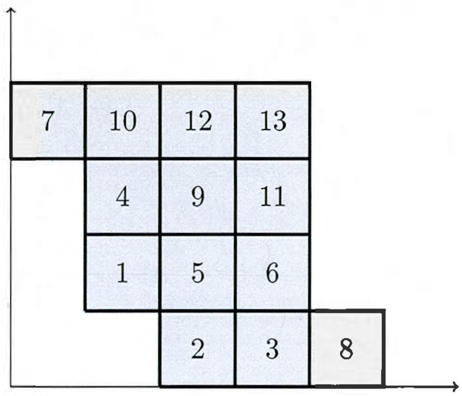


Figure 1.15 A standard Young tableau of shape $(5, 4, 4, 4)/(2, 1, 1)$.

For any non-empty convex diagram D there is an infinite number of semistandard Young tableaux of shape D. However the number of semistandard Young tableaux of shape D and fixed content α is always finite. A remarkable well-known result is that for every β resulting from permuting the entries of α, and for every skew partition λ/μ, the number of semistandard Young tableaux of shape λ/μ and content α is equal to the number of semistandard Young tableaux of shape λ/μ and content β. This can be shown with a tricky bijection and also as a result of the symmetry of skew Schur functions. We make this fact evident in Section 1.4.3, in a simple way without the need of symmetric functions.

Definition 22. A *standard Young tableau* $\tau : D \to \mathbb{N}$, where D is any convex diagram, is a semistandard Young tableau whose entries are the numbers $1, 2, \ldots, n = |D|$, each appearing exactly once (see Figure 1.15). In other words, a standard Young tableau is a semistandard Young tableau of content $(1, 1, \ldots, 1)$.

Counting the number of standard Young tableaux of a given convex shape D has since long been an especially important problem in algebraic combinatorics. When D is a partition, this is the dimension of the irreducible representation indexed by λ on the symmetric group $\mathfrak{S}_n$. For some special shapes it counts geodesics in a graph (Adin, King, and Roichman, 2011). The following is the well-known hook and determinant formulae for the number of standard Young tableaux of shape λ, where λ is a partition:

Theorem 4 (Frame, de B., and Thrall, 1954). *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition with $\lambda_k > 0$. For each cell $(i,j) \in \lambda$, define* $\mathrm{hook}(i,j)$ *to be the number of cells in λ that are above (i,j) in the same column, or to the right of (i,j) in the same row, including (i,j) itself. The number of standard Young tableaux of shape λ is equal to:*

$$f^\lambda = \frac{n! \prod\limits_{1\le i<j\le k} (\lambda_i - \lambda_j + j - i)}{\prod\limits_{i=1}^{k} (\lambda_i + k - i)!} = \frac{n!}{\prod\limits_{(i,j)\in\lambda} \mathrm{hook}(i,j)} \tag{1.63}$$

We deduce this, along with several other determinant-like tableau-enumerating formulae in Section 1.4.4.

1.4.3 Bijections with Collections of Lattice Paths

In this section we present a bijection between families of semistandard Young tableaux of certain convex shapes, and collections of lattice paths. The ultimate goal is to derive, in the next section, enumerative formulae for these families of tableaux. The basic method for our derivation is well-known, however we have not found any source in the literature which presents it in the general form of this work. Our lattice paths may be horizontally or vertically flipped when compared to the ones in other sources. This is done in order to obtain north-east lattice paths.

Let $\beta = (\beta_1, \dots, \beta_k)$, $\gamma = (\gamma_1, \dots, \gamma_k)$ be compositions with $\gamma \subseteq \beta$ and suppose that β/γ is convex. Consider a filling

$$\tau : \beta/\gamma \to \mathbb{N}^+,$$

and suppose that τ is weakly increasing on every row. Define the north-east lattice paths $p_1, \dots, p_k$ from τ as follows:

1. p_i starts at the point $(\gamma_i - i, 0)$ and ends at $(\beta_i - i, \infty)$.

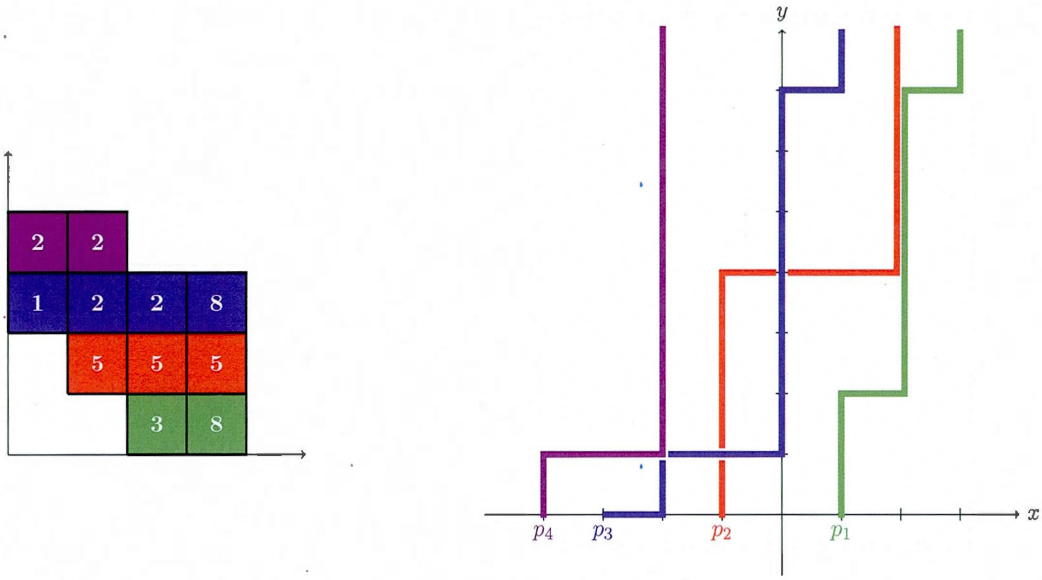


Figure 1.16 A filling of $\beta/\gamma = (4,4,4,2)/(2,1)$ with weakly increasing rows and the corresponding lattice paths.

2. The number of east steps of p_i that are contained in the horizontal line $y = y_0$ is equal to the number of entries equal to $y_0 + 1$ in the i-th row of τ ($i = 1, \ldots, k$, $y_0 \geq 0$).

See Figure 1.16 for an example of this construction: Clearly these paths are uniquely defined by these two conditions, and the filling τ may be obtained from $p_1, \ldots, p_k$ by counting the east steps of each p_i in each horizontal line of the upper half plane, and using the assumption that τ is weakly increasing on each row. Recall that $\tau(i,j)$ denotes the entry of τ on the i-th row (upwards) and j-th column (from left to right), so the entries of the i-th row of τ are

$$\tau(i, \gamma_i + 1) \leq \tau(i, \gamma_i + 2) \leq \cdots \leq \tau(i, \beta_i).$$

We claim that the following three statements are equivalent for all i, j with $1 \leq i \leq k-1$ and $\max\{\gamma_i, \gamma_{i+1}\} + 1 \leq j \leq \min\{\beta_i, \beta_{i+1}\}$:

a. $\tau(i,j) < \tau(i+1,j)$.

b. The $(j-\gamma_i)$-th east step of p_i is strictly below the $(j-\gamma_{i+1})$-th east step of p_{i+1}.

c. The path p_i does not intersect, and is strictly below, the path p_{i+1} in the vertical line $x = j - i - 1$.

Indeed, statements **a** and **b** are equivalent because, by construction, $\tau(i,j) - 1$ (the $(j-\gamma_i)$-th entry of the i-th row of τ) is the y-coordinate of the $(j-\gamma_i)$-th east step of p_i, and $\tau(i+1,j)-1$ is the y-coordinate of the $(j-\gamma_{i+1})$-th east step of p_{i+1}. Statements **b** and **c** are equivalent because the line $x = j-i-1$ contains both the ending point of the $(j-\gamma_{i+1})$-th east step of p_{i+1}, and the starting point of the $(j-\gamma_i)$-th east step of p_i.

By quantifying statements **a** and **c** for all j between $\max\{\gamma_i, \gamma_{i+1}\}+1$ and $\min\{\beta_i, \beta_{i+1}\}$, we obtain that the following two statements are equivalent:

a'. $\tau(i,j) < \tau(i+1,j)$ for $\max\{\gamma_i,\gamma_{i+1}\}+1 \le j \le \min\{\beta_i,\beta_{i+1}\}$.

c'. The path p_i does not intersect, and is strictly below, the path p_{i+1} in each of the vertical lines $x = x_0$ for $\max\{\gamma_i,\gamma_{i+1}\} - i \le x_0 \le \min\{\beta_i,\beta_{i+1}\} - i - 1$.

Now by letting i free in the set $\{1,\ldots,k-1\}$, we obtain that the following two statements are equivalent:

A. τ is a semistandard Young tableau.

C. For $i = 1,\ldots,k-1$, the path p_i does not intersect, and is strictly below, the path p_{i+1} in each of the vertical lines $x = x_0$ for $\max\{\gamma_i,\gamma_{i+1}\}-i \le x_0 \le \min\{\beta_i,\beta_{i+1}\}-i-1$.

The study of semistandard Young tableaux of shape β/γ is then "reduced" to the study of sequences of paths satisfying statement **C**. We refer to any sequence $(p_1,\ldots,p_k)$ of paths between the points $(\gamma_i - i, 0)$ and $(\beta_i - i, N-1)$ $(i = 1,\ldots,k)$, as a *network of shape* β/γ, and we say that such network is *semistandard* if it satisfies statement **C**. Thus there is a bijective correspondence between semistandard Young tableaux of shape

β/γ and semistandard networks of the same shape. The rest of this section is dedicated to finding meaningful interpretations of statement **C** for specific compositions β and γ. In particular it is interesting to review the cases for which this statement implies that the paths $p_1, \ldots, p_k$ are non-intersecting.

Recall that p_i starts at the line $x = \gamma_i - i$ and ends at the line $x = \beta_i - i$, while p_{i+1} starts at the line $x = \gamma_{i+1} - i - 1$ and ends at the line $x = \beta_{i+1} - i - 1$. Thus:

Remark. If $(p_1, \ldots, p_k)$ is a network of shape β/γ, then the vertical lines simultaneously visited by paths p_i and p_{i+1} are those of the form $x = x_0$ for

$$\max\{\gamma_i, \gamma_{i+1} - 1\} - i \leq x_0 \leq \min\{\beta_i + 1, \beta_{i+1}\} - i - 1.$$

One of our main goals is to compare this range to that of statement **C**. In fact these two ranges are very similar, differing on either side by at most 1. If both β and γ are partitions (β/γ is a skew partition), then these two ranges are the same. Hence the well-known equivalence:

Lemma 13. *If λ/μ is a skew partition, then any network $(p_1, \ldots, p_k)$ of shape λ/μ is semistandard if and only if the paths $p_1, \ldots, p_k$ are non-intersecting.*

See Figure 1.17 for an example of a semistandard Young tableau of skew shape and the corresponding non-intersecting lattice paths. Next we review more general cases for β and γ. The following lemma highlights the main difference between the skew-partition case and all other cases;

Lemma 14. *If the diagram of β/γ is connected, and either β or γ is not a partition, then every semistandard network $(p_1, \ldots, p_k)$ of shape β/γ is intersecting. More precisely a network $(p_1, \ldots, p_k)$ of shape β/γ is semistandard if and only if it satisfies the three properties below for every $i \in \{1, \ldots, k-1\}$;*

- $\gamma_i < \gamma_{i+1} \Leftrightarrow$ *the paths p_i and p_{i+1} intersect in the line $x = \gamma_{i+1} - i - 1$.*
- $\beta_i < \beta_{i+1} \Leftrightarrow$ *the paths p_i and p_{i+1} intersect in the line $x = \beta_i - i$.*

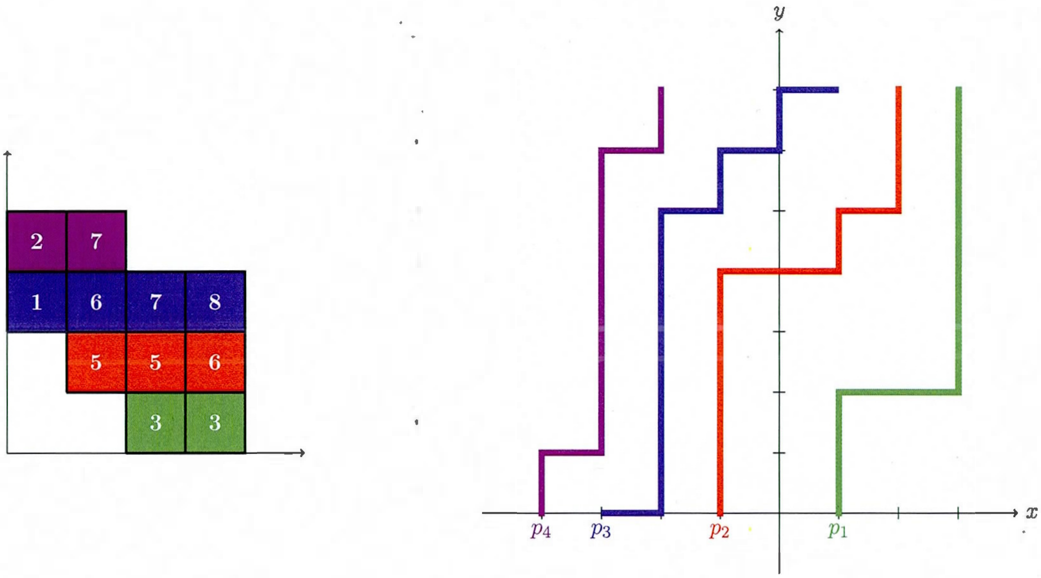


Figure 1.17 A semistandard Young tableau of shape $\lambda/\mu = (4,4,4,2)/(2,1)$ and the corresponding semistandard network of shape λ/μ.

- *Elsewhere in the plane these paths do not intersect, and inside every other vertical line simultaneously visited by both paths, p_i is strictly below p_{i+1}.*

Proof. Assume first that $(p_1, \ldots, p_k)$ is semistandard.

If $\gamma_i < \gamma_{i+1}$ for some $i \in \{1, \ldots, k-1\}$, then $\gamma_i - i \leq \gamma_{i+1} - i - 1$, and so the path p_i starts to the left of p_{i+1}. Thus p_i must either intersect or be strictly above p_{i+1} in the line $x = \gamma_{i+1} - i - 1$, but we know by **C** that p_i is strictly below p_{i+1} in the line $x = \gamma_{i+1} - i$. Hence they must intersect in $x = \gamma_{i+1} - i - 1$. Similarly, if $\beta_i < \beta_{i+1}$ for some $i \in \{1, \ldots, k-1\}$, then $\beta_i - i \leq \beta_{i+1} - i - 1$, and so the path p_i ends to the left of p_{i+1}. Thus p_i must either intersect or be strictly above p_{i+1} in the line $x = \beta_i - i$, but we know by **C** that p_i is strictly below p_{i+1} in the line $x = \beta_i - i - 1$. Hence they must intersect in $x = \beta_i - i$.

Now if p_i and p_{i+1} intersect in the line $x = \gamma_{i+1} - i - 1$, then p_i must start to the left of this line; $\gamma_i - i \leq \gamma_{i+1} - i - 1$, or equivalently $\gamma_i < \gamma_{i+1}$. Similarly, if p_i and p_{i+1}

intersect in the line $x = \beta_i - i$, then $\beta_{i+1} - i - 1 \geq \beta_i - i$, or equivalently, $\beta_i < \beta_{i+1}$.

Finally, the last claim of the lemma is equivalent to statement **C**, as a result of the remark above Lemma 13. □

We are now ready to use the correspondence between semistandard Young tableaux and semistandard networks, along with Theorems 1 and 2 of Section 1.2, to derive enumeration formulae for important families of semistandard and standard Young tableaux.

1.4.4 Enumeration Formulae

Next we use the theorems of section 1.2 to obtain several existing and some new enumeration formulae for certain families of standard and semistandard Young tableaux of convex shapes. The bijection from the previous section between semistandard Young tableaux and semistandard networks is repeatedly used as the first step for each derivation. It is important to remark that while this bijection is well-known for most important shapes, the unifying enumeration approach presented here had not been previously implemented because the Lindstrom-Gessel-Viennot and Stembridge's lemmas (Corollaries 1 and 2) only allow for restrictions on paths which may be accounted for by modifications of the graph G. This limitation is resolved by employing our natural extensions of these lemmas (Theorems 1 and 2).

We start by providing an elementary proof of the hook length formula (Theorem 4) for the number of standard Young tableaux of a partition shape. This proof was devised independently by the author. However, it is essentially the same as the one presented by Eriksson (1993), although lattice paths are not mentioned explicitly by Eriksson, and instead the equivalent concept of *rat races* was used in his work, along with a proper involution.

Proof of Theorem 4. Let τ be a standard Young tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. Let $(p_1, \ldots, p_k)$ be the semistandard network corresponding to τ, so that p_i starts at the point $(-i,0)$ and ends at the point $(\lambda_i - i, \infty)$. By Lemma 13, the paths $p_1, \ldots, p_k$ are non-intersecting. The condition that the entries of τ are the integers $1, \ldots, n$, each appearing exactly once, is equivalent to the condition that every horizontal line of equation $y = y_0$ ($y_0 = 0, \ldots, n-1$) contains exactly one east step from the paths $p_1, \ldots, p_k$. In particular this means that these paths are vertical above the line $y = n-1$, and so we may assume that the ending point of p_i is $(\lambda_i - i, n-1)$ rather than $(\lambda_i - i, \infty)$, for $i = 1, \ldots, k$.

For $\sigma \in \mathfrak{S}_k$, let T_σ denote the set of all tuples $(p_1, \ldots, p_k)$ of (possibly intersecting) lattice paths between the vertices $(-i, 0)$ and $(\lambda_{\sigma_i} - \sigma_i, n-1)$ for $i = 1, \ldots, k$, respectively, with n east steps in total; exactly one contained in each of the horizontal lines $y = y_0$ ($y_0 = 0, \ldots, n-1$). These sets are clearly disjoint. Define T as the union of T_σ over all $\sigma \in \mathfrak{S}_n$. The function $F : T \to \mathfrak{S}_k$ defined by $F(t) = \sigma$ for $t \in T_\sigma$ may be easily verified to be a k-arrangement. Let T_0 denote the set of all non-intersecting tuples in T. By the above observation, $|T_0|$ is the number of standard Young tableaux of shape λ. Theorem 1 yields;

$$|T_0| = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sign}(\sigma) |T_\sigma|.$$

For any $t = (p_1, \ldots, p_k) \in T_\sigma$, define

$$A_i = A_i(t) := \{l \in \{1, \ldots, n\} : p_i \text{ has an east step in the line } y = l - 1\}, \tag{1.64}$$

for $i = 1, \ldots, k$. Clearly $(A_1, \ldots, A_k)$ determines t. The fact that $t \in T_\sigma$ is equivalent to the conditions;

$$\bigsqcup_{i=1}^{k} A_i = \{1, \ldots, n\}, \tag{1.65}$$

and;

$$|A_i| = \lambda_{\sigma_i} + i - \sigma_i, \quad i = 1, \ldots, k, \tag{1.66}$$

Thus 1.64 provides a bijection between T_σ and the family of all k-tuples $(A_1, \ldots, A_k)$ of sets satisfying 1.65 and 1.66. Hence

$$|T_\sigma| = \binom{n}{\lambda_{\sigma_1} + 1 - \sigma_1, \ldots, \lambda_{\sigma_k} + k - \sigma_k},$$

where the right-hand side multinomial coefficient is assumed to be 0 whenever any of the numbers $\lambda_{\sigma_i} + i - \sigma_i$ $(i = 1, \ldots, k)$ is negative. Therefore:

$$\begin{aligned}
|T_0| &= \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sign}(\sigma) \binom{n}{\lambda_{\sigma_1} + 1 - \sigma_1, \ldots, \lambda_{\sigma_k} + k - \sigma_k} \\
&= n! \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sign}(\sigma) \prod_{i=1}^{k} \frac{(\lambda_{\sigma_i} + k - \sigma_i)_{k-i}}{(\lambda_{\sigma_i} + k - \sigma_i)!} \\
&= \frac{n!}{\prod_{i=1}^{k} (\lambda_i + k - i)!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sign}(\sigma) \prod_{i=1}^{k} (\lambda_{\sigma_i} + k - \sigma_i)_{k-i} \\
&= \frac{n!}{\prod_{i=1}^{k} (\lambda_i + k - i)!} \det[(\lambda_j + k - j)_{k-i}]_{1 \le i,j \le k},
\end{aligned}$$

where $(a)_b$ denotes the descending factorial $a(a-1)\cdots(a-(b-1))$. Since $(\lambda_j + k - j)_{k-i}$ is a monic polynomial of degree $k-i$ evaluated in $\lambda_j + k - j$, we may apply row operations to obtain

$$\begin{aligned}
|T_0| &= \frac{n!}{\prod_{i=1}^{k} (\lambda_i + k - i)!} \det[(\lambda_j + k - j)^{k-i}]_{1 \le i,j \le k} \\
&= \frac{n!}{\prod_{i=1}^{k} (\lambda_i + k - i)!} \prod_{1 \le i < j \le k} (\lambda_i - \lambda_j + j - i),
\end{aligned}$$

where the last equality is the well-known determinant of Vandermonde's matrix. □

A similar idea provides a formula for the Kostka coefficient $K_{\lambda/\mu,\alpha}$ which counts the number of semistandard Young tableaux of shape λ/μ and content α.

Theorem 5. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be partitions with $\mu \subseteq \lambda$, and let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a composition with $|\alpha| = |\lambda/\mu| = n$. The Kostka coefficient $K_{\lambda/\mu,\alpha}$, which counts the number of semi-standard Young tableaux of shape λ/μ and content α, is given by the formula:*

$$\begin{aligned}
K_{\lambda/\mu,\alpha} &= \sum_{\sigma \in \mathfrak{S}_k} \mathit{sign}(\sigma) \binom{\{1^{\alpha_1}, \ldots, m^{\alpha_m}\}}{\lambda_{\sigma_1} + 1 - \sigma_1 - \mu_1, \ldots, \lambda_{\sigma_k} + k - \sigma_k - \mu_k} \\
&= \sum_{\sigma \in \mathfrak{S}_k} \mathit{sign}(\sigma) \binom{\{1^{\alpha_1}, \ldots, m^{\alpha_m}\}}{\lambda_1 + \sigma_1 - 1 - \mu_{\sigma_1}, \ldots, \lambda_k + \sigma_k - k - \mu_{\sigma_k}}
\end{aligned}$$

where $\binom{A}{a_1, \ldots, a_k}$ denotes the number of ways of writing the multiset A as the ordered disjoint union of k multisets of cardinalities $a_1, \ldots, a_k$ respectively. This number is assumed to be 0 when any of the a_i's is negative.

Proof. Observe that a network $(p_1, \ldots, p_k)$ of shape λ (p_i starts at $(\mu_i - i, 0)$ and ends at $(\lambda_i - i, \infty)$ for $i = 1, \ldots, k$) corresponds to a semistandard Young tableau of content α if and only if the paths $p_1, \ldots, p_k$ are non-intersecting and contain, among all of them, exactly α_{y_0+1} east steps inside the horizontal line $y = y_0$ for $y_0 = 0, \ldots, m-1$. Since these paths are vertical above the line $y = m-1$, we may assume that p_i ends at $(\lambda_i - i, m-1)$ rather than $(\lambda_i - i, \infty)$ for $i = 1, \ldots, k$.

For $\sigma \in \mathfrak{S}_k$, let T_σ denote the set of all tuples $(p_1, \ldots, p_k)$ of (possibly intersecting) lattice paths between the vertices $(\mu_i - i, 0)$ and $(\lambda_{\sigma_i} - \sigma_i, m-1)$ for $i = 1, \ldots, k$, respectively, with n east steps in total; exactly α_{y_0+1} of them contained in the horizontal line $y = y_0$ for $y_0 = 0, \ldots, n-1$. These sets are clearly disjoint. Define T as the union of T_σ over all $\sigma \in \mathfrak{S}_n$. The function $F : T \to \mathfrak{S}_k$ defined by $F(t) = \sigma$ for $t \in T_\sigma$ may be easily verified to be a k-arrangement. Let T_0 denote the set of all non-intersecting tuples in T. By the above observation, $|T_0|$ is the number of semistandard Young tableaux of shape λ/μ and content α. Theorem 1 yields;

$$|T_0| = \sum_{\sigma \in \mathfrak{S}_k} \text{sign}(\sigma)|T_\sigma|.$$

For any $t = (p_1, \ldots, p_k) \in T_\sigma$, define

$$A_i = A_i(t) := \bigsqcup_{l=1}^{m} \{l^{r_l}\}, \quad i = 1, \ldots, k, \tag{1.67}$$

where r_l is the number of east steps of p_i in the line $y = l-1$, and $\{l^{r_l}\}$ denotes the multiset containing l exactly r_l times. Clearly $(A_1, \ldots, A_k)$ determines t. The fact that $t \in T_\sigma$ is equivalent to the conditions;

$$\bigsqcup_{i=1}^{k} A_i = \{1^{\alpha_1}, \ldots, m^{\alpha_m}\}, \tag{1.68}$$

and;

$$|A_i| = \lambda_{\sigma_i} + i - \sigma_i - \mu_i, \quad i = 1, \ldots, k, \tag{1.69}$$

Thus 1.67 provides a bijection between T_σ and the family of all k-tuples $(A_1, \ldots, A_k)$ of multisets satisfying 1.68 and 1.69. Hence

$$|T_\sigma| = \binom{\{1^{\alpha_1}, \ldots, m^{\alpha_m}\}}{\lambda_{\sigma_1} + 1 - \sigma_1 - \mu_1, \ldots, \lambda_{\sigma_k} + k - \sigma_k - \mu_k},$$

as wanted. $\square$

We underline that our proof of this result appears to be more direct than the one presented by Lederer (2006) for the case $\mu = \emptyset$, and the symmetric function approach he mentions. However, our proof shares with Lederer's proof the advantage of being entirely elementary. The case $\alpha = (1, 1, \ldots, 1) = (1^n)$ yields a determinant formula for the number of standard Young tableaux of skew shape λ/μ;

$$K_{\lambda/\mu,(1^n)} = \frac{n!}{\prod_{i=1}^{k}(\lambda_i + k - i)!} \det\left[(\lambda_j + k - j)_{k-i+\mu_i}\right]_{1\leq i,j\leq k} \tag{1.70}$$

As evidenced by Lemma 14, semistandard networks of shapes other than partitions or skew partitions, do not consist entirely of non-intersecting lattice paths. However in the case of a shifted diagram η^*, we can obtain non-intersecting paths by applying a simple transformation to each network of shape η^*, without losing any information on the network. Details are in the proof of the next result. Recall that $\binom{A}{a_1,\ldots,a_k}$ denotes the number of ways of writing the multiset A as the ordered disjoint union of k multisets of cardinalities $a_1, \ldots, a_k$ respectively.

Theorem 6. *Let $\eta = (\eta_1, \ldots, \eta_k)$ (k even) be a strictly decreasing partition and let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a composition with $|\eta| = |\alpha| = n$. Then the number of semistandard Young tableaux of shape $\eta^* = (\eta_1, \eta_2 + 1, \ldots, \eta_k + k - 1)/(0, 1, \ldots, k-1)$ and content α is given by;*

$$\sum_{\pi\in M_k} \operatorname{sign}(\pi) \sum \prod_{\{i,j\}\in\pi,\, i<j} \left(\binom{A^*_{i,j}}{\eta_i - 1, \eta_j} - \binom{A^*_{i,j}}{\eta_i, \eta_j - 1}\right),$$

where $A^* := A - \{\max(A)\}$ for every multiset A, and the second sum is over all families $\{A_{i,j} : \{i,j\} \in \pi,\ i < j\}$ of multisets satisfying $|A_{i,j}| = \eta_i + \eta_j$ ($\{i,j\} \in \pi$, $i < j$) and $\bigsqcup A_{i,j} = \{0^{\alpha_m}, 1^{\alpha_{m-1}}, \ldots, (m-1)^{\alpha_1}\}$.

Proof. First recall that any network of shape η^* consists of paths p_i starting at $(-1, 0)$ and ending at $(\eta_i - 1, \infty)$ for $i = 1, \ldots, k$. Suppose that the network $(p_1, \ldots, p_k)$ corresponds to a semistandard Young tableau of shape η^* and content α. Thus each path p_i $(i = 1, \ldots, k)$ is vertical above the line $y = m - 1$ and so we may assume that its ending point is $(\eta_i - 1, m - 1)$. Lemma 14 states that the semistandard condition is equivalent to the condition that these paths intersect only in the vertical line $x = -1$. Hence by removing all of their initial vertical steps we obtain a tuple $(p'_1, \ldots, p'_k)$ of non-intersecting lattice paths such that;

- p'_i $(i = 1, \ldots, k)$ starts with a horizontal step at the segment
$$x = -1, \quad 0 \leq y \leq m - 1,$$
- p'_i $(i = 1, \ldots, k)$ ends at $(\eta_i - 1, m - 1)$,
- $p_1, \ldots, p_k$ possess in total exactly α_{y_0+1} east steps contained in the line $y = y_0$ for $y_0 = 1, \ldots, m - 1$.

By translating and rotating these paths 180^o, we obtain a tuple $(q_1, \ldots, q_k)$ of non-intersecting lattice paths such that;

- q_i $(i = 1, \ldots, k)$ starts at $(-\eta_i, 0)$,
- q_i $(i = 1, \ldots, k)$ ends with a horizontal step at the segment
$$x = 0, \quad 0 \leq y \leq m - 1,$$
- $q_1, \ldots, q_k$ possess in total exactly α_{m-y_0} east steps contained in the line $y = y_0$ for $y_0 = 1, \ldots, m - 1$.

The map $(p_1, \ldots, p_k) \mapsto (q_1, \ldots, q_k)$ is clearly invertible. Denote by T the set of all tuples $(q_1, \ldots, q_k)$ of (possibly intersecting) lattice paths satisfying the three properties above. We wish to count $|T_0|$, where T_0 is the set of all non-intersecting tuples in T. Set $I := (v_1, \ldots, v_k)$ where $v_i = (0, i - 1)$ (this is the segment from the second property above). Clearly T is I-stable. Suppose that k is even. Thus by Theorem 1;

$$|T_0| = \sum_{\pi \in M_k} \operatorname{sign}(\pi) |T_\pi|$$

Where T_π is the set of tuples $(q_1, \ldots, q_k) \in T$ such that q_i, q_j are non-intersecting for $\{i,j\} \in \pi$. Fix some $\pi \in M_k$. For $t = (q_1, \ldots, q_k) \in T_\pi$ and each $\{i,j\} \in \pi$ with $i < j$, denote by $A_{i,j} = A_{i,j}(t)$ the multiset of all y-coordinates of east steps of q_i and q_j. We will refer to this set as the *east step height set* of (q_i, q_j). Since q_l has η_l east steps for $l = 1, \ldots, k$, then

$$|A_{i,j}| = \eta_i + \eta_j \text{ for all } \{i,j\} \in \pi,\ i < j. \tag{1.71}$$

Also;

$$\bigsqcup_{\{i,j\} \in \pi,\, i<j} A_{i,j} = \{0^{\alpha_m}, 1^{\alpha_{m-1}}, \ldots, (m-1)^{\alpha_1}\}. \tag{1.72}$$

Now for $1 \leq i < j \leq k$, and any multiset $A \subseteq \{0^{\alpha_m}, 1^{\alpha_{m-1}}, \ldots, (m-1)^{\alpha_1}\}$ with $|A| = \eta_i + \eta_j$, let $T_{i,j}(A)$ denote the set of all pairs (q_i, q_j) of lattice paths starting at $(-\eta_i, 0)$, $(-\eta_j, 0)$ respectively, ending at I with a horizontal step, and having east step height set A. Let $T^0_{i,j}(A)$ denote the subset of all non-intersecting pairs in $T_{i,j}(A)$. Thus;

$$|T_\pi| = \sum \prod_{\{i,j\} \in \pi} |T^0_{i,j}(A_{i,j})|,$$

where the sum is over all families $\{A_{i,j}\}_{\{i,j\} \in \pi,\, i<j}$ of multisets satisfying equations 1.71 and 1.72 above. A simple involutive argument yields;

$$T^0_{i,j}(A) = \binom{A^*}{\eta_i - 1, \eta_j} - \binom{A^*}{\eta_i, \eta_j - 1},$$

This is because the first term counts the number of elements $(p_i, p_j) \in T_{i,j}(A)$ such that p_i ends weakly above p_j, while the second term counts the number of elements $(p_i, p_j) \in T_{i,j}(A)$ such that p_i ends weakly below p_j. In this last case the paths p_i, p_j must necessarily intersect, and so the second term counts exactly the intersecting elements from the first term. Therefore;

$$|T_0| = \sum_{\pi \in M_k} \operatorname{sign}(\pi) \sum_{\substack{\{A_{i,j} : \{i,j\} \in \pi,\, i<j\} \\ |A_{i,j}| = \eta_i + \eta_j \\ \bigsqcup A_{i,j} = \{0^{\alpha_m}, \ldots, (m-1)^{\alpha_1}\}}} \prod_{\{i,j\} \in \pi,\, i<j} \left(\binom{A^*_{i,j}}{\eta_i - 1, \eta_j} - \binom{A^*_{i,j}}{\eta_i, \eta_j - 1} \right),$$

as wanted. For the standard case we have $\alpha = (1, 1, \ldots, 1) = (1^n)$ and so the generalized binomial coefficients above only depend on the cardinalities $|A^*_{i,j}| = |A_{i,j}| - 1 = \eta_i + \eta_j - 1$, and not on the sets $A_{i,j}$, which yields;

$$
\begin{aligned}
|T_0| &= \sum_{\pi \in M_k} \operatorname{sign}(\pi) \frac{n!}{\displaystyle\prod_{\{i,j\}\in\pi,\, i<j} (\eta_i + \eta_j)!} \prod_{\{i,j\}\in\pi,\, i<j} \left(\frac{(\eta_i - \eta_j)(\eta_i + \eta_j - 1)!}{\eta_i! \eta_j!} \right) \\
&= \frac{n!}{\displaystyle\prod_{i=1}^{k} \eta_i!} \operatorname{pfaff} \left[\frac{\eta_i - \eta_j}{\eta_i + \eta_j} \right]_{1 \le i < j \le k} \\
&= \frac{n! \displaystyle\prod_{1 \le i < j \le k} (\eta_i - \eta_j)}{\left(\displaystyle\prod_{i=1}^{k} \eta_i! \right) \left(\displaystyle\prod_{1 \le i < j \le k} (\eta_i + \eta_j) \right)}
\end{aligned}
$$

□

CHAPTER II

SL$_K$-TILINGS

In this chapter we study two intimately related objects, namely T-systems (Di Francesco, 2010; Di Francesco and Kedem, 2009) and SL_k-tilings (Bergeron and Reutenauer, 2010). We start by defining these objects and providing a unified notation for both of them. For this we first recall the octahedron recurrence:

$$T(m,i,j)T(m-2,i,j) = T(m-1,i-1,j)T(m-1,i+1,j) - T(m-1,i,j-1)T(m-1,i,j+1), \tag{2.1}$$

defined over a three dimensional array

$$T : D \to R,$$

with values (entries) in some zero-characteristic field R, and whose domain D is a subset of

$$\{(a,b,c) \in \mathbb{Z}^3 : a+b+c \equiv 0 \,(\mathrm{mod}\ 2)\}.$$

We call these subsets *3-dimensional grids.* Suppose that the values of some entries of T are initially known (we refer to these values as *boundary conditions* or *initial values*), and assume that such values, along with the octahedron recurrence, are sufficient to compute all entries of T in its domain D. It is immediate from equation 2.1, that all entries of T must then be rational functions on the initial values. It is often the case, for convenient 3-dimensional grids and properly positioned boundary conditions, that all entries of T may in fact be written, in terms of the initial values, as Laurent polynomials. In the related context of cluster algebras, this is referred to as the *Laurent phenomenon* by Fomin and Zelevinsky (2002a). See also Berenstein, Fomin, and Zelevinsky (2005); Fomin and Zelevinsky (2002b, 2003, 2007). The Laurent polynomials appearing in cluster algebras appear to satisfy the property that their coefficients are all nonnegative integers. This positivity property in total generality is still only a conjecture, and combinatorial arguments are often used to prove special cases.

We now proceed to review the notion of T-systems. Given an integer $k \geq 1$, a *T-system of height k* is an array

$$T : \{(a, b, c) \in \mathbb{Z}^3 : 0 \leq a \leq k,\ a + b + c \equiv 0 \,(\mathrm{mod}\ 2)\} \to R, \tag{2.2}$$

satisfying the octahedron recurrence (2.1) in its domain, along with the additional condition;

$$T(a, b, c) = 1 \text{ whenever } a \text{ is } 0 \text{ or } k. \tag{2.3}$$

Under the assumption that

$$T(a, b, c) \neq 0 \text{ for all } a, b, c \text{ in the domain of } T, \tag{2.4}$$

the following relation results from inductively applying the octahedron recurrence and the Desnanot-Jacobi identity (Proposition 1);

$$T(m, i, j) = \det[T(1, i - m - 1 + p + q, j + p - q)]_{1 \leq p,q \leq m}. \tag{2.5}$$

Indeed, for $0 \leq m \leq 1$ the equality is evident. Assuming it is true for $m - 1$ and $m - 2$, we obtain;

$$\begin{aligned}
& T(m, i, j) \\
= \;& \frac{T(m-1, i-1, j)T(m-1, i+1, j) - T(m-1, i, j-1)T(m-1, i, j+1)}{T(m-2, i, j)} \\
= \;& \frac{\det A_{[1,m-1][1,m-1]} \det A_{[2,m][2,m]} - \det A_{[1,m-1][2,m]} \det A_{[2,m][1,m-1]}}{\det A_{[2,m-1][2,m-1]}} \\
= \;& \det(A),
\end{aligned}$$

where $A = [T(1, i-m-1+p+q, j+p-q)]_{1\leq p,q\leq m}$, as claimed. In particular for $m = k$, equation 2.5 becomes;

$$1 = \det[T(1, i-k-1+p+q, j+p-q)]_{1\leq p,q\leq k}. \tag{2.6}$$

By setting $P_{ij} := T(1, i+j-1, i-j)$, we obtain an $\mathbb{Z}\times\mathbb{Z}$ matrix P whose adjacent minors of order $\leq k$ are the entries of T. In particular all adjacent minors of P of order $\leq k$ are non-zero (by 2.4), and all of its adjacent $k \times k$ minors are equal to 1 (by 2.6).

Conversely, given any matrix P such that every one of its adjacent $(k-1)\times(k-1)$ minors are equal to 1, and every one of its minors of smaller order are non-zero, we recover a T-system of height k by setting $T(1, i, j) := P_{(\frac{1+i+j}{2})(\frac{1+i-j}{2})}$.

Following Bergeron and Reutenauer (2010), we define an SL_k*-tiling* to be a $\mathbb{Z}\times\mathbb{Z}$ matrix $P = [p_{ij}]_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$ such that all of its $k\times k$ adjacent minors are equal to 1. It is important to highlight that Bergeron and Reutenauer only restrict SL_k-tilings on having non-zero $(k-1)\times(k-1)$, allowing for a wide spectrum of tilings not considered by Di Francesco and Kedem.

Borrowing terminology from mathematical physics, a *boundary condition* for a non-zero T-system of height k, is simply an equation of the form $T(a, b, c) = x_{a,b,c}$ for (a, b, c) in the domain from 2.2, and $x_{a,b,c}$ in $R^* = R - \{0\}$. A *solution* to a T-system under a collection B of boundary conditions is a non-zero T-system (satisfying equation 2.4) T which satisfies all of the conditions in B. In terms of the SL_k-tiling $P = [T(1, i+j-1, i-j)]_{i,j\in\mathbb{Z}}$, boundary conditions are just equations of the form $\det(P_{IJ}) = x_{I,J}$ ($x_{I,J} \in R$), where I, J are equipotent sets of consecutive integers (i.e., $\det(P_{IJ})$ is an adjacent minor of P). In the rest of this work we focus on the matrix terminology of SL_k tilings, and only mention that of T-systems when necessary as a reference to Di Francesco's results.

We are interested in some particularly well-behaved SL_k-tiling boundary conditions which correspond to Di Francesco's (2010) "arbitrary" boundary conditions for T-systems. Our boundary conditions are defined in terms of a sequence (called *dress*, see Definition 24) of subsets of the integer plane (called *fringes*, see Definition 23), which we introduce in the next section.

In Section 2.1 we state the main results of this chapter. In Section 2.2 we provide all the proofs that are missing from Section 2.1, along with some new definitions and lemmas which become necessary to complete those proofs.

2.1 General boundary conditions

We proceed to introduce the combinatorial notions of *fringes* and *dresses*. As we outline later, each dress defines a general collection of boundary conditions determining an unique SL_k tiling. The entries of this SL_k tiling turn out, as we prove later, to be positive Laurent polynomials in terms of the field elements from the boundary conditions. We also give a general proof for the minors in the case $k = 2$ and conjecture a combinatorial model for all k which may lead to a general proof of Laurent positivity.

Picture the integer plane $\mathbb{Z} \times \mathbb{Z}$ in matrix form, so that the first coordinate increases downwards while the second one increases to the right. Informally, a *fringe* is an infinite subset of the plane which resembles a staircase of constant width. See for example Figure 2.1, where the dots represent the elements of F (For instance, the leftmost dot means that $(3, -2) \in F$). The set F in this figure is a 3-*fringe*. The formal definition follows.

Definition 23. For $m \geq 1$, define an m-*fringe* to be a subset F of the integer plane

$$\mathbb{Z} \times \mathbb{Z} = \{(i, j) : i, j \in \mathbb{Z}\},$$

satisfying the following properties:

1. **Diagonal property:** For all $r \in \mathbb{Z}$, the intersection between F and the r-*th main diagonal*

   $$D_r := \{(i, j) : j - i = r\},$$

   is a set of m consecutive points of D_r. More formally, there exist $i_r, j_r \in \mathbb{Z}$ such

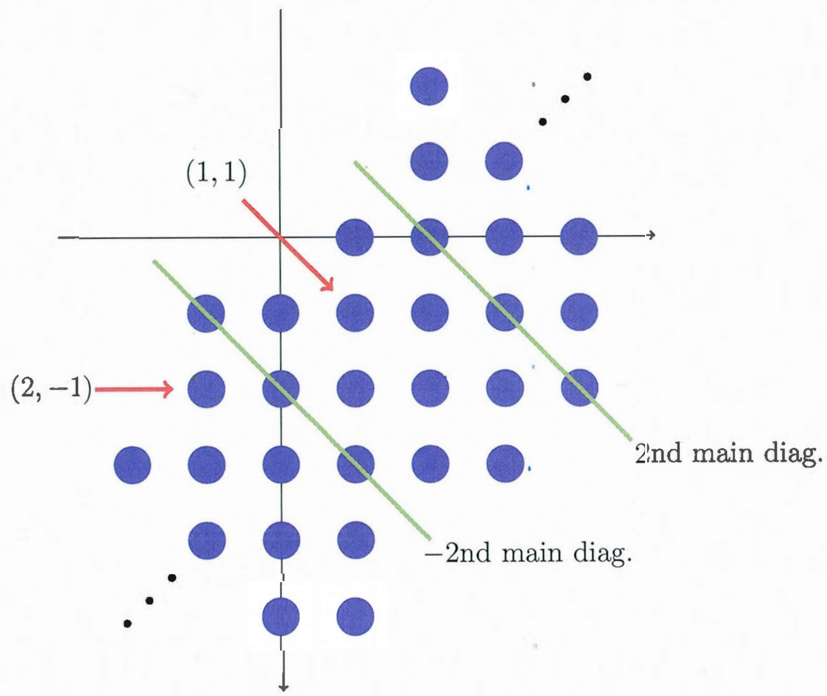


Figure 2.1 Part of a 3-fringe.

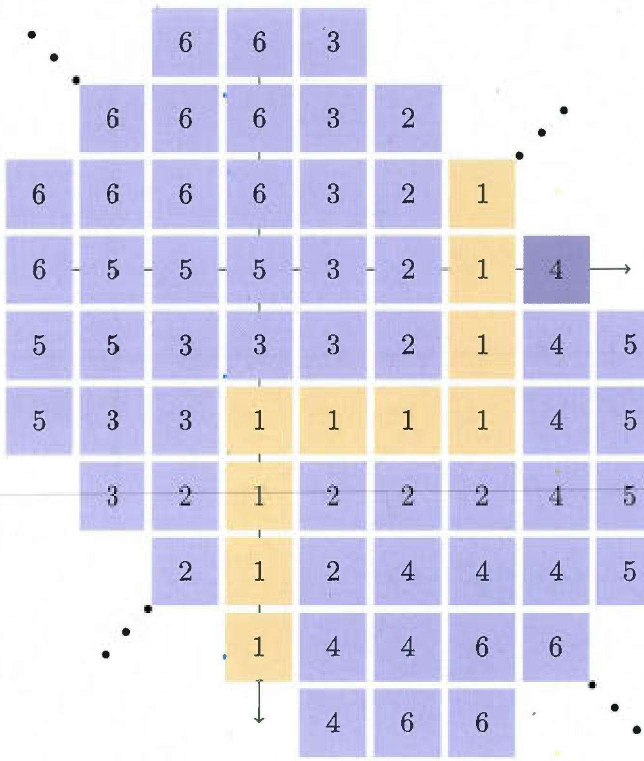


Figure 2.2 Part of a 6-dress whose 1-fringe $F^{(1)}$ has been highlighted.

that $j_r - i_r = r$ and

$$F_r := F \cap D_r = \{(i_r, j_r), (i_r + 1, j_r + 1), \ldots, (i_r + m - 1, j_r + m - 1)\}.$$

The set F_r is called the *r-th diagonal* of F, and the number i_r is called the *r-th handle* of F. In the 3-fringe of Figure 2.1, only the -5-th, -4-th, -3-rd, -2-nd, -1-st, 0-th, 1-st, 2-nd, 3-rd and 4-th diagonals are visible. Some handles of this 3-fringe are; $i_{-2} = 1$, $i_{-1} = 1$, $i_0 = 1$, $i_1 = 0$, $i_2 = 0$, $i_3 = -1$, $i_4 = -2$.

2. **Adjacency property:** For all $r \in \mathbb{Z}$;

$$i_{r+1} \in \{i_r - 1, i_r\}. \tag{2.7}$$

In particular, this means that the sequence $\{i_r\}_{r\in\mathbb{Z}}$ is non-increasing.

3. **Staircase property:** There are arbitrarily large and arbitrarily small integers r for which $i_{r+1} = i_r - 1$, as well as arbitrarily large and arbitrarily small integers r for which $i_{r+1} = i_r$. In simple terms this means that a fringe is not eventually vertical or eventually horizontal in any direction.

Observe that a set F satisfying the diagonal property above is an m-fringe if and only if the set $\{(i_r, i_r + r) : r \in \mathbb{Z}\}$ is a 1-fringe. Before we continue with the definition of *dresses*, let us state and prove a result highlighting an important algebraic connection between SL_k-tilings and fringes.

Lemma 15. *Let $P := [P_{ab}]_{a,b\in\mathbb{Z}}$ be an infinite matrix over a field of characteristic zero, let $F := \{(i_r, i_r + r) : r \in \mathbb{Z}\}$ be a 1-fringe, and let $k \geq 1$ be an integer. Suppose that;*

- *The minor $\det[P_{i_r+a, i_r+b+r}]_{0\leq a,b\leq k-1}$ is equal to 1 for all $r \in \mathbb{Z}$.*
- *Every $(k+1) \times (k+1)$ connected minor of P is equal to zero.*

Then P is an SL_k-tiling.

Proof. The first hypothesis above may be rewritten as;

- P is an SL_k-tiling over H (i.e., it satisfies the SL_k condition when restricted to H), where H is the k-fringe whose handles are the same as those of F.

Let r be an integer so that $i_{r+1} = i_r$ and $i_{r-1} = i_r + 1$. Clearly the set H' obtained from H by replacing i_r with $i_r + 1$ is also a k-fringe. We claim that P is an SL_k-tiling over H' as well. Indeed, by the Desnanot-Jacobi identity;

$$\det[P_{i_r+1+a,i_r+1+b+r}]_{0\leq a,b\leq k}$$
$$= \frac{\det[P_{i_{r-1}+a,i_{r-1}+b+r-1}]_{0\leq a,b\leq k} \det[P_{i_{r+1}+a,i_{r+1}+b+r+1}]_{0\leq a,b\leq k} + 0}{\det[P_{i_r+a,i_r+b+r}]_{0\leq a,b\leq k}}$$
$$= 1.$$

Similarly, if r is such that $i_{r+1} = i_r - 1$ and $i_{r-1} = i_r$, then P is an SL_k-tiling over the k-fringe H', resulting from H by replacing i_r with $i_r - 1$. It is evident by the staircase property of F that by applying these transformations successively, we may obtain a new fringe containing any desired point of the plane. Therefore P must be an SL_k-tiling over the whole plane, as wanted. $\square$

We now introduce a new object which may be regarded as a finite increasing sequence (meaning that every term is contained in its successor) of fringes.

Definition 24. A *k-dress* ($k \geq 2$) is a function

$$f : \mathbb{Z} \times \mathbb{Z} \to \{1, \ldots, k\},$$

such that the set

$$f^{-1}(\{1, \ldots, m\}) \subseteq \mathbb{Z} \times \mathbb{Z}$$

is an m-fringe for $m = 1, \ldots, k-1$. We identify a k-dress with the increasing sequence

$$\{F^{(m)}\}_{1\leq m\leq k-1}$$

of fringes $F^{(m)} := f^{-1}(\{1, \ldots, m\})$. The function f is constantly equal to k outside these fringes.

Following the notation from the above definition of fringes, for $m = 1, \ldots, k-1$, $r \in \mathbb{Z}$; let $F_r^{(m)} = D_r \cap F^{(m)}$ denote the r-th diagonal of $F^{(m)}$, and let $i_r^{(m)}$ denote its r-th handle, so that;

$$F_r^{(m)} = \{(i_r^{(m)}, i_r^{(m)} + r), (i_r^{(m)} + 1, i_r^{(m)} + 1 + r), \ldots, (i_r^{(m)} + m - 1, i_r^{(m)} + m - 1 + r)\}.$$

The main motivation of our research is trying to show the following conjecture;

Conjecture 1. *Let $f := \{F^{(m)}\}_{1 \le m \le k-1}$ be a k-dress with handles $\left\{i_r^{(m)}\right\}_{1 \le m \le k-1}$, and let $X = \left\{x_r^{(m)}\right\}_{\substack{1 \le m \le k-1 \\ r \in \mathbb{Z}}}$ be a family of algebraically independent (over the rationals) formal variables. There exists an unique SL_k-tiling $P = [p_{ij}]_{i,j \in \mathbb{Z}}$ whose entries are rational functions in X, such that for each $m = 1, \ldots, k-1$ and each $r \in \mathbb{Z}$, the following "boundary condition" is satisfied;*

$$\det \left[p_{\left(i_r^{(m)}+a\right)\left(i_r^{(m)}+r+b\right)} \right]_{0 \le a,b \le m-1} = x_r^{(m)}, \tag{2.8}$$

Moreover, each minor of this SL_k-tiling, of order smaller than k, is a non-zero Laurent polynomial with non-negative integer coefficients in X.

When written in terms of the non-zero T-system $T(1, i, j) := P_{(\frac{1+i+j}{2})(\frac{1+i-j}{2})}$, this conjecture is equivalent to Di Francesco's (2010) Corollary 4.13. However, Di Francesco's proof is partial in the sense that it only shows the Laurent non negativity for determinants of sub matrices which are weakly below or above $F^{(1)}$, while disregarding those containing entries from both regions of the plane. Additionally, his results are obtained under the additional condition that every minor of order $< k$ is non-zero. Bergeron and Reutenauer (see Proposition 9) lift this restriction and prove Conjecture 1 when the boundary conditions satisfy $i_r^{(m)} = i_r^{(1)}$ for $m = 1, \ldots, k-1$ and for determinants of sub matrices which are weakly below $F^{(1)}$.

There are three aspects to this conjecture, namely **uniqueness**, **existence**, and **Laurent non negativity**. The first one is proven below using induction and simple algebraic arguments, and the last two are proven at the end of this section for $k = 2$ employing constructive combinatorial arguments involving weighted tuples of non-intersecting paths within certain particularly complicated graphs denoted G^- and G^+, which are defined from the 2-dress f. Our proposed combinatorial model appears to hold for every k and it might provide a first step towards a complete proof of this conjecture.

Proof of the uniqueness statement of Conjecture 1. Along with uniqueness, we also show that every adjacent minor of order smaller than or equal to k must necessarily be a non-zero rational function in X, whose numerator and denominator have nonnegative integer coefficients. For this we denote those minors as follows;

$$M_{i,j,m} := \det[p_{(i+a)(j+b)}]_{0 \leq a,b \leq m-1},$$

and define a partial order $\leq$ on the set of indices

$$\{(i,j,k) : i,j \in \mathbb{Z},\ 1 \leq m \leq k\},$$

by;

$$\begin{aligned}
(i,j,m) \leq (i',j',m') \quad \text{if;} \quad & (i,j,m) = (i',j',m') \\
\text{or;} \quad & m' > m = 0 \\
\text{or;} \quad & m' < m = k \\
\text{or;} \quad & m, m' < k \text{ and } i \geq i_r^{(m)} \text{ and } i' \geq i_r^{(m')} \text{ and } i \leq i' \text{ and } j \leq j', \\
\text{or;} \quad & m, m' < k \text{ and } i \leq i_r^{(m)} \text{ and } i' \leq i_r^{(m')} \text{ and } i \geq i' \text{ and } j \geq j',
\end{aligned}$$

where $r := j - i$ and $r' := j' - i'$. This partial order is inductive, as a result of the defining properties of fringes, and its minimal elements are those triplets (i,j,m) such that $m \in \{0,k\}$, or, $m < k$ and $(i,j) = (i_r^{(m)}, i_r^{(m)} + r)$ for some $r \in \mathbb{Z}$. For these minimal elements we have $M_{i,j,m} = 1$ and $M_{i,j,m} = x_r^{(m)}$ respectively. Let (i,j,m) be an element which is not minimal. Thus $0 < m < k$ and $i \neq i_r^{(m)}$. If $i < i_r^{(m)}$, then;

$$i + 1 \leq i_r^{(m)}$$

$$i \leq i_r^{(m)} - 1 \leq i_{r+1}^{(m)}$$

$$i + 1 \leq i_r^{(m)} \leq i_{r-1}^{(m)},$$

$$i \leq i_r^{(m)} - 1 \leq i_r^{(m+1)},$$

$$i+1 \le i_r^{(m)} \le i_r^{(m-1)}.$$

These inequalities imply respectively that each of the triplets $(i+1, j+1, m), (i, j+1, m), (i+1, j, m), (i, j, m+1), (i+1, j+1, m-1)$ is $\le (i, j, m)$, and so the equation;

$$M_{i,j,m} = \frac{M_{i,j+1,m}M_{i+1,j,m} + M_{i,j,m+1}M_{i+1,j+1,m-1}}{M_{i+1,j+1,m}}, \tag{2.9}$$

resulting from the Desnanot-Jacobi identity, writes $M_{i,j,m}$ as a fraction of positive integer polynomials on adjacent minors indexed by triplets which precede (i, j, m) in the order $\le$. Similarly, if $i > i_r^{(m)}$, then;

$$i-1 \ge i_r^{(m)}$$

$$i \ge i_r^{(m)} + 1 \ge i_{r-1}^{(m)}$$

$$i-1 \ge i_r^{(m)} \ge i_{r+1}^{(m)},$$

$$i-1 \ge i_r^{(m)} \ge i_r^{(m+1)},$$

$$i \ge i_r^{(m)} + 1 \ge i_r^{(m-1)}.$$

These inequalities imply respectively that each of the triplets $(i-1, j-1, m), (i, j-1, m), (i-1, j, m), (i-1, j-1, m+1), (i, j, m-1)$ is $\le (i, j, m)$, and so the equation;

$$M_{i,j,m} = \frac{M_{i,j-1,m}M_{i-1,j,m} + M_{i-1,j-1,m+1}M_{i,j,m-1}}{M_{i-1,j-1,m}}, \tag{2.10}$$

also resulting from the Desnanot-Jacobi identity, again writes $M_{i,j,m}$ as a fraction of positive integer polynomials on adjacent minors indexed by preceding triplets. Hence we have shown by induction that each one of these minors is uniquely determined by equation 2.8, and is a non-zero rational function with positive integer coefficients in X. In particular this is true for the minors $M_{i,j,1} = p_{ij}$ $(i, j \in \mathbb{Z})$. $\square$

The inductive argument from the uniqueness proof above fails to conclude the Laurent nonnegativity of the entries of P, due to the inconvenient fact that a quotient of nonnegative-coefficient polynomials is not necessarily nonnegative as well. For example the polynomials a^3+b^3 and $a+b$ are both nonnegative, but their quotient a^2-ab+b^2 is not. In fact, it appears that due to this type of complications, basic inductive arguments are virtually inexistent in published proofs of Laurent non negativity for systems where this phenomenon is not trivial.

The rest of this section is intended to describe a proposed combinatorial model for the minors of Conjecture 1. We prove that this model holds for $k = 2$ and conclude the Laurent non negativity statement for this case. Subsequently, in Section 2.3, in an attempt to break the spell against simple inductive arguments for Laurent non negativity, we present another proof of this phenomenon, albeit only for the entries p_{ij} (and some other particular minors), which is based entirely on the Desnanot-Jacobi identity and two of its corollaries. Although that proof does not tackle every possible minor of P, it has the advantage of providing a simple algorithm for expanding the entries of P in terms of the variables X.

Our combinatorial model is constructed from a $(k+1)$-dress, rather than a k-dress, so our first step is to consider an arbitrary k-fringe $F^{(k)}$ containing $F^{(k-1)}$, so that the sequence $g := \{F^{(m)}\}_{1\leq m\leq k}$ is a $(k+1)$-dress. A trivial way to define this k-dress is by letting the r-th handle of $F^{(k)}$ be $i_r^{(k)} := i_r^{(k-1)}$, where $i_r^{(k-1)}$ is the r-th handle of the $(k-1)$-fringe $F^{(k-1)}$. But we do not limit ourselves to this particular extension, allowing g to be any $(k+1)$-dress which coincides with f up to $F^{(k-1)}$. All of the definitions and results that follow are in terms of a generic $(k+1)$-dress $g = \{F^{(m)}\}_{1\leq m\leq k}$, although we should keep in mind that it was constructed by extending the k-dress f as just described.

Lemma 16. *Let g be a $(k+1)$-dress. For all $r \in \mathbb{Z}$, $m \in \{1, \ldots, k\}$, there exists exactly one point $(i,j) \in D_r$ such that $g(i,j) = m$.*

Proof. We just need to show that the set $D_r \cap g^{-1}(m)$ has exactly one element. Indeed:

$$\begin{aligned} D_r \cap g^{-1}(m) &= D_r \cap \left(g^{-1}(\{1,\ldots,m\}) - g^{-1}(\{1,\ldots,m\})\right) \\ &= D_r \cap \left(F^{(m)} - F^{(m-1)}\right) = F_r^{(m)} - F_r^{(m-1)}, \end{aligned}$$

where we set $F_r^{(0)} = F^{(0)} := \emptyset$. Recall that $F^{(m-1)} \subseteq F^{(m)}$ and so $F_r^{(m-1)} \subseteq F_r^{(m)}$. Furthermore, $\left|F_r^{(m-1)}\right| = m-1$ and $\left|F_r^{(m)}\right| = m$. Therefore $\left|F_r^{(m)} - F_r^{(m-1)}\right| = 1$, as

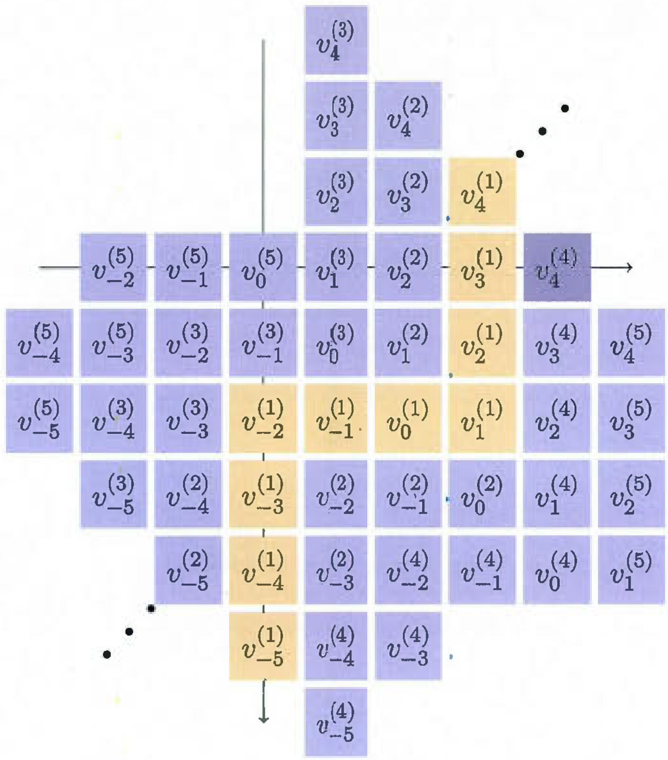


Figure 2.3 Positions of the points $v_r^{(m)}$ for the 6-dress of Figure 2.2.

wanted. $\square$

As per the Lemma above, for $r \in \mathbb{Z}$, $m \in \{1, \ldots, k\}$, let $v_r^{(m)}$ denote the only element of D_r which satisfies $g\left(v_r^{(m)}\right) = m$. We refer to the sequence $\{v_r^{(m)}\}_{\substack{r\in\mathbb{Z}\\ 1\leq m\leq k}}$ as the *defining sequence of* g.

The positions of the points $v_r^{(m)}$ $(r \in \mathbb{Z},\ 1 \leq m \leq 5)$ for the 6-dress of Figure 2.2, may be observed in Figure 2.3. The original 6-dress may be retrieved from Figure 2.3 by erasing everything but the numbers in brackets, and then filling up the rest of the plane with 6's. Although somewhat redundant, this new way of drawing k-dresses in terms of its defining sequence may be preferable to the original one of Figure 2.2, since it is now easier to locate the diagonals D_r $(r \in \mathbb{Z})$, and because we have omitted the implicit k's (in this case 6's) around $F^{(k-1)}$.

It is essential for our combinatorial model to split the complement $\mathbb{Z} \times \mathbb{Z} - F^{(1)}$ of the 1-fringe $F^{(1)}$, into two sections, namely its *lower complement* $F^{(1)-}$, consisting of all points of the plane which are strictly below $F^{(1)}$, and its *upper complement* $F^{(1)+}$, consisting of all points of $\mathbb{Z} \times \mathbb{Z}$ which are strictly above $F^{(1)}$.

Although rarely used in our work for $m \neq 1$, we extend this notation to every fringe $F^{(m)}$ $(1 \leq m \leq k)$, by letting $F^{(m)-}, F^{(m)+}$ denote the sets of all points of $\mathbb{Z} \times \mathbb{Z}$ which are strictly below or strictly above $F^{(m)}$, respectively. Furthermore, set;

$$\overline{F^{(m)-}} := F^{(m)} \cup F^{(m)-},$$

$$\overline{F^{(m)+}} := F^{(m)} \cup F^{(m)+},$$

and for all $r \in \mathbb{Z}$;

$$F_r^{(m)-} := D_r \cap F^{(m)-},$$

$$F_r^{(m)+} := D_r \cap F^{(m)+},$$

$$\overline{F_r^{(m)-}} := D_r \cap \overline{F^{(m)-}},$$

$$\overline{F_r^{(m)+}} := D_r \cap \overline{F^{(m)+}},$$

It is immediate by definition that;

$$\begin{array}{ll} F_r^{(m)-} \subseteq F_r^{(m-1)-}, & \overline{F_r^{(m)-}} \subseteq \overline{F_r^{(m-1)-}}, \\ F_r^{(m)+} \subseteq F_r^{(m-1)+}, & \overline{F_r^{(m)+}} \subseteq \overline{F_r^{(m-1)+}}, \end{array} \qquad \text{for} \quad m = 2, \ldots, k \tag{2.11}$$

Also the same relations are true if we omit the subindices r. An important fact resulting from these relations is the following:

Lemma 17. *Every $(k+1)$-dress $g = \{F^{(m)}\}_{1 \leq m \leq k}$ is non-decreasing in both coordinates within $\overline{F^{(1)-}}$. Moreover it is strictly increasing within $\overline{F_r^{(1)-}} \cap F^{(k)}$. Also it is non-increasing in both coordinates within $\overline{F^{(1)+}}$ and strictly decreasing within $\overline{F_r^{(1)+}} \cap F^{(k)}$.*

Proof. We need to show that for all $(i, j) \in \overline{F^{(1)-}}$;

$$g(i, j) \leq g(i + 1, j),$$

and

$$g(i,j) \leq g(i,j+1).$$

This is evident if $g(i,j) = 1$. Otherwise, if $g(i,j) = m > 1$, then $(i,j) \notin F^{(m-1)}$, or equivalently, $(i,j) \in F^{(m-1)-} \cup F^{(m-1)+}$. Since $F^{(m-1)+} \subseteq F^{(1)+}$ is disjoint with $\overline{F^{(1)-}}$, we deduce that $(i,j) \in F^{(m-1)-}$, which by definition implies that $(i+1,j),(i,j+1) \in F^{(m-1)-} \subseteq \mathbb{Z} \times \mathbb{Z} - F^{(m-1)} = g^{-1}(\{m, m+1, m+2, \ldots\})$. Thus $g(i+1,j) \geq m$ and $g(i,j+1) \geq m$, as wanted. Similarly we deduce that g is non-increasing within $\overline{F^{(1)+}}$.

For the strict monotonicity, we argue that g takes different values within $\overline{F_r^{(1)-}} \cap F^{(k)}$ and within $\overline{F_r^{(1)+}} \cap F^{(k)}$. In fact we claim that g takes different values within the union of these two sets. Indeed;

$$\left(\overline{F_r^{(1)-}} \cap F^{(k)}\right) \cup \left(\overline{F_r^{(1)+}} \cap F^{(k)}\right) = D_r \cap F^{(k)} = F_r^{(k)} = \{v_r^{(1)}, \ldots, v_r^{(k)}\},$$

and $g\left(v_r^{(m)}\right) = m$ for $m = 1, \ldots, k$. □

We are now ready to introduce the graphs $G^-(g)$ and $G^+(g)$, which are the main combinatorial objects from our model. The definition of these graphs relies on the notion of neighbouring points: We say that two points $v, v' \in \mathbb{Z} \times \mathbb{Z}$ are *neighbours* or *neighbouring*, if they differ by 1 at only one coordinate. For example $(1,-25)$ and $(2,-25)$ are neighbours, while $(1,2)$ and $(2,3)$ are not.

Given the $(k+1)$-dress g, define an infinite acyclic digraph $G^-(g) := (V(g), E^-(g))$ with vertex set $V(g) := \{v_r^{(m)}\}_{\substack{r\in\mathbb{Z} \\ 1\leq m\leq k}} = F^{(k)}$, so that every one of its edges goes from $F_r^{(k)}$ to $F_{r+1}^{(k)}$ for some $r \in \mathbb{Z}$, by the following rule: For $r \in \mathbb{Z}$, $m, m' \in \{1, \ldots, k\}$, the edge $v_r^{(m)} \to v_{r+1}^{(m')}$ is in $E^-(g)$ if and only if there exist $q, q' \in \{1, \ldots, k+1\}$ and neighbouring points $v \in \overline{F_r^{(1)-}}$, $v' \in \overline{F_{r+1}^{(1)-}}$ satisfying $g(v) = q$, $g(v') = q'$, such that either $q \leq m \leq m' \leq q'$ or $q \geq m \geq m' \geq q'$.

Similarly define another acyclic digraph $G^+(g) := (V(g), E^+(g))$ with the same vertex set $V(g) := \{v_r^{(m)}\}_{\substack{r\in\mathbb{Z} \\ 1\leq m\leq k}} = F^{(k)}$, but with every edge going from $F_{r+1}^{(k)}$ to $F_r^{(k)}$ for some $r \in \mathbb{Z}$, by the analogous rule: For $r \in \mathbb{Z}$, $m, m' \in \{1, \ldots, k\}$, the edge $v_{r+1}^{(m)} \to v_r^{(m')}$

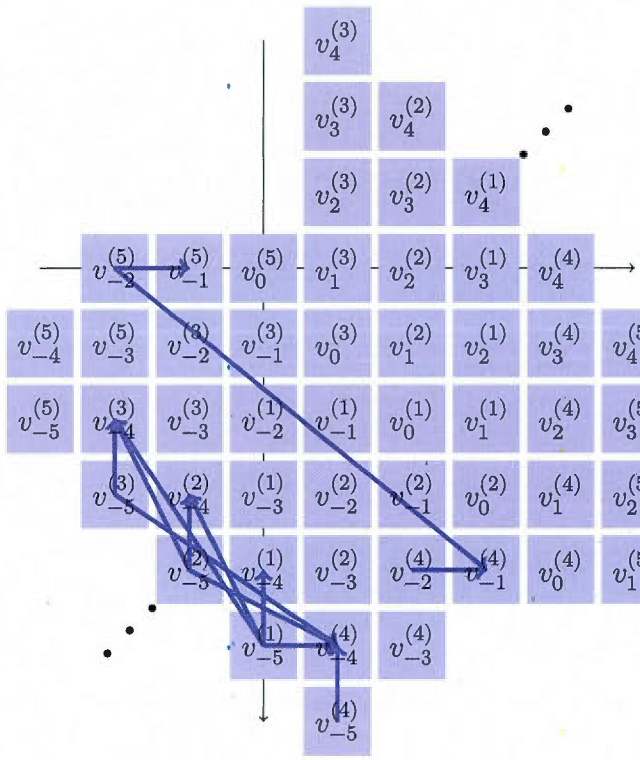


Figure 2.4 Some edges of $G^{-}(g)$ for a 6-dress g.

is in $E^{+}(g)$, if and only if there exist $q, q' \in \{1, \ldots, k+1\}$ and neighbouring points $v \in \overline{F_{r+1}^{(1)+}}$, $v' \in \overline{F_{r}^{(1)+}}$ satisfying $g(v) = q$, $g(v') = q'$, such that either $q \leq m \leq m' \leq q'$ or $q \geq m \geq m' \geq q'$.

We exemplify this construction by deducing a few edges of the digraph $G^{-}(g)$ where g is the 6-dress from Figure 2.4. The fact that $v := v_{-5}^{(1)} \in \overline{F_{-5}^{(1)-}}$ and $v' = v_{-4}^{(4)} \in \overline{F_{-4}^{(1)-}}$ are neighbours satisfying $g(v) = 1$ and $g(v') = 4$, implies that $v_{-5}^{(m)} \to v_{-4}^{(m')}$ is an edge of $G^{-}(g)$ for all m, m' with $1 \leq m \leq m' \leq 4$. These are the ten edges between $F_{-5}^{(5)}$ and $F_{-4}^{(5)}$ which are visible in Figure 2.4. Also the fact that the point $v := (5, 3) \in \overline{F_{-2}^{(1)-}}$ with $g(v) = 6$ neighbours the point $v' := v_{-1}^{(4)} \in \overline{F_{-1}^{(1)-}}$ with $g(v') = 4$, implies that $v_{-2}^{(m)} \to v_{-1}^{(m')}$ is an edge of $G^{-}(g)$ for all $m, m' \in \{1, 2, 3, 4, 5\}$ with $6 \geq m \geq m' \geq 4$. These are the three edges between $F_{-2}^{(5)}$ and $F_{-1}^{(5)}$ which are visible in the figure.

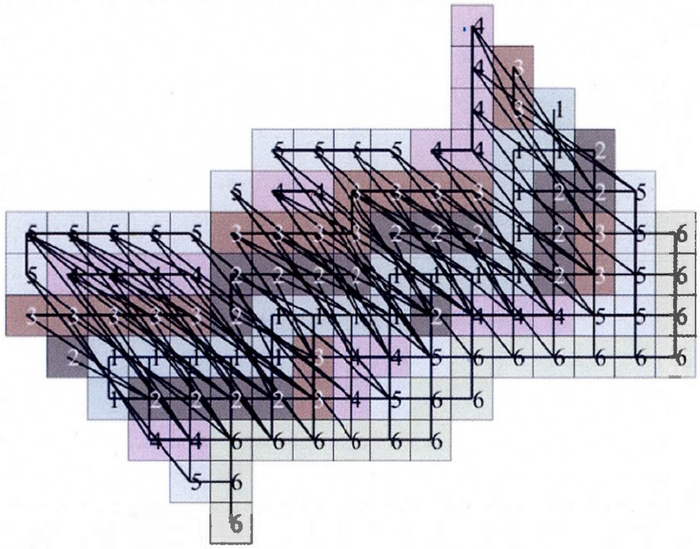


Figure 2.5 Part of a 6-dress g and the corresponding section of $G^-(g)$.

The two digraphs $G^-(g)$ and $G^+(g)$ are often far from planar, and their shape is generally very intricate, although they have some very regular properties which we review next. See Figure 2.5 for a Maple-generated example of $G^-(g)$ (arrows omitted) for certain 6-dress g.

We introduce a new simpler notation for paths of the digraphs $G^-(g), G^+(g)$ as follows;

$$\overrightarrow{\begin{pmatrix} m_r & m_{r+1} & m_{r+2} & \cdots & m_{r'} \\ r & r+1 & r+2 & \cdots & r' \end{pmatrix}} := v_r^{(m_r)} \to v_{r+1}^{(m_{r+1})} \to v_{r+2}^{(m_{r+2})} \to \cdots \to v_{r'}^{(m_{r'})} \in G^-(g)$$

$$\overleftarrow{\begin{pmatrix} m_r & m_{r+1} & m_{r+2} & \cdots & m_{r'} \\ r & r+1 & r+2 & \cdots & r' \end{pmatrix}} := v_{r'}^{(m_{r'})} \to v_{r'-1}^{(m_{r'-1})} \to v_{r'-2}^{(m_{r'-2})} \to \cdots \to v_r^{(m_r)} \in G^+(g)$$

Observe that paths of $G^+(g)$ are read from right to left in this notation. This is indicated by the arrow on top. We may often omit this arrow as long as it is obvious whether the path is in $G^-(g)$ or $G^+(g)$. For any path ρ of $G^-(g)$ or $G^+(g)$, denote by $\pi^g(\rho)$ (or simply $\pi(\rho)$ when the dress g is implicit) the set of all integers s for which there exists some $m \in \{1, \dots, k\}$ such that $v_s^{(m)} \in \rho$. In the notation above, this is just the set $\{r, r+1, \dots, r'\}$ of all integers in the second line. Moreover, for any $s \in \pi^g(\rho)$, denote by $\rho[s]$ the only number such that $v_s^{(\rho[s])} \in \rho$. In our new notation, this is the number placed immediately above s. A simple example of these notations follows;

$$\rho := \begin{pmatrix} 1 & 1 & 2 & 4 & 2 \\ -3 & -2 & -1 & 0 & 1 \end{pmatrix} = v_{-3}^{(1)} \to v_{-2}^{(1)} \to v_{-1}^{(2)} \to v_0^{(4)} \to v_1^{(2)}$$

$$\pi(\rho) = \{-3, -2, -1, 0, 1\}$$

$$\rho[-3] = \rho[-2] = 1, \quad \rho[-1] = \rho[1] = 2, \quad \rho[0] = 4.$$

We are interested in paths of $G^-(g)$ and $G^+(g)$ which start and end at some particular vertices in $F^{(1)}$, described below.

For $(a, b) \in \overline{F^{(1)-}}$, denote by;

$\blacktriangleleft(a, b)$: The rightmost point of $F^{(1)}$ which is in the same horizontal line and weakly to the left of (a, b).

$\blacktriangle(a, b)$: The bottommost point of $F^{(1)}$ which is in the same vertical line and weakly below (a, b).

For $(a, b) \in \overline{F^{(1)+}}$, denote by;

$\blacktriangleright(a, b)$: The leftmost point of $F^{(1)}$ which is in the same horizontal line and weakly to the right of (a, b).

$\blacktriangledown(a, b)$: The topmost point of $F^{(1)}$ which is in the same vertical line and weakly above (a, b).

See Figure 2.6 for examples of this notation. Notice that for each point (a, b) in the 1-fringe $F^{(1)} = \overline{F^{(1)-}} \cap \overline{F^{(1)+}}$;

$$\blacktriangleleft(a, b) = \blacktriangle(a, b) = \blacktriangleright(a, b) = \blacktriangledown(a, b). \tag{2.12}$$

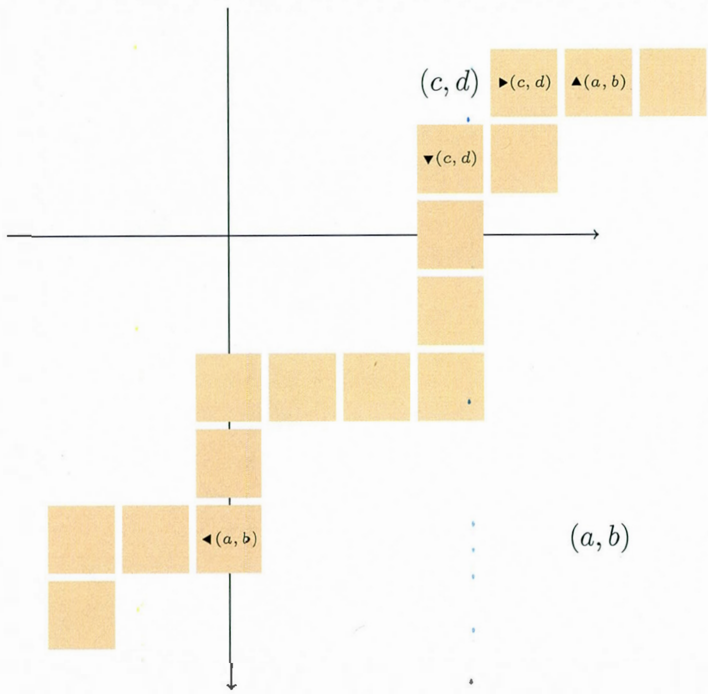


Figure 2.6 Two points $(a, b) = (4, 5)$, $(c, d) = (-2, 3)$ along with their images under $\blacktriangleleft$, $\blacktriangle$ and $\blacktriangleright$, $\blacktriangledown$ respectively, inside the 1-fringe $F^{(1)}$.

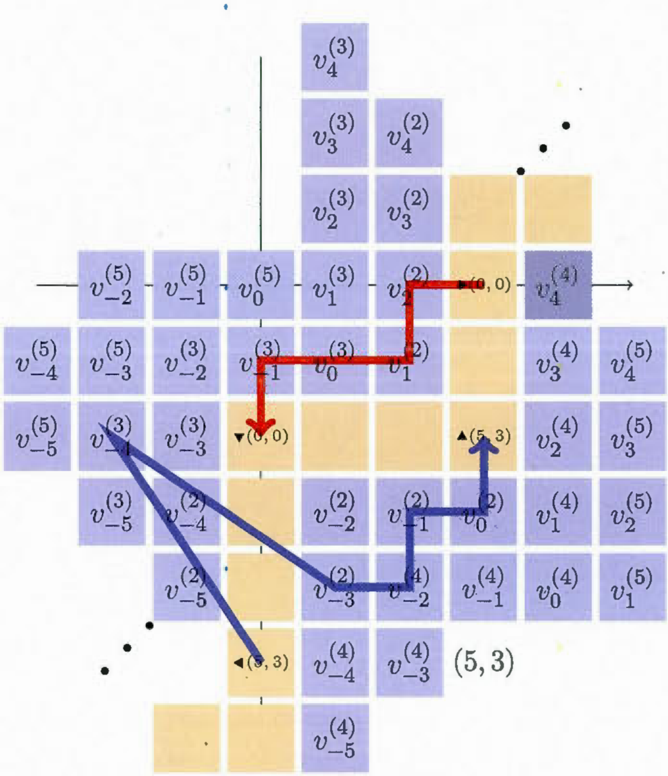


Figure 2.7 A 6-dress whose 1-fringe $F^{(1)}$ has been highlighted, along with two paths in $\nearrow(5,3)$ and $\swarrow\left(v_0^{(5)}\right) = \swarrow(0,0)$ respectively.

For $(a,b) \in \overline{F^{(1)-}}$, denote by $\nearrow(a,b)$ the set of all paths in $G^-(g)$ between $\blacktriangleleft(a,b)$ and $\blacktriangle(a,b)$ (see Figure 2.7). Similarly, for $(a,b) \in \overline{F^{(1)+}}$, denote by $\swarrow(a,b)$ the set of all paths in $G^+(g)$ between $\blacktriangleright(a,b)$ and $\blacktriangledown(a,b)$. Observe from (2.12) that for $(a,b) \in F^{(1)}$;

$$\nearrow(a,b) = \swarrow(a,b) = \{(a,b)\}, \tag{2.13}$$

where the (a,b) on the right hand side denotes the length-zero path between (a,b) and itself.

It is convenient to distinguish the vertices of $V(g)$ which are below or above $F^{(1)}$. To this end, denote;

$$V^+(g) := F^{(1)+} \cap F^{(k)},$$

$$V^-(g) := F^{(1)-} \cap F^{(k)},$$

$$\overline{V^+(g)} := \overline{F^{(1)+}} \cap F^{(k)},$$

$$\overline{V^-(g)} := \overline{F^{(1)-}} \cap F^{(k)}.$$

We now proceed to define weights w^-, w^+ for the graphs $G^-(g)$ and $G^+(g)$, over the ring $R := \mathbb{Z}[X, X^{-1}]$ of Laurent polynomials with integer coefficients in the set of algebraically independent (over the rationals) formal variables $X = \{x_r^{(m)}\}_{\substack{r\in\mathbb{Z} \\ 1\leq m\leq k-1}}$. We call them respectively the *lower weight* and the *upper weight associated to g over* $\mathbb{Z}[X, X^{-1}]$. We first define these weights on the vertex set $V(g) = F^{(k)}$, and later on the edge sets $E^-(g), E^+(g)$.

Define $y : V(g) \to R$ by;

$$y\left(v_r^{(m)}\right) := \frac{x_r^{(m)}}{x_r^{(m-1)}},$$

for $r \in \mathbb{Z}$, $m \in \{1, \ldots, k\}$, where we set $x_r^{(0)} = x_r^{(k)} := 1$ for all $r \in \mathbb{Z}$. Now define functions $z^-, z^+ : \mathbb{Z} \times \mathbb{Z} \to R$, as follows;

$$z^-(i,j) := \begin{cases} y(i,j) & \text{if} \quad (i,j) \in F^{(1)}, \\ \dfrac{y(i,j)y(i-1,j-1)^{1-\theta^-(i,j)}}{y(i-1,j)y(i,j-1)} & \text{if} \quad (i,j) \in V^-(g), \\ 1 & \text{otherwise,} \end{cases}$$

$$z^+(i,j) := \begin{cases} y(i,j) & \text{if} \quad (i,j) \in F^{(1)}, \\ \dfrac{y(i,j)y(i+1,j+1)^{1-\theta^+(i,j)}}{y(i+1,j)y(i,j+1)} & \text{if} \quad (i,j) \in V^+(g), \\ 1 & \text{otherwise,} \end{cases}$$

where $\theta^-(i,j)$ and $\theta^+(i,j)$ denote the cardinalities of the sets $F^{(1)} \cap \{(i-1,j),(i,j-1)\}$ and $F^{(1)} \cap \{(i+1,j),(i,j+1)\}$ respectively. Observe that $\theta^-(i,j), \theta^+(i,j) \in \{0,1,2\}$ for all $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. Finally for $(i,j) \in V(g)$, set;

$$w^-(i,j) = \begin{cases} \displaystyle\prod_{t=1}^{\infty} z^-(i-t, j-t) & \text{if} \quad (i,j) \in \overline{V^-(g)} \\ \dfrac{1}{w^+(i,j)} & \text{if} \quad (i,j) \in V^+(g) \end{cases} \tag{2.14}$$

$$w^{+}(i,j)=\begin{cases}\displaystyle\prod_{t=1}^{\infty} z^{+}(i+t,j+t) & \text{if} \quad (i,j)\in\overline{V^{+}(g)}\\[2ex] \dfrac{1}{w^{-}(i,j)} & \text{if} \quad (i,j)\in V^{-}(g)\end{cases}\tag{2.15}$$

Despite referencing each other, the two definitions above are not conflicting, since $V^{-}(g)\subseteq\overline{V^{-}(g)}$ and $V^{+}(g)\subseteq\overline{V^{+}(g)}$. Observe that for $(i,j)=v_r^{(1)}\in F^{(1)}$;

$$w^{-}(i,j)=w^{+}(i,j)=x_r^{(1)},\tag{2.16}$$

and in general for $(i,j)=v_r^{(m)}\in V(g)$;

$$w^{-}(i,j)w^{+}(i,j)=1.$$

We still need to define w^{-},w^{+} on edges. For any choice $\pm$ of sign, we set $w^{\pm}(e)$ to be equal to 1 whenever $e\in E^{\pm}-\left(F^{(1)}\times F^{(1)}\right)$, and we set;

$$w^{\pm}(u\to v):=\begin{cases}\dfrac{1}{y(u)} & \text{if} \quad u\to v \quad \text{is horizontal,}\\[2ex] \dfrac{1}{y(v)} & \text{if} \quad u\to v \quad \text{is vertical,}\end{cases}$$

for every edge $u\to v\in E^{\pm}\cap\left(F^{(1)}\times F^{(1)}\right)$.

As stated before, we use the weight w^{-} for paths of $G^{-}(g)$, and the weight w^{+} for paths of $G^{+}(g)$. More precisely we are interested in the weighted sums $|\nearrow(a,b)|_{w^{-}}$ $\left((a,b)\in\overline{F^{(1)-}}\right)$ and $|\nearrow(a,b)|_{w^{+}}$ $\left((a,b)\in\overline{F^{(1)+}}\right)$. For $(i,j)\in\mathbb{Z}\times\mathbb{Z}$ we conveniently denote;

$$\mathcal{P}^{g}(i,j):=\begin{cases}\nearrow(i,j) & \text{if} \quad (i,j)\in\overline{F^{(1)-}},\\ \swarrow(i,j) & \text{if} \quad (i,j)\in\overline{F^{(1)+}},\end{cases}$$

and for each $\rho\in\mathcal{P}^{g}(i,j)$ we set;

$$w(\rho):=\begin{cases}w^{-}(\rho) & \text{if} \quad (i,j)\in\overline{F^{(1)-}},\\ w^{+}(\rho) & \text{if} \quad (i,j)\in\overline{F^{(1)+}},\end{cases}$$

Equations (2.13) and (2.16) ensure that the set $\mathcal{P}^{g}(i,j)$ $((i,j)\in\mathbb{Z}\times\mathbb{Z})$ and the weight $w(\rho)$ $(\rho\in\mathcal{P}^{g}(i,j))$ are well defined. Following the same weighted cardinality notation from before, we denote;

$$|\mathcal{P}^{g}(i,j)|_{w}:=\sum_{\rho\in\mathcal{P}^{g}(i,j)} w(\rho),$$

for $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. Note that w is a weight on paths, and unlike w^-,w^+, it is not well defined on vertices or edges. We call w the *weight associated to g over $R = \mathbb{Z}[X, X^{-1}]$*.

Next we state a new conjecture relating the matrix from Conjecture 1 and the combinatorial objects described above. We then proceed to state some results which will lead to the proof of both conjectures for the case $k = 2$.

Conjecture 2. *Let f be a k-dress and let g be a $(k+1)$-dress obtained from g by adding a k-fringe $F^{(k)}$. Then the matrix*

$$P^{X,g} := [|\mathcal{P}^g(a,b)|_w]_{(a,b)\in\mathbb{Z}\times\mathbb{Z}},$$

satisfies the determinantal equations from Conjecture 1, including the SL_k*-condition (the condition that every $k \times k$ connected minor of $P^{X,g}$ is equal to 1).*

To show Conjectures 1 and 2 for $k = 2$ it is necessary to introduce a broader notion of intersection of paths from

$$\mathcal{P}^g := \bigcup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} \mathcal{P}^g(a,b).$$

We introduce this broader notion later, and proceed now to state some general properties of the graphs $G^-(g)$,$G^+(g)$ and their paths, for all values of k. The following property, which we use often, is immediate from the definition of these graphs.

Lemma 18. *Let g be the $(k+1)$-dress with defining sequence $\{v_r^{(m)}\}_{\substack{r\in\mathbb{Z}\\ 1\leq m\leq k}}$. Suppose that $m, m' \in \{1,\ldots,k\}$ and let l, l' be integers such that either $m \leq l \leq l' \leq m'$ or $m \geq l \geq l' \geq m'$. If $\overrightarrow{\left(\begin{smallmatrix} m & m' \\ r & r+1 \end{smallmatrix}\right)} \in E^-(g)$, then $\overrightarrow{\left(\begin{smallmatrix} l & l' \\ r & r+1 \end{smallmatrix}\right)} \in E^-(g)$. Similarly, if $\overleftarrow{\left(\begin{smallmatrix} m & m' \\ r & r+1 \end{smallmatrix}\right)} \in E^+(g)$, then $\overleftarrow{\left(\begin{smallmatrix} l & l' \\ r & r+1 \end{smallmatrix}\right)} \in E^-(g)$.*

A less immediate property is the following:

Lemma 19. *Let g be the $(k+1)$-dress with defining sequence $\{v_r^{(m)}\}_{\substack{r\in\mathbb{Z}\\ 1\leq m\leq k}}$. For all $r \in \mathbb{Z}$, $m \in \{1,\ldots,k\}$;*

$$\overrightarrow{\begin{pmatrix} m & m \\ r & r+1 \end{pmatrix}} \in G^-(g)$$

$$\overleftarrow{\begin{pmatrix} m & m \\ r & r+1 \end{pmatrix}} \in G^+(g)$$

Proof. We show this only for $G^-(g)$, as the result for $G^+(g)$ is analogous. Consider the set $A := \overline{F^{(1)-}} \cap (D_r \cup D_{r+1})$. Order the elements $v_1, v_2, v_3, \ldots$ of A increasingly by the sum of their coordinates. Thus for all $i \geq 1$; The neighbouring points v_i, v_{i+1} are on different diagonals among D_r, D_{r+1}, and satisfy $g(v_i) \leq g(v_{i+1})$ (since g is non-decreasing within $\overline{F^{(1)-}}$). Moreover, $g(v_1) = 1$, and for n large enough, $g(v_n) = k+1$. Hence for all $m \in \{1, \ldots, k\}$, there exists $i \geq 1$ such that $g(v_i) \leq m \leq g(v_{i+1})$. More precisely, there exist q, q' such that $q \leq m \leq q'$ and either $v_r^{(q)}, v_{r+1}^{(q')}$ or $v_{r+1}^{(q)}, v_r^{(q')}$ are neighbours. In either case we obtain by definition that $v_r^{(m)} \to v_{r+1}^{(m)} \in G^-(g)$, as wanted. □

Using the same idea we get the next two lemmas;

Lemma 20. *Let g be a $(k+1)$-dress. For all $r \in \mathbb{Z}$, $l, l', m, m' \in \{1, \ldots, k\}$ with either $l < m < l' < m'$ or $l > m > l' > m'$, and for any fixed choice of sign $\pm$;*

$$\begin{pmatrix} l & l' \\ r & r+1 \end{pmatrix}, \begin{pmatrix} m & m' \\ r & r+1 \end{pmatrix} \in G^{\pm}(g) \Leftrightarrow \begin{pmatrix} l & m' \\ r & r+1 \end{pmatrix}, \begin{pmatrix} m & l' \\ r & r+1 \end{pmatrix} \in G^{\pm}(g).$$

Lemma 21. *Let g be a $(k+1)$-dress. For all $r \in \mathbb{Z}$, $l, l', m, m' \in \{1, \ldots, k\}$ with $l < l'$, $m > m'$, and*

$$|\{l, l+1, \ldots, l'\} \cap \{m, m-1, \ldots, m'\}| \geq 2,$$

and for any fixed choice of sign $\pm$, the two statements below cannot happen simultaneously;

$$\begin{pmatrix} l & l' \\ r & r+1 \end{pmatrix} \in G^{\pm}(g),$$

$$\begin{pmatrix} m & m' \\ r & r+1 \end{pmatrix} \in G^{\pm}(g),$$

Some of the results that follow are proven in the next section, as a way to keep this section lighter and easier to read.

Lemma 22 (Proof in page 106). *Let g be a $(k+1)$-dress. The restriction of $G^-(g)$ to $\overline{V^-(g)}$ is the north-east lattice graph on $\overline{V^-(g)}$. Similarly, the restriction of $G^+(g)$ to $\overline{V^+(g)}$ is the south-west lattice graph on $\overline{V^+(g)}$.*

Lemma 23 (Proof in page 110). *Let g be a $(k+1)$-dress. If two points of $\overline{V^-(g)}$ are in the same horizontal line or in the same vertical line, then there is exactly one path in $G^-(g)$ joining them. Similarly, if two points of $\overline{V^+(g)}$ are in the same horizontal line or in the same vertical line, then there is exactly one path in $G^+(g)$ joining them. In either case, this unique path is a straight (horizontal or vertical) segment between the two points.*

Lemma 24. *Let g be a $(k+1)$-dress. For $(a,b) \in V(g)$, the only path in $\mathcal{P}^g(a,b)$ which visits (a,b) is the the path ρ satisfying that $\rho(\rightarrow (a,b))$ is horizontal and $\rho((a,b)\rightarrow)$ is vertical.*

Proof. This is a direct result of Lemma 23, since for each $(a,b) \in \overline{V^-(g)}$, the two points $(a,b), \blacktriangleleft(a,b) \in \overline{V^-(g)}$ are in the same horizontal line, while the two points $(a,b), \blacktriangle(a,b) \in \overline{V^-(g)}$ are in the same vertical line. Also for $(a,b) \in \overline{V^+(g)}$, the two points $(a,b), \blacktriangleright(a,b) \in \overline{V^-(g)}$ are in the same horizontal line, and the two points $(a,b), \blacktriangledown(a,b) \in \overline{V^-(g)}$ are in the same vertical line. $\square$

Lemma 25 (Proof in page 112). *Let g be a $(k+1)$-dress. Let ρ be a path in $\mathcal{P}^g(a,b)$ for some $(a,b) \in \mathbb{Z}\times\mathbb{Z}$. Then $g(c,d) \leq g(a,b)$ for all $(c,d) \in \rho$.*

We are ready to introduce our broader notion of path intersection between paths of $\mathcal{P}^g = \cup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}}\mathcal{P}^g(a,b)$. We do this in three different definitions. The first one is the usual intersection (sharing one common vertex) with an additional restriction. The other two notions, namely those of *crossing paths* and *bonding* paths, are remarkably different.

We need some new notation. For $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, observe that for $\rho \in \mathcal{P}^g(a,b)$, the set $\pi^g(\rho)$ depends only on (a,b). We thus use the notation $\pi^g(a,b)$ for this set. Recall that the set $\pi^g(a,b)$ is an integer interval (a sequence of consecutive integers). More precisely: If either $(a,b) \in \overline{F^{(1)-}}$, $(i,j) = \blacktriangleleft(a,b)$, $(i',j') = \blacktriangle(a,b)$, or $(a,b) \in \overline{F^{(1)-}}$, $(i,j) = \blacktriangledown(a,b)$, $(i',j') = \blacktriangleright(a,b)$, then;

$$\pi^g(a,b) := \{j-i, j-i+1, j-i+2, \ldots, j'-i'\}.$$

For example, for the points $(a,b) = (4,5)$, $(c,d) = (-2,3)$ of Figure 2.6, we have $\pi^g(a,b) = \{-4,-3,-2,-1,0,1,2,3,4,5,6,7\}$ and $\pi^g(c,d) = \{4,5,6\}$. Observe that $\pi^g(a,b) = \{b-a\}$ for all $(a,b) \in F^{(1)}$. Furthermore;

$$|\pi^g(a,b)| = 1 \Leftrightarrow |\pi^g(a,b)| \leq 2 \Leftrightarrow (a,b) \in F^{(1)}.$$

Definition 25. Let $g = \{F^{(m)}\}_{1 \leq m \leq k}$ be a $(k+1)$-dress. Let $(a,b),(c,d)$ be points in $\mathbb{Z} \times \mathbb{Z}$. Suppose that $|\pi^g(a,b) \cap \pi^g(c,d)| \geq 2$ and either; $(a,b),(c,d) \in F^{(1)-}$, or; $(a,b),(c,d) \in F^{(1)+}$. Then two paths $\rho \in \mathcal{P}^g(a,b)$, $\eta \in \mathcal{P}^g(c,d)$ are said to be *intersecting* if they share a common vertex. If they intersect at a point $v_r^{(m)}$ for some $m \in \{1,\ldots,k\}$, then we say that they *intersect in the diagonal* $D_r = \{(i,j) : j-i = r\}$. See Figure 2.8 for an example of two intersecting paths.

Definition 26. Let $g = \{F^{(m)}\}_{1 \leq m \leq k}$ be the $(k+1)$-dress with defining sequence $\{v_r^{(m)}\}_{\substack{r \in \mathbb{Z} \\ 1 \leq m \leq k}}$. Suppose that $(a,b),(c,d) \in F^{(1)\pm}$ for some fixed choice $\pm$ of sign. Then two paths $\rho \in \mathcal{P}^g(a,b)$, $\eta \in \mathcal{P}^g(c,d)$ are said to be (mutually) *crossing* if there exist integers $r \in \mathbb{Z}$, $l, l', m, m' \in \{1,\ldots,k\}$, satisfying the following conditions;

- $\begin{pmatrix} l & l' \\ r & r+1 \end{pmatrix} \in \rho$,
- $\begin{pmatrix} m & m' \\ r & r+1 \end{pmatrix} \in \eta$,
- $\begin{pmatrix} l & m' \\ r & r+1 \end{pmatrix}, \begin{pmatrix} m & l' \\ r & r+1 \end{pmatrix} \in G^{\pm}(g)$.

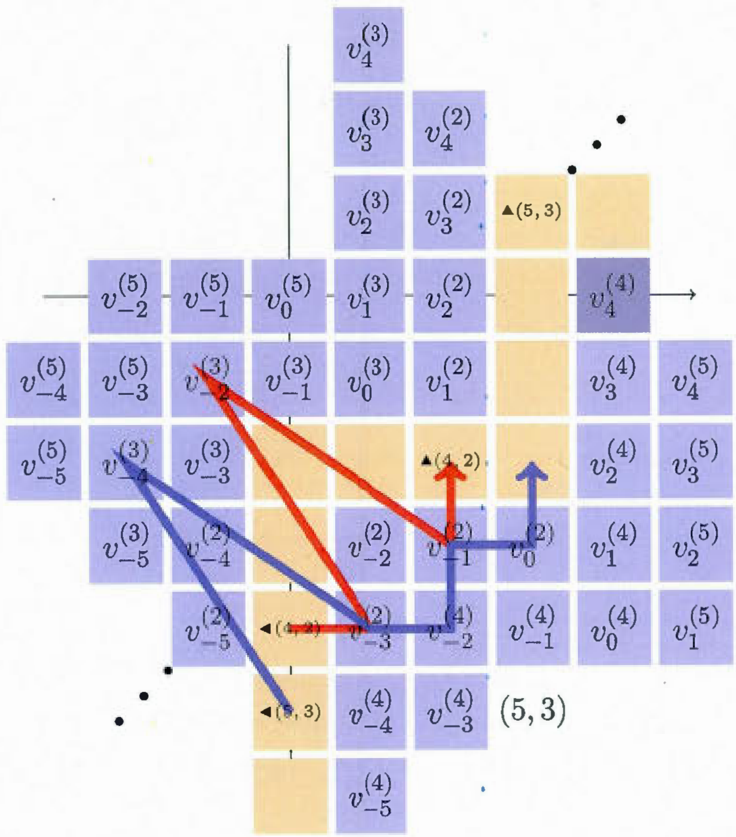


Figure 2.8 Intersecting paths in $\rho \in \mathcal{P}^g(4,2) = \mathcal{J}(4,2)$ and $\eta \in \mathcal{P}^g(5,3) = \mathcal{J}(5,3)$. These paths intersect at $v_{-3}^{(2)} = (4,1)$ in D_{-3} and at $v_{-1}^{(2)} = (3,2)$ in D_{-1}.

We say that these two paths *cross between* D_r *and* D_{r+1}. See Figure 2.9 for an example of a pair of crossing paths.

Definition 27. Let $g = \{F^{(m)}\}_{1\leq m\leq k}$ be the $(k+1)$-dress with defining sequence $\{v_r^{(m)}\}_{\substack{r\in\mathbb{Z}\\ 1\leq m\leq k}}$. Suppose that $(a,b),(c,d) \in \mathbb{Z}\times\mathbb{Z}$. We say that two paths $\rho \in \mathcal{P}^g(a,b)$, $\eta \in \mathcal{P}^g(c,d)$ are *bonding*, if $v_r^{(1)} \in \rho, \eta$ (equivalently $\rho[r] = \eta[r] = 1$) for all $r \in \pi^g(a,b) \cap \pi^g(c,d)$. In other words, the paths ρ and η remain within $F^{(1)}$ across all the diagonals D_r which they both visit. See figure 2.10 for an example of bonding paths.

Definition 28. Let g be a $(k+1)$-dress. A tuple $(\rho_1,\dots,\rho_n)$ of paths from the set $\mathcal{P}^g = \cup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}}\mathcal{P}^g(a,b)$ is said to be *non-intersecting, non-crossing* or *not-bonding*, if no pair of paths among $\rho_1,\dots,\rho_n$ is intersecting, crossing, or bonding, respectively.

Lemma 26 (Proof in page 125). *Let $P^{X,g}$ be as in Conjecture 2. Let a,b,c,d be integers with $a<c$, $b<d$. The minor;*

$$\det(P^{X,g}_{\{a,c\}\{b,d\}}),$$

is equal to the weighted sum of all non-intersecting, non-crossing and non-bonding pairs $(\rho,\eta) \in \mathcal{P}^g(a,b)\times\mathcal{P}^g(c,d)$.

Theorem 7 (Proof in page 125). *Conjectures 1 and 2 hold for $k=2$. More formally; let $F^{(1)}$ be a 1-fringe with handles $\left\{i_r^{(1)}\right\}_{r\in\mathbb{Z}}$, and let $X = \left\{x_r^{(1)}\right\}_{r\in\mathbb{Z}}$ be a family of algebraically independent (over the rationals) formal variables. There exists an unique SL_2-tiling $P = [p_{ij}]_{i,j\in\mathbb{Z}}$ whose entries are rational functions in X, such that for each $r \in \mathbb{Z}$;*

$$p_{\left(i_r^{(1)}\right)\left(i_r^{(1)}+r\right)} = x_r^{(1)}. \tag{2.17}$$

Moreover, each entry and each 2×2 minor of this SL_2-tiling is a non-zero Laurent polynomial with non-negative integer coefficients in X.

2.2 Proofs of results from section 2.1

We note, since we often use arguments based on symmetries, that each one of the defining properties of fringes is invariant under any translation of the points of F, and

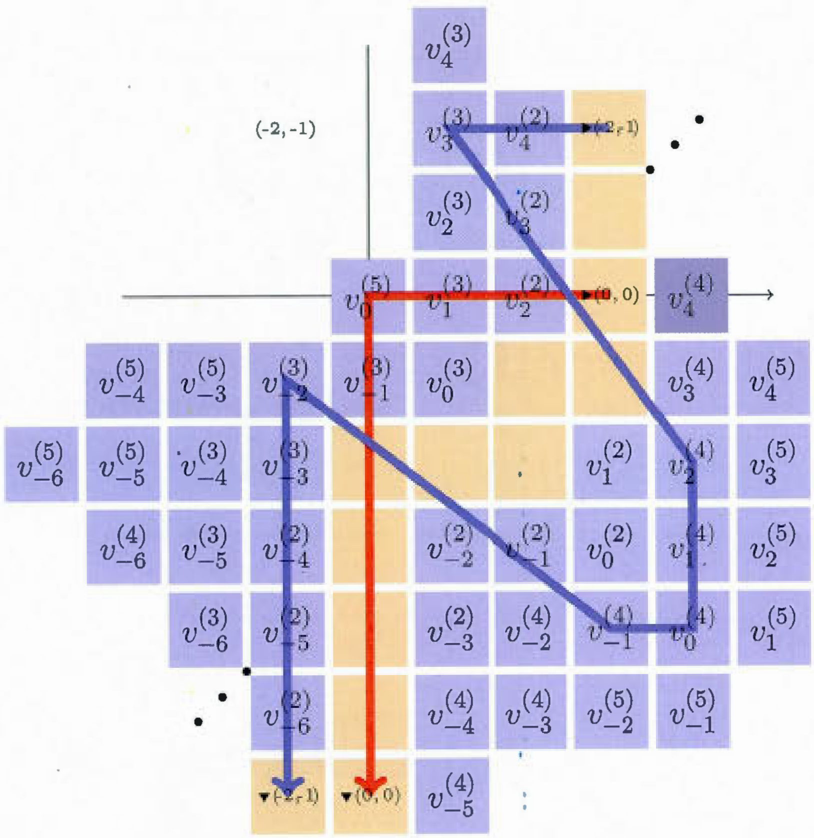


Figure 2.9 Drifting paths $\rho \in \mathcal{P}^g(-2,-1) = \swarrow(-2,-1)$ and $\eta \in \mathcal{P}^g(0,0) = \swarrow(0,0)$. These paths cross between D_1 and D_0 because $v_1^{(4)} \to v_0^{(4)} \in \rho$, $v_1^{(3)} \to v_0^{(5)} \in \eta$, $v_1^{(4)} \to v_0^{(5)} \in G^+(g)$, $v_1^{(3)} \to v_0^{(4)} \in G^+(g)$. Similarly they cross between D_0 and D_{-1}. They do not cross between D_{-1} and D_{-2}, since the condition $v_{-1}^{(4)} \to v_{-2}^{(1)} \in G^-(g)$ does not hold (all the other conditions hold). Observe that these paths are not intersecting.

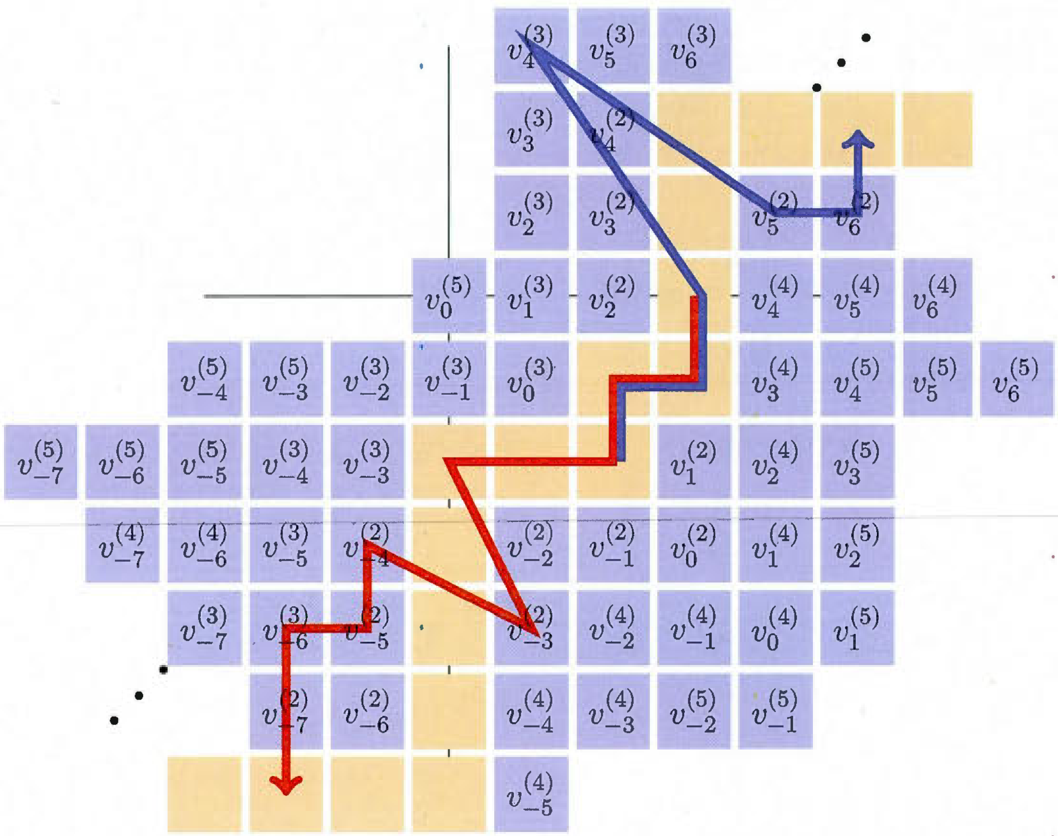


Figure 2.10 Cuddling paths $\rho \in \mathcal{P}^g(2,5) = \nearrow(2,5)$ and $\eta \in \mathcal{P}^g(0,-2) = \swarrow(0,-2)$.

also under the transposition $(i,j) \mapsto (j,i)$ and the 180° rotation $(i,j) \mapsto (-i,-j)$. We denote by $\operatorname{Tr} F$ and $\mathrm{R}_{180^\circ} F$ the transpose and the 180° rotation of F, respectively. We denote the r-th diagonal of $\operatorname{Tr} F$ by $[\operatorname{Tr} F]_r$. This must not be confused with $\operatorname{Tr} F_r$, which is just the transpose of the set F_r. Similarly we denote the r-th diagonal of $\mathrm{R}_{180^\circ} F$ by $[\mathrm{R}_{180^\circ} F]_r$. It is not difficult to see that

$$[\operatorname{Tr} F]_r = \operatorname{Tr} F_{-r} = \{(j,i) : (i,j) \in F_{-r}\},$$

and that

$$[\mathrm{R}_{180^\circ} F]_r = \mathrm{R}_{180^\circ} F_{-r} = \{(-i,-j) : (i,j) \in F_{-r}\}.$$

Let $[\operatorname{Tr} F]^+$ and $[\operatorname{Tr} F]^-$ denote respectively the upper and lower complement of the m-fringe $\operatorname{Tr} F$. Similarly define $[\mathrm{R}_{180^\circ} F]^+$ and $[\mathrm{R}_{180^\circ} F]^-$, and naturally set $\overline{[\operatorname{Tr} F]^+} := (\operatorname{Tr} F) \cup [\operatorname{Tr} F]^+$, $\overline{[\operatorname{Tr} F]^-} := (\operatorname{Tr} F) \cup [\operatorname{Tr} F]^-$, $\overline{[\mathrm{R}_{180^\circ} F]^+} := (\mathrm{R}_{180^\circ} F) \cup [\mathrm{R}_{180^\circ} F]^+$, and $\overline{[\mathrm{R}_{180^\circ} F]^-} := (\mathrm{R}_{180^\circ} F) \cup [\mathrm{R}_{180^\circ} F]^-$. Observe that;

$$[\operatorname{Tr} F]^- = \operatorname{Tr} F^-,$$

$$[\operatorname{Tr} F]^+ = \operatorname{Tr} F^+,$$

$$[\mathrm{R}_{180^\circ} F]^- = \mathrm{R}_{180^\circ} F^+,$$

$$[\mathrm{R}_{180^\circ} F]^+ = \mathrm{R}_{180^\circ} F^-.$$

Moreover;

$$\overline{[\operatorname{Tr} F]^-} = \operatorname{Tr} \overline{F^-},$$

$$\overline{[\operatorname{Tr} F]^+} = \operatorname{Tr} \overline{F^+},$$

$$\overline{[\mathrm{R}_{180^\circ} F]^-} = \mathrm{R}_{180^\circ} \overline{F^+},$$

$$\overline{[\mathrm{R}_{180^\circ} F]^+} = \mathrm{R}_{180^\circ} \overline{F^-}.$$

For a $(k+1)$-dress $g := \{F^{(m)}\}_{1 \le m \le k}$ new $(k+1)$-dresses;

$$\operatorname{Tr} g := \{\operatorname{Tr} F^{(m)}\}_{1 \le m \le k},$$

$$\mathrm{R}_{180^\circ} g := \{\mathrm{R}_{180^\circ} F^{(m)}\}_{1 \le m \le k},$$

These transformations correspond to the functional compositions $g \circ \mathrm{Tr}$ and $g \circ \mathrm{R}_{180^\circ}$, since both Tr and R_{180° are their own inverse.

We may also apply these transformations to graphs. For any graph G with vertices in $\mathbb{Z} \times \mathbb{Z}$, define $\mathrm{Tr}\, G$ and $\mathrm{R}_{180^\circ}\, G$ respectively to be the graphs resulting from relabelling the vertices of G by the corresponding transformation, and denote by $\mathrm{rev}\, G$ the graph resulting from reversing the direction of each one of G's edges. A careful yet straightforward examination of the definitions of the graphs $G^-(g)$ and $G^+(g)$ reveals that;

$$G^-(\mathrm{Tr}\, g) = \mathrm{rev}\,\mathrm{Tr}\, G^-(g)$$

$$G^+(\mathrm{Tr}\, g) = \mathrm{rev}\,\mathrm{Tr}\, G^+(g)$$

$$G^-(\mathrm{R}_{180^\circ}\, g) = \mathrm{R}_{180^\circ}\, G^+(g)$$

$$G^+(\mathrm{R}_{180^\circ}\, g) = \mathrm{R}_{180^\circ}\, G^-(g)$$

Now let $G^{NE}(g)$ denote the graph with vertex set $\overline{V^-(g)}$ and whose edges are all the unit north steps and all the unit east steps between vertices of $\overline{V^-(g)}$. Analogously, let $G^{SW}(g)$ denote the graph with vertex set $\overline{V^+(g)}$ and whose edges are all the unit south steps and all the unit west steps between vertices of $\overline{V^+(g)}$. We have;

$$G^{NE}(\mathrm{Tr}\, g) = \mathrm{rev}\,\mathrm{Tr}\, G^{NE}(g)$$

$$G^{SW}(\mathrm{Tr}\, g) = \mathrm{rev}\,\mathrm{Tr}\, G^{SW}(g)$$

$$G^{NE}(\mathrm{R}_{180^\circ}\, g) = \mathrm{R}_{180^\circ}\, G^{SW}(g)$$

$$G^{SW}(\mathrm{R}_{180^\circ}\, g) = \mathrm{R}_{180^\circ}\, G^{NE}(g)$$

Proof of Lemma 22 (Section 2.1, page 99). Consider any $(i,j) = v_r^{(m)} \in V^-(g)$. Assume that $(i,j+1), (i-1,j) \in V^-(g)$ (all the other cases, for example $(i,j+1) \in V^-(g)$, $(i-1,j) \notin V^-(g)$, are quite similar). We need to show that the only edges of $G^-(g)$ departing from (i,j) and ending within $\overline{V^+(g)}$, are $(i,j) \to (i,j+1)$ and $(i,j) \to (i-1,j)$. These are in fact edges of $G^-(g)$ by definition (simply set $v := (i,j)$, $v' \in \{(i,j+1),(i-1,j)\}$, $q := m = g(v)$ and $q' = m' := g(v')$).

Suppose that for some $m' \in \{1,\ldots,k\}$, the vertex $v_{r+1}^{(m')}$ is in $V^-(g)$ and the edge $v_r^{(m)} \to v_{r+1}^{(m')}$ is in $E^-(g)$. We need to prove that $v_{r+1}^{(m')}$ is either $(i,j+1)$ or $(i-1,j)$, or equivalently, that m' is either $g(i,j+1)$ or $g(i-1,j)$. By definition of $G^-(g)$ we know that there exist $q,q' \in \{1,\ldots,k+1\}$ with either $q \le m \le m' \le q'$ or $q \ge m \ge m' \ge q'$, such that $g(v) = q$, $g(v') = q'$ for some neighbouring vertices $v \in \overline{F_r^{(1)-}}$, $v' \in \overline{F_{r+1}^{(1)-}}$. Clearly there is some integer t such that $v = (i+t, j+t)$, and $v' \in \{(i+t, j+t+1), (i+t-1, j+t)\}$.

Let us first consider the case $v' = (i+t, j+t+1)$:

If $t < 0$, then $g(i+t,j+t) \le g(i+t,j+t+1) \le g(i-1,j) \le g(i,j)$ (since g is non-decreasing within $\overline{F^{(1)-}}$) and so $q \le q' \le g(i-1,j) \le m \le m'$. Thus $m = m' = q' = g(i-1,j)$, as wanted.

If $t > 0$, then $g(i,j) \le g(i,j+1) \le g(i+t,j+t) \le g(i+t,j+t+1)$ and so $m \le g(i,j+1) \le q \le q'$. Thus $m = q = g(i,j+1)$, which means that $t = 0$, a contradiction.

If $t = 0$, then $m = q = g(i,j) \le g(i,j+1) = q'$ and so $g(i,j) \le m' \le g(i,j+1)$. Thus $g(i-1,j) \le m' \le g(i,j+1)$. This means that $v_{r+1}^{(m')}$ is located between $(i-1,j)$ and $(i,j+1)$, but these are consecutive points of D_{r+1}. Hence $v_{r+1}^{(m')}$ is equal to one of them, as wanted.

Let us now consider the case $v' = (i+t-1, j+t)$:

If $t < 0$, then $g(i+t-1,j+t) \le g(i+t,j+t) \le g(i-1,j) \le g(i,j)$ and so $q' \le q \le g(i-1,j) \le m$. Thus $m = q' = g(i-1,j)$, which means that $t = 0$, a contradiction.

If $t > 0$, then $g(i,j) \le g(i,j+1) \le g(i+t-1,j+t) \le g(i+t,j+t)$ and so $m' \le m \le g(i,j+1) \le q' \le q$. Thus $m = m' = q' = g(i,j+1)$, as wanted.

If $t = 0$, then $m = q = g(i,j) \ge g(i-1,j) = q'$, and so $g(i,j) \ge m' \ge g(i-1,j)$. Thus $g(i,j+1) \ge m' \ge g(i-1,j)$. This means that $v_{r+1}^{(m')}$ is located between $(i,j+1)$ and $(i-1,j)$, so as before, it must be equal to one of them, as wanted.

We have shown that the restriction of $G^-(g)$ to $\overline{V^-(g)}$ is equal to $G^{NE}(g)$. We now show that the restriction of $G^+(g)$ to $\overline{V^+(g)}$ is equal to $G^{SW}(g)$. Recall that $G^-(\mathrm{R}_{180^\circ}\, g) = \mathrm{R}_{180^\circ}\, G^+(g)$, $G^{NE}(\mathrm{R}_{180^\circ}\, g) = \mathrm{R}_{180^\circ}\, G^{SW}$, and $\overline{V^-(\mathrm{R}_{180^\circ}\, g)} = \mathrm{R}_{180^\circ}\, \overline{V^+(g)}$. Thus the proven claim for the $(k+1)$-dress $\mathrm{R}_{180^\circ}\, g$ reads;

The restriction of $\mathrm{R}_{180^\circ}\, G^+(g)$ to $\mathrm{R}_{180^\circ}\, \overline{V^-(g)}$ is equal to $\mathrm{R}_{180^\circ}\, G^{SW}(g)$.

Equivalently;

The restriction of $G^+(g)$ to $\overline{V^-(g)}$ is equal to $G^{SW}(g)$,

as wanted. ☐

The following Lemma is helpful in the proofs that succeed it.

Lemma 27. *Let i, j, l, r be integers with $g(i, i+r) < g(j, j+r)$ (respectively $\leq$). If $(j, j+r) \in \overline{V^-(g)}$ and $(i, i+r) \to (l, l+r+1) \in E^-(g)$, then $g(l, l+r+1) < g(j, j+r+1)$ (respectively $\leq$). Similarly, if $(j, j+r) \in \overline{V^+(g)}$ and $(i, i+r) \to (l, l+r-1) \in E^+(g)$, then $g(l, l+r-1) < g(j, j+r-1)$ (respectively $\leq$).*

Proof. Let us show the first claim. We may assume that

$$(j, j+r+1) \in V(g) = F^{(k)},$$

since otherwise we would easily obtain $g(j, j+r+1) = k+1 > k \geq g(l, l+r+1)$. Set $m := g(i, i+r)$ and $m' := g(l, l+r+1)$, so that $(i, i+r) = v_r^{(m)}$ and $(l, l+r+1) = v_{r+1}^{(m')}$. If $m' \leq m$, then;

$$g(l, l+r+1) \leq g(i, i+r) < g(j, j+r) \leq g(j, j+r+1),$$

respectively,

$$g(l, l+r+1) \leq g(i, i+r) \leq g(j, j+r) \leq g(j, j+r+1),$$

because g is non-decreasing within $\overline{F^{(1)-}}$. Otherwise, if $m' > m$, there exist neighbouring vertices $v \in \overline{F_r^{(1)-}}$, $v' \in \overline{F_{r+1}^{(1)-}}$ with $g(v) = q \leq m < m' \leq q' = g(v')$. The inequality $g(v) < g(v')$ implies that v' is to the right (not above) v, so there must be an integer t

such that $v = (j+t, j+t+r)$ and $v' = (j+t, j+t+r+1)$. Moreover, t must be < 0 (respectively ≤ 0), since otherwise the following contradiction would arise;

$$m = g(i, i+r) < g(j, j+r) \leq g(j+t, j+t+r) = q \leq m,$$

respectively,

$$m = g(i, i+r) <\leq g(j, j+r) < g(j+t, j+t+r) = q \leq m.$$

Therefore

$$g(j+t, j+t+r) \leq g(j+t, j+t+r+1) < g(j, j+r+1),$$

respectively,

$$g(j+t, j+t+r) \leq g(j+t, j+t+r+1) \leq g(j, j+r+1)$$

(since g is strictly increasing within $D_r \cap \overline{V^-(g)} = \overline{F_r^{(1)-}} \cap F^{(k)}$), and so;

$$q \leq q' < g(j, j+r+1),$$

respectively,

$$q \leq q' \leq g(j, j+r+1),$$

which implies that $m' < g(j, j+r+1)$, respectively $m' \leq g(j, j+r+1)$, as wanted.

Finally we deduce the second claim from the first one by considering the $(k+1)$-dress $\mathrm{R}_{180^\circ}\, g$. Indeed the first claim for this dress reads;

Let i, j, l, r be integers with $g \circ \mathrm{R}_{180^\circ}\,(i, i+r) < g \circ \mathrm{R}_{180^\circ}\,(j, j+r)$ (respectively $\leq$).
If $(j, j+r) \in \mathrm{R}_{180^\circ}\,\overline{V^+(g)}$ and $(i, i+r) \to (l, l+r+1) \in \mathrm{R}_{180^\circ}\,E^+(g)$,
then $g \circ \mathrm{R}_{180^\circ}\,(l, l+r+1) < g \circ \mathrm{R}_{180^\circ}\,(j, j+r+1)$ (respectively $\leq$)

This is the same as;

Let i, j, l, r be integers with $g(-i, -i-r) < g(-j, -j-r)$ (respectively $\leq$).
If $(-j, -j-r) \in \overline{V^+(g)}$ and $(-i, -i-r) \to (-l, -l-r-1) \in E^+(g)$,
then $g(-l, -l-r-1) < g(-j, -j-r-1)$ (respectively $\leq$),

which is equivalent to the second statement under the map $(i, j, l, r) \mapsto (-i, -j, -l, -r)$.

□

Proof of Lemma 23 (Section 2.1, page 99). Let us show the first claim (the one involving points of $\overline{V^-(g)}$ and paths in $G^-(p)$). Recall that the restriction of $G^-(p)$ to $\overline{V^-(g)}$ is the graph $G^{NE}(g)$, consisting of all north and east unit steps between points of $\overline{V^-(g)}$. Thus if two points of $\overline{V^-(g)}$ are in the same horizontal line, or in the same vertical line, there is a straight path between them, consisting of vertices of $\overline{V^-(g)}$ and either only north steps, or only east steps. Next we show the uniqueness statement.

Suppose that $(j, j+p), (j, j+q) \in \overline{V^-(g)}$ $(p < q')$. Thus $(j, j+r) \in \overline{V^-(g)}$ for all r between p and q. Let

$$\rho := v_p \to v_{p+1} \to \cdots \to v_q \quad (v_p = (j, j+p),\, v_q = (j, j+q))$$

be a path between $(j, j+p)$ and $(j, j+q)$ in $G^-(g)$, and let $v_r \to v_{r+1} = (i, i+r) \to (l, l+r+1)$ be any step of this path. From Lemma 27 we have;

$$g(v_r) \leq g(j, j+r) \Rightarrow g(v_{r+1}) \leq g(j, j+r+1),$$

Hence we deduce inductively that $g(v_r) \leq g(j, j+r)$ for $r = p, p+1, \ldots, q$. Also from Lemma 27 we have;

$$g(v_r) < g(j, j+r) \Rightarrow g(v_{r+1}) < g(j, j+r+1),$$

which we write;

$$g(v_{r+1}) \geq g(j, j+r+1) \Rightarrow g(v_r) \geq g(j, j+r),$$

Hence we deduce, also inductively, that $g(v_r) \geq g(j, j+r)$ for $r = p, \ldots, q$. Therefore $g(v_r) = g(j, j+r)$, and so $v_r = (j, j+r)$ for $r = p, \ldots, q$, as wanted.

We now show the same for points in the same vertical line. Above we proved;

For $(j, j+p), (j, j+q) \in \overline{V^-(g)}$ with $p < q$, the only path between $(j, j+p)$ and $(j, j+q)$ in $G^-(g)$, is $(j, j+p) \to (j, j+p+1) \to \cdots \to (j, j+q)$.

We write this statement in terms of the $(k+1)$-dress $\operatorname{Tr} g$;

For $(j, j+p), (j, j+q) \in \operatorname{Tr} \overline{V^-(g)}$ with $p < q$, the only path between $(j, j+p)$ and $(j, j+q)$ in $\operatorname{rev} \operatorname{Tr} G^-(g)$, is $(j, j+p) \to (j, j+p+1) \to \cdots \to (j, j+q)$.

Equivalently;

> For $(j+p,j),(j+q,j) \in \overline{V^{-}(g)}$ with $p<q$, the only path between $(j+q,j)$ and $(j+p,j)$ in $G^{-}(g)$, is $(j+q,j) \to (j+q-1,j) \to \cdots \to (j+p)$,

as wanted.

For the second claim (the one involving points of $\overline{V^{+}(g)}$ and paths in $G^{+}(p)$), we write the first claim in terms of the $(r+1)$-dress $\mathrm{R}_{180^\circ}\, g$;

> If two points of $\mathrm{R}_{180^\circ}\, \overline{V^{+}(g)}$ are in the same horizontal line or in the same vertical line, then there is exactly one path in $\mathrm{R}_{180^\circ}\, G^{+}(p)$ joining them.

Equivalently;

> If two points of $\overline{V^{+}(g)}$ are in the same horizontal line or in the same vertical line, then there is exactly one path in $G^{+}(p)$ joining them.

Moreover this unique path is either consisting entirely of west steps, or of south steps, respectively, as claimed in the lemma's statement. $\square$

The functions $\blacktriangleright$, $\blacktriangle$, $\blacktriangleleft$, $\blacktriangledown$, $\curvearrowright$ and $\curvearrowleft$, introduced in the previous section, depend on the $(k+1)$-fringe g. To avoid ambiguity we denote them also by $\blacktriangleright_g$, $\blacktriangle_g$, $\blacktriangleleft_g$, $\blacktriangledown_g$, $\curvearrowright_g$ and $\curvearrowleft_g$ when the $(k+1)$-fringe being used is not evident. Next we characterize these functions for the $(k+1)$-fringes $\mathrm{Tr}\, g$ and $\mathrm{R}_{180^\circ}\, g$. All of the equalities below are straightforward.

$$\blacktriangleright_{\mathrm{Tr}\, g}(a,b) = \mathrm{Tr}\ \blacktriangledown_g\, \mathrm{Tr}\,(a,b) = \mathrm{Tr}\ \blacktriangledown_g\,(b,a),$$

$$\blacktriangle_{\mathrm{Tr}\, g}(a,b) = \mathrm{Tr}\ \blacktriangleleft_g\, \mathrm{Tr}\,(a,b) = \mathrm{Tr}\ \blacktriangleleft_g\,(b,a),$$

$$\blacktriangleleft_{\mathrm{Tr}\, g}(a,b) = \mathrm{Tr}\ \blacktriangle_g\, \mathrm{Tr}\,(a,b) = \mathrm{Tr}\ \blacktriangle_g\,(b,a),$$

$$\blacktriangledown_{\mathrm{Tr}\, g}(a,b) = \mathrm{Tr}\ \blacktriangleright_g\, \mathrm{Tr}\,(a,b) = \mathrm{Tr}\ \blacktriangleright_g\,(b,a).$$

For $(a,b) \in \overline{[\mathrm{Tr}\, F^{(1)}]^{-}} = \mathrm{Tr}\, F^{(1)-}$ (i.e., $(b,a) \in F^{(1)-}$);

$$\curvearrowright_{\mathrm{Tr}\, g}(a,b) = \mathrm{rev}\,\mathrm{Tr}\ \curvearrowright_g\, \mathrm{Tr}\,(a,b) = \mathrm{rev}\,\mathrm{Tr}\ \curvearrowright_g\,(b,a).$$

For $(a,b) \in \overline{[\mathrm{Tr}\, F^{(1)}]^{+}} = \mathrm{Tr}\, F^{(1)+}$ (i.e., $(b,a) \in F^{(1)+}$);

$$\curvearrowleft_{\mathrm{Tr}\, g}(a,b) = \mathrm{rev}\, \mathrm{Tr} \curvearrowleft_g \mathrm{Tr}\,(a,b) = \mathrm{rev}\, \mathrm{Tr} \curvearrowleft_g (b,a).$$

Furthermore:

$$\blacktriangleright_{\mathrm{R}_{180^\circ}\, g}(a,b) = \mathrm{R}_{180^\circ} \blacktriangleleft_g \mathrm{R}_{180^\circ}\,(a,b) = \mathrm{R}_{180^\circ} \blacktriangleleft_g (-a,-b),$$

$$\blacktriangle_{\mathrm{R}_{180^\circ}\, g}(a,b) = \mathrm{R}_{180^\circ} \blacktriangledown_g \mathrm{R}_{180^\circ}\,(a,b) = \mathrm{R}_{180^\circ} \blacktriangledown_g (-a,-b),$$

$$\blacktriangleleft_{\mathrm{R}_{180^\circ}\, g}(a,b) = \mathrm{R}_{180^\circ} \blacktriangleright_g \mathrm{R}_{180^\circ}\,(a,b) = \mathrm{R}_{180^\circ} \blacktriangleright_g (-a,-b),$$

$$\blacktriangledown_{\mathrm{R}_{180^\circ}\, g}(a,b) = \mathrm{R}_{180^\circ} \blacktriangle_g \mathrm{R}_{180^\circ}\,(a,b) = \mathrm{R}_{180^\circ} \blacktriangle_g (-a,-b),$$

For $(a,b) \in \overline{[\mathrm{R}_{180^\circ}\, F^{(1)}]^{-}} = \mathrm{R}_{180^\circ}\, F^{(1)+}$ (i.e., $(-a,-b) \in F^{(1)+}$);

$$\curvearrowright_{\mathrm{R}_{180^\circ}\, g}(a,b) = \mathrm{R}_{180^\circ} \curvearrowleft_g \mathrm{R}_{180^\circ}\,(a,b) = \mathrm{R}_{180^\circ} \curvearrowleft_g (-a,-b).$$

For $(a,b) \in \overline{[\mathrm{R}_{180^\circ}\, F^{(1)}]^{+}} = \mathrm{R}_{180^\circ}\, F^{(1)-}$ (i.e., $(-a,-b) \in F^{(1)-}$);

$$\curvearrowleft_{\mathrm{R}_{180^\circ}\, g}(a,b) = \mathrm{R}_{180^\circ} \curvearrowright \mathrm{R}_{180^\circ}\,(a,b) = \mathrm{R}_{180^\circ} \curvearrowright (-a,-b).$$

Proof of Lemma 25 (Section 2.1, page 99). The result is evident for $(a,b) \in \mathbb{Z} \times \mathbb{Z} - V(g) = g^{-1}(k+1)$. Let us show the statement for $(a,b) \in D_r \cap \overline{V^{-}(g)}$. Suppose that $\blacktriangleleft(a,b) \in D_p$ and $\blacktriangle(a,b) \in D_q$, where clearly $p \leq r \leq q$. Observe that $(a,b) = (a, a+r)$, $\blacktriangleleft(a,b) = (a, a+p)$, and $\blacktriangle(a,b) = (a+r-q, a+r)$. Set $\rho := v_p \to v_{p+1} \to \cdots \to v_q$, where $v_p = \blacktriangleleft(a,b)$, $v_q = \blacktriangle(a,b)$. From Lemma 27, we obtain inductively that $g(v_s) \leq g(a, a+s)$ for $s = p, \ldots, r$. Since g is non-decreasing within $\overline{F^{(1)-}}$, we have;

$$g(v_s) \leq g(a,b) \text{ for } s = 1, \ldots, r. \tag{2.18}$$

Consider the path $\mathrm{rev}\, \mathrm{Tr}\, \rho = \mathrm{Tr}\, v_q \to \mathrm{Tr}\, v_{q-1} \to \cdots \to \mathrm{Tr}\, v_p$ in $\curvearrowright_{\mathrm{Tr}\, g}(a+r, a)$. This path is between $\mathrm{Tr}\,(a+r-q, a+r) = (a+r, a+r-q)$ and $\mathrm{Tr}\,(a, a+p) = (a+p, a)$. Again from Lemma 27 we obtain that $\mathrm{Tr}\, g(\mathrm{Tr}\, v_{q-s}) \leq \mathrm{Tr}\, g(a+r, a+r-q+s)$ for $s = 0, \ldots, q-r$. Thus by the non-decreasingness of $\mathrm{Tr}\, g$ within $\mathrm{Tr}\, \overline{F^{(1)-}}$;

$$\mathrm{Tr}\, g(\mathrm{Tr}\, v_{q-s}) \leq \mathrm{Tr}\, g(b,a) \text{ for } s = 0, \ldots, q-r,$$

or equivalently;

$$g(v_t) \leq g(a,b) \text{ for } t = r, \ldots, q. \tag{2.19}$$

Hence from 2.18 and 2.19, we obtain the desired inequalities.

To prove the statement for $(a,b) \in \overline{V^+(g)}$, write the original statement for the $(k+1)$-dress $\mathrm{R}_{180^\circ}\, g$;

If $(a,b) \in \mathrm{R}_{180^\circ}\, \overline{V^+(g)}$ and $\rho \in \mathrm{R}_{180^\circ} \swarrow (-a,-b)$, then $g(-c,-d) \leq g(-a,-b)$ for $(c,d) \in \rho$.

Equivalently;

$$\text{If } (-a,-b) \in \overline{V^+(g)} \text{ and } \mathrm{R}_{180^\circ}\, \rho \in \swarrow(-a,-b),$$
$$\text{then } g(-c,-d) \leq g(-a,-b) \text{ for } (-c,-d) \in \mathrm{R}_{180^\circ}\, \rho,$$

as wanted. □

The rest of this section is dedicated to providing some weight-preserving involutions on pairs of intersecting, crossing, and bonding paths. This is done to extend the usual notion of *switching two paths at a common vertex* (see page 11).

The involution X_r ($r \in \mathbb{Z}$) on intersecting pairs of paths: This is the usual "switching" involution on paths. Assume that (a,b) and (c,d) are both in $F^{(1)-}$ or both in $F^{(1)+}$ with $|\pi^g(a,b) \cap \pi^g(c,d)| \geq 2$. Let ρ, η be paths in $\mathcal{P}^g(a,b)$, $\mathcal{P}^g(c,d)$ respectively. If they are intersecting at a point $v_r^{(m)}$ in D_r, set $\mathsf{X}_r(\rho,\eta) := (\rho', \eta') \in \mathcal{P}^g(a,d) \times \mathcal{P}^g(b,c)$, where;

$$\rho' := \rho(\to v_r^{(m)})\eta(v_r^{(m)} \to)$$

$$\eta' := \eta(\to v_r^{(m)})\rho(v_r^{(m)} \to)$$

This is clearly an involution and it is weight preserving, since the vertices and edges of (ρ', η') are the same as those of (ρ, η), and because $(a,b), (c,d) \in F^{(1)-}$ if and only if $(a,d), (c,b) \in F^{(1)-}$ as well (similarly for $F^{(1)+}$). Notice also that because of this preservation of vertices and edges;

$$\pi^g(a,b) \cap \pi^g(c,d) = \pi^g(a,d) \cap \pi^g(c,b),$$

and;

$$\pi^g(a,b) \cup \pi^g(c,d) = \pi^g(a,d) \cup \pi^g(c,b).$$

The involution $\mathsf{X}_{r+\frac{1}{2}}$ ($r \in \mathbb{Z}$) on crossing pairs of paths: Assume that (a,b) and (c,d) are both in $\overline{F^{(1)-}}$. Let ρ, η be paths in $\mathcal{P}^g(a,b)$, $\mathcal{P}^g(c,d)$ respectively. If these paths cross between D_r and D_{r+1} with $v_r^{(l)} \to v_{r+1}^{(l')} \in \rho$ and $v_r^{(m)} \to v_{r+1}^{(m')} \in \eta$, then set $\mathsf{X}_{r+\frac{1}{2}}(\rho,\eta) := (\rho',\eta') \in \mathcal{P}^g(a,d) \times \mathcal{P}^g(b,c)$, where;

$$\rho' := \rho(\to v_r^{(l)})\,(v_r^{(l)} \to v_{r+1}^{(m')})\,\eta(v_{r+1}^{(m')} \to)$$

$$\eta' := \eta(\to v_r^{(m)})\,(v_r^{(m)} \to v_{r+1}^{(l')})\,\eta(v_{r+1}^{(l')} \to)$$

Similarly, if $(a,b),(c,d)$ are both in $\overline{F^{(1)+}}$ with $v_{r+1}^{(l)} \to v_r^{(l')} \in \rho$ and $v_{r+1}^{(m)} \to v_r^{(m')} \in \eta$, then set $\mathsf{X}_{r+\frac{1}{2}} := (\rho',\eta') \in \mathcal{P}^g(a,d) \times \mathcal{P}^g(b,c)$, where;

$$\rho' := \rho(\to v_{r+1}^{(l)})\,(v_{r+1}^{(l)} \to v_r^{(m')})\,\eta(v_r^{(m')} \to)$$

$$\eta' := \eta(\to v_{r+1}^{(m)})\,(v_{r+1}^{(m)} \to v_r^{(l')})\,\eta(v_r^{(l')} \to)$$

As before this is a weight preserving involution on crossing pairs of paths.

The involution $\mathsf{X}_{\mathrm{bond}}$ on bonding pairs of paths: For every path α, let $V(\alpha)$ denote the set of vertices of α. The following three lemmas are essential to define $\mathsf{X}_{\mathrm{bond}}$.

Lemma 28. *Let g be a $(k+1)$-dress. If $\rho \in \mathcal{P}^g(a,b)$, $\eta \in \mathcal{P}^g(c,d)$ are bonding, then for all $r \in \pi^g(a,b) \cup \pi^g(c,d)$, the set $V(\rho) \cup V(\eta)$ contains exactly one point from the diagonal D_r.*

Proof. Let $v_r^{(l)}, v_r^{(m)}$ be elements of $V(\rho) \cap V(\eta)$. If they are both in the same set $V(\rho)$ or $V(\eta)$, then clearly $l = m$. Otherwise, if they are on different sets, then $r \in \pi^g(a,b) \cap \pi^g(c,d)$ and so $m = l = 1$. $\square$

Lemma 29. *Let g be a $(k+1)$-dress. Let (a,b) be a point of the plane $\mathbb{Z} \times \mathbb{Z}$, let s,t be nonnegative integers, and suppose that the set*

$$A := \{(a+s,b), (a+s-1,b), \ldots, (a,b), (a,b+1), \ldots, (a,b+t)\}$$

is entirely contained in $F^{(1)}$. *Let* ρ *be any path in* $G^-(g)$ *which visits* D_r *for* $r = b-a-s, b-a-s+1, \ldots, b+t-a$. *Then;*

$$(a,b) \in \rho \Rightarrow A \subseteq V(\rho). \tag{2.20}$$

Similarly, if we suppose that the set

$$B := \{(a,b-t),(a,b-t+1),\ldots,(a,b),(a-1,b),\ldots,(a-s,b)\}$$

is entirely contained in $F^{(1)}$, *and that* ρ *is any path in* $G^+(g)$ *which visits* D_r *for* $r = b-t-a, b-t-a+1, \ldots, b-a+s$, *then;*

$$(a,b) \in \rho \Rightarrow B \subseteq V(\rho). \tag{2.21}$$

Proof. Let us show (2.20). Assume $t > 1$. We show that $(a, b+u) \in V(\rho)$ for $0 \le u \le t$ by induction on u. The base case $u = 0$ is given. Suppose that $(a, b+u) \in V(\rho)$ for some $u < t$. Since $b+u+1-a \le b+t-a$, then ρ visits a vertex in $D_{b+u+1-a}$. Let $(a+l, b+u+l+1)$ be that vertex, for some $l \in \mathbb{Z}$. By the definition of $G^-(g)$, there must exist neighbouring vertices $v \in D_{b+u-a} \cap \overline{V^-(g)}$, $v' \in D_{b+u+1-a} \cap \overline{V^-(g)}$ such that either;

$$g(v) \ge g(a,b+u) = 1 \ge g(a+l, b+u+l+1) \ge g(v'),$$

or;

$$g(v) \le g(a,b+u) = 1 \le g(a+l, b+u+l+1) \le g(v').$$

In the first case we have $g(a+l, b+u+l+1) = 1$, and so by Lemma 16 we deduce that $l = 0$, as wanted. In the second case we have $g(v) = 1$, and since $v \in D_{b+u-a}$, we deduce from Lemma 16 that $v = (a, b+u)$, but the only neighbour of v in $D_{b+u+1-a} \cap \overline{V^-(g)}$ is $(a, b+u+1)$. Hence $v' = (a, b+u+1)$ and $g(v') = 1$. This yields $g(a+l, b+u+l+1) = 1$, and again by Lemma 16 we obtain that $l = 0$, as wanted.

Now assume $s > 0$. We show that $(a+u, b) \in V(\rho)$ for $0 \le u \le s$ by induction on u. As before, the base case $u = 0$ is given. Suppose that $(a+u, b) \in V(\rho)$ for some $u < s$. Since $b-a-u-1 \ge b-a-s$, then ρ visits a vertex in $D_{b-a-u-1}$. Let $(a+u+l+1, b+l)$ be that vertex, for some $l \in \mathbb{Z}$. By the definition of $G^-(g)$, there must exist neighbouring vertices $v \in D_{b-a-u-1} \cap \overline{V^-(g)}$, $v' \in D_{b-a-u} \cap \overline{V^-(g)}$ such that either;

$$g(v) \leq g(a+u+l+1, b+l) \leq g(a+u, b) = 1 \leq g(v'),$$

or;

$$g(v) \geq g(a+u+l+1, b+l) \geq g(a+u, b) = 1 \geq g(v').$$

In the first case we have $g(a+u+l+1, b+l) = 1$, and so by Lemma 16 we deduce that $l = 0$, as wanted. In the second case we have $g(v') = 1$, and since $v' \in D_{b-a-u}$, we deduce from Lemma 16 that $v = (a+u, b)$, but the only neighbour of v' in $D_{b-u-a-1} \cap \overline{V^-(g)}$ is $(a+u+1, b)$. Hence $v = (a+u+1, b)$ and $g(v) = 1$. This yields $g(a+u+l+1, b+l) = 1$, and again by Lemma 16 we obtain that $l = 0$, as wanted.

Analogous arguments yield (2.21). □

Lemma 30. *Let g be a $(k+1)$-dress. Suppose that $\rho \in \mathcal{P}^g(a,b)$, $\eta \in \mathcal{P}^g(c,d)$ are bonding paths. Then there is exactly one path $\rho' \in \mathcal{P}^g(a,d)$ such that;*

$$\begin{aligned} V(\rho') := & \left\{ v_r^{(m)} \in V(\rho) \cup V(\eta) : r \in \pi^g(a,d) \cap (\pi^g(a,b) \cup \pi^g(c,d)) \right\} \\ & \cup \left\{ v_r^{(1)} : r \in \pi^g(a,d) - (\pi^g(a,b) \cup \pi^g(c,d)) \right\} \end{aligned} \tag{2.22}$$

Proof. Let S denote the right hand side of (2.22). We start by verifying that S contains no more than one vertex within every diagonal D_r for $r \in \pi^g(a,d)$. This is evident by definition for every r in $\pi^g(a,d) - (\pi^g(a,b) \cup \pi^g(c,d))$. It is also true for r in $\pi^g(a,d) \cap (\pi^g(a,b) \cup \pi^g(c,d))$, as a result of Lemma 28. Now for all $r \in \pi^g(a,d)$, let $m_r \in \{1, \ldots, k\}$ be the unique integer such that $v_r^{(m_r)} \in S$. To complete the proof we split into cases, most of which are straightforward:

Case 1. If $(a,d) \in \overline{F^{(1)-}}$, we need to show that for all r with $r, r+1 \in \pi^g(a,d)$, the edge $v_r^{(m_r)} \to v_{r+1}^{(m_{r+1})}$ is in $G^-(g)$. The following sub cases refer to the possible values of r.

Case 1.1. If $r, r+1 \in \pi^g(a,d) - (\pi^g(a,b) \cup \pi^g(c,d))$, then $m_r = m_{r+1} = 1$, and so from Lemma 19 we know that $v_r^{(m_r)} \to v_{r+1}^{(m_{r+1})} \in G^-(g)$, as wanted.

Case 1.2. If $r \in \pi^g(a,d) - (\pi^g(a,b) \cup \pi^g(c,d))$ and $r+1 \in \pi^g(a,d) \cap (\pi^g(a,b) \cup \pi^g(c,d))$, then $m_r = 1$, and $r+1$ must be an initial point of one of the integer intervals $\pi^g(a,b)$ or $\pi^g(c,d)$, from where we deduce that $m_{r+1} = 1$ as well (because the paths ρ, η begin at $F^{(1)}$), and so $v_r^{(m_r)} \to v_{r+1}^{(m_{r+1})} \in G^-(g)$, as wanted. The same argument holds when $r+1 \in \pi^g(a,d) - (\pi^g(a,b) \cup \pi^g(c,d))$ and $r \in \pi^g(a,d) \cap (\pi^g(a,b) \cup \pi^g(c,d))$, except this time r must be an ending point of either $\pi^g(a,b)$ or $\pi^g(c,d)$.

Case 1.3. If $r, r+1 \in \pi^g(a,d) \cap (\pi^g(a,b) \cup \pi^g(c,d))$, we have three more subcases;

Case 1.3.1. If $r, r+1 \in \pi^g(a,b) \cap \pi^g(c,d)$, then since ρ, η are bonding, $m_r = m_{r+1} = 1$. As before, from Lemma 19, we know that $v_r^{(m_r)} \to v_{r+1}^{(m_{r+1})} \in G^-(g)$.

Case 1.3.2. If $r \in \pi^g(a,b) - \pi^g(c,d)$ and $r+1 \in \pi^g(c,d) - \pi^g(a,b)$ (or vice versa), then by the same argument from **Case 1.2**, the points $v_r^{(m_r)}, v_{r+1}^{(m_{r+1})}$ are endpoints of ρ or η, and so $m_r = m_{r+1} = 1$.

Case 1.3.3. If $r, r+1 \in \pi^g(v)$ for some $v \in \{(a,b),(c,d)\}$. Let α denote the path among ρ, η which is in $\mathcal{P}^g(v)$, so that $v_r^{(m_r)}, v_{r+1}^{(m_{r+1})} \in \alpha$. Observe that v is either in the same vertical line, or in the same horizontal line as (a,d). We have two more subcases;

Case 1.3.3.1. If $v \in \overline{F^{(1)-}}$, then $\alpha \in \mathcal{J}(v)$, and so $v_r^{(m_r)} \to v_{r+1}^{(m_{r+1})} \in \alpha$, which implies that $v_r^{(m_r)} \to v_{r+1}^{(m_{r+1})} \in G^-(g)$, as wanted.

Case 1.3.3.2. If $v \in F^{(1)+}$, then the intersection $\pi^g(a,d) \cap \pi^g(v)$ is either empty, or consists of one common endpoint between the integer intervals $\pi^g(a,d)$, $\pi^g(v)$. This contradicts that $r, r+1 \in \pi^g(a,d) \cap \pi^g(v)$.

Case 2. If $(a,d) \in F^{(1)+}$, we need to show that for all r with $r, r+1 \in \pi^g(a,d)$, the edge $v_{r+1}^{(m_{r+1})} \to v_r^{(m_r)}$ is in $G^+(g)$. As before we consider some sub cases for the value of r. **Cases 2.1, 2.2, 2.3.1, 2.3.2** are the same as **Cases 1.1, 1.2, 1.3.1, 1.3.2** above, and their reasoning is similar (only the direction of the edges are inverted, and the graph $G^+(g)$ is used instead of $G^-(g)$). **Cases 2.3.3.1, 2.3.3.2** are also similar to **Cases 1.3.3.1, 1.3.3.2** above, with the additional modification that $\overline{F^{(1)-}}$ is replaced by $\overline{F^{(1)+}}$ and $F^{(1)+}$ is replaced by $F^{(1)-}$.

We have shown that ρ' is a path of $G^-(g)$ if $(a,d) \in \overline{F^{(1)-}}$ and a path of $G^+(g)$ if $(a,d) \in \overline{F^{(1)+}}$. Also it is clear by definition that this path visits D_r for all $r \in \pi^g(a,d)$. In order to conclude that $\rho' \in \mathcal{P}^g(a,d)$, it remains for us to show that ρ' starts and ends at $F^{(1)}$. More precisely, if $s := \min \pi^g(a,d)$, $t := \max \pi^g(a,d)$, we need to show that $m_s = m_t = 1$. We split again into a few cases for s:

If $(a,d),(a,b) \in F^{(1)-}$, since they are in the same horizontal line, then $s = \min \pi^g(a,b)$ and so $v_s^{(m_s)} \in F^{(1)}$, as wanted.

If $(a,d) \in F^{(1)-}$, $(a,b) \in \overline{F^{(1)+}}$ and $\pi^g(a,b) \cap \pi^g(a,d) \neq \emptyset$, then $s = \min \pi^g(a,d) = \max \pi^g(a,b)$, and so $v_s^{(m_s)} \in F^{(1)}$, as wanted.

If $(a,d) \in F^{(1)-}$, $(a,b) \in \overline{F^{(1)+}}$, $\pi^g(a,b) \cap \pi^g(a,d) = \emptyset$, and $c < a$, then $s = \min \pi^g(a,d) \notin \pi^g(a,b) \cup \pi^g(c,d)$, and so $v_s^{(m_s)} \in F^{(1)}$, as wanted.

If $(a,d) \in F^{(1)-}$, $(a,b) \in \overline{F^{(1)+}}$, $\pi^g(a,b) \cap \pi^g(a,d) = \emptyset$, and $c \geq a$, then $s \in \pi^g(c,d)$ and $(i,j) := \max \pi^g(a,b) \in \pi^g(c,d)$. Thus since ρ, η are bonding; $v_{j-i}^{(1)} \in \rho, \eta$, and since $v_s^{(1)}$ is in $F^{(1)}$ in the same horizontal line and to the right of $v_{j-i}^{(1)}$, we deduce by Lemma 29 that $v_s^{(1)} \in \eta$, and so $m_s = 1$, as wanted.

We have shown that $m_s = 1$ when $(a,d) \in F^{(1)-}$. Analogously, by symmetry; $m_s = m_t = 1$ whenever $(a,d) \notin F^{(1)}$.

If $(a,d) \in F^{(1)}$, then $s = t = d - a$. If $s \in \pi^g(a,b)$, then since $(a,b),(a,d)$ are in the same horizontal line; $s \in \{\min \pi^g(a,b), \max \pi^g(a,b)\}$, and so $m_s = 1$, as wanted. The same holds if $s \in \pi^g(c,d)$, because $(c,d),(a,d)$ are in the same vertical line. Finally if $s \notin \pi^g(a,b) \cup \pi^g(c,d)$, then $m_s = 1$ by the definition of ρ', as wanted. $\square$

We are now ready to define X_{bond}. For two bonding paths $\rho \in \mathcal{P}^g(a,b)$, $\eta \in \mathcal{P}^g(c,d)$, set $\mathsf{X}_{\text{bond}}(\rho,\eta) := (\rho',\eta')$, where ρ',η' are the unique (see Lemma 30 above) paths in

$\mathcal{P}^g(a,d)$, $\mathcal{P}^g(c,b)$ respectively, satisfying;

$$\begin{aligned} V(\rho') &:= \left\{v_r^{(m)} \in V(\rho) \cup V(\eta) : r \in \pi^g(a,d) \cap (\pi^g(a,b) \cup \pi^g(c,d))\right\} \\ &\quad \cup \left\{v_r^{(1)} : r \in \pi^g(a,d) - (\pi^g(a,b) \cup \pi^g(c,d))\right\} \\ V(\eta') &:= \left\{v_r^{(m)} \in V(\rho) \cup V(\eta) : r \in \pi^g(c,b) \cap (\pi^g(a,b) \cup \pi^g(c,d))\right\} \\ &\quad \cup \left\{v_r^{(1)} : r \in \pi^g(c,b) - (\pi^g(a,b) \cup \pi^g(c,d))\right\} \end{aligned}$$

Later (Lemmas 33, 35 and 36) we show that $\mathsf{X}_{\mathrm{bond}}$ is a weight-preserving involution. We support these claims with a few other lemmas for which we need some new notation. For $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, define $\partial\pi^g(a,b) := \{\min \pi^g(a,b), \max \pi^g(a,b)\}$. In particular for $(a,b) \in F^{(1)}$ we have $\partial\pi^g(a,b) = \pi^g(a,b) = \{b-a\}$, and for $(a,b) \notin F^{(1)}$ we have $|\partial\pi^g(a,b)| = 2$.

Lemma 31. *Let g be a $(k+1)$-dress. For $a,b,c,d \in \mathbb{Z}$;*

$$\pi^g(a,b) \cap \pi^g(a,d) \cap \pi^g(c,b) \subseteq \pi^g(c,d) \cup \partial\pi^g(a,b).$$

Proof. Observe that $(a,b), (a,d)$ are on the same horizontal line. It is easy to check that if they are on different sides of $F^{(1)}$ (one is in $\overline{F^{(1)-}}$ while the other one is in $\overline{F^{(1)+}}$), then $\pi^g(a,b) \cap \pi^g(a,d)$ is either empty or consisting of a single common endpoint of the intervals $\pi^g(a,b), \pi^g(a,d)$. In that case $\pi^g(a,b) \cap \pi^g(a,d) \subseteq \partial\pi^g(a,b)$. The same holds if $(a,b), (c,b)$ are on different sides of $F^{(1)}$. Thus we may assume that the three points $(a,b), (a,d), (c,b)$ are in $F^{(1)\pm}$, for some choice of $-$ or $+$. In the rest of this proof we use '$\pm$' for this same choice and '$\mp$' for the opposite choice. If $(c,d) \in \overline{F^{(1)\mp}}$, then clearly $\pi^g(a,d) \cap \pi^g(c,b) = \emptyset$ or $\pi^g(a,d) \cap \pi^g(c,b) = \pi^g(c,d)$ (the last one holds under certain conditions when $(c,d) \in F^{(1)}$), so we may assume that $(c,d) \in F^{(1)\pm}$ as well. If $\pm a \geq \pm c$, then (a,d) is closer to $F^{(1)}$ than (c,d) and so $\pi^g(a,d) \subseteq \pi^g(c,d)$. Also if $\pm b \geq \pm d$, then (c,b) is closer to $F^{(1)}$ than (c,d) and so $\pi^g(c,b) \subseteq \pi^g(c,d)$. Hence we may assume that $\pm a < \pm c$ and $\pm b < \pm d$. Considering both choices $+$ or $-$ it is easy to verify that $\pi^g(a,d) \cap \pi^g(c,b) = \pi^g(c,d)$. □

Lemma 32. *Let g be a $(k+1)$-dress. If $\rho \in \mathcal{P}^g(a,b), \eta \in \mathcal{P}^g(c,d)$ are bonding and $r \in \pi^g(a,b) - (\pi^g(a,d) \cup \pi^g(c,b) \cup \pi^g(c,d))$ for some $a,b,c,d \in \mathbb{Z}$, then $v_r^{(1)} \in \rho$.*

Proof. By symmetry we may assume without loss of generality that $(a, b) \in \overline{F^{(1)-}}$. The result is obvious for $(a, b) \in F^{(1)}$, so we assume that $(a, b) \in F^{(1)-}$. We must have $c < a$ and $d < b$ since otherwise we would obtain $\pi^g(a, b) \subseteq \pi^g(a, d) \cup \pi^g(c, b)$. Thus

$$\min \pi^g(a, d) \leq \min \pi^g(c, d) \leq \max \pi^g(c, d) \leq \max \pi^g(c, b),$$

and

$$\min \pi^g(a, d) \leq \min \pi^g(a, b) \leq \max \pi^g(a, b) \leq \max \pi^g(c, b).$$

These inequalities imply that any number in $\pi^g(a, b) - (\pi^g(a, d) \cup \pi^g(c, b) \cup \pi^g(c, d))$ must be between $\max \pi^g(a, d)$ and $\min \pi^g(c, d)$ or between $\max \pi^g(c, d)$ and $\min \pi^g(c, b)$. If r is between $\max \pi^g(a, d)$ and $\min \pi^g(c, d)$, then necessarily $(a, d) \in \overline{F^{(1)-}}$, $(c, d) \in \overline{F^{(1)+}}$, and $v^{(1)}_{\min \pi^g(c,d)} \in \rho$ (this last relation happens because ρ, η are bonding and $\min \pi^g(c, d) \in \pi^g(a, b) \cap \pi^g(c, d)$). Also the point $v_r^{(1)}$ is in $F^{(1)}$ below $v^{(1)}_{\min \pi^g(c,d)}$. Hence by Lemma 29; $v_r^{(1)} \in \rho$, as wanted. Similarly, if r is between $\max \pi^g(c, d)$ and $\min \pi^g(c, b)$, then $v^{(1)}_{\max \pi^g(c,d)}$ is in ρ, and $v_r^{(1)}$ is in $F^{(1)}$ to its right. Hence, again by Lemma 29; $v_r^{(1)} \in \rho$, as wanted. □

Lemma 33. *If $\rho \in \mathcal{P}^g(a, b)$, $\eta \in \mathcal{P}^g(c, d)$ are bonding paths, then $\mathrm{X}_{\text{bond}}(\rho, \eta)$ is a pair of bonding paths as well.*

Proof. Set $(\rho', \eta') := \mathrm{X}_{\text{bond}}(\rho, \eta)$. For every $r \in \pi^g(a, d)$, let m_r be the unique integer in $\{1, \dots, k\}$ such that $v_r^{(m_r)} \in \rho'$, and for every $r \in \pi^g(c, b)$, let n_r be the unique integer in $\{1, \dots, k\}$ such that $v_r^{(n_r)} \in \eta'$. Consider any $s \in \pi^g(a, d) \cap \pi^g(c, b)$. We need to show that $m_s = n_s = 1$. By hypothesis;

$$\{v_s^{(m_s)}, v_s^{(n_s)}\} \subseteq V(\rho') \cup V(\eta') \subseteq V(\rho) \cup V(\eta) \cup \{v_r^{(1)} : r \notin \pi^g(a, b) \cup \pi^g(c, d)\}.$$

The sets $V(\rho) \cup V(\eta)$ and $\{v_r^{(1)} : r \notin \pi^g(a, b) \cup \pi^g(c, d)\}$ are disjoint. In fact, no diagonal D_r contains points from both of them. Thus we have either;

$$\{v_s^{(m_s)}, v_s^{(n_s)}\} \subseteq \{v_r^{(1)} : r \notin \pi^g(a, b) \cup \pi^g(c, d)\},$$

in which case $m_s = n_s = 1$, or;

$$\{v_s^{(m_s)}, v_s^{(n_s)}\} \subseteq V(\rho) \cup V(\eta).$$

In this case $s \in \pi^g(a,b) \cup \pi^g(c,d)$, and from Lemma 30; $m_s = n_s$. To simplify the rest of this proof, set $m := m_s = n_s$. If $s \in \pi^g(a,b) \cap \pi^g(c,d)$, then since ρ, η are bonding, we deduce that $m = 1$, as wanted.

It remains for us to consider the case $s \in \pi^g(a,b) - \pi^g(c,d)$ (the case $s \in \pi^g(c,d) - \pi^g(a,b)$ is analogous). This means that $v_s^{(m)} \in V(\rho) - V(\eta)$. Since $s \in \pi^g(a,b) \cap \pi^g(a,d) \cap \pi^g(c,b)$ and $s \notin \pi^g(c,d)$, by Lemma 31, we have that $s \in \partial\pi^g(a,b)$, from where we obtain that $m_s = 1$. □

Lemma 34. *Let g be a $(k+1)$-dress. If $\rho \in \mathcal{P}^g(a,b), \eta \in \mathcal{P}^g(c,d)$ are bonding and $(\rho', \eta') = \chi_{\text{bond}}(\rho, \eta)$, then*

$$V(\rho) \cup V(\eta) - F^{(1)} = V(\rho') \cup V(\eta') - F^{(1)}. \tag{2.23}$$

Proof. Let us first show the inclusion $\supseteq$. Take any $v_r^{(m)} \in V(\rho')$ with $m > 1$. We need to show that $v_r^{(m)} \in V(\rho) \cup V(\eta)$. Since $r \in \pi^g(a,d)$, it would suffice to show that $r \in \pi^g(a,b) \cup \pi^g(c,d)$. Suppose the opposite;

$$r \in \pi^g(a,d) - (\pi^g(a,b) \cup \pi^g(c,d)),$$

in order to obtain a contradiction. Notice that $r \notin \pi^g(c,b)$, since otherwise r would be in $\pi^g(a,d) \cap \pi^g(c,b)$ and so m would be equal to 1. Thus;

$$r \in \pi^g(a,d) - (\pi^g(a,b) \cup \pi^g(c,d) \cup \pi^g(c,b))$$

By Lemma 32, we obtain that $v_r^{(1)} \in \rho'$ and so $m = 1$, which contradicts the assumption $m > 1$, concluding the proof.

We now show the inclusion $\subseteq$. Take any $v_r^{(m)} \in V(\rho)$ with $m > 1$. We need to show that $v_r^{(m)} \in V(\rho') \cup V(\eta')$, for which it would suffice to prove that $r \in \pi^g(a,d) \cup \pi^g(c,b)$. Suppose the opposite;

$$r \in \pi^g(a,b) - (\pi^g(a,d) \cup \pi^g(c,b)).$$

Notice that $r \notin \pi^g(c,d)$, since otherwise r would be in $\pi^g(a,b) \cap \pi^g(c,d)$ and so m would be equal to 1. Thus;

$$r \in \pi^g(a,b) - (\pi^g(a,d) \cup \pi^g(c,b) \cup \pi^g(c,d)).$$

By Lemma 32, we obtain that $v_r^{(1)} \in \rho$ and so $m = 1$, which again contradicts the assumption $m > 1$, concluding the proof.

Moreover it is easy to check that r must be in $\pi^g(i,j)$ for some $(i,j) \in \{(a,d),(c,b)\} \cap F^{(1)-}$, since otherwise, if $(i,j) \in \overline{F^{(1)+}}$, then $\pi^g(a,b) \cap \pi^g(i,j) \subseteq \partial\pi^g(a,b)$, which again yields the contradiction $m = 1$. As a result, the vertices from (2.23) also preserve their weights along both sides of the equality (recall that the weight of a vertex from $V(\rho) \cup V(\eta)$ or $V(\rho') \cup V(\eta')$ depends on the path it inhabits, by the definition of w in terms of w^-, w^+). $\square$

Lemma 35. χ_{bond} *is an involution on pairs of bonding paths. More formally;*

$$\chi_{\text{bond}}(\chi_{\text{bond}}(\rho,\eta)) = (\rho,\eta).$$

Proof. To be consistent with the previous lemmas, assume that $\rho \in \mathcal{P}^g(a,b), \eta \in \mathcal{P}^g(c,d)$ for some $a,b,c,d \in \mathbb{Z}$, and set $(\rho',\eta') := \chi_{\text{bond}}(\rho,\eta)$. Define ρ'',η'' to be the only paths in $\mathcal{P}^g(a,b), \mathcal{P}^g(c,d)$ respectively so that;

$$\begin{aligned} V(\rho'') \;:=\; & \left\{v_r^{(m)} \in V(\rho') \cup V(\eta') : r \in \pi^g(a,b) \cap (\pi^g(a,d) \cup \pi^g(c,b))\right\} \\ & \cup \left\{v_r^{(1)} : r \in \pi^g(a,b) - (\pi^g(a,d) \cup \pi^g(c,b))\right\} \end{aligned} \tag{2.24}$$

$$\begin{aligned} V(\eta'') \;:=\; & \left\{v_r^{(m)} \in V(\rho') \cup V(\eta') : r \in \pi^g(c,d) \cap (\pi^g(a,d) \cup \pi^g(c,b))\right\} \\ & \cup \left\{v_r^{(1)} : r \in \pi^g(c,d) - (\pi^g(a,d) \cup \pi^g(c,b))\right\} \end{aligned} \tag{2.25}$$

We need to show that $\rho'' = \rho$ and $\eta'' = \eta$. We only prove the first equality, since the second one is analogous. We show the equivalent equality $V(\rho'') = V(\rho)$. Recall from Lemma 34 that $V(\rho) \cup V(\eta) - F^{(1)} = V(\rho'') \cup V(\eta'') - F^{(1)}$. Choose any $v_r^{(m)} \in V(\rho'') - F^{(1)}$. Thus $v_r^{(m)} \in V(\rho'') \cup V(\eta'')$ and by Lemma 28, since $r \in \pi^g(a,b)$, we deduce that $v_r^{(m)} \in \rho''$. More specifically $v_r^{(m)} \in V(\rho'') - F^{(1)}$ (because $m > 1$). Hence

$V(\rho) - F^{(1)} \subseteq V(\rho'') - F^{(1)}$ and similarly $V(\rho'') - F^{(1)} \subseteq V(\rho) - F^{(1)}$. Therefore $V(\rho) - F^{(1)} = V(\rho'') - F^{(1)}$ and since ρ, ρ'' visit the same diagonals; $V(\rho) = V(\rho'')$, as wanted. □

Lemma 36. X_{bond} *is weight-preserving. More formally;*

$$w(\mathsf{X}_{\text{bond}}(\rho, \eta)) = w(\rho, \eta).$$

Proof. Lemma 34, along with the remark from the last paragraph of its proof, imply that the equality

$$w(\rho, \eta) = w(\rho', \eta'),$$

holds if and only if the equality

$$w(\alpha(a, b), \alpha(c, d)) = w(\alpha(a, d), \alpha(c, b)),$$

also holds, where $\alpha(i, j)$ is the path in $\mathcal{P}^g(i, j)$ whose vertices are all in $F^{(1)}$. We thus proceed to show the second equality. It is easier to write this equality as

$$\frac{w(\alpha(a, b))}{w(\alpha(a, d))} = \frac{w(\alpha(c, b))}{w(\alpha(c, d))}.$$

This is equivalent to the statement that the left hand side of the equality does not depend on the value of a. It is easy to check by definition of w, that for $(i, j) \in F^{(1)\pm}$;

$$w(\alpha(i, j)) = w^{\pm}(\alpha(i, j)) = \frac{\prod\limits_{r \in (A^{\pm}(i,j) \cap \pi^g(i,j)) \cup \partial \pi^g(i,j)} x_r^{(1)}}{\prod\limits_{r \in A^{\mp}(i,j) \cap \pi^g()} x_r^{(1)}},$$

where;

$$A^{\pm} := \{j' - i' : (i', j'), (i' \pm 1, j'), (i', j' \pm 1) \in F^{(1)}\}$$

It is thus straightforward to verify that for $b > d$;

$$\frac{w(\alpha(a, b))}{w(\alpha(a, d))} = \frac{\prod\limits_{r \in A(b,d)} x_r^{(m)}}{\prod\limits_{r \in B(b,d)} x_r^{(m)}}$$

where $A(b,d), B(b,d)$ are the following sets;

$$\{j-i : (i,j) \in F^{(1)},\ d < j < b\} \cup \{d - \min\{i : (i,d) \in F^{(1)}\}, b - \max\{i : (i,b) \in F^{(1)}\}\},$$

$$\{j-i : (i,j) \in F^{(1)},\ d < j < b\} \cup \{d - \min\{i : (i,d) \in F^{(1)}\}, b - \max\{i : (i,b) \in F^{(1)}\}\},$$

respectively, which do not depend on a, as wanted. □

We have shown that the functions X_r $(r \in \frac{1}{2}\mathbb{Z})$, X_{bond} are weight preserving involutions.

Lemma 37. *Let $g = \{F^{(m)}\}_{1 \le m \le k}$ be a $(k+1)$-dress. Suppose that $(a,b),(c,d) \in F^{(1)-}$ or $(a,b),(c,d) \in F^{(1)+}$, and that ρ, η are paths in $\mathcal{P}^g(a,b), \mathcal{P}^g(c,d)$, respectively. If ρ, η are non-intersecting and non-crossing, then either $\rho[r] > \eta[r]$ for all $r \in \pi^g(\rho) \cap \pi^g(\eta)$, or $\rho[r] < \eta[r]$ for all $r \in \pi^g(\rho) \cap \pi^g(\eta)$.*

Proof. Assume that $\rho \in \mathcal{P}^g(a,b)$, $\eta \in \mathcal{P}^g(c,d)$ are non-intersecting and non-crossing. Suppose, in order to obtain a contradiction, that there exists some $r \in \pi^g(\rho) \cap \pi^g(\eta)$ such that $\rho[r] < \eta[r]$ and $\rho[r+1] > \eta[r+1]$. We consider the following six possible cases;

$$\rho[r] < \eta[r] \le \eta[r+1] < \rho[r+1],$$

$$\eta[r+1] \le \rho[r] \le \rho[r+1] \le \eta[r],$$

$$\eta[r+1] < \rho[r+1] \le \rho[r] < \eta[r],$$

$$\rho[r] \le \eta[r+1] < \eta[r] \le \rho[r+1],$$

$$\eta[r+1] \le \rho[r] < \eta[r] \le \rho[r+1],$$

$$\rho[r] \le \eta[r+1] < \rho[r+1] \le \eta[r].$$

Each of the first three cases implies, by Lemma 18, that ρ, η are crossing, producing a contradiction, while each of the last three cases are impossible by Lemma 21. □

Lemma 38. *Let g be a $(k+1)$-dress. If $a,b,c,d \in \mathbb{Z}$ are such that $a < c$ and $b > d$, then every pair of paths $(\rho,\eta) \in \mathcal{P}^g(a,b) \times \mathcal{P}^g(c,d)$ is either intersecting, crossing, or bonding.*

Proof. If $(a,b),(c,d) \in F^{(1)-}$, then by the inequalities $a < c$, $b > d$, we know that;

$$\min \pi^g(c,d) < \min \pi^g(a,b),$$

$$\max \pi^g(c,d) < \max \pi^g(a,b).$$

If $|\pi^g(a,b) \cap \pi^g(c,d)| \leq 1$, then ρ, η are trivially bonding. Otherwise we have;

$$\min \pi^g(c,d) < \min \pi^g(a,b) < \max \pi^g(c,d) < \max \pi^g(a,b).$$

Thus $\rho[r] = 1 \leq \eta[r]$ for $r = \min \pi^g(a,b)$ and $\rho[r'] \geq 1 = \eta[r']$ for $r' = \max \pi^g(c,d)$, and so by Lemma 37, ρ, η must be intersecting or crossing. A similar argument yields the statement for $(a,b),(c,d) \in F^{(1)+}$. If $(a,b),(c,d)$ are on different sides of $F^{(1)}$, it is easy to see that $\pi^g(a,b) \cap \pi^g(c,d) = \emptyset$ and so ρ, η are trivially bonding. $\square$

Proof of Lemma 26 (Section 2.1, page 102). Write;

$$\det(P^{X,g}_{\{a,c\}\{b,d\}}) = |\mathcal{P}^g(a,b) \times \mathcal{P}^g(c,d)|_w - |\mathcal{P}^g(a,d) \times \mathcal{P}^g(c,b)|_w.$$

Observe that by Lemma 38, every pair in $\mathcal{P}^g(a,d) \times \mathcal{P}^g(c,b)$ is either intersecting, crossing or bonding. Thus we just need to find a weight-preserving involution f between $\mathcal{P}^{X,g}(a,d) \times \mathcal{P}^{X,g}(c,b)$ and the set of all intersecting, crossing, or bonding pairs from $\mathcal{P}^{X,g}(a,b) \times \mathcal{P}^{X,g}(c,d)$.

For $(a,b),(c,d) \in F^{(1)-}$ or $(a,b),(c,d) \in F^{(1)+}$ and for every pair $(\rho,\eta) \in \mathcal{P}^{X,g}(a,d) \times \mathcal{P}^{X,g}(c,b)$, let r be the smallest element of $\frac{1}{2}\mathbb{Z}$ such that X_r is defined on the pair (ρ,η). Set $f(\rho,\eta) := \mathsf{X}_r(\rho,\eta)$.

For $(a,b) \in \overline{F^{(1)+}}$, $(c,d) \in \overline{F^{(1)-}}$ we simply set $f := \mathsf{X}_{\text{bond}}$ concluding the proof. $\square$

Proof of Theorem 7 (Section 2.1, page 102). Let us first prove Conjecture 2 for $k = 2$. Condition 2.8 holds trivially for $m = 1$, since the only path in $\mathcal{P}^g(i_r^{(1)}, i_r^{(1)} + r)$ is the point $(i_r^{(1)}, i_r^{(1)} + r)$ itself, whose weight is $x_r^{(1)}$. For $m = 2$; it is immediate from the definition of bonding paths and Lemma 24, that the only non-intersecting, non-crossing, non-bonding pair from

$$\mathcal{P}^g(i_r^{(1)}, i_r^{(1)} + r) \times \mathcal{P}^g(i_r^{(2)}, i_r^{(2)} + r),$$

is $((i_r^{(1)}, i_r^{(1)} + r), \rho)$, where ρ is the unique path from $\mathcal{P}^g(i_r^{(2)}, i_r^{(2)} + r)$ which visits $(i_r^{(2)}, i_r^{(2)}+r)$. From the definition of the path weight w we obtain that $w(\rho) = x_r^{(2)}/x_r^{(1)} = 1/x_r^{(1)}$ and so $w((i_r^{(1)}, i_r^{(1)} + r), \rho) = 1$ as wanted.

For the SL_2 condition; consider $(a, b), (a + 1, b + 1) \in F^{(1)-}$ (the case $\in F^{(1)+}$ is analogous). From Lemma 37, any pair (ρ, η) of non-intersecting, non-crossing, non-bonding paths from $\mathcal{P}^g(a, b) \times \mathcal{P}^g(a + 1, b + 1)$ satisfies that $\rho[r] < \eta[r]$ for all $r \in \pi^g(a, b) \cap \pi^g(a+1, b+1)$. Since $\pi^g(a, b) \subset \pi^g(a+1, b+1)$, this means that $\rho[r] = 1$ for all $r \in \pi^g(a, b)$ and from Lemma 29; $\eta[r] = 1$ for all $r \in \partial\pi^g(a+1, b+1)$ and $\eta[r] = 2$ for all $r \in \pi^g(a+1, b+1) - \partial\pi^g(a+1, b+1)$. These properties describe the paths ρ, η uniquely, and a straightforward calculation yields $w(\rho, \eta) = w(\rho)w(\eta) = 1$, as wanted.

We have proven Conjecture 2 for $k = 2$. Conjecture 1 is then immediate by Lemma 26, since we have already proved the uniqueness statement.

□

2.3 Inductive approach

As mentioned before, one of our main motivations is to step closer to a complete proof of Conjecture 1. Several authors (Di Francesco, 2010; Di Francesco and Kedem, 2009; Speyer, 2007) have provided very general partial proofs of this conjecture. In particular the two reviewed partial proofs imply the Laurent positivity property for all the entries (1×1 minors) of an SL_k-tiling under the general boundary conditions of Conjecture 1. In this section we provide a new proof of this property. This proof has the advantage that it is purely inductive, and thus provides a fast algorithm to compute entries of an SL_k-tiling in terms of boundary conditions.

The idea is to prove a series of determinant identities that recursively allow us to isolate each entry in terms of preceding ones according to some inductive partial ordering. To understand these identities we need to examine some properties of a generic dress $f = \{F^{(m)}\}_{1 \leq m \leq}$, and to introduce some new parameters.

Let $f = \{F^{(m)}\}_{1\leq m\leq k-1}$ be a k-dress as in Conjecture 1. Let g be a $(k+1)$-dress obtained from f by adding a k-fringe $F^{(k)}$. As in the previous sections, denote by $v_r^{(m)}$ the elements of the defining sequence of g. For $r \in \mathbb{Z}$, $m \in \{1,\ldots,k\}$, define $u_r^{(m)}$ to be the first entry of $v_r^{(m)}$, so that $v_r^{(m)} = (u_r^{(m)}, u_r^{(m)} + r)$, and define the sequence $\{\epsilon_r^{(m)}\}_{\substack{m\geq 1,\\ r\in\mathbb{Z}}}$ in terms of the sequence $\{i_r^{(m)}\}_{\substack{m\geq 1\\ r\in\mathbb{Z}}}$ as explained next. Observe that for $r \in \mathbb{Z}$, $m \in \{1,\ldots,k\}$;

$$\{u_r^{(1)},\ldots,u_r^{(m)}\} = \{i_r^{(m)},\ldots,i_r^{(m)} + m - 1\}.$$

This means that for all $r \in \mathbb{Z}, m \geq 1$, the set $\{u_r^{(1)},\ldots,u_r^{(m)}\}$ consists of consecutive integers, which forces $u_r^{(m)}$ to be equal to either $\max\{u_r^{(1)},\ldots,u_r^{(m)}\} = i_r^{(m)} + m - 1$ or $\min\{u_r^{(1)},\ldots,u_r^{(m)}\} = i_r^{(m)}$. Set $\epsilon_r^{(m)} := 0$ in the first case, and $\epsilon_r^{(m)} := 1$ otherwise. One could easily verify that for $r \in \mathbb{Z}, m \geq 2$;

$$\epsilon_r^{(m)} = i_r^{(m-1)} - i_r^{(m)}.$$

Let us exemplify these parameters on the dress of Figure 2.2. For $r = -1$ we have;

$$\begin{array}{ccc}
(\; i_r^{(1)},\; i_r^{(2)},\; i_r^{(3)},\; i_r^{(4)},\; i_r^{(5)} \;) & = & (\; 1,\; 0,\; -1,\; -1,\; -1 \;) \\
(\; u_r^{(1)},\; u_r^{(2)},\; u_r^{(3)},\; u_r^{(4)},\; u_r^{(5)} \;) & = & (\; 1,\; 0,\; -1,\; 2,\; 3 \;) \\
(\; \epsilon_r^{(1)},\; \epsilon_r^{(2)},\; \epsilon_r^{(3)},\; \epsilon_r^{(4)},\; \epsilon_r^{(5)} \;) & = & (\; 0,\; 1,\; 1,\; 0,\; 0 \;)
\end{array}$$

It is straightforward from a successive application of the Desnanot-Jacobi identity, that every $(k-1) \times (k-1)$ connected minor of the matrix P from Conjecture 1, must be different from zero. Again by the Desnanot-Jacobi identity, this implies that every $(k+1) \times (k+1)$ connected minor of the matrix P from Conjecture 1, is equal to 0. Thus P has rank k and so every $(k+1) \times (k+1)$ minor of P is equal to 0. We extend the $(k+1)$-dress g to an ∞-dress by adding fringes $F^{(m)}$ for $m > k$. This also extends all the sequences above to $m > k$. To be consistent with this extension we set $x_r^{(m)} = 0$ for $m > k$.

Even though it makes for redundant notation, it is convenient to consider the bijections $U_r : \mathbb{N}^+ \to \mathbb{Z}$ $(r \in \mathbb{Z})$, defined by

$$U_r(m) := u_r^{(m)} \text{ for } m \geq 1,$$

and $V_r : \mathbb{N}^+ \to \mathbb{Z}$ ($r \in \mathbb{Z}$), given by

$$V_r(m) := U_r(m) + r = u_r^{(m)} + r \text{ for } m \geq 1.$$

We may then rewrite equation (2.8) as;

$$\det(P_{U_r\{1,\ldots,m\}\,V_r\{1,\ldots,m\}}) = x_r^{(m)} \quad (m \leq k). \tag{2.26}$$

Recall that for any bijection σ and any subset I of its domain, the inversion number of σ over I, denoted $\mathrm{inv}_I(\sigma)$, is the number of pairs $(a, b) \in I \times I$ such that $a < b$ and $\sigma(a) > \sigma(b)$. Recall also the notation $\mathrm{sign}_I(\sigma) := (-1)^{\mathrm{inv}_I(\sigma)}$. It can be shown that for $I = \{m_1, \ldots, m_d\} \subseteq \mathbb{N}^+$ with $1 \leq m_1 < \cdots < m_d$;

$$\mathrm{inv}_I(U_r) = \mathrm{inv}_I(V_r) = \sum_{t=1}^{d} (t-1)\epsilon_r^{(m_d)}.$$

We wish to prove that under equation (2.8), the entries of P are Laurent polynomials with positive integer coefficients in the $x_r^{(m)}$. We first show that certain particular family of minors of P are positive Laurent polynomials by showing that they satisfy some convenient recurrence relations. Let $r \in \mathbb{Z}$, $m, i, j \in \{1, \ldots, k\}$ be integers with $i, j \geq m + 1$. Define;

$$P(r, m, i, j) := \det(P_{U_r\{1,\ldots,m,i\}\,V_r\{1,\ldots,m,j\}})$$

The following is immediate from (2.26). It is conveniently stated as a lemma for the purpose of future reference.

Lemma 39. *For $r \in \mathbb{Z}$, $m \in \{1, \ldots, k-1\}$, we have $P(r, m, m+1, m+1) = x_r^{(m)}$.*

The following lemma allows for the expansion of some of these minors in terms of other minors.

Lemma 40. *For $r \in \mathbb{Z}$, $m, i, j \in \{1, \ldots, k\}$ with $i, j > m + 1$;*

$$\begin{aligned} &P(r, m, i, j) \\ &= \frac{1}{x_r^{(m+1)}} P(r, m, m+1, j) P(r, m, i, m+1) \\ &\quad + (-1)^{\epsilon_r^{(i)} + \epsilon_r^{(j)}} \frac{x_r^{(m)}}{x_r^{(m+1)}} P(r, m+1, i, j) \end{aligned} \tag{2.27}$$

Proof. By the Desnanot-Jacobi identity (Proposition 1) and Lemma 5 (with the permutations U_r and V_r) we have;

$$\begin{aligned}
&\det(P_{U_r\{1,\ldots,m,i\}\, V_r\{1,\ldots,m,j\}}) \det(P_{U_r\{1,\ldots,m+1\}\, V_r\{1,\ldots,m+1\}}) \\
&\quad - \det(P_{U_r\{1,\ldots,m+1\}\, V_r\{1,\ldots,m,j\}}) \det(P_{U_r\{1,\ldots,m,i\}\, V_r\{1,\ldots,m+1\}}) \\
&= (-1)^{\epsilon_r^{(i)}+\epsilon_r^{(j)}} \det(P_{U_r\{1,\ldots,m\}\, V_r\{1,\ldots,m\}}) \det(P_{U_r\{1,\ldots,m+1,i\}\, V_r\{1,\ldots,m+1,j\}}),
\end{aligned}$$

which is equivalent to 2.27. ☐

This recurrence, however, is not sufficient for expanding $P(r,m,i,j)$ in terms of the $x_r^{(m)}$, since the right hand side contains minors of the form $P(r,m,i',j')$ which do not satisfy $i',j' > m+1$. This situation is partially addressed in the next lemma.

Lemma 41. *Let $r \in \mathbb{Z}$, $m, i \in \{1, \ldots, k\}$ be integers with $i > m+1$, $m \geq 1$, and $\epsilon_r^{(i)} = 0$. Set*

$$i' := U_{r-1}^{-1}(u_r^{(i)}), \quad j' := U_{r-1}^{-1}(u_r^{(m+1)} + 1), \quad j'' := U_{r-1}^{-1}(i_r^{(m)} + m),$$

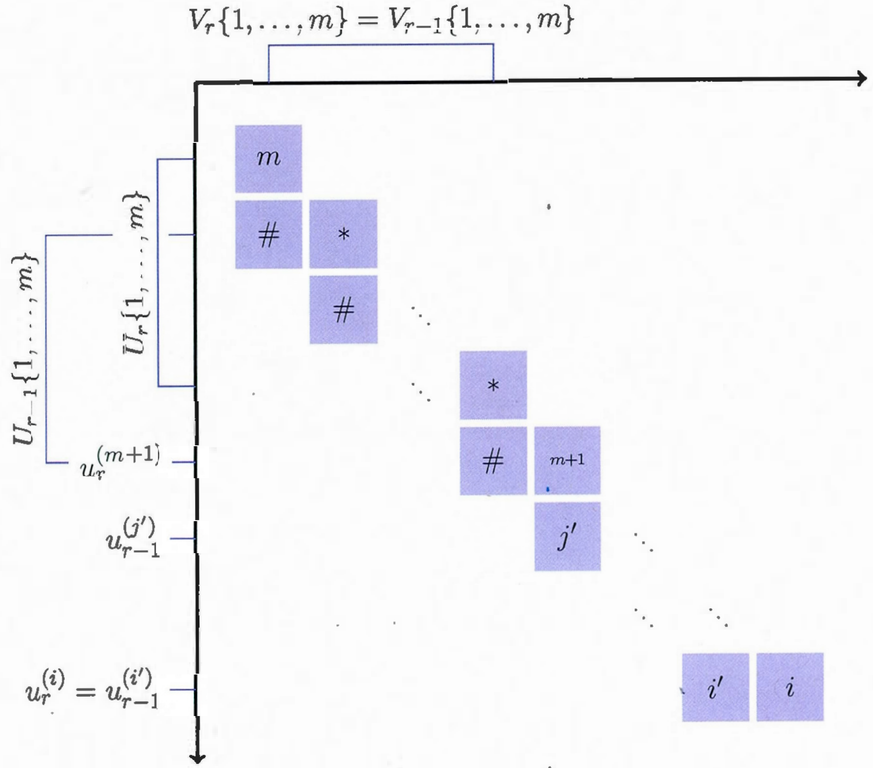
$$M := \max(U_r\{1, \ldots, m\}).$$

Then P(r,m,i,m+1) is equal to;

$$\begin{cases}
\frac{x_r^{(m+1)}}{x_{r-1}^{(m)}} P(r, m-1, i, m) + \frac{x_r^{(m)}}{x_{r-1}^{(m)}} P(r-1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_r^{(m)} = i_{r-1}^{(m)} - 1 \\ \epsilon_r^{(m)} = 1 \end{cases} \\
\frac{x_r^{(m+1)}}{x_{r-1}^{(m)}} P(r-1, m-1, i', m) + \frac{x_r^{(m)}}{x_{r-1}^{(m)}} P(r-1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_r^{(m)} = i_{r-1}^{(m)} - 1 \\ \epsilon_r^{(m)} = 0 \end{cases} \\
\frac{x_r^{(m+1)}}{x_{r-1}^{(m+1)}} P(r-1, m, i', m+1) + \frac{x_r^{(m)}}{x_{r-1}^{(m+1)}} P(r-1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_r^{(m)} = i_{r-1}^{(m)} \end{cases} \\
x_{r-1}^{(m-1)} & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_r^{(m)} = i_{r-1}^{(m)} - 1 \\ u_r^{(i)} = M+1 \end{cases} \\
\frac{x_{r-1}^{(m+1)}}{x_{r-1}^{(m)}} P(r, m-1, i, m) + \frac{x_r^{(m)}}{x_{r-1}^{(m)}} P(r-1, m, i', m+1) & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_r^{(m)} = i_{r-1}^{(m)} - 1 \\ u_r^{(i)} > M+1 \\ \epsilon_r^{(m)} = 1 \end{cases} \\
\frac{x_{r-1}^{(m+1)}}{x_{r-1}^{(m)}} P(r-1, m-1, i', m) + \frac{x_r^{(m)}}{x_{r-1}^{(m)}} P(r-1, m, i', m+1) & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_r^{(m)} = i_{r-1}^{(m)} - 1 \\ u_r^{(i)} > M+1 \\ \epsilon_r^{(m)} = 0 \end{cases} \\
P(r-1, m, i', j'') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_r^{(m)} = i_{r-1}^{(m)} \end{cases}
\end{cases}$$

Moreover, in each of the cases that i' appears, it is $\geq m+1$ and satisfies $\epsilon_{r-1}^{(i')} = 0$. Similarly for j' and j''.

Proof. **Case 1:** With the given conditions, the r-th and $(r-1)$-th diagonals of the dress f are as follows;



(i' could be equal to j'), where the $*$'s represent all the numbers from 1 to $m-1$, and the $\#$'s represent all the numbers from 1 to m in some order. From equation 1.18;

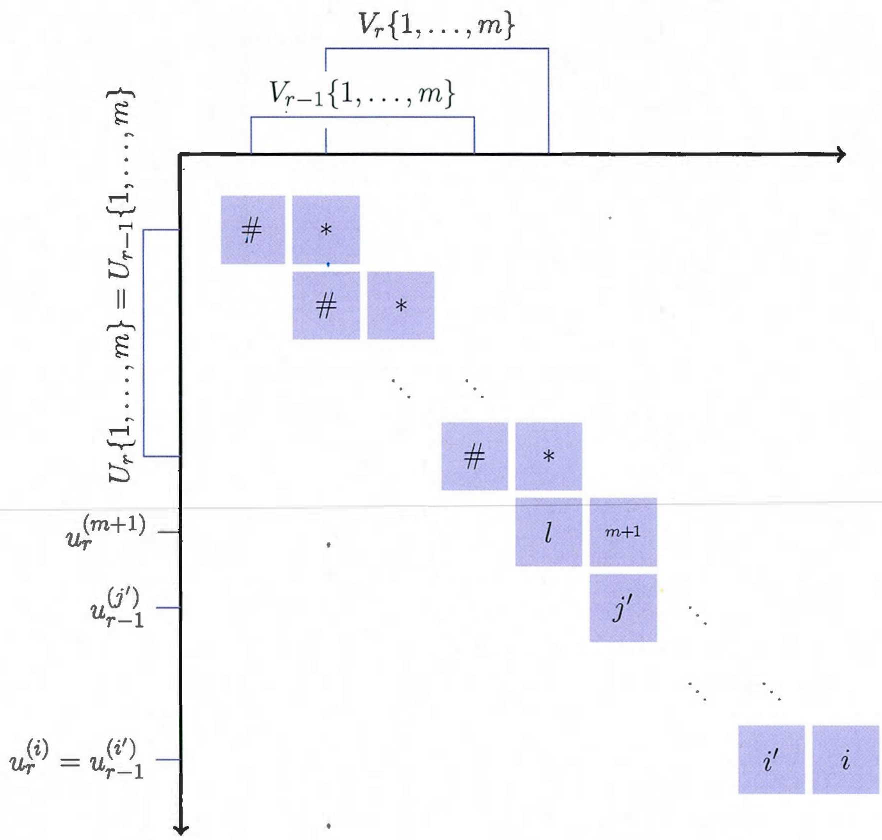
$$\begin{aligned}
&\det(P_{U_{r-1}\{1,\dots,m\}\,V_{r-1}\{1,\dots,m\}})\det(P_{U_r\{1,\dots,m,i\}\,V_r\{1,\dots,m+1\}}) \\
&- \det(P_{U_r\{1,\dots,m+1\}\,V_r\{1,\dots,m+1\}})\det(P_{U_r\{1,\dots,m-1,i\}\,V_r\{1,\dots,m\}}) \\
&= \det(P_{U_r\{1,\dots,m\}\,V_r\{1,\dots,m\}})\det(P_{U_{r-1}\{1,\dots,m,i'\}\,V_{r-1}\{1,\dots,m,j'\}}),
\end{aligned}$$

which is equivalent to the desired recurrence.

Case 2: The picture is the same as the one for **Case 1**, except that the m is now on the position of the bottom $*$. This forces an m at the position of the bottom $\#$ (otherwise the adjacency property of fringes would be contradicted). Again from 1.18;

$$\det(P_{U_{r-1}\{1,\ldots,m\}\,V_{r-1}\{1,\ldots,m\}})\det(P_{U_r\{1,\ldots,m,i\}\,V_r\{1,\ldots,m+1\}})$$
$$-\det(P_{U_r\{1,\ldots,m+1\}\,V_r\{1,\ldots,m+1\}})\det(P_{U_{r-1}\{1,\ldots,m-1,i'\}\,V_{r-1}\{1,\ldots,m\}})$$
$$=\det(P_{U_r\{1,\ldots,m\}\,V_r\{1,\ldots,m\}})\det(P_{U_{r-1}\{1,\ldots,m,i'\}\,V_{r-1}\{1,\ldots,m,j'\}}).$$

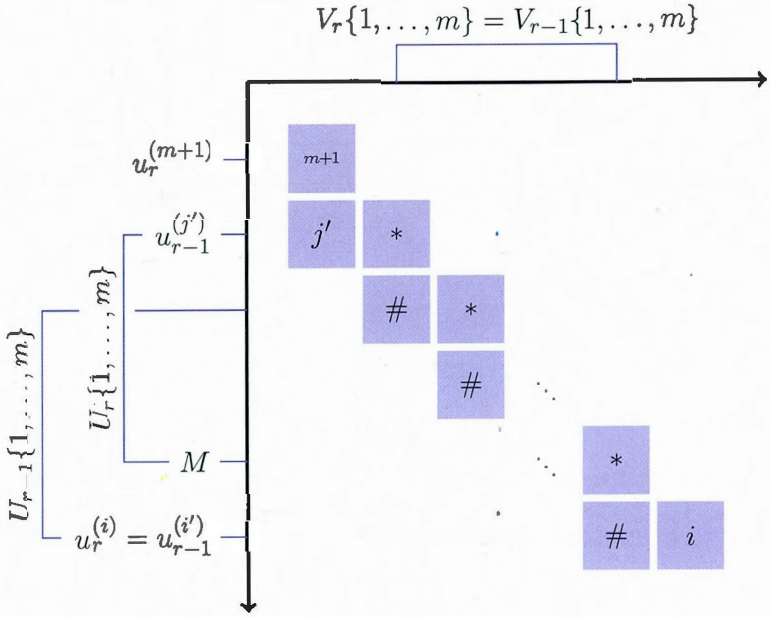
Case 3: In this case the $(r-1)$-th and r-th diagonals of f are as follows;



(as before, i' could be equal to j'), where both the $*$'s and the $\#$'s represent all the numbers from 1 to m in some order, and l is some integer greater than m. In fact, l has to be equal to $m+1$, since otherwise the adjacency property would be contradicted. The Desnanot-Jacobi identity hence yields;

$$\det(P_{U_{r-1}(\{1,\ldots,m+1\})\,V_{r-1}(\{1,\ldots,m+1\})})\det(P_{U_r(\{1,\ldots,m,i\})\,V_r(\{1,\ldots,m+1\})})$$
$$-\det(P_{U_r(\{1,\ldots,m+1\})\,V_r(\{1,\ldots,m+1\})})\det(P_{U_{r-1}(\{1,\ldots,m,i'\})\,V_{r-1}(\{1,\ldots,m+1\})})$$
$$=\det(P_{U_r(\{1,\ldots,m\})\,V_r(\{1,\ldots,m\})})\det(P_{U_{r-1}(\{1,\ldots,m+1,i'\})\,V_{r-1}(\{1,\ldots,m+1,j'\})}),$$

Case 4: The r-th and $(r-1)$-th diagonals of the dress f are as follows;

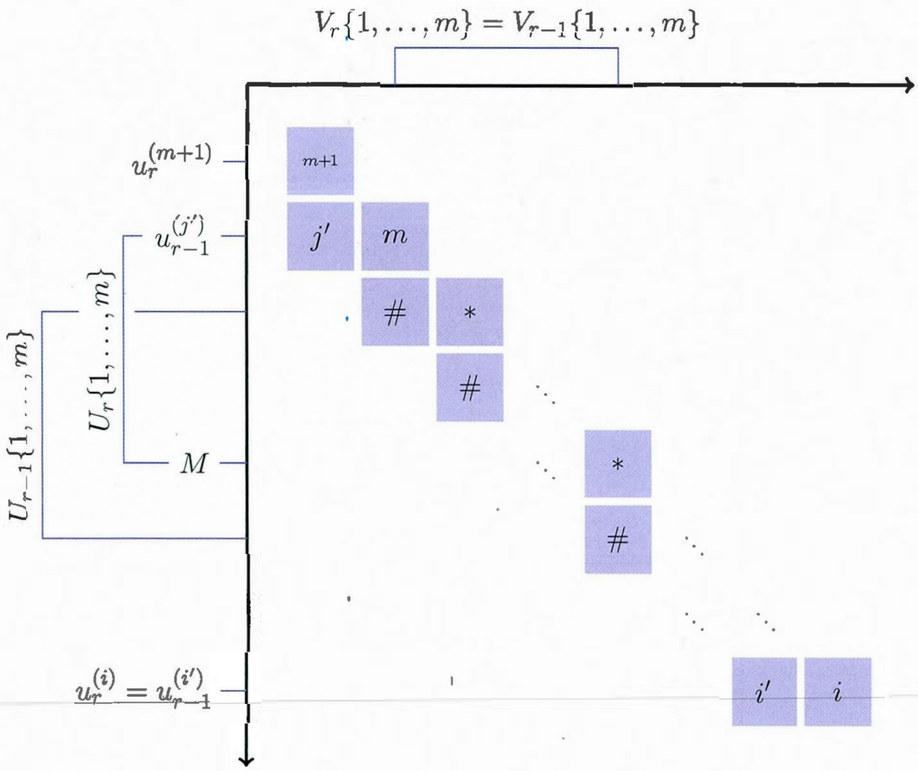


where both the $*$'s and the #'s represent all the numbers from 1 to m in some order. Notice that j' has to be equal to $m+1$ to avoid contradicting the adjacency property. Hence;

$$\det(P_{U_r\{1,\ldots,m,i\}\,V_r\{1,\ldots,m+1\}}) = \det(P_{U_{r-1}\{1,\ldots,m+1\}\,V_{r-1}\{1,\ldots,m+1\}}) = x_{r-1}^{(m+1)},$$

as wanted.

Case 5: The r-th and $(r-1)$-th diagonals of the dress f are as follows;



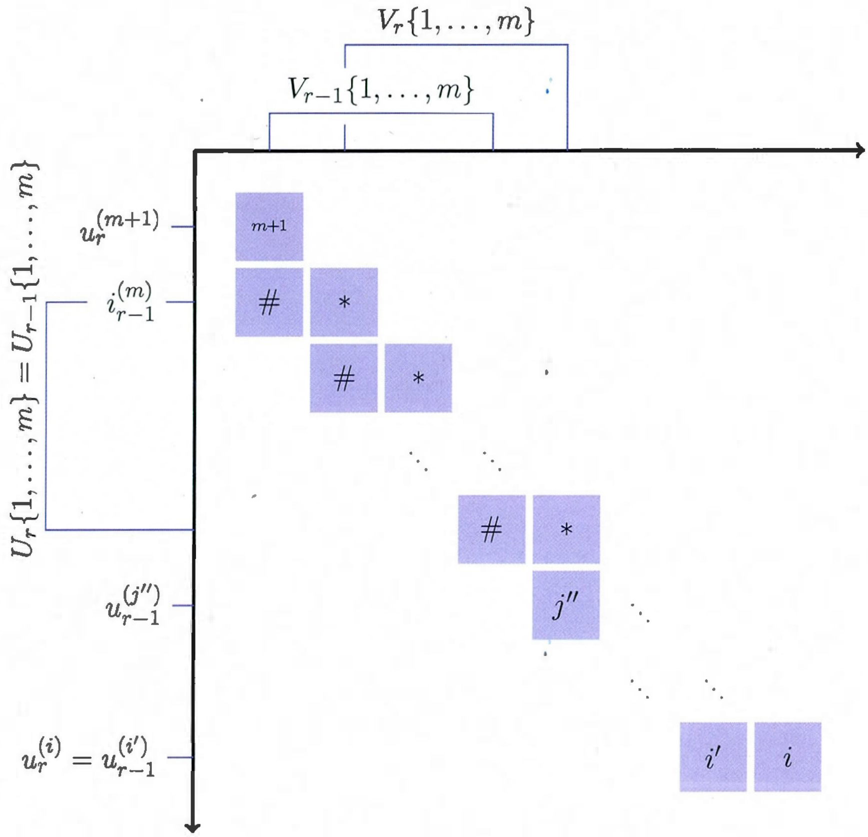
where the $*$'s represent all the number from 1 to $m-1$, and the $\#$'s represent all the numbers from 1 to m in some order. By the adjacency property we have that $j' = m+1$. From the identity 1.19;

$$\begin{aligned}
&\det(P_{U_{r-1}\{1,\ldots,m\}\,V_{r-1}\{1,\ldots,m\}})\det(P_{U_r\{1,\ldots,m,i\}\,V_r\{1,\ldots,m+1\}}) \\
&-\det(P_{U_{r-1}\{1,\ldots,m+1\}\,V_{r-1}\{1,\ldots,m+1\}})\det(P_{U_r\{1,\ldots,m-1,i\}\,V_r\{1,\ldots,m\}}) \\
&=\det(P_{U_r\{1,\ldots,m\}\,V_r\{1,\ldots,m\}})\det(P_{U_{r-1}\{1,\ldots,m,i'\}\,V_{r-1}\{1,\ldots,m+1\}}).
\end{aligned}$$

Case 6: The picture in this case is the same as the one for **Case 5**, except that the m is now on the position of the bottom $*$. This forces an m at the bottom $\#$, since the opposite would contradict the adjacency property. As before $j' = m+1$. Thus the same identity yields;

$$\det(P_{U_{r-1}\{1,\dots,m\}\,V_{r-1}\{1,\dots,m\}})\det(P_{U_r\{1,\dots,m,i\}\,V_r\{1,\dots,m+1\}})$$
$$-\det(P_{U_{r-1}\{1,\dots,m+1\}\,V_{r-1}\{1,\dots,m+1\}})\det(P_{U_{r-1}\{1,\dots,m-1,i'\}\,V_{r-1}\{1,\dots,m\}})$$
$$=\det(P_{U_r\{1,\dots,m\}\,V_r\{1,\dots,m\}})\det(P_{U_{r-1}\{1,\dots,m,i'\}\,V_{r-1}\{1,\dots,m+1\}}).$$

Case 7: The r-th and $(r-1)$-th diagonals of f are as follows;



(i' could be equal to j''). Thus;

$$\det(P_{U_r\{1,\dots,m,i\}\,V_r\{1,\dots,m+1\}}) = \det(P_{U_{r-1}\{1,\dots,m,i'\}\,V_{r-1}\{1,\dots,m,j''\}}),$$

as wanted. □

The next three lemmas are proven using the same identities case by case as those from the previous proof. For Lemma 42, the minors from each identity are transposed. For

Lemma 43 they are reflected along the anti-diagonal. For Lemma 44, they are reflected along both diagonals.

Lemma 42. *Let r, m, j be integers with $j > m+1$, $m \geq 1$, and $\epsilon_r^{(j)} = 0$. Set*

$$j' := U_{r+1}^{-1}(u_r^{(j)} - 1), \quad i' := U_{r+1}^{-1}(u_r^{(m+1)}), \quad i'' := U_{r+1}^{-1}(i_r^{(m)} + m - 1),$$

$$M := \max(U_r\{1, \ldots, m\}).$$

Then P(r,m,m+1,j) is equal to;

$$\begin{cases}
\frac{x_r^{(m+1)}}{x_{r+1}^{(m)}} P(r, m-1, m, j) + \frac{x_r^{(m)}}{x_{r+1}^{(m)}} P(r+1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r+1}^{(m)} = i_r^{(m)} \\ \epsilon_r^{(m)} = 1 \end{cases} \\
\frac{x_r^{(m+1)}}{x_{r+1}^{(m)}} P(r+1, m-1, m, j') + \frac{x_r^{(m)}}{x_{r+1}^{(m)}} P(r+1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r+1}^{(m)} = i_r^{(m)} \\ \epsilon_r^{(m)} = 0 \end{cases} \\
\frac{x_r^{(m+1)}}{x_{r+1}^{(m+1)}} P(r+1, m, m+1, j') + \frac{x_r^{(m)}}{x_{r+1}^{(m+1)}} P(r+1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r+1}^{(m)} = i_r^{(m)} - 1 \end{cases} \\
x_{r+1}^{(m-1)} & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r+1}^{(m)} = i_r^{(m)} \\ u_r^{(j)} = M+1 \end{cases} \\
\frac{x_{r+1}^{(m+1)}}{x_{r+1}^{(m)}} P(r, m-1, m, j) + \frac{x_r^{(m)}}{x_{r+1}^{(m)}} P(r+1, m, m+1, j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r+1}^{(m)} = i_r^{(m)} \\ u_r^{(j)} > M+1 \\ \epsilon_r^{(m)} = 1 \end{cases} \\
\frac{x_{r+1}^{(m+1)}}{x_{r+1}^{(m)}} P(r+1, m-1, m, j') + \frac{x_r^{(m)}}{x_{r+1}^{(m)}} P(r+1, m, m+1, j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r+1}^{(m)} = i_r^{(m)} \\ u_r^{(j)} > M+1 \\ \epsilon_r^{(m)} = 0 \end{cases} \\
P(r+1, m, i'', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r+1}^{(m)} = i_r^{(m)} - 1 \end{cases}
\end{cases}$$

Moreover, in each of the cases that i' appears, it is $\geq m+1$ and satisfies $\epsilon_{r+1}^{(i')} = 0$. Similarly for j' and i''.

Lemma 43. *Let r, m, j be integers with $j > m+1$, $m \geq 1$, and $\epsilon_r^{(j)} = 1$. Set*

$$j' := U_{r-1}^{-1}(u_r^{(j)} + 1), \quad i' := U_{r-1}^{-1}(u_r^{(m+1)}), \quad i'' := U_{r-1}^{-1}(i_r^{(m)}).$$

Then P(r,m,m+1,j) is equal to;

$$\begin{cases}
\frac{x_r^{(m+1)}}{x_{r-1}^{(m)}} P(r, m-1, m, j) + \frac{x_r^{(m)}}{x_{r-1}^{(m)}} P(r-1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r-1}^{(m)} = i_r^{(m)} \\ \epsilon_r^{(m)} = 0 \end{cases} \\
\frac{x_r^{(m+1)}}{x_{r-1}^{(m)}} P(r-1, m-1, m, j') + \frac{x_r^{(m)}}{x_{r-1}^{(m)}} P(r-1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r-1}^{(m)} = i_r^{(m)} \\ \epsilon_r^{(m)} = 1 \end{cases} \\
\frac{x_r^{(m+1)}}{x_{r-1}^{(m+1)}} P(r-1, m, m+1, j') + \frac{x_r^{(m)}}{x_{r-1}^{(m+1)}} P(r-1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r-1}^{(m)} = i_r^{(m)} + 1 \end{cases} \\
x_{r-1}^{(m-1)} & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r-1}^{(m)} = i_r^{(m)} \\ u_r^{(j)} = i_r^{(m)} - 1 \end{cases} \\
\frac{x_{r-1}^{(m+1)}}{x_{r-1}^{(m)}} P(r, m-1, m, j) + \frac{x_r^{(m)}}{x_{r-1}^{(m)}} P(r-1, m, m+1, j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r-1}^{(m)} = i_r^{(m)} \\ u_r^{(j)} < i_r^{(m)} - 1 \\ \epsilon_r^{(m)} = 1 \end{cases} \\
\frac{x_{r-1}^{(m+1)}}{x_{r-1}^{(m)}} P(r-1, m-1, m, j') + \frac{x_r^{(m)}}{x_{r-1}^{(m)}} P(r-1, m, m+1, j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r-1}^{(m)} = i_r^{(m)} \\ u_r^{(j)} < i_r^{(m)} - 1 \\ \epsilon_r^{(m)} = 1 \end{cases} \\
P(r-1, m, i'', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r-1}^{(m)} = i_r^{(m)} + 1 \end{cases}
\end{cases}$$

Moreover, in each of the cases that i' appears, it is $\geq m+1$ and satisfies $\epsilon_{r-1}^{(i')} = 0$. Similarly for j' and i''.

Lemma 44. *Let r, m, i be integers with $i > m+1$, $m \geq 1$, and $\epsilon_r^{(i)} = 1$. Set*

$$i' := U_{r+1}^{-1}(u_r^{(i)}), \quad j' := U_{r+1}^{-1}(u_r^{(m+1)} - 1), \quad j'' := U_{r+1}^{-1}(i_r^{(m)} - 1).$$

Then P(r,m,i,m+1) is equal to;

$$\begin{cases}
\frac{x_r^{(m+1)}}{x_{r+1}^{(m)}} P(r, m-1, i, m) + \frac{x_r^{(m)}}{x_{r+1}^{(m)}} P(r+1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r+1}^{(m)} = i_r^{(m)} - 1 \\ \epsilon_r^{(m)} = 0 \end{cases} \\
\frac{x_r^{(m+1)}}{x_{r+1}^{(m)}} P(r+1, m-1, i', m) + \frac{x_r^{(m)}}{x_{r+1}^{(m)}} P(r+1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r+1}^{(m)} = i_r^{(m)} - 1 \\ \epsilon_r^{(m)} = 1 \end{cases} \\
\frac{x_r^{(m+1)}}{x_{r+1}^{(m+1)}} P(r+1, m, i', m+1) + \frac{x_r^{(m)}}{x_{r+1}^{(m+1)}} P(r+1, m, i', j') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 1, \\ i_{r+1}^{(m)} = i_r^{(m)} \end{cases} \\
x_{r+1}^{(m-1)} & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r+1}^{(m)} = i_r^{(m)} - 1 \\ u_r^{(i)} = i_r^{(m)} - 1 \end{cases} \\
\frac{x_{r+1}^{(m+1)}}{x_{r+1}^{(m)}} P(r, m-1, i, m) + \frac{x_r^{(m)}}{x_{r+1}^{(m)}} P(r+1, m, i', m+1) & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r+1}^{(m)} = i_r^{(m)} - 1 \\ u_r^{(i)} < i_r^{(m)} - 1 \\ \epsilon_r^{(m)} = 1 \end{cases} \\
\frac{x_{r+1}^{(m+1)}}{x_{r+1}^{(m)}} P(r+1, m-1, i', m) + \frac{x_r^{(m)}}{x_{r+1}^{(m)}} P(r+1, m, i', m+1) & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r+1}^{(m)} = i_r^{(m)} - 1 \\ u_r^{(i)} < i_r^{(m)} - 1 \\ \epsilon_r^{(m)} = 1 \end{cases} \\
P(r+1, m, i', j'') & \text{if } \begin{cases} \epsilon_r^{(m+1)} = 0, \\ i_{r+1}^{(m)} = i_r^{(m)} \end{cases}
\end{cases}$$

Moreover, in each of the cases that i' appears, it is $\geq m+1$ and satisfies $\epsilon_{r+1}^{(i')} = 1$. Similarly for j' and j''.

Observe that Lemmas 39, 40, 41, 42 write each minor of the forms

$$\begin{gathered}P(r,m,i,j),\ P(r,m,i,m+1),\ P(r,m,m+1,j),\ P(r,m,m+1,m+1)\\(i,j>m+1,\ \epsilon_r^{(i)}=\epsilon_r^{(j)}=0),\end{gathered}\tag{2.28}$$

either as a Laurent monomial in the variables $x_r^{(m)}$ or as a sum of other such minors times Laurent monomials in the $x_r^{(m)}$. On the other hand Lemmas 39, 40, 43, 44 do the same for minors of the form;

$$\begin{gathered}P(r,m,i,j),\ P(r,m,i,m+1),\ P(r,m,m+1,j),\ P(r,m,m+1,m+1)\\(i,j>m+1,\ \epsilon_r^{(i)}=\epsilon_r^{(j)}=1).\end{gathered}\tag{2.29}$$

We use the first observation to show that every minor of the forms 2.28 is a Laurent polynomial with positive integer coefficients in the $x_r^{(m)}$. An analogous argument works for minors of the forms 2.29.

Lemmas 40-42 write every minor $\det(P_{IJ})$ of the forms 2.28 in terms of determinants of submatrices $P_{I'J'}$ for $I' \times J'$ contained in the region;

$$R(I,J) := \{(a,b) \in F^{(f(\max(I),\max(J)))} : a \leq \max(I),\ b \leq \max(J)\},$$

The region $R(I,J)$ is always finite (and thus it contains a finite number of submatrices) as a result of the staircase property of the fringe $F^{(f(\max(I),\max(J)))}$. Hence one cannot apply these Lemmas indefinitely, and so every minor of the forms 2.28 may be written (by iteration of these Lemmas) as a Laurent polynomial in the $x_r^{(m)}$ with positive integer coefficients. Similarly we also conclude that every minor of the forms 2.29 is a Laurent polynomial in the $x_r^{(m)}$ with positive integer coefficients.

Consider now any $(a,b) \in \mathbb{Z} \times \mathbb{Z}$. Set $r := b-a$, so that $(a,b) \in D_r$. If $U_r^{-1}(a) = 1$ (equivalently $(a,b) \in F^{(1)}$), then $(a,b) = (u_r^{(1)}, u_r^{(1)} + r)$ and so $P_{ab} = x_r^{(1)}$. Otherwise, if $U_r^{-1}(a) = i > 1$ (equivalently $(a,b) \in F^{(i)} - F^{(i-1)}$), then $(a,b) = (u_r^{(i)}, u_r^{(i)} + r)$. Notice that the statement $\epsilon_r^{(i)} = 0$ means that $(a,b) \in F^{-}$. Next we prove that for all $(a,b) \in F^{-}$, p_{ab} is a Laurent polynomial with positive integer coefficients in the $x_r^{(m)}$. The same result for $(a,b) \in F^{+}$ follows from an analogous argument. Write p_{ab} as follows;

$$\begin{aligned} p_{ab} &= \frac{1}{p_{u_r^{(1)}(u_r^{(1)}+r)}}\left(p_{u_r^{(1)}(u_r^{(1)}+r)}p_{ab} - p_{a(u_r^{(1)}+r)}P_{u_r^{(1)}b} + P_{a(u_r^{(1)}+r)}p_{u_r^{(1)}b}\right) \\ &= \frac{1}{x_r^{(1)}}\left(P(r,1,i,i) + p_{a(u_r^{(1)}+r)}p_{u_r^{(1)}b}\right). \end{aligned}$$

Clearly $(a, u_r^{(1)}+r), (u_r^{(1)}, b) \in F^- \cup F^{(1)}$, and $R(a, u_r^{(1)}+r), R(u_r^{(1)}, b) \subset R(a,b)$. Hence this recurrence may not be applied indefinitely, and by iteration we can write every p_{ab} ($(a,b) \in F^-$) as a Laurent polynomial with positive integer coefficients in X. Analogously for $(a,b) \in F^+$. Therefore every entry of P is a positive integral Laurent polynomial in X, as wanted.

CHAPTER III

CONCLUSIONS AND FUTURE WORK

This chapter is intended as a short conclusion for the main portion of this thesis; Chapter 2, as we consider the original results from Chapter 1 to be self-contained and less relevant to our ongoing research.

As mentioned several times in this work, one of our main motivations has been to step closer to a proof of Conjecture 1. Several authors (Bergeron and Reutenauer, 2010; Di Francesco, 2010; Di Francesco and Kedem, 2009; Speyer, 2007) have provided very general partial proofs of this conjecture. Our main result of the previous chapter, Theorem 7, is less general in the sense that it only deals with the case $k = 2$, but it considers minors which the other mentioned proofs ignore, namely those which enclose the entries from the boundary conditions of the SL_2-tiling. Our combinatorial model provides hope for a general proof for any value of k, which would likely require a stronger definition of path intersection extending the notion of bonding paths.

Our inductive calculation (Section 2.3) of the Laurent positive entries of an SL_k-tiling under general boundary conditions involves division only by monomials, which is generally more efficient than a division by polynomials of more than one term. This may be a first step towards a complete inductive proof of Conjecture 1. Such proof may be achieved in the future with the use of identities similar to those arising from Theorem 3.

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