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RÉSUMÉ

Nous étudions un problème de chirurgie de Dehn, à savoir la caractérisation des noeuds dans les espaces lenticulaires qui admettent des chirurgies intégrales homéomorphes à $S^1 \times S^2$. Nous montrons que ces nœuds sont fibrés et qu'ils bordent des surfaces de Seifert planaires. De façon équivalente, les nœuds induits dans $S^1 \times S^2$ sont isotopes à des tresses.

Le principal outil que nous avons utilisé est l'homologie de Heegaard-Floer, un ensemble d'invariants de type théorie de jauge développés par Ozsváth-Szabó à partir de 2000. En outre, nous montrons que ces nœuds sont simples au sens de Floer, donc conjecturalement simples. Compte tenu de cette dernière conjecture, nous avons initié une étude de nœuds simples dans les espaces lenticulaires appropriés et nous avons donné une liste potentiellement complète de tous les nœuds simples avec des chirurgies intégrales $S^1 \times S^2$. Ces nœuds se révèlent être les nœuds induits dans les espaces lenticulaires obtenues en effectuant une chirurgie de Dehn sur certains nœuds doublement primitifs dans $S^1 \times S^2$, exactement ceux construits par Baker.

Mots clés: chirurgie de Dehn, espace lenticulaire, homologie de Heegaard-Floer, nœud fibré.

ABSTRACT

We study a Dehn surgery problem, namely the characterisation of knots in lens spaces which admit longitudinal $S^1 \times S^2$ surgeries. We prove that such knots are fibred and rationally bound planar Seifert surfaces. Equivalently, the induced knots in $S^1 \times S^2$ are isotopic to braids.

The main tool we used is Heegaard-Floer homology, a gauge-theoretic package of invariants developed by Ozsváth-Szabó from 2000 onwards. We further show that these knots are Floer simple, hence conjecturally simple. In view of this conjecture, we initiate a study of simple knots in the relevant lens spaces and give a potentially complete list of all simple knots with longitudinal $S^1 \times S^2$ surgeries. These knots turn out to be exactly the knots in lens spaces obtained by performing Dehn surgery on some doubly primitive knots in $S^1 \times S^2$, as constructed by Baker.

Keywords: Dehn surgery, lens space, Heegaard-Floer homology, fibred knot.

INTRODUCTION

Dehn surgery is the process of removing a tubular neighbourhood of a knot inside a three-manifold and gluing it back via a different attaching map. All closed, orientable three-manifolds can be constructed by surgery on a link (collection of knots) inside the three-sphere, the complexity of the manifolds being reflected in the complexity of the link. It is interesting to know when some fixed manifold M can be obtained from another manifold N by performing surgery on a single knot and, if possible, determine all the knots with this property. A particularly intriguing setting is when M and N are homeomorphic, in which case the surgery is called cosmetic.

Even when M and N are simple (with respect to some notion of complexity), this is a very challenging problem. Some famous success stories in this regard are: the knot complement problem (Gordon and Luecke, 1989), Property "P" (Kronheimer and Mrowka, 2004), Property "R" (Gabai, 1987), cosmetic surgeries on the solid torus (Gabai, 1989), (Gabai, 1990), (Berge, 1991). A very important question in the field is the Berge Conjecture, which gives a possibly complete list of all knots in S^3 which admit lens space surgeries (Berge, 1984).

It is a curious feature of the examples above that the knots encountered tend to have 'small' genus (to be made precise in what follows) and are fibred.

A three-manifold M is said to *fibre* over the circle with fibre Σ if there exists a surjective submersion $\rho\colon M \longrightarrow S^1$ such that the preimage of every $x \in S^1$ is an embedded 2-dimensional submanifold of M homeomorphic to Σ.

It was observed by Gordon that all knots which are known to admit surgeries with finite fundamental group (or equivalently are covered by S^3) are fibred. This fact has been afterwards proved using the powerful new gauge theoretic techniques developed by Ozsváth-Szabó from 2002 onwards, collectively called Heegaard-Floer homology (Ozsváth and Szabó, 2006c),(Ozsváth and Szabó, 2004c).

Heegaard-Floer homology grew out of the search for methods of computation for Monopole-Floer homology, and, while achieving that goal, it developed into a new theory with unique applications. Among the most celebrated results we mention the detection of the Thurston norm of three-manifolds (Ni, 2009) and of fibredness (Ni, 2007).

The main motivation behind this work was to investigate and extend these type of results to more general Dehn surgery problems. We achieved this goal in two directions. Firstly we proved a fibredness theorem in the setting of orbifolds and orbifold Dehn surgery, and secondly, we provided several restrictions on knots in lens spaces which admit $S^1 \times S^2$ surgeries. Conjecturally, these restrictions are strong enough to characterise all such knots.

0.1 Orbi-lens spaces and Berge-Gabai knots

An important research direction in hyperbolic geometry is the study of *commensurability classes* of manifolds.

Two manifolds are said commensurable if they have a common finite cover. This relation is easily seen to be an equivalence relation and its classes are called commensurability classes.

In the general category of hyperbolic three-manifolds, describing these commensurability classes seems a very difficult problem. It is natural to restrict ones attention to special classes of manifolds, for example complements of hyperbolic knots in S^3. Here, one is led to further distinguish between two very different situations, according to whether or not the knot complements admit *hidden symmetries.*

A hidden symmetry of a manifold M is a symmetry of a finite cover $\overline{M}$ of M which is not the lift of a symmetry of M.

An *orbi-lens space* is the quotient of S^3 by a cyclic group of (orientation preserving) isometries. The underlying manifold of an orbi-lens space is a lens space. The singular locus (when non-empty) consists of one or both of the cores of the solid tori in the Heegaard genus one decomposition of the lens space.

In the non-hidden symmetries case, in (Boileau et al., 2011) it was shown that non-trivial commensurability classes of knot complements are obtained from knots in orbi-lens spaces having non-trivial orbi-lens space surgeries. Here the knots are assumed disjoint from the singular locus.

The situation should be contrasted with the similar problem of a knot complement covering another knot complement, which was reduced to the Berge Conjecture by González-Acuña and Whitten in (González-Acuña and Whitten, 1992).

More precisely, a knot K (whose complement admits no hidden symmetries) which is not unique in its commensurability class, is the lift of a knot $\widetilde{K}$ in an orbi-lens space which admits a non-trivial orbi-lens space surgery (Boileau et al., 2011, Proposition 4.13).

In the case when the orbi-lens space is a manifold, the knot $\widetilde{K}$ is fibred by a Heegaard-Floer argument due to Rasmussen (Boileau et al., 2011, Theorem 6.5). It immediately follows that K is fibred as well, by pulling back the fibration of $\widetilde{K}$.

A *Berge-Gabai* knot is a knot in the solid torus $S^1 \times D^2$ which admits a non-trivial cosmetic surgery.

Gabai showed that such a knot is a 1-bridge braid, i.e. it can be isotoped to be everywhere transverse to the D^2 fibres and lie in the boundary of the solid torus, except for the *bridge*, which is an unknotted arc in the interior of the solid torus.

In the case when the orbi-lens space (say L) has connected, non-empty singular locus, $\widetilde{K}$ is necessarily a *Berge-Gabai* knot in the exterior of the singular locus. It is important to observe that the natural fibration by punctured disks of $\widetilde{K}$ does not extend to $|L|$ for homological reasons, but there is a fibration on $L \setminus \widetilde{K}$ given by the same Heegaard-Floer argument. However, due to the presence of the singular locus, this fibration cannot be lifted *a priori* to $S^3 \setminus K$.

Theorem (Boileau et al., 2011, Theorem 6.1). Let $\widetilde{K}$ be a 1-bridge braid on n strands in a solid torus V. For any essential simple closed curve C on ∂V whose algebraic winding number in V is coprime to n there is a locally trivial fibring of the exterior of $\widetilde{K}$ in V by surfaces whose intersection with ∂V has n components, each a curve parallel to C.

This theorem proves that the exterior of a Berge-Gabai fibres over the circle in more than one way, and indeed one of these fibrations can be lifted to $S^3 \setminus K$.

The aforementioned fibration theorem in the context of orbifolds is a corollary of the previous result.

Theorem. Let K be a knot in an orbi-lens space L which is primitive in $|L|$. If K admits a non-trivial orbi-lens space surgery, then the exterior of K admits a fibring by 2-orbifolds with base the circle.

Note that a fibration $\rho\colon M \longrightarrow S^1$ determines by pull-back a cohomology class $\rho^*(d\theta) \in H^1(M;\mathbb{R})$, where $d\theta$ is the angle form on S^1. We call $\rho^*(d\theta)$ the *direction* of the fibration.

The previous theorem provides new examples of hyperbolic manifolds which fibre over the circle in every possible direction allowed by the Thurston norm theory (Thurston, 1986).

Corollary (Boileau et al., 2011, Proposition 1.6) Let M be the exterior of a hyperbolic 1-bridge braid in a solid torus V. Then each top-dimensional face of the Thurston norm ball in $H^1(M;\mathbb{R}) \cong \mathbb{R}^2$ is a fibred face.

Equivalently, the set of directions $\{\rho^*(d\theta) \;:\; \rho\colon M \longrightarrow S^1 \text{ fibration}\}$ is dense in $H^1(M;\mathbb{R})$.

Furthermore, theorem 6.1 of (Boileau et al., 2011) gives more information about the fibration of the induced Berge knots in lens spaces, namely in the standard 1-bridge position, the induced Berge knot's fibration is (up to isotopy) transverse to one of the cores of the Heegaard solid tori, (conjecturally to both).

0.2 Knots in lens spaces admitting $S^1 \times S^2$ surgeries

Consider a Berge-Gabai knot $K \subset V$ where V is a solid torus. By embedding V in $S^1 \times S^2$ in the canonical way, i.e. as one of the Heegaard solid tori in the unique Heegaard decomposition of $S^1 \times S^2$ of genus 1, $K \subset S^1 \times S^2$ will admit (longitudinal) lens space surgeries. We call this process the *Berge-Gabai construction.*

This surgery exhibits an interesting property of the lens spaces obtained, namely they bound smooth, rational homology four-balls, by classical handle theory (Gompf and Stipsicz, 1999). These lens spaces were classified by Lisca using gauge-theoretic methods (Lisca, 2007), and Rasmussen observed that the list he obtained coincides with the list of lens spaces obtained through the Berge-Gabai construction above (Greene, 2010).

In the same paper (Greene, 2010), Greene conjectured that this is the only way in which lens spaces can be obtained from $S^1 \times S^2$ by Dehn surgery.

A *doubly primitive* knot in M is a knot which can be isotoped to lie in a Heegaard surface of genus 2 of M with the extra property that it carries a free generator of the fundamental group of each handlebody.

Berge proved that doubly primitive knots in any three-manifold have lens space surgeries.

It turns out that Greene's conjecture is false. Baker (Baker, 2012) constructed more examples of knots in $S^1 \times S^2$ with lens space surgeries. All of his knots are *doubly primitive* in $S^1 \times S^2$. It was checked that in $S^1 \times S^2$ they can be isotoped to be braids.

A *simple* knot in a lens space L is a knot which can be decomposed into two arcs which are contained in the meridian disks of the two solid tori forming the genus 1 Heegaard splitting of L.

Baker's knots have the remarkable property that the induced knots in the lens spaces are *simple.*

Theorem Let $K \subset L(p,q)$ be a knot in a lens space which admits longitudinal $S^1 \times S^2$ surgeries. Then K is fibred and the generalised Seifert surface of K is a m-punctured disk, where $m^2 = p$.

From the point of view of $S^1 \times S^2$,

Theorem If $K \subset S^1 \times S^2$ admits longitudinal lens space surgeries, then K is isotopic to a braid.

Corollary A doubly-primitive knot in $S^1 \times S^2$ is a braid.

The restriction to longitudinal surgeries is not drastic, by the Cyclic Surgery Theorem (Culler et al., 1987), which states that if $K \subset M$ with $\pi_1(M)$ cyclic is a knot whose exterior is not Seifert fibred, then any other surgery on K which gives a manifold with cyclic fundamental group, is longitudinal.

Knots with Seifert fibred exteriors in lens spaces are classified, and so are surgeries on them.

The proofs of these results rely heavily on Heegaard-Floer homology. Indeed, lens spaces are L spaces, manifolds with the smallest Heegaard-Floer homology possible.

An L space Y is a rational homology three-sphere with $\widehat{HF}(Y;\mathbb{Z})$ free of rank $\#H_1(Y;\mathbb{Z})$.

There is a corresponding notion for knots, we say that a knot $K \subset Y$ is *Floer simple* if $rk(\widehat{HFK}(Y,K)) = rk(\widehat{HF}(Y))$.

Theorem Let $K \subset L(p,q)$ be a knot in a lens space which admits a longitudinal $S^1 \times S^2$ surgery. Then K is Floer simple.

It has been conjectured (Rasmussen, 2007) that Floer simple knots in lens spaces are simple. In view of this, it is natural to ask which simple knots in lens spaces admit $S^1 \times S^2$ surgeries.

We only give partial results here, namely

Theorem For the first two families of lens spaces which bound rational homology four-balls, (out of 4), the simple knots which admit $S^1 \times S^2$ surgeries are exactly the knots induced by doing Dehn surgery on the doubly-primitive knots in $S^1 \times S^2$ constructed by Baker.

We conjecture that this is true for the other families as well. A positive resolution of this conjecture would imply that the doubly-primitive knots constructed by Baker are *all* the doubly-primitive knots in $S^1 \times S^2$.

As evidence for this conjecture, we mention that we verified it for lens spaces of orders up to $p = m^2 = 500^2$ and we also provide a technique for proving it, which we successfully used for the first two families.

0.3 Organisation

The rest of this thesis is organised as follows:

In Chapter 1 we prove the orbifold fibredness theorem and describe the relevance for the commensurability problem. In Chapter 2 we provide the necessary background on Heegaard-Floer homology. In Chapter 3 we prove the results about the knots in lens spaces with $S^1 \times S^2$ surgeries. Finally, in Chapter 4 we make an analysis of simple knots in lens spaces which bound rational homology four-balls.

CHAPTER I

KNOT COMMENSURABILITY AND FIBREDNESS

1.1 Dehn surgery on knots

In this section we present some background material concerning basic three-manifold topology and Dehn surgery, in view of completeness and establishing the notation used throughout the thesis.

A *slope* in the torus $S^1 \times S^1$ is an isotopy class of unoriented simple closed curves in $S^1 \times S^1$. We will identify slopes with $\pm$ primitive homology classes in $H_1(S^1 \times S^1)$.

The distance between two slopes α and β is the minimal number of (transverse) intersections of the curves representing α, resp. β.

A *knot* manifold M is a compact oriented three-dimensional manifold with one boundary component homeomorphic to $S^1 \times S^1$.

A knot $K \subset Y$ where Y is a closed, oriented three-manifold is a smooth embedding of S^1 in Y. The *exterior* of K, denoted $Ext(K)$ or $Y \setminus\setminus K$ is the knot manifold $Y \setminus Int(N(K))$ where $N(K)$ is a tubular neighbourhood of K.

Dehn filling a knot manifold M along a slope α on ∂M is the process of gluing a solid torus $D^2 \times S^1$ to ∂M with the gluing map that identifies ∂D^2 with α.

Dehn surgery on a knot $K \subset Y$ along a slope α on $\partial(Y \setminus\setminus K)$ is the process of removing an open tubular neighbourhood of K and Dehn filling along the slope α.

For $K \subset Y$, there is a distinguished slope μ on $\partial(Y \setminus\setminus K)$, called the *meridian* of K, which is characterised by the fact that $\mu = \partial D^2$ where D^2 is a properly embedded disk in $N(K)$ which cannot be isotoped into $\partial N(K)$.

A slope λ at distance 1 from μ is called a *longitudinal* or *integral* slope, and Dehn surgery on a longitudinal slope is called longitudinal surgery or Morse surgery.

A *rational homology three-sphere* Y is an oriented, closed three-manifold with $H_*(Y;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$.

A knot $K \subset Y$ is (rationally) null-homologous if the homology class that it represents, denoted by $[K]$, is 0 in $H_1(Y;\mathbb{Z})$, resp. $H_1(Y,\mathbb{Q})$.

A *Seifert surface* F properly embedded in $Ext(K)$ (in some ambient three-manifold Y) is a smooth surface with ∂F a collection of simple closed curves in $\partial Ext(K)$, none of which bounds a disk in $\partial Ext(K)$. One can always arrange that these curves are parallel (including orientation).

1.2 Berge-Gabai knots and cyclic commensurability

In a joint project with M. Boileau, S. Boyer, and G. S. Walsh (Boileau et al., 2011), we investigate commensurability classes of hyperbolic knot complements in S^3. In this chapter we present a fibredness theorem for knots which are not unique in their *cyclic commensurability class.* The material is all taken from (Boileau et al., 2011) with minor modifications.

Definition 1.2.1. *Two oriented orbifolds are commensurable if they have orientation-preserving homeomorphic finite sheeted covers. If the covers are cyclic, we say that the orbifolds are cyclically commensurable.*

For knot complements, we say (abusively) that the knots are commensurable if their complements are. The *commensurability class* of $K \subset S^3$ is the set

$$\mathcal{C} = \{K' \subset S^3 \colon K' \text{ commensurable with } K\}$$

It is natural to restrict attention to *non-arithmeticknots without hidden symmetries.*

Definition 1.2.2. *We say that K has no hidden symmetries if all the symmetries of any finite sheeted cover of $S^3 \setminus K$ are lifts of symmetries of $S^3 \setminus K$.*

We will not define what *(non)-arithmetic* knots are, we simply recall a clasic result of Margulis which gives an equivalent condition, namely that there exists a unique minimal orbifold in the commensurability class of $S^3 \setminus K$.

An outstanding question in the field is the Reid-Walsh Conjecture

Conjecture 1.2.3. *(Reid and Walsh, 2008) For a hyperbolic knot $K \subset S^3$, $|\mathcal{C}(K)| \leq 3$.*

The main theorem of (Boileau et al., 2011) is the following

Theorem 1.2.4. *(Boileau et al., 2011)[Theorem 1.4]*

1. *Knots without hidden symmetries which are commensurable are cyclically commensurable.*
2. *A cyclic commensurability class contains at most three hyperbolic knot complements.*

The cyclic commensurability class of $K \setminus S^3$ is denoted as follows

$$\mathcal{CC}(K) = \{K' \subset S^3 \colon K' \text{ cyclically commensurable with } K\}$$

We also provide several obstructions for a knot which is not unique in its cyclic commensurability class

Theorem 1.2.5. *(Boileau et al., 2011)[Theorem 1.7] Let $K \subset S^3$ be a hyperbolic knot. If $|\mathcal{CC}| \geq 2$, then*

1. *K is fibred.*

2. *the genus of K is the same as the genus of any $K' \subset CC$.*

3. *the volume of K is different from that of any $K' \in CC \setminus K$. In particular, the only mutant of K contained in CC is K.*

4. *K is chiral and not commensurable with its mirror image.*

We will focus here on the proof of 1, more precisely for the case where K is *periodic*:

Definition 1.2.6. *We say that K is periodic if it admits a non-free symmetry with an axis disjoint from K.*

The quotient of S^3 by this symmetry is an *orbi-lens space*:

Definition 1.2.7. *An orbi-lens space is the quotient orbifold of S^3 by a finite cyclic subgroup of $SO(4)$.*

We denote the singular set of an orbifold $\mathcal{O}$ by $\Sigma(\mathcal{O})$ and by $|O|$ its underlying manifold.

The first homology group of an orbifold is the abelianisation of its fundamental group.

A knot in an orbi-lens space $\mathcal{L}$ is *primitive* if it carries a generator of $H_1(\mathcal{L})$.

Lemma 1.2.8. *(Boileau et al., 2011)[Corollary 3.2] A 3-orbifold $\mathcal{L}$ is an orbi-lens space if and only if $|\mathcal{L}|$ is a lens space which admits a genus one Heegaard splitting $|\mathcal{L}| = V_1 \cup V_2$ such that $\Sigma(\mathcal{L})$ is a closed submanifold of the union of the cores C_1, C_2 of V_1, V_2, and there are coprime positive integers $b_1, b_2 \geq 1$ such that a point of C_j has isotropy group $\mathbb{Z}/b_j$. In the latter case, $\pi_1(\mathcal{L}) \cong \mathbb{Z}/(b_1 b_2 |\pi_1(|\mathcal{L}|)|)$.*

We will use $\mathcal{L}(p, q; b_1, b_2)$ to denote the orbifold described in the lemma. As we are mainly concerned with the case $b_1 = 1$ and $b_2 = a$, we use $\mathcal{L}(p, q; a)$ to denote $\mathcal{L}(p, q; 1, a)$. When $a = 1$, $\mathcal{L}(p, q; a)$ is just $L(p, q)$.

Recall that a *cusp* of a complete, finite volume, orientable, hyperbolic 3-orbifold is of the form $T^2 \times \mathbb{R}$, where T^2 is a Euclidean the two-dimensional torus.

A *slope* r in a torus cusp of a complete, non-compact, finite volume hyperbolic 3-orbifold $\mathcal{O}$ is a cusp isotopy class of essential simple closed curves which lie on some torus section of the cusp.

The positive solution of the Smith conjecture implies that $\mathrm{Isom}^+(S^3 \setminus K)$ is cyclic or dihedral and the subgroup of $\mathrm{Isom}^+(S^3 \setminus K)$ which acts freely on K is cyclic of index at most 2. We denote this subgroup by $Z(K)$ and the quotient $(S^3 \setminus K)/Z(K)$ by $\mathcal{Z}(K)$.

Call $r(K)$ the projection of the meridian of K (a slope in the cusp of $S^3 \setminus K$) to the cusp of $\mathcal{Z}(K)$.

Proposition 1.2.9. *(Boileau et al., 2011)[Proposition 4.13] A commensurability class contains cyclically commensurable knot complements $S^3 \setminus K$ and $S^3 \setminus K'$ where $K' \neq K$ if and only if it contains the complement of a knot $\bar{K}$ in an orbi-lens space $\mathcal{L}$ such that $\bar{K}$ is primitive in $\mathcal{L}$ and $\mathcal{L}$ admits an orbi-lens space surgery $\mathcal{L}'$ along $\bar{K}$ of slope $r' \neq r(K)$. Furthermore, we may assume that $\mathcal{L} \setminus \bar{K} = \mathcal{Z}_K$ and if $\pi'\colon S^3 \longrightarrow \mathcal{L}'$ is the universal covering and $\bar{K}' \subset \mathcal{L}'$ is the core of the r'-Dehn filling of $\mathcal{Z}_K$, then $K' = \pi^{-1}(\bar{K}')$.*

Definition 1.2.10. A *Berge-Gabai knot in a solid torus* is a 1-bridge braid in a solid torus which admits a non-trivial cosmetic surgery slope.

Definition 1.2.11. *(Boileau et al., 2011)[Definition 5.5]*

1. *Let w, p, q, a be integers with $w, a, p \geq 1$ and $\gcd(p, q) = \gcd(w, ap) = 1$. A Berge-Gabai knot $\bar{K}$ of winding number w in $\mathcal{L}(p, q; a)$ consists of a knot $\bar{K} \subset \mathcal{L}(p, q; a)$ and a genus one Heegaard splitting $V_1 \cup V_2$ of $|\mathcal{L}(p, q; a)|$ such that $\bar{K}$ is a Berge-Gabai knot of winding number w in V_1 and $\Sigma(\mathcal{L}(p, q; a))$ is a closed submanifold of the core of V_2.*

2. *A $(p, q; a)$-unwrapped Berge-Gabai knot in S^3 is a knot in S^3 which is the inverse image of a Berge-Gabai knot in $\mathcal{L}(p, q; a)$ under the universal cover $S^3 \to \mathcal{L}(p, q; a)$.*

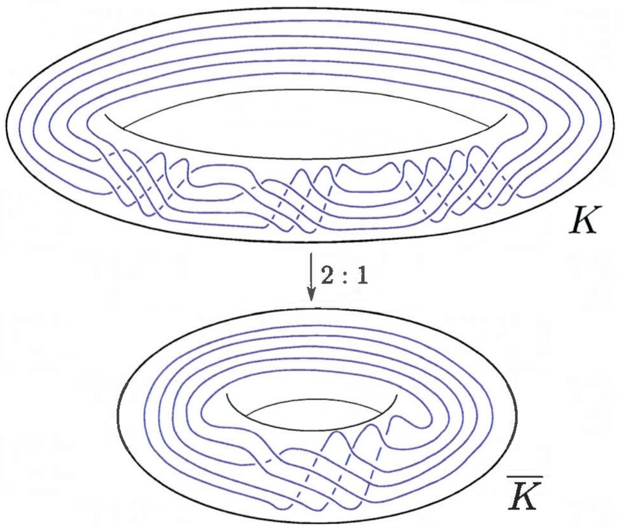

Figure 1.1 A Berge-Gabai knot $\overline{K} \subset L(2,1)$ and the unwrapped Berge-Gabai knot in S^3

Note that the inverse image in S^3 of a Berge-Gabai knot in $\mathcal{L}(p,q;a)$ is a knot (i.e. connected) as its winding number w is coprime to ap.

Theorem 1.2.12. *(Boileau et al., 2011)[Theorem 6.1] Let K be a 1-bridge braid on n strands in a solid torus V. For any essential simple closed curve C on ∂V whose algebraic winding number in V is coprime to n there is a locally trivial fibring of the exterior of K in V by surfaces whose intersection with ∂V has n components, each a curve parallel to C.*

Corollary 1.2.13. *(Boileau et al., 2011)[Corollary 6.2] An unwrapped Berge-Gabai knot is a fibred knot.*

Proof of Corollary 1.2.13. Let K be an unwrapped Berge-Gabai knot in S^3. Then K is the inverse image in S^3 of a Berge-Gabai knot $\bar{K} \subset \mathcal{L}(p,q;a)$ of winding number n, say, under the universal cover $S^3 \to \mathcal{L}(p,q;a)$. Thus there is a genus one Heegaard splitting $V_1 \cup V_2$ of $|\mathcal{L}(p,q;a)|$ such that $\bar{K}$ is a Berge-Gabai knot of winding number n in V_1 and $\Sigma(\mathcal{L}(p,q;a))$ is a closed submanifold of the core C_2 of V_2. As $|\mathcal{L}(p,q;a)| = L(p,q)$, the algebraic intersection number of a meridian curve of V_1 with one of V_2 is $\pm p$. By definition, $\gcd(p,n) = 1$, so Theorem 1.2.12 implies that there is a locally trivial fibring of the exterior of $\bar{K}$ by surfaces which intersect ∂V in curves parallel to the meridian of V_2. Therefore we can extend the fibration over the exterior of K in $L(p,q) = |\mathcal{L}(p,q;a)|$ in such a way that it is everywhere transverse to $\Sigma(\mathcal{L}(p,q;a))$. Hence the fibration lifts to a fibring of the exterior of K. □

Proof of Theorem 1.2.12. Let K be the closed 1-bridge braid contained in the interior of a solid torus V determined by the three parameters:

- n, the braid index of K;
- b, the bridge index of K;
- t, the twisting number of K.

See (Gabai, 1990) for an explanation of these parameters and Figure 1.2 for an example. (Our conventions differ from those of (Gabai, 1990) by mirroring and changing orientation. This modification is convenient for presenting the knot's fundamental group.)

Number the braid's strands successively $\overline{0}$ to $\overline{n-1}$ and let σ_i denote the i^{th} elementary braid in which the i^{th} strand passes over the $(i+1)^{st}$. The braid associated to K has the following form: $\beta(K) = \sigma_{b-1}\cdots\sigma_0\delta^t$ where $\delta = \sigma_{n-2}\cdots\sigma_0$ is the positive $2\pi/n$ twist. Denote by π the permutation of $\mathbb{Z}/n$ determined by $\beta(K)$. It has the following simple form:

$$\pi(\bar{a}) = \begin{cases} \overline{a+t+1} & \text{if } 0 \leq a < b \\ \bar{t} & \text{if } a = b \\ \overline{a+t} & \text{if } b < a < n \end{cases} \tag{1.1}$$

for some $a \in \bar{a}$. As K is a knot, π is an n-cycle.

Let $T_1 = \partial V$ and $T_2 = \partial N(K)$ the boundary of a closed tubular neighborhood of K in $\mathrm{int}(V)$. There is a meridian class $\mu_1 \in H_1(T_1)$ well-defined up to ± 1 and represented by the boundary of a meridian disk of V_1. Let $\lambda_1 \in H_1(T_1)$ be any class which forms a basis of $H_1(T_1)$ with μ_1. Then λ_1 generates $H_1(V)$.

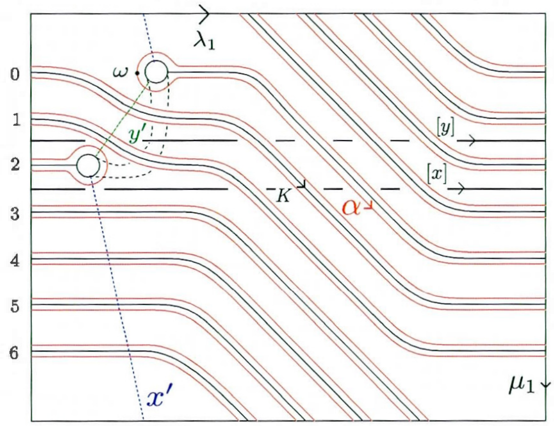

Figure 1.2 The Fintushel-Stern knot ($n = 7$, $b = 2$, $t = 4$). The curve x' is obtained from the arc labelled x' by closing it in the boundary of the tunnel with an arc parallel to the bridge and y' is obtained similarly by closing the arc y' in the boundary of the tunnel.
Here R is: $y\ x\ y\ x\ x\ y\ x\ x\ y^{-1}x^{-1}y^{-1}x^{-1}x^{-1}y^{-1}x^{-1}x^{-1}$.

Let M denote the exterior of K in V and fix an essential simple closed curve C on ∂V. We are clearly done if C is a meridian curve of V, so assume that this is not the case. Then we can orient C and find coprime integers $p \geq 1, q$ so that

$$[C] = q\mu_1 + p\lambda_1 \in H_1(T_1)$$

Note that p is the algebraic winding number of C in V. Assuming that $\gcd(p, n) = 1$ we must show that there is a locally trivial fibring of M by surfaces which intersect ∂V in curves parallel to C. The tools we use to prove this are Brown's theorem (Brown, 1987) and Stallings' fibration criterion (Stallings, 1962). See also (Ozsváth and Szabó, 2005) where a similar argument is invoked; our proof is only slightly more involved. Brown's theorem gives necessary and sufficient conditions under which a homomorphism from a two-generator one-relator group to $\mathbb{Z}$ has finitely generated kernel and Stallings' theorem produces a fibration of a 3-manifold given such a homomorphism of its fundamental group. More precisely:

Theorem 1.2.14. (Theorem 4.3 and Proposition 3.1 of (Brown, 1987)) *Let* $G = \langle x, y :$

$R\rangle$ be a two-generator one-relator group with $R = R_1R_2 \dots R_m$, $R_i \in \{x, x^{-1}, y, y^{-1}\}$, a cyclically reduced and non-trivial relator. Let $S_1, \dots S_m$ be the proper initial segments of the relator R, i.e. $S_i = R_1 \dots R_{i-1}$. Finally let $\varphi : G \to \mathbb{R}$ be a non-zero homomorphism. If $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$, then $\ker(\varphi)$ is finitely generated if and only if the sequence $\{\varphi(S_i)\}_{i=1}^m$ assumes its maximum and minimum values exactly once.

It is easy to see that the exterior M of K is homeomorphic to a genus 2 handlebody with a 2-handle attached to it. Start with a solid torus $U' \subset \text{int}(V)$ obtained by removing a small open collar of T_1 in V. Denote $\partial U'$ by T_3. As K is 1-bridge, it can be isotoped into U' so that the bridge is a properly embedded arc and its complement, γ say, is contained in T_3. Fix a disk neighborhood $D \subset T_3$ of γ and let $\alpha = \partial D$. Let U be the exterior of the bridge in U', a genus two handlebody. We can assume that $T_3 \setminus \partial U \subset \text{int}(D)$ and therefore $\alpha \subset \partial U$. By construction, α bounds a 2-disk properly embedded in $\overline{V \setminus U}$ (i.e. a copy of D isotoped rel ∂D into $\overline{V \setminus U}$). It is easy to see that M is a regular neighborhood of the union of U and this disk.

The fundamental group of U is free on two generators x, y represented by two curves in T_3 representing λ_1. (See Figure 1.2.) There are a pair of dual curves $x', y' \subset \partial U$ to these generators. This means that

- x' and y' bound disks in U;
- x intersects x' transversely in one point and is disjoint from y';
- y intersects y' transversely in one point and is disjoint from x'.

See Figure 1.2. The word $R \in \pi_1(U)$ in x, y represented by the curve α can be read off in the usual way: each signed intersection of α with x', resp. y', contributes $x^{\pm 1}$, resp. $y^{\pm 1}$, while travelling around α.

We introduce the auxiliary function $f : \mathbb{Z}/n \setminus \{\bar{b}\} \to \{x, y\}$ given by:

$$f(\bar{a}) = \begin{cases} y & \text{if } 0 \leq a < b \\ x & \text{if } b < a < n \end{cases} \tag{1.2}$$

for some $a \in \bar{a}$. Let $w_j = f(\pi^j(\bar{b}))$ and consider the word $w = w_1 w_2 \dots w_{n-1}$. Then $R = ywxy^{-1}w^{-1}x^{-1}$. To see this, start with y from the base point ω (c.f. Figure 1.2); then follow the knot until the b strand, which contributes w; then turn at the lower foot of the handle, which contributes xy^{-1}; then walk along the knot in the opposite direction until the strand b is reached, which contributes w^{-1}; then close by passing x', which contributes to the final x^{-1}. Notice that R is cyclically reduced. It follows that

$$\pi_1(M) = \langle x, y : ywxy^{-1}w^{-1}x^{-1} \rangle$$

Let $\mu_2 \in H_1(T_2)$ be a meridinal class of K. The reader will verify that we can choose the longitudinal class λ_1 for V, a longitudinal class $\lambda_2 \in H_1(T_2)$ for K, and possibly replace μ_1 by $-\mu_1$ so that in $H_1(M)$:

- $n\lambda_1 = \lambda_2$;
- $\mu_1 = n\mu_2$;
- $[yx^{-1}] = \mu_2$ (i.e. $[yx^{-1}]$ is represented by a meridian of K at the bridge);
- $\lambda_1 + t\mu_2 = [x]$ (i.e. λ_1 and $[x]$ co-bound an annulus in V which K punctures t times).

Consider the homomorphism $\pi_1(U) \to \mathbb{Z}$ which sends x to $pt - nq \neq 0$ and y to $pt - nq + p \neq 0$. Since the exponent sum of both x and y in R is zero, it induces a homomorphism $\varphi : \pi_1(M) \to \mathbb{Z}$. Since $\gcd(p, nq) = 1$, φ is surjective. From the above, it can then be verified that $\varphi(\lambda_1) = -nq$ and $\varphi(\mu_1) = np$. Hence $\varphi(\mu_1^q \lambda_1^p) = 0$.

Lemma 1.2.15. *Let $S_1, S_2, \dots, S_{2n+2}$ be the proper initial segments of $R = ywxy^{-1}w^{-1}x^{-1} = R_1 R_2 \dots R_{2n+2}$ where $R_i \in \{x, x^{-1}, y, y^{-1}\}$. Then the sequence $\{\varphi(S_i)\}_{i=1}^{2n+2}$ achieves its maximum and minimum values exactly once.*

Proof. By construction, $\varphi(x) \neq 0$, $\varphi(y) \neq 0$, and $\varphi(y) > \varphi(x)$. The conclusion of the lemma is easily seen to hold when $\varphi(x)$ and $\varphi(y)$ have the same sign, so assume that $\varphi(x) < 0 < \varphi(y)$.

Set $S = \max\{\varphi(S_i) : 1 \leq i \leq 2n+2\}$ and $s = \min\{\varphi(S_i) : 1 \leq i \leq 2n+2\}$.

Since $\varphi(x) < 0 < \varphi(y)$ we have

$$\begin{cases} s \leq \varphi(S_{n+2}) < \varphi(S_{n+1}) < \varphi(S_n) \leq S \\ \\ s \leq \varphi(S_{n+i}) = \varphi(S_{n-i+2}) + \varphi(x) - \varphi(y) < \varphi(S_{n-i+2}) \leq S \text{ for } 3 \leq i \leq n+1 \\ \\ s \leq \varphi(S_{2n+1}) = \varphi(S_{2n+2}) + \varphi(x) < \varphi(S_{2n+2}) = 0 < \varphi(y) = \varphi(S_1) \leq S \end{cases} \tag{1.3}$$

Thus the maxima of $\{\varphi(S_i)\}_{i=1}^{2n+2}$ can only occur in the sequence $\varphi(S_1), \varphi(S_2), \ldots, \varphi(S_n)$ and the minima in $\varphi(S_{n+2}), \varphi(S_{n+3}), \ldots, \varphi(S_{2n+1})$.

We look at the maxima of $\{\varphi(S_i)\}_{i=1}^{2n+2}$ first. Suppose that $1 \leq l < r \leq n$. We claim that $\varphi(R_{l+1}) + \cdots + \varphi(R_r) \not\equiv 0 \pmod{n}$. If so, $\varphi(S_l) \neq \varphi(S_r)$ and therefore S occurs precisely once amongst the values $\{\varphi(S_i)\}_{i=1}^{n}$.

Let $\overline{\varphi}$ be the reduction of φ modulo n. Since $\gcd(p, n) = 1$, we can define

$$\hat{\varphi} = \overline{p}^{-1}\overline{\varphi} : \pi_1(M) \to \mathbb{Z}/n$$

Then $\hat{\varphi}(x) = \overline{t}$ and $\hat{\varphi}(y) = \overline{t+1}$ and therefore

$$\hat{\varphi}(f(\overline{a})) = \pi(\overline{a}) - \overline{a}$$

for all $a \in \mathbb{Z}/n \backslash \{\overline{b}\}$. Hence $\hat{\varphi}(R_{l+1}) + \cdots + \hat{\varphi}(R_r) = \hat{\varphi}(w_l) + \cdots + \hat{\varphi}(w_{r-1}) = \hat{\varphi}(f(\pi^l(\overline{b}))) + \cdots + \hat{\varphi}(f(\pi^{r-1}(\overline{b}))) = (\pi^{l+1}(\overline{b}) - \pi^l(\overline{b})) + \cdots + (\pi^r(\overline{b}) - \pi^{r-1}(\overline{b})) = \pi^r(\overline{b}) - \pi^l(\overline{b})$. Since π is an n-cycle and $1 \leq l < r \leq n$ we see that $\pi^r(\overline{b}) \neq \pi^l(\overline{b})$. It follows that $\varphi(R_{l+1}) + \cdots + \varphi(R_r) \not\equiv 0 \pmod{n}$.

The uniqueness of the minimum follows along the same lines. We saw above that the minima of $\{\varphi(S_i)\}_{i=1}^{2n+2}$ only occur in $\varphi(S_{n+2}), \varphi(S_{n+3}), \ldots, \varphi(S_{2n+1})$. As before,

$\varphi(R_{l+1}) + \cdots + \varphi(R_r) \not\equiv 0 \pmod{n}$ for all $n+2 \leq l < r \leq 2n+1$ and therefore $\varphi(S_{n+2}), \varphi(S_{n+3}), \ldots, \varphi(S_{2n+1})$ are pairwise distinct. This implies the desired conclusion. □

We can now complete the proof of Theorem 1.2.12. The previous lemma couples with Theorem 1.2.14 to show that the kernel of φ is finitely generated. Stallings' fibration criterion (Stallings, 1962) implies that M admits a locally trivial surface fibration with fibre F such that $\pi_1(F) = \ker(\varphi)$. Since $\varphi(\mu_1) = np \neq 0$ while $\varphi(\mu_1^q \lambda_1^p) = 0$, $\ker(\varphi|_{\pi_1(T_1)})$ is the infinite cyclic subgroup of $\pi_1(T_1)$ generated by $[C]$. Hence the fibration meets T_1 in curves parallel to C. To complete the proof, we must show that the intersection of a fibre F with T_1 has n components.

To that end, note that as φ is surjective we can orient F so that for each $\zeta \in H_1(M)$ we have $\varphi(\zeta) = \zeta \cdot [F]$. Let $\phi_1 \in H_1(M)$ be the class represented by the cycle $F \cap T_1$ with the induced orientation. Clearly, $\phi_1 = \pm|F \cap T_1|[C]$. Since $\varphi(\lambda_1) = -nq$ and $\varphi(\mu_1) = np$, $\varphi(\pi_1(T_1)) = n\mathbb{Z}$. Thus if $\zeta \in H_1(M)$ is represented by a dual cycle to $[C]$ on T_1, then

$$n = \varphi(\zeta) = \zeta \cdot [F] = |\zeta \cdot \phi_1| = ||F \cap T_1|\zeta \cdot [C]| = |F \cap T_1|$$

This completes the proof. □

An interesting consequence of the proof above is the following

Proposition 1.2.16. *(Boileau et al., 2011)[Proposition 1.6] Let M be the exterior of a hyperbolic 1-bridge braid in a solid torus V. Then each top-dimensional face of the Thurston norm ball in $H_2(M, \partial M; \mathbb{R})$ is a fibred face.*

Proof. Let K be a hyperbolic 1-bridge braid on n strands in a solid torus V. We use the notation developed in the proof of Theorem 1.2.12. In particular, M is the exterior of K in V and $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$ with basis λ_1, μ_2. By construction there are classes $\xi_1, \xi_2 \in H_2(M, \partial M)$ such that if $\partial : H_2(M, \partial M) \to H_1(\partial M)$ is the connecting homomorphism, then $\partial\xi_1 = \mu_1 - n\mu_2$ and $\partial\xi_2 = n\lambda_1 - \lambda_2$. Since $|\lambda_1 \cdot \xi_j| = \delta_{1j}$ and $|\mu_2 \cdot \xi_j| = \delta_{2j}$, $\{\xi_1, \xi_2\}$ is a basis for $H_2(M, \partial M) \cong H^1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Consider the homomorphism ψ given by the composition $H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) = H_1(T_1) \oplus H_1(T_2) \to H_1(T_1)$. Then $\psi(a\xi_1 + b\xi_2) = a\mu_1 + nb\lambda_1$, and therefore ψ is injective.

Let p, q be coprime integers such that $\gcd(n, p) = 1$. According to Theorem 1.2.12, there is a fibre F in M which can be oriented so that $\psi([F]) = [F \cap T_1] = nq\mu_1 + np\lambda_1 = \psi(nq\xi_1 + p\xi_2)$. Hence $[F] = nq\xi_1 + p\xi_2$ so that $nq\xi_1 + p\xi_2$ is a fibre class in $H_2(M, \partial M)$.

Fix coprime integers a, b and consider the class $\xi = a\xi_1 + b\xi_2$. The proposition will follow if we can show that the projective class of ξ can be arbitrarily closely approximated by fibre classes (Thurston, 1986, Theorem 2). By the previous paragraph ξ is a fibre class when $a = 0$, so suppose this is not the case. It suffices to show that $\frac{b}{a} = \lim_m \frac{b_m}{a_m}$ where $a_m\xi_1 + b_m\xi_2$ are fibre classes. This is easy to verify: for each integer $m > 0$ set $p_m = nmba + 1$ and $q_m = mb^2$. Then $\gcd(p_m, nq_m) = 1$ and from the previous paragraph we see that $nq_m\xi_1 + p_m\xi_2$ is a fibre class. Finally, $\lim_m \frac{nq_m}{p_m} = \frac{b}{a}$, which completes the proof. □

Here is a curious consequence, more precisely a reformulation of Theorem 1.2.12

Proposition 1.2.17. *(Boileau et al., 2011)[Theorem 1.5] Let K be a knot in an orbi-lens space (with non-empty singular set) L which is primitive in $|L|$. If K admits a non-trivial orbi-lens space surgery, then the exterior of K admits a fibring by 2-orbifolds with base the circle.*

Proof. Suppose $\mathcal{L} = \mathcal{L}(p, q; a, b)$. Set $L_0 = \mathcal{L}(p, q; a, b) \setminus N(\Sigma(\mathcal{L}(p, q; a, b)))$ and

$$L_0 \cong \begin{cases} S^1 \times D^2 & \text{if } |\Sigma(\mathcal{L}(p, q; a, b))| = 1 \\ S^1 \times S^1 \times [0, 1] & \text{if } |\Sigma(\mathcal{L}(p, q; a, b))| = 2 \end{cases}$$

Since K admits a non-trivial orbi-lens space surgery in $\mathcal{L}$, L_0 admits a non-trivial cosmetic surgery . Because $S^1 \times S^1 \times I$ has no nontrivial cosmetic surgeries (Boileau et al., 2011)[Lemma 5.1], $L_0 \cong S^1 \times D^2$ (so we can suppose that $b = 1$) and K is a Berge-Gabai knot in L_0 . Let n be the winding number of K in L_0. Our hypotheses imply that $\gcd(p, n) = 1$. Thus Theorem 1.2.12 implies that there is a locally trivial fibring of the exterior of K in L_0 by surfaces which intersect ∂L_0 in curves parallel to the meridian slope of the solid torus $\overline{N(\Sigma(\mathcal{L}(p,q;a)))}$. Therefore we can extend the fibration over the exterior of K in $\mathcal{L}(p,q;a)$ in such a way that it is everywhere transverse to $\Sigma(\mathcal{L}(p,q;a))$. We endow each fibre F of this surface fibration with the structure of a 2-orbifold by declaring each point of $F \cap \Sigma(\mathcal{L}(p,q;a))$ to be a cone point of order a. In this way the exterior of K in $\mathcal{L}(p,q;a)$ admits an orbifold fibring with base the circle. □

It is known that any knot in S^3 which admits a lens space surgery (and more generally a L-space surgery) is fibred (Ni, 2007). As an application of Theorem 1.2.12, we can say something more about the fibration of a Berge knot. Recall that the Berge knots are the doubly primitive knots in S^3 and conjecturally they are all the knots with lens space surgeries. It was proved by Berge that the induced knot in the surgered lens space is simple (Berge, 1984) In particular, it is a 1-bridge knot.

Proposition 1.2.18. *Let K' be the induced knot in a lens space $L(p,q)$ obtained by surgery on a doubly primitive $K \subset S^3$. Then, when K' is supported in a tubular neighborhood of the Heegaard torus T of $L(p,q)$ as a simple knot, the cores of the Heegaard solid tori determined by T can be isotoped (perhaps not simultaneously) to be transverse to the fibration of $L \setminus \overset{\circ}{K'}$.*

Proof. Call U_1 and U_2 the Heegaard solid tori bounding T. View K' in U_1 first. Then K' is a braid in U_1 and because of the primitivity of K', the winding number $w(K')$ is coprime to p. We can apply theorem 1.2.12 to conclude that there is a fibration of $U_1 \setminus \overset{\circ}{N}(K')$ which meets U_2 in meridian disks, hence the fibration can be extended to the fibration of $L(p,q) \setminus K$.

Now repeat this argument for U_2. □

Remark 1.2.19. *We conjecture nevertheless that the two cores of $L(p,q)$ can be simultaneously isotoped to transverse positions with respect to the fibration of K.*

CHAPTER II

BACKGROUND ON HEEGAARD-FLOER HOMOLOGY

In this chapter we describe the construction of Heegaard-Floer homology, introduced by Ozsváth-Szabó in 2000, which will play an essential role in the proof of our main theorems.

2.1 Heegaard splittings

Throughout this chapter, Y will denote a closed, connected, oriented three-manifold.

Definition 2.1.1. *A* Heegaard splitting (decomposition) *of* Y *is a tuple* (Σ, U_0, U_1), *where* $\Sigma \subset Y$ *is a separating, closed, oriented surface,* $Y \setminus \Sigma = U_0 \cup U_1$ *with each* U_i *an open handlebody.* Σ *is called a* Heegaard surface.

We will always assume that Y, Σ, U_0, U_1 are oriented using the following convention: $\Sigma = \partial U_0 = -\partial U_1$ and the orientation on Y coincides with the orientations of U_0 and U_1. Two Heegaard splittings (Σ, U_0, U_1), (Σ', U_0', U_1') of Y, resp. Y', are (orientation preserving) *homeomorphic* if there exists an (orientation preserving) homeomorphism $\varphi : Y \to Y'$ such that $\varphi(\Sigma) = \Sigma'$ and $\varphi(U_i) = U_i'$.

Any closed, connected, orientable three-manifold admits Heegaard splittings. To construct one, take U_0 to be a regular neighborhood of the 1-dimensional skeleton of a triangulation of Y (which always exists. See e.g. (Moise, 1977)). See (Scharlemann, 2000) for a survey of Heegaard splittings. The genus of Σ is by definition the *genus of the splitting* and the smallest genus among all splittings of Y is the *(Heegaard) genus* of

Y.

The genus gives a measure of the complexity of three-manifolds. There is one manifold of genus 0, namely S^3. Manifolds of genus 1 form a simple family: they are lens spaces and $S^1 \times S^2$.[1] All genus 1 manifolds have cyclic fundamental group and except for $S^1 \times S^2$ have spherical geometry. The manifolds of genus 2 form a much more complicated class, which in particular contains hyperbolic manifolds. A complete classification is currently out of reach.

Given two manifolds Y, Y' with some Heegaard splittings (Σ, U_0, U_1), resp. (Σ', U_0', U_1'), one can construct a Heegaard splitting on the connected sum $Y \# Y'$ by choosing the three-balls B, resp B', on which the sum is performed such that $B \cap \Sigma \cong D^2$, resp. $B' \cap \Sigma' \cong D^2$ and ∂B is identified with $\partial B'$ by a homeomorphism φ such that $\varphi(\Sigma \cap B) = \Sigma' \cap B'$.

Observe that $\Sigma \# \Sigma'$ will be a Heegaard surface in $Y \# Y'$. We call this Heegaard splitting the *connected sum* of the two splittings, we can write it as: $(\Sigma, U_0, U_1) \# (\Sigma', U_0', U_1') = (\Sigma \# \Sigma', U_0 \#_\partial U_0', U_1 \#_\partial U_1')$.

From a given splitting (Σ, U_0, U_1) of Y, one can obtain new splittings by: *isotopy* and *(de)stabilisation*. Isotopy refers to the ambient isotopy of Σ in Y, whereas stabilisation is the connect sum $(\Sigma, U_0, U_1) \# (T^2, V_0, V_1)$, where (T^2, V_0, V_1) is the unique (up to isotopy) genus 1 splitting of S^3. Conversely, we say that (Σ, U_0, U_1) was obtained from $(\Sigma, U_0, U_1) \# (T^2, V_0, V_1)$ by destabilisation.

It is a classical theorem of Reidemeister (Reidemeister, 1933) and Singer (Singer, 1933) that any two Heegaard splittings of Y become isotopic after a finite number of stabilisations.

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[1] We adopt the convention that $S^1 \times S^2$ is not a lens space.

2.2 Heegaard diagrams and Morse theory

Given a decomposition (Σ, U_0, U_1) of Y, one can specify the handlebodies U_i by *complete sets of attaching circles*:

Definition 2.2.1. *A* complete sets of attaching circles *for* Σ *- an oriented compact surface of genus* g *- is a collection* $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ *of essential simple closed curves in* Σ *which are linearly independent in* $H_1(\Sigma; \mathbb{Z})$.

To describe U_0, say, one chooses g curves $\alpha_1, \dots, \alpha_g$ which bound disks in U_0, the condition that they are linearly independent in $H_1(\Sigma, \mathbb{Z})$ implies that $\Sigma \backslash \{\cup_i \alpha_i\}$ is a punctured sphere. Note that the curves $\alpha_1, \dots, \alpha_g$ are not unique (up to isotopy).

Conversely, starting with Σ and $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$, one can construct a handlebody in which the α_i curves will bound disks by attaching to $\Sigma \times \{0\} \subset \Sigma \times I$ 2-handles along $\alpha_i \times \{0\}$ and gluing a three ball to the sphere boundary component of the resulting manifold.

Heegaard splittings can be specified up to (oriented) homeomorphism by *Heegaard diagrams*:

Definition 2.2.2. *A* Heegaard diagram *is a tuple* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *where* Σ *is a compact, oriented surface of genus* g *and* $\boldsymbol{\alpha}$, *resp.* $\boldsymbol{\beta}$ *are* complete sets of attaching circles. *Two Heegaard diagrams* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *and* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ *are* diffeomorphic *if there exists an orientation preserving diffeomorphism* $\varphi : \Sigma \to \Sigma'$ *such that* $\varphi(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'$ *and* $\varphi(\boldsymbol{\beta}) = \boldsymbol{\beta}'$.

From a Heegaard diagram one constructs a splitting by attaching 2-handles along $\alpha_i \times \{0\} \subset \Sigma \times I$ and 2-handles along $\beta_i \times \{1\} \subset \Sigma \times I$ and gluing 2 three-balls along the resulting two-sphere boundary components. By convention, we orient the manifold thus obtained consistently with the product orientation on $\Sigma \times I$.

The theory of Heegaard diagrams is equivalent to Kirby calculus (also called handle attachment calculus) in dimension 3, see (Gompf and Stipsicz, 1999), (Milnor, 1965) for the theory of handle attachment in general, which in its turn is closely related to Morse theory and Cerf theory. See (Milnor, 1963), (Cerf, 1970) as references for the latter.

Endow Y with a Riemannian metric. Choose a self-indexing Morse function $f : Y \to [0,3]$, with one index 0, resp. one index 3 critical point. Then the level set $\Sigma = f^{-1}(\frac{3}{2})$ is a Heegaard surface of Y, $U_0 = f^{-1}[0, \frac{3}{2}]$, $U_1 = f^{-1}[\frac{3}{2}, 3]$. Moreover, the ascending manifolds (under the flow of the negative gradient of f) of the index 1 critical points intersect Σ in a complete set of attaching circles, denoted $\boldsymbol{\alpha}$ and similarly the descending manifolds of index 2 critical points intersect Σ in $\boldsymbol{\beta}$. Then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ becomes a Heegaard splitting of Y. We say that f is *compatible* with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Conversely, given a Heegaard diagram, there exists a compatible Morse function (Milnor, 1965).

One defines several *moves* on Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$:

- *isotopy* : replace $\boldsymbol{\alpha} =: \boldsymbol{\alpha}_0$, a complete sets of attaching circles, with $\boldsymbol{\alpha}_1$, where $\boldsymbol{\alpha}_t, t \in [0,1]$, is a (smooth) isotopy such that for all $t \in [0,1]$, $\boldsymbol{\alpha}_t$ is a complete set of attaching circles; the same for $\boldsymbol{\beta}$.

- *handleslide* the set of attaching circles $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ is replaced by the set of attaching circles $\boldsymbol{\alpha}' = \{\alpha_1', \dots, \alpha_g\}$, where α_1, α_1' and α_2, bound a *pair of pants*, i.e. a thrice punctured sphere in $\Sigma - \{\alpha_3 \cup \ldots \cup \alpha_g\}$; similarly for $\boldsymbol{\beta}$.

- *stabilisation* replace Σ with $\Sigma' = \Sigma \# T^2$, $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ with $\boldsymbol{\alpha}' = \{\alpha_1, \dots, \alpha_g, \alpha_{g+1}\}$ and $\boldsymbol{\beta}' = \{\beta_1, \dots, \beta_g, \beta_{g+1}\}$ where $\alpha_{g+1}, \beta_{g+1} \subset T^2$ are two simple closed curves intersecting transversely in one point and disjoint from the disk on which the connected sum is performed. The inverse operation is called *destabilisation.*

Any two Heegaard diagrams representing the same manifold become diffeomorphic after applying a finite number of moves, by classical Cerf theory (Cerf, 1970).

2.3 Heegaard-Floer homology

Heegaard-Floer homology is a package of invariants of smooth three- and four- dimensional manifolds, developed by Ozsváth and Szabó from 2000 onwards. It was conjectured right from the beginning to be equivalent to the Seiberg-Witten-Floer homology developed by Kronheimer and Mrowka (Kronheimer and Mrowka, 2007) and the motivation for its construction was to provide ways to compute the latter. The conjecture was recently proved by two independent groups (Colin, Ghiggini and Honda, 2011), (Kutluhan, Lee and Taubes, 2011).

Heegaard-Floer homology is indeed more easily computable and was extended to objects which had no corresponding monopole invariant, for example knots in the three-sphere, where the invariant (discovered independently by Rasmussen (Rasmussen, 2003)), is a categorification of the Alexander polynomial. It is known to detect geometric properties of three-manifolds, such as the minimal genus of embedded surfaces in a given homology class, in particular the genus of a knot. Also, Donaldson's diagonalisation theorem (Donaldson, 1983) can be proved within the framework of Heegaard-Floer homology (Ozsváth and Szabó, 2003a).

We will give a summary of the construction for the objects we are interested in, the reader is referred to the original papers for a complete account (Ozsváth and Szabó, 2004c),(Ozsváth and Szabó, 2004b),(Ozsváth and Szabó, 2004a),(Rasmussen, 2003), see also the expository papers (Ozsváth and Szabó, 2006c), (Ozsváth and Szabó, 2006a).

As the name suggests, Heegaard-Floer homology is defined using a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of Y, with an additional *basepoint* $z \in \Sigma - (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$.

Definition 2.3.1. *A* pointed Heegaard diagram *is a tuple* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, *where* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *is a Heegaard diagram for* Y *and* $z \in \Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$. *Two pointed Heegaard diagrams* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ *and* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z')$ *are* diffeomorphic *if the underlying Heegaard diagrams* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, *resp.* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ *are diffeomorphic by a diffeomorphism which respects the basepoints.*

There is a natural notion of *pointed moves* for pointed Heegaard diagrams. These are just the moves on Heegaard diagrams described above with the extra conditions that:

- isotopies are supported in the complement of the basepoint,
- the pair of pants in the definition of handleslide does not contain the basepoint,
- the connected sum with T^2 in the stabilisation move is made on a disk not containing the basepoint.

It is shown in (Ozsváth and Szabó, 2004c, Proposition 7.1) that any two pointed Heegaard diagrams of Y become diffeomorphic after a finite sequence of pointed moves.

2.3.1 The construction of Heegaard-Floer homology

Heegaard-Floer homology is a version of Lagrangian-Floer homology in the g-fold symmetric product of Σ : $\mathrm{Sym}^g(\Sigma) = \Sigma^{\times g}/_{S_g}$, where $\Sigma^{\times g} = \underbrace{\Sigma \times \ldots \times \Sigma}_{g-\text{times}}$, S_g denotes the symmetric group on g elements and the action on $\Sigma^{\times g}$ is the natural one - permutation of the factors. Denote by $\pi : \Sigma^{\times g} \to \mathrm{Sym}^g(\Sigma)$ the canonical projection.

The usual setup for Lagrangian-Floer homology is a symplectic manifold (M, ω) with a generic almost complex structure J and a pair of Lagrangian submanifolds (L_0, L_1). One then analyses the moduli space of holomorphic disks with certain boundary conditions. See (Gromov, 1985),(Floer, 1988) and (McDuff and Salamon, 2004) for an introduction to this field.

Choose a Kähler structure on Σ. The product complex structure on $\Sigma^{\times g}$ descends to the symmetric product, making π a holomorphic map. The proof that $\mathrm{Sym}^g(\Sigma)$ is a complex manifold follows from the fact that $\mathrm{Sym}^g(\mathbb{C})$ is a complex manifold biholomorphic to $\mathbb{C}^g$. This biholomorphism is constructed by associating to a monic degree g polynomial (which can be seen as a vector of $\mathbb{C}^g$ - the coordinates being the non-dominant coefficients) its unordered set of roots (with multiplicities) - an element of $\mathrm{Sym}^g(\mathbb{C})$.

The two sets of attaching circles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ give rise to the tori $\mathbb{T}_\alpha$, resp. $\mathbb{T}_\beta \subset \mathrm{Sym}^g(\Sigma)$:

$\mathbb{T}_\alpha = \pi(\alpha_1 \times \dots \times \alpha_g)$, $\mathbb{T}_\beta = \pi(\beta_1 \times \ldots \times \beta_g)$. Note that π is a branched cover with singular locus the *diagonal* $\Delta \subset \Sigma^{\times g}$, where by definition $(z_1, \ldots, z_g) \in \Delta \iff z_i = z_j$ for some $i \neq j$. Since the α_i curves are disjoint, $\alpha_1 \times \ldots \times \alpha_g \cap \Delta = \emptyset$, therefore $\mathbb{T}_\alpha$ is an embedded torus in $\mathrm{Sym}^g(\Sigma)$, and similarly $\mathbb{T}_\beta$. We will suppose that they intersect transversely, which is equivalent to the transversality of each α_i with each β_j.

Intersection points of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ have an interpretation in terms of the Heegaard diagram: let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Then $\mathbf{x}$ is an unordered g-tuple of points $(x_1, \ldots, x_g)$ with $x_i \in \Sigma$, where $x_i \neq x_j$ if $i \neq j$ (since $\mathbb{T}_\alpha \cap \Delta = \emptyset$) and each x_i belongs to an α curve and a β curve. By relabelling the elements x_i, we can suppose that $x_i \in \alpha_i$, and so $x_i \in \alpha_i \cap \beta_{\varphi(i)}$ for some permutation $\sigma \in S_g$. In words, an intersection point between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ is a choice of intersection points between the α and β curves, where each curve in the Heegaard diagram appears exactly once.

$\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ will play the role of the Lagrangian submanifolds. At the time of writing of (Ozsváth and Szabó, 2004c), there was no known way to push forward the product Kähler structure (in particular the symplectic form) on $\Sigma^{\times g}$ to $\mathrm{Sym}^g(\Sigma)$. This was done later by Perutz (Perutz, 2008). The two tori are totally real with respect to an almost complex structure coming from the product complex structure on $\Sigma^{\times g}$. Ozsváth and Szabó were able to adapt the Floer homology techniques to this case. We will not go into the analytical details, we just note that work of Perutz (Perutz, 2008), shows that Heegaard-Floer homology can be viewed as a classic Lagrangian-Floer homology. See also (Lipshitz, 2006) for a 'cylindrical' reformulation of Heegaard-Floer homology, where the ambient symplectic manifold is $[0,1] \times \mathbb{R} \times \Sigma$, but one allows pseudo-holomorphic curves of higher genus.

It is convenient to suppose that Σ has genus $g > 2$, see (Ozsváth and Szabó, 2004c, Section 2.4) and below. This is not an essential restriction since one can always stabilise a Heegaard diagram. We note that for genus 1 and 2 Heegaard diagrams one can still compute the Floer homology of the manifold, but some definitions require modifications.

Given two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, one is interested in the moduli space of pseudo-holomorphic disks connecting $\mathbf{x}$ to $\mathbf{y}$. These disks are analogous to the trajectories between critical points of a Morse function in classic Morse theory. In the Floer homology setting, the existence of these trajectories is homologically obstructed. As a result, the invariant splits according to Spin^c structures on Y. This is made more precise in what follows.

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and e_1 be the arc of $\partial\mathbb{D}$ of points with positive real part, and e_2 the arc in $\partial\mathbb{D}$ with negative real part.

Denote by $\Omega(\mathbf{x},\mathbf{y})$ the set of maps

$$\left\{ u\colon \mathbb{D} \longrightarrow \mathrm{Sym}^g(\Sigma) \;\middle|\; \begin{array}{l} u(-i) = x, u(i) = y, \\ u(e_1) \subset \mathbb{T}_\alpha, u(e_2) \subset \mathbb{T}_\beta \end{array} \right\}$$

Such a disk u is called a *Whitney disk connecting* $\mathbf{x}$ *to* $\mathbf{y}$. Two Whitney disks u_0 and u_1 are said to be homotopic if there is a continuous one-parameter family $(u_t)_{t\in[0,1]}$ of Whitney disks interpolating them. The set of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$ will be denoted by $\pi_2(\mathbf{x},\mathbf{y})$.

There is a natural splicing operation $*\colon \pi_2(\mathbf{x},\mathbf{y}) \times \pi_2(\mathbf{y},\mathbf{z}) \longrightarrow \pi_2(\mathbf{x},\mathbf{z})$ which simply concatenates two Whitney disks.

The structure of $\pi_2(\mathbf{x},\mathbf{y})$ is determined by algebraic-topological data on the triple $(\mathrm{Sym}^g(\Sigma), \mathbb{T}_\alpha, \mathbb{T}_\beta)$ which in its turn can be rephrased in terms of the homology of Y.

Proposition 2.3.2. *(Ozsváth and Szabó, 2004c, Sections 2.3 and 2.4) We have the following isomorphisms:*

$$\frac{H_1(\mathrm{Sym}^g(\Sigma))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma)}{[\alpha_1], ..., [\alpha_g], [\beta_1], ..., [\beta_g]} \cong H_1(Y;\mathbb{Z}).$$

Given an element u of $\pi_2(\mathbf{x},\mathbf{y})$, one sees that the cycle $u(e_1) - u(e_2)$ is zero in $H_1(\mathrm{Sym}^g(\Sigma);\mathbb{Z})$. The image of this cycle in $H_1(\mathrm{Sym}^g(\Sigma))/(H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)$ is independent of the arcs $u(e_1), u(e_2)$, in particular it is independent of u. This motivates the following:

Definition 2.3.3. *(Ozsváth and Szabó, 2004c, Definition 2.11) Let* $a\colon [0,1] \longrightarrow \mathbb{T}_\alpha$, $b\colon [0,1] \longrightarrow \mathbb{T}_\beta$ *be two arcs from* $\mathbf{x}$ *to* $\mathbf{y}$ *in* $\mathrm{Sym}^g(\Sigma)$*. Define* $\varepsilon(\mathbf{x},\mathbf{y})$ *to be the image* *of the cycle $a - b$ in $H_1(Y;\mathbb{Z})$ under the isomorphisms from Proposition 2.3.2. This quantity is sometimes referred to as the ε* grading.

It is immediate from the definition of ε that $\varepsilon(\mathbf{x},\mathbf{y}) + \varepsilon(\mathbf{y},\mathbf{z}) = \varepsilon(\mathbf{x},\mathbf{z})$, for $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

The ε grading is the aforementioned obstruction to the existence of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$.

Proposition 2.3.4. *(Ozsváth and Szabó, 2004c, Proposition 2.15) When $g > 2$, $\pi_2(\mathbf{x},\mathbf{y})$ is nonempty $\iff$ $\varepsilon(\mathbf{x},\mathbf{y}) = 0$. If this happens,*

$$\pi_2(\mathbf{x},\mathbf{y}) \cong \mathbb{Z} \oplus H^1(Y;\mathbb{Z})$$

as principal $\mathbb{Z} \oplus H^1(Y;\mathbb{Z})$ spaces.

By the above discussion, intersection points of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are partitioned into equivalence classes in affine bijection with $H_1(Y;\mathbb{Z})$, by declaring $\mathbf{x}$ and $\mathbf{y}$ to be equivalent if $\varepsilon(\mathbf{x},\mathbf{y}) = 0$. These equivalence classes turn out to be in natural bijection with Spin^c structures on Y, once we fix a basepoint for the Heegaard diagram.

2.4 Spin^c structures

Recall that the Lie group $\mathrm{Spin}^{\mathbb{C}}(\mathrm{n}), (n \geq 3)$, is the quotient $(\mathrm{Spin}(\mathrm{n}) \times \mathrm{Spin}(2)) /_{\mathbb{Z}_2}$, where the generator of $\mathbb{Z}_2$ acts on each $\mathrm{Spin}(\mathrm{r})_{\mathrm{r=n,2}}$ factor by the nontrivial deck transformation of the cover $\mathrm{Spin}(\mathrm{r}) \to \mathrm{SO}(\mathrm{r})$. Note that there is a canonical homomorphism $\mathrm{Spin}^{\mathbb{C}}(\mathrm{n}) \to \mathrm{SO}(\mathrm{n})$, see (Gompf and Stipsicz, 1999) for details.

Endow our three-manifold Y with a Riemannian metric g and consider the principal $SO(3)$ bundle of orthonormal oriented frames $f_Y \colon Fr \to Y$.

Definition 2.4.1. *A* Spin^c *structure on (Y, g) is a lift of the $SO(3)$ bundle f_Y to a principal* Spin$^{\mathbb{C}}$ *bundle.*

Two Spin^c structures s_0, resp. s_1 on (Y, g_0), resp. (Y, g_1) are said to be equivalent if there is a 1-parameter family of metrics $(g_t)_{t\in[0,1]}$ and a continuous 1-parameter family of Spin^c structures s_t on (Y, g_t). Therefore, equivalence classes of Spin^c structures on Y do not depend on any particular metric, they are associated to the manifold itself. Abusively, we will call these equivalence classes simply Spin^c structures and we will denote by Spin$^c(Y)$ the set of Spin^c structures on Y.

In dimension 3, Spin^c structures admit a topological interpretation, due to Turaev (Turaev, 1997).

Definition 2.4.2. *Two non-zero vector fields v_1, v_2 on Y are* homologous *if they are homotopic in the complement of a three-ball (or equivalently in the complement of a finite number of three-balls) in Y.*

Proposition 2.4.3. *(Turaev, 1997)* Spin^c *structures on Y are in natural bijection with homology classes of vector fields.*

The homology classes of vector fields form an affine space over $H^2(Y,\mathbb{Z})$. To see this, choose a trivialisation of the tangent bundle of Y, $\tau\colon TY \longrightarrow Y\times\mathbb{R}^3$; this way one can identify unit vector fields on Y with maps $v : Y \to S^2 \subset \mathbb{R}^3$. Then homotopy classes of vector fields are in one-to-one correspondence with homotopy classes of maps $v\colon Y \to S^2$. The homology classes of vector fields are uniquely determined by the induced maps $v^*\colon H^2(S^2;\mathbb{Z}) \longrightarrow H^2(Y;\mathbb{Z})$, hence, after fixing a generator of $H^2(S^2;\mathbb{Z})$, they are in one-to-one correspondence with elements of $H^2(Y;\mathbb{Z})$. This correspondence is not canonical, since it depends on τ. However, the difference between the corresponding elements in $H^2(Y;\mathbb{Z})$ is independent of the trivialisation (Ozsváth and Szabó, 2004c, Section 2.6), hence there is a well-defined difference between two Spin^c structures, which is an element of $H^2(Y;\mathbb{Z})$. This shows that Spin$^c(Y)$ is an affine space over $H^2(Y;\mathbb{Z})$.

In a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, an intersection point $\mathbf{x}$ of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ determines a Spin^c structure on Y in the following way: suppose $\mathbf{x}$ consists of the g-tuple $(x_1, \ldots, x_g)$, where $x_i \in \alpha_i \cap \beta_{\varphi(i)}$ for some permutation $\varphi \in S_g$.

Fix a Morse function $f\colon Y \longrightarrow [0,3]$ compatible with the Heegaard diagram. Consider the vector field $v := -\nabla(f)$. Each intersection point $x_i \in \mathbf{x}$ determines a trajectory from the index 2 critical point corresponding to $\beta_{\varphi(i)}$ to the index 1 critical point corresponding to α_i. A regular neighborhood of this trajectory is a ball, in which one can isotope v to a non-zero vector field v'. This is always possible when the indexes of the two critical points in the ball have different parity. After performing this operation for each $x_{i,i\in\{1,\ldots,g\}}$, we obtain a vector field which we still call v'. The basepoint z determines a trajectory from the index 3 critical point to the index 0 critical point. A regular neighborhood of this trajectory is again a three-ball, in which one can isotope v' to a non-zero vector field v''. Notice that by definition, the homology class of v'' does not depend on the particular isotopies performed in the above three-balls. Hence we have a well-defined map $s_z\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c(Y)$, sending $\mathbf{x}$ to the homology class of v''.

The following result justifies the splitting of the intersection points between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ according to Spin^c structures on Y.

Proposition 2.4.4. *(Ozsváth and Szabó, 2004c, Lemma 2.19) For* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

$$s_z(\mathbf{y}) - s_z(\mathbf{x}) = PD[\varepsilon(\mathbf{x}, \mathbf{y})],$$

where $PD[\gamma]$ *is the Poincaré dual of* $\gamma \in H_1(Y;\mathbb{Z})$.

A Spin^c structure has a well-defined Chern class, an element of $H^2(Y;\mathbb{Z})$.

Definition 2.4.5. *For a* Spin^c *structure* ξ *on* Y*, given as the homology class of the vector field* v*, one defines its Chern class by* $c_1(\xi) = [v] - [-v]$.

An equivalent formulation which will be useful later is the following:

Proposition 2.4.6. *(Ozsváth and Szabó, 2004c, Section 2.6) The Chern class of the* Spin^c *structure* $[v]$ *is equal to the Euler class of the orthogonal complement of* v*, an oriented rank* 2 *vector bundle, or its first Chern class when viewed as a complex line bundle.*

For $\xi \in \mathrm{Spin}^c(Y)$ represented by the homology class $[v]$, the Spin^c structure $\overline{\xi} := [-v]$ is called the *conjugate* Spin^c structure. It is obvious from the definitions that $c_1(\xi) = -c_1(\overline{\xi})$.

2.5 The invariants

There is one more ingredient in the definition of the homology groups, namely *orientation systems*. They do not play an essential role, in general, since it was proved by Ozsváth-Szabó in (Ozsváth and Szabó, 2004b)[Theorem 10.12] that there is a canonical choice of a (equivalence class of) orientation system for a three-manifold. This is why they are generally omitted from the notation of the Floer homology groups. See Section 3 of (Ozsváth and Szabó, 2004c) for a complete discussion. They arise at a certain point in our proofs, so we include a brief introduction.

Definition 2.5.1. *(Ozsváth and Szabó, 2004b)[Definition 3.11] For a fixed $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, a* coherent system of orientations $\mathfrak{o}$ *is a choice of non-vanishing sections $\mathfrak{o}(\phi)$ of the determinant line bundle of the linearisation of the $\bar{\partial}$ operator over each $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ for each $\mathbf{x}, \mathbf{y}$ representing $\mathfrak{s}$ and each $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, which are compatible with respect to gluing:*

$$\mathfrak{o}(\phi_1) \wedge \mathfrak{o}(\phi_2) = \mathfrak{o}(\phi_1 * \phi_2)$$

and

$$\mathfrak{o}(u * S) = \mathfrak{o}(u)$$

where $\wedge$ denotes the splicing of Whitney disks and S is the holomorphic sphere generating $\pi_2'(\mathrm{Sym}^g(\Sigma))$.

In order to orient the moduli spaces of holomorphic representatives of Whitney disks, one chooses an orientation system. Unless otherwise specified, this will always be the canonical one given in (Ozsváth and Szabó, 2004b)[Theorem 10.12].

As mentioned above, there are several versions of Heegaard-Floer homology. The difference lies in the role played by the basepoint z. We are interested in this work mostly in the HF^+ and $\widehat{HF}$ versions.

The simplest invariant defined in (Ozsváth and Szabó, 2004c) is $\widehat{HF}$. It is the homology of a complex $\widehat{CF}$, defined in terms of a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, and a path of almost complex structures in $\mathrm{Sym}^g(\Sigma)$. We will not go into the analytical details regarding moduli spaces of pseudo-holomorphic disks, almost complex structures, but we will define the main concepts and state the necessary theorems which justify the definition of Floer homology. The reader is referred to Section 3 of (Ozsváth and Szabó, 2004c) for the complete account.

In classical Morse theory, one analyses the moduli space of trajectories (under the gradient flow) from one critical point to another. In Heegaard-Floer theory, the critical points are replaced by the intersection points between the two tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, and the trajectories are the Whitney disks which are moreover pseudo-holomorphic maps.

The expected dimension of the moduli space $\mathcal{M}(\mathbf{x}, \mathbf{y})$ of holomorphic Whitney disks in a given homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is given by the *Maslov index* of ϕ - denoted by $\mu(\phi)$.

The unit disk $\mathbb{D}$ in $\mathbb{C}$ has a one-parameter family of automorphisms preserving the points i and $-i$. These are easily seen as vertical translations in a biholomorphic model for $\mathbb{D}$, namely the band $\{z \in \mathbb{C} | -1 < Re(z) < 1\}$, for which $\pm i$ correspond to $\pm\infty$. Therefore one is mainly interested in 1-dimensional moduli spaces of holomorphic Whitney disks, for which the *unparametrised* moduli spaces

$$\widehat{\mathcal{M}}(\phi) = \frac{\mathcal{M}(\mathbf{x}, \mathbf{y})}{\mathbb{R}}$$

have dimension 0.

Theorem 2.5.2. *(Ozsváth and Szabó, 2004c, Theorem 3.18) For $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in general position and for generic choices of (paths of) almost-complex structures the following are true: there is no non-constant holomorphic Whitney disk in any homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) = 0$; for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) = 1$, $\widehat{\mathcal{M}}(\phi)$ is a compact, zero-dimensional manifold.*

Recall (cf. Proposition 2.3.4) that for $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\pi_2(\mathbf{x}, \mathbf{y})$ is an affine space over $\mathbb{Z} \oplus H^1(Y; \mathbb{Z})$. The action of the $\mathbb{Z}$ factor on $\pi_2(\mathbf{x}, \mathbf{y})$ can be seen as the action of $\pi_2(\mathrm{Sym}^g(\Sigma))$ by gluing spheres to the Whitney disks. Consequently, one can compute the change in the Maslov index when changing the homotopy class of Whitney disks.

To identify the elements of $\pi_2(\mathrm{Sym}^g(\Sigma))$, one employs the subvariety $V_z = z \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$ in view of the fact that the generator of $\pi_2(\mathrm{Sym}^g(\Sigma))$ intersects V_z transversely once (Ozsváth and Szabó, 2004c, Proposition 2.7).

Definition 2.5.3. *Let $n_z \colon \pi_2(\mathbf{x}, \mathbf{y}) \longrightarrow \mathbb{Z}$ be the function defined by $n_z(\phi) = \#(u \cap V_z)$, for some $u \in \phi$.*

Lemma 2.5.4. *(Ozsváth and Szabó, 2004c, Lemma 3.3) Let $S \in \pi_2(\mathrm{Sym}^g(\Sigma))$ be the positive generator. Then for $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$,*

$$\mu(\phi + k[S]) = \mu(\phi) + 2k.$$

The chain complex $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is freely generated over $\mathbb{Z}$ by intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

One defines a relative grading (degree) on the generators:

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_z(\phi)$$

for some $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. This quantity is independent of ϕ, as a consequence of Lemma 2.5.4, together with the fact that $\langle c_1(\mathrm{Sym}^g(\Sigma), [S] \rangle = 1$ (Ozsváth and Szabó, 2004c, Lemma 2.8) and the excision principle for the Maslov index (McDuff and Salamon, 2004).

The differential is the map $\partial : \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \longrightarrow \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ given by:

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1, n_z(\phi) = 0\}} \# \left(\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \right) \mathbf{y}$$

and extended to $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ by linearity. Note that if $\mathbf{y}$ appears in $\partial \mathbf{x}$, then necessarily $\mathrm{gr}(\mathbf{x}, \mathbf{y}) = 1$.

It is proved by Ozsváth-Szabó in (Ozsváth and Szabó, 2004c, Theorem 4.1) when $b_1(Y) = 0$ and (Ozsváth and Szabó, 2004c, Theorem 4.15) in general that ∂ above is a differential, i.e $\partial \circ \partial = 0$ and the homology groups of $(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), \partial)$ are denoted by $\widehat{HF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. Moreover, they prove that the isomorphism class of these homology groups does not depend on the choice of (paths of) complex structures (Ozsváth and Szabó, 2004c, Theorem 6.1) and on the topological choices: the Heegaard surface Σ, the complete sets of attaching circles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and the basepoint z. This is achieved by showing that the the complexes corresponding to two pointed Heegaard diagrams related by pointed Heegaard moves are chain homotopic (Ozsváth and Szabó, 2004c, Sections 7-11). This is true for the other versions HF^+, HF^-, HF^∞ and HF_{red}, see below.

One must make the observation that when computing the Heegaard-Floer homology of a manifold Y with $b_1(Y) > 0$, one must impose additional admissibility assumptions on the Heegaard diagram (Ozsváth and Szabó, 2004c, Definition 4.10). We will not go into the details, we simply note that any pointed Heegaard diagram for Y is isotopic to an admissible one (Ozsváth and Szabó, 2004c, Lemma 5.4).

More refined versions of the homology theory are defined by allowing the holomorphic disks in the definition of ∂ to intersect V_z.

The $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ complex (Ozsváth and Szabó, 2004c, Equation 11) is the chain complex freely generated over $\mathbb{Z}$ by pairs $[\mathbf{x}, i]$, with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i \in \mathbb{Z}$ with the differential $\partial^\infty : CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \longrightarrow CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ given by:

$$\partial^\infty[\mathbf{x}, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi)=1\}} \# \left(\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})\right) [\mathbf{y}, i - n_z(\phi)]$$

Note that there is a natural chain map $U : CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \longrightarrow CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ which sends $[\mathbf{x}, i]$ to $[\mathbf{x}, i-1]$, thus lowering the degree by 2.

Because the (transverse) intersection of a holomorphic disk $u \in \Omega(\mathbf{x}, \mathbf{y})$ with the submanifold V_z is positive, we have that $[\mathbf{y}, j]$ can be a term in the sum defining $\partial^\infty[x, i]$ only if $j \leq i$. This allows Ozsváth-Szabó to consider the subcomplex $(CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), \partial^\infty) \subset CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), z), \partial^\infty)$ and the induced quotient complex $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)/CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.

Their homologies are denoted by $CF^{\pm}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.

In view of their independence on analytical and topological choices (Ozsváth and Szabó, 2004c, Theorem 11.1), the homology groups above are in fact topological invariants of Y itself. The discussion in the previous section shows that ∂ and ∂^{∞} respect the splitting of the generators with respect to Spin^c structures, hence, for a fixed Spin^c structure $\mathfrak{s}$ on Y, one can speak of the Floer homology groups $\widehat{HF}(Y, \mathfrak{s}), HF^{\pm}(Y, \mathfrak{s})$ and $HF^{\infty}(Y, \mathfrak{s})$.

It is an algebraic consequence of the definitions that the homology theories above are related by the long exact sequences (Ozsváth and Szabó, 2004c, Theorem 11.1):

$$\cdots \longrightarrow HF^{-}(Y, \mathfrak{s}) \longrightarrow HF^{\infty}(Y, \mathfrak{s}) \longrightarrow HF^{+}(Y, \mathfrak{s}) \longrightarrow \cdots$$

and

$$\cdots \widehat{HF}(Y, \mathfrak{s}) \xrightarrow{U} HF^{+}(Y, \mathfrak{s}) \xrightarrow{U} HF^{+}(U, \mathfrak{s}) \longrightarrow \cdots$$

In particular, it follows that $\widehat{HF}(Y, \mathfrak{s})$ is non-zero if and only if $HF^{+}(Y, \mathfrak{s})$ is non-zero (Ozsváth and Szabó, 2004b, Proposition 2.1).

2.6 Four-manifolds

The invariants defined above are functorial with respect to cobordisms, turning Heegaard Floer homology into a version of topological quantum field theory (TQFT). The maps associated to cobordisms are defined using counts of holomorphic triangles.

An arbitrary cobordism between two three-manifolds Y and Y' can be decomposed into a number of simpler cobordisms, corresponding to longitudinal surgeries on knots in Y.

Recall that given a knot $K \subset Y$ and a framing λ, there is a canonical cobordism W_{λ} from Y to $Y_{\lambda}(K)$, constructed in the following way: thicken Y to $Y \times [0, 1]$ and add a four-dimensional two-handle $D^2 \times D^2$ to $Y \times 1$ with the attaching map specified by the framing λ.

In this setting Ozsváth-Szabó define a map in Floer homology which is an invariant of the cobordism and splits according to Spin^c structures on W_{λ}.

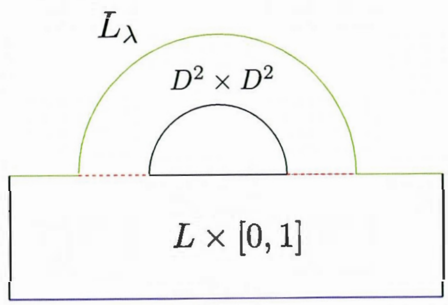

Figure 2.1 A schematic representation for the cobordism W_λ

There is a topological interpretation of Spin^c structures on four-manifolds due to Turaev (Turaev, 1997), analogous to the three-dimensional interpretation, so we will use this interpretation as definition, as in (Ozsváth and Szabó, 2004c).

Definition 2.6.1. *(Ozsváth and Szabó, 2004c)[Section 8.1] A* Spin^c *structure on a four manifold W is an equivalence class of almost complex structures J defined on $W \setminus A$, where $A \subset W$ is a finite set of points and the equivalence relation is the following: J defined on $W \setminus A$ is equivalent to J' defined on $W \setminus A'$ if there exists a compact one-dimensional manifold with boundary B such that $A \cup A' \subset B$ and J is isotopic to J' on $W \setminus B$.*

Definition 2.6.2. *The Chern class of a* Spin^c *structure represented by the complex structure J on $W \setminus A$ is the (unique) extension of the first Chern class of the induced complex tangent bundle of $W \setminus A$.*

Remark 2.6.3. *Similarly to the three-dimensional case, J on $W \setminus A$ can be thought of as an oriented 2-dimensional plane field, which together with its orthogonal, allow one to define a complex multiplication (up to isotopy). This is one way to see the restriction of a* Spin^c *structure on a four manifold to its boundary, if any.*

2.7 Chern class formulae

In order to prove equation 3.2, we will use the formulas for the evaluation of the Chern class of a Spin^c structure on a three, resp. four-dimensional manifold against a two-dimensional homology class represented by a periodic domain. Fix a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.

Definition 2.7.1. *(Ozsváth and Szabó, 2004c) A* region *is a connected component of* $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. *A* domain *is a (finite) formal sum of regions with integer coefficients.*

A domain has a naturally defined boundary which consists of linear combinations of arcs of the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves. The coefficient of a region $\mathcal{R}$ in $\mathcal{P}$ is called the *multiplicity* of $\mathcal{R}$ in $\mathcal{P}$.

Definition 2.7.2. *(Ozsváth and Szabó, 2004c) A periodic domain $\mathcal{P}$ is a domain whose boundary is a combination of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves and the region containing the base point z has multiplicity* 0 *in $\mathcal{P}$.*

Periodic domains are in one-to-one correspondence with elements of $H_2(Y, \mathbb{Z})$ (by seeing the periodic domain as a two-chain in Y, to which one adds capping disks along the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves in the boundary of the domain).

More precisely, there is an oriented two-manifold with boundary F and a map $\Phi \colon F \longrightarrow \Sigma$. One defines the *Euler measure* of $\mathcal{P}$ by

$$\chi(\mathcal{P}) = \langle c_1(\Phi^* T\Sigma; \partial), F \rangle$$

One defines the multiplicity $\overline{n}_x(\mathcal{P})$ of a point $x \in \Sigma$ with respect to a domain $\mathcal{P} = \sum_i a_i \mathcal{R}_i$ (Ozsváth and Szabó, 2004c, Section 7.1)

$$\overline{n}_x(\sum_i a_i \mathcal{R}_i) = \begin{cases} 1 & \text{if } x \text{ is in the interior of } \mathcal{R}_i \\ \frac{1}{2} & \text{if } x \text{ is in the interior of some edge of } \mathcal{R}_i \\ & \text{or two vertices of } \mathcal{R}_i \text{ are identified with } x \\ \frac{1}{4} & \text{if } x \text{ is a vertex of } \mathcal{R}_i \\ 0 & \text{if } x \notin \mathcal{R}_i \end{cases}$$

Lemma 2.7.3. *(Ozsváth and Szabó, 2004b, Proposition 7.5) Consider a class $\mathcal{A} \in H_2(Y, \mathbb{Z})$, represented by the periodic domain $\mathcal{P}$. An intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ together with the basepoint z give rise to a* Spin^c *structure $s_z(\mathbf{x})$. Then:*

$$\langle c_1(s_z(\mathbf{x})), \mathcal{A}\rangle = \chi(\mathcal{P}) + 2\sum_{x_i \in \mathbf{x}} \overline{n}_{x_i}(\mathcal{P}).$$

In a four dimensional manifold X given by a pointed Heegaard triple diagram, there is an analogous equation (Ozsváth and Szabó, 2006b, Section 6.1). Let $\mathcal{P}$ be a triply periodic domain.

An ingredient in the formula is the *dual spider number* $\sigma(u, \mathcal{P})$ of a Whitney triangle $u\colon \Delta \longrightarrow \mathrm{Sym}^g(\Sigma)$ and the triply periodic domain $\mathcal{P}$.

$$\sigma(u, \mathcal{P}) = n_{u(x)}(\mathcal{P}) + \#(a \cap \partial'_\alpha \mathcal{P}) + \#(b \cap \partial'_\beta \mathcal{P}) + \#(c \cap \partial'_\gamma \mathcal{P})$$

Proposition 2.7.4. *Given a Whitney triangle u and a triply periodic domain $\mathcal{P}$ which represents the two-dimensional homology class $H(\mathcal{P}) \in H_2(X; \mathbb{Z})$, we have the following formula:*

$$\langle c_1(\mathfrak{s}_z(u)), H(\mathcal{P})\rangle = \chi(\mathcal{P}) + \#(\partial\mathcal{P}) - 2n_z(\mathcal{P}) + 2\sigma(u, \mathcal{P})$$

2.8 Triple cobordisms and induced maps

The maps induced by cobordisms are defined with the help of Heegaard *triple* diagrams, which are simply surfaces Σ with *three* sets of attaching circles $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, for the handlebodies U_α, U_β and U_γ.

We can form the three-manifolds $Y_{\alpha,\beta} = U_\alpha \cup U_\beta$, $Y_{\beta,\gamma} = U_\beta \cup U_\gamma$ and $Y_{\alpha,\gamma} = U_\alpha \cup U_\gamma$. Moreover, there is a natural four-dimensional manifold associated to this diagram:

Consider Δ to be the two-simplex with vertices $v_\alpha, v_\beta, v_\gamma$ in clockwise order and let e_i be the edge opposite to v_i, for $i = \alpha, \beta, \gamma$. Then define

$$X_{\alpha,\beta,\gamma} = \frac{(\Delta \times \Sigma) \coprod (U_\alpha \times e_\alpha) \coprod (U_\beta \times e_\beta) \coprod (U_\gamma \times e_\gamma)}{(e_\alpha \times \Sigma) \sim (e_\alpha \times \partial U_\alpha), (e_\beta \times \Sigma) \sim (e_\beta \times \partial U_\beta), (e_\gamma \times \Sigma) \sim (e_\gamma \times \partial U_\gamma)}$$

where the quotient is by the identifications in the denominator.

Note that $\partial X_{\alpha,\beta,\gamma} = -Y_{\alpha,\beta} \cup -Y_{\beta,\gamma} \cup Y_{\alpha,\gamma}$. It is useful to know how to compute from this data the relevant homology groups of $X_{\alpha,\beta,\gamma}$.

Proposition 2.8.1. *(Ozsváth and Szabó, 2004c)[Proposition 8.2] For $X_{\alpha,\beta,\gamma}$ as above, we have*

$$H_2(X_{\alpha,\beta,\gamma};\mathbb{Z}) \cong \operatorname{Ker}\left(Span(\alpha_i)_i \oplus Span(\beta_i)_i \oplus Span(\gamma_i)_i\right) \longrightarrow H_1(\Sigma;\mathbb{Z})$$

and

$$H_1(X_{\alpha,\beta,\gamma};\mathbb{Z}) \cong \operatorname{Coker}\left(Span(\alpha_i)_i \oplus Span(\beta_i)_i \oplus Span(\gamma_i)_i\right) \longrightarrow H_1(\Sigma;\mathbb{Z}).$$

□

Several notions generalise in a straightforward manner from the three-dimensional case.

A triple Heegaard diagram with an additional basepoint z in the complement of the attaching circles is called a *pointed Heegaard diagram*, denoted $(\Sigma,\alpha,\beta,\gamma,z)$. A two-chain in $(\Sigma,\alpha,\beta,\gamma,z)$ which vanishes at the basepoint is called a *triply periodic domain.*

As stated before, maps are constructed using holomorphic triangles.

Definition 2.8.2. *(Ozsváth and Szabó, 2004c)[Section 8.1] Let* $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ *and* $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. *A map*

$$u\colon \Delta \longrightarrow \mathrm{Sym}^g(\Sigma)$$

satisfying the extra conditions $u(v_\gamma) = \mathbf{x}$, $u(v_\alpha) = \mathbf{y}$, $u(v_\beta) = \mathbf{w}$ *and* $u(e_\alpha) \subset \mathbb{T}_\alpha$, $u(e_\beta) \subset \mathbb{T}_\beta$, $u(e_\gamma) \subset \mathbb{T}_\gamma$, *is called a Whitney triangle connecting* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{w}$.

Two Whitney triangles connecting $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{w}$ *are* homotopic *if they are homotopic through maps which are also Whitney triangles connecting* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{w}$. *The set of homotopy classes of Whitney triangles connection* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{w}$ *is denoted by* $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$.

Given $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{w}$ as above, there is a homological obstruction to the existence of a Whitney disk connecting them. It takes the form of a map

$$\epsilon\colon (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times (\mathbb{T}_\beta \cap \mathbb{T}_\gamma) \times (\mathbb{T}_\alpha \cap \mathbb{T}_\gamma) \longrightarrow H_1(X;\mathbb{Z})$$

constructed as follows: Choose an a arc in $\mathbb{T}_\beta$ (equivalently a multiple arc in the β_i curves) connecting $\mathbf{x}$ to $\mathbf{y}$, an arc b in $\mathbb{T}_\gamma$ connecting $\mathbf{y}$ and $\mathbf{w}$ and an arc $c \subset \mathbb{T}_\alpha$ connecting $\mathbf{w}$ to $\mathbf{x}$. Then $\epsilon(\mathbf{x},\mathbf{y},\mathbf{w}) = [a+b+c]$ - the class of the cycle in $H_1(X;\mathbb{Z})$.

Proposition 2.8.3. *(Ozsváth and Szabó, 2004c)[Proposition 8.3] For* $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ *and* $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$,

$$\pi_2(\mathbf{x},\mathbf{y},\mathbf{x}) \neq \emptyset \iff \epsilon(\mathbf{x},\mathbf{y},\mathbf{z}) = 0.$$

Moreover, for $g(\Sigma) > 1$, *if* $\epsilon(\mathbf{x},\mathbf{y},\mathbf{w}) = 0$, *then*

$$\pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \cong \mathbb{Z} \oplus H_2(X;\mathbb{Z})$$

as principal spaces over $\mathbb{Z} \oplus H_2(X;\mathbb{Z})$.

Remark 2.8.4. *The action of* $\mathbb{Z}$ *on* $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ *corresponds to splicing a number of spheres generating* $H_2(\mathrm{Sym}^g(\Sigma))$ *(it is recorded by* $n_z(u)$*) and the action of* $H_2(X;\mathbb{Z})$ *is by adding triply periodic domains.*

There is a natural map $\mathfrak{s}_z : \pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \longrightarrow \mathrm{Spin}^c(X)$ defined in (Ozsváth and Szabó, 2004c)[Section 8.1] which we won't describe here, we only mention that it is analogous to the corresponding map for three-dimensional manifolds, and from its definition it is immediate to see how $\mathfrak{s}_z(u)$ restricts to $\mathrm{Spin}^c(Y_{\alpha,\beta}) \times \mathrm{Spin}^c(Y_{\beta,\gamma}) \times \mathrm{Spin}^c(Y_{\alpha,\gamma})$

$$\mathfrak{s}_z(u)|_{\mathrm{Spin}^c(Y_{\alpha,\beta})} = \mathfrak{s}_z(\mathbf{x})$$

and the same for $\mathbf{y},\mathbf{w}$.

There is an analogous notion of admissibility for triple Heegaard diagrams, which we won't define, it is sufficient to know that any triple Heegaard diagram can be modified to become admissible by Heegaard moves.

The moduli space of *holomorphic* Whitney triangles in a given homotopy class $\psi \in \pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ is denoted by $\mathcal{M}(\psi)$. For an admissible Heegaard triple diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},z)$ and a Spin^c structure $\mathfrak{s}$ on X, Ozsváth-Szabó define a map:

$$f^\infty(\cdot,\mathfrak{s}) \colon CF^\infty(Y_{\alpha,\beta},\mathfrak{s}_{\alpha,\beta}) \otimes CF^\infty(Y_{\beta,\gamma},\mathfrak{s}_{\beta,\gamma}) \longrightarrow CF^\infty(Y_{\alpha,\gamma},\mathfrak{s}_{\alpha,\gamma})$$

by

$$f^{\infty}([\mathbf{x},i]\otimes[\mathbf{y},j];\mathfrak{s})=\sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma}\ \sum_{\{\psi\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})\colon\, \mathfrak{s}_z(\psi)=0,\mu(\psi)=0\}}(\#\mathcal{M}(\psi))\cdot[\mathbf{w},i+j-n_z(\psi)]$$

It is proved in (Ozsváth and Szabó, 2004c)[Theorem 8.12] that the map f^{∞} induces a well-defined map

$$F^{+}\colon(\cdot,\mathfrak{s})\colon HF^{+}(Y_{\alpha,\beta},\mathfrak{s}_{\alpha,\beta})\otimes HF^{\leq 0}(Y_{\beta,\gamma},\mathfrak{s}_{\beta,\gamma})\longrightarrow HF^{+}(Y_{\alpha,\gamma},\mathfrak{s}_{\alpha,\gamma})$$

which is invariant under perturbations of the complex structure on $\mathrm{Sym}^g(\Sigma)$ and isotopies of the attaching curves.

Using these triple cobordisms, Ozsváth-Szabó define in (Ozsváth and Szabó, 2006b) maps associated to cobordisms between the Heegaard-Floer homologies of two 3-manifolds, by decomposing the cobordism into handle attachments. The maps split naturally according to Spin^c structures on the cobordism.

For torsion Spin^c structures $\mathfrak{s}$, one can define an absolute grading on $HF(Y,\mathfrak{s})$ (any variant) and the cobordism maps $F(W,\mathfrak{r})$ shift this absolute grading by the quantity

$$\frac{c_1(\mathfrak{r})^2-2\chi(W)-3\sigma(W)}{4}$$

see (Ozsváth and Szabó, 2006b) for details.

2.9 The integral surgeries long exact sequence

One of the most important properties of Heegaard-Floer homology is the surgery long exact sequence (Ozsváth and Szabó, 2004b)[Theorem 9.1] which relates the Floer homologies of three manifolds obtained by Dehn filling a knot manifold, such that the slopes are respectively at distance 1 from each other. The sequence has been generalised in several ways, see (Ozsváth and Szabó, 2004b)[Theorems 9.12, 9.14 9.19] and (Ozsváth and Szabó, 2008b)[Theorem 3.1]. We reproduce here Theorem 9.19 of that paper, since it is most useful for our purposes.

Theorem 2.9.1. *(Ozsváth and Szabó, 2004b)[Theorem 9.19] Let Y be an* integer *homology sphere, $K\subset Y$ a knot, Y_0, resp. Y_p ($p\in\mathbb{N}$) the manifold obtained by Dehn surgery* *along K with slope* 0*, resp. p. There exists a surjective map $Q\colon \mathrm{Spin}^c(Y_0) \longrightarrow \mathrm{Spin}^c(Y_p)$ with the property that for each* Spin^c *structure $\mathfrak{t} \in Y_p$, we have a $U-$equivariant exact sequence*

$$\cdots \xrightarrow{F_1} HF^+(Y_0,[\mathfrak{t}]) \xrightarrow{F_2} HF^+(Y_p,\mathfrak{t}) \xrightarrow{F_3} HF^+(Y) \longrightarrow \cdots$$

2.10 Knot Floer homology

Ozsváth-Szabó (Ozsváth and Szabó, 2004a) and independently Rasmussen (Rasmussen, 2003) extended the Heegaard-Floer package to (rationally null-homologous) knots in three manifolds. The invariant takes the form of a filtration of the chain complex computing the Heegaard-Floer homology of the underlying three-manifold. The associated graded object is a bi-graded abelian group, denoted by $\widehat{HFK}$.

We will in introduce the necessary material by following (Ozsváth and Szabó, 2004a) and (Ozsváth and Szabó, 2011). Note that in the former reference, where only null-homologous knots are considered, relative Spin^c structures are defined as absolute Spin^c structures on the 0-surgery on the knot, whereas in the latter, relative Spin^c structures are defined entirely within the knot complement. We will follow the second approach, though we note that it is easy to see that for null-homologous knots the two definitions describe essentially the same objects (Ozsváth and Szabó, 2011, Section 3.1)

The data needed to define $\widehat{HFK}$ is that of a *doubly pointed Heegaard diagram.*

Definition 2.10.1. *(Ozsváth and Szabó, 2004a, Definition 2.4) A doubly pointed Heegaard diagram describing a knot $K \subset Y$ is a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, such that (Σ, α, β) is a Heegaard diagram for Y and w, z determine the knot K in the following way: choose a properly embedded arc γ_α in the U_α handlebody with endpoints w and z, oriented from z to w and disjoint from the α cutting disks. Similarly, choose a properly embedded arc $\gamma_\beta \subset U_\beta$ with $\partial\gamma_b = z - w$. Then the resulting knot will be $K := \gamma_\alpha \cup \gamma_\beta$ with the induced orientation.*

Note that the two arcs are uniquely determined, up to isotopy, by the doubly pointed

Heegaard diagram, hence K is well-defined. It is proved (Ozsváth and Szabó, 2004a, Proposition 3.5) that any pair (Y, K) admits a doubly pointed Heegaard diagram and any two doubly pointed Heegaard diagrams differ by a finite sequence of Heegaard moves, natural analogues of the pointed Heegaard moves.

The various Heegaard-Floer type invariants for (Y, K) are constructed from the complex $CFK^{\infty}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ - the free abelian group generated by pairs $[\mathbf{x}, i, j]$ where $\mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ is an intersection point between the totally real tori in $\mathrm{Sym}^{g}(\Sigma)$ and $i, j \in \mathbb{Z}$. The differential ∂^{∞} is defined as follows: (Ozsváth and Szabó, 2004a, Section 3.1)

$$\partial^{\infty}[\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1\}} \# \left(\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \right) [\mathbf{y}, i - n_w(\phi), j - n_z(\phi)]$$

where, as in the absolute case, $\# \left(\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \right)$ denotes a count of elements in the zero-dimensional moduli space of unparametrised pseudo-holomorphic disks connecting $\mathbf{x}$ and $\mathbf{y}$. The analytic details are entirely omitted, we just note that the discussion is similar to the one in the absolute case, see (Ozsváth and Szabó, 2004a, Theorem 3.1 and its proof).

One sees that the indices i, j keep track of the intersection of holomorphic disks with the subvarieties $V_w = w \times \mathrm{Sym}^{g-1}(\Sigma)$, resp. $V_z = w \times \mathrm{Sym}^{g-1}(\Sigma)$.

These intersection numbers $n_z(\phi) = \#\phi \cap V_z$ resp. $n_w(\phi) = \#\phi \cap V_w$ are non-negative since both these manifolds are (pseudo) holomorphic, hence there is a $\mathbb{Z} \times \mathbb{Z}$ filtration $\mathcal{F}$ on $CFK^{\infty}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ given by $\mathcal{F}[\mathbf{x}, i, j] = (i, j)$.

As in the absolute case, the existence of a holomorphic disk connecting two intersection points is homologically obstructed, as a result, the complex $CFK^{\infty}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ splits, and it turns out that the resulting summands are in one-to-one correspondence with relative Spin^c structures on $Y \setminus K$:

Let $\mathcal{V}(Y, K)$ be the set of non-vanishing vector fields on $Y \setminus \overset{\circ}{N}(K)$ whose restriction to $\partial N(K)$ belongs to v_T. Declare two vector fields in $\mathcal{V}(Y, K)$ to be *homologous* if they are isotopic in the complement of a finite number of three-balls supported in $Y \setminus N(K)$.

Definition 2.10.2. *(Ozsváth and Szabó, 2011, Section 2.2) and (Ozsváth and Szabó, 2008a, Section 3.2) (Note that in the former reference the behaviour on $\partial N(K)$ differs slightly) The equivalence classes of vector fields in $\mathcal{V}(Y, K)$ under the above relation are called* relative Spin^c structures, *and they form the set denoted by* $\underline{\text{Spin}}^c(Y, K)$.

$\underline{\text{Spin}}^c(Y, K)$ is an affine space $H^2(Y, K; \mathbb{Z})$ by the same construction as in the absolute case. The Chern class of a relative Spin^c structure $[v]$, with $v \in \mathcal{V}(Y, K)$, is the cohomology class $c_1([v]) = [v] - [-v] \in H^2(Y, K; \mathbb{Z})$. Equivalently, the Chern class of $[v]$ can be defined as follows: choose a Riemannian metric on Y and consider the oriented plane field $v^\perp$. This plane field has a canonical non-zero section on $\partial N(K)$, namely the outward-pointing unit vector field u_T in $v^\perp$. Then $c_1([v]) = e(v^\perp, u_T)$, i.e. it is the relative Euler class of $v^\perp$ with respect to the trivialisation u_t.

The relationship between $\underline{\text{Spin}}^c$ structures and intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is the following (Ozsváth and Szabó, 2008a, Section), (Ozsváth and Szabó, 2011, Section 2.4):

The Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ is obtained as the $\frac{3}{2}$ level of a self-indexing Morse function $f \colon Y \longrightarrow [0, 3]$, where, as in the absolute case, the α curves are the intersection of the ascending submanifolds of the index 1 critical points with the Heegaard surface, and the β curves the intersection of the descending submanifolds of the index 2 critical points with Σ. The knot K is then the union of the trajectories under $-\nabla(f)$ containing w and z. Now given an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, one can construct a non-zero vector field on $Y \setminus\setminus K$: suppose $\mathbf{x} = \{x_{1,}, \ldots, x_g\}$. In a neighborhood of x_i, one can modify $\nabla(f)$ such that the new vector field is non-zero (in that neighborhood). Let V be a tubular neighborhood of K, there is a standard procedure to modify the $-\nabla(f)$ on V to a nowhere zero vector field v, uniquely characterized by the property that v is everywhere transverse to the meridian disks of V and K (as an oriented curve) is a trajectory of v.

This construction provides a non-zero vector $v_{\mathbf{x}}$ field on $Y \setminus\setminus K$ with the restriction to $\partial N(K)$ the vector field v_T. Therefore $v_{\mathbf{x}}$ determines a relative Spin^c structure $[v_{\mathbf{x}}]$.

Note that $[v_{\mathbf{x}}]$ doesn't depend on the choices made in its definition, hence there is a well-defined map $\underline{\mathfrak{s}}_{w,z}\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \underline{\mathrm{Spin}}^c(Y,K)$, given by $\underline{\mathfrak{s}}_{w,z}(\mathbf{x}) = [v_{\mathbf{x}}]$.

As in the absolute case, there is a map which quantifies the obstruction to the existence of holomorphic disks between two intersection points $\mathbf{x},\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$: there exist paths $a\colon [0,1] \longrightarrow \mathbb{T}_\alpha$, $b\colon [0,1] \longrightarrow \mathbb{T}_\beta$ such that $\partial a = \partial b = \mathbf{x} - \mathbf{y}$. These paths can be seen in Σ as a collection of g paths with the images in $\alpha_1 \cup \cdots \cup \alpha_g$, resp. $\beta_1 \cup \ldots \beta_g$. Then the closed multicurve $a-b$ is a cycle in $H_1(Y\setminus K,\mathbb{Z})$, then define $\underline{\epsilon}\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow H_1(Y\setminus K,\mathbb{Z})$ by: $\underline{\epsilon}(\mathbf{x},\mathbf{y}) = [a-b]$. It is easily seen that $\underline{\epsilon}$ is well-defined, i.e. different choices of a and b lead to the same homology class.

We have the following:

Lemma 2.10.3. *(Ozsváth and Szabó, 2011, Lemma 2.1) For $\mathbf{x},\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have :*

$$\underline{\mathfrak{s}}_{w,z}(\mathbf{y}) - \underline{\mathfrak{s}}_{w,z}(\mathbf{x}) = PD[\underline{\epsilon}(\mathbf{x},\mathbf{y})]$$

This implies that, if there is a holomorphic disk $\phi \in \pi_2(\mathbf{x},\mathbf{y})$, Then

$$\underline{\mathfrak{s}}_{w,z}(\mathbf{x}) - \underline{\mathfrak{s}}_{w,z}(\mathbf{y}) = (n_z(\phi) - n_w(\phi)) \cdot PD[\mu]$$

Consequently, there is a splitting of $CFK^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$ into subcomplexes associated to relative Spin^c structures on $Y \setminus K$: fix $\xi \in \underline{\mathrm{Spin}}^c(Y,K)$ and consider the subgroup $CFK^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z,\xi)$ of $CFK^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$ freely generated by the elements $[\mathbf{x},i,j]$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $\underline{\mathfrak{s}}_{w,z}(\mathbf{x}) + (i-j)\cdot PD[\mu] = \xi$. It is immediate from Lemma 2.10.3 that indeed $\partial(CFK^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z,\xi)) \subset CFK^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$.

It is proved in (Ozsváth and Szabó, 2004a) and (Ozsváth and Szabó, 2011) that the filtered chain homotopy type of $CFK^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z,\xi)$ is an invariant of the pair (Y,K) and of the Spin^c structure ξ, i.e. it does not depend on the doubly-pointed Heegaard diagram and on the analytical choices made in its definition. Therefore, this complex will be denoted by $CFK^\infty(Y,K,\xi)$.

One can form the associated graded object, namely the induced quotient complex denoted $\widehat{CFK}(Y,K,\xi)$ generated by the elements $[\mathbf{x},0,0] \in CFK^\infty(Y,K,\xi)$ with the induced differential $\widehat{\partial}$. Its homology, $H_*(\widehat{CFK}(Y,K,\xi))$ is denoted by $\widehat{HFK}(Y,K,\xi)$ and is called the Knot Floer homology of K in the Spin^c structure ξ.

For the case of knots in S^3, there is a natural identification of relative Spin^c structures on $Y \setminus K$ with integers: note that in this case, relative Spin^c structures are determined by their Chern class, since there is no 2-torsion in $H_1(Y \setminus K;\mathbb{Z})$, (recall that $c_1(\xi+h) = c_1(\xi)+2h$). Also, since $H_1(Y \setminus K;\mathbb{Z}) \cong \mathbb{Z}$, the evaluation of this Chern class on the Seifert surface of K gives the above mentioned identification. Therefore, one denotes by $\widehat{HFK}(S^3,K,i)$ the group $\widehat{HFK}(S^3,K,\xi)$, where $\xi \in \underline{\mathrm{Spin}}^c(S^3,K)$ is the unique Spin^c structure ξ with $\langle c_1(\xi),[F]\rangle = 2i-1$.

Remark 2.10.4. *The term -1 in this equation does not appear in (Ozsváth and Szabó, 2004a), it is a consequence of the different notion of relative* Spin^c *structure for null-homologous knots (Ozsváth and Szabó, 2011, Section 3.1).*

With this notation, the Euler characteristic of $\widehat{HFK}(S^3,K)$ takes the following remarkable form:

Theorem 2.10.5. *(Ozsváth and Szabó, 2004a,) Let $K \subset S^3$. Then:*

$$\sum_{i\in\mathbb{Z}} \chi(\widehat{HFK}(S^3,K,i)) \cdot T^i = \Delta_K(T)$$

where $\Delta_K(T) = \sum_i a_i \cdot T^i$ is the Alexander polynomial of K, normalized such that $a_i = a_{-i}$.

Remark 2.10.6. *Note that the sum above is finite since $\widehat{HFK}$ is finitely generated.*

CHAPTER III

KNOTS IN LENS SPACES HAVING $S^1 \times S^2$ SURGERIES

3.1 The Berge-Gabai construction

As in (Boileau et al., 2011, Definition 5.4), we call the knots in $S^1 \times D^2$ which admit a nontrivial cosmetic surgery - *Berge-Gabai* knots. We will call the slope of this surgery a *distinguished slope.*

It was proved by Gabai in (Gabai, 1989) that such a knot must necessarily be a 1-bridge braid with respect to both the initial solid torus and the surgered solid torus.

Here is one way to obtain $S^1 \times S^2$ by surgery on a knot in a lens space: start with a solid torus V with meridian μ and a *Berge-Gabai* knot $K \subset V$. There is a slope $\alpha \in H_1(\partial N(K))$ such that $V' := V_\alpha(K)$ is another solid torus, with meridian μ'. Do *Dehn* filling on V along μ' to obtain a lens space L. Then $K \subset L$ has an $S^1 \times S^2$ surgery: indeed $L_\alpha(K)$ has a genus 1 Heegaard splitting in which the meridians of the two solid tori coincide (this common meridian is μ').

It is a pleasant fact that these knots are embedded in a very particular way in the lens space:

Definition 3.1.1. *A* simple knot *in a lens space L is a 1-bridge knot which can be isotoped such that the* 2 *bridges are contained respectively in the meridian disks of the* 2 *Heegaard solid tori.*

Theorem 3.1.2. *Let $K \subset V$ be a Berge-Gabai knot with distinguished slope α in the* *solid torus V. Assume that V is further embedded in a lens space L as a Heegaard torus. If $L_\alpha(K) \cong S^1 \times S^2$, then K is a simple knot in L.*

Proof. As before, call μ the meridian of V. $K \subset V$ is a braid of index n - say. There exists a closed n-punctured disk $D \subset V \setminus N(K)$ with boundary components μ and n copies of the meridian of K. Let $\overline{D}$ be the meridinal disk of V containing D. Let K' be the induced knot in the surgered solid torus $V' := V_\alpha(K)$. $K' \subset V'$ is also a braid, hence there is an analogous n'-punctured disk $D' \subset V' \setminus N(K')$. It is a fact proved by Gabai (Gabai, 1990, Corollary 3.3) that $n = n'$; also, in the case when K is not a torus knot, the slope α is a longitude of K (Gabai, 1989, Lemma 2.3).

Consider the graph of intersection between D and D' : (as in (Gabai, 1989, Lemma 2.3) where the analysis of this intersection is employed to show that these knots are 1-bridge).

After an isotopy of D and D', we can suppose that their intersection consists of n^2 arcs, all with endpoints on ∂V, resp. $\partial N(K)$. The orientation on these arcs is by convention from the endpoints on $\partial N(K)$ to the endpoints on ∂V. As in the proof of (Gabai, 1989, Lemma 2.3), there is a boundary component m' of D' such that all of the n arcs incident to it are parallel (in D'); let these n arcs be $e_1, \dots, e_n$, labelled by their appearance on m' when walking along the oriented knot K. Since α is a longitude, K is isotopic to m'; m' is isotopic to the reunion $e_1 \cup f_0 \cup (-e_n) \cup g_0$, where f_0 is the arc in μ between e_1 and e_n and g_0 is the arc in m' between e_n and e_1 with respect to the given orientation on m'. The isotopy sweeps the squares in D' realising the parallelism between the e_i's.

The arc $-e_n \cup g_0$ has its endpoints on the same meridinal disk D and winds once around the solid torus V. It can be isotoped rel endpoints in V to a union of arcs $f_1 \cup e$ where: f_1 is the arc in ν between the endpoint of e_n and the next (as walking along ν with the orientation inherited from K) point of intersection between ν and D; e is an arc in $\overline{D}$ joining the end of f_1 to the start of e_1. This isotopy is sweeping the rectangle formed by a continuous family of segments joining the points of g_0 and f_1 belonging to the same D^2 fibre of $V = S^1 \times D^2$. The segment in the $\overline{D}$ fibre is e_n.

Now $f_0 \cup f_1$ will be the first bridge of the isotoped K, which can be pushed in the meridinal disk of V and $e \cup e_1$ is the other bridge, contained in $\overline{D}$ - the meridinal disk of V.

$\square$

The notion of *simple knot* appeared in (Hedden, 2011) as part of a program to prove the Berge conjecture:

Conjecture 3.1.3. *The only knots in S^3 having lens space surgeries are* doubly primitive knots, *i.e. knots in the genus* 2 *Heegaard surface of S^3 which represent generators of the fundamental group of both handlebodies.*

It was proved by Berge that the knot induced in the lens space by surgery on a doubly primitive knot in S^3 is a simple knot. In fact, the Berge conjecture can be rephrased in terms of the induced knot in the lens space and it is equivalent to:

Conjecture 3.1.4. *Let K be a knot in a lens space L which admits an S^3 surgery. Then K is simple.*

In the rest of this chapter we will analyse the Knot Floer homology of knots in lens spaces which admit an $S^1 \times S^2$ surgery.

3.2 Topological preliminaries

Consider a knot K in a lens space $Y = L(p, q)$ which has a $S^1 \times S^2$ surgery along a slope $\pm[\lambda]$. Knots in lens spaces whose exteriors admit Seifert fibred structures have been classified, see (Brin, 2007) for example, and surgeries on them are well understood.

We make the assumption that $Y \setminus\setminus K$ is irreducible and not Seifert fibred, hence by the Cyclic Surgery Theorem (Culler et al., 1987), the slope $\pm[\lambda]$ is at distance 1 from the meridian of K. Denote by $Y_\lambda(K)$ the result of Dehn surgery along $\pm[\lambda]$.

We fix an orientation on K and denote by $\overline{K}$ the resulting oriented knot. $\overline{K}$ determines a class $[\overline{K}]$ in $H_1(Y) \cong \mathbb{Z}/p$ whose order k is by definition the *order of* K $(ord(K))$. From now on we will consider λ to be oriented coherently with $\overline{K}$. We investigate under which conditions $Y_\lambda(K)$ is a homology $S^1 \times S^2$.

Definition 3.2.1. *The rational longitude of $K \subset Y$ is the unique slope in $\partial N(K)$ which is 0 in $H_1(Y \setminus\setminus K; \mathbb{Q})$.*

The proof that there is a unique such slope is a straightforward application of Poincaré duality, see for example (Hatcher, 2007). Note that only surgery along the rational longitude of K can produce a *non* $\mathbb{Q}$-homology sphere, so in view of the above discussion, we will only consider knots for which the rational longitude is a longitude.

Lemma 3.2.2. *A knot $K \subset Y$, where Y is a $\mathbb{Q}$-homology sphere of order p , with $ord(K) = m$, has an integer surgery which is a homology $S^1 \times S^2$ $\Longleftrightarrow$ the rational longitude of K is a longitude and $p = m^2$.*

Proof. Consider the long exact sequence associated to the pair $(Y, N(K))$:

$$0 \longrightarrow H_2(Y, N(K)) \longrightarrow H_1(N(K)) \longrightarrow H_1(Y) \longrightarrow H_1(Y, N(K)) \longrightarrow 0$$

One sees from the sequence that $\#H_1(Y, N(K)) = p/m$. Denote this group by G. Write the long exact sequence for the pair $(M, \partial M)$

$$\dots \xrightarrow{0} H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \longrightarrow H_1(M) \longrightarrow H_1(M, \partial M) \longrightarrow 0$$

By Poincaré duality and the universal coefficients theorem, $H_1(M) \cong \mathbb{Z} \oplus G$. Consider the base of $H_1(\partial M)$ formed by $[\mu]$ (the meridian of K) and $[\lambda]$ and observe that the connecting homomorphism ∂ has image the subgroup $Span([m\lambda]) \subset \mathbb{Z} \oplus \mathbb{Z}$, hence $\mathbb{Z} \oplus \mathbb{Z}/m \overset{\alpha}{\hookrightarrow} \mathbb{Z} \oplus G$, so $m \leq p/m$.

For the direct implication, since $H_1(Y_\lambda(K)) \cong H_1(M)/Im([\lambda]) \cong \mathbb{Z}$, we must have $\mathbb{Z}/m \overset{\alpha}{\twoheadrightarrow} G$, so $m = p/m$.

For the converse, since $\#G = m$, the map $\alpha|_{\mathbb{Z}/m} \hookrightarrow G$ must be bijective, hence $H_1(Y_\lambda(K)) \cong \mathbb{Z}$. □

Remark 3.2.3. *The order of a lens space which bounds a $\mathbb{Q}$-homology 4-ball is a perfect square. See (Lisca, 2007) for instance.*

3.3 An integral surgery long exact sequence

In this section we present a long exact sequence for integral surgeries on a knot in a rational homology three-sphere, defined in Theorem 6.2 of (Ozsváth and Szabó, 2011), with additional refinements given by Spin^c structures as in Theorem 9.19 of (Ozsváth and Szabó, 2004b).

Let $K \subset Y$ be an oriented knot, where Y is a rational homology three sphere. Let λ be the rational longitude of K, which we will suppose to be a longitudinal slope and oriented coherently with K. Suppose also that $H_1(Y_\lambda(K)) \cong \mathbb{Z}$. Pick a minimal genus Seifert surface F, oriented such that $\partial[F] = m \cdot \lambda$, for some $m > 0$; m is thus the order of the knot. (One can always assume that the boundary of the Seifert surface consists of m coherently oriented copies of the same simple closed curve representing λ by tubing any consecutive components of ∂F with opposite orientation.) Fix also the meridian μ of K, oriented such that $\mu \cdot F > 0$. Finally, consider an integer $p > 0$. We will be interested in the manifold $Y_{p\mu+\lambda}(K)$, denoted from now on as Y_p, and, of course, in $Y_\lambda := Y_\lambda(K)$.

Lemma 3.3.1. *(Ozsváth and Szabó, 2004b)[Lemma 9.2] One can construct a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$ of genus g with the following properties:*

1. *The Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is a Heegaard diagram for $K \subset Y$, as in Definition 2.10.1, with the extra property that β_g is a meridian of K. We will assume $\beta_g = \mu$.*
2. *The curves β_i, γ_i and δ_i, for $i \in \{1, \dots, g-1\}$, are small isotopic translates of each other and intersect each other transversely in two points, and the isotopies do not cross the basepoints.*

3. *the curve γ_g is isotopic in the boundary of the knot complement, i.e. in the manifold described by the Heegaard diagram $(\Sigma, (\alpha_i)_{i=1}^{g}, (\beta_i)_{i=1}^{g-1})$ to λ*

4. *the curve δ_g is isotopic to the juxtaposition of δ_g and p copies of β_g, i.e. δ_g is the $p\mu + \lambda$ slope.*

5. *The diagram is admissible.*

Proof. Apply the proof of Lemma 9.2 of (Ozsváth and Szabó, 2004b) and stabilize the Heegaard diagram obtained in a neighbourhood of a point of K. See Figure 3.4 for the part of the Heegaard diagram containing μ, called the *the winding region.* □

For $\mathfrak{t} \in \mathrm{Spin}^c(Y_p)$, consider the following set

$$\Lambda_{Y_p}(\mathfrak{t}) = \{\mathfrak{k} \in \mathrm{Spin}^c(Y_p) : \mathfrak{k} - \mathfrak{t} \in \mathrm{Span}([\lambda])\}$$

where $[\lambda]$ is seen here as an element of $H_1(Y_p;\mathbb{Z})$.

Remark 3.3.2. Spin^c *structures (for all objects: three-manifolds, four-manifolds, knot complements) form affine spaces over H^2, in what follows we will sometimes find it more convenient to use homology classes rather than cohomology classes, the two are identified of course by the version of Poincaré duality relevant for each case.*

Similarly, we define $\Lambda_Y(\mathfrak{s})$ to be the orbit of $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ under the action of $\mathrm{Span}([\lambda]) \subset H_1(Y;\mathbb{Z})$. Note that $\#(\Lambda_{Y_p}(\mathfrak{t})) = \#(\Lambda_Y(\mathfrak{s})) = m$.

Consider the cobordism W_p obtained by reversing the two-handle attachment corresponding to the Morse surgery on K with slope $p\mu + \lambda$. Note that there exists a unique $\mathrm{Span}([\lambda])$ orbit $\Lambda_Y(\mathfrak{b}) \subset \mathrm{Spin}^c(Y)$ which is cobordant in W_p to $\mathfrak{t}$, for some $\mathfrak{b} \in \mathrm{Spin}^c(Y)$. This follows from the fact that Spin^c structures on Ywhich are cobordant to a fixed Spin^c structure on Y_p form an affine space over

$$\mathrm{Im}[(H_2(W_p, Y;\mathbb{Z})) \longrightarrow H_1(Y;\mathbb{Z})]$$

and this image is $\mathrm{Span}([\lambda]) \subset H_1(Y;\mathbb{Z})$.

We have now all the ingredients to write the aforementioned long exact sequence.

Theorem 3.3.3. *(Ozsváth and Szabó, 2004b, Essentially Theorem 9.19) There is a map* $Q\colon \mathrm{Spin}^c(Y_\lambda(K)) \longrightarrow \mathrm{Spin}^c(Y_p)/_{\mathrm{Span}(\lambda)}$ *such that for any* $\mathfrak{t} \in \mathrm{Spin}^c(Y_p)$ *there is a* $U-$*equivariant long exact sequence*

$$\cdots \xrightarrow{F_1} HF^+(Y_\lambda, Q^{-1}(\Lambda_{Y_p}(\mathfrak{t}))) \xrightarrow{F_2} HF^+(Y_p, \Lambda_{Y_p}(\mathfrak{t})) \xrightarrow{F_3} HF^+(Y, \Lambda_Y(\mathfrak{b})) \longrightarrow \cdots$$

where $\mathfrak{b}$ *is a* Spin^c *structure on* Y *cobordant to* $\mathfrak{t}$ *in* W_p.

Proof. The construction of the long exact sequence is a generalisation of the long exact sequence of (Ozsváth and Szabó, 2004b)[Theorem 9.19], it appeared also in (Ozsváth and Szabó, 2008b) for the case of null-homologous knots. We sketch the proof of (Ozsváth and Szabó, 2004b)[Theorem 9.19] focusing on the parts which need to be slightly modified for our situation, i.e. for knots which are only rationally null-homologous, with the extra property that their rational longitude is a longitude.

Start with the Heegaard diagram from the previous lemma. For $i \in \{1, \ldots, g-1\}$, we denote the intersection points of β_i, γ_i and δ_i by

$$y_i^\pm = \beta_i \cap \gamma_i, v_i^\pm = \gamma_i \cap \delta_i, w_i^\pm = \beta_i \cap \delta_i$$

with the sign indicating the intersection sign. Also,

$$y_g = \beta_g \cap \gamma_g, w_g = \beta_g \cap \delta_g.$$

Note that there are p intersection points between γ_g and δ_g. We choose one - call it v_g, which will be fixed by Claim 3.3.4 below.

Let $\Theta_{\beta,\gamma} = \{y_1^+, \ldots, y_{g-1}^+, y_g\}$, $\Theta_{\gamma,\delta} = \{v_1^+, \ldots, v_{g-1}^+, v_g\}$ and $\Theta_{\beta,\delta} = \{w_1^+, \ldots, w_{g-1}^+, w_g\}$.

Then the elements $\theta_{\beta,\gamma} = [\Theta_{\beta,\gamma}, 0]$, $\theta_{\gamma,\delta} = [\Theta_{\gamma,\delta}, 0]$ and $\theta_{\beta,\delta} = [\Theta_{\beta,\delta}, 0]$ are cycles in $CF^\infty(\mathbb{T}_\beta, \mathbb{T}_\gamma)$, $CF^\infty(\mathbb{T}_\gamma, \mathbb{T}_\delta)$ and $CF^\infty(\mathbb{T}_\beta, \mathbb{T}_\delta)$, respectively (Ozsváth and Szabó, 2004b)[Proposition 9.3].

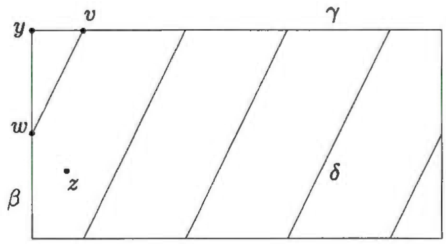

Figure 3.1 The intersection point v of γ and δ

Note that $Y_{\gamma,\delta}$, i.e. the manifold described by the Heegaard diagram $(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta})$ is the lens space $L(p,1)$.

Claim 3.3.4. *(Ozsváth and Szabó, 2004b)[Proposition 9.15] There is a choice of $v_g \in \gamma_g \cap \delta_g$ such that there are homotopy classes of triangles $\{\psi_k^{\pm}\}_{k=1}^{\infty} \in \pi_2(\Theta_{\beta,\gamma}, \Theta_{\gamma,\delta}, \Theta_{\beta,\delta})$ satisfying the following properties:*

- $\mu(\psi_k^{\pm}) = 0$
- $n_z(\psi_k^+) = n_z(\psi_k^-)$
- $n_z(\psi_k^+) < n_z(\psi_{k+1}^+)$

Moreover, each triangle in $\pi_2(\Theta_{\beta,\gamma}, \Theta_{\gamma,\delta}, \Theta_{\beta,\delta})$ is Spin^c *equivalent to some $\psi_k^{\pm}$. There are also choices of perturbations of the complex structure on* $\mathrm{Sym}^g(\Sigma)$ *such that for $\psi \in \pi_2(\mathbf{x}, \Theta_{\gamma,\delta}, \Theta_{\beta,\delta})$, where $\mathbf{x} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ with $\mu(\psi) = 0$, we have*

$$\#\mathcal{M}(\psi) = \begin{cases} \pm 1 & , \text{if } \psi \in \{\psi_k^{\pm}\}_{k=1}^{\infty} \\ 0, & \text{otherwise} \end{cases}$$

Proof of claim. This claim is the analogue of (Ozsváth and Szabó, 2004b)[Proposition 9.5]. There one is interested in $\frac{1}{q}$ surgeries and it is β_g and δ_g that intersect more than once. By the gluing result of Theorem 9.4 of the same paper, it is enough to establish the claim for a Heegaard diagram $(E, \beta, \gamma, \delta, z)$ of genus 1, with the three curves β, γ, δ in the same position as our curves $\beta_g, \gamma_g, \delta_g$.

Figure 3.3 shows how to choose the intersection point v of γ and δ with respect to the basepoint z such that the homotopy classes of triangles exist. Note that $n_z(\psi_k^{\pm}) = p \cdot \frac{k(k+1)}{2}$. Since our Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$ is an iterated stabilisation of $(E, \beta, \gamma, \delta, z)$, the proof of Proposition 9.5 of (Ozsváth and Szabó, 2004b) applies without any other changes. $\square$

Now choose our v_g in the same position as v above.

Using the Heegaard diagram defined above, we define two maps via counts of holomorphic triangles, as in (Ozsváth and Szabó, 2004b)[Theorem 9.19]. Let $t_{\gamma,\delta}$ be the Spin^c structure in which $\Theta_{\gamma,\delta}$ is supported.

For a given $t' \in \mathrm{Spin}^c(Y_\lambda)$, there is a unique λ orbit $\Lambda_{Y_p}(t)$, for some $t \in \mathrm{Spin}^c(Y_p)$ with the property that there exists a Spin^c structure $\mathfrak{s}_{\alpha,\gamma,\delta}$ on the triple cobordism $X_{\alpha,\gamma,\delta}$ determined by the Heegaard diagram $(\Sigma, \alpha, \gamma, \delta)$ which extends t', t and $t_{\gamma,\delta}$. Then, by definition, $Q(t) = \Lambda_{Y_p}(t)$.

The map $f_2 \colon CF^\infty(\Sigma, \alpha, \gamma, z) \longrightarrow CF^\infty(\Sigma, \alpha, \delta, z)$ is obtained by counting holomorphic triangles in the triple Heegaard diagram $(\Sigma, \alpha, \gamma, \delta, z)$. More precisely,

$$f_2(\xi) = \sum_{\{\mathfrak{s} \in \mathrm{Spin}^c(X_{\alpha,\gamma,\delta}) \colon \mathfrak{s}_{Y_\lambda} = t, \mathfrak{s}_{Y_{\gamma,\delta}} = t_{\gamma,\delta}\}} f^+_{\alpha,\gamma,\delta}(\xi \otimes \Theta_{\gamma,\delta}, \mathfrak{s})$$

where $Y_{\gamma,\delta}$ is the three-manifold determined by the Heegaard diagram (Σ, γ, δ).

Similarly, the map $f_3 \colon CF^\infty(\Sigma, \alpha, \delta, z) \longrightarrow CF^\infty(\Sigma, \alpha, \beta, z)$ is defined by

$$f_3(\xi) = \sum_{\{\mathfrak{s} \in \mathrm{Spin}^c(X_{\alpha,\delta,\beta}) \colon \mathfrak{s}_{Y_p} \in \Lambda_{Y_p(t)}\}} f^+_{\alpha,\delta,\beta}(\xi \otimes \Theta_{\delta,\beta}, \mathfrak{s})$$

Denote by F_2, resp. F_3 the maps induced in homology by f_2, resp. f_3. Note that the image of F_3 is supported on a Λ_Y orbit of some $\mathfrak{b} \in \mathrm{Spin}^c(Y)$ cobordant to t in W_p seen here as the filling by $\#^{g-1} S^1 \times D^3$ along the (Y, δ, β) part of the triple cobordism $X_{\alpha,\delta,\beta}$.

The maps defined by triple cobordism satisfy an associativity property, (Ozsváth and

Szabó, 2004c, Theorem 8.16) which in our context states that the composition $F_3 \circ F_2$ factors through the sum of functions $F^{\leq 0}_{\gamma,\delta,\beta}(\_, \mathfrak{s}_{\gamma,\delta,\beta})$ applied to $\Theta_{\gamma,\delta} \otimes \Theta_{\delta,\beta}$. But this element is 0 by Claim 3.3.4 since

$$\sum_{\mathfrak{s} \in S_{\gamma,\delta,\beta}} F^{\leq 0}_{\gamma,\delta,\beta}(\Theta_{\gamma,\delta} \otimes \Theta_{\delta,\beta}, \mathfrak{s}) = \sum_{k=1}^{\infty} \left[\Theta_{\gamma,\beta}, -p\frac{k(k+1)}{2}\right] \pm \left[\Theta_{\gamma,\beta}, -p\frac{k(k+1)}{2}\right]$$

where $S_{\gamma,\delta,\beta} = \{\mathfrak{s}_{\gamma,\delta,\beta} \in \mathrm{Spin}^c(X_{\gamma,\delta,\beta})\ \mathfrak{s}_{\gamma,\delta,\beta}|_{Y_{\gamma,\delta}} = t_{\gamma,\delta}, \mathfrak{s}_{\gamma,\delta,\beta}|_{Y_{\delta,\beta}} = t_{\delta,\beta}\}$.

The signs in the sum above depend on the orientation systems on the four-manifold given by the triple diagram $(\Sigma, \gamma, \delta, \beta)$ which can be chosen to be always " $-$ " since the triangles belong to different $\delta H^1(Y_{\alpha,\delta}) + \delta H^1(Y_{\gamma,\beta})$ orbits in $\mathrm{Spin}^c(X_{\alpha,\gamma,\delta,\beta})$.

The curve δ_g is isotopic to the juxtaposition of γ_g and p copies of β_g, denote by $\delta_g(s)$, $s \in [0,1]$ the isotopy. Then the intersections of the curve $\delta_g(s)$, for s close enough to 1, partition into two sets, according to the curve they are most close to, γ_g or β_g. As in (Ozsváth and Szabó, 2004b)[Theorem 9.19], we define a map

$$\imath\colon CF^+(Y_\lambda, Q^{-1}(\Lambda_{Y_p}(t))) \longrightarrow CF^+(Y_p, \Lambda_{Y_p}(t))$$

by sending an intersection point between α and γ to the unique nearby intersection point between α and δ. Similarly, we define a map

$$\pi\colon CF^+(Y_p, \Lambda_{Y_p}(t)) \longrightarrow CF^+(Y, \Lambda_Y(\mathfrak{b}))$$

by sending an intersection point of α and δ to the nearby intersection point between α and β. Since we are fixing a $\Lambda_{Y_p(t)}$ orbit, only one of the p corresponding intersection points between α and δ is taken into consideration. These two facts imply that there is a short exact sequence

$$0 \longrightarrow CF^+(Y_\lambda, Q^{-1}(\Lambda_{Y_p}(t))) \xrightarrow{\imath} CF^+(Y_p, \Lambda_{Y_p(t)}) \xrightarrow{\pi} CF^+(Y, \Lambda_Y(t)) \longrightarrow 0$$

which has a splitting map R, since the last group is free. These maps are not necessarily chain maps, but with their help one can construct two such maps which will determine by simple homological algebra the desired long exact sequence.

When $\delta_g(s)$ is sufficiently close to the juxtaposition of β_g and γ_g, one can define area filtrations on $CF^+(Y)$, $CF^+(Y_\lambda)$ and $CF^+(Y_p)$ which are strictly decreasing for boundary maps, such that the maps defined above f_2 and f_3 decompose as $f_2 = \imath +$ lower order terms and also $f_3 = \pi +$ lower order terms. Also $f_3 \circ f_2$ is chain homotopic to 0 by a $U-$ equivariant homotopy $H\colon CF^+(Y_0, Q^{-1}(\Lambda_{Y_p}(t))) \longrightarrow CF^+(Y, \Lambda_Y(\mathfrak{b}))$ obtained by counting holomorphic squares in the quadruple cobordism given by the Heegaard diagram $(\Sigma, \alpha, \beta, \gamma, \delta)$ which moreover decreases the filtration. More precisely,

$$H([\mathbf{x}, i]) = \sum_{\square \in \pi_2(\mathbf{x}, \Theta_{\beta,\gamma}, \Theta_{\gamma,\delta}, \mathbf{y}), \mu(\square)=0} [\mathbf{y}, -n_z(\square)]$$

Then one defines

$$R' = R \circ \sum_{k=0}^{\infty} (Id - f_3 \circ R)^{\circ k}$$

and

$$g_2 := f_2 - (\partial(R' \circ H) + (R' \circ H)\partial)$$

Then Ozsváth-Szabó show that f_2 is chain homotopic to g_2 and the maps fit into the short exact sequence

$$0 \longrightarrow CF^+(Y_\lambda, Q^{-1}(\Lambda_{Y_p}(t))) \xrightarrow{g_2} CF^+(Y_p, \Lambda_{Y_p(t)}) \xrightarrow{f_3} CF^+(Y, \Lambda_Y(t)) \longrightarrow 0$$

which then gives the long exact sequence in the statement of the theorem. $\square$

Remark 3.3.5. *It is important to observe that the map g_2 also has the property that $g_2 = \imath +$ lower order terms w. r. t. the filtration.*

3.4 The top grading in Knot Floer homology

Theorem 3.4.1. *Let $K \subset L(p, q)$ be a knot with a longitudinal $S^1 \times S^2$ surgery. Then $g(K) \leq 1$.*

Proof. We want to show that our knot K has genus 0 or 1. We will suppose that it has genus $g > 1$ and arrive at a contradiction. The proof is modelled on Corollary 4.5 in (Ozsváth and Szabó, 2004a) for the case of knots in S^3.

The idea is that the 'top grading' in the knot Floer homology of K is identified with the Floer homology (to be made precise shortly) of $Y_\lambda(K)$ in a certain Spin^c structure.

The top grading in $\widehat{HFK}(Y,K)$ is related to the genus of K by work of Ni (Ni, 2009). There it is proved more generally that Heegaard Floer homology detects the Thurston norm of a three-manifold. We recall Ni's theorem with the necessary background:

An Alexander grading is defined on the space of relative Spin^c structures on $Y \setminus N(K)$ by the following formula:

$$\mathfrak{H}(\xi) = \frac{c_1(\xi) - PD[\mu]}{2}$$

Recall that we have chosen the orientations of the Seifert surface and of the meridian such that the Seifert surface F is oriented coherently with $\overline{K}$ and $\mu \cap F > 0$.

For a rational homology class $h \in H_2(M, \partial M; \mathbb{Q})$ one defines the function:

$$y(h) = \max_{\{\xi \in \underline{\mathrm{Spin}}^c(Y,K) | \widehat{HFK}(Y,k,\xi) \neq 0\}} \langle \mathfrak{H}(\xi), h \rangle$$

then we have:

Theorem 3.4.2. *(Ni, 2009) Fix a minimal genus Seifert surface F for K and denote by $h = [F] \in H_2(M, \partial M; \mathbb{Z})$ where here M is the exterior of K. Then:*

$$-\chi(F) + |h \cdot [\mu]| = 2y(h)$$

Let g be the genus of F and ξ_M be a relative $\underline{\mathrm{Spin}}^c$ structure for which $y(h) = \langle \mathfrak{H}(\xi_M), h \rangle$ We find that $\langle c_1(\xi_M), h \rangle = 2g - 2 + 3m$. Denote by $\widehat{F}$ the closed surface in $Y_\lambda(K)$ obtained by capping off F and by S the surface obtained by capping off F in W_p. Note that $[S]^2 = -m^2 p$ (the self intersection number).

By choosing p large enough, we can suppose that there is a unique Spin^c structure $\overline{\xi}_M$ on $Y_\lambda(K)$ which restricts (in the canonical way, see below) to ξ_M on $Ext(K)$ with $HF^+(Y_\lambda(K), \overline{\xi}_M) \neq 0$.

To find $\overline{\xi}_M$, we take v (a non-zero vector field on M) as a representative for ξ with the restriction on ∂M to be the translation invariant vector field on $S^1 \times S^1$ (unique up to isotopy) and extend it canonically over $Y_\lambda(K)$ such that the induced knot in $Y_\lambda(K)$ is an oriented trajectory. This extension process is described in (Ozsváth and Szabó, 2008a). By our orientation conventions, we have:

$$\langle c_1(\overline{\xi}_M), \widehat{F}\rangle = \langle c_1(\xi_M), F\rangle - \langle PD[\mu], F\rangle = 2g - 2 + 2m. \tag{3.1}$$

We will use the shorthand notation $C_{\xi_M} = CFK^\infty(Y, K, \xi_M)$

We can find a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for our knot K such that the meridian is the curve β_g and w, z are on either side of μ as in figure 3.4. Also, the slopes δ and λ intersect β_g transversely once as in the figure. We will fix this diagram in what follows.

Consider now the following short exact sequence:

$$0 \longrightarrow C_{\xi_M}\{i < 0 \text{ and } j \geq -1\} \longrightarrow C_{\xi_M}\{i \geq 0 \text{ or } j \geq -1\} \longrightarrow C_{\xi_M}\{i \geq 0\} \longrightarrow 0$$

Observe that $H_* C_{\xi_M}\{i < 0 \text{ and } j \geq -1\} \cong \widehat{HFK}(Y, K, \xi_M)$ because it is the top-dimensional summand.

Also $H_*(C_{\xi_M}\{i \geq 0 \text{ or } j \geq -1\}) \cong HF^+(Y_p, t_\xi)$ for some $t_\xi \in \text{Spin}^c(Y_p)$ by (Ozsváth and Szabó, 2011, Section 4). These groups are identified with the Floer homologies of 'large enough' surgeries on K.

The natural projection $C_{\xi_M}\{i \geq 0 \text{ or } j \geq -1\} \xrightarrow{v} C_{\xi_M}\{i \geq 0\}$ is in fact modelled on the cobordism map $F^+_{\alpha,\delta,\beta} : HF^+(Y_p, t_\xi) \longrightarrow HFK^+(Y, \xi_M - \mu)$ in a certain Spin^c structure $\mathfrak{x}$, to be made precise below.

Ozsváth-Szabó define the following map (Ozsváth and Szabó, 2011) $\Phi : CF^+(Y_p, t_\xi) \longrightarrow CFK^+(Y, K, \xi_M)$ given by

$$\Phi[x, i] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\psi \in \pi_2(\mathbf{x}, \Theta_{\delta,\beta}, \mathbf{y})\ E(\mathfrak{s}_w(\psi)) = \xi_M\ \mu(\psi) = 0\}} (\#\mathcal{M}(\psi))[y, i - n_w(\psi), i - n_z(\psi)]$$

for a triangle $\psi \in \pi_2(\mathbf{x}, \Theta_{\delta,\beta}, \mathbf{y})$ where $\mathbf{x}$, resp. $\mathbf{y}$ are generators in $CF(\Sigma, \alpha, \delta)$ resp. $CF(\Sigma, \alpha, \beta)$. Here E: $\text{Spin}^c(W_p) \longrightarrow \underline{\text{Spin}}^c(Y, K)$ is a restriction map on Spin^c structures defined in (Ozsváth and Szabó, 2011, Proposition 2.2) by

$$\psi \in \pi_2(\mathbf{x}, \Theta_{\delta,\beta}, \mathbf{y}) \to \mathfrak{s}_{w,z}(\mathbf{x}) + (n_w(\psi) - n_z(\psi))\mu$$

for $\mathbf{x}$, $\mathbf{y}$ generators in $CF(\Sigma, \alpha, \beta)$ resp. $CF(\Sigma, \alpha, \delta)$.

Then $F^+_{\alpha,\delta,\beta}$ is Φ followed by the vertical projection v given by $v[\mathbf{y}, i, j] = [\mathbf{y}, i]$.

We have the following formula, for a triangle ψ with $E(\mathfrak{s}_w(\psi)) = \xi_M$, analogous to equation (14) of (Ozsváth and Szabó, 2004a).

$$\langle c_1(s_{\xi_M}), [\widehat{F}]\rangle - 2m(n_w(\psi) - n_z(\psi)) = \langle c_1(s_w(\psi)), [S]\rangle + pm \tag{3.2}$$

For the small triangle this is an application of the first Chern class formula (in 3 and 4 dimensions of Section 2.7). Adding a domain $\phi' \in \pi_2(\mathbf{x}'_1, \mathbf{x}')$ leaves the formula unchanged since the points w and z are separated by a β curve, and also adding a homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ leaves the formula true by a simple calculation. Adding a triply periodic domain corresponding to $[S]$ also leaves the equation unchanged.

From Equation 3.2, we have that $\langle c_1(\mathfrak{x}), S\rangle = 2g - 2 - pm$. We write the long exact sequence of Theorem 3.3.3 for the Λ_{Y_p} orbit of t_ξ.

$$\cdots \longrightarrow HF^+(Y_\lambda(K), \overline{\xi}_M - \mu) \longrightarrow HF^+(Y_p, \Lambda_{Y_p}(t_\xi)) \xrightarrow{F_3} HF^+(Y, \Lambda_Y(\xi_M - \mu)) \longrightarrow \cdots$$

and we compare it to the sum of long exact sequences induced by the short exact sequence

$$0 \to C_{\Lambda_Y(\xi_M)}\{i < 0 \text{ and } j \geq -1\} \to C_{\Lambda_Y(\xi_M)}\{i \geq 0 \text{ or } j \geq -1\} \xrightarrow{\phi} C_{\Lambda_Y(\xi_M)}\{i \geq 0\} \to 0$$

where

$$C_{\Lambda_Y(\xi_M)}\{i < 0 \text{ and } j \geq -1\} = \bigoplus_{\xi \in \Lambda_Y(\xi_M)} C_\xi\{i < 0 \text{ and } j \geq -1\}$$

similarly for the others, and the maps are simply the direct sums of the respective maps.

The second map ϕ is the sum of the maps induced by Spin^c structures of type $\mathfrak{x}$ with the same Chern class. In fact one can see that these Spin^c structures differ by torsion elements of $H^2(W_p; \mathbb{Z})$.

A part of the Heegaard diagram for the triple cobordism $L\ L_\lambda\ L_p$

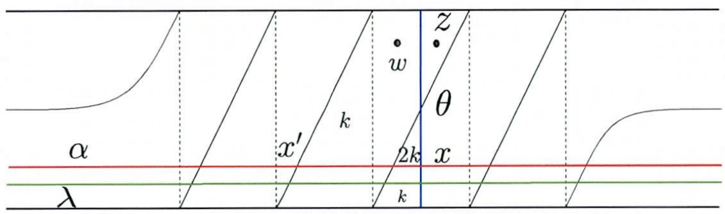

Figure 3.2 The winding region, Note that the periodic domain associated to $\overline{F}$ has λ as a boundary component with multiplicity m.

It is easy to see that ϕ is surjective in homology, because Y is an L-space and in large enough degrees, ϕ is an isomorphism.

We verify that $\mathfrak{x}$ induces the map with the highest shift in the absolute grading among all other Spin^c with the same restrictions on Y, resp. Y_p.

Let Ω be the core of the two-handle which gives W_p. For $\mathfrak{t}_n := \mathfrak{x} + n \cdot PD[\Omega] \in S(\xi)$ we have:

$$c_1(\mathfrak{t}_n) = c_1(\mathfrak{x}) + 2n \cdot [\Omega]$$

and therefore,

$$c_1(\mathfrak{t}_n)^2 - c_1(\mathfrak{x})^2 = 4n \langle c_1(\mathfrak{x}), PD[\Omega] \rangle + 4n^2 \left(PD[\Omega] \cdot PD[\Omega] \right) \qquad (3.3)$$

We multiply this equation by m and use the fact that $[m\Omega]$ is the image of $[S]$ under the natural map $H_2(W) \longrightarrow H_2(W, \partial W)$. Then

$$m \left(c_1(\mathfrak{t}_n)^2 - c_1(\mathfrak{x})^2 \right) = 4n \left(2g - 2 - pm \right) - 4n^2 pm$$

One sees that indeed all other Spin^c structures shift the absolute grading with a smaller amount $(\frac{2g-2}{m})$ than $\mathfrak{x}$. Then the map F_3 is a sum of maps on different Spin^c structures, the highest of them (in terms of c_1^2 being the sum of the induced maps in the $\mathfrak{x}'s$ Spin^c structures).

We use now another filtration in Heegaard-Floer homology, namely the one given by the absolute grading, to conclude that F_3 has essentially the same behaviour as its top grading component, i.e. it is also surjective and the kernel of F_3 is identified with the kernel of (the sum of) $\mathfrak{x}$.

Therefore from the previous long exact sequence, one can deduce $\widehat{HFK}(Y, K, \Lambda_Y(\xi_M)) \cong HF^+(Y_\lambda(K), \overline{\xi}_M - \mu)$. Note that $\langle c_1(\overline{\xi}_M - \mu, S^2)\rangle = 2g-2 \neq 0$. Since $HF^+(S^1 \times S^2, \mathfrak{s}) = 0$ for all Spin^c structures with non-zero Chern class, we must have $g(K) \leq 1$. □

3.5 The genus 1 case

The proof above breaks down for knots K of genus 1. The Chern class formula does not give the required filtration, also $HF^+(S^1 \times S^2, \mathfrak{s}_0) \ncong 0$, where $\mathfrak{s}_0$ is the unique Spin^c structure of $S^1 \times S^2$ with Chern class 0. In fact, this group is not even finitely generated, so we cannot hope for an isomorphism with a subgroup of $\widehat{HFK}(Y, K)$.

However, as we will see below, a similar statement is true, provided that we use Heegaard-Floer homology groups with twisted coefficients.

3.5.1 Heegaard-Floer homology with twisted coefficients

Fix a manifold Y given by an admissible pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. Heegaard-Floer homology is already a version of the more general Lagrangian-Floer homology with twisted coefficients. In the latter theory, for two Lagrangian manifolds L_0, resp. L_1 of a symplectic manifold M, the universal coefficient system for $HF^*(M, L_0, L_1)$ is $\pi_1(\Omega(L_0, L_1))$. In the Heegaard-Floer setting, $\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)) \cong \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$, when $g(\Sigma) > 1$ and $\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)) \hookrightarrow \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$ if $g(\Sigma) = 1$ (Ozsváth and Szabó, 2004c, Proposition 2.15). The basepoint z in the Heegaard diagram, together with the morphism $n_z \colon \pi_2(\mathbf{x}, \mathbf{y}) \longrightarrow \mathbb{Z}$ account for the $\mathbb{Z}$ summand in the coefficient system above. For manifolds Y with $b_1 > 0$ there is a variant of Heegaard-Floer homology which corresponds to the universally twisted Lagrangian-Floer homology (Ozsváth and Szabó, 2004b, Remark 8.1). This variant, denoted $\underline{HF}(Y)$, was introduced by Ozsváth-Szabó in (Ozsváth and Szabó, 2004b) and was used by Ni (Ni, 2007) for proving that Heegaard Floer homology detects fibred manifolds with genus 1 fibres.

We give a brief description of the theory by following closely Section 8 of (Ozsváth and Szabó, 2004b). The coefficient system is constructed with the help of an *additive assignment* $A\colon \pi_2(\mathbf{x},\mathbf{y}) \longrightarrow H^1(Y,\mathbb{Z})$ and a *complete set of paths* for the Spin^c structures of Y.

Definition 3.5.1. *(Ozsváth and Szabó, 2004c, Definition 2.12) An additive assignment is a collection of functions* $\{A_{\mathbf{x},\mathbf{y}}\colon \pi_2(\mathbf{x},\mathbf{y}) \longrightarrow \mathbb{Z}\}$ *with the property that:*

$$A_{\mathbf{x},\mathbf{y}}(\phi) + A_{\mathbf{y},\mathbf{z}}(\psi) = A_{\mathbf{x},\mathbf{z}}(\phi * \psi)$$

for any $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ *and* $\psi \in \pi_2(\mathbf{y},\mathbf{z})$.

Definition 3.5.2. *(Ozsváth and Szabó, 2004c, Definition 3.12) Let* $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, *where* Y *is presented as a pointed Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. *A complete set of paths for* $\mathfrak{s}$ *is an enumeration* $\mathcal{S} = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$ *of all the intersection points between* $\mathbb{T}_\alpha$ *and* $\mathbb{T}_\beta$ *representing* $\mathfrak{s}$ *together with a collection of homotopy classes* $\theta_i \in \pi_2(\mathbf{x}_0, \mathbf{x}_i)$, *for* $i = 1, \ldots, m$, *with* $n_z(\theta_i) = 0$.

A complete set of paths gives rise to identifications

$$\pi_2(\mathbf{x}_i, \mathbf{x}_j) \cong \pi_2(\mathbf{x}_0, x_0)$$

by the following convention

$$\theta_i * \pi_2(\mathbf{x}_i, \mathbf{x}_j) \cong \pi_2(\mathbf{x}_0, \mathbf{x}_0) * \theta_j$$

Since $\pi_2(\mathbf{x}_0, \mathbf{x}_0) \cong \mathbb{Z} \oplus H^1(Y;\mathbb{Z})$, with the automorphism given by the n_z function, when $g(\Sigma) > 1$, one can define a map $A\colon \pi_2(\mathbf{x}_i, \mathbf{x}_j) \longrightarrow H^1(Y;\mathbb{Z})$ by $\pi_2(\mathbf{x}_i, \mathbf{x}_j) \cong \pi_2(\mathbf{x}_0, \mathbf{x}_0) \longrightarrow H^1(Y;\mathbb{Z})$ where the second map is the canonical projection onto the second factor. The associativity of $*$ implies that A is an additive assignment.

Pick a formal parameter e and let

$$\underline{CF^\infty}(Y,\mathfrak{s}) = CF^\infty(Y,\mathfrak{s}) \otimes_{\mathbb{Z}} \mathbb{Z}[H^1(Y,\mathbb{Z})]$$

The differential is given by:

$$\underline{\partial}^{\infty}[\mathbf{x}, i] = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \left(\sum_{\phi\in\pi_2(\mathbf{x},\mathbf{y})} \#\mathcal{M}(\phi) e^{A(\phi)} [\mathbf{y}, i - n_z(\phi)] \right)$$

Then the Heegaard Floer homology with twisted coefficients $\underline{HF^{\infty}}(Y, \mathfrak{s}) = H_*(\underline{CF^{\infty}})$ is an invariant of the manifold Y and of the Spin^c structure $\mathfrak{s}$ (Ozsváth and Szabó, 2004b, Subsection 8.2.3).

As in the untwisted case, there are several versions of the theory: $\underline{HF^+}$, $\underline{HF^-}$, etc.

3.5.2 A particular coefficient system

For our purposes we will choose as coefficient system a Novikov ring V, in fact a field, as in (Ai and Ni, 2008).

$$V = \left\{ \sum_{r\in\mathbb{R}} a_r e^r : \ \#\{a_r \mid a_r \neq 0, r \leq c\} < \infty, \forall c \in \mathbb{R} \right\}$$

Given a cohomology class $[\omega] \in H^2(Y; \mathbb{R})$ and a representative cocycle ω, one can define the additive assignment

$$A(\phi) = \int_\phi \omega.$$

Also, V can can be endowed with the structure of a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$-module by the ring homomorphism

$$\mathbb{Z}[H^1(Y;\mathbb{Z})] \ni \sum a_h \cdot h \longrightarrow \sum a_h \cdot e^{\langle h\cup\omega, [Y]\rangle} \in V$$

V with this module structure is denoted by V_ω.

Then $\underline{HF^{\infty}}(Y; V_\omega)$ is the homology of the chain complex $\underline{CF^{\infty}}(Y; V_\omega) = CF^{\infty}(Y) \otimes V_\omega$ with the differential

$$\underline{\partial}^{\infty}[\mathbf{x}, i] = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \left(\sum_{\phi\in\pi_2(\mathbf{x},\mathbf{y}),\mu(\phi)=1} \#\mathcal{M}(\phi) e^{\int_\phi \omega} [\mathbf{y}, i - n_z(\phi)] \right)$$

and similarly for $\underline{HF^+}(Y; V_\omega)$, etc.

Remark 3.5.3. *Since V is a field, the modules above are vector spaces.*

There are maps induced by cobordisms for the twisted version as well. They are somewhat simpler to define for our particular coefficient system V_ω than in general, see (Ozsváth and Szabó, 2004b) for the full generality and (Ozsváth and Szabó, 2003b) for $V\omega$.

As before, to a four-dimensional 2-handle attachment W (i.e. Morse surgery on some knot $K \subset Y$), one can associate a Heegaard triple diagram $(\Sigma, \alpha, \beta, \gamma, z)$, where (Σ, α, β) defines Y, (Σ, β, γ) describes a connected sum of $S^1 \times S^2$ and (Σ, α, γ) describes the surgered manifold $Y_K(\lambda)$ for some integral slope λ. Consider a cocycle ω representing $[\omega] \in H^2(W; \mathbb{R})$.

Then the map induced by W in twisted Floer homology

$$\underline{f}^\infty(\cdot, \mathfrak{s})\colon \underline{CF}^\infty(Y_{\alpha,\beta}, \mathfrak{s}_{\alpha,\beta}) \longrightarrow \underline{CF}^\infty(Y_{\alpha,\gamma}, \mathfrak{s}_{\alpha,\gamma})$$

is given by

$$\underline{f}^\infty([\mathbf{x}, i]; \mathfrak{s}) = \sum_{\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\{\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{w})\colon \mathfrak{s}_z(\psi)=0, \mu(\psi)=0\}} (\#\mathcal{M}(\psi)) \cdot e^{\int_\psi \omega} \cdot [\mathbf{w}, i - n_z(\psi)]$$

It is proved in (Ozsváth and Szabó, 2006b) that $\underline{f}^\infty$ induces a well-defined map

$$\underline{F}^+\colon (\cdot, \mathfrak{s})\colon \underline{HF}^+(Y_{\alpha,\beta}, \mathfrak{s}_{\alpha,\beta}, V_\omega) \longrightarrow \underline{HF}^+(Y_{\alpha,\gamma}, \mathfrak{s}_{\alpha,\gamma}; V_\omega)$$

which is invariant under perturbations of the complex structure on $\mathrm{Sym}^g(\Sigma)$ and isotopies of the attaching curves.

3.5.3 The long exact sequence for twisted coefficients

Associativity follows the same way, and we can adapt the long exact sequence of Theorem 3.3.3:

Theorem 3.5.4. *With the setup from section 3.3, let $[\omega] = PD[\mu \times I] \in H^2(X, \mathbb{R})$ be the Poincaré dual of the cylinder over the meridian of K and $\mathfrak{t} \in \mathrm{Spin}^c(Y_p)$, there is a*

$U-$equivariant long exact sequence

$$\cdots \xrightarrow{\underline{F_1}} \underline{HF^+}(Y_\lambda, Q^{-1}(\Lambda_{Y_p}(\mathfrak{t})); V_\omega) \xrightarrow{\underline{F_2}} \underline{HF^+}(Y_p, \Lambda_{Y_p}(\mathfrak{t}); V_\omega) \xrightarrow{\underline{F_3}} \underline{HF^+}(Y, \Lambda_Y(\mathfrak{b}); V_\omega) \longrightarrow$$

where $\mathfrak{b}$ *is a* Spin^c *structure on* Y *cobordant to* $\mathfrak{t}$ *in* W_p.

Proof. The proof is similar to the untwisted case, for associativity the count of holomorphic triangles reduces to the untwisted case as in (Ai and Peters, 2006, Theorem 3.1) and the rest follows similarly. □

3.5.4 The top grading in Knot Floer homology and twisted coefficients

In the case of rational homology three-spheres, the Heegaard Floer homology with twisted coefficients is essentially the same as the untwisted version. More precisely, $\underline{HF^+}(Y, V_\omega) \cong HF^+(Y, \mathbb{Z}) \otimes V_\omega$. Then the previous long exact sequence reads

$$\cdots \xrightarrow{\underline{F_1}} \underline{HF^+}(Y_\lambda, Q^{-1}(\Lambda_{Y_p}(\mathfrak{t})); V_\omega) \xrightarrow{\underline{F_2}} \underline{HF^+}(Y_p, \Lambda_{Y_p}(\mathfrak{t})) \otimes V_\omega \xrightarrow{\underline{F_3}} \underline{HF^+}(Y, \Lambda_Y(\mathfrak{b})) \otimes V_\omega \longrightarrow$$

and the map $\underline{F_3}$ is related to the untwisted F_3 (in a fixed Spin^c structure) by

$$\underline{(F_3, \mathfrak{s})} = \pm e^c(F_3, \mathfrak{s}) \cdot e^{\langle c_1(\mathfrak{s}) \cup PD[\mu \times I], [W] \rangle}$$

As before, the absolute grading shows that the map $\underline{F_3}$ has the same behaviour (surjectivity and the same kernel) as its top grading, but this time, because the genus of K is 1, equation 3.3 shows that there are two (for a fixed Spin^c structure on Y and $2m$ in a $Span(\lambda)$ orbit) Spin^c structures with largest shift in absolute grading, namely $\mathfrak{x}$ and $\mathfrak{y} = \mathfrak{x} - PD[\Omega]$.

Note that we can write

$$\underline{F_3} = (\underline{F_3}, \mathfrak{x}) + e^{-m}(\underline{F_3}, \mathfrak{x}) + \text{ lower order}$$

This situation was studied in (Ai and Ni, 2008), Lemmas 5.1 and 5.2, it is proved that $\ker(\underline{F_3}) \cong \ker(F_3) \otimes V_\omega$ and both these maps are surjective. But it is also proved in (Ai and Peters, 2006) that $\underline{HF^+}(S^1 \times S^2, 0) \cong 0$ which then implies that $\widehat{HFK}(Y, K, top) \cong 0$ so K cannot have genus 1.

3.6 More information from Floer homology

From the long exact sequence above it follows that the map F_3 is an isomorphism (with V coefficients, hence also with $\mathbb{Q}$ coefficients). This implies that Y_p is almost an L-space, i.e. it has the smallest rank possible in Floer homology, though there could be torsion. This situation was studied in (Ozsváth and Szabó, 2005), where it is proved that K must have $\widehat{HFK}(Y, K, \xi) \cong \mathbb{Q}$ or 0, for any relative Spin^c structure ξ.

Theorem 3.6.1. *A knot K in a lens space Y with a longitudinal $S^1 \times S^2$ surgery is Floer simple (with rational coefficients).*

Proof. Since $\chi(\widehat{HFK}(Y, K, \mathfrak{s})) = 1$ for $\mathfrak{s}$ an *absolute* Spin^c structure on Y, we see that there must be an *odd* number of relative Spin^c structures ξ which restrict to $\mathfrak{s}$ with $\widehat{HFK}(Y, K, \xi) \neq 0$. But for two such relative Spin^c structures $\xi_1 \neq \xi_2$,

$$|\langle c_1(\xi_1) - c_1(\xi_2), [F]\rangle| \geq 2m$$

and if there are at least 3 such Spin^c structures, then

$$\max_{\xi_1, \xi_2 \text{ extend } \mathfrak{s}} |\langle c_1(\xi_1) - c_1(\xi_2), [F]\rangle| \geq 4m$$

contradiction with 3.1 for knots of genus 0. (just apply equation 3.1 to ξ and $\overline{\xi}$) □

3.7 Fibredness

One of our main results is the following

Theorem 3.7.1. *Let $K \subset L$ be a knot in a lens space which admits a longitudinal $S^1 \times S^2$ surgery. Then K is fibred.*

Proof. Recent work of Ni and Wu (Ni and Wu, 2012) shows, using the absolute grading in Knot Floer homology, that in an arbitrary lens space, Floer simple knots have monic Floer homology if and only if the simple knots in the same homology have monic Floer homology, monic meaning that $\widehat{HFK}(Y, K, top) \cong \mathbb{Q}$. In the next Chapter we will see that indeed simple knots of the relevant order in lens spaces with $S^1 \times S^2$ surgeries are fibred. $\square$

CHAPTER IV

SIMPLE KNOTS IN LISCA'S FAMILIES OF LENS SPACES

In this chapter we investigate the simple knots K in lens spaces $Y = L(m^2, q)$ belonging to Lisca's families (see below) with the property that $[K]$ is an element of order m in $H_1(Y;\mathbb{Z})$. This condition is necessary for K to admit a longitudinal $S^1 \times S^2$ surgery, cf. Lemma 3.2.2.

Remark 4.0.2. *A computer experimentation showed that simple knots of order m with $m < 500$ in lens spaces $L(m^2, q)$, with q arbitrary (of course satisfying $\gcd(m^2, q) = 1$) are fibred.*

Based on this, we formulate the following

Question 4.0.3. *Is any simple knot of order m in a lens space of order m^2 fibred?*

Below we show that the answer to this question is 'yes' for Lisca's families of lens spaces. As the reader will see, the extra conditions on q, described in Definition 4.0.4 below, play an essential role in the proof.

We describe here Lisca's family of lens spaces and the main theorem of his paper (Lisca, 2007):

Definition 4.0.4. *(Lisca, 2007, Definition 1.1) Let $\mathbb{Q}_{>0} := \{x \in \mathbb{Q} : x > 0\}$, and define the maps $f, g : \mathbb{Q}_{>0} \longrightarrow \mathbb{Q}_{>0}$ by setting, for $\frac{p}{q} \in \mathbb{Q}_{>0}$, with $p > q > 0$ and $(p, q) = 1$,*

$$f\left(\frac{p}{q}\right) = \frac{p}{p-q}, \quad g\left(\frac{p}{q}\right) = \frac{p}{q'},$$

where $p > q' > 0$ and $qq' \equiv 1 \pmod p$. Define $\mathcal{R} \subset \mathbb{Q}_{>0}$ to be the smallest subset of $\mathbb{Q}_{>0}$ such that $f(\mathcal{R}) \subseteq \mathcal{R}$, $g(\mathcal{R}) \subseteq \mathcal{R}$ and $\mathcal{R}$ contains the set of rational numbers $\frac{p}{q}$ such that $p > q > 0$, $(p,q) = 1$, $p = m^2$ for some $m \in \mathbb{N}$ and q is one of the following types:

1. *$md \pm 1$ with $m > d > 0$ and $(m,d) = 1$;*
2. *$md \pm 1$ with $m > d > 0$ and $(m,d) = 2$;*
3. *$d(m \pm 1)$, where $d > 1$ divides $2m \mp 1$;*
4. *$d(m \pm 1)$, where $d > 1$ is odd and divides $m \pm 1$.*

Remark 4.0.5. *It is easy to see that a lens space Y which admits a longitudinal $S^1 \times S^2$ surgery along some knot $K \subset Y$ bounds a smooth rational homology four-ball.*

Remark 4.0.6. *Family (2) does not appear explicitly in (Lisca, 2007, Definition 1.1), we learned about it from Ken Baker (Baker, 2012).*

Theorem 4.0.7. *(Lisca, 2007, Theorem 1.2) Let $p > q > 0$ be coprime integers. Then, the following statements are equivalent:*

1. *The lens space $L(p,q)$ smoothly bounds a rational homology ball.*
2. *There exist:*
   - *(a) A surface with boundary Σ, homeomorphic to a disk if p is odd and to the disjoint union of a disk and a Möbius band if p is even;*
   - *(b) A ribbon immersion $i \colon \Sigma \looparrowright S^3$ with $i(\partial\Sigma) = K(p,q)$.*
3. *$\frac{p}{q}$ belongs to $\mathcal{R}$.*

Remark 4.0.8. *Condition 2 refers to a naturally associated two-bridge link $K(p,q)$ to a lens space with the same parameters, see (Rolfsen, 1990) for details. We will have no use for $K(p,q)$, it was included only for completeness.*

The rest of this section will be devoted to the proof of the following

Theorem 4.0.9. *Simple knots of order m in a lens space $L(m^2, q)$ belonging to any of the Lisca's families are fibred.*

The general strategy is to use Brown's algorithm combined with Stallings' fibration theorem, since a simple knot's complement in a lens space admits a genus 2 Heegaard splitting, or, equivalently, in terms of Heegaard-Floer homology, it can be described by a doubly pointed Heegaard diagram of genus 1, for which it is trivial to compute its Knot Floer homology.

Remark 4.0.10. *This method has been used by Ozsváth-Szabó in (Ozsváth and Szabó, 2005, Section 5) to prove that Berge knots are fibred. They show more generally that any* primitive *simple knot in a lens space is fibred, so this result does not apply to our situation. The presentation for the fundamental group is nevertheless the same, we include it here for the reader's convenience. We also use different notation and conventions.*

Remark 4.0.11. *It can be easily seen that using either Brown's theorem or Knot Floer homology, the calculations turn out to be identical. More precisely, the sequence of numbers in Brown's theorem, see below, is identical to the sequence of evaluations of Chern classes of relative* Spin^c *structures of $Y \setminus K$ in which $\widehat{HFK}(Y, K)$ is supported, against the Seifert surface of K. Then both theories guarantee the existence of a fibration of $Y \setminus K$ as soon as this common sequence of numbers assumes its maximum and minimum exactly once. Also, both theories exhibit a formula for the genus of K in terms of the width (i.e. the difference between the maximum and the minimum) of this sequence. See the proof of Theorem 4.8.5 for more details.*

We will give a proof for each family in Definition 4.0.4, but first we will fix some notation and state some facts which are independent of the special form of q.

We will denote classes modulo m^2 by $\overline{n}$ and classes modulo m by $\widehat{n}$.

It is easy to observe that $K \subset L(m^2, q)$ has order m if and only if $[K] = \overline{k \cdot m}$, for some integer $k \in \{1, \dots, m-1\}$ with $\gcd(k, m) = 1$. Given k as above, there is essentially

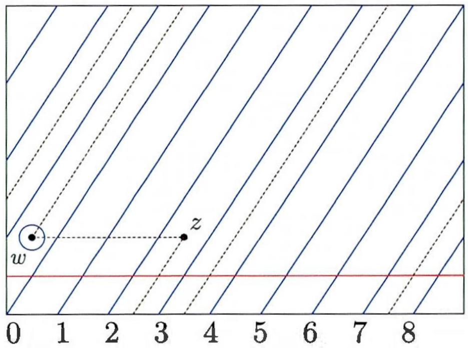

Figure 4.1 A Heegaard diagram for the simple knot $K(9,4,3) \subset L(9,4)$

one doubly pointed Heegaard diagram $(T^2, \alpha, \beta, w, z)$ specifying K, where (T^2, α, β) represents the standard Heegaard genus 1 diagram of $L(m^2, q)$ (with the α and β curves being geodesics for a Euclidean metric on T^2) and the basepoints are situated slightly above the α curve (see figure 4). The generator $\overline{1} \in \mathbb{Z}/_{m^2}$ is taken to be the homology class of the core of the α handlebody, oriented 'upwards'. Let α be oriented from left to right and let γ be an arc parallel to α connecting w and z and oriented such that $\partial\gamma = z - w$.

Note that there are two essentially different ways of choosing γ, one which is coherently oriented with α and one which is oppositely oriented, and their union forms a circle parallel to α. We will denote the shortest one (measured by the number of intersection points with β) by γ and the other one by γ'.

The relative position of w and z in the Heegaard diagram is determined by k. More precisely, $\#(\gamma \cap \beta) \equiv k \cdot q \pmod p$ and given the properties of k and the discussion about γ, we obtain $\#(\gamma \cap \beta) = t \cdot m$, with $t \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor\}$ and $\gcd(t, m) = 1$.

Remark 4.0.12. *Given a simple knot K in a lens space $L(p,q)$, described by a doubly* *pointed genus 1 Heegaard diagram as above, the quantity* $\theta = \#(\gamma \cap \beta)$ *determines* K *up to isotopy, so we can denote* K *by* $K(p, q, \theta)$.

Denote by U_α the solid torus bounded by the α curve (hence having a meridinal disk D_α with boundary α) and let γ_1 be a properly embedded arc in U_α connecting w and z which doesn't intersect D_α. Such a γ_1 can be obtained by pushing the interior of γ into U_α. The exterior of γ_1 in U_α is then a genus 2 handlebody, call it U'_α. The exterior of K in $L(m^2, q)$ is then homeomorphic to the handlebody U'_α with a 2-handle attached along the β curve.

Therefore $\pi_1(Y \setminus K)$ admits a presentation with two generators and one relation. The fundamental group of U'_α is the free group on two generators which we can choose to be two simple closed curves, x and y, supported on $\partial U'_\alpha$ in such a way that $\#(x \cap \gamma) = 0$, $\#(x \cap \gamma') = 1$, resp. $\#(y \cap \gamma = 1), \#(y \cap \gamma' = 0)$. Moreover, we will suppose that both x and y are isotopic in U_α to the core of U_α and oriented coherently (i.e. upwards, which by the above choices means that $[x] = [y] = \overline{1} \in H_1(L(m^2, q))$.) The relator -call it R - is represented by the β curve, which we will also assume to be oriented upwards.

It is easy now to write the presentation for $\pi_1(Y \setminus K)$: simply follow the β curve and record the intersection points with the γ and γ' arcs - for each intersection with γ add a y to the relator, and for each intersection with γ' add an x. We obtain then: $\pi_1(Y \setminus K) \cong \langle x, y \mid R \rangle$.

It is immediate to see that the resulting word has m^2 letters, it is cyclically reduced and that changing the position on the β curve from which we start indexing the intersection points with the γ arcs has the effect of replacing R with a cyclic permutation of its letters.

Since w and z were chosen slightly above α, for each intersection point between β and γ, (resp. γ') there is a nearby intersection point of β with α - simply follow β downwards until it meets α. Then R contains tm letters y and $(m-t)m$ letters x.

To check fibredness, we must write the morphism $\phi\colon \pi_1(Y \setminus K) \longrightarrow \mathbb{Z}$, represented geometrically by the algebraic intersection of loops in $Y \setminus K$ with the Seifert surface F of K. Since ϕ factors through $H_1(Y \setminus K) \cong \mathbb{Z} \oplus \mathbb{Z}/m$ (cf. 3.2.2) and ϕ is surjective (F is nonseparating) ϕ must be then (up to sign) the map

$$\pi_1(Y \setminus K) \xrightarrow{Ab} H_1(Y \setminus K) \longrightarrow H_1(Y \setminus K)/_{Tors} \cong \mathbb{Z},$$

where the second map is the obvious one.

In the presentation that we found, ϕ can be written as:

$$\begin{cases} \phi(x) = -t \\ \phi(y) = m - t \end{cases}$$

It is convenient at this point to number the intersection points of α and β with numbers (in fact classes modulo p) starting at the right of the w point and continuing towards the right along α, (again, see figure 4). This way we identify the intersection points with $\mathbb{Z}/p$. The intersections of β with γ correspond to $\{\overline{0}, \ldots, \overline{tm-1}\}$ and the intersections of β with γ' correspond to $\{\overline{tm}, \ldots, \overline{p-1}\}$.

For $i \in \mathbb{Z}$, the quantity $\widehat{\left\lfloor \frac{i}{m} \right\rfloor}$ (recall that $\widehat{n}$ denotes the class of n modulo m) depends only on the class of i modulo m^2, so we define the function $\mathbb{Z}/_{m^2} \longrightarrow \mathbb{Z}/m$ by $\overline{i} \longrightarrow \widehat{\left\lfloor \frac{i}{m} \right\rfloor}$, for some $i \in \overline{i}$. From now on, we will denote this class modulo m by $\left[\frac{i}{m}\right]$.

Then the intersections of β with γ (resp. γ') correspond to the classes $\overline{i} \in \mathbb{Z}/p$ with $\left[\frac{i}{m}\right] \in \left\{\widehat{0}, \ldots, \widehat{t-1}\right\}$ $\left(\text{resp. } \left[\frac{i}{m}\right] \in \left\{\widehat{t}, \ldots, \widehat{m-1}\right\}\right)$.

The following function will be useful: Let $f\colon \mathbb{Z}/m \longrightarrow \{x, y\}$ be given by:

$$f(\widehat{a}) = \begin{cases} y & \text{if } 0 \leq a < t \\ x & \text{if } t \leq a < m \end{cases}$$

for some $a \in \widehat{a}$.

Our relator R becomes

$$R = f(\widehat{0})\, f\left(\left[\frac{q}{m}\right]\right) \cdots f\left(\left[\frac{(p-1)q}{m}\right]\right) \tag{4.1}$$

Let ψ be the map $(\phi \circ f)\colon \mathbb{Z}/m \longrightarrow \mathbb{Z}$. Observe that $\psi(\widehat{0}) + \cdots + \psi(\widehat{m-1}) = 0$.

With this notation, Brown's theorem, 1.2.14, coupled with Stallings fibration criterion (Stallings, 1962) say that K is fibred if and only if the sequence:

$$\psi(\widehat{0}),\ \psi(\widehat{0}) + \psi\left(\left[\frac{q}{m}\right]\right),\ \cdots,\ \psi(\widehat{0}) + \cdots + \psi\left(\left[\frac{(p-1)\cdot q}{m}\right]\right)$$

achieves its maximum and minimum exactly once.

Call the sums above S_i, i.e. $S_i = \sum_{j=0}^{i} \psi\left(\left[\frac{j \cdot q}{m}\right]\right)$, for $i \in \{0, \cdots, p-1\}$.

Remark 4.0.13. *We will also use classes modulo p to index the sums above, with the obvious interpretation. This is unambiguous because $S_{p-1} = 0$ ($S_{p-1} = \phi(R) = 0$ by definition).*

Now we will prove a series of lemmas concerning R and the sequence $(S_i)_{i=0}^{p-1}$.

Lemma 4.0.14. *For all $\overline{i}, \overline{j} \in \mathbb{Z}/p$, we have that $S_{\overline{i}} \equiv S_{\overline{j}} \pmod{m} \iff i \equiv j \pmod{m}$.*

Proof. Note that $\phi(x) \equiv \phi(y) \equiv -t \pmod{m}$, so $S_i \equiv i \cdot (-t) \pmod{m}$ and since $\gcd(t, m) = 1$, the conclusion follows. $\square$

Lemma 4.0.15. *The sequence $(S_i)_{i=0}^{p-1}$ achieves its maximum only once if and only if it achieves its minimum only once.*

Proof. We claim that for any $\overline{i} \in \mathbb{Z}/p$,

$$\psi\left(\left[\frac{i}{m}\right]\right) = \psi\left(\left[\frac{tm-1-i}{m}\right]\right) \tag{4.2}$$

Since $\overline{i} \in \{\overline{0}, \ldots, \overline{tm-1}\} \iff \overline{tm-1-i} \in \{\overline{0}, \ldots, \overline{tm-1}\}$, $\forall \overline{i} \in \mathbb{Z}/p$, we have

$$f\left(\left[\frac{i}{m}\right]\right) = y \iff f\left(\left[\frac{tm-1-i}{m}\right]\right) = y$$

and the claim is proved.

Let $l \in \{\overline{1}, \dots, \overline{p-1}\}$ be the unique number with the property $l \cdot \overline{q} = \overline{tm-1}$. Then, for $\overline{i} \in \mathbb{Z}/p$:

$$S_i = \sum_{j=0}^{i} \psi\left(\left[\frac{jq}{m}\right]\right) = \sum_{j=0}^{i} \psi\left(\left[\frac{lq-jq}{m}\right]\right) = S_{\overline{l}} - S_{\overline{l-i-1}}$$

The conclusion of the lemma follows from the equalities:

$$\#\max\left((S_{\overline{i}})_{\overline{i}=\overline{0}}^{\overline{p-1}}\right) = \#\max\left((S_{\overline{l}} - S_{\overline{l-i-1}})_{\overline{i}=\overline{0}}^{\overline{p-1}}\right) = \#\min\left((S_{\overline{l-i-1}})_{\overline{i}=\overline{0}}^{\overline{i}=\overline{p-1}}\right),$$

where by $\#\max(X_i)$ (resp. $\#\min(X_i)$) we denote the number of maxima (resp. minima) of the sequence X_i.

□

4.1 Lisca's family (1)

Theorem 4.1.1. *The sequence $(S_i)_{i=0}^{p-1}$ for a lens space $L(p,q)$ belonging to Lisca's family* (1) *achieves its maximum only once.*

Proof. We know that $q = d \cdot m \pm 1$, with $0 < d < m$ and $\gcd(d,m) = 1$. After a possible change $q \longrightarrow (p-q)$, which has the effect of changing the orientation of $L(p,q)$, we can suppose $q = dm + 1$.

Then, for $j, i \in \{0, \dots, m-1\}$,

$$\left[\frac{(jm+i)q}{m}\right] = \widehat{j+id} \tag{4.3}$$

This implies that $\left\{\left[\frac{(jm+i)q}{m}\right] : i \in \{0, \dots, m-1\}\right\} = \left\{\widehat{0}, \dots, \widehat{m-1}\right\}$, and in particular

$$S_{jm+m-1} = 0. \tag{4.4}$$

By lemma 4.0.14, the numbers $S_{jm+0}, S_{jm+1}, \dots, S_{jm+m-1}$ are all distinct and exactly one of them is the maximum of this sequence, say S_{m_j}, for some $m_j \in \{0, \dots, m-1\}$.

Let $S^j_{\hat{i}} := S_{jm+i}$ for $j, i \in \{0, \ldots, m-1\}$ and let $\widehat{d'} := \widehat{d}^{-1}$.

By equations 4.3 and 4.4,

$$S^j_{\hat{i}} = \psi\left(\hat{j}\right) + \cdots + \psi\left(\widehat{j+id}\right) = \psi\left(\widehat{(d'j)d}\right) + \cdots + \psi\left(\widehat{(d'j+i)d}\right) = -S^0_{\widehat{d'j-1}} + S^0_{\widehat{d'j+i}} \tag{4.5}$$

Remark 4.1.2. *In the above formula we can index the sequences S^j by classes modulo m because $S_{jm+m-1} = 0$ cf. eq 4.4.*

We deduce that $\widehat{m_j} = \widehat{m_0} - \widehat{d'j}$.

By Lemma 4.0.14, the maxima $(S_{m_j})_{j=0}^{m-1}$ are all distinct, hence exactly one of them is the maximum of the sequence $(S_i)_{i=0}^{p-1}$. □

Let $W_i = \left[\frac{iq}{m}\right]$, for $i \in \{0, \ldots, p-1\}$ i.e. we can write the relator $R = f(W_0), \ldots, f(W_{p-1})$. It will also be convenient to denote the m subsequences of $(W_i)_{i=0}^{p-1}$ by $W^j_i = W_{jm+i}$, for $i, j \in \{0, \ldots, m-1\}$, we will also frequently use classes modulo m to index the subsequences $W^j_{\hat{i}}$, with the obvious interpretation: $W^j_{\hat{i}} = W^j_i$ where $i \in \hat{i}$ is the canonical representative $i \in \{0, \ldots, m-1\}$.

4.1.1 An example

Let $m = 5, d = 2$, i.e. $(p, q) = (25, 11)$ and fix also $t = 2$, then $\phi(x) = -2$ and $\phi(y) = 3$.

The intersections of the β curve with the α curve are:

$$\overline{0}, \overline{11}, \overline{22}, \overline{8}, \overline{19}, \overline{5}, \overline{16}, \overline{2}, \overline{13}, \overline{24}, \overline{10}, \overline{21}, \overline{7}, \overline{18}, \overline{4}, \overline{15}, \overline{1}, \overline{12}, \overline{23}, \overline{9}, \overline{20}, \overline{6}, \overline{17}, \overline{3}, \overline{14}$$

Then W becomes $\hat{0}, \hat{2}, \hat{4}, \hat{1}, \hat{3}, \hat{1}, \hat{3}, \hat{0}, \hat{2}, \hat{4}, \hat{2}, \hat{4}, \hat{1}, \hat{3}, \hat{0}, \hat{3}, \hat{0}, \hat{2}, \hat{4}, \hat{1}, \hat{4}, \hat{1}, \hat{3}, \hat{0}, \hat{2}$.

Below we see on the left the values of the word $(W^j_i)_i$ (hats omitted) and on the right the sequences $(S^j_i)_i$, arranged as matrices. Since $\widehat{d'} = \hat{3}$, the maxima in the sequences

$(S_i^j)_i$ occur in positions: $0, 2, 4, 1, 3$ respectively.

$$W_i^j = \begin{pmatrix} 0 & 2 & 4 & 1 & 3 \\ 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \end{pmatrix} \quad S_i^j = \begin{pmatrix} 3 & 1 & -1 & 2 & 0 \\ 3 & 1 & 4 & 2 & 0 \\ -2 & -4 & -1 & -3 & 0 \\ -2 & 1 & -1 & -3 & 0 \\ -2 & 1 & -1 & 2 & 0 \end{pmatrix}$$

4.1.2 The width of the knot's Heegaard-Floer homology

We compute here the width of the sequence $(S_i)_{i=0}^{p-1}$, since this quantity determines the genus of K. We will use this information to give a complete classification of *simple* knots in Lisca's family 1 with an $S^1 \times S^2$ surgery.

Definition 4.1.3. *The* width *of a finite sequence of (integer) numbers* (S_i) *is* $w(S_i) = \max S_i - \min S_i$.

Theorem 4.1.4. *Let* $K = K(m^2, dm + 1, tm)$ *(see Remark 4.0.12 for this notation) be a simple knot of order* m *in a lens space* $L(m^2, q)$ *in Lisca's family* (1)*. Then*

$$w(K) = 2 \cdot w(K(m, d, t)).$$

For the proof we state an easy result which will be useful for the other families as well. Recall the definition of W given before the example in section 4.1.1.

Proposition 4.1.5. *Let* $L(m^2, q)$ *be a lens space given by a Heegaard diagram as in Section 4. Then*

$$\widehat{W_{\hat{i}}^{j+1}} = \widehat{W_{\hat{i}}^{j}} + \hat{q}$$

Proof.

$$\widehat{W_{\hat{i}}^{j+1}} = \left\lceil \frac{((j+1)m + i)q}{m} \right\rceil = \left\lceil \frac{(jm + i)q + mq}{m} \right\rceil = \widehat{W_{\hat{i}}^{j}} + \hat{q}$$

□

Proof of Theorem 4.1.4. By Equation 4.3, $W^{0}_{\widehat{i}} = \widehat{i \cdot d}$, in particular

$$W^{\widehat{0}}_{\widehat{i+1}} - W^{\widehat{0}}_{\widehat{i}} = \widehat{d}.$$

By the previous proposition, the same is true for all the sequences $S^{\widehat{j}}$. This says that the sequences $S^{\widehat{j}}$ are all the cyclic permutations of the sequence $S^{\widehat{0}}$.

By Equation 4.5, $w(S^{\widehat{j}}) = w(S^{\widehat{0}})$, $\forall j \in \mathbb{Z}/m$ and also note that $w(S^{\widehat{0}}) = w(K(m,d,t))$.

Also by Equation 4.5, we see that

$$A := \left\{ S^{\widehat{j}}_{\widehat{i}} : \widehat{i}, \widehat{j} \in \mathbb{Z}/m \right\} = \left\{ S^{\widehat{0}}_{\widehat{r}} - S^{\widehat{0}}_{\widehat{l}} \, : \, \widehat{l}, \widehat{r} \in \mathbb{Z}_{/m} \right\} =: B$$

We have

$$\begin{cases} \max((S_i)_{i=0}^{p-1}) & = \max(A) & = \max(B) & = w(S^{\widehat{0}}) \\ \min((S_i)_{i=0}^{p-1}) & = \min(A) & = \min(B) & = -w(S^{\widehat{0}}) \end{cases}$$

hence $w((S_i)_{i=0}^{p-1}) = 2w(S^{\widehat{0}}) = 2w(K(m,d,t))$. □

4.2 Lisca's family (2)

In this case, $q = dm + 1$ with $\gcd(d, m) = 2$. As for family (1), there is no essential difference between the cases $q = dm + 1$ and $q = dm - 1$ (one can interchange d with $m - d$).

Theorem 4.2.1. *The sequence $(S_i)_{i=0}^{p-1}$ for a lens space $L(p,q)$ belonging to Lisca's family (2) achieves its maximum only once.*

In this case, the m subsequences $W^{\widehat{j}}$ of W do not contain all the classes modulo m, thus $S_{im+m-1} \neq 0$ generically. However, we can cyclically permute W so that this desired property becomes true.

Lemma 4.2.2. *Let $W'_{\overline{i}} = W_{\overline{i}+\frac{m}{2}}$, $\forall \overline{i} \in \mathbb{Z}_{/p}$. Then $W'^{\widehat{j}}$ contains all the classes modulo m.*

Proof. Since m is even, it makes sense to speak about even (resp. odd) classes modulo m.

$W^{\widehat{0}}$ has the following form:

$$W_i^{\widehat{0}} = \widehat{d \cdot i},$$

for $i \in \{0, \dots, m-1\}$.

Since $\gcd(d, m) = 2$, $W_{\widehat{i}}^{\widehat{0}} = W_{\widehat{i'}}^{\widehat{0}} \iff \widehat{i} = \widehat{i'}$ or $\widehat{i} - \widehat{i'} = \widehat{\frac{m}{2}}$. This means that $W^{\widehat{0}}$ contains all the even classes modulo m, and each such class appears exactly twice in $W^{\widehat{0}}$.

By Proposition 4.1.5, $W^{\widehat{j}}$ contains all even (resp. odd) classes modulo m (and only those) exactly when $\widehat{j}$ is even, (resp. odd).

Again by Proposition 4.1.5, we see that $W'^{\widehat{j}}$ contains all classes modulo m, and moreover

$$W'^{\widehat{j}}_{i+\frac{m}{2}} = W'^{\widehat{j}}_i + \widehat{1},$$

for $i \in \{0, \dots, \frac{m}{2} - 1\}$. See 4.2.1 below for a concrete example. $\square$

4.2.1 Example

Let $m = 6, d = 4$, then $p = 36, q = 25$ and the intersections of the α and β are $\overline{0}, \overline{25}, \overline{14}, \overline{3}, \overline{28}, \overline{17}, \dots$, hence $W = \widehat{0}, \widehat{4}, \widehat{2}, \widehat{0}, \widehat{4}, \widehat{2}, \dots$

Then (hats omitted):

$$W_i^j = \begin{pmatrix} 0 & 4 & 2 & 0 & 4 & 2 \\ 1 & 5 & 3 & 1 & 5 & 3 \\ 2 & 0 & 4 & 2 & 0 & 4 \\ 3 & 1 & 5 & 3 & 1 & 5 \\ 4 & 2 & 0 & 4 & 2 & 0 \\ 5 & 3 & 1 & 5 & 3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 4 & 2 & 1 & 5 & 3 \\ 1 & 5 & 3 & 2 & 0 & 4 \\ 2 & 0 & 4 & 3 & 1 & 5 \\ 3 & 1 & 5 & 4 & 2 & 0 \\ 4 & 2 & 0 & 5 & 3 & 1 \\ 5 & 3 & 1 & 0 & 4 & 2 \end{pmatrix} = W'^j_i$$

Take $t = 1$ and then the sums $S_i'^j$ (see below) are the following

$$\begin{pmatrix} 5 & 4 & 3 & 2 & 1 & 0 \\ -1 & -2 & -3 & -4 & 1 & 0 \\ -1 & 4 & 3 & 2 & 1 & 0 \\ -1 & -2 & -3 & -4 & -5 & 0 \\ -1 & -2 & 3 & 2 & 1 & 0 \\ -1 & -2 & -3 & 2 & 1 & 0 \end{pmatrix}$$

Remark 4.2.3. *We will work from now on with W'. Consequently we will adapt the notation of objects relating to W by adding the symbol ' to the analogous object relating to W'. For example, the sequence to which we will apply Brown's algorithm will be*

$$S_i' := \sum_{j=0}^{i} \psi(W_j')$$

By the previous lemma, $S'_{sm+m-1} = 0$, for $s \in \{0, \dots, m-1\}$, and like in the family (1) case, we will analyse each sequence $S'^{\widehat{j}}$ separately, and then compare the 'local' maxima obtained.

Lemma 4.2.4. *For $\widehat{j} \in \mathbb{Z}/_m$,*

$$S'^{\widehat{j}}_{\frac{m}{2}-1} = \begin{cases} \frac{m}{2}, & \textit{if } \widehat{j} \textit{ is even} \\ -\frac{m}{2}, & \textit{if } \widehat{j} \textit{ is odd} \end{cases}$$

Proof. Note that t is odd, since $\gcd(t, m) = 1$. Therefore

$$\#\{a : \ 0 \leq a < t, a \text{ even }\} = \frac{t+1}{2}$$

respectively

$$\#\{a : \ 0 \leq a < t, a \text{ odd }\} = \frac{t-1}{2}$$

Let $\widehat{j} \in \mathbb{Z}_{/m}$ even. Then

$$S'^{\widehat{j}}_{\frac{m}{2}-1} = \left(\frac{t+1}{2}\right) \cdot (m-t) + \left(\frac{m}{2} - \frac{t+1}{2}\right) \cdot (-t) = \frac{m}{2}$$

and for $\widehat{j}$ odd,

$$S'^{\widehat{j}}_{\frac{m}{2}-1} = \left(\frac{t-1}{2}\right) \cdot (m-t) + \left(\frac{m}{2} - \frac{t-1}{2}\right) \cdot (-t) = -\frac{m}{2}.$$

$\square$

Lemma 4.2.5. *The maximum of some sequence $S'^{\widehat{j}}$, for $\widehat{j} \in \mathbb{Z}/_m$ can be the maximum of $(S'_i)_{i=0}^{p-1}$ only if $\widehat{j}$ is even.*

Proof.

Remark 4.2.6. *It's easy to see that proposition 4.1.5 applies to any cyclic permutation of W, in particular to W'.*

Then, using Proposition 4.1.5, $S'^{\widehat{j+2}} = S'^{\widehat{j}} + \widehat{2}$. Also, from the definition of W', we see that for $0 \leq i < \frac{m}{2} - 2$, resp. $\frac{m}{2} \leq i < m-2$,

$$W'^{\widehat{j}}_{i+1} - W'^{\widehat{j}}_{i} = \widehat{d}.$$

These two facts imply that

- if $\widehat{j}$ is even, the sequences $(W'^{\widehat{j}}_i)_{i=0}^{\frac{m}{2}-1}$, resp. $(W'^{\widehat{j}}_i)_{i=\frac{m}{2}}^{m-1}$ are cyclic permutations of $(W'^{\widehat{0}}_i)_{i=0}^{\frac{m}{2}-1}$, resp. $(W'^{\widehat{0}}_i)_{i=\frac{m}{2}}^{m-1}$
- if $\widehat{j}$ is odd, $(W'^{\widehat{j}}_i)_{i=0}^{\frac{m}{2}-1}$, resp. $(W'^{\widehat{j}}_i)_{i=\frac{m}{2}}^{m-1}$ are cyclic permutations of $(W'^{\widehat{1}}_i)_{i=0}^{\frac{m}{2}-1}$, resp. $(W'^{\widehat{1}}_i)_{i=\frac{m}{2}}^{m-1}$.

Also from the definition of W', for $\widehat{j} \in \mathbb{Z}/_m$ and $i \in \{0, \ldots, \frac{m}{2} - 1\}$, we have that

$$W'^{\widehat{j}}_{i+\frac{m}{2}} = W'^{\widehat{j+1}}_{i} \tag{4.6}$$

Suppose that $\max((S'_i)_{i=0}^{p-1}) = \max S'^{\widehat{j}}$, with $\widehat{j}$ odd. Then $\max S'^{\widehat{j}} = S'^{\widehat{j}}_i$ for some $i \in \{0, \ldots, m-1\}$.

We have two cases:

1. $i \in \{0, \ldots, \frac{m}{2} - 1\}$.

   By Equation 4.6,

$$S'^{\widehat{j}}_i = S'^{\widehat{j-1}}_{i+\frac{m}{2}} - S'^{\widehat{j-1}}_{\frac{m}{2}-1}$$

   but since $\widehat{j}$ is odd, $S'^{\widehat{j-1}}_{\frac{m}{2}-1} = \frac{m}{2}$ (by Lemma 4.2.4), we get the contradiction $S'^{\widehat{j-1}}_{i+\frac{m}{2}} > S'^{\widehat{j}}_i$.

2. $i \in \{\frac{m}{2}, \ldots, m-1\}$.

   Again by Equation 4.6,

$$S'^{\widehat{j}}_i = S'^{\widehat{j+1}}_{i-\frac{m}{2}} + S'^{\widehat{j}}_{\frac{m}{2}-1}$$

   and since $\widehat{j}$ is odd, $S'^{\widehat{j}}_{\frac{m}{2}-1} = -\frac{m}{2}$ (by Lemma 4.2.4) and we arrive at a contradiction with $S'^{\widehat{j}}_i$ being the maximum of the sequence $(S'_i)_{i=0}^{p-1}$.

□

Proof of Theorem 4.2.1. We need to consider only the partial sums $S'^{\widehat{j}}$ with $\widehat{j}$ even. By Lemma 4.0.14, for $i, i' \in \{0, \ldots, m-1\}$, $S'^{\widehat{j}}_i = S'^{\widehat{j}'}_{i'}$ implies $i = i'$. Suppose that $\max((S'_i)_{i=0}^{p-1}) = S'^{\widehat{j_0}}_{l-1}$ for some $l \in \{1, \ldots, m\}$ and $\widehat{j_0} \in 2\mathbb{Z}/_m$. Then other potential maxima can only occur as the numbers $S'^{\widehat{j}}_{l-1}$ for $j \in 2\mathbb{Z}/_m$.

As before, we have two different cases

1. $l \in \{1, \ldots, \frac{m}{2}\}$
2. $l \in \{\frac{m}{2} + 1, \ldots, m\}$.

We introduce some terminology first:

Definition 4.2.7. *Let A be a non-empty set and $n \in \mathbb{Z}$.*

1. *A* circular sequence *$M_{\widetilde{i}}$ indexed by $\mathbb{Z}/_n$ is a function $M \colon \mathbb{Z}/_n \longrightarrow A$.*

2. *for $l \in \{1, \dots, n\}$, a subsequence N_i of $M_{\widetilde{i}}$ of length l is a function $M \circ N\colon \{0, \dots, l-1\} \longrightarrow A$, where $N\colon \{0, \dots, l-1\} \longrightarrow \mathbb{Z}/_n$ is given by $N(i) = \widetilde{a} + \widetilde{i}$, for some $\widetilde{a} \in \mathbb{Z}/_n$. We say that N_i starts at $\widetilde{a}$.*

3. *Given a function $\psi\colon A \longrightarrow \mathbb{Z}$, the* sum of the subsequence N_i by ψ *is* $\Sigma(N_i) := \sum_{i=0}^{l-1} \psi(N_i)$.

Remark 4.2.8. *In our setting, we denote by $\widetilde{a}$ classes modulo $\frac{m}{2}$.*

Remark 4.2.9. *Since the functions $\psi, \widetilde{\psi}$ (see below) are fixed throughout the proof, we will not mention them in the text and simply say 'the sum of a subsequence' instead of 'the sum of a subsequence by $\widetilde{\psi}$.'*

We will suppose that $M := \max((S_i')_{i=0}^{p-1})$ is achieved twice and we'll arrive at a contradiction. Each case will be dealt with separately.

In case 1, let $\widetilde{W_{\widetilde{i}}}$ be the circular sequence indexed by $\mathbb{Z}/_{\frac{m}{2}}$, given by $\widetilde{W_{\widetilde{i}}} = W_i'^{\widehat{0}}$, for $i \in \{0, \dots, \frac{m}{2} - 1\}$ and the function $\widetilde{\psi}\colon \mathbb{Z}/_{\frac{m}{2}} \longrightarrow \mathbb{Z}$ defined by

$$\widetilde{\psi}(\widetilde{\alpha}) = \psi(W_\alpha'^{\widehat{0}}),$$

$\forall \alpha \in \{0, \dots, \frac{m}{2} - 1\}$. The sequences $(W_i'^{\widehat{j}})_{i=0}^{l}$ are, by the proof of the previous lemma, all the subsequences of length l of the circular sequence $\widetilde{W_{\widetilde{i}}}$, hence their sums by ψ, namely $S_{l-1}'^{\widehat{j}}$, are the sums by $\widetilde{\psi}$ of the subsequences of length l of $\widetilde{W_{\widetilde{i}}}$, by the definition of $\widetilde{\psi}$.

By our assumption, there are two subsequences of length l with sum equal to M. Call the two subsequences $(A_i)_{i=0}^{l-1}$ and $(B_i)_{i=0}^{l-1}$. Suppose that A starts at $\widetilde{a}$ and B starts at $\widetilde{b}$.

It is, of course, natural to think of the numbers $\widetilde{W_{\widetilde{i}}}$ as being arranged in a circle, with indices ordered clockwise, say, see figure 4.2.1

Modulo switching A with B, there are essentially two possible relative positions for the indices $\widetilde{a}, \widetilde{b}, \widetilde{a+l}, \widetilde{b+l}$ in $\mathbb{Z}/_{\frac{m}{2}}$:

(i) The indices appear in the order $\widetilde{a}, \widetilde{b}, \widetilde{a+l}, \widetilde{b+l}$

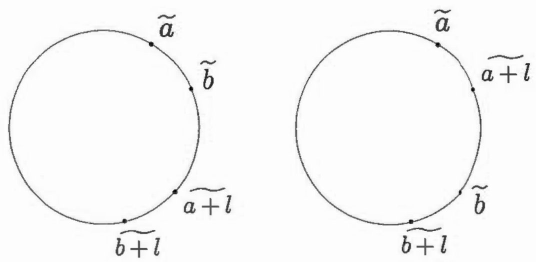

Figure 4.2 The two orderings on circular sequences

(ii) The indices appear in the order $\widetilde{a}, \widetilde{a+l}, \widetilde{b}, \widetilde{b+l}$

Given two classes $\widetilde{i}, \widetilde{j} \in \mathbb{Z}/\frac{m}{2}$, we will denote by $\widetilde{W_{\widetilde{i},\widetilde{j}}}$ the subsequence $\widetilde{W_{\widetilde{i}}}, \widetilde{W_{\widetilde{i+1}}}, \ldots, \widetilde{W_{\widetilde{j-1}}}$ and by $\Sigma_{\widetilde{i},\widetilde{j}}$ its sum.

With this notation, $\Sigma_{\widetilde{a},\widetilde{a+l}} = \Sigma_{\widetilde{b},\widetilde{b+l}} = M$ and

$$\Sigma_{\widetilde{\alpha},\widetilde{\beta}} + \Sigma_{\widetilde{\beta},\widetilde{\alpha}} = \Sigma(\widetilde{W_{\widetilde{i}}}) = \frac{m}{2} \tag{4.7}$$

$\forall \widetilde{\alpha}, \widetilde{\beta} \in \mathbb{Z}/\frac{m}{2}$, by Lemma 4.2.4.

In case (i),

- if $\Sigma_{\widetilde{a},\widetilde{b}} > 0$, then $\Sigma_{\widetilde{a},\widetilde{b+l}} = \Sigma_{\widetilde{a},\widetilde{b}} + \Sigma_{\widetilde{b},\widetilde{b+l}} > M$ - contradiction.
- if $\Sigma_{\widetilde{a},\widetilde{b}} < 0$, then $\Sigma_{\widetilde{b},\widetilde{a+l}} = \Sigma_{\widetilde{a},\widetilde{a+l}} - \Sigma_{\widetilde{a},\widetilde{b}} > M$ - contradiction.
- $\Sigma_{\widetilde{a},\widetilde{b}} = 0$ is impossible by Lemma 4.0.14

In case ii,

- if $\Sigma_{\widetilde{a},\widetilde{b}} > 0$, then $\Sigma_{\widetilde{a},\widetilde{b+l}} = \Sigma_{\widetilde{a},\widetilde{b}} + \Sigma_{\widetilde{b},\widetilde{b+l}} > M$ - contradiction.
- if $\Sigma_{\widetilde{b},\widetilde{a}} > 0$, then $\Sigma_{\widetilde{b},\widetilde{a+l}} = \Sigma_{\widetilde{b},\widetilde{a}} + \Sigma_{\widetilde{a},\widetilde{a+l}} > M$ - contradiction.

By Equation 4.7, at least one of the situations above arise, and we obtain a contradiction.

In case 2, again from the proof of Lemma 4.2.5, we have

$$S'^{\widehat{j}}_{l-1} = \frac{m}{2} + S'^{\widehat{j+1}}_{l-\frac{m}{2}-1}$$

As before, the sequences $(W'^{\widehat{j+1}}_i)_{i=0}^{l-\frac{m}{2}-1}$ are the subsequences of length $l - \frac{m}{2}$ of the circular sequence $\widetilde{W^1_{\widetilde{i}}} := W'^{\widehat{1}}_i$, for $i \in \{0, \ldots, \frac{m}{2} - 1\}$. We define the function $\widetilde{\psi^1} \colon \mathbb{Z}/_{\frac{m}{2}} \longrightarrow \mathbb{Z}$ by

$$\widetilde{\psi^1}(\widetilde{\alpha}) = \psi(W'^{\widehat{1}}_{\alpha})$$

and then the sums $S'^{\widehat{j+1}}_{l-\frac{m}{2}-1}$ become the sums by $\widetilde{\psi}^1$ of the subsequences of length $l - \frac{m}{2}$ of $\widetilde{W^1_{\widetilde{i}}}$.

Let $l' = l - \frac{m}{2}$. We suppose again that there are two subsequences A and B of length l' of $\widetilde{W^1_i}$ where A starts at $\widetilde{a}$ and B starts at $\widetilde{b}$.

We denote the sum of a subsequence of $\widetilde{W^1_i}$ by $\widetilde{\psi}^1$ with Σ^1, for example,

$$\Sigma^1_{\widetilde{a},\widetilde{a+l'}} = \Sigma^1_{\widetilde{b},\widetilde{b+l'}} = M - \frac{m}{2}.$$

As in case 1, we distinguish the two subcases,

(i) The indices appear in the order $\widetilde{a}, \widetilde{b}, \widetilde{a+l'}, \widetilde{b+l'}$

(ii) The indices appear in the order $\widetilde{a}, \widetilde{a+l'}, \widetilde{b}, \widetilde{b+l'}$

Subcase (i) is analogous to subcase i of case 1

- if $\Sigma^1_{\widetilde{a},\widetilde{b}} > 0$, then $\Sigma^1_{\widetilde{a},\widetilde{b+l}} = \Sigma^1_{\widetilde{a},\widetilde{b}} + \Sigma^1_{\widetilde{b},\widetilde{b+l}} > M - \frac{m}{2}$ - contradiction.
- if $\Sigma^1_{\widetilde{a},\widetilde{b}} < 0$, then $\Sigma^1_{\widetilde{b},\widetilde{a+l}} = \Sigma^1_{\widetilde{a},\widetilde{a+l}} - \Sigma^1_{\widetilde{a},\widetilde{b}} > M - \frac{m}{2}$ - contradiction.

Subcase (ii) is more involved than subcase (ii) of case 1

- if $\Sigma^1_{\widetilde{a},\widetilde{b}} > 0$, then $\Sigma^1_{\widetilde{a},\widetilde{b+l}} = \Sigma^1_{\widetilde{a},\widetilde{b}} + \Sigma^1_{\widetilde{b},\widetilde{b+l}} > M - \frac{m}{2}$ - contradiction.

- if $\Sigma^1_{\widetilde{b},\widetilde{a}} > 0$, then $\Sigma^1_{\widetilde{b},\widetilde{a+l}} = \Sigma^1_{\widetilde{b},\widetilde{a}} + \Sigma^1_{\widetilde{a},\widetilde{a+l}} > M - \frac{m}{2}$ - contradiction.

Since

$$\Sigma^1_{\widetilde{a},\widetilde{b}} + \Sigma^1_{\widetilde{b},\widetilde{a}} = -\frac{m}{2}$$

a priori both sums can be negative.

Nevertheless, we prove that this can't happen.

Suppose that $\Sigma^1_{\widetilde{a},\widetilde{b}}, \Sigma^1_{\widetilde{b},\widetilde{a}} < 0$. By the previous equation, we obtain

$$-\frac{m}{2} < \Sigma^1_{\widetilde{a},\widetilde{b}}, \Sigma^1_{\widetilde{b},\widetilde{a}} < 0. \tag{4.8}$$

Recall the circular sequence $\widetilde{W_{\widetilde{i}}}$ from case 1.

By Lemma 4.1.5 we have $\widetilde{W^1_{\widetilde{i}}} = \widetilde{W_{\widetilde{i}}} + \widehat{1}$, $\forall\, \widetilde{i} \in \mathbb{Z}/\frac{m}{2}$. This implies that

$$\widetilde{\psi}(\widetilde{W_{\widetilde{i}}}) = \begin{cases} \widetilde{\psi}^1(\widetilde{W^1_{\widetilde{i}}}) & \text{, if } \widetilde{W_{\widetilde{i}}} \neq \widehat{t-1} \\ \widetilde{\psi}^1(\widetilde{W^1_{\widetilde{i}}}) + m & \text{, if } \widetilde{W_{\widetilde{i}}} = \widehat{t-1} \end{cases} \tag{4.9}$$

Let $\widetilde{\iota} \in \mathbb{Z}/\frac{m}{2}$ be the class with $\widetilde{W_{\widetilde{\iota}}} = \widehat{t-1}$.

We have two (equivalent) cases:

- $\widetilde{\iota} \in \widetilde{W^1_{\widetilde{a},\widetilde{b}}}$: By Equation 4.9,

$$\Sigma_{\widetilde{a},\widetilde{b+l}} = \Sigma_{\widetilde{a},\widetilde{b}} + \Sigma_{\widetilde{b},\widetilde{b+l}} = \Sigma^1_{\widetilde{a},\widetilde{b}} + m + \Sigma^1_{\widetilde{b},\widetilde{b+l}}$$

  but by Equation 4.8,

$$\Sigma^1_{\widetilde{a},\widetilde{b}} > -\frac{m}{2},$$

  hence

$$\Sigma_{\widetilde{a},\widetilde{b+l}} > \frac{m}{2} + \Sigma^1_{\widetilde{b},\widetilde{b+l}}$$

  contradiction with the fact that $\frac{m}{2} + \Sigma^1_{\widetilde{b},\widetilde{b+l}}$ was the maximum of the sequence $(S'_i)_{i=0}^{p-1}$.

- $\widetilde{\iota} \in \widetilde{W}^1_{\widetilde{b},\widetilde{a}}$: replace b with a and apply the previous argument.

$\square$

4.2.2 The width of the knot's Heegaard Floer homology

We will not compute the width of all the simple knots of order m in Lisca's family 2, it is enough for our purposes to verify when the width is minimal.

Lemma 4.2.10. *Let $K = K(m^2, q, mt)$ a simple knot of order m in a lens space belonging to Lisca's family 2. Then*

- *if $t = 1$, $w(K) = 2(m-1)$*
- *if $t \neq 1$, $w(K) \geq 2m$*

Proof. If $t = 1$, it is trivial to verify that $w(K) = 2(m-1)$.

If $t > 1$, we have

$$S'^{\widehat{0}}_{\frac{m}{2}} = S'^{\widehat{0}}_{\frac{m}{2}-1} + \psi(\widehat{1}) = \frac{m}{2} + m - t$$

by Lemmas 4.2.4, 4.1.5 and Equation 4.6.

Choose now $\widehat{j} \in \mathbb{Z}/_m$ odd with the property that $S'^{\widehat{j}}_{\frac{m}{2}} = \widehat{m-1}$ Then

$$S'^{\widehat{j}}_{\frac{m}{2}} = -\frac{m}{2} + (-t)$$

by the same argument as above.

It follows that $w(K) \geq 2m$. $\square$

4.3 Lisca's family (3_+)

For the lens spaces in Lisca's family 3, the situation is not symmetric when we change the sign in the definition, i.e. $q = d(m \pm 1)$. Therefore, we treat each case separately.

Suppose then that $L(m^2, q)$ is a lens space with $q = d(m+1)$ for some $d \in \{1, \dots, m-1\}$ which divides $2m-1$. This family of lens spaces is denoted by 3_+.

Theorem 4.3.1. *The sequence $(S_i)_{i=0}^{p-1}$ for a lens space $L(p,q)$ belonging to Lisca's family (3_+) achieves its maximum only once.*

As before, generically, the sequences $W^{\widehat{j}}$ (we use the notation from 4.1) do not contain all the classes modulo m, so the partial sums $S^{\widehat{j}}$ are not zero.

Lemma 4.3.2. *There is a cyclic permutation W' of W such that the sequences $W'^{\widehat{j}}$ contain all the classes modulo m, in particular $S'^{\widehat{j}} = 0$, $\forall\, \widehat{j} \in \mathbb{Z}/_m$.*

Proof. Let $i_0 = \frac{d'+1}{2}$.

Then the word W' defined by

$$W'_{\overline{i}} = W_{\overline{i}+\overline{i_0}}$$

for $\overline{i} \in \mathbb{Z}/_p$, has the required property.

We verify that the classes $\left[\frac{i_0 q}{m}\right], \left[\frac{i_0 q+q}{m}\right], \dots, \left[\frac{i_0 q+(m-1)q}{m}\right]$ are all distinct.

Since $q = dm + d$, then

$$W'_i - W'_{i-1} \in \{\widehat{d}, \widehat{d+1}\}$$

for $i \in \{1, \dots, p-1\}$.

Claim 4.3.3. *Let $i \in \{1, \dots, m+i_0-1\}$. Then*

$$W_i - W_{i-1} = \widehat{d+1} \iff i \in \{i_0, 2i_0, 3i_0 - 1, 4i_0 - 1, \dots, m+1-i_0, m\}. \tag{4.10}$$

The set above can be written as $A = \left\{ri_0 - \lfloor\frac{r-1}{2}\rfloor \,|1 \leq r \leq d\right\}$.

Proof of claim. By the formula for $q = dm + d$, we have that

$$\left[\frac{iq}{m}\right] - \left[\frac{(i-1)q}{m}\right] = \widehat{d+1} \iff \widehat{iq} \in \{\widehat{0}, \dots, \widehat{d-1}\} \tag{4.11}$$

A simple computation shows that, for $r \in \{1, \dots, d\}$,

$$rm \leq \left(ri_0 - \left\lceil \frac{r-1}{2} \right\rceil\right) \cdot d < rm + d \tag{4.12}$$

and this implies that exactly for those values of i that belong to A, $W_i - W_{i-1} = \widehat{d+1}$. □

Translating by i_0, we obtain

$$W_i' - W_{i-1}' = \widehat{d+1} \iff i \in \{i_0, 2i_0 - 1, 3i_0 - 1, \ldots, m - i_0\} =: B$$

for $i \in \{0, \ldots, m-1\}$.

Consider now the numbers $0, d, 2d, \ldots (m-2)d$ and let $a_0, a_1, \ldots, a_{m-2} \in \{0, \ldots m-2\}$ be the canonical representatives of these numbers modulo $m-1$, i.e. $a_i \equiv d \cdot i \mod (m-1)$.

Form the sequence $\widehat{a_0}, \widehat{a_1}, \ldots, \widehat{a_{m-2}}, \widehat{a_{m-1}}$ where by definition $\widehat{a_{m-1}} = \widehat{m-1}$.

Claim 4.3.4. *We have*

$$\widehat{a_i} - \widehat{a_{i-1}} = \begin{cases} \widehat{d}, & \text{if } i \notin B \\ \widehat{d+1}, & \text{if } i \in B \end{cases}$$

Proof of claim. Note that

$$B = \left\{ri_0 - \left\lfloor \frac{r}{2} \right\rfloor \mid r \in \{1, \ldots, d-1\}\right\}.$$

Denote by $b_r = ri_0 - \lfloor \frac{r}{2} \rfloor$, $r \in \{1, \ldots, d-1\}\}$ For $r \in \{1, \ldots, d-1\}$,

$$b_r \cdot d = rm + r\frac{d-1}{2} - \left\lfloor \frac{r}{2} \right\rfloor \cdot d$$

hence

$$a_{b_r} = r + r\frac{d-1}{2} - \left\lfloor \frac{r}{2} \right\rfloor d = \begin{cases} \frac{r}{2}, & \text{if } r \text{ is even} \\ \frac{d+r}{2}, & \text{if } r \text{ is odd} \end{cases}$$

and then

$$a_{b_r - 1} = \begin{cases} \frac{r}{2} - d + m - 1, & \text{if } r \text{ is even} \\ \frac{d+r}{2} - d + m - 1, & \text{if } r \text{ is odd} \end{cases}$$

and for $i \in B$, the claim is proved. Notice that the values a_i for $i \in \{b_1, \ldots, b_{d-1}\}$ are distinct and they belong to the set $\{1, \ldots, d-1\}$ hence for all $i \notin B$, $a_i \geq d$ and then it follows that $a_i - a_{i-1} = d$.

Also note that $a_{m-2} = (m-2)d - (m-1)(d-1) = m-d-1$ and the claim is proved. $\square$

Putting together the claims 4.3.4 and 4.3.3 gives the conclusion of the lemma. $\square$

Thanks to the above lemma, the problem of the maximum of the sequence $(S_i')_{i=0}^{m-1}$ is reduced to finding the maxima of the length m subsequences $(S_i')_{i=rm}^{rm+m-1}$, for $r \in \{0, \ldots, m-1\}$. By Proposition 4.1.5, there is one sequence W'^{j_0} which starts with $\widehat{0}$.

Remark 4.3.5. *For the rest of this section we will denote (abusing slightly the language) by W'^j the sequence of length m of W' which starts with $\widehat{j}$, but its elements will be the canonical representative of the classes modulo m in the segment $\{0, \ldots, m-1\}$. We will also use indices modulo m to denote the sequences $W'^j = (W_{\widehat{i}}'^j)_{i=0}^{m-1}$ and their sums $S_{\widehat{i}}'^j$. By the previous lemma, this is unambiguous.*

Claims 4.3.4 and 4.3.3 imply that

$$W_i'^0 = a_i \tag{4.13}$$

for $i \in \{0, \ldots, m-1\}$.

Proof of Theorem 4.3.1. We will prove that $\max(S_i')_{i=0}^{m-1}$ is achieved only once and is equal to $w(S'^0)$. To this end, we compare the sequences W'^j with corresponding cyclic permutations of W'^0.

Claim 4.3.6. *Let $j \in \{1, \ldots, m-1\}$ and $i_j \in \{1, \ldots, m-1\}$ be the index of j in W'^0,* *i.e.* $W'^0_{i_j} = j$. *Then*

$$W'^j_{\hat{i}} = \begin{cases} W'^0_{\widehat{i+i_j}}, & \text{if } i < m - i_j \text{ and } W'^0_{i+i_j} \geq j \\ W'^0_{\widehat{i+i_j}} - 1, & \text{if } i < m - i_j \text{ and } W'^0_{i+i_j} < j \\ W'^0_{\widehat{i+i_j+1}}, & \text{if } i \geq m - i_j \text{ and } W'^0_{i+i_j+1} > j \\ W'^0_{\widehat{i+i_j+1}} - 1, & \text{if } i \geq m - i_j \text{ and } W'^0_{i+i_j+1} \leq j \end{cases} \tag{4.14}$$

Proof of claim. By equation 4.13, for $i \in \{0, \ldots, m - i_j - 1\}$,

$$W'^0_{i+i_j} \equiv W'^0_i + j \pmod{m-1}$$

hence

$$W'^0_i = \begin{cases} W'^0_{i+i_j} - j, & \text{if } W'^0_{i+i_j} \geq j \\ W'^0_{i+i_j} + m - 1 - j, & \text{if } W'^0_{i+i_j} < j \end{cases}$$

By Lemma 4.1.5, W'^j_i is the canonical representative modulo m of $W'^0_i + j$, hence by the previous equation,

$$W'^j_i = \begin{cases} W'^0_{i+i_j}, & \text{if } W'^0_{i+i_j} \geq j \\ W'^0_{i+i_j} - 1, & \text{if } W'^0_{i+i_j} < j \end{cases}$$

for $i \in \{0, \ldots, m - i_j - 1\}$.

For $i \in \{m - i_j, \ldots, m-2\}$, the number $i + i_j$ is out of the range $\{0, \ldots, m-1\}$ and we must use classes modulo m. Since the numbers $a_0, a_1, \ldots, a_{m-2}$ are the canonical representatives modulo $m-1$ of the numbers $0, d, \ldots, (m-2)d$, by equation 4.13,

$$W'^0_{i+i_j-(m-1)} \equiv W'^0_i + j \mod (m-1)$$

hence

$$W'^0_i = \begin{cases} W'^0_{i+i_j-(m-1)} - j, & \text{if } W'^0_{i+i_j-(m-1)} \geq j \\ W'^0_{i+i_j-(m-1)} + m - 1 - j, & \text{if } W'^0_{i+i_j-(m-1)} < j \end{cases}$$

Passing to classes modulo m, we obtain

$$W'^j_i = \begin{cases} W'^0_{\widehat{i+i_j+1}}, & \text{if } W'^0_{\widehat{i+i_j+1}} \geq j \\ W'^0_{\widehat{i+i_j+1}} - 1, & \text{if } W'^0_{\widehat{i+i_j+1}} < j \end{cases}$$

for $i \in \{m - i_j, \dots, m-2\}$.

Finally, note that $W'^j_{m-1} = j - 1$, and the claim is proved. $\square$

Let $w > 0$ be the width of the sequence $(S'^0)_{i=0}^{m-1}$. Since $\psi(W'^0_i) \equiv -t \pmod{m}$, the numbers $(S'^0_i)_{i=0}^{m-1}$ are all distinct, hence S'^0 has a unique maximum and a unique minimum. Let i_{min}, resp. i_{max} denote the index of $\min(S'^0)$, resp. of $\max(S'^0)$. By the definition of w, it follows that

$$\max(S'^0_{\widehat{i+i_j}})_{i=0}^{m-1} \leq w$$

for all $j \in \{0, \dots, m-1\}$ and the equality is obtained only for $i_j = i_{min} + 1$.

Claim 4.3.7. *For $j \in \{0, \dots, m-1\}$,*

$$\max(S'^j_i)_{i=0}^{m-1} \leq w$$

Remark 4.3.8. *The i'th partial sum of the sequence $(S'^0_{\widehat{i+i_j}})_{i=0}^{m-1}$ can be written as $S'^0_{i+i_j} - S'^0_{i_j-1}$.*

Proof of claim. We distinguish two cases:

1. $j < t$

   In this situation, notice that if $j' \leq j$, then $\psi(\widehat{j'}) = m - t$. Then, for $i \in \{0, \dots, m - i_j - 1\}$, $\psi(W'^j_i) = \psi(W'^0_{i+i_j})$, by equation 4.14, hence

   $$S'^j_i = S'^0_{i+i_j} - S'^0_{i_j-1} \tag{4.15}$$

   for $i \in \{0, \dots, m - i_j - 1\}$.

   Then $\psi(W'^j_{m-i_j}) \leq m - t = S'^0_0$ hence we can extend equation 4.15 to an inequality

   $$S'^j_i \leq S'^0_{i+i_j} - S'^0_{i_j-1}$$

   for $i \in \{0, \dots, m - i_j\}$.

For $i \in \{m - i_j, \ldots, m-1\}$, we have

$$\psi(W_i'^j) = \psi(W'^0_{i+i_j+1}) \tag{4.16}$$

and from this,

$$S_i'^j - S'^j_{m-i_j-1} = \sum_{k=m-i_j}^{i} \psi(W_k'^j) \leq \psi(0) + \sum_{k=m-i_j}^{i-1} \psi(W_k'^j) = \sum_{k=m-i_j}^{i} \psi(W'^0_{k+i_j})$$

where the last equality comes from equation 4.16.

But

$$\sum_{k=m-i_j}^{i} \psi(W'^0_{k+i_j}) = \sum_{k=m}^{i+i_j} \psi(W_k'^0) = S'^0_{i+i_j} - S'^0_{m-1}$$

Now use equation 4.15 for $i = m - i_j - 1$ and the conclusion is proved.

Remark 4.3.9. *(a) Note that we have proved something stronger than the claim, namely*

$$S_i'^j \leq S'^0_{\widehat{i+i_j}} - S'^0_{i_j-1}$$

for $i \in \{0, \ldots, m-1\}$.

(b) From equation 4.16, we can write directly

$$S_i'^j = S'^0_{\widehat{i+i_j+1}} - S'^0_{\widehat{i_j-1}} - \psi(W'^0_{\widehat{i_j+(m-i_j)}})$$

for $i \in \{m - i_j, \ldots, m-1\}$. Since $\psi(W_0'^0) = 0$, we have that

$$S_i'^j < S'^0_{\widehat{i+i_j+1}} - S'^0_{\widehat{i_j-1}}$$

for $i \in \{m - i_j, \ldots, m-1\}$, hence the partial sum $S_i'^j$ is strictly *smaller than the sum of a (circular) subsequence of W'^0.*

2. $j \geq t$

In this case, we will prove the following inequality:

$$S_i'^j \leq S'^0_{\widehat{i+i_{j-1}+1}} - S'^0_{\widehat{i_{j-1}}} \tag{4.17}$$

for $i \in \{0, \ldots, m-1\}$, with equality if $i \geq m - i_{j-1} - 1$.

For $i \in \{0, \ldots, m - i_{j-1} - 1\}$, by applying equation 4.14, we obtain:

$$W_i'^{j-1} = \begin{cases} W'^0_{\widehat{i+i_{j-1}}}, & \text{if } W'^0_{\widehat{i+i_{j-1}}} \geq j-1 \\ W'^0_{\widehat{i+i_{j-1}}} - 1, & \text{if } W'^0_{\widehat{i+i_{j-1}}} < j-1 \end{cases}$$

but by Lemma 4.1.5,

$$W_i'^{j-1} \equiv W_i'^j - 1 \pmod{m}.$$

Then

$$W_i'^j = \begin{cases} W'^0_{\widehat{i+i_{j-1}}} + 1, & \text{if } W'^0_{\widehat{i+i_{j-1}}} \geq j-1 \\ W'^0_{\widehat{i+i_{j-1}}}, & \text{if } W'^0_{\widehat{i+i_{j-1}}} < j-1 \end{cases}$$

for $i \in \{0, \ldots, m - i_{j-1} - 2\}$ and $W'^j_{\widehat{m-i_{j-1}-1}} = 0$.

Note that for $i \in \{1, \ldots, m - i_{j-1} - 2\}$, $\psi(W_i'^j) = \psi(W'^0_{\widehat{i+i_{j-1}}})$ since $j \geq t$ and $\widehat{i+i_{j-1}} \neq \widehat{t-1}, \widehat{m-1}$. Therefore

$$S_i'^j - \psi(\widehat{j}) = S'^0_{\widehat{i+i_{j-1}}} - S'^0_{\widehat{i_{j-1}}}$$

for $i \in \{1, \ldots, m - i_{j-1} - 2\}$.

Remark 4.3.10. *Since $\psi(\widehat{j}) < 0$, the last equality implies that*

$$S_i'^j < S'^0_{\widehat{i+i_{j-1}}} - S'^0_{\widehat{i_{j-1}}},$$

i.e. the partial sum $S_i'^j$ of W'^j is strictly *smaller than the sum of a subsequence of W'^0, for $i \in \{0, \ldots, m - i_{j-1} - 2\}$.*

In particular,

$$S'^j \leq S'^0_{\widehat{i+i_{j-1}+1}} - S'^0_{\widehat{i_{j-1}}}.$$

Since $\psi(\widehat{m-1}) = \psi(\widehat{j})$,

$$S'^j_{\widehat{m-i_{j-1}-1}} = S_0'^0 - S'^0_{\widehat{i_{j-1}}}. \tag{4.18}$$

Let $i \in \{m - i_{j-1}, \ldots, m-1\}$. By equation 4.14,

$$W'^{j-1} = \begin{cases} W'^0_{\widehat{i+i_{j-1}+1}}, & \text{if } W'^0_{\widehat{i+i_{j-1}+1}} > j-1 \\ W'^0_{\widehat{i+i_{j-1}+1}} - 1, & \text{if } W'^0_{\widehat{i+i_{j-1}+1}} \leq j-1 \end{cases}$$

By Lemma 4.1.5,

$$W_i'^j = \begin{cases} W'^0_{i+\widehat{i_{j-1}}+1} + 1, & \text{if } W'^0_{i+\widehat{i_{j-1}}+1} > j-1 \\ W'^0_{i+\widehat{i_{j-1}}+1}, & \text{if } W'^0_{i+\widehat{i_{j-1}}+1} \leq j-1 \end{cases} \tag{4.19}$$

since $W'^0_{i+\widehat{i_{j-1}}+1} \neq m-1$ (because $i > m - i_{j-1} - 1$). Note that

$$\psi(W_i'^j) = \psi(W'^0_{i+\widehat{i_{j-1}}+1})$$

for $i \in \{m - i_{j-1}, \ldots, m-1\}$. This implies that

$$S_i'^j - S'^j_{m-\widehat{i_{j-1}}-1} = S'^0_{i+\widehat{i_{j-1}}+1} - S_0'^0 \tag{4.20}$$

for $i \in \{m - i_{j-1}, \ldots, m-1\}$.

Now simply add the equations 4.18 and 4.20 to get the conclusion of the claim. □

Claim 4.3.11. *There is at most one pair of values $(i,j) \in \{0, \ldots, m-1\}^2$ with $S_i'^j = w$.*

Proof of claim. Suppose that there are two such values (i_1, j_1) and (i_2, j_2). By Lemma 4.0.14, the two sequences have the same length, i.e. $i_1 = i_2$.

If both $j_1, j_2 < t$ then the previous claim shows that

$$w = S'^0_{i_1+i_{j_1}} - S'^0_{i_{j_1}-1} = S'^0_{i_2+i_{j_2}} - S'^0_{i_{j_2}-1},$$

which is a contradiction because w is the sum of a *unique* subsequence of W'^0.

Similarly, for $j_1, j_2 \geq t$, we obtain that

$$w = S'^0_{i_1+\widehat{i_{j_1-1}}+1} - S'^0_{\widehat{i_{j_1-1}}} = S'^0_{i_2+\widehat{i_{j_2-1}}+1} - S'^0_{\widehat{i_{j_2-1}}}$$

which is again a contradiction by the same argument.

If $j_1 < t$ and $j_2 \geq t$, then the equality cases in the claim, more precisely Remarks 4.3.9 and 4.3.10 imply that the width w is realised as a partial sum of a subsequence of W'^0 which contains $W_0'^0$ for the case (i_2, j_2) and it doesn't contain $W_0'^0$ for (i_1, j_1) - contradiction. □

We prove now that w is realised as the maximum of S'^j, for some $j \in \{0, \ldots, m-1\}$. We distinguish again two cases

1. $i_{min} < i_{max}$

   In this case, simply consider $j = W'^0_{i_{min}+1}$, hence $i_j = i_{min} + 1$. By the minimality of min, we get $j < t$ and then

   $$\psi(W_i'^j) = \psi(W'^0_{i+i_j})$$

   for $i \in \{0, \ldots, m - i_j - 1\}$ which in particular contains the index $i_{max} - i_{min} - 1$. We obtain $S'^j_{i_{max}-i_{min}-1} = S'^0_{i_{max}} - S'^0_{i_{min}} = w$.

2. $i_{max} < i_{min}$

   Let $j = W'_{i_{min}} + 1$. By the minimality of $S'^0_{i_{min}}$, we obtain $j \geq t$ and $j > m - d$ and we are in the situation analysed in the previous claim. Its proof implies that

   $$S'^j_{\widehat{i_{max}-i_{min}}-1} = S'^0_{i_{max}} - S'^0_{i_{min}} = w(S'^0)$$

   Remark 4.3.12. *The class $\widehat{i_{max} - i_{min}} - 1$ is represented by $i_{max} - i_{min} + m - 1 \geq m - i_{j-1} - 1 = i_{min} - 1$, hence we are in the case of equality in Claim 4.3.7.*

□

4.3.1 An example

Let $m = 5$, $d = 3$ (hence $(p, q) = (25, 18)$ and $d' = 3$)

The intersections of the α and β curves are: $0, 18, 11, 4, 22, 15, 8, 1, 19, 12, 5, \ldots, 14$ and the words W, resp. W' are listed below

$$W_i^j = \begin{pmatrix} 0 & 3 & 2 & 0 & 4 \\ 3 & 1 & 0 & 3 & 2 \\ 1 & 4 & 3 & 1 & 0 \\ 4 & 2 & 1 & 4 & 3 \\ 2 & 0 & 4 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 4 & 3 & 1 \\ 0 & 3 & 2 & 1 & 4 \\ 3 & 1 & 0 & 4 & 2 \\ 1 & 4 & 3 & 2 & 0 \\ 4 & 2 & 1 & 0 & 3 \end{pmatrix} = W_i'^j$$

Consider $t = 2$. Then $f(\widehat{0}) = f(\widehat{1}) = y = 3$ and $f(\widehat{2}) = f(\widehat{3}) = f(\widehat{4}) = x = -2$ and the sums $S_i'^j$ are

$$\begin{pmatrix} -2 & 1 & -1 & -3 & 0 \\ 3 & 1 & -1 & 2 & 0 \\ -2 & 1 & 4 & 2 & 0 \\ 3 & 1 & -1 & -3 & 0 \\ -2 & -4 & -1 & 2 & 0 \end{pmatrix}$$

4.4 Lisca's family (3_-)

This family is somewhat similar to family (3_+), but the situation is slightly more complicated.

Let $L(p, q)$ be a lens space belonging to Lisca's family (3) with $p = m^2$ and $q = dm - d$ for some $d < m$ a divisor of $2m + 1$. Let $d' = \frac{2m+1}{d}$.

Theorem 4.4.1. *The sequence $(S_i)_{i=0}^{p-1}$ for a lens space $L(p, q)$ in Lisca's family (3_-) achieves its maximum only once.*

Remark 4.4.2. *We will use the same techniques and notations from the previous case: $W_i^j, W_i'^j, S_i'^j$ etc.*

Observation 4.4.3. *Generically, the sequences W^j of W do not contain all the classes modulo m, so their sum is non-zero in general.*

Lemma 4.4.4. *There exist a cyclic permutation W' of W such that the sequences W'^j, for $j \in \{0, \ldots, m-1\}$, contain all the classes modulo m, therefore $S'^j = 0$.*

Proof. Choose $i_0 = \frac{d'+1}{2}$ and let

$$W_{\overline{i}}' = W_{\overline{i+i_0}}$$

Then

$$W' = \left[\frac{i_0 \cdot q}{m}\right], \left[\frac{(i_0+1) \cdot q}{m}\right], \ldots, \left[\frac{(i_0+p-1) \cdot q}{m}\right].$$

We will prove that the first m letters of W' are distinct, and this together with Lemma 4.1.5 will give the desired conclusion.

Since $q = dm - d$,

$$W_i - W_{i-1} \in \{\widehat{d}, \widehat{d-1}\},$$

for $i \in \{1, \dots, p\}$.

Claim 4.4.5. *For $i \in \{2, \dots, m+1\}$,*

$$W_i - W_{i-1} = \widehat{d-1} \iff i \in \{i_0, 2i_0 - 1, 3i_0 - 1, \dots, m+1\}.$$

Proof of claim. Denote by A the set on the right. Then

$$A = \left\{ r \cdot i_0 - \left\lfloor \frac{r}{2} \right\rfloor \mid r \in \{1, \dots, d\} \right\}$$

Since $q = dm - d$,

$$\left\lceil \frac{iq}{m} \right\rceil - \left\lceil \frac{(i-1)q}{m} \right\rceil = \widehat{d-1} \iff \widehat{iq} \in \{-\widehat{1}, \dots, -\widehat{d}\}. \tag{4.21}$$

For $r \in \{1, \dots, d\}$, we have

$$\left(r \cdot i_0 - \left\lfloor \frac{r}{2} \right\rfloor \right) \cdot d = \begin{cases} rm + \frac{r+d}{2}, & \text{if } r \text{ is odd} \\ rm + \frac{r}{2}, & \text{if } r \text{ is even} \end{cases}$$

Note that

$$\left\{ \widehat{W_i} \mid i \in A \right\} = \left\{ \widehat{-1}, \dots, \widehat{-d} \right\},$$

which implies that for $i \in A$, $W_i - W_{i-1} = \widehat{d-1}$ and for any other value of $i \in \{2. \dots, m+1\}$, $W_i - W_{i-1} = \widehat{d}$. □

By Lemma 4.1.5, we have that for $i \in \{1, \dots, m + i_0 - 1\}$,

$$W_i - W_{i-1} = \widehat{d-1} \iff i \in \{1, i_0, 2i_0 - 1, \dots, m+1\} = \{1\} \cup A$$

Translating by i_0, we obtain

$$W'_i - W'_{i-1} = \widehat{d-1} \iff i \in \{i_0 - 1, 2i_0 - 1, \dots, m+1-i_0\} \tag{4.22}$$

for $i \in \{1, \ldots, m-1\}$. Call this set B. Then

$$B = \left\{ r \cdot i_0 - \left\lfloor \frac{r+1}{2} \right\rfloor \mid r \in \{1, \ldots, d-1\} \right\} \tag{4.23}$$

Denote by $b_r = r \cdot i_0 - \lfloor \frac{r+1}{2} \rfloor$, $r \in \{1, \ldots, d-1\}$.

Let a_i be the canonical representatives modulo $m+1$ of the classes $d - 1 + i \cdot d$, for $i \in \{0, \ldots, m-1\}$; in other words $a_i \equiv d - 1 + i \cdot d \pmod{m+1}$ and $a_i \in \{0, \ldots, m\}$.

Remark 4.4.6. *Since d is a divisor of $2m+1$, $\gcd(d, m+1) = 1$, so the numbers a_i are all distinct.*

Claim 4.4.7. *For $i \in \{1, \ldots, m\}$,*

$$\widehat{a_i} - \widehat{a_{i-1}} = \begin{cases} \widehat{d-1}, & \text{if } i \in B. \\ \widehat{d}, & \text{if } i \notin B. \end{cases}$$

Proof of claim. It is immediate to see that

$$\widehat{a_i} - \widehat{a_{i-1}} = \widehat{d-1} \iff a_i \in \{0, \ldots, d-1\}$$

Let $i \in B$, i.e. $i = r \cdot i_0 - \lfloor \frac{r+1}{2} \rfloor$ for some $r \in \{1, \ldots, d-1\}$. A simple computation shows that

$$a_i = \begin{cases} \frac{d-r}{2} - 1, & \text{if } r \text{ odd} \\ d - 1 - \frac{r}{2}, & \text{if } r \text{ even} \end{cases}$$

which shows that, for $i \in B$, $\widehat{a_i} - \widehat{a_{i-1}} = \widehat{d-1}$. Note that

$$\{a_i \mid i \in B\} = \{0, \ldots, d-2\}$$

and $a_0 = d - 1$. Together with Remark 4.4.6, this implies that for all indices $i \in \{1, \ldots, m-1\} \setminus B$, $\widehat{a_i} - \widehat{a_{i-1}} = \widehat{d}$. $\square$

The previous claim, coupled with equation 4.22, show that

$$W_i'^{d-1} = \widehat{a_i}, \tag{4.24}$$

for $i \in \{0, \dots, m-1\}$.

Note that $a_m = m - d$, hence $a_i \neq m$, for $i \in \{0, \dots, m-1\}$, which implies that the classes $\widehat{a_i}$ are distinct, for $i \in \{0, \dots, m-1\}$. □

It is convenient to extend the sequence $(a_i)_{i=0}^{m-1}$ with $a_m = m$, by definition. We can write then

$$a_i \equiv d - 1 + d \cdot i \pmod{m+1} \tag{4.25}$$

for $i \in \{0, \dots, m\}$.

As in the previous argument, we will compare the sequences W'^j with cyclic permutations of the sequence W'^{d-1}.

Remark 4.4.8. *From now on, by abusing language, we will denote by $W_i'^j$ the canonical representative modulo m of the class $W_i'^j$ and we will frequently use indices modulo m with the obvious interpretation, i.e. $W_{\widehat{i}}'^j = W_i'^j$ for $i \in \{0, \dots, m-1\}$. Finally, we will denote by i_j the index of the number j in W'^{d-1}, i.e. $W_i'^{d-1} = j$.*

Lemma 4.4.9. *For $i, j \in \{0, \dots, m-1\}$ with $j \geq d-1$,*

$$W_i'^j = \begin{cases} W_{\widehat{i+i_j}}'^{d-1}, & \text{if } i < m - i_j \text{ and } W_{\widehat{i+i_j}}'^{d-1} \geq j - (d-1) \\ W_{\widehat{i+i_j}}'^{d-1} + 1, & \text{if } i < m - i_j \text{ and } W_{\widehat{i+i_j}}'^{d-1} < j - (d-1) \\ 0, & \text{if } i = m - i_j \\ W_{\widehat{i+i_j-1}}'^{d-1}, & \text{if } i > m - i_j \text{ and } W_{\widehat{i+i_j-1}}'^{d-1} \geq j - (d-1) \\ W_{\widehat{i+i_j-1}}'^{d-1} + 1, & \text{if } i > m - i_j \text{ and } W_{\widehat{i+i_j-1}}'^{d-1} < j - (d-1) \end{cases}$$

Proof. By Claim 4.4.7, for $i \in \{0, \dots, m - i_j - 1\}$,

$$W_{i+i_j}'^{d-1} \equiv W_i'^{d-1} + j - (d-1) \pmod{m+1}.$$

Therefore,

$$W_{i+i_j}'^{d-1} = \begin{cases} W_i'^{d-1} + j - (d-1), & \text{if } W_i'^{d-1} \leq m - j + (d-1) \\ W_i'^{d-1} + j - d - m, & \text{if } W_i'^{d-1} > m - j + (d-1) \end{cases} \tag{4.26}$$

Equivalently,

$$W_i^{\prime d-1} = \begin{cases} W_{i+i_j}^{\prime d-1} - j + (d-1), & \text{if } W_{i+i_j}^{\prime d-1} \geq j - (d-1) \\ W_{i+i_j}^{\prime d-1} - j + d + m, & \text{if } W_{i+i_j}^{\prime d-1} < j - (d-1) \end{cases} \tag{4.27}$$

By Lemma 4.1.5,

$$W_i^{\prime j} \equiv W_i^{\prime d-1} + j - (d-1) \pmod m$$

hence

$$W_i^{\prime j} = \begin{cases} W_i^{\prime d-1} + j - (d-1), & \text{if } W_i^{\prime d-1} < m - j + (d-1) \\ W_i^{\prime d-1} + j - (d-1) - m, & \text{if } W_i^{\prime d-1} > m - j + (d-1) \end{cases}$$

Remark 4.4.10. *A simple computation shows that*

$$W_{m-i_j}^{\prime d-1} = m - j + (d-1),$$

therefore we can use strict inequalities in the previous formula, for $i \in \{0, \ldots, m - i_j - 1\}$.

Together with equations 4.26 and 4.27, this gives

$$W_i^{\prime j} = \begin{cases} W_{i+i_j}^{\prime d-1}, & \text{if } i < m - i_j \text{ and } W_{i+i_j}^{\prime d-1} \geq j - (d-1) \\ W_{i+i_j}^{\prime d-1} + 1, & \text{if } i < m - i_j \text{ and } W_{i+i_j}^{\prime d-1} < j - (d-1) \end{cases} \tag{4.28}$$

For $i = m - i_j$, by Remark 4.4.10 and Lemma 4.1.5,

$$W_{m-i_j}^{\prime j} = 0. \tag{4.29}$$

By Remark 4.4.10,

$$W_{m-i_j+1}^{\prime d-1} - m + j - d \equiv W_0^{\prime d-1} \pmod{m+1}$$

and by finite induction on $i \in \{m - i_j + 1, \ldots, m-1\}$, we deduce

$$W_{\widehat{i+i_j-1}}^{\prime d-1} \equiv W_{\widehat{i}}^{\prime d-1} + j - (d-1) \pmod{m+1}, \tag{4.30}$$

hence

$$W_{\widehat{i+i_j-1}}^{\prime d-1} = \begin{cases} W_{\widehat{i}}^{\prime d-1} + j - (d-1), & \text{if } W_{\widehat{i}}^{\prime d-1} \leq m - j + (d-1) \\ W_{\widehat{i}}^{\prime d-1} + j - d - m, & \text{if } W_{\widehat{i}}^{\prime d-1} > m - j + (d-1) \end{cases}$$

and

$$W_i'^{d-1} = \begin{cases} W'^{d-1}_{\widehat{i+i_j-1}} - j + (d-1), & \text{if } W'^{d-1}_{\widehat{i+i_j-1}} \geq j - (d-1) \\ W'^{d-1}_{\widehat{i+i_j-1}} - j + d + m, & \text{if } W'^{d-1}_{\widehat{i+i_j-1}} < j - (d-1) \end{cases}$$

These two equations and Lemma 4.1.5 imply

$$W'^{j}_{\widehat{i}} = \begin{cases} W'^{d-1}_{\widehat{i+i_j-1}}, & \text{if } i > m - i_j \text{ and } W'^{d-1}_{\widehat{i+i_j-1}} \geq j - (d-1) \\ W'^{d-1}_{\widehat{i+i_j-1}} + 1, & \text{if } i > m - i_j \text{ and } W'^{d-1}_{\widehat{i+i_j-1}} < j - (d-1) \end{cases} \tag{4.31}$$

for $i \in \{m - i_j + 1, \ldots, m - 1\}$. □

Lemma 4.4.11. *For $i, j \in \{0, \ldots, m-1\}$ with $j < d - 1$,*

$$W'^{j}_{\widehat{i}} = \begin{cases} W'^{d-1}_{i+i_j}, & \text{if } i < m - i_j \text{ and } W'^{d-1}_{i+i_j} < m + j - (d-1) \\ W'^{d-1}_{i+i_j} - 1, & \text{if } i < m - i_j \text{ and } W'^{d-1}_{i+i_j} > m + j - (d-1) \\ m - 1, & \text{if } i = m - i_j \\ W'^{d-1}_{\widehat{i+i_j-1}}, & \text{if } i > m - i_j \text{ and } W'^{d-1}_{\widehat{i+i_j-1}} < m + j - (d-1) \\ W'^{d-1}_{\widehat{i+i_j-1}} - 1, & \text{if } i > m - i_j \text{ and } W'^{d-1}_{\widehat{i+i_j-1}} > m + j - (d-1) \end{cases}$$

Proof. The proof is similar to the proof of Lemma 4.4.11.

By Claim 4.4.7, for $i \in \{0, \ldots, m - i_j - 1\}$,

$$W'^{d-1}_{i+i_j} \equiv W_i'^{d-1} + j - (d-1) \pmod{m+1}.$$

Therefore,

$$W'^{d-1}_{i+i_j} = \begin{cases} W_i'^{d-1} + j - (d-1), & \text{if } W_i'^{d-1} \geq (d-1) - j \\ W_i'^{d-1} + j - (d-1) + m + 1, & \text{if } W_i'^{d-1} < (d-1) - j \end{cases} \tag{4.32}$$

Equivalently,

$$W_i'^{d-1} = \begin{cases} W'^{d-1}_{i+i_j} - j + (d-1), & \text{if } W'^{d-1}_{i+i_j} < m + j - (d-1) \\ W'^{d-1}_{i+i_j} - j + (d-1) + m + 1, & \text{if } W'^{d-1}_{i+i_j} \geq m + j - (d-1) \end{cases} \tag{4.33}$$

By Lemma 4.1.5,

$$W'^{j}_{i} \equiv W'^{d-1}_{i} + j - (d-1) \pmod m$$

hence

$$W'^{j}_{i} = \begin{cases} W'^{d-1}_{i} + j - (d-1), & \text{if } W'^{d-1}_{i} \geq (d-1) - j \\ W'^{d-1}_{i} + j - (d-1) + m, & \text{if } W'^{d-1}_{i} < (d-1) - j \end{cases} \tag{4.34}$$

By the previous equation and equations 4.32 and 4.33,

$$W'^{j}_{i} = \begin{cases} W'^{d-1}_{i+i_j}, & \text{if } i < m - i_j \text{ and } W'^{d-1}_{i+i_j} < m + j - (d-1) \\ W'^{d-1}_{i+i_j} - 1, & \text{if } i < m - i_j \text{ and } W'^{d-1}_{i+i_j} > m + j - (d-1) \end{cases} \tag{4.35}$$

Remark 4.4.12. *Once again, the equality case in the previous equation does not occur for $i \in \{0, \ldots, m - i_j - 1\}$, since it is easy to observe that*

$$W'^{d-1}_{i_j-1} = m + j - (d-1). \tag{4.36}$$

For $i = m - i_j$, by Remark 4.4.10 and Lemma 4.1.5,

$$W'^{j}_{m-i_j} = m - 1. \tag{4.37}$$

By equation 4.30,

$$W'^{d-1}_{\widehat{i+i_j-1}} \equiv W'^{d-1}_{\widehat{i}} + j - (d-1) \pmod{m+1},$$

for $i \in \{m - i_j + 1, \ldots, m-1\}$ hence

$$W'^{d-1}_{\widehat{i+i_j-1}} = \begin{cases} W'^{d-1}_{\widehat{i}} + j - (d-1), & \text{if } W'^{d-1}_{\widehat{i}} \geq (d-1) - j \\ W'^{d-1}_{\widehat{i}} + j - (d-1) + m, & \text{if } W'^{d-1}_{\widehat{i}} < (d-1) - j \end{cases}$$

and

$$W'^{d-1}_{i} = \begin{cases} W'^{d-1}_{\widehat{i+i_j-1}} - j + (d-1), & \text{if } W'^{d-1}_{\widehat{i+i_j-1}} < m + j - (d-1) \\ W'^{d-1}_{\widehat{i+i_j-1}} - j + (d-1) - m, & \text{if } W'^{d-1}_{\widehat{i+i_j-1}} \geq m + j - (d-1) \end{cases}$$

The two equations and Lemma 4.1.5, or more precisely equation 4.34, give the last part of the conclusion of the lemma:

$$W'^{j}_{\widehat{i}} = \begin{cases} W'^{d-1}_{\widehat{i+i_j-1}}, & \text{if } i > m - i_j \text{ and } W'^{d-1}_{\widehat{i+i_j-1}} < m + j - (d-1) \\ W'^{d-1}_{\widehat{i+i_j-1}} - 1, & \text{if } i > m - i_j \text{ and } W'^{d-1}_{\widehat{i+i_j-1}} > m + j - (d-1) \end{cases} \tag{4.38}$$

for $i \in \{m - i_j + 1, \ldots, m - 1\}$.

Remark 4.4.13. *There is no equality case in the formula above by remark 4.4.10.*

□

The two lemmas allow us to compare the sequences W'^j with W'^{d-1} and estimate their maxima.

Proof of Theorem 4.4.1. As above, there are two different situations, depending on j and d. We treat them separately.

1. $j > d - 1$

   Depending on $t \in \{2, \ldots, m/2\}$, there are two different possibilities:

   (a) $t \geq j - (d - 1)$

   Using Lemma 4.4.9 we can write

$$\psi(W'^j_{\hat{i}}) = \begin{cases} \psi(W'^{d-1}_{\widehat{i+i_j}}), & \text{if } i < m - i_j \\ \psi(0), & \text{if } i = m - i_j \\ \psi(W'^{d-1}_{\widehat{i+i_j-1}}), & \text{if } i > m - i_j \end{cases} \tag{4.39}$$

   for $i \in \{0, \ldots, m - 1\}$.

   Remark 4.4.14. *The condition $t \geq j - (d - 1)$ implies that $j - d < t$, i.e. $\psi(W'^{d-1}_{i_j - 1}) = \psi(0)$.*

   Claim 4.4.15. *In this situation, for $i \in \{0, \ldots, m - 1\}$, we have*

$$S'^j_i = \begin{cases} S'^{d-1}_{i+i_j} - S'^{d-1}_{i_j-1}, & \text{if } i < m - i_j \\ S'^{d-1}_{\widehat{i+i_j-1}} - S'^{d-1}_{\widehat{i_j-2}}, & \text{if } i \geq m - i_j \end{cases}$$

   Moreover, for $i < m - i_j$,

$$S'^j_i < S'^{d-1}_{i+i_j} - S'^{d-1}_{i_j-2}.$$

Proof of claim. For $i < m - i_j$, by equation 4.39, we have

$$S_i^{\prime j} = S_{i+i_j}^{\prime d-1} - S_{i_j-1}^{\prime d-1}$$

and since $\psi(W_{i_j-1}^{\prime d-1}) = \psi(0) > 0$,

$$S_{i_j-1}^{\prime d-1} > S_{i_j-2}^{\prime d-1},$$

and the inequality in the last part of the claim follows.

Since $\psi(0) = \psi(W_{i_j-1}^{\prime d-1})$, we can extend the previous equation to:

$$S_{m-i_j}^{\prime j} = S_{m-1}^{\prime d-1} - S_{i_j-2}^{\prime d-1}$$

and then continue by finite induction on $i \in \{m - i_j + 1, \ldots, m-1\}$ using equation 4.39 to extend the equality to

$$S_i^{\prime j} = S_{\widehat{i+i_j-1}}^{\prime d-1} - S_{\widehat{i_j-2}}^{\prime d-1}.$$

$\square$

(b) $t < j - (d-1)$

In this case, $\psi(W_{i_j-1}^{\prime d-1}) < 0$, and of course, $\psi(W_{i_j}^{\prime d-1}) < 0$.

Claim 4.4.16. *With the hypothesis above, for $i \in \{0, \ldots, m-1\}$, we have*

$$S_i^{\prime j} < w.$$

Proof of claim. Consider first the case $j < m - 1$. By Lemma 4.4.9,

$$W_i^{\prime j+1} = \begin{cases} W_{\widehat{i+i_{j+1}}}^{\prime d-1}, & \text{if } i < m - i_{j+1} \text{ and } W_{\widehat{i+i_{j+1}}}^{\prime d-1} \geq j+1-(d-1) \\ W_{\widehat{i+i_{j+1}}}^{\prime d-1} + 1, & \text{if } i < m - i_{j+1} \text{ and } W_{\widehat{i+i_{j+1}}}^{\prime d-1} < j+1-(d-1) \\ 0, & \text{if } i = m - i_{j+1} \\ W_{\widehat{i+i_{j+1}-1}}^{\prime d-1}, & \text{if } i > m - i_{j+1} \text{ and } W_{\widehat{i+i_{j+1}-1}}^{\prime d-1} \geq j+1-(d-1) \\ W_{\widehat{i+i_{j+1}-1}}^{\prime d-1} + 1, & \text{if } i > m - i_{j+1} \text{ and } W_{\widehat{i+i_{j+1}-1}}^{\prime d-1} < j+1-(d-1) \end{cases}$$

By Lemma 4.1.5,

$$W_i'^j = \begin{cases} W'^{d-1}_{\widehat{i+i_{j+1}}} - 1, & \text{if } i < m - i_{j+1} \text{ and } W'^{d-1}_{\widehat{i+i_{j+1}}} \geq j+1-(d-1) \\ W'^{d-1}_{\widehat{i+i_{j+1}}}, & \text{if } i < m - i_{j+1} \text{ and } W'^{d-1}_{\widehat{i+i_{j+1}}} < j+1-(d-1) \\ m-1, & \text{if } i = m - i_{j+1} \\ W'^{d-1}_{\widehat{i+i_{j+1}-1}} - 1, & \text{if } i > m - i_{j+1} \text{ and } W'^{d-1}_{\widehat{i+i_{j+1}-1}} \geq j+1-(d-1) \\ W'^{d-1}_{\widehat{i+i_{j+1}-1}}, & \text{if } i > m - i_{j+1} \text{ and } W'^{d-1}_{\widehat{i+i_{j+1}-1}} < j+1-(d-1) \end{cases}$$

Since $t < j-(d-1)$, the previous equation gives

$$\psi(W_i'^j) = \begin{cases} \psi(W'^{d-1}_{\widehat{i+i_{j+1}}}), & \text{if } i < m - i_{j+1} \\ \psi(m-1), & \text{if } i = m - i_{j+1} \\ \psi(W'^{d-1}_{\widehat{i+i_{j+1}-1}}), & \text{if } i > m - i_{j+1} \end{cases}$$

From this, it is immediate to see that

$$S_i'^j = S'^{d-1}_{\widehat{i+i_{j+1}}} - S'^{d-1}_{\widehat{i_{j+1}-1}} \tag{4.40}$$

for $i < m - i_{j+1}$.

Then,

$$S'^j_{m-i_{j+1}} = S'^{d-1}_{\widehat{m-1}} - S'^{d-1}_{\widehat{i_{j+1}-1}} + \psi(m-1) < S'^{d-1}_{\widehat{m-1}} - S'^{d-1}_{\widehat{i_{j+1}-1}},$$

and finally, by induction on $i \in \{m-i_{j+1}, \ldots, m-1\}$, we have

$$S_i'^j = S'^{d-1}_{\widehat{i+i_{j+1}-1}} - S'^{d-1}_{\widehat{i_{j+1}-1}} + \psi(m-1).$$

Since $\psi(W'^{d-1}_{i_{j+1}}) < 0$, we have $S_i'^j < w$, for all $i \in \{0, \ldots, m-1\}$.

For $j = m-1$, from Lemma 4.1.5,

$$W'^{m-1}_{i+1} \equiv W'^{d-1}_{i+1} - d \pmod{m}$$

for $i \in \{0, \ldots, m-1\}$.

Also, from equation 4.24, we know that

$$W_i'^{d-1} \equiv W'^{d-1}_{i+1} - d \pmod{m+1}$$

for $i \in \{0, \ldots, m-2\}$. These two equations imply

$$W_i'^{d-1} = \begin{cases} W_{i+1}'^{m-1}, & \text{if } W_{i+1}'^{m-1} < m-d \\ W_{i+1}'^{m-1} + 1, & \text{if } W_{i+1}'^{m-1} \geq m-d \end{cases}$$

for $i \in \{0, \ldots, m-2\}$ (see also 4.3.6 for a similar argument).

The condition $t < (m-1) - (d-1)$ implies that

$$\psi(W_i'^{d-1}) = \psi(W_{i+1}'^{m-1})$$

for $i \in \{0, \ldots, m-2\}$, hence

$$S_i'^{m-1} - \psi(W_0'^{m-1}) = S_{i-1}'^{d-1}$$

for $i \in \{1, \ldots, m-1\}$. Since $\psi(W_0'^{d-1}) < 0$, we obtain

$$S_i'^{m-1} < S_{i-1}'^{d-1}$$

for $i \in \{1, \ldots, m-1\}$ and the claim is proved. □

2. $j < d-1$

   (a) $t \leq m + j - (d-1)$

   Note that this implies that $\psi(W_{i_j-1}'^{d-1}) < 0$. Also, by Lemma 4.4.11,

$$\psi(W_i'^{j}) = \begin{cases} \psi(W_{i+i_j}'^{d-1}), & \text{if } i < m - i_j \\ \psi(W_{m-1}'^{d-1}), & \text{if } i = m - i_j \\ \psi(W_{i+i_j-1}'^{d-1}), & \text{if } i > m - i_j \end{cases} \tag{4.41}$$

   Claim 4.4.17. *For $i \in \{0, \ldots, m-1\}$, we have*

$$S_i'^{j} = \begin{cases} S_{i+i_j}'^{d-1} - S_{i_j-1}'^{d-1}, & \text{if } i < m - i_j \\ S_{i+i_j-1}'^{d-1} - S_{i_j-2}'^{d-1}, & \text{if } i \geq m - i_j \end{cases}$$

   Also, for $i \geq m - i_j$,

$$S_i'^{j} < w.$$

Proof of claim. From equation 4.41, we immediately see that

$$S_i'^j = S_{i+i_j}'^{d-1} - S_{i_j-1}'^{d-1}$$

for $i < m - i_j$.

For $i = m - i_j$,

$$S_i'^j = S_{i+i_j}'^{d-1} - S_{i_j-1}'^{d-1} + \psi(W_{m-1}'^{d-1}) < S_{i+i_j}'^{d-1} - S_{i_j-1}'^{d-1}$$

and for $i > m - i_j$, also from equation 4.41,

$$S_i'^j = S_{\widehat{i+i_j}-1}'^{d-1} - S_{i_j-1}'^{d-1} + \psi(W_{m-1}'^{d-1}) \tag{4.42}$$

for $j \in \{m - i_j + 1, \ldots, m - 1\}$. Now use the observation that $\psi(W_{i_j-1}'^{d-1}) = \psi(W_{m-1}'^{d-1})$ to get the equality stated in the claim. By equation 4.42, the inequality in the claim is also established. $\square$

(b) $t > m + j - (d - 1)$

In this case $\psi(W_{i_j-1}'^{d-1}) > 0$.

Claim 4.4.18. *For $j > 0$ we have the following formula*

$$S_i'^j = \begin{cases} S_{i+i_{j-1}}'^{d-1} - S_{i_{j-1}-1}'^{d-1}, & \text{if } i < m - i_{j-1} \\ S_{\widehat{i+i_{j-1}}}'^{d-1} - S_{\widehat{i_{j-1}-2}}'^{d-1}, & \text{if } i \geq m - i_{j-1} \end{cases}$$

Moreover, for $i < m - i_{j-1}$,

$$S_i'^j < S_{i+i_{j-1}}'^{d-1} - S_{i_{j-1}-2}'^{d-1}.$$

For $j = 0$,

$$S_i'^j = S_{i+m-d}'^{d-1} - S_{m-d-1}'^{d-1}$$

$\forall i \in \{0, \ldots, m-1\}$.

Proof of claim. Suppose $j > 0$ first. By Lemma 4.4.11 applied to $j-1$, we obtain

$$W_i'^{j-1} = \begin{cases} W'^{d-1}_{i+i_{j-1}}, & \text{if } i < m - i_{j-1} \text{ and } W'^{d-1}_{i+i_{j-1}} < m+j-d \\ W'^{d-1}_{i+i_{j-1}} - 1, & \text{if } i < m - i_{j-1} \text{ and } W'^{d-1}_{i+i_{j-1}} > m+j-d \\ m-1, & \text{if } i = m - i_{j-1} \\ W'^{d-1}_{i+\widehat{i_{j-1}-1}}, & \text{if } i > m - i_{j-1} \text{ and } W'^{d-1}_{i+\widehat{i_{j-1}-1}} < m+j-d \\ W'^{d-1}_{i+\widehat{i_{j-1}-1}} - 1, & \text{if } i > m - i_{j-1} \text{ and } W'^{d-1}_{i+\widehat{i_{j-1}-1}} > m+j-d \end{cases}$$

Using Lemma 4.1.5,

$$W_i'^{j} = \begin{cases} W'^{d-1}_{i+i_{j-1}} + 1, & \text{if } i < m - i_{j-1} \text{ and } W'^{d-1}_{i+i_{j-1}} < m+j-d \\ W'^{d-1}_{i+i_{j-1}}, & \text{if } i < m - i_{j-1} \text{ and } W'^{d-1}_{i+i_{j-1}} > m+j-d \\ 0, & \text{if } i = m - i_{j-1} \\ W'^{d-1}_{i+\widehat{i_{j-1}-1}} + 1, & \text{if } i > m - i_{j-1} \text{ and } W'^{d-1}_{i+\widehat{i_{j-1}-1}} < m+j-d \\ W'^{d-1}_{i+\widehat{i_{j-1}-1}}, & \text{if } i > m - i_{j-1} \text{ and } W'^{d-1}_{i+\widehat{i_{j-1}-1}} > m+j-d \end{cases}$$

and using the inequality $t > m + j - (d-1)$,

$$\psi(W_i'^{j}) = \begin{cases} \psi(W'^{d-1}_{i+i_{j-1}}), & \text{if } i < m - i_{j-1} \\ 0, & \text{if } i = m - i_{j-1} \\ \psi(W'^{d-1}_{i+\widehat{i_{j-1}-1}}), & \text{if } i > m - i_{j-1} \end{cases}$$

It follows that

$$S_i'^{j} = S'^{d-1}_{i+i_{j-1}} - S'^{d-1}_{i_{j-1}-1}$$

for $i \in \{0, \ldots, m - i_{j-1}\}$ and also, since $\psi(W'^{d-1}_{i_{j-1}-1}) = \psi(W'^{d-1}_{i_{j-1}}) > 0$,

$$S_i'^{j} < S'^{d-1}_{i+i_{j-1}} - S'^{d-1}_{i_{j-1}-2}$$

for i in the same range. For $i = m - i_{j-1}$, by the previous observation,

$$S_i'^{j} = S'^{d-1}_{i+i_{j-1}} - S'^{d-1}_{i_{j-1}-2}.$$

Continuing by induction on $i \in \{m - i_{j-1}, \dots, m-1\}$, we obtain

$$S_i'^j = S'^{d-1}_{i+i_{j-1}} - S'^{d-1}_{i_{j-1}-2}.$$

for $i \geq m - i_{j-1}$ and the first part of the claim is proved.

Suppose now $j = 0$.

By Lemma 4.1.5,

$$W_i'^0 \equiv W_i'^{d-1} - (d-1) \pmod m$$

We also know from equation 4.24 that

$$W_i'^{d-1} - d \equiv W'^{d-1}_{i-1} \pmod{m+1}$$

for $i \in \{1, \dots, m-1\}$.

Together, these two equations allow us to write

$$W_i'^0 = \begin{cases} W'^{d-1}_{i-1} + 1, & \text{if } W'^{d-1}_{i-1} < m + 1 - d \\ W'^{d-1}_{i-1}, & \text{if } W'^{d-1}_{i-1} \geq m + 1 - d \end{cases}$$

for $i \in \{1, \dots, m-1\}$.

By the hypothesis on t, we have

$$\psi(W_i'^0) = \psi(W'^{d-1}_{i-1})$$

for i in the same range.

By equation 4.24, we know that $W'^{d-1}_{m-1} = m - d$ hence $\psi(W'^{d-1}_{m-1}) > 0$.

Use now the fact that $\psi(W_0'^0) = \psi(W'^{d-1}_{m-1})$ to conclude the claim. □

The previous four claims imply that

$$\max\left(S'\right) \leq w.$$

We prove now that the width w is realised as $S_i'^j$ for some $i, j \in \{0, \dots, m-1\}$.

By definition, $w = \max(S'^{d-1}) - \min(S'^{d-1})$. Suppose that $\max(S'^{d-1}) = S'^{d-1}_{i_{max}}$ and $\min(S'^{d-1}) = S'^{d-1}_{i_{min}}$ for some $i_{max}, i_{min} \in \{0, \dots, m-1\}$.

Then

$$w = S'^{d-1}_{i_{max}} - S'^{d-1}_{i_{min}}$$

Also, by Lemma 4.0.14 there is only one value of $i \in \{0, \ldots, m-1\}$ for which $S'^j_i = w$, (independently of j).

We distinguish again several possibilities

1. $i_{min} < i_{max}$

   In this case we observe that $W'^{d-1}_i = 0$ for some $i \in \{i_{min}+1, \ldots, i_{max}\}$, because otherwise we can replace the sequence $W'^{d-1}_{i_{min}+1}, \ldots, W'^{d-1}_{i_{max}}$ with $W'^{d-1}_{i_{min}+1} - 1, \ldots, W'^{d-1}_{i_{max}} - 1$ to obtain a subsequence of W'^{d-1} with sum $\geq w$. (This new sequence of numbers is indeed a subsequence of W'^{d-1} by equation 4.24).

   Consider the number $j = W'^{d-1}_{i_{min}+1}$, equivalently $i_j = i_{min} + 1$. Claims 4.4.15 and 4.4.16 imply that one cannot have $j > d-1$, because otherwise we can find a subsequence of W'^{d-1} with sum strictly greater than w :

   Remark 4.4.19. *If we are in the case of Claim 4.4.15, the fact that $i_{min} < i_{max}$ implies that $w = S'^{d-1}_{i+i_j} - S'^{d-1}_{i_j-1}$ for some $i < m - i_j$, more precisely $i = i_{max} - i_{min} - 1$, hence we are in the branch with the strict inequality.*

   Then we must have $j \leq d-1$.

   Note that $j \neq d-1$ because otherwise we would have $i_{min} = m-1$.

   Then $j+1 \leq d-1$ and for $j+1 < d-1$ Claim 4.4.18 (applied to $j+1$) implies that we cannot have $t > m + j - (d-1)$ because of the maximality of w.

   For $j = d-2$, we can apply Claim 4.4.17 to conclude that w is achieved as $S'^{d-2}_{i_{max}-i_{min}-1}$.

   Otherwise we must have $t \leq m + j - (d-1)$ and again Claim 4.4.17 gives us a $i \in \{0, \ldots, m - i_j - 1\}$ with $w = S'^j_i$.

2. $i_{min} > i_{max}$

Here, consider $j = W'^{d-1}_{i_{min}+2}$ or $i_j = i_{min} + 2$.

If $j > d-1$, Claim 4.4.15 applies and gives a unique pair (i, j) with $i \geq m - i_j$ for which $S'^j_i = w$.

If $j < d-1$, then if $j+1 < d-1$, Claim 4.4.18 applies (to $j+1$) and again there is a unique pair (i, j) with $i \geq m - i_j$ and $S'^{j+1} = w$.

There are two values which are not covered: $j = d-1$ and $j = d-2$. We will show however that they cannot occur, i.e. S'^{d-1}_{j-2} cannot be the minimum of S'^{d-1}.

- $j = d-1$.

  Then $i_{min} = m-2$, and since S'^{d-1}_{m-2} is the minimum of S'^{d-1}, we must have $\psi(W'^{d-1}_{m-1}) > 0$, and since $W'^{d-1}_{m-1} = m-d$, we must have $t > m-d$. In particular, $d > \frac{m}{2}$. But d is a proper divisor of $2m+1$, hence $d' = 3$. We can then compute

  $$W'^{d-1}_{m-3} \equiv m - 3d \pmod{m+1}$$

  i.e. $W'^{d-1}_{m-3} = 0$. But

  $$S'^{d-1}_{i_{min}-2} = S'^{d-1}_{i_{min}} - S'^{d-1}_{i_{min}-1} - S'^{d-1}_{i_{min}} \leq S'^{d-1}_{i_{min}} - (m-t) + t$$

  contradiction with $S'^{d-1}_{i_{min}}$ being the minimum of S'^{d-1}.

- $j = d-2$

  In this case one can compute $W'^{d-1}_{i_j-1} = m-1$, hence $i_j - 2$ cannot be i_{min}, since $S'^{d-1}_{i_j-1} = S'^{d-1}_{i_j-2} + \psi(m-1)$.

For the uniqueness, consider again the two cases:

1. $i_{min} < i_{max}$

   Then the only claim that applies is 4.4.17, in the other situations there is a strict inequality $S'^j_i < w$ for $i < m - i_j$. But for different $j < d-1$,

   $$S'^j_i = S'^{d-1}_{i+i_j} - S'^{d-1}_{i_j-1}$$

   hence only for one j is the right hand side equal to w.

2. $i_{min} > i_{max}$

Here only Claims 4.4.15 and 4.4.18 can achieve w and one can see that they are mutually exclusive. One cannot have simultaneously $j > d-1$ and $j+1 < d-1$.

We have shown that w cannot be achieved as $S_i'^j$ twice for $j \neq d-1$. For $j = d-1$, $i_{min} = m-1 > i_{max}$ and then only Claims 4.4.15 and 4.4.18 can achieve w. But at a careful examination one sees that for equality in Claim 4.4.15 one would have to take $i_j = 1$ and there is practically no case $i \geq m - i_j$, and for Claim 4.4.18 one needs to take $j = 2d$, but the hypothesis of the claim is $j < d-1$. □

4.4.1 An example

The first interesting example is $m = 7$, $d = 3$ (hence $(p, q) = (49, 18)$ and $d' = 5$)

The intersections of the α and β curves are: $0, 18, 36, 5, 23, 41, 10, 28, \ldots, 31$ and the words W, resp. W' are given below

$$W_i^j = \begin{pmatrix} 0 & 2 & 5 & 0 & 3 & 5 & 1 \\ 4 & 6 & 2 & 4 & 0 & 2 & 5 \\ 1 & 3 & 6 & 1 & 4 & 6 & 2 \\ 5 & 0 & 3 & 5 & 1 & 3 & 6 \\ 2 & 4 & 0 & 2 & 5 & 0 & 3 \\ 6 & 1 & 4 & 6 & 2 & 4 & 0 \\ 3 & 5 & 1 & 3 & 6 & 1 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 3 & 5 & 1 & 4 & 6 & 2 \\ 4 & 0 & 2 & 5 & 1 & 3 & 6 \\ 1 & 4 & 6 & 2 & 5 & 0 & 3 \\ 5 & 1 & 3 & 6 & 2 & 4 & 0 \\ 2 & 5 & 0 & 3 & 6 & 1 & 4 \\ 6 & 2 & 4 & 0 & 3 & 5 & 1 \\ 3 & 6 & 1 & 4 & 0 & 2 & 5 \end{pmatrix} = W_i'^j$$

Consider $t = 3$. The sums $S_i'^j$ are

$$\begin{pmatrix} 4 & 1 & -2 & 2 & -1 & -4 & 0 \\ -3 & 1 & 5 & 2 & 6 & 3 & 0 \\ 4 & 1 & -2 & 2 & -1 & 3 & 0 \\ -3 & 1 & -2 & -5 & -1 & -4 & 0 \\ 4 & 1 & 5 & 2 & -1 & 3 & 0 \\ -3 & 1 & -2 & 2 & -1 & -4 & 0 \\ -3 & -6 & -2 & -5 & -1 & 3 & 0 \end{pmatrix}$$

4.5 Lisca' family (4_+)

Here, $p = m^2$ and $q = dm + d$ where $dd' = m + 1$ and d odd.

Theorem 4.5.1. *The sequence $(S_i)_{i=0}^{p-1}$ for a lens space $L(p, q)$ belonging to Lisca's family (4_+) achieves its maximum only once.*

As before, we will gather information about the relator W and in this particular case, we will notice that W' has similar properties with the word W' of family (3_+) and the same argument applies.

Lemma 4.5.2. *There is a cyclic permutation W' of W such that the partial sums $(S_i')_{i=mr}^{mr+m-1} = 0$, where $r \in \{0, \ldots, m-1\}$.*

Proof. Define

$$W'_{\bar{i}} = W_{\bar{i}+\bar{i_0}}$$

where $i_0 = \frac{m-d'+1}{2}$.

We will verify that the classes $\left[\frac{i_0 q}{m}\right], \left[\frac{i_0 q+q}{m}\right], \ldots, \left[\frac{i_0 q+(m-1)q}{m}\right]$ are all distinct.

Since $q = dm + d$,

$$W_i' - W_{i-1}' \in \{\widehat{d}, \widehat{d+1}\}$$

for $i \in \{1, \ldots, m-1\}$.

Claim 4.5.3. *For* $i \in \{1, \ldots, m\}$,

$$W_i - W_{i-1} = \widehat{d+1} \iff i \in \left\{d', 2d', \ldots, (d-1)d'\right\} \cup \{m\} =: A \tag{4.43}$$

Proof of claim. For $r \in \{1, \ldots, d-1\}$,

$$(rd')q \equiv (rd')d \equiv r \pmod{m}$$

and, of course,

$$mq \equiv 0 \pmod{m}$$

hence, by equation 4.11, exactly for those values of $i \in A$, $W_i - W_{i-1} = \widehat{d+1}$. □

Let

$$\widetilde{A} = \left\{i \in \{1, \ldots, m^2\} \mid \exists \imath \in A \text{ such that } i \equiv \imath \pmod{m}\right\}$$

By Lemma 4.1.5, for $i \in \{1, \ldots, m^2\}$,

$$W_i - W_{i-1} = \widehat{d+1} \iff i \in \widetilde{A}$$

After cyclically permuting by i_0, for $i \in \{1, \ldots, m-1\}$,

$$W'_i - W'_{i-1} = \widehat{d+1} \iff i \in \left(\widetilde{A} - i_0\right) \cap \{1, \ldots, m-1\} =: B$$

This set can be written as

$$B = \left\{d', 2d', \ldots, \frac{d-1}{2}d', \frac{d+1}{2}d' - 1, \left(\frac{d+1}{2} + 1\right)d' - 1, \ldots, (d-1)d' - 1\right\}$$

As for family 3_+, we will compare this sequence of numbers with the corresponding sequence for $(a_i)_{i=0}^m$, where

$$a_i \equiv d \cdot i \pmod{m-1}$$

and $a_i \in \{0, \ldots, m-2\}$, for $i \in \{0, \ldots, m-2\}$ and $a_{m-1} = m-1$ by definition.

Claim 4.5.4. *For* $i \in \{1, \ldots, m-1\}$,

$$a_i - a_{i-1} \equiv d+1 \pmod{m} \iff i \in B.$$

Proof of claim. Since $a_i - a_{i-1} \equiv d \pmod{m-1}$, for $i \in \{1, \ldots, m-1\}$, we have

$$a_i - a_{i-1} \equiv d+1 \pmod{m} \iff a_i \in \{0, \ldots, d-1\}.$$

By hypothesis, d is an *odd* divisor of $m+1$, hence $\gcd(d, m-1) = 1$, therefore the numbers a_i, for $i \in \{1, \ldots, m-1\}$, are distinct and non-zero. Since $\#(B) = d-1$, it suffices to show that for $i \in B$, $a_i \in \{1, \ldots, d-1\}$.

We can write

$$B = \left\{d'r \mid r \in \left\{1, \ldots, \frac{d-1}{2}\right\}\right\} \cup \left\{d'r - 1 \mid r \in \left\{\frac{d+1}{2}, \ldots, d-1\right\}\right\},$$

and we treat each component separately.

1. For $r \in \left\{1, \ldots, \frac{d-1}{2}\right\}$,

   $$a_{(d'r)} \equiv dd'r \equiv 2r \pmod{m-1}$$

   hence $a_{(d'r)} \in \{1, \ldots, d-1\}$.

2. For $r \in \left\{\frac{d+1}{2}, \ldots, d-1\right\}$,

   $$a_{(d'r-1)} \equiv (d'r-1)d \equiv 2r - d \pmod{m-1}$$

   and again $a_{(d'r-1)} \in \{1, \ldots, d-1\}$.

□

Putting together the claims above, we obtain

$$W_i'^0 = a_i, \tag{4.44}$$

for $i \in \{0, \ldots, m-1\}$ and the lemma is proved. □

Proof of Theorem 4.5.1. By Lemma 4.1.5, the subsequence W'^0 of W' determines the whole word W'.

By a careful investigation of the proof of Theorem 4.3.1, we observe that this special form of W'^0, described in equation 4.44 above, is all the hypothesis used, so that proof applies verbatim to our W'. □

4.5.1 An example

Let $m = 8$, $d = 3$ (hence $(p,q) = (64,27)$ and $d' = 3$) Then

$$W_i^j = \begin{pmatrix} 0 & 3 & 6 & 2 & 5 & 0 & 4 & 7 \\ 3 & 6 & 1 & 5 & 0 & 3 & 7 & 2 \\ 6 & 1 & 4 & 0 & 3 & 6 & 2 & 5 \\ 1 & 4 & 7 & 3 & 6 & 1 & 5 & 0 \\ 4 & 7 & 2 & 6 & 1 & 4 & 0 & 3 \\ 7 & 2 & 5 & 1 & 4 & 7 & 3 & 6 \\ 2 & 5 & 0 & 4 & 7 & 2 & 6 & 1 \\ 5 & 0 & 3 & 7 & 2 & 5 & 1 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 5 & 0 & 4 & 7 & 3 & 6 & 1 \\ 5 & 0 & 3 & 7 & 2 & 6 & 1 & 4 \\ 0 & 3 & 6 & 2 & 5 & 1 & 4 & 7 \\ 3 & 6 & 1 & 5 & 0 & 4 & 7 & 2 \\ 6 & 1 & 4 & 0 & 3 & 7 & 2 & 5 \\ 1 & 4 & 7 & 3 & 6 & 2 & 5 & 0 \\ 4 & 7 & 2 & 6 & 1 & 5 & 0 & 3 \\ 7 & 2 & 5 & 1 & 4 & 0 & 3 & 6 \end{pmatrix} = W_i'^j$$

Consider $t = 3$. The sums $S_i'^j$ are

$$\begin{pmatrix} 5 & 2 & 7 & 4 & 1 & -2 & -5 & 0 \\ -3 & 2 & -1 & -4 & 1 & -2 & 3 & 0 \\ 5 & 2 & -1 & 4 & 1 & 6 & 3 & 0 \\ -3 & -6 & -1 & -4 & 1 & -2 & -5 & 0 \\ -3 & 2 & -1 & 4 & 1 & -2 & 3 & 0 \\ 5 & 2 & -1 & -4 & -7 & -2 & -5 & 0 \\ -3 & -6 & -1 & -4 & 1 & -2 & 3 & 0 \\ -3 & 2 & -1 & 4 & 1 & 6 & 3 & 0 \end{pmatrix}$$

4.6 Lisca's family (4_-)

Here, we have $q = dm - d$ for some $d > 1$ odd and divisor of $m - 1$, and let $d' > 0$ be the quotient $d' = \frac{m-1}{d}$. This family is similar to family 3_-.

Lemma 4.6.1. *There is a cyclic permutation W' of W such that the partial sums $(S_i')_{i=mr}^{mr+m-1} = 0$, where $r \in \{0, \ldots, m-1\}$.*

Proof. Let $i_0 = \frac{m-d'+1}{2}$. We will verify that the classes

$$\left[\frac{i_0 q}{m}\right], \left[\frac{(i_0+1)q}{m}\right], \ldots, \left[\frac{(i_0+(m-1))q}{m}\right]$$

are distinct. Since $q = dm - d$, $W_i - W_{i-1} \in \left\{\widehat{d}, \widehat{d-1}\right\}$.

Claim 4.6.2. *For* $i \in \{1, \ldots, m\}$,

$$W_i - W_{i-1} = \widehat{d-1} \iff i \in \left\{1, d'+1, 2d'+1, \ldots, m-d'\right\} =: A.$$

Proof of claim. By equation (4.21), the conclusion is equivalent to

$$W_i \in \{-\widehat{1}, \ldots, -\widehat{d}\} \iff i \in A.$$

Note that the cardinality of these sets is d and since the classes $(W_i)_{i=1}^m$ are distinct, it is sufficient to show one inclusion.

Let $i \in A$. Then $i = rd' + 1$, for some $r \in \{0, \ldots, d-1\}$. We have

$$(rd'+1)(dm-d) \equiv -(rd'+1)d \equiv r - d \pmod{m}$$

and the claim is proved. □

As before, define

$$\widetilde{A} = \left\{i \in \{1, \ldots, m^2\} \mid \exists \imath \in A \text{ such that } i \equiv \imath \pmod{m}\right\}$$

Then, by Lemma 4.1.5, we have

$$W_i - W_{i-1} = \widehat{d-1} \iff i \in \widetilde{A}.$$

Consider now W' defined by $W'_{\overline{i}} = W_{\overline{i}+\overline{i_0}}$, for $i \in \left\{0, \ldots, m^2-1\right\}$. From the previous observation we deduce that for $i \in \{1, \ldots, m-1\}$,

$$W'_i - W'_{i-1} = \widehat{d-1} \iff i \in \widetilde{A} \cap \{1, \ldots, m-1\} =: B. \tag{4.45}$$

Explicitly, the set on the right is

$$B = \left\{d', 2d', \ldots, \frac{d-1}{2}d', \frac{d+1}{2}d' + 1, \left(\frac{d+1}{2} + 1\right)d' + 1, \ldots, (d-1)d' + 1\right\}.$$

Consider now the sequence of number $(a_i)_{i=0}^{m} \in \{0, \dots, m\}$ defined by:

$$\begin{cases} a_i \equiv (d-1) + i \cdot d \pmod{m+1}, & \text{if } i < m \\ a_m = m \end{cases}$$

Since $a_i - a_{i-1} \equiv d \pmod{m+1}$, we deduce that $a_i - a_{i-1} \equiv d$ or $d-1 \pmod{m}$.

Claim 4.6.3. *For $i \in \{1, \dots, m-1\}$,*

$$a_i - a_{i-1} \equiv d-1 \pmod{m} \iff i \in B.$$

Proof of claim. It is easy to see that

$$a_i - a_{i-1} \equiv d-1 \pmod{m} \iff a_i \in \{0, \dots, d-1\}$$

Note that the numbers a_i, for $i \in \{0, \dots, m\}$, are distinct, since $\gcd(d, m+1) = 1$ (because d is a divisor of $m-1$ and d is odd).

Since $a_0 = d-1$, $a_i \in \{0, \dots, d-2\}$, for $i \in \{1, \dots, m-1\}$.

Note that $\#B = d-1$ hence by the previous observations, it is enough to verify that $a_i \in \{0, \dots, d-2\}$, for $i \in B$.

Since

$$B = \left\{ rd' \mid r \in \left\{1, \dots, \frac{d-1}{2}\right\}\right\} \cup \left\{ rd' + 1 \mid r \in \left\{\frac{d+1}{2}, \dots, d-1\right\}\right\}$$

we distinguish two cases:

1. $r \in \{1, \dots, \frac{d-1}{2}\}$

   Then, by the definition of a_i,

   $$a_{(rd')} = (d-1) - 2r \in \{0, \dots, d-2\}$$

2. $r \in \{\frac{d+1}{2}, \dots, d-1\}$

   In this case,

   $$a_{(rd'+1)} = 2d - 1 - 2r \in \{0, \dots, d-2\}$$

□

The previous claim and equation (4.45) imply that

$$W_i'^{d-1} \equiv (d-1) + i \cdot t \pmod{m+1} \tag{4.46}$$

for $i \in \{0, \ldots, m-1\}$. □

Theorem 4.6.4. *The sequence $(S_i')_{i=0}^{m-1}$ corresponding to the word W' for a lens space in Lisca's family 4_- achieves its maximum only once.*

Proof. By the proof of the previous Lemma, we have found that the subsequence W'^{d-1} has exactly the same form (cf.. equations (4.45) and (4.24)) as the subsequence W'^{d-1} of family 3_-. By examining the proof of Theorem 4.4.1, we see that this is what we use from the hypothesis $dd' = 2m+1$, except for dealing with the special values $j = d-1, d-2$, so that proof applies without changes to our sequence W'^{d-1}. We treat the two cases here, the point is to show that $i_{min} \neq i_j - 2$.

- $j = d-1$

  Then $\psi(W_{m-1}'^{d-1}) > 0$, i.e. $m - d < t$ and in particular $d > \frac{m}{2}$. But this is impossible since d is a proper divisor of $m-1$.

- $j = d-2$

  Again, $W_{i_j-1}'^{d-1} = m-1$, so $i_j - 2$ cannot be i_{min} since $S_{i_j-1}'^{d-1} < S_{i_j-2}'^{d-1}$.

□

4.6.1 An example

The first interesting example is $m = 7, d = 3$, but the lens space obtained, namely $L(49, 18)$, belongs to family 3_- as well.

We illustrate then the next example $m = 10, d = 3$, which gives the lens space $L(100, 27)$.

$$W_i^j = \begin{pmatrix} 0 & 2 & 5 & 8 & 0 & 3 & 6 & 8 & 1 & 4 \\ 7 & 9 & 2 & 5 & 7 & 0 & 3 & 5 & 8 & 1 \\ 4 & 6 & 9 & 2 & 4 & 7 & 0 & 2 & 5 & 8 \\ 1 & 3 & 6 & 9 & 1 & 4 & 7 & 9 & 2 & 5 \\ 8 & 0 & 3 & 6 & 8 & 1 & 4 & 6 & 9 & 2 \\ 5 & 7 & 0 & 3 & 5 & 8 & 1 & 3 & 6 & 9 \\ 2 & 4 & 7 & 0 & 2 & 5 & 8 & 0 & 3 & 6 \\ 9 & 1 & 4 & 7 & 9 & 2 & 5 & 7 & 0 & 3 \\ 6 & 8 & 1 & 4 & 6 & 9 & 2 & 4 & 7 & 0 \\ 3 & 5 & 8 & 1 & 3 & 6 & 9 & 1 & 4 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 3 & 6 & 8 & 1 & 4 & 7 & 9 & 2 & 5 \\ 7 & 0 & 3 & 5 & 8 & 1 & 4 & 6 & 9 & 2 \\ 4 & 7 & 0 & 2 & 5 & 8 & 1 & 3 & 6 & 9 \\ 1 & 4 & 7 & 9 & 2 & 5 & 8 & 0 & 3 & 6 \\ 8 & 1 & 4 & 6 & 9 & 2 & 5 & 7 & 0 & 3 \\ 5 & 8 & 1 & 3 & 6 & 9 & 2 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 & 3 & 6 & 9 & 1 & 4 & 7 \\ 9 & 2 & 5 & 7 & 0 & 3 & 6 & 8 & 1 & 4 \\ 6 & 9 & 2 & 4 & 7 & 0 & 3 & 5 & 8 & 1 \\ 3 & 6 & 9 & 1 & 4 & 7 & 0 & 2 & 5 & 8 \end{pmatrix} = W_i'^j$$

Consider $t = 3$. The sums $S_i'^j$ are

$$\begin{pmatrix} 7 & 4 & 1 & -2 & 5 & 2 & -1 & -4 & 3 & 0 \\ -3 & 4 & 1 & -2 & -5 & 2 & -1 & -4 & -7 & 0 \\ -3 & -6 & 1 & 8 & 5 & 2 & 9 & 6 & 3 & 0 \\ 7 & 4 & 1 & -2 & 5 & 2 & -1 & 6 & 3 & 0 \\ -3 & 4 & 1 & -2 & -5 & 2 & -1 & -4 & 3 & 0 \\ -3 & -6 & 1 & -2 & -5 & -8 & -1 & -4 & -7 & 0 \\ 7 & 4 & 1 & 8 & 5 & 2 & -1 & 6 & 3 & 0 \\ -3 & 4 & 1 & -2 & 5 & 2 & -1 & -4 & 3 & 0 \\ -3 & -6 & 1 & -2 & -5 & 2 & -1 & -4 & -7 & 0 \\ -3 & -6 & -9 & -2 & -5 & -8 & -1 & 6 & 3 & 0 \end{pmatrix}$$

4.7 Proof of the fibredness theorem for simple knots

Summing up the analysis for each family of lens spaces above, we arrive at the

Proof of Theorem 4.0.9. For each of Lisca's families of lens spaces, we proved (cf. Theorems 4.1.1, 4.2.1, 4.3.1, 4.4.1, 4.5.1 and 4.6.4) that the sequence $(S_i)_{i=0}^{p-1}$ (or the induced sequence of sums associated to a circular permutation of W) achieves its maximum exactly once. By Lemma 4.0.15, the sequence achieves its minimum exactly once. Now apply Brown's Theorem 1.2.14 and Stallings' Theorem (Stallings, 1962) to get the desired conclusion. □

4.8 Towards the classification of simple knots of genus 0 in Lisca's lens spaces

In this section we gather our partial results regarding the genus of the simple knots in Lisca's families above and speculate about the general picture.

Conjecture 4.8.1. *Given a lens space $L(m^2, q)$ belonging to one of Lisca's families above, the simple knot $K(m^2, q, tm)$ (see Remark 4.0.12 for the explanation of this notation) has a planar (genus 0) Seifert surface if and only if:*

- *Family 1.* $t \in \{1, d, m-1, m-d\}$
- *Family 2.* $t \in \{1, m-1\}$
- *Family* 3_+. $t \in \{1, d, m-1, m-d\}$
- *Family* 3_-.
  - $t \in \{1, d, m-1, m-d\}$ *or*
  - $(m, d, t) \in \{(7, 3, 2), (7, 5, 3)\}$
- *Family* 4_+.
  - $t \in \{1, d, m-1, m-d\}$ *or*
  - $m = 2d - 1$ *and* $t \in \{2, m-2\}$
- *Family* 4_-.
  - $t \in \{1, d, m-1, m-d\}$ *or*
  - $m = 2d + 1$ *and* $t \in \{2, m-2\}$

Remark 4.8.2. *It is interesting to compare the list above with the knots obtained by Baker (Baker, 2012) as the induced knots from performing surface-framed surgery on doubly-primitive knots in $S^1 \times S^2$.*

We present now some evidence supporting Conjecture 4.8.1.

A brief computer experimentation showed that

Proposition 4.8.3. *For $m \leq 500$, the conjecture is true.*

Proof. Simple knots are combinatorial objects, described as above by 3 natural numbers: p, q and t, cf. Remark 4.0.12. The sequence used in Brown's theorem is algorithmically computable from p, q and t by modular arithmetic. Also, claims 4.8.6 and 4.8.7 below show that the genus of the simple knot is also encoded in this sequence. It is straightforward to enumerate all simple knots in lens spaces up to some fixed order and check for each the genus ok the simple knots in the relevant homology classes. We used code written in C. □

Remark 4.8.4. *Note that the proposition applies for lens spaces of order up to* 250000.

Theorem 4.8.5. *For Lisca's families* 1 *and* 2, *Conjecture 4.8.1 is true.*

Proof. We begin by explaining the similarity between the Brown algorithm and Heegaard-Floer homology in establishing the fibredness of simple knots in lens spaces, cf. Remark 4.0.11.

Recall the setup from Section 4, where we consider a genus 1 Heegaard diagram for $Y := L(p, q)$, with two curves α and β intersecting transversely in p points, denoted $\overline{0}, \ldots, \overline{p-1}$. These points also represent generators of the complex $\widehat{CFK}(Y)$. When considered in the *doubly-pointed* Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$, these points represent generators for $\widehat{CFK}(Y, K)$ and even for $\widehat{HFK}(Y, K)$, since there are no differentials. For consistency with the established notation in Chapter 2, we will also denote these points by $\mathbf{x}_i$, $i \in \{0, \ldots, p-1\}$ or even $\mathbf{x}_{\bar{i}}$ with the obvious interpretation. Refer to figure 4 for a concrete example.

By the discussion done in Section 4, Brown's algorithm leads us to the analysis of the following sequence:

$$S_i = \sum_{j=0}^{i} \theta(\overline{jq})$$

where $\theta \colon \mathbb{Z}/_p \longrightarrow \mathbb{Z}$ is given by

$$\theta(\bar{i}) = \begin{cases} m-t & \text{if } \bar{i} \in \{\bar{0}, \ldots, \overline{tm-1}\} \\ -t & \text{if } \bar{i} \in \{\overline{tm}, \ldots, \overline{p-1}\} \end{cases} \tag{4.47}$$

Equivalently, with the notation from Section 4, $\theta(\bar{i}) = \psi\left(\left[\frac{\bar{i}}{m}\right]\right)$.

On the Heegaard-Floer side, we will use the ϵ grading 2.3.3 to compute the evaluation (up to an overall additive constant) of the relative Chern classes in which the points $\mathbf{x}_i$ are supported on the Seifert surface of K.

Note that the points $\mathbf{x}_i$, when recorded in the order in which they appear on the β curve, form the sequence $\mathbf{x}_{\bar{0}}, \mathbf{x}_{\bar{q}}, \ldots, \mathbf{x}_{\overline{(p-1)q}}$.

Fix $\bar{i} \in \mathbb{Z}/_p$. By definition 2.3.3, $\epsilon(\mathbf{x}_{\bar{i}}, \mathbf{x}_{\overline{i+q}})$ is the homology class (in $H_1(Y \setminus K)$) of a path in $\alpha \cup \beta$ starting at $\mathbf{x}_{\bar{i}}$, walking along α until $\mathbf{x}_{\overline{i+1}}$ and returning along β. But by the definition of x and y in $\pi_1(Y \setminus K)$, we have

$$\epsilon(\mathbf{x}_{\bar{i}}, \mathbf{x}_{\overline{i+q}}) = \begin{cases} -y, & \text{if } i \in \{0, \ldots, tm-1\} \\ x, & \text{if } i \in \{tm, \ldots, p-1\} \end{cases}$$

It is known (see the discussion in (Boileau et al., 2011)) that for two Spin^c structures $\mathfrak{s}_1, \mathfrak{s}_2 \in \text{Spin}^c(M)$, where M is a compact, oriented three-manifold with boundary (if any) consisting of tori, we have

$$c_1(\mathfrak{s}_2) - c_1(\mathfrak{s}_1) = 2 \cdot (\mathfrak{s}_2 - \mathfrak{s}_1).$$

By Poincaré duality,

$$\begin{cases} \langle [F], x\rangle = -t \\ \langle [F], y\rangle = m - t \end{cases}$$

These values are the images of x, resp. y in $\mathbb{Z} \cong H_1(Y \setminus K)/_{tors}$. The previous three equations give

$$\langle c_1(s_{w,z}(\mathbf{x}_{\overline{i+q}})), [F]\rangle - \langle c_1(s_{w,z}(\mathbf{x}_{\bar{i}})), [F]\rangle = \begin{cases} 2(m-t), & \text{if } i \in \{0, \dots, tm-1\} \\ -2t, & \text{if } i \in \{tm, \dots, p-1\} \end{cases}$$

Compare this to equation (4.47) to obtain

$$w\left((S_i)_{i=0}^{p-1}\right) = w\left(\widehat{HFK}(Y, K)\right) \tag{4.48}$$

by Lemma 2.7.3, we can compute the genus of a simple knot in a lens space:

Claim 4.8.6. *Let $K(p,q,s) \subset L(p,q)$ be a simple knot in the lens space $Y = L(p,q)$. Let k be the order of K and let F be a minimal genus Seifert surface for K. Then*

$$\chi(F) = k - w\left((S_i)_i\right) \tag{4.49}$$

where $(S_i)_{i=0}^{p-1}$ is the sequence of partial sums obtained by applying Brown's algorithm to K, as above.

Proof of claim. By equation (4.48),

$$w\left((S_i)_i\right) = \max_{\left\{\mathfrak{s} \mid \widehat{HFK}(Y,K,\mathfrak{s}) \neq 0\right\}} \langle c_1(\mathfrak{s}), [F]\rangle - \min_{\left\{\mathfrak{s} \mid \widehat{HFK}(Y,K,\mathfrak{s}) \neq 0\right\}} \langle c_1(\mathfrak{s}), [F]\rangle$$

where max, resp. min are taken over the set of relative Spin^c structures $\underline{\mathrm{Spin}}^c(Y, K)$.

Let $\xi_M \in \mathrm{Spin}^c(Y, K)$ be a relative Spin^c structure which realises the maximum evaluation above.

By Theorem 1.1 of (Ni, 2009),

$$-\chi(F) + k = \langle c_1(\xi_M), [F]\rangle - k.$$

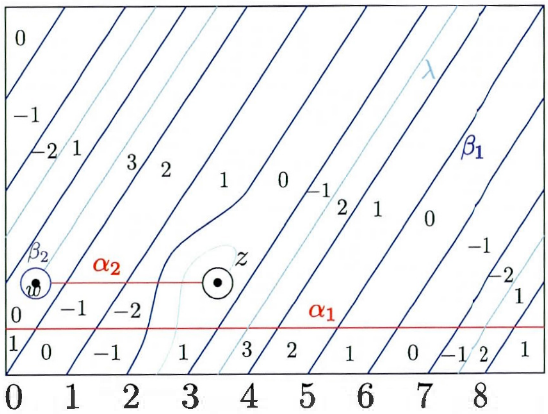

Figure 4.3 Brown's algorithm and Spin^c structures

Also, by Lemma 4.1 of the same paper, $\xi_m := J(\xi_M) + PD[\mu]$ realises the minimum. Now, by definition,

$$w\left(\widehat{HFK}(Y,K)\right) = \frac{\langle c_1(\xi_M),[F]\rangle - \langle c_1(\xi_m),[F]\rangle}{2}$$

and since

$$\langle c_1(J(\xi_M) + PD[\mu]),[F]\rangle = \langle c_1(J(\xi_M)) + 2\cdot PD[\mu],[F]\rangle = -\langle c_1(\xi_M),[F]\rangle + 2k$$

we obtain

$$\langle c_1(\xi_M),[F]\rangle = w(\widehat{HFK}(Y,K)) + k$$

and inserting this into equation (4.49), we obtain the desired formula. See figure 4.8 for a concrete example.

□

We can apply the formula above in two situations that are particularly relevant to our problem:

Claim 4.8.7. *Let $K(m^2,q,tm) \subset L(m^2,q)$ be a simple knot of order m in a lens space of order m^2 with the rational longitude a longitude. Then*

$$g(K) = 0 \iff w\left((S_i)_i\right) = 2m-2$$

where $(S_i)_{i=0}^{m^2-1}$ is the sequence of partial sums given by Brown's algorithm.

Proof of claim. Simply apply equation (4.49) to K. The hypothesis that the rational longitude of K is a longitude implies that the Seifert surface of K has m boundary components. □

Claim 4.8.8. *Let $K(m,d,t) \subset L(m,d)$ be a* primitive *simple knot in a lens space. Then*

$$K \text{ is a core of } L(m,d) \iff w\left((S_i)_i\right) = m-1.$$

where, as above, $(S_i)_{i=0}^{m-1}$ is the sequence of partial sums given by Brown's algorithm.

Proof of claim. The direct implication is trivially verified. For the converse, by the width hypothesis, we obtain that K has a Seifert surface of Euler characteristic 1, which can only be a disk. Also, primitive simple knots are fibred, (Ozsváth and Szabó, 2005) hence K must be a core of $L(p,q)$. □

We can now finish the proof of Theorem 4.8.5. For family 1, by Theorem 4.1.4 and the two claims above

$$K(m^2, dm+1, tm) \text{ has genus } 0 \iff K(m,d,t) \text{ is a core } of L(m,d).$$

Now just observe that the cores of $L(m,d)$ are $K(m,d,1)$ and $K(m,d,d)$ (and by changing orientation also $K(m,d,m-1)$ and $K(m,d,m-d)$).

For family 2, Lemma 4.2.10 together with Claim 4.8.7 give the conclusion.

□

We summarise now the relevant results proved in our analysis of the words W_i' for the families 3 and 4.

Lemma 4.8.9. *Let $Y := L(m^2,q)$ be a lens space belonging to Lisca's families 3_+ or 4_+, i.e. $q = dm + d$ for some d with some divisibility properties cf. 4.0.4. Let $(a_i)_{i=0}^{m-1}$ be a sequence of numbers with the properties:*

- $a_i \in \{0,\ldots,m-2\}$, *for* $i \in \{0,\ldots,m-2\}$

- $a_i \equiv i \cdot d \pmod{m-1}$, $i \in \{0, \ldots, m-2\}$
- $a_{m-1} = m-1$

For $t \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor\}$, *with* $\gcd(m,t) = 1$, *let* $\phi\colon \{0, \ldots, m-1\} \longrightarrow Z$ *be the function*

$$\phi(s) = \begin{cases} m-t, & \text{if } s \in \{0, \ldots, t-1\} \\ -t, & \text{if } s \in \{t, \ldots, m-1\} \end{cases}$$

and let w *be the width of the sequence* $(\Sigma_i)_{i=0}^{m-1}$ *given by*

$$\Sigma_i = \sum_{j=0}^{i} \phi(a_i).$$

Then

$$w(\widehat{HFK}(Y, K(m^2, q, tm))) = 2w.$$

Remark 4.8.10. *The sequence above does not come from applying Brown's algorithm to a simple knot in a lens space, but we observed that it is related to the Brown sequence for the knot* $K(m-1, d, t)$. *We hope to come back to this question in a future work.*

Proof of Lemma 4.8.9. It follows from (the proof of the) Theorems 4.3.1 and 4.5.1 that

$$\max{(S_i)_{i=0}^{p-1}} = w$$

We show now that this implies that

$$\min{(S_i)_{i=0}^{p-1}} = -w.$$

This happens because of a symmetry satisfied by the numbers a_i above. More precisely, we will prove that for $j \in \{0, \ldots, m-1\}$, $\exists\, j' \in \{0, \ldots, m-1\}$ with the property

$$\psi(W_i'^{j}) = \psi(W_{m-2-i}'^{j'}).$$

Claim 4.8.11. *Recall the following set defined in 4.3 for Lisca's family* 3_+ *and in 4.5 for family* 4_+.

$$B = \left\{ i \in \{1, \ldots, m-1\} \,\middle|\, a_i - a_{i-1} \equiv d+1 \pmod{m} \right\}.$$

For both of the families the following is true:

$$i \in B \iff m - i \in B.$$

Proof of claim. We treat each family separately:

1. Family 3_+

   Recall that $i_0 = \frac{d'+1}{2}$ where $dd' = 2m - 1$ and

   $$B = \left\{ ri_0 - \left\lfloor \frac{r}{2} \right\rfloor \; r \in \{1, \ldots, d-1\} \right\}$$

   Now note that for $r \in \{1, \ldots, d-1\}$, $d - r \in \{1, \ldots, d-1\}$ and

   $$ri_0 - \left\lfloor \frac{r}{2} \right\rfloor + (d-r)i_0 - \left\lfloor \frac{d-r}{2} \right\rfloor = di_0 - \frac{d-1}{2} = m$$

2. Family 4_+

   Here $i_0 = \frac{m-d'+1}{2}$ where $dd' = m + 1$ and d is odd.

   $$B = \left\{ d'r \,|r \in \left\{1, \ldots, \frac{d-1}{2}\right\}\right\} \cup \left\{ d'r - 1 \,|r \in \left\{\frac{d+1}{2}, \ldots, d-1\right\}\right\}.$$

   and a simple computation shows that, for $r \in \{1, \ldots, \frac{d-1}{2}\}$,

   $$d'r + d'(d-r) - 1 = dd' - 1 = m.$$

□

Note that this implies that, for $i \in \{0, \ldots, m-1\}$ and $j \in \{-i, -i+1, \ldots, 0, 1, \ldots, m-1-i\}$,

$$W'^0_{i+j} - W'^0_i = W'^0_{m-1-i} - W'^0_{m-1-i-j}$$

by finite induction on j.

The same argument as in the proof of Lemma 4.0.15 shows that $\psi(\widehat{a}) = \psi(\widehat{t-1-a})$ for $a \in \mathbb{Z}$.

Now fix $j \in \{0, \dots, m-1\}$ and let $\imath_0 \in \{0, \dots, m-1\}$ be the index of $\widehat{0}$ in W'^j. There exists a unique $j' \in \{0, \dots, m-1\}$ with the property $W'^{j'}_{m-1-\imath_0} = \widehat{t-1}$. Together with the previous equality, this implies that

$$\psi(W'^j_i) = \psi(W'^{j'}_{m-1-i}) \tag{4.50}$$

for $i \in \{0, \dots, m-1\}$. To see this, note that it is true for $i = \imath_0$ and then use induction on $i - \imath_0$.

Then,

$$S'^j_i = \sum_{k=0}^{i} \psi(W'^j_k) = \sum_{k=m-1-i}^{m-1} \psi(W'^{j'}_k) = -S'^{j'}_{m-2-i}.$$

□

Similarly, we have:

Lemma 4.8.12. *Let $Y := L(m^2, q)$ be a lens space belonging to Lisca's families 3_- or 4_-, i.e. $q = dm - d$ for some d cf. 4.0.4. Let $(a_i)_{i=0}^{m-1}$ be a sequence of numbers with the properties:*

- *$a_i \in \{0, \dots, m-1\}$, for $i \in \{0, \dots, m-1\}$*
- *$a_i \equiv i \cdot (d-1) + i \cdot d \pmod{m+1}$, $i \in \{0, \dots, m-1\}$*

For $t \in \{1, \dots, \lfloor \frac{m}{2} \rfloor\}$, with $\gcd(m, t) = 1$, let $\phi\colon \{0, \dots, m-1\} \longrightarrow Z$ be the function

$$\phi(s) = \begin{cases} m - t, & \textit{if } s \in \{0, \dots, t-1\} \\ -t, & \textit{if } s \in \{t, \dots, m-1\} \end{cases}$$

and let w be the width of the sequence $(S_i)_{i=0}^{m-1}$ given by

$$S_i = \sum_{j=0}^{i} \phi(a_i).$$

Then

$$w(\widehat{HFK}(Y, K(m^2, q, tm))) = 2w.$$

Proof. As before, it follows from (the proof of the) Theorems 4.4.1 and 4.6.4 together with equation (4.48) that $\max(S_i) = w$. We will show, using the same argument as in the previous lemma, that $\min(S_i) = -w$.

Claim 4.8.13. *Consider the sets B defined in 4.4 and 4.6. Then*

$$i \in B \iff m - i \in B.$$

Proof of claim.

1. Family 3_-

   Cf. Section 4.4, $dd' = 2m + 1$, and $i_0 = \frac{d'+1}{2}$. By equation (4.23),

   $$B = \left\{ r \cdot i_0 - \left\lfloor \frac{r+1}{2} \right\rfloor \mid r \in \{1, \ldots, d-1\} \right\}$$

   and we observe that

   $$ri_0 - \lfloor \frac{r+1}{2} \rfloor + (d-r)i_0 - \lfloor \frac{d-r+1}{2} \rfloor = di_0 - \frac{d+1}{2} = m$$

2. Family 4_-

   In this case, $dd' = m - 1$, $i_0 = \frac{m-d'+1}{2}$ From section 4.6,

   $$B = \left\{ rd' \mid r \in \left\{1, \ldots, \frac{d-1}{2}\right\}\right\} \cup \left\{ rd' + 1 \mid r \in \left\{\frac{d+1}{2}, \ldots, d-1\right\}\right\}$$

   we obtain, for $r \in \{1, \ldots, \frac{d-1}{2}\}$,

   $$rd' + (d-r)d' + 1 = dd' + 1 = m.$$

□

Now simply apply the argument from Lemma 4.8.9 to obtain the desired conclusion.

□

Proposition 4.8.14. *A fibred knot $K \subset Y = L(p, q)$ has an $S^1 \times S^2$ surgery if and only if $g(K) = 0$.*

Proof. Let F be a Seifert surface for K. Recall that only surgery along the slope $\lambda = \partial F$ yields a non-rational homology three-sphere.

For the direct implication, let $(F_t)_{t \in S^1}$ be the generic fibre in the fibration of $Y \setminus N(K)$. Note that $g(F_t) = g(F), \forall t \in S^1$. We cap off every fibre F_t in $Y_\lambda(K)$ with meridinal disks of the surgery solid torus, and obtain a fibration $\overline{F}_t$ of $S^1 \times S^2$. Then $\overline{F}_t$ has to be the two-sphere.

The converse follows similarly, $Y_\lambda(K)$ becomes an oriented S^2 bundle over S^1, hence it is homeomorphic to $S^1 \times S^2$. $\square$

CONCLUSION

From the results in Chapters 3 and 4, a similarity with the Berge Conjecture emerges. Knots in lens spaces which admit integer S^3 or $S^1 \times S^2$ surgeries are fibred, have small genus and simple Floer homology. By this work and work of Baker, we can say that their status is roughly the same: in both cases we have strong enough restrictions so that conjecturally the restrictions determine the knots, they are both implied by the conjecture that Floer simple knots in lens spaces are simple, hence it is somewhat natural to expect that they are simultaneously true or false.

We remark here that an arbitrary Berge-Gabai knot standardly embedded in one of the Heegaard tori of a lens space has non-trivial lens space surgeries. A brief computer experimentation using Brown's algorithm showed that 'most of the time' the knot in the lens space is *not* fibred. For example, the Berge-Gabai knot $B(5,2,3)$ is not fibred when standardly embedded in $L(15,11)$.

We also remark that arbitrary simple knots in lens spaces are not fibred. There are two special situations however. One is the case of primitive knots, which were shown to be fibred by Ozsváth-Szabó, and the other is the case when K is a knot of order m in a lens space of order m^2. It seems plausible that these knots are again fibred. This special case deserves some more analysis in our opinion. We plan to investigate the problem further.

In another direction, it may be true that a knot K in an L-space which admits longitudinal $S^1 \times S^2$ surgeries is fibred, and hence it is a braid in $S^1 \times S^2$.

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