UNIVERSITÉ DU QUÉBEC À MONTRÉAL

COMBINATOIRE DES ESPACES COINVARIANTS TRIVARIÉS DU GROUPE SYMÉTRIQUE

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RÉSUMÉ

Ce travail traite principalement de l'énumération d'extensions de structures combinatoires classiques appelées chemins de Dyck et fonctions de stationnement. Ces structures, très étudiées en raison de leur rôle fondamental dans de multiples contextes combinatoires, sont aussi étroitement liées à la théorie de la représentation, à la théorie des fonctions symétriques et à la géométrie algébrique, entre autres. En particulier, elles sont liées à l'étude combinatoire des espaces coinvariants diagonaux $\mathcal{DR}_{k,n}$, introduits par Garsia et Haiman, qui sont des représentations du groupe symétrique $\mathfrak{S}_n$ sur des espaces de polynômes en k jeux de n variables.

Dans le cas bivarié, il a été conjecturé que les séries de Hilbert des espaces coinvariants diagonaux « augmentés » $\mathcal{DR}^m_{2,n}$ et de leur sous-représentation signe $\mathcal{DR}^{m\ \varepsilon}_{2,n}$ sont respectivement égales à une somme de statistiques sur les fonctions de m-stationnement et sur les chemins de m-Dyck. Il existe également une conjecture plus générale pour la série de Frobenius de ces espaces qui s'appelle la conjecture « shuffle » (Haglund et al., 2005).

Dans le cas trivarié, Haiman a conjecturé en 1994 les dimensions suivantes :

$$\dim(\mathcal{DR}_{3,n}{}^{\varepsilon}) = \frac{2}{n(n+1)}\binom{4n+1}{n-1} \qquad \text{et} \qquad \dim(\mathcal{DR}_{3,n}) = 2^n(n+1)^{n-2}.$$

D'autre part, en 2006, Chapoton a démontré que les intervalles dans le treillis de Tamari sont comptés par $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$. Motivé par ses travaux sur le cas trivarié et par les deux conjectures précédentes, Bergeron a introduit le treillis de m-Tamari et étendu certaines questions concernant les chemins de m-Dyck et les fonctions de m-stationnement pour rendre compte du cas trivarié. Il a conjecturé que

$$\dim(\mathcal{DR}^m_{3,n}{}^{\varepsilon}) = \frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1},$$

que

$$\dim(\mathcal{DR}^m_{3,n}) = (m+1)^n(mn+1)^{n-2},$$

et que ces deux cardinalités comptent respectivement les intervalles et les intervalles de stationnement du treillis de m-Tamari.

Dans cette thèse, nous démontrons une généralisation commune de ces deux conjectures énumératives que nous avons énoncée avec Bergeron. Plus précisément, avec Mireille Bousquet-Mélou et Guillaume Chapuy, nous avons démontré que la série de Frobenius d'une certaine représentation combinatoire sur les intervalles de stationnement

du treillis de m-Tamari est donnée par :

$$\sum_{\lambda=(\lambda_1,..,\lambda_\ell)\vdash n} (mn+1)^{\ell-2} \prod_{1\leq i\leq \ell} \binom{(m+1)\lambda_i}{\lambda_i} \frac{p_\lambda}{z_\lambda}.$$

Cette démonstration équivaut à résoudre un nouveau type d'équations différentielles à variable catalytique. Toujours avec Bergeron, nous avons conjecturé que le produit tensoriel de cette représentation combinatoire et de la représentation signe ε est isomorphe à $\mathcal{DR}^m_{3,n}$. Nous avons également formulé une généralisation de la conjecture « shuffle » en proposant une formule combinatoire explicite pour la série de Frobenius graduée de $\mathcal{DR}^m_{3,n}$. Ceci renforce notre hypothèse que l'étude des intervalles du treillis de m-Tamari est bel et bien en lien avec l'étude des espaces $\mathcal{DR}^m_{3,n}$.

Mots clés : combinatoire algébrique, combinatoire énumérative, représentations du groupe symétrique, fonctions génératrices, statistiques.

COMBINATORICS OF THE TRIVARIATE DIAGONAL COINVARIANT SPACES OF THE SYMMETRIC GROUP

ABSTRACT

This work is concerned mainly with the enumeration of classical combinatorial structures called Dyck paths and parking functions. These structures, well studied because of their fundamental role in multiple combinatorial contexts, are closely related to certain problems of representation theory, symmetric function theory and algebraic geometry, among others. In particular, they are closely related to the combinatorial study of the coinvariant spaces $\mathcal{DR}_{k,n}$, introduced by Garsia and Haiman, which are representations of the symmetric group $\mathfrak{S}_n$ on spaces of polynomials in k sets of n variables.

In the bivariate case, it is conjectured that the Hilbert series of the "higher" coinvariant spaces of the symmetric group $\mathcal{DR}^m_{2,n}$ and its set of alternates $\mathcal{DR}^{m\ \varepsilon}_{2,n}$ are respectively equal to a sum of statistics on m-parking functions and m-Dyck paths. There also exists a more general conjecture for the Frobenius series of these spaces which is called the "shuffle" conjecture (Haglund et al., 2005).

In the trivariate case, Haiman conjectured in 1994 the following dimensions :

$$\dim(\mathcal{DR}_{3,n}{}^{\varepsilon}) = \frac{2}{n(n+1)}\binom{4n+1}{n-1} \qquad \text{and} \qquad \dim(\mathcal{DR}_{3,n}) = 2^n(n+1)^{n-2}.$$

In parallel, in 2006, Chapoton showed that the intervals in the m-Tamari lattice are counted by $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$. Motivated by his work on the trivariate case and by the two preceding conjectures, Bergeron introduced the m-Tamari lattice and extended some constructions concerning m-Dyck paths and m-parking functions to account for the trivariate case. He conjectured that

$$\dim(\mathcal{DR}^{m\ \varepsilon}_{3,n}) = \frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1},$$

that

$$\dim(\mathcal{DR}^m_{3,n}) = (m+1)^n(mn+1)^{n-2},$$

and that these two numbers respectively count the intervals and the labelled intervals in the m-Tamari lattice.

In this thesis, we demonstrate a common generalization of these two enumerative conjectures that we have proposed together with Bergeron. More precisely, with Mireille Bousquet-Mélou and Guillaume Chapuy, we have shown that the Frobenius series of a certain combinatorial representation on the labelled intervals of the m-Tamari lattice is given by :

$$\sum_{\lambda=(\lambda_1,..,\lambda_\ell)\vdash n} (mn+1)^{\ell-2} \prod_{1\leq i\leq \ell} \binom{(m+1)\lambda_i}{\lambda_i} \frac{p_\lambda}{z_\lambda}.$$

This demonstration involves the solution of a new kind of differential-catalytic equations. Still with Bergeron, we have conjectured that the tensorial product of this combinatorial representation and the sign representation ε is isomorphic to $\mathcal{DR}^m_{3,n}$, and proposed a "shuffle-like" conjecture for the graded Frobenius series of $\mathcal{DR}^m_{3,n}$. This reinforces our hypothesis that the study of intervals in the m-Tamari lattice is closely linked to the study of the spaces $\mathcal{DR}^m_{3,n}$.

Keywords : algebraic combinatorics, enumerative combinatorics, representations of the symmetric group, generating functions, statistics.

CHAPITRE I

NOTIONS DE BASE

Le matériel de ce chapitre s'inspire des présentations classiques qu'on peut trouver dans les livres (Goulden et Jackson, 2004), (Sagan, 2001), (Stanley, 1997) et (Stanley, 1999). Tous les énoncés sont bien connus.

1.1 Groupe symétrique

Soit $n \in \mathbb{N}$, où $\mathbb{N}$ est l'ensemble des nombres entiers ≥ 0. On dénote par $\mathfrak{S}_n$ le *groupe symétrique à n éléments*. Ce dernier est composé des bijections, appelées permutations, partant de l'ensemble des nombres $\{1, 2, ..., n-1, n\}$ sur lui-même. L'opération sur ce groupe est la composition de fonctions. Il est trivial que la cardinalité $|\mathfrak{S}_n|$ de l'ensemble $\mathfrak{S}_n$ est égale à $n!$. Nous utiliserons parfois la notation en ligne d'une permutation et celle en produit de cycles disjoints. Par exemple, soit la permutation $\sigma \in \mathfrak{S}_n$ donnée par $\sigma(1) = 2, \sigma(2) = 5, \sigma(3) = 3, \sigma(4) = 6, \sigma(5) = 1, \sigma(6) = 4$. Alors la notation en ligne de σ est donnée par 253614 et son produit en cycles disjoints par $(1, 2, 5)(3)(4, 6)$. Soit $\mathrm{Mat}_{\mathfrak{S}_n}$ l'ensemble des matrices de dimensions $n \times n$ dont les éléments appartiennent à l'ensemble $\{0, 1\}$ et dont l'élément 1 apparaît une seule fois dans chaque colonne et dans chaque ligne. Il n'est pas difficile de voir que l'ensemble $\mathrm{Mat}_{\mathfrak{S}_n}$, avec comme opération le produit de matrices, est un groupe isomorphe à $\mathfrak{S}_n$.

Pour définir les fonctions symétriques, nous aurons besoin du *groupe symétrique infini* $\mathfrak{S}$. Ce dernier est composé des bijections partant de l'ensemble des nombres entiers

strictement positifs $\mathbb{P}$ vers lui-même, et laissant tous les éléments fixés sauf un nombre fini. L'opération sur ce groupe est également la composition de fonctions.

1.2 Fonctions génératrices

Soit $\{F_n\}_{n\geq 0}$ une suite de familles d'objets tels que $|F_n| < \infty$ pour tous les $n \in \mathbb{N}$. Soit t une indéterminée. On peut encoder de l'information concernant les F_n en utilisant des séries formelles en t. Une façon est de garder en information les cardinalités des F_n de la manière suivante :

$$F(t) = \sum_{n\geq 0} |F_n|\, t^n.$$

Nous appelons cette série $F(t)$ la *fonction génératrice ordinaire* des $\{F_n\}$ (ou série ordinaire). On considère aussi la *fonction génératrice exponentielle* (ou série exponentielle) :

$$F(t) = \sum_{n\geq 0} |F_n| \frac{t^n}{n!}.$$

Ces fonctions génératrices permettent de traduire des récurrences satisfaites par des structures combinatoires en termes d'équations fonctionnelles.

1.3 Inversion de Lagrange et théorème de Puiseux

Soit K un corps commutatif de caractéristique 0. Les *séries formelles sur* K en t que l'on dénote $K[[t]]$ sont les séries de la forme $\sum_{i\geq 0} k_i t^i$ où les $k_i \in K$. Le coefficient k_i de t^i de la série formelle $\sum_{i\geq 0} k_i t^i$ est dénoté par $[t^i] \sum_{i\geq 0} k_i t^i$. Le théorème d'inversion de Lagrange s'énonce ainsi :

Théorème 1 (inversion de Lagrange) *Supposons que* $G(t) \in K[[t]]$ *telle que* $G(0) \neq 0$ *et soit* $f(t)$ *définie implicitement par* $f(t) = tG(f(t))$. *Alors pour chaque* $H(t) \in K[[t]]$ *on a*

$$n[t^n]H(f(t)) = [t^{n-1}]H'(t)G(t)^n. \tag{1.1}$$

Soit $\mathbb{Z}$ l'ensemble des entiers. Plus tard, nous aurons besoin des *séries formelles de Laurent sur* K en t. Ces dernières sont dénotées par $K((t))$. Elles sont les séries en t qui

s'écrivent de la forme $\sum_{i\geq l} k_i t^i$, pour un certain $l \in \mathbb{Z}$. Comme auparavant, $[t^j]\sum_{i\geq l} k_i t^i$ symbolise le coefficient k_j de t^j.

Supposons en plus que K est algébriquement clos. Soit $K^{\text{fra}}((t)) = K((t))[t^{1/2}, t^{1/3}, ...]$ les séries fractionnaires de Laurent en t (sur K). Le théorème de Puiseux s'énonce ainsi :

Théorème 2 (théorème de Puiseux) *$K^{\text{fra}}((t))$ est algébriquement clos.*

1.4 Fonctions symétriques

Cette section n'est qu'un aperçu très sommaire de la théorie des fonctions symétriques (pour plus d'informations voir le chapitre 7 du livre (Stanley, 1999)).

Avant de définir les fonctions symétriques, nous introduisons les compositions et les partages. Soit $n \in \mathbb{N}$. Une *composition de n* est une somme ordonnée $n = \alpha_1 + \alpha_2 + ... + \alpha_\ell$ dont les parts sont des entiers strictement positifs. Nous utilisons la notation $\alpha = (\alpha_1, ..., \alpha_\ell)$ pour dénoter les compositions, et $\alpha \models n$ symbolise que α est une composition de n. Une *composition généralisée de n* est une somme ordonnée d'entiers ≥ 0 (possiblement infinie). Nous utiliserons la notation $\alpha = (\alpha_1, \alpha_2, \alpha_3, ...)$ pour les compositions généralisées. Par exemples, (4,0,1), (0,2,3) et (0,2,0,1,2,0,0,$\bar{0}$,...) sont des compositions généralisées de 5.

Les partages sont des cas particuliers des compositions. Un *partage* de n est une composition $\alpha = (\alpha_1, ..., \alpha_\ell)$ dont la liste des parts décroît ($\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_\ell \geq 1$). Nous utilisons souvent λ ou μ comme notation pour les partages. Nous décrivons les partages de deux façons. Pour la premiére, soit $\lambda = (\lambda_1, \lambda_2, ..., \lambda_\ell)$ tel que $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_\ell \geq 1$ et $\lambda_1 + \lambda_2 + ... + \lambda_\ell = n$. Le nombre de parts de λ est désigné par $\ell(\lambda)$ ($= \ell$). À titre d'exemple, les partages de 5 sont les suivants : $(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1)$ et $(1,1,1,1,1)$. Le nombre de parts de $(2,1,1,1)$ est de $\ell(2,1,1,1) = 4$. La notation $\lambda \vdash n$ signifie que λ est un partage de n. Pour $\lambda = (\lambda_1, \lambda_2, ..., \lambda_\ell)$, une deuxième description consiste à écrire $\lambda = \langle 1^{m_1} 2^{m_2} ... n^{m_n} \rangle$, avec $m_i = |\{\lambda_j = i : 1 \leq j \leq \ell\}|$. Pour une composition généralisée α, $\lambda(\alpha)$ symbolise l'unique partage qui réordonne les parts strictement

positives de α de manière décroissante, éliminant les parts '0".

Soit $w = w_1, w_2, w_3, ...$ une liste infinie d'indéterminées. Une *fonction symétrique homogène de degré n*, sur un anneau commutatif R (avec identité), est une série formelle

$$S(w) = \sum_{\alpha} c_\alpha w^\alpha$$

où la somme a lieu sur l'ensemble des compositions généralisées $\alpha = (\alpha_1, \alpha_2, ...)$ de n, avec

1. $c_\alpha \in R$,
2. w^α désigne le monôme $w_1^{\alpha_1} w_2^{\alpha_2} w_3^{\alpha_3} ...$,
3. $f(w_1, w_2, w_3, ...) = f(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}, ...)$ pour toutes les permutations σ appartenant au groupe symétrique infini $\mathfrak{S}$.

On désigne par Λ_R^n l'ensemble des fonctions symétriques sur R (avec les indéterminées w) de degré n et par Λ_R l'ensemble de toutes les fonctions symétriques sur R. On a alors $\Lambda_R = \bigoplus_{n \geq 0} \Lambda_R^n$ et cet ensemble constitue une algèbre (graduée) lorsque qu'il est muni des opérations d'addition et de multiplication usuelles sur les séries formelles. Pour le reste de ce texte, on utilise $R = \mathbb{C}$, et pour simplifier la notation, on écrit Λ à la place de $\Lambda_{\mathbb{C}}$ et Λ^n à la place de $\Lambda_{\mathbb{C}}^n$.

Les bases des fonctions symétriques présentées ci-dessous nous seront utiles plus tard. Nous renvoyons le lecteur à (Stanley, 1999) pour les preuves et autres propriétés de ces dernières. Pour $\lambda \vdash n$, la *fonction symétrique monomiale* $m_\lambda(w) \in \Lambda^n$ est définie par

$$m_\lambda(w) = \sum_{\alpha} w^\alpha,$$

où la somme a lieu sur l'ensemble des compositions généralisées $\alpha = (\alpha_1, \alpha_2, ...)$ (il y en a un nombre infini) telles que $\lambda(\alpha) = \lambda$. Il est facile de montrer que

Théorème 3 (classique) *L'ensemble* $\{m_\lambda(w) : \lambda \vdash n\}$ *est une base de* Λ^n *(en tant qu'espace vectoriel) et donc sa dimension* $\dim(\Lambda^n)$, *est égale au nombre de partages de* n.

Pour $\lambda = (\lambda_1, \lambda_2, ..., \lambda_\ell)$, la *fonction symétrique élémentaire* $e_\lambda(w) \in \Lambda^n$ est définie par

$$e_\lambda(w) = e_{\lambda_1}(w)...e_{\lambda_\ell}(w),$$

où

$$e_n(w) = \begin{cases} m_{1^n}(w) = \sum_{1 \le i_1 < i_2 < ... < i_n} w_{i_1} w_{i_2} ... w_{i_n}, & \text{si } n \ge 1, \\ 1, & \text{si } n = 0. \end{cases}$$

Ceci est équivalent à dire que $e_n(w) = [t^n] \prod_{i \ge 1} (1 + w_i t)$.

La *fonction symétrique complète* $h_\lambda(w) \in \Lambda^n$ est définie par

$$h_\lambda(w) = h_{\lambda_1}(w)...h_{\lambda_\ell}(w),$$

où

$$h_n(w) = \begin{cases} \sum_{\lambda \vdash n} m_\lambda(w) = \sum_{1 \le i_1 \le i_2 \le ... \le i_n} w_{i_1} w_{i_2} ... w_{i_n}, & n \ge 1, \\ 1, & n = 0. \end{cases}$$

Ceci est équivalent à dire que $h_n(w) = [t^n] \prod_{i \ge 1} \frac{1}{1 - w_i t}$.

La *fonction symétrique de puissance* $p_\lambda(w) \in \Lambda^n$ est définie par

$$p_\lambda(w) = p_{\lambda_1}(w)...p_{\lambda_\ell}(w),$$

où

$$p_n(w) = \begin{cases} \sum_{i \ge 1} w_i^n, & n \ge 1, \\ 1, & n = 0. \end{cases}$$

Les fonctions symétriques de Schur sont un peu plus compliquées à définir et il y a plusieurs manières de le faire. Nous en donnons ici une définition combinatoire. Nous devons d'abord introduire la notion de tableau semi-standard.

Le diagramme de Ferrers d'un partage $\lambda = (\lambda_1, ..., \lambda_\ell) \vdash n$ est le sous-ensemble de $\mathbb{N}^2$ contenant les couples $\{(i, j) : 0 \le i \le \lambda_j - 1\}$. Un tel sous-ensemble peut être symbolisé par un agencement de cases dont la première rangée à partir du bas a longueur λ_1, la deuxième rangée a longueur λ_2, ..., et la $\ell^{\text{ème}}$ rangée a longueur λ_ℓ, et tel que les premières cases de chaque rangée appartiennent à une même colonne (voir la figure

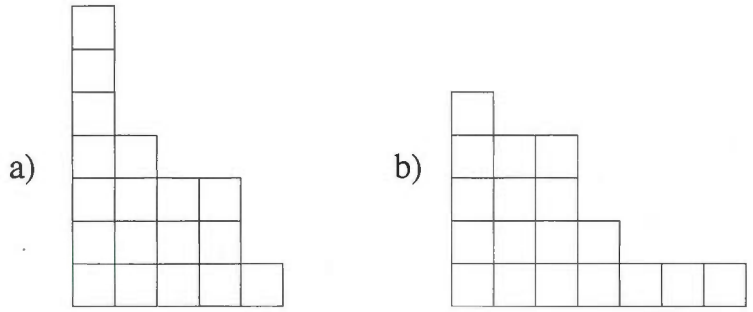

Figure 1.1 a) Diagramme de Ferrers du partage $\lambda = (5,4,4,2,1,1,1)$. b) Diagramme de Ferrers de $\lambda' = (7,4,3,3,1)$.

7			
4	7		
2	6	6	6
1	3	3	4

Figure 1.2 Tableau semi-standard de forme $\lambda = (4,4,2,1)$.

1.1 pour un exemple de diagramme de Ferrers sous forme d'un agencement de cases). Le partage conjugué de λ est la liste (décroissante) des hauteurs des colonnes dans le diagramme de Ferrers de λ, lorsque ce dernier est symbolisé par un agencement de cases (voir la figure 1.1 pour un exemple de partage conjugué).

Un tableau semi-standard de forme λ est un étiquetage des cases du diagramme de Ferrers de forme λ avec des nombres strictement positifs de manière qu'il y ait croissance stricte sur les colonnes (de bas en haut) et croissance sur les lignes de gauche à droite (voir la figure 1.2). Un tableau de forme $\lambda \vdash n$ est dit standard s'il est semi-standard et s'il est rempli avec toutes les valeurs dans {1,2,...,n} (il y a donc croissance stricte sur les lignes également). L'ensemble de ces tableaux standards est dénoté par SYD(λ) et celui de ces tableaux semi-standards par SSYD(λ). Un exemple de tableau semi-standard est donné à la figure 1.2.

5			
4	4	9	
2	3	5	5

Figure 1.3 Tableau semi-standard, avec comme monôme associé $w_2 w_3 w_4^2 w_5^3 w_9$.

Soit $T \in \mathrm{SSYD}(\lambda)$, alors on dénote w^T par

$$w^T := \prod_{i \in T} w_i.$$

Nous donnons un exemple à la figure 1.3.

La *fonction (symétrique) de Schur* $s_\lambda(w)$ se définit de la manière suivante :

$$s_\lambda(w) := \sum_{T \in \mathrm{SSYD}(\lambda)} w^T.$$

Il n'est pas difficile de démontrer combinatoirement que les fonctions de Schur ainsi définies sont symétriques. Nous énonçons le théorème suivant sans preuve.

Théorème 4 (classique) *Les ensembles* $\{m_\lambda(w) : \lambda \vdash n\}$, $\{e_\lambda(w) : \lambda \vdash n\}$, $\{h_\lambda(w) : \lambda \vdash n\}$, $\{p_\lambda(w) : \lambda \vdash n\}$ *et* $\{s_\lambda(w) : \lambda \vdash n\}$ *forment tous des bases de* Λ^n. *Autrement dit, en tant qu'algèbre,* Λ *est librement engendrée par* $\{e_k(w) : k \geq 1\}$, *ou* $\{h_k(w) : k \geq 1\}$, *ou* $\{p_k(w) : k \geq 1\}$.

Pour terminer cette section, nous présentons une involution ω sur les fonctions symétriques et deux identités bien simples qui relient les $h_\lambda(w)$ et les $e_\lambda(w)$ aux $p_\lambda(w)$. À partir de maintenant, nous omettrons d'écrire les indéterminées dans les fonctions symétriques lorsque cela sera clair.

L'involution ω est l'automorphisme d'algèbre définit en posant

$$\omega(p_k) = \begin{cases} (-1)^{k-1} p_k, & \forall k \geq 1 \\ 1, & \text{si } k = 0. \end{cases} \tag{1.2}$$

On trouve facilement que

$$h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda},$$

où $z_\lambda = \prod_{1 \leq i \leq n} i^{m_i} m_i!$ (avec $\lambda = \langle 1^{m_1} 2^{m_2} ... n^{m_n} \rangle$), puisque

$$\begin{aligned} h_n &= [t^n] \prod_{i \geq 1} \frac{1}{1 - w_i t} = [t^n] \exp\left(\log \prod_{i \geq 1} \frac{1}{1 - w_i t} \right) = [t^n] \exp\left(\sum_{i \geq 1} \sum_{j \geq 1} \frac{(w_i t)^j}{j} \right) \\ &= [t^n] \exp\left(\sum_{j \geq 1} \sum_{i \geq 1} \frac{(w_i t)^j}{j} \right) = [t^n] \exp\left(\sum_{j \geq 1} \frac{p_j}{j} t^j \right) \\ &= \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}. \end{aligned}$$

De même manière, on trouve que

$$e_n = \sum_{\lambda \vdash n} (-1)^{n - l(\lambda)} \frac{p_\lambda}{z_\lambda}.$$

Il en résulte que

$$\omega(h_\lambda) = e_\lambda.$$

1.4.1 Produit scalaire sur les fonctions symétriques

Le *produit scalaire de Hall* sur les fonctions symétriques est la forme bilinéaire symétrique $\Lambda \times \Lambda \to \mathbb{C}$ définie par

$$\langle s_\lambda, s_\beta \rangle = \delta_{\lambda = \beta}, \tag{1.3}$$

où λ, β sont des partages et pour n'importe quel énoncé logique L

$$\delta_L = \begin{cases} 1 & \text{si } L \text{ est vrai} \\ 0 & \text{si } L \text{ est faux.} \end{cases}$$

Nous notons sans démonstration (voir (Stanley, 1999)) que cette définition du produit scalaire (de Hall) est équivalente à chacune des deux suivantes

$$\langle h_\lambda, m_\beta \rangle = \delta_{\lambda = \beta}, \tag{1.4}$$

$$\langle p_\lambda, p_\beta \rangle = z_\lambda \delta_{\lambda = \beta}. \tag{1.5}$$

1.5 Fonctions quasi-symétriques

Une *fonction quasi-symétrique* sur les indéterminées $w = w_1, w_2, ...$ est une série formelle $q(w) \in \mathbb{C}[[w_1, w_2, ...]]$ telle que $\forall a_1, ..., a_k \in \mathbb{P}$ et $\forall k \in \mathbb{P}$:

$$[w_{i_1}^{a_1} ... w_{i_k}^{a_k}]q(w) = [x_{j_1}^{a_1} ... x_{j_k}^{a_k}]q(w), \ (\forall\ 1 \leq i_1 < ... < i_k \ et \ \forall\ 1 \leq j_1 < ... < j_k).$$

Soit $\alpha = (\alpha_1, ..., \alpha_\ell) \models n$. La *fonction quasi-symétrique monomiale de type* α est définie par

$$M_\alpha(w) = \sum_{i_1 < ... < i_\ell} w_{i_1}^{\alpha_1} ... w_{i_\ell}^{\alpha_\ell}$$

Il est clair que l'ensemble $\{M_\alpha(w) : \alpha \models n\}$ est une base de l'espace vectoriel des fonctions quasi-symétriques homogènes de degré n.

Soit $S_\alpha = \{\alpha_1, \alpha_1+\alpha_2, ..., \alpha_1+...+\alpha_{\ell-1}\}$. *Les fonctions quasi-symétriques fondamentales de type* α sont définies par [1]

$$Q_\alpha(w) = \sum_{\substack{i_1 \leq i_2 \leq ... \leq i_n \\ i_j < i_{j+1} \ \forall j \in S_\alpha}} w_{i_1} ... w_{i_n}.$$

En combinant l'argument qui permet de voir que ces séries formelles sont des fonctions quasi-symétriques et le principe d'inclusion-exclusion, il n'est pas difficile de vérifier que l'ensemble $\{Q_\alpha : \alpha \models n\}$ est une base des fonctions quasi-symétriques homogènes de degré n. Cette base sera utile pour énoncer une conjecture plus tard.

1.6 Représentations de groupes finis

Soit $d \in \mathbb{P}$. Dans toute cette thèse, G sera un groupe fini et GL_d symbolisera le groupe des matrices $(x_{ij})_{d \times d}$ sur $\mathbb{C}$, inversibles sous la multiplication. Une *représentation de* G est un homomorphisme de groupe $X : G \to GL_d$. Notons que GL_d agit sur $\mathbb{C}^d$ par multiplication sur la gauche.

---

1. Stanley utilise la notation L_α pour ces dernières alors que Haglund utilise la notation Q_{n,S_α}.

Pour $d_1, d_2 \in \mathbb{P}$, soit $X : G \to GL_{d_1}$ et $Y : G \to GL_{d_2}$ des représentations de G. Ces représentations X et Y sont dites *isomorphes* si $d_1 = d_2$ et s'il existe une matrice $C \in GL_{d_1}$ telle que $CX(g)C^{-1} = Y(g)$ pour tous les $g \in G$.

De part le fait que le groupe GL_d est isomorphe au groupe $GL(V)$ des automorphismes linéaires sur V, où V est un $\mathbb{C}$-espace vectoriel de dimension d, on peut définir une représentation de G comme étant un homomorphisme de groupe $\rho : G \to GL(V)$. Cette définition équivalente permet de définir une représentation lorsque V est de dimension infinie. Soit V' un sous-espace vectoriel de $\mathbb{C}^d$ et $G(V')$ défini par $G(V') = \rho(G)(V') = \{\rho(g)(v') : g \in G, v' \in V'\}$. Un sous-espace V' de $\mathbb{C}^d$ (muni de l'action $\rho(G)$ sur ce dernier) tel que $G(V') = V'$ est dit une *sous-représentation* de la représentation ρ sur $V = \mathbb{C}^d$. Une représentation est dite *irréductible* si pour tout V' un sous-espace vectoriel de $\mathbb{C}^d$ tel que $G(V') = V'$, alors $V' = \{0\}$ ou $V' = \mathbb{C}^d$. Les théorèmes suivants sont classiques.

Théorème 5 *Toute représentation se décompose de manière unique en une somme directe de représentations irréductibles (à isomorphisme près, et à l'ordre près des composantes).*

Théorème 6 *Le nombre de représentations irréductibles distinctes (à isomorphisme près) d'un groupe G est égal au nombre de classes de conjugaison du groupe.*

Rappellons que les classes de conjugaison de $\mathfrak{S}_n$ sont naturellement caractérisées par les partages de n, ces partages décrivant la structure cyclique. Il existe une façon d'indexer les classes d'isomorphies des représentations irréductibles de $\mathfrak{S}_n$ par les partages de n. Selon cette indexation, on a (pour plus de détails, voir (Sagan, 2001)) :

$$\dim(S_\lambda) = |\mathrm{SYD}(\lambda)|, \tag{1.6}$$

où $\dim(S_\lambda)$ est la dimension de S_λ. Dans le cas du partage (1^n), S_{1^n} est appelée la représentation signe et nous utiliserons parfois le symbole ε comme notation équivalente.

Ainsi si R est une représentation du groupe symétrique, alors nous utiliserons la notation R^ε pour représenter la sous-représentation signe de R.

Une représentation combinatoire est une représentation de $\mathfrak{S}_n$ isomorphe à une représentation de la forme $X : \mathfrak{S}_n \to \mathrm{Mat}_{\mathfrak{S}_d}$ pour un certain $d \in \mathbb{P}$.

1.6.1 Caractère d'une représentation de dimension finie

Pour le reste de ce texte, G désigne un groupe fini. Pour une matrice M carrée, $\mathrm{Tr}(M)$ désigne la trace de M. Une *fonction de classe* sur G est une fonction $\psi : G \to \mathbb{C}$ constante sur les classes de conjugaison de G. Soit $Y : G \to GL_d$ une représentation de G. Le *caractère* de cette représentation est donné par la fonction $\chi : G \to \mathbb{C}$ définie en posant que $\chi(g) = \mathrm{Tr}(Y(g))$. En prenant A, B des matrices carrées de mêmes dimensions, il est bien connu que $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Il en résulte que le caractère d'une représentation est constant à isomorphisme près. Le théorème suivant montre l'équivalence entre les caractères et les représentations.

Théorème 7 *Deux représentations de G sont isomorphes si et seulement si elles ont le même caractère.*

1.7 Séries de Hilbert

Soit V un espace vectoriel (sur $\mathbb{C}$). Une *graduation* de V est une décomposition de V en une somme directe de sous-espaces indexés par l'ensemble $\mathbb{N}^r$ pour un certain $r \in \mathbb{N}$. Dans notre cas, ces sous-espaces sont de dimensions finies. On écrit alors $V = \bigoplus_{s \in \mathbb{N}^r} V_s$. On associe à un espace gradué la série de Hilbert de V donnée par

$$\mathcal{H}(V; q) := \sum_{s \in \mathbb{N}^r} \dim(V_s) q^s,$$

où $q^s = q_1^{s_1} q_2^{s_2} \ldots q_r^{s_r}$.

1.8 Séries de Frobénius graduées

La *caractéristique de Frobenius* (voir (Sagan, 2001)) transforme les fonctions de classe de $\mathfrak{S}_n$ en fonctions symétriques de degré n. Plus spécifiquement, pour ψ une fonction de classe sur $\mathfrak{S}_n$, on pose

$$\mathrm{ch}^n(\psi) := \sum_{\mu \vdash n} \psi_\mu \frac{p_\mu(w)}{z_\mu},$$

où ψ_μ est la valeur de la fonction de classe ψ sur une permutation de type μ. Nous utiliserons la terminologie *série de Frobenius d'une représentation de* $\mathfrak{S}_n$ pour la caractéristique de Frobenius de son caractère. Ainsi, si R est une représentation de $\mathfrak{S}_n$ et son caractère est donné par χ, alors la série de Frobenius de R est :

$$\mathcal{F}(R) := \mathrm{ch}^n(\chi).$$

Le théorème suivant est classique.

Théorème 8 *Soit $n \in \mathbb{N}$. La caractréristique de Frobenius est un isomorphisme linéaire qui envoie les caractères des représentations irréductibles de $\mathfrak{S}_n$ vers les fonctions symétriques de Schur. Plus spécifiquement,*

$$\mathcal{F}(S_\lambda) = s_\lambda,$$

où λ est un partage de n.

Soit R une représentation graduée de $\mathfrak{S}_n$, $R = \bigoplus_{s \in \mathbb{N}^r} R_s$, où tous les R_s sont des représentations de $\mathfrak{S}_n$ de dimensions finies. La *série de Frobénius graduée* de R est définie comme

$$\mathcal{F}(R; q) := \sum_{s \in \mathbb{N}^r} \mathcal{F}(R_s) q^s,$$

où $q^s = q_1^{s_1} q_2^{s_2} \ldots q_r^{s_r}$.

Soit $(Y_n)_{n \geq 0}$ une séquence (infinie) où Y_n est une représentation de $\mathfrak{S}_n$. Nous appellons l'expression suivante la *série raffinée de Frobenius* (non-graduée) :

$$\sum_{n \geq 0} \mathcal{F}(Y_n) t^n.$$

Cette dernière sera utile pour calculer résursivement la série de Frobenius d'une certaine représentation combinatoire, et obtenir par le fait même des théorèmes d'énumération difficiles.

1.9 Ensembles partiellement ordonnés et treillis

Un *ensemble partiellement ordonné (poset)* est un ensemble P muni d'une relation d'ordre $\leq$. Soient $x, y, z \in P$. On dit que x est minimum (resp. maximum) si $x \leq y$ (resp. $x \geq y$) pour tous les $y \in P$. S'il existe, le minimum (resp. maximum) est unique et est dénoté par $\hat{0}$ (resp. $\hat{1}$). Une *chaîne* de longueur k dans un poset P est une séquence $x_0 < x_1 < ... < x_k$ où les x_i sont des éléments de P. On dit que y *couvre* x s'il n'existe aucun z tel que $x < z < y$. Il est clair qu'un poset est engendré par la fermeture transitive de ses relations de couverture. Une *multichaîne* de longueur k dans P est une séquence $x_0 \leq x_1 \leq ... \leq x_k$. Une multichaîne de longueur 1 dans P est un *intervalle* de P. Deux posets P et Q sont dits *isomorphes* s'il existe une bijection entre les éléments de P et de Q telle que les relations soient préservées par cette bijection. Le *dual* P^* d'un poset P est le poset obtenu en munissant l'ensemble sous-jacent à P de la relation d'ordre telle que $y \leq x$ dans P^* si et seulement si $x \leq y$ dans P. Un poset est dit *auto-dual* s'il est isomorphe à son dual.

Soit $x, y \in P$. Une *borne supérieure* (resp. inférieure) de x et y est un élément $z \in P$ tel que $z \geq x$, $z \geq y$, et tel que $t \geq z$ (resp. $t \leq z$) pour tout t pour lequel $t \geq x$ et $t \geq y$ (resp. $t \leq x$ et $t \leq y$). Si elle existe, il est clair que cette dernière est unique et elle est dénotée par $x \vee y$ (resp. $x \wedge y$). On dit d'un poset P qu'il est un treillis s'il existe une borne inférieure et une borne supérieure pour toute paire d'éléments de P.

1.10 Structures combinatoires

Les principaux résultats de cette thèse consistent à énumérer des structures combinatoires qui en généralisent d'autres appartenant à la famille des « structures de Catalan ». Ces dernières ont été étudiées de façon extensive. Les chemins de m-Dyck et les fonc-

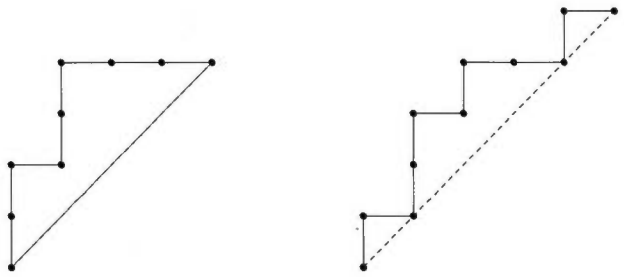

Figure 1.4 Un chemin de Dyck de hauteur 4 et un autre de hauteur 5.

tions de m-stationnement, définis ci-dessous, font partie de cette famille des structures de Catalan (voir (Stanley, 1999) pour une liste de plus d'une centaine de structures de Catalan).

1.10.1 Chemins de m-Dyck et fonctions de m-stationnement

Les *chemins de Dyck* sont des chemins dans $\mathbb{N} \times \mathbb{N}$, constitués de pas $(0,1)$ (aussi appelés pas nord) et de pas $(1,0)$ (aussi appelés pas est), qui commencent en $(0,0)$, se terminent en (n,n) et ne descendent jamais sous la droite $x = y$. On dénote par Dyck(n) l'ensemble des chemins de Dyck constitués de n pas nord (voir la figure 1.4). Les chemins appartenant à Dyck(n) sont dits de hauteur n.

Une formule élégante permet d'énumérer ces derniers. Elle peut s'obtenir via le lemme cyclique (Riordan, 1969) dont en voici une explication très sommaire d'un cas particulier. Les chemins de Dyck de hauteur n peuvent être mis en bijection avec des mots circulaires comportant n lettres N et $n+1$ lettres E. Cette bijection envoie la séquence de pas nord et de pas est d'un chemin de Dyck vers une séquence de lettres N et E telle que chacun des pas nord (resp. est) est remplacé par la lettre N (resp. E), en ajoutant une lettre E à la fin. L'ordre linéaire sur les lettres est ensuite remplacé par un ordre cyclique. Il n'est pas difficile de démontrer que ces mots circulaires sont comptés par

$$\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

Ces nombres sont appellés les *nombres de Catalan*. Ils comptent plusieurs structures

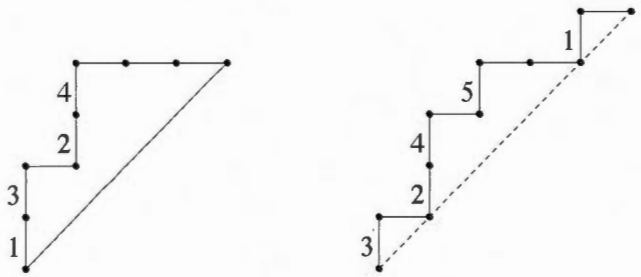

Figure 1.5 Une fonction de stationnement de hauteur 4 et une autre de hauteur 5.

combinatoires, et interviennent en combinatoire algébrique de plusieurs manières. Nous reviendrons sur une de ces instances dans les chapitres suivants.

Les *fonctions de stationnement* (dénotées par l'ensemble Park(n)) s'obtiennent en étiquetant bijectivement les n pas nord d'un chemin de Dyck de hauteur n par les entiers dans $\{1, 2, ..., n\}$, de manière à ce que les étiquettes sur des pas nord consécutifs croissent (voir la figure 1.5).

Ces fonctions de stationnement peuvent être également comptées par le lemme cyclique. En utilisant une transformation similaire à celle sur les chemins de Dyck donnée précédemment, on peut construire une bijection entre les fonctions de stationnement et les ordres circulaires de $n+1$ boîtes qui contiennent les entiers $\{1, 2, ..., n\}$.

Ces derniers sont facilement comptés par

$$(n+1)^{n-1}.$$

Les *chemins de m-Dyck* de hauteur n (dénotés par l'ensemble $\text{Dyck}_m(n)$) sont les chemins dans le quart de plan constitués de pas nord et de pas est qui commencent en $(0,0)$ se terminent en (mn, n), et ne descendent jamais sous la droite $x = my$ (voir la figure 1.6).

Le lemme cyclique permet encore de démontrer que les chemins de m-Dyck sont comptés

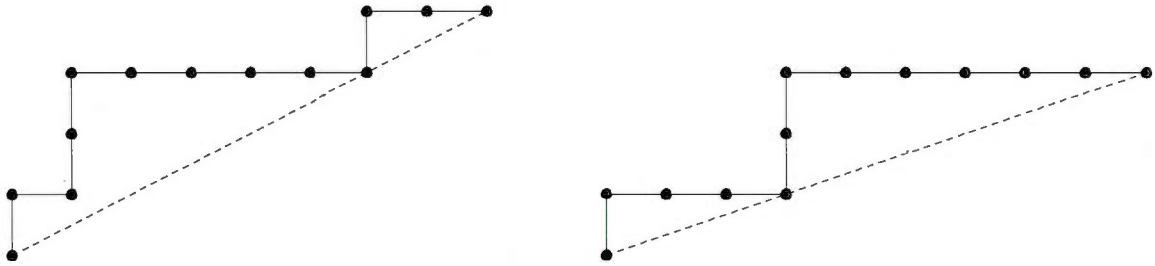

Figure 1.6 Le membre de gauche appartient à $\mathrm{Dyck}_2(4)$ et celui de droite à $\mathrm{Dyck}_3(3)$.

par

$$\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n} = \frac{1}{mn+1}\binom{(m+1)n}{n}. \tag{1.7}$$

Soit

$$f(t) = \sum_{n\geq 0} |\mathrm{Dyck}_m(n)|t^n$$

la fonction génératrice (ordinaire) des chemins de m-Dyck. Alors il n'est pas trop difficile d'établir l'équation fonctionnelle

$$f(t) = \frac{1}{1-tf(t)^m}. \tag{1.8}$$

En utilisant le théorème d'inversion de Lagrange, on peut obtenir facilement une autre preuve que les chemins de m-Dyck sont comptés par la formule 1.7. Un chemin de m-Dyck est dit *primitif*, s'il ne touche la droite $x = my$ qu'à ses deux extrémités. Il n'est pas difficile de démontrer que ces derniers sont comptés par

$$\frac{1}{(m+1)n-1}\binom{(m+1)n-1}{n}.$$

Les *fonctions de m-stationnement* (dénotées par l'ensemble $\mathrm{Park}_m(n)$) sont définies de la même manière que les fonctions de stationnement (voir la figure 1.7 pour un exemple) et elles sont comptées par

$$(mn+1)^{n-1}. \tag{1.9}$$

Soit $g(t)$ la fonction génératrice (exponentielle) des fonctions de m-stationnement. De

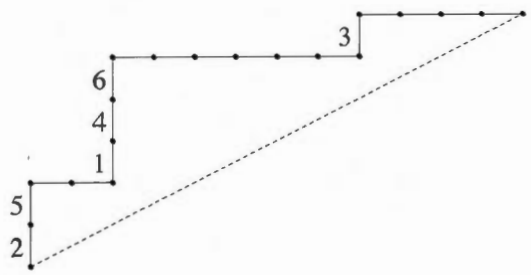

Figure 1.7 Une fonction de 2-stationnement appartenant à $\mathrm{Park}_2(6)$.

manière similaire au cas précédent, on peut obtenir l'équation fonctionnelle

$$g(t) = e^{t(g(t))^m}. \tag{1.10}$$

De part l'inversion de Lagrange, on obtient une autre preuve de la formule 1.9. Les fonctions de m-stationnement *primitives* sont les fonctions de m-stationnement définies sur des chemins de m-Dyck primitifs. Ces dernières sont comptées par la formule

$$(mn-1)^{n-1}.$$

1.10.2 Treillis de Tamari

L'ensemble partiellement ordonné (poset) de Tamari a comme éléments les chemins de Dyck et est caractérisé par les relations de couverture suivantes : Soit $D \in \mathrm{Dyck}(n)$ tel que $D = SERF$, où S, R, F sont des mots[2] en $\{N, E\}$ et R est un (sous-)chemin de Dyck primitif (non nul). Soit $D' = SREF$. Dans ce poset, le chemin de Dyck D' couvre D. Tamari a prouvé que ce poset est en fait un treillis (Friedman et Tamari, 1967; Huang et Tamari, 1972). La figure 1.8 montre le treillis de Tamari sur les chemins de Dyck de hauteur 4.

En 2006, Chapoton (Chapoton, 2006) a démontré que le nombre d'intervalles (multi-chaînes de longueur 1) dans le treillis de Tamari est égal à

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}. \tag{1.11}$$

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2. La lettre N représente un pas nord et la lettre E représente un pas Est.

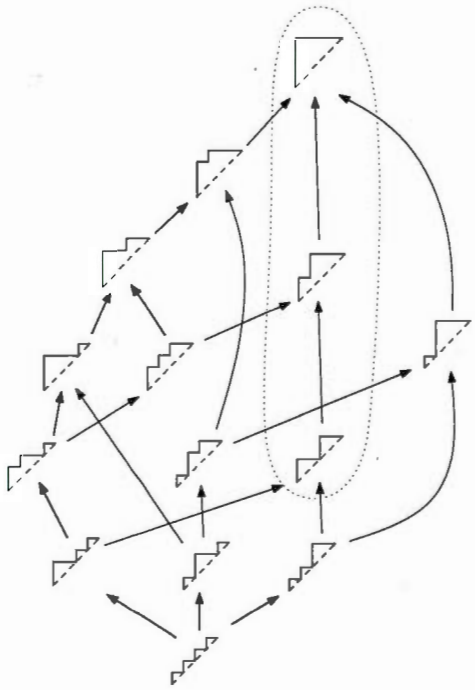

Figure 1.8 Treillis de Tamari sur $Dyck(4)$.

L'étude de ces intervalles et de certaines de leurs extensions est le sujet principal de cette thèse.

Afin de trouver une récurrence pour l'énumération de ces intervalles et de leurs extensions, il est nécessaire d'introduire une variable supplémentaire appellée variable catalytique, ainsi qu'un opérateur linéaire sur cette dernière.

Pour l'énumération envisagée, on est amené à considérer la fonction génératrice

$$F(t;x) := \sum_{n\geq 0} t^n \sum_{D_1 \in \mathrm{Dyck(n)}} x^{\mathrm{contacts(D_1)}} \sum_{D_1 \leq D_2} 1,$$

où $contacts(D_1)$ est le nombre de « contacts » de D_1 avec la droite de pente 1 passant par l'origine et la somme de droite est indexée par les chemins de Dyck plus grands ou égaux à D_1 dans le treillis de Tamari. Cette fonction génératrice satisfait une équation

fonctionnelle exprimée à l'aide de l'opérateur Δ, définit comme

$$\Delta(R(x)) = \frac{R(x) - R(1)}{x - 1}, \tag{1.12}$$

où $R(x) = R(t; x) \in \mathbb{C}[x][[t]]$. Dans ce contexte, la variable x est dite *catalytique* (voir (Bousquet-Mélou et Jehanne, 2006) pour les équations à variable catalytique). Nous avons démontré que

$$F(t; x) = \left(\frac{1}{1 - txF(t; x)\Delta}\right)(x), \tag{1.13}$$

le membre de droite de l'équation précédente doit être interprété comme l'itéré de l'opérateur $txF(t; x)\Delta$ appliqué à x. L'équation 1.13 est clairement équivalente à

$$F(t; x) = x + txF(t; x)\Delta(F(t; x)).$$

Autrement dit :

$$F(t; x) = x + txF(t; x)\frac{F(t; x) - F(t; 1)}{x - 1}. \tag{1.14}$$

Le treillis de Tamari peut être également considéré comme un poset sur l'ensemble des arbres binaires, via une bijection classique entre les chemins de Dyck et les arbres binaires. Chapoton avait obtenu une équation équivalente à 1.14 par le biais d'une récurrence sur les arbres binaires. Comme nous le rappelons dans le prochain paragraphe, une version équivalente de cette équation avait déjà été résolue précédemment dans le contexte de l'énumération d'une famille de cartes planaires enracinées. Notons que l'inversion de Lagrange ne s'applique pas directement à ce type d'équation. Nous présentons deux méthodes connues pour la résoudre.

La première approche consiste à « extraire » de l'équation une paramétrisation de $F(t; 1)$ (on retrouve cette approche à la page 150 du livre (Goulden et Jackson, 2004)). Elle a l'avantage d'être constructive. Cependant, cette approche ne fonctionne que pour résoudre certaines de ces équations à variable catalytique. En particulier, elle ne fonctionne pas pour la plupart des nouvelles équations que nous avons résolues dans cette thèse. Partant de l'équation précédente et en complétant le carré en $F(t; x)$ (remplaçant $F(t; 1)$ par f, et $F(t; x)$ par F), on obtient

$$(2txF - (txf + x - 1))^2 = (txf + x - 1)^2 + 4tx^2(1 - x)$$

Par le théorème de Puiseux, il existe $\alpha(t)$ une série de Laurent fractionnaire en t qui annule le membre de droite de l'équation précédente lorsque $x = \alpha(t)$. En prenant la dérivée partielle de cette même équation par rapport à x et en remplaçant x par $\alpha := \alpha(t)$, on obtient une deuxième équation satisfaite par $x = \alpha$ en fonction de t et f. Ces deux équations sont

$$(t\alpha f + \alpha - 1)^2 + 4t\alpha^2(1-\alpha) = 0,$$

$$(tf+1)(t\alpha f + \alpha - 1) - 12\alpha^2 t + 8\alpha t = 0.$$

Par élimination, il est possible de déduire les deux équations suivantes

$$t = \alpha^{-4}(\alpha - 1),$$

$$tf = \alpha^{-2}(1-\alpha)(\alpha - 2).$$

Le changement de variable $\alpha = \frac{1}{1-z}$ donne alors

$$t = z(1-z)^3 \quad , \quad F(t;1) = f = \frac{1-2z}{(1-z)^3}.$$

Via l'inversion de Lagrange, il est facile de démontrer que le nombre d'intervalles est bien donné par la formule 1.11.

La seconde méthode consiste à « deviner » de bons changements de variables et une paramétrisation en fonction de ces derniers, à savoir

$$t = z(1-z)^3 \quad , \quad x = \frac{1+u}{(1+zu)^2},$$

$$F(t;x) = \frac{(1+u)}{(1+zu)(1-z)^3}\Big(\frac{u - 2zu - z^2u^2}{u}\Big).$$

On verifie alors que cette paramétrisation satisfait bien l'équation 1.14 (à l'aide des changements de variables pour t et x).

Notons que u tend vers 0 lorsque $x = 1$, on a donc la spécialisation

$$F(t;1) = \frac{1-2z}{(1-z)^3}.$$

Nous allons expliquer d'où viennent ces changements de variables et cette paramétrisation dans les chapitres 4 et 5.

1.11 q-analogues et statistiques combinatoires

Grosso modo, un *q-analogue* d'une expression (ou identité) est une généralisation de cette expression (ou identité) contenant le paramètre q et qui se spécialise en l'expression de départ lorsque $q \to 1$. Nous nous intéresserons ici à l'énumération de structures combinatoires avec certaines statistiques supplémentaires. Soit S un ensemble de structures combinatoires. Une *statistique* stat sur S est une fonction qui associe à chacun des éléments de S un élément de $\mathbb{N}$. Nous allons considérer des raffinements d'énumérations de la forme

$$\sum_{s \in S} q^{\text{stat(s)}}.$$

Cette somme se spécialise clairement en $|S|$ lorsque $q = 1$.

Un cas bien connu est l'énumération des permutations selon le nombre d'inversions. Une *inversion* d'une permutation $\sigma \in \mathfrak{S}_n$ est un couple (i, j), tel que $1 \leq i < j \leq n$ et $\sigma(i) > \sigma(j)$. La statistique $\text{inv}(\sigma)$ « compte » le nombre d'inversions de σ. Il n'est pas difficile de montrer qu'on a le q-analogue

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = (1+q)(1+q+q^2)...(1+q+..+q^{n-1})$$

de $n!$.

Plusieurs de nos théorèmes et conjectures prennent la forme d'égalités entre des séries de Hilbert (et de Frobénius) graduées, pour certaines représentations de $\mathfrak{S}_n$, et des q-analogues de structures combinatoires. Nous discuterons de conjectures de ce type reliant les séries de Hilbert (ou de Frobenius) d'espaces coinvariants diagonaux trivariés, à l'énumération de structures combinatoires généralisant les intervalles dans le treillis de Tamari. Nous définirons les espaces pertinents dans le chapitre suivant.

CHAPITRE II

APERÇU DE TRAVAUX ANTÉRIEURS SUR LES ESPACES COINVARIANTS DIAGONAUX DE $\mathfrak{S}_n$

Dans ce chapitre, nous suivons d'assez près la présentation de Haglund dans le livre (Haglund, 2008) et celle de Bergeron dans le livre (Bergeron, 2009). Aucune de nos contributions n'apparaît dans ce chapitre.

2.1 Les espaces coinvariants diagonaux de $\mathfrak{S}_n$

Avec $n, k \in \mathbb{P}$, soit $X = (x_{i,j})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}}$ une matrice d'indéterminées et $\mathbb{C}[X]$ l'anneau des polynômes en ces indéterminées. Pour $1 \leq i \leq k$, chacun des ensembles $\{x_{i,j} : 1 \leq j \leq n\}$ est appelé un *jeu de variables*. Soit $r = (r_1, ..., r_k) \in \mathbb{N}^k$, on désigne par $\mathbb{C}[X]_r$ l'ensemble des polynômes homogènes de degré r_j en le $j^{\text{ème}}$ jeu de variables pour tous les $1 \leq j \leq k$. On obtient ainsi une graduation sur $\mathbb{C}[X]$:

$$\mathbb{C}[X] = \bigoplus_{r \in \mathbb{N}^k} \mathbb{C}[X]_r.$$

On définit une action du groupe symétrique $\mathfrak{S}_n$ sur la matrice X en permutant les colonnes comme suit

$$\sigma(X) = (x_{i,\sigma(j)})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}}.$$

On utilise cette action des permutations sur la matrice d'indéterminées X pour définir une représentation de $\mathfrak{S}_n$ sur les polynôme $f(X) \in \mathbb{C}[X]$ en posant

$$\sigma(f(X)) = f(\sigma(X)).$$

Il est clair que cette action est compatible avec la graduation ci-haut.

Les polynômes diagonalement symétriques sont les polynômes invariants par cette action. On dénote par $\mathrm{DSym}(X)$ le sous-anneau de $\mathbb{C}[X]$ constitué de tous ces polynômes diagonalent symétriques. Soit $\mathcal{J}$ l'idéal engendré par les polynômes diagonalement symétriques sans terme constant. L'idéal $\mathcal{J}$ est homogène. Il hérite de la graduation sur $\mathbb{C}[X]$, avec $\mathcal{J}_r = \mathcal{J} \cap \mathbb{C}[X]_r$. On obtient donc

$$\mathcal{J} = \bigoplus_{r \in \mathbb{N}^k} \mathcal{J}_r.$$

Les espaces coinvariants diagonaux du groupe symétrique sont définis comme étant le quotient (gradué)

$$\mathcal{DR}_{k,n} = \mathbb{C}[X]/\mathcal{J}.$$

Lorsque $k = 1$, ces derniers sont appelés simplement espaces coinvariants classiques (voir (Humphreys, 1990; Haglund, 2008)). Nous reviendrons un peu sur l'historique des ces espaces en fonction du paramètre k dans les sections suivantes. Étant donné que $\mathcal{J}$ est invariant par l'action de $\mathfrak{S}_n$, il est clair que ce quotient est en fait une représentation graduée de $\mathfrak{S}_n$. Il n'est cependant pas trivial que ces espaces sont de dimensions finies. La graduation de $\mathbb{C}[X]$ passe à $\mathcal{DR}_{k,n}$ pour donner

$$\mathcal{DR}_{k,n} = \bigoplus_{r \in \mathbb{N}^k} \mathbb{C}[X]_r/\mathcal{J}_r.$$

Chacun des espaces $\mathbb{C}[X]_r/\mathcal{J}_r$ dans la somme précédente est invariant pour l'action de $\mathfrak{S}_n$. Nous utilisons les variables $q_1, q_2, ...q_k$ pour garder trace du degré de chacun des jeux de variables, dans les séries de Hilbert et de Frobenius graduées. Ces dernières sont alors données par

$$\mathcal{H}(\mathcal{DR}_{k,n}; q) = \sum_{r \in \mathbb{N}^k} \dim(\mathbb{C}[X]_r/\mathcal{J}_r)\, q^r, \tag{2.1}$$

$$\mathcal{F}(\mathcal{DR}_{k,n}; q) = \sum_{r \in \mathbb{N}^k} \mathcal{F}(\mathbb{C}[X]_r/\mathcal{J}_r)\, q^r, \tag{2.2}$$

où $q^r = q_1^{r_1}...q_k^{r_k}$. Nous nous intéresserons également à $\mathcal{H}(\mathcal{DR}_{k,n}{}^{\varepsilon}; q)$, où comme mentionné auparavant ε représente la représentation signe de $\mathfrak{S}_n$, aussi dénotée par S_{1^n}. Nous rappelons également que $\mathcal{DR}_{k,n}{}^{\varepsilon}$ symbolise la sous-représentation signe de $\mathcal{DR}_{k,n}$.

Les espaces $\mathcal{DR}_{k,n}$ se généralisent de la façon suivante. Un polynôme $f(X) \in \mathbb{C}[X]$ est dit *alternant* si $\forall \sigma \in \mathfrak{S}_n$ alors $\sigma(f(X)) = (-1)^{\mathrm{inv}(\sigma)} f(X)$. Soit $\mathcal{A}$ l'idéal engendré[1] par les polynômes alternants dans $\mathbb{C}[X]$. Soit $m \in \mathbb{P}$. Les *espaces coinvariants diagonaux augmentés* (ce sont également des représentations de $\mathfrak{S}_n$) généralisent les espaces coinvariants précédents, et se définissent par

$$\mathcal{DR}^m_{k,n} = \varepsilon^{m-1} \otimes (\mathcal{A}^{m-1}/\mathcal{J}\mathcal{A}^{m-1}),$$

$\mathcal{H}(\mathcal{DR}^m_{k,n}; q)$ et $\mathcal{F}(\mathcal{DR}^m_{k,n}; q)$ sont les extensions évidentes du cas $m = 1$ donné par les équations 2.1 et 2.2.

Cette thèse porte spécifiquement sur l'étude et l'énumération d'objets combinatoires qui sont en fait des extensions des intervalles du treillis de Tamari. Nous présenterons des résultats qui suggèrent fortement que ces objects sont intimement liés aux espaces $\mathcal{DR}^m_{3,n}$ et $\mathcal{DR}^{m\ \varepsilon}_{3,n}$.

2.2 Les polynômes harmoniques diagonaux

Pour $f(X) \in \mathbb{C}[X]$, $f(\partial X)$ symbolise l'opérateur obtenu en remplaçant les indéterminées $x_{i,j}$ dans $f(X)$ par les dérivées partielles $\partial x_{i,j}$. L'espace des polynômes harmoniques diagonaux est la représentation de $\mathfrak{S}_n$ définie par

$$\mathcal{DH}_{k,n} = \{f(X) \in \mathbb{C}[X] : g(\partial X) f(X) = 0, \forall g(X) \in \mathcal{J}\}.$$

Comme pour les espaces coinvariants diagonaux, une version augmentée de ces représentations est indexée par un paramètre $m \in \mathbb{P}$. Les espaces des *polynômes harmoniques diagonaux augmentés* sont définis par

$$\mathcal{DH}^m_{k,n} = (\mathcal{A}^{m-1}) \cap \{f(X) \in \mathbb{C}[X] : g(\partial X) f(X) = 0, \forall g(X) \in \mathcal{A}^{m-1}\mathcal{J}\}.$$

Il a été démontré que (voir (Steinberg, 1964; Bergeron, 2011))

$$\mathcal{DH}^m_{k,n} \cong \mathcal{DR}^m_{k,n},$$

comme représentations de $\mathfrak{S}_n$ graduées.

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1. $(-1)^{\mathrm{inv}(\sigma)}$ est le signe de la permutation σ.

2.3 Résultats et conjectures connus en fonction du nombre de jeux de variables

Dans cette section, nous énonçons quelques théorèmes et conjectures apparaissant dans la littérature, sur les espaces coinvariants diagonaux lorsque le nombre de jeux de variables $k \leq 3$. Cette liste est loin d'être exhaustive. Cette présentation vise à bien situer le contexte de nos travaux. Les résultats de ce chapitre ne sont pas nos résultats : nous présentons les nôtres dans les prochains chapitres.

2.3.1 1 jeu de variables

Lorsqu'il n'y a qu'un seul jeu de variables, $\mathcal{DR}_{1,n}$ simplement dénoté par $\mathcal{R}_n$ est l'espace coinvariant classique, et $\mathcal{H}_n$ $(= \mathcal{DH}_{1,n})$ est l'espace des polynômes harmoniques (classiques). Il a été démontré, autour des années 1950, que $\mathcal{R}_n$ est isomorphe à la représentation régulière[2]. De plus,

$$\mathcal{H}(\mathcal{R}_n; q_1) = (1+q_1)...(1+q_1+..+q_1^{n-1}) = \sum_{\sigma \in \mathfrak{S}_n} q_1^{\mathrm{inv}(\sigma)},$$

$$\mathcal{F}(\mathcal{R}_n; q_1) = \sum_{\lambda \vdash n} s_\lambda(w) \sum_{T \in \mathrm{SYD}(\lambda)} q_1^{\mathrm{charge(T)}},$$

où charge(T) est une certaine statistique sur les tableaux standards. Une base de $\mathcal{R}_n$ est donnée par

$$\left\{ \prod_{1 \leq i \leq n} x_i^{\alpha_i} : 0 \leq \alpha_i \leq i-1 \right\}.$$

Rappelons que le polynôme de Vandermonde est donné par

$$\mathrm{Vand}_n(X) = \det\left((x_i^{j-1})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \right) = \prod_{1 \leq i < j \leq n} (x_j - x_i),$$

où pour une matrice carrée M, $\det(M)$ symbolise le déterminant de M. On peut montrer que $\mathcal{H}_n$ est l'espace vectoriel engendré par $\mathrm{Vand}_n(X)$ et ses dérivées partielles en X. À partir de cette description, il n'est pas trop difficile de constater que $\dim(\mathcal{H}_n{}^{\varepsilon}) = 1$ et que cet espace est engendré par $\mathrm{Vand}_n(X)$.

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2. En fait ces résultats, ainsi que les suivants dans cette section, sont un cas spécial de théorèmes sur les groupes de réflexions complexes (Humphreys, 1990).

2.3.2 2 jeux de variables

Les *polynômes de Macdonald modifiés* $\tilde{H}_\lambda = \tilde{H}_\lambda[X; q_1, q_2]$, que nous ne définissons pas ici, sont des fonctions symétriques avec coefficients dans $\mathbb{C}[q_1, q_2]$ indexées par les partage $\lambda \vdash n$. Ils constituent une base des fonctions symétriques avec coefficients dans $\mathbb{C}[q_1, q_2]$. Ce sont des fonctions propres d'un opérateur ∇, introduit par Bergeron et Garsia, défini par

$$\nabla(\tilde{H}_\lambda) = q_1^{n(\lambda)} q_2^{n(\lambda')} \tilde{H}_\lambda, \tag{2.3}$$

où $n(\lambda) = \sum_{1 \le i \le \ell} (i-1)\lambda_i$. Le théorème suivant, prouvé par Mark Haiman (à l'aide de la géométrie algébrique), est une des contributions majeures dans ce domaine.

Théorème 9 (Haiman)

$$\mathcal{F}(\mathcal{DR}^m_{2,n}; q_1, q_2) = \nabla^m(e_n(w)).$$

Cette identité permet de relier l'étude d'espaces coinvariants diagonaux à plusieurs questions combinatoires. Il en découle (voir (Garsia et Haiman, 1996)) les formules

Corollaire 1

$$\dim(\mathcal{DR}^{m\ \varepsilon}_{2,n}) = \frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n},$$
$$\dim(\mathcal{DR}^m_{2,n}) = (mn+1)^{n-1},$$

exprimées en terme du nombre de chemins de m-Dyck et du nombre de fonctions de m-stationnement, respectivement.

Une version plus fine du corollaire précédent, et plusieurs conjectures, nécessitent l'introduction de deux statistiques sur les chemins de m-Dyck, et sur les fonctions de m-stationnement. Soit $D \in \mathrm{Dyck}_m(n)$. La statistique $\mathrm{area}(D)$ est définie comme le nombre de cases situées entièrement au-dessus de la droite dans le quart de plan d'équation $x = my$, et en dessous du chemin D. Elle est définie pour les fonctions de m-stationnement de la même manière (sans se soucier des étiquettes).

Soit Ω l'opérateur :

$$\Omega(v(t;q_1)) = v(tq_1;q_1), \tag{2.4}$$

où $v(t;q_1) \in \mathbb{C}[q_1][[t]]$. Il n'est pas difficile de montrer que les généralisations suivantes des équations 1.8 et 1.10 permettent de tenir compte de la statistique area (comme puissance du paramètre q_1) pour les chemins de m-Dyck et pour les fonctions de m-stationnement respectivement[3] [4] :

$$f(t;q_1) = 1 + t(f(t;q_1)\Omega)^m(1),$$

$$g(t;q_1) = e^{t(g(t;q_1)\Omega)^m}(1).$$

Chacun des membres de droite de ces équations correspond à appliquer un opérateur à 1. Définissons maintenant la deuxième statistique appelée « dinv » , due à Haiman. Pour chacun des pas nord dans D, soit (x,y) la coordonnée de son point inférieur (pour le premier pas nord, cette valeur est de (0,0)). On obtient donc n points de la forme $(x_1,0),(x_2,1),(x_3,2),...,(x_{n-1},n-2),(x_n,n-1)$, où $x_1 = 0$ et $(x_i,i-1)$ est le point associé au $i^{\text{ème}}$ pas nord. Pour $1 \leq i \leq n$, soit $a_i = m(i-1) - x_i$. Pour chacun des paires de pas nord de la forme (i,j) où $1 \leq i < j \leq n$, cette paire contribue

$$\max(0, m - |a_i - a_j|, m - |a_i - a_j - 1|).$$

La statistique dinv(D) est la somme de ces contributions. La figure 2.1 montre un chemin de Dyck, avec ses statistiques area et dinv, ainsi que la séquence des a_i associée.

Cette statistique se définit de manière similaire sur les fonctions de m-stationnement. Soit $P \in \text{Park}_m(n)$. Soient a_i et x_i définis de la même manière qu'au paragraphe

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3. la première équation est clairement équivalente à

$$f(t;q_1) = 1 + tf(t;q_1)f(tq_1;q_1)f(tq_1^2;q_1)...f(tq_1^m;q_1)$$

4. La série $f(t;q_1)$ (resp. $g(t;q_1)$) donne la série génératrice pondérée des chemins de m-Dyck (resp. des fonctions de m-stationnement) selon la statistique area.

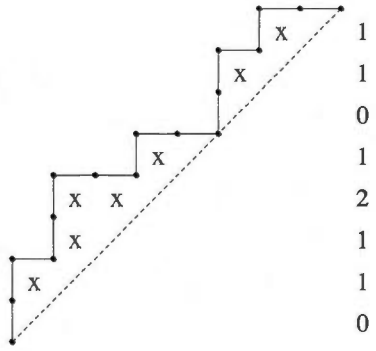

Figure 2.1 Un chemin de Dyck D avec $\text{area}(D) = 7, \text{dinv}(D) = 17$. Les nombres dans la colonne de droite constituent la séquence des a_i associée à D (de bas en haut).

précédent sur le chemin de m-Dyck de P. Soit b_i l'étiquette du $i^{\text{ème}}$ pas nord. Pour chacun des paires de pas nord de la forme (i, j) où $1 \leq i < j \leq n$, cette paire contribue

$$\begin{cases} \max(0, m - |a_i - a_j|) & \text{si } b_i < b_j, \\ \max(0, m - |a_i - a_j - 1|) & \text{si } b_i > b_j. \end{cases}$$

La statistique $\text{dinv}(P)$ est la somme de ces contributions.

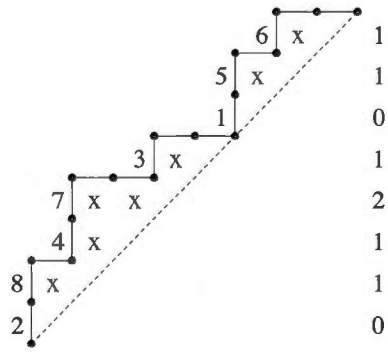

Figure 2.2 Une fonction de stationnement avec $\text{area}(P) = 7$, $\text{dinv}(P) = 11$. La colonne de droite est la séquence des a_i associée à P (de bas en haut).

Nous sommes maintenant en mesure d'énoncer plusieurs conjectures et certains théorèmes exprimant les séries de Hilbert et de Frobenius des espaces coinvariants diagonaux bivariés en termes de l'énumération pondérée de chemins de m-Dyck et de fonctions de m-stationnement. En raison des formules 1.4 et 1.6 et du théorème 9, nous avons

l'égalité :

$$\mathcal{H}(\mathcal{DR}_{2,n}{}^{\varepsilon}; q_1, q_2) = \langle \nabla(e_n(w)), s_{1^n}(w) \rangle$$

Théorème 10 (Garsia-Haglund)

$$\langle \nabla(e_n(w)), s_{1^n}(w) \rangle = \sum_{D \in \text{Dyck(n)}} q_1^{\text{area(D)}} q_2^{\text{dinv(D)}}. \tag{2.5}$$

Le théorème précédent a été démontré pour la première fois par Haglund et Garsia (voir (Garsia et Haglund, 2002)) à l'aide de la paire de statistiques (bounce,area) sur les chemins de Dyck[5], où bounce est une autre statistique sur les chemins de Dyck. Elle a été redémontrée (avec d'autres techniques) plus récemment par Garsia, Xin et Zabrocki (Garsia, Xin et Zabrocki, pear). Ils ont démontré qu'une récurrence satisfaite par les objects combinatoires (en tenant compte des statistiques area et dinv) était également satisfaite par certaines fonctions symétriques C_α ($\alpha = (\alpha_1, ..., \alpha_\ell) \models n$). Plus précisemment, ils ont démontré que

$$\langle \nabla C_\alpha, s_{1^n} \rangle = \sum_{\text{touch(D)}=\alpha} q^{\text{dinv(D)}} t^{\text{area(D)}}, \tag{2.6}$$

où touch$(D) = \alpha$ symbolise que D « touche » la diagonale exactement en les points $0, \alpha_1, \alpha_1 + \alpha_2, ..., n$.[6]

Notons que la définition de $\mathcal{DR}_{2,n}$ entraîne évidemment que
$\mathcal{H}(\mathcal{DR}_{2,n}{}^{\varepsilon}; q_1, q_2) = \mathcal{H}(\mathcal{DR}_{2,n}{}^{\varepsilon}; q_2, q_1)$. Le théorème 2.5 implique qu'en échangeant le rôle des deux jeux de variables que :

Corollaire 2

$$\sum_{D \in \text{Dyck(n)}} q_1^{\text{area(D)}} q_2^{\text{dinv(D)}} = \sum_{D \in \text{Dyck(n)}} q_1^{\text{dinv(D)}} q_2^{\text{area(D)}}.$$

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5. Il existe une bijection sur les chemins de Dyck qui envoie le couple de statistiques (area,dinv) vers le couple (bounce,area) (voir (Haglund, 2008)).

6. Ils ont également démontré qu'une autre famille de fonctions symétriques indexées par les $\alpha \models n$ était reliée aux chemins de Dyck qui touchent la diagonale au moins aux points $0, \alpha_1, \alpha_1 + \alpha_2, ..., n$.

La seule preuve connue du corollaire précédent utilise le théorème 9. La question suivante devient alors naturelle.

Problème 1 *Démontrer directememt, à partir des objets combinatoires, que*

$$\sum_{D\in \mathrm{Dyck(n)}} q_1^{\mathrm{area(D)}} q_2^{\mathrm{dinv(D)}} = \sum_{D\in \mathrm{Dyck(n)}} q_1^{\mathrm{dinv(D)}} q_2^{\mathrm{area(D)}}.$$

Le théorème 9 implique immédiatement que

$$\mathcal{H}(\mathcal{DR}_{2,n}^{m\ \varepsilon}; q_1, q_2) = \langle \nabla^m(e_n(w)), s_{1^n}(w)\rangle \tag{2.7}$$

La conjecture suivante reste cependant ouverte.

Conjecture 1 $\forall m > 1$, *démontrer à partir de l'identité 2.7 que*

$$\mathcal{H}(\mathcal{DR}_{2,n}^{m\ \varepsilon}; q_1, q_2) = \sum_{D\in \mathrm{Dyck_m(n)}} q_1^{\mathrm{area(D)}} q_2^{\mathrm{dinv(D)}}.$$

Conjecture 2 *Démontrer directement, à partir des objets combinatoires, que*

$$\sum_{D\in \mathrm{Dyck_m(n)}} q_1^{\mathrm{area(D)}} q_2^{\mathrm{dinv(D)}} = \sum_{D\in \mathrm{Dyck_m(n)}} q_1^{\mathrm{dinv(D)}} q_2^{\mathrm{area(D)}}.$$

En utilisant encore les formules 1.4 et 1.6 et le théorème 9, on obtient

$$\mathcal{H}(\mathcal{DR}_{2,n}^{m}; q_1, q_2) = \langle \nabla^m(e_n(w)), h_{1^n}(w)\rangle\,. \tag{2.8}$$

On peut alors se poser les mêmes questions pour $\mathcal{H}(\mathcal{DR}_{2,n}^{m})$ que celles pour $\mathcal{H}(\mathcal{DR}_{2,n}^{m\ \varepsilon})$.

Conjecture 3

$$\mathcal{H}(\mathcal{DR}_{2,n}^{m}; q_1, q_2) = \sum_{P\in \mathrm{Park_m(n)}} q_1^{\mathrm{area(P)}} q_2^{\mathrm{dinv(P)}}.$$

Conjecture 4 *Démontrer directement, à partir des objets combinatoires, que*

$$\sum_{P\in \mathrm{Park_m(n)}} q_1^{\mathrm{area(P)}} q_2^{\mathrm{dinv(P)}} = \sum_{P\in \mathrm{Park_m(n)}} q_1^{\mathrm{dinv}}(P) q_2^{\mathrm{area(P)}}.$$

Il existe une représentation combinatoire simple associée à l'étude de $\mathcal{DR}^m_{2,n}$. Soit $\sigma \in \mathfrak{S}_n$ et $P \in \mathrm{Park}_m(n)$. La permutation σ peut agir sur P en permutant les étiquettes de P puis en réordonnant ces dernières pour qu'elles soient croissantes sur les pas nord consécutifs de P. Nous donnons un exemple de cette action à la figure 2.3. Cette action de $\mathfrak{S}_n$ sur $\mathrm{Park}_m(n)$ est une représentation de $\mathfrak{S}_n$ que l'on dénote par $R_{\mathrm{Park}_m(n)}$. La représentation $R_{\mathrm{Park}_m(n)}$ est reliée aux espaces coinvariants $\mathcal{DR}^m_{2,n}$ de la manière suivante (voir (Garsia et Haiman, 1996)) :

Théorème 11 (Haiman)

$$\mathcal{DR}^m_{2,n} \cong \varepsilon \otimes R_{\mathrm{Park_m(n)}}. \tag{2.9}$$

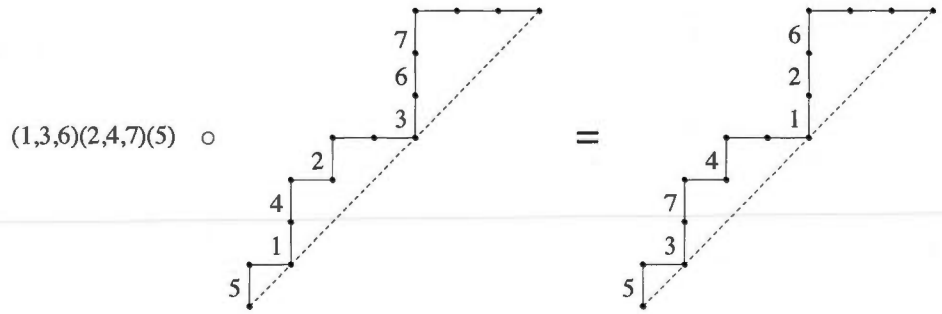

Figure 2.3 Un exemple de la représentation parking $R_{\mathrm{Park_1(n)}}$.

On s'intéresse maintenant à la série de Frobenius de $\mathcal{DR}^m_{2,n}$. Voici une des principales questions ouvertes dans le domaine.

Problème 2 *Pour chacun des partages $\lambda \vdash n$, trouver une description combinatoire des polynômes*

$$\langle \mathcal{F}(\mathcal{DR}^m_{2,n}; q_1, q_2), s_\lambda(w) \rangle.$$

Pour le cas $m = 1$, ce problème a été résolu par Haglund pour les partages en forme d'équerre. Ces derniers sont des partages de la forme $((n-r), 1^r)$, pour un certain $r \in \mathbb{N}$. Cette description combinatoire est basée sur des objects combinatoires appelés

les chemins de Schröder. Ce sont des chemins dans le quart de plan partant de (0,0), terminant en (n, n), constitués de pas nord, est et nord-est (des pas de la forme (1,1)) et qui ne descendent jamais sous la droite $x = y$. Des généralisations des statistiques area et dinv s'étendent aux chemins de Schröder.

Il existe une conjecture générale qui donne une formule combinatoire de $\mathcal{F}(\mathcal{DR}^m_{2,n})$, mais cette dernière utilise les fonctions quasi-symétriques. Pour énoncer cette conjecture, nous avons besoin d'associer une composition à chaque fonction de m-stationnement. Pour $P \in \mathrm{Park}_m(n)$, on définit la permutation read(P) comme étant la lecture des éléments de P sur chacune des diagonales de pente $1/m$, d'en haut à droite à en bas à gauche, en commençant par la diagonale la plus haute et en descendant (voir la figure 2.4 pour un exemple).

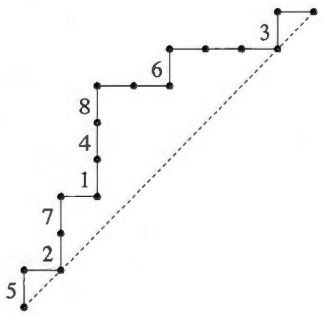

Figure 2.4 Une fonction de stationnement dont read$(P) = 8, 6, 4, 1, 7, 3, 2, 5$.

Soit $V = \{i_1, i_2, ..., i_{h-1}\}$ l'ensemble des descentes de $(\mathrm{read}(P))^{-1}$ (avec $1 \leq i_1 < i_2 < ... < i_{h-1} < n$ et $h \geq 1$). La composition co(P) est définie comme étant

$$\mathrm{co}(P) = (i_1, i_2 - i_1, i_3 - i_2, ..., i_{h-1} - i_{h-2}, n - i_{h-1}).$$

Conjecture 5 (Haglund-Haiman-Loehr-Remmel-Ulyanov)

$$\mathcal{F}(\mathcal{DR}^m_{2,n}; q_1, q_2) = \sum_{P \in \mathrm{Park_m(n)}} q_1^{\mathrm{area(P)}} q_2^{\mathrm{dinv(P)}} Q_{\mathrm{co(P)}}(w) \tag{2.10}$$

La conjecture précédente s'appelle la conjecture des mélanges [7]. Notons que cette conjecture implique les conjectures 1 et 3 et le théorème d'Haglund en lien avec les chemins de Schröder mentionnés auparavant.

Le but ultime serait de trouver une base explicite des espaces $\mathcal{DR}^m_{2,n}$ (et $\mathcal{DH}^m_{2,n}$) et ainsi pouvoir démontrer la conjecture 2 en « lisant » les éléments de cette base. On pourrait également espérer que cette base aiderait à résoudre d'autres conjectures mentionnées auparavant.

2.3.3 3 jeux de variables et plus

Le contenu de cette section est le point de départ des résultats obtenus dans cette thèse.

En 1994 , Haiman (Haiman, 1994) a conjecturé [8] que

$$\dim(\mathcal{DR}_{3,n^\varepsilon}) = \frac{2}{n(n+1)}\binom{4n+1}{n-1}, \tag{2.11}$$

$$\dim(\mathcal{DR}_{3,n}) = 2^n(n+1)^{n-2}. \tag{2.12}$$

Il a cependant noté en 2001 (Haiman, 2001; Haiman, 2002) que les techniques utilisées dans sa preuve pour le cas $k = 2$ ne peuvent pas s'appliquer pour $k \geq 3$. D'autre part, comme énoncé dans le chapitre 1, Chapoton a démontré en 2006 que les intervalles du treillis de Tamari sont comptés par la formule 2.11. Inspiré par la conjecture de Haiman et le théorème de Chapoton, François Bergeron a conjecturé en 2008 que

Conjecture 6 *Le nombre d'intervalles étiquetés dans le treillis de Tamari est donné par*

$$2^n(n+1)^{n-2}. \tag{2.13}$$

---

7. En anglais, elle s'appelle la conjecture « shuffle ».

8. Pour être plus précis, en 1994, il n'avait pas utilisé le terme conjecture car il n'avait calculé que quelques petites valeurs de n. Il n'utilisera le terme conjecture que par la suite.

Ces intervalles étiquetés sont des intervalles dans le treillis de Tamari dont la borne supérieure est une fonction de stationnement.

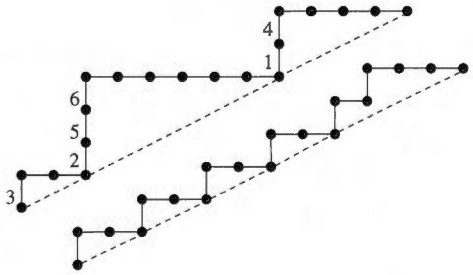

Figure 2.5 Un intervalle étiqueté dans le treillis de 2-Tamari, dont la borne supérieure est une fonction de 2-stationnement et la borne inférieure est un chemin de 2-Dyck.

Bergeron a ensuite étendu sa conjecture au cas du treillis de m-Tamari, correspondant à un ordre sur les chemins de m-Dyck que nous présentons un peu plus tard dans cette section. Il a ainsi énoncé les quatre conjectures suivantes :

Conjecture 7

$$\dim(\mathcal{DR}_{3,n}^{m\ \varepsilon}) = \frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}. \tag{2.14}$$

Conjecture 8

$$\dim(\mathcal{DR}_{3,n}^{m}) = (m+1)^n(mn+1)^{n-2}. \tag{2.15}$$

Conjecture 9 *Les intervalles dans le treilis de m-Tamari sont comptés par*

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}. \tag{2.16}$$

Conjecture 10 *Les intervalles étiquetés dans le treillis de m-Tamari sont comptés par*

$$(m+1)^n(mn+1)^{n-2}. \tag{2.17}$$

Ces conjectures ont été présentées par Bergeron dans plusieurs conférences au Canada, en France et aux États-Unis avant que nous résolvions les conjectures 9 et 10 en 2010-2011. La généralisation de ces conjectures et leur démonstration constituent l'une de nos

deux principales contributions dans cette thèse. Nous y reviendrons dans les prochains chapitres.

Pour terminer cette section, nous présentons la généralisation du treillis de Tamari en fonction du paramètre m due a Bergeron. Le *treillis de m-Tamari* a comme éléments les chemins de m-Dyck et est engendré par les relations de couverture suivantes : soit $D \in \mathrm{Dyck}_m(n)$ tel que $D = SERF$, où S, R, F sont des mots en $\{E, N\}$ et R est un chemin de m-Dyck primitif. Soit $D' = SREF$. Dans ce poset le chemin de m-Dyck D' est un couvert de D. Nous donnons un exemple d'intervalle étiqueté dans le treillis de 2-Tamari à la figure 2.5. Il n'est pas difficile de voir que ce poset est en fait un treillis. Nous montrerons des exemples de ces treillis dans les articles contenus dans les chapitres suivants.

Dans les trois prochains chapitres, nous détaillons brièvement nos articles produits pendant cette thèse, en suivant l'ordre chronologique de leur soumission sur arXiv. Chaque article (en anglais) est précédé d'un court résumé (en français) où les principaux résultats sont énoncés. Une liste de problèmes que nous n'avons pas encore été en mesure de résoudre est présentée dans le dernier chapitre.

CHAPITRE III

PREMIER ARTICLE : HIGHER TRIVARIATE DIAGONAL HARMONICS VIA GENERALIZED TAMARI POSETS

Cet article a été accepté par *Journal of combinatorics*[1]. Nous y proposons, entre autres, une nouvelle statistique sur les intervalles étiquetés et non-étiquetés dans le treillis de m-Tamari qui semble rendre compte (conjointement avec la statistique dinv) du degré de deux jeux de variables dans les séries de Hilbert et de Frobenius des espaces coinvariants trivariés augmentés. Ceci renforce l'argument que ces intervalles sont bel et bien liés aux espaces $\mathcal{DR}^m_{3,n}$.

Soit $D_1, D_2 \in \mathrm{Dyck}_m(n)$ tels que $D_1 \leq D_2$ dans le treillis de m-Tamari. On dénote un tel intervalle par $[D_1, D_2]$. Nous considérons la statistique distance, dénotée par $d(D_1, D_2)$, comme étant la longueur de la plus grande chaîne partant de D_1 allant vers D_2 dans le treillis de m-Tamari[2]. On généralise les conjectures du chapitre précédent en tenant compte de ces deux statistiques.

Conjecture 11

$$\mathcal{H}(\mathcal{DR}^{m\ \epsilon}_{3,n}; q_1, q_2, 1) = \sum_{[D_1,D_2]} q_1^{d(D_1,D_2)} q_2^{\mathrm{dinv}(D_2)}, \tag{3.1}$$

où l'on somme sur tous les intervalles de hauteur n dans le treillis de m-Tamari.

---

1. Je remercie *Journal of combinatorics* de m'avoir autorisé d'insérer cet article dans cette thèse.

2. Cette statistique est définie de la même manière sur les intervalles étiquetés.

Conjecture 12

$$\mathcal{H}(\mathcal{DR}^m_{3,n}; q_1, q_2, 1) = \sum_{[D_1,D_2]} \sum_{P \in \mathrm{Park}_m(D_2)} q_1^{d(D_1,D_2)} q_2^{\mathrm{dinv}(P)}. \tag{3.2}$$

Conjecture 13

$$\mathcal{F}(\mathcal{DR}^m_{3,n}; q_1, q_2, 1) = \sum_{[D_1,D_2]} \sum_{P \in \mathrm{Park}_m(D_2)} q_1^{d(D_1,D_2)} q_2^{\mathrm{dinv(P)}} Q_{\mathrm{co}(P)}(w). \tag{3.3}$$

On munit l'ensemble des intervalles étiquetés de hauteur n dans le treillis de m-Tamari d'une action par permutation de $\mathfrak{S}_n$, en posant pour $\sigma \in \mathfrak{S}_n$, que la permutation σ agit sur un intervalle étiqueté de la même manière que la représentation $R_{\mathrm{Park}_m(n)}$ (en laissant l'extrémité inférieure intacte). Soit $R_{\mathrm{Tam}_m(n)}$ cette représentation combinatoire. Nous avons conjecturé que

Conjecture 14

$$\mathcal{DR}^m_{3,n} \cong \varepsilon \otimes R_{\mathrm{Tam}_m(n)}. \tag{3.4}$$

Nous avons également conjecturé que la série de Frobénius non-graduée de $R_{\mathrm{Tam}_m(n)}$ est donnée par

Conjecture 15

$$\mathcal{F}(R_{\mathrm{Tam}_m(n)}) = \sum_{\lambda=(\lambda_1,..,\lambda_\ell)\vdash n} (mn+1)^{\ell-2} \prod_{1\le i\le \ell} \binom{(m+1)\lambda_i}{\lambda_i} \frac{p_\lambda(w)}{z_\lambda}. \tag{3.5}$$

Notons que le cas $\lambda = (1, 1, .., 1)$ de cette conjecture implique la conjecture 10 de Bergeron. La preuve de la conjecture 15 est la principale contribution de cette thèse. Elle fait l'objet du troisième article que nous incluons dans cette thèse.

Nous avons également obtenu plusieurs autres formules et calculs en lien avec les espaces trivariés. Nous renvoyons le lecteur à l'article qui suit pour plus de détails.

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Higher Trivariate Diagonal Harmonics via Generalized Tamari Posets*

FRANCOIS BERGERON AND LOUIS-FRANCOIS PRÉVILLE-RATELLE

We consider the graded $\mathbb{S}_n$-modules of higher diagonally harmonic polynomials in three sets of variables (the trivariate case), and show that they have interesting ties with generalizations of the Tamari poset and parking functions. In particular we get several nice formulas for the associated Hilbert series and graded Frobenius characteristics. This also leads to entirely new combinatorial formulas.

AMS 2000 SUBJECT CLASSIFICATIONS: Primary 05E10; secondary 05A19.
KEYWORDS AND PHRASES: Trivariate harmonics, r-Tamari poset.

1. Introduction

Our global aim here is to give an explicit description of the space $\mathcal{H}_n$ of **trivariate diagonal harmonic polynomials**. We make interesting progress along these lines, and propose a combinatorial description of the graded Frobenius characteristic for this space. This description encompasses the "shuffle conjecture" of [15], and suggests new combinatorial identities

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linked to the combinatorics of the Tamari lattice. In fact it has lead us to introduce a new notion of r-Tamari poset, and these identities may be stated in terms of nonnegative integer polynomials in the variable r. The first drafts of this paper has prompted other researchers to consider these combinatorial questions, and some have already been settled (see [5] and [6]).

More explicitly, in the ring $\mathcal{R}_n := \mathbb{C}[X]$ of polynomials in the three sets of variables

$$X := \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{pmatrix},$$

$\mathcal{H}_n$ is the subspace of polynomial zeros of the **polarized power sums** differential operators

$$P_\alpha(\partial X) := \sum_{j=1}^{n} \partial x_j^a \partial y_j^b \partial z_j^c, \tag{1}$$

for $\alpha = (a, b, c)$ running thought the set of vectors of **norm** $|\alpha| := a + b + c$, with $1 \leq |\alpha| \leq n$. Here, ∂v denotes derivation with respect to v. It is clear from the definition that $\mathcal{H}_n$ is closed under derivation. For one set of variables, say $\mathbf{x} = x_1, \ldots, x_n$, $\mathcal{H}_n$ is entirely described by classical theorems (see [19, section 3.6]) as the span of all partial derivatives of the Vandermonde determinant

$$\Delta_n(\mathbf{x}) = \det\left(x_i^j\right)_{1 \leq i \leq n,\ 0 \leq j \leq n-1},$$

It is of dimension $n!$, and isomorphic to the regular representation of $\mathbb{S}_n$, for the action that permutes variables. The description of the case of two sets of variables took close to 15 years to be finalized (see [17]), and much remains to be understood. It is of dimension $(n+1)^{n-1}$. It has been observed by Haiman (see [16]) that the dimension of $\mathcal{H}_n$ seems to be given by the formula

$$\dim \mathcal{H}_n = 2^n (n+1)^{n-2}, \tag{2}$$

but, until very recently, almost no one had further studied the trivariate case. We have endeavoured to do so, considering more general spaces $\mathcal{H}_n^{(r)}$ (see definition (12)) for which we have experimentally found out that

$$\dim \mathcal{H}_n^{(r)} = (r+1)^n (rn+1)^{n-2}. \tag{3}$$

Symmetric group action

We turn both $\mathcal{R}_n$ and $\mathcal{H}_n$ into $\mathbb{S}_n$-module by considering the **diagonal action** of symmetric group $\mathbb{S}_n$ on variables. Recall that, for $\sigma \in \mathbb{S}_N$, the polynomial $\sigma \cdot f(X)$ is obtained by replacing the variables in $f(X)$ by

$$\begin{pmatrix} x_{\sigma(1)} & x_{\sigma(2)} & \cdots & x_{\sigma(n)} \\ y_{\sigma(1)} & y_{\sigma(2)} & \cdots & y_{\sigma(n)} \\ z_{\sigma(1)} & z_{\sigma(2)} & \cdots & z_{\sigma(n)} \end{pmatrix}.$$

We denote by X^A the monomial

$$X^A := x_1^{a_1} \cdots x_n^{a_n} \cdot y_1^{b_1} \cdots y_n^{b_n} \cdot z_1^{c_1} \cdots z_n^{c_n},$$

for a matrix of integers

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}.$$

Writing A_j for the j^{th}-column of A, we may define the **degree** of a monomial X^A to be the vector

$$\deg(X^A) := \sum_{j=1}^{n} A_j,$$

in $\mathbb{N}^3$. Clearly the diagonal action preserves degree. The **total degree** $\operatorname{tdeg}(X^A)$ of a monomial X^A is the sum of the components of $\deg(X^A)$. An interesting subspace of $\mathcal{H}_n$ is the space $\mathcal{A}_n$ af **alternating** polynomials in $\mathcal{H}_n$. Recall that these are the polynomials such that $\sigma \cdot f(X) = \operatorname{sign}(\sigma)\, f(X)$. It is also worth recalling that, in the case of two sets of variables, this subspace has dimension equal to the Catalan number

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

Another observation of Haiman (*loc. sit.*) is that, in the trivariate case, we seem to have

$$\dim \mathcal{A}_n = \frac{2}{n(n+1)}\binom{4n+1}{n-1}. \tag{4}$$

More generally, our experimental calculations suggest that the alternating component $\mathcal{A}_n^{(r)}$ of $\mathcal{H}_n^{(r)}$ has dimension

$$\dim \mathcal{A}_n^{(r)} = \frac{(r+1)}{n\,(rn+1)} \binom{(r+1)^2\, n + r}{n-1}. \tag{5}$$

It is intriguing[1] that this formula bears a resemblance to a formula that appears in [20].

Degree grading

The space $\mathcal{H}_n$ is graduated by degree, and its **homogeneous components** $\mathcal{H}_{n,\mathbf{d}}$ (with $d \in \mathbb{N}^3$) are S_n-invariant, hence $\mathcal{A}_n$ is also graded. There corresponds direct sum decompositions

$$\mathcal{H}_n = \bigoplus_{d \in \mathbb{N}^3} \mathcal{H}_{n,\mathbf{d}}, \qquad \text{and} \qquad \mathcal{A}_n = \bigoplus_{d \in \mathbb{N}^3} \mathcal{A}_{n,\mathbf{d}}. \tag{6}$$

As discussed in Section 2, the space $\mathcal{H}_n$ is finite dimensional. In fact, we will see that the homogeneous component $\mathcal{H}_{n,\mathbf{d}}$ is non vanishing only if

$$|\mathbf{d}| = d_1 + d_2 + d_3 \leq \binom{n}{2},$$

where $\mathbf{d} = (d_1, d_2, d_3)$. We can thus consider the **Hilbert series**:

$$\mathcal{H}_n(\mathbf{q}) := \sum_{|\mathbf{d}|=0}^{\binom{n}{2}} \dim(\mathcal{H}_{n,\mathbf{d}})\, \mathbf{q}^{\mathbf{d}}, \qquad \text{and} \qquad \mathcal{A}_n(\mathbf{q}) := \sum_{|\mathbf{d}|=0}^{\binom{n}{2}} \dim(\mathcal{A}_{n,\mathbf{d}})\, \mathbf{q}^{\mathbf{d}} \tag{7}$$

of $\mathcal{H}_n$ and $\mathcal{A}_n$, writing $\mathbf{q}^{\mathbf{d}}$ for $q_1^{d_1} q_2^{d_2} q_3^{d_3}$. The Hilbert series $\mathcal{H}_n(\mathbf{q})$ is a symmetric polynomial in q_1, q_2 and q_3. In fact, it is always Schur positive as discussed in Section 3. This is to say that it expands as a nonnegative integer coefficient linear combination of the Schur polynomials $s_\mu(\mathbf{q})$ in the variables $\mathbf{q}$.

For example, we easily calculate that $\mathcal{H}_2$ affords the linear basis

$$\{1, x_2 - x_1, y_2 - y_1, z_2 - z_1\}, \tag{8}$$

[1] Thanks to the anonymous referee for this observation.

whose subset $\{x_2 - x_1, y_2 - y_1, z_2 - z_1\}$ is clearly a basis of $\mathcal{A}_2$. We thus get the Hilbert series

$$\mathcal{H}_2(\mathbf{q}) = 1+(q_1+q_2+q_3) = 1+s_1(\mathbf{q}), \qquad \text{and} \qquad \mathcal{A}_2(\mathbf{q}) = q_1+q_2+q_3 = s_1(\mathbf{q}).$$

To calculate $\mathcal{H}_n(\mathbf{q})$ for larger n, we exploit the fact $\mathcal{H}_n$ is closed under taking derivatives, as well as applying the operators (see Section 2)

$$E^{(k)}_{\mathbf{uv}} := \sum_{i=1}^{n} u_i \partial v_i^k, \qquad (E_{\mathbf{uv}} := E^{(1)}_{\mathbf{uv}}).$$

Here $\mathbf{u}$, $\mathbf{v}$ stand for any two of the three sets of n variables:

$$\mathbf{x} = x_1, \ldots, x_n, \qquad \mathbf{y} = y_1, \ldots, y_n, \qquad \text{and} \qquad \mathbf{z} = z_1, \ldots, z_n.$$

It easy to verify (see Section 2) that the Vandermonde determinant $\Delta_n(\mathbf{u})$ belongs to $\mathcal{H}_n$, whether $\mathbf{u}$ be $\mathbf{x}$, $\mathbf{y}$ of $\mathbf{z}$. Using all this, we can calculate that

$$\begin{aligned}\mathcal{H}_3(\mathbf{q}) \;=\; & 1 + 2\,(q_1 + q_2 + q_3) \\ & +2\,(q_1^2 + q_2^2 + q_3^2) + 3\,(q_1q_2 + q_1q_3 + q_2q_3) \\ & +q_1^3 + q_2^3 + q_3^3 + q_1^2q_2 + q_1^2q_3 + q_1q_2^2 + q_1q_3^2 + q_2^2q_3 + q_2q_3^2 + q_1q_2q_3\end{aligned}$$

by checking directly that we have the following respective bases $\mathcal{B}_d$ for each $\mathcal{H}_{n,\mathbf{d}}$:

$$\begin{aligned}
&\mathcal{B}_{300} = \{\Delta_3(\mathbf{x})\}, \quad \mathcal{B}_{200} = \{\partial_{x_1}\Delta_3(\mathbf{x}), \partial_{x_2}\Delta_3(\mathbf{x})\}, \quad \mathcal{B}_{100} = \{\partial^2_{x_1}\Delta_3(\mathbf{x}), \partial_{x_1}\partial_{x_2}\Delta_3(\mathbf{x})\},\\
&\mathcal{B}_{030} = \{\Delta_3(\mathbf{y})\}, \quad \mathcal{B}_{020} = \{\partial_{y_1}\Delta_3(\mathbf{y}), \partial_{y_2}\Delta_3(\mathbf{y})\}, \quad \mathcal{B}_{010} = \{\partial^2_{y_1}\Delta_3(\mathbf{y}), \partial_{y_1}\partial_{y_2}\Delta_3(\mathbf{y})\},\\
&\mathcal{B}_{003} = \{\Delta_3(\mathbf{z})\}, \quad \mathcal{B}_{002} = \{\partial_{z_1}\Delta_3(\mathbf{z}), \partial_{z_2}\Delta_3(\mathbf{z})\}, \quad \mathcal{B}_{001} = \{\partial^2_{z_1}\Delta_3(\mathbf{z}), \partial_{x_1}\partial_{z_2}\Delta_3(\mathbf{z})\},\\
&\mathcal{B}_{210} = \{E_{\mathbf{yx}}\Delta_3(\mathbf{x})\}, \quad \mathcal{B}_{120} = \{E_{\mathbf{yx}}E_{\mathbf{yx}}\Delta_3(\mathbf{x})\}, \quad \mathcal{B}_{201} = \{E_{\mathbf{zx}}\Delta_3(\mathbf{x})\},\\
&\mathcal{B}_{102} = \{E_{\mathbf{zx}}E_{\mathbf{zx}}\Delta_3(\mathbf{x})\}, \quad \mathcal{B}_{021} = \{E_{\mathbf{zy}}\Delta_3(\mathbf{y})\}, \quad \mathcal{B}_{012} = \{E_{\mathbf{zy}}E_{\mathbf{zy}}\Delta_3(\mathbf{y})\},\\
&\mathcal{B}_{110} = \{E_{\mathbf{yx}}\partial_{x_1}\Delta_3(\mathbf{x}), E_{\mathbf{yx}}\partial_{x_2}\Delta_3(\mathbf{x}), E^{(2)}_{\mathbf{yx}}\Delta_3(\mathbf{x})\},\\
&\mathcal{B}_{101} = \{E_{\mathbf{zx}}\partial_{x_1}\Delta_3(\mathbf{x}), E_{\mathbf{zx}}\partial_{x_2}\Delta_3(\mathbf{x}), E^{(2)}_{\mathbf{zx}}\Delta_3(\mathbf{x})\},\\
&\mathcal{B}_{011} = \{E_{\mathbf{zy}}\partial_{y_1}\Delta_3(\mathbf{y}), E_{\mathbf{zy}}\partial_{y_2}\Delta_3(\mathbf{y}), E^{(2)}_{\mathbf{zy}}\Delta_3(\mathbf{y})\},\\
&\mathcal{B}_{000} = \{1\}, \quad \mathcal{B}_{111} = \{E_{\mathbf{zx}}E_{\mathbf{yx}}\Delta_3(\mathbf{x})\}
\end{aligned}$$

Collecting previous observations, and doing some further explicit calcula-

tions, we get that

(9)

$$\begin{aligned}
\mathcal{H}_1(\mathbf{q}) &= 1,\\
\mathcal{H}_2(\mathbf{q}) &= 1+s_1(\mathbf{q}) \quad .\\
\mathcal{H}_3(\mathbf{q}) &= 1+2\,s_1(\mathbf{q})+2\,s_2(\mathbf{q})+s_{11}(\mathbf{q})+s_3(\mathbf{q}),\\
\mathcal{H}_4(\mathbf{q}) &= 1+3\,s_1(\mathbf{q})+5\,s_2(\mathbf{q})+3\,s_{11}(\mathbf{q})+6\,s_3(\mathbf{q})+5\,s_{21}(\mathbf{q})+s_{111}(\mathbf{q})\\
&\quad +5\,s_4(\mathbf{q})+4\,s_{31}(\mathbf{q})+3\,s_5(\mathbf{q})+s_{41}(\mathbf{q})+s_6(\mathbf{q}).
\end{aligned}$$

If we specialize one of the parameters (say q_3) to 0, we get back the graded Hilbert series of bivariate diagonal harmonics (of overall dimension $(n+1)^{n-1}$) which has received a lot of attention in recent years (see [8, 10, 13, 14, 15, 18, 21]). In other words, the $\mathbf{z}$-free (or $\mathbf{x}$-free, or $\mathbf{y}$-free) component of $\mathcal{H}_n$ coincides with the "usual" space of diagonal harmonics (in the bivariate case, as previously considered in the literature). Hence, all of our formulas involving the parameters $\mathbf{q}$ specialize to known formulas by the simple device of setting one of the three parameters in $\mathbf{q}$ equal to 0. Evaluating $\mathcal{H}_n(\mathbf{q})$ at q_i equal to 1, we clearly get the overall dimension of $\mathcal{H}_n$. For this evaluation, the value of (9) agrees with formula (2).

Graded character

We refine the dimension analysis by taking into account the decomposition into irreducibles of the homogeneous components of $\mathcal{H}_n$, see (6). This is all encompassed into the **graded Frobenius characteristic** of $\mathcal{H}$:

$$\mathcal{H}_n(\mathbf{q};\mathbf{w}) := \sum_{d\in\mathbb{N}^3} \mathbf{q}^{\mathbf{d}}\mathcal{F}_{\mathcal{H}_{n;\mathbf{d}}}(\mathbf{w}).$$

Recall that the Frobenius characteristic $\mathcal{F}_{\mathcal{V}}(\mathbf{w})$, of a $\mathbb{S}_n$-module $\mathcal{V}$, is the symmetric function (in auxiliary variables $\mathbf{w}=w_1,w_2,\ldots$) whose expansion in terms of the Schur functions $S_\lambda(\mathbf{w})$ records the multiplicity of irreducibles in $\mathcal{V}$. This is to say that we have

$$\mathcal{F}_{\mathcal{V}}(\mathbf{w}) = \sum_{\lambda\vdash n} n_\lambda\, S_\lambda(\mathbf{w}),$$

with the sum being over partitions of n, which classify irreducible representations of $\mathbb{S}_n$. For reasons also discussed in [2], the expansion of $\mathcal{H}_n(\mathbf{q};\mathbf{w})$ in terms of the $S_\lambda(\mathbf{w})$ affords Schur positive coefficients in the $s_\mu(\mathbf{q})$. This

is to say that

$$\mathcal{H}_n(\mathbf{q};\mathbf{w}) = \sum_{\lambda\vdash n}\left(\sum_{\mu} n_{\lambda,\mu} s_\mu(\mathbf{q})\right) S_\lambda(\mathbf{w}), \qquad n_{\lambda,\mu}\in\mathbb{N}. \tag{10}$$

Hence there are two kinds of Schur function playing a role here. To emphasize this, we denote those in the $\mathbf{q}$-variables by a lower case "s", and drop the variables $\mathbf{q}$. For example, we have

(11)

$$\begin{aligned}
\mathcal{H}_1(\mathbf{w};\mathbf{q}) &= S_1(\mathbf{w})\\
\mathcal{H}_2(\mathbf{w};\mathbf{q}) &= S_2(\mathbf{w}) + s_1 S_{11}(\mathbf{w})\\
\mathcal{H}_3(\mathbf{w};\mathbf{q}) &= S_3(\mathbf{w}) + (s_1+s_2)\,S_{21}(\mathbf{w}) + (s_{11}+s_3)\,S_{111}(\mathbf{w})\\
\mathcal{H}_4(\mathbf{w};\mathbf{q}) &= S_4(\mathbf{w}) + (s_1+s_2+s_3)\,S_{31}(\mathbf{w}) + (s_2+s_{21}+s_4)\,S_{22}(\mathbf{w})\\
&\quad +(s_{11}+s_3+s_{21}+s_4+s_{31}+s_5)\,S_{211}(\mathbf{w})\\
&\quad +(s_{111}+s_{31}+s_{41}+s_6)\,S_{1111}(\mathbf{w})\\
\mathcal{H}_5(\mathbf{w};\mathbf{q}) &= S_5(\mathbf{w}) + (s_1+s_2+s_3+s_4)\,S_{41}(\mathbf{w})\\
&\quad +(s_2+s_3+s_{21}+s_4+s_{31}+s_{22}+s_5+s_{41}+s_6)\,S_{32}(\mathbf{w})\\
&\quad +(s_{11}+s_3+s_{21}+s_4+2\,s_{31}+2\,s_5+s_{41}\\
&\qquad +s_{32}+s_6+s_{51}+s_7)\,S_{311}(\mathbf{w})\\
&\quad +(s_{21}+s_4+s_{31}+s_{22}+s_{211}+s_5+2\,s_{41}+s_{32}\\
&\qquad +s_{311}+s_6+2\,s_{51}+s_{42}+s_7+s_{61}+s_8)\,S_{221}(\mathbf{w})\\
&\quad +(s_{111}+s_{31}+s_{211}+2\,s_{41}+s_{32}+s_{311}+s_6+2\,s_{51}\\
&\qquad +s_{42}+s_{411}+s_{33}+s_7+2\,s_{61}+s_{52}+s_8+s_{71}+s_9)\,S_{2111}(\mathbf{w})\\
&\quad +(s_{311}+s_{42}+s_{411}+s_{61}+s_{511}+s_{43}\\
&\qquad +s_{71}+s_{62}+s_{81}+s_{10})\,S_{11111}(\mathbf{w})
\end{aligned}$$

In these expressions, the Schur functions $s_\mu = s_\mu(\mathbf{q})$ are to be evaluated in the three parameters $\mathbf{q} = q_1, q_2, q_3$. The Schur function $S_\lambda(\mathbf{w})$ act as placeholder for irreducible representations of $\mathbb{S}_n$. Thus the multiplicities of the irreducible representations of $\mathbb{S}_n$ in $\mathcal{H}_n$, classified by the partitions μ of n, appear as the coefficients of $S_\mu(\mathbf{w})$ through the evaluation of (11) at $q_1 = 1$, $q_2 = 1$ and $q_3 = 1$. In particular, the value obtained this way, as coefficient of $S_{11\cdots1}(\mathbf{w})$, agree formula (4). Furthermore, the expressions in (9) may be directly calculated from (11) by replacing each $S_\lambda(\mathbf{w})$ by the number f_μ of standard tableaux of shape μ (see [1]).

Generalized harmonics

Using notions further discussed in Section 2, all of these considerations generalize to the spaces of **higher diagonal harmonics**

$$\mathcal{H}_n^{(r)} := (\mathcal{A}_n^{r-1}) \cap (\mathcal{A}_n^{r-1}\mathcal{I}_n)^{\perp}, \tag{12}$$

where $\mathcal{A}_n$ (resp. $\mathcal{I}_n$) stands for the ideal generated by **alternating** polynomials (resp. **invariant**) polynomials in $\mathcal{R}$, and the orthogonal complement is with respect to the $\mathbb{S}_n$-invariant scalar product defined by

$$\langle X^A, X^B \rangle := \begin{cases} A! & \text{if } A = B, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Here $A!$ stands for the product of factorials:

$$A! := a_1! \cdots a_n! \cdot b_1! \cdots b_n! \cdot c_1! \cdots c_n!.$$

When r is even, the diagonal action on $\mathcal{H}_n^{(r)}$ is twisted by the sign representation. For $n = 2$, the space $\mathcal{H}_2^{(r)}$ decomposes as a direct sum of its isotypic **invariant** component, $\mathcal{I}_2^{(r)}$, and **alternant** component, $\mathcal{A}_2^{(r)}$:

$$\mathcal{H}_2^{(r)} = \mathcal{I}_2^{(r)} \oplus \mathcal{A}_2^{(r)}, \tag{14}$$

which afford the respective bases

$$\begin{aligned} \mathcal{B}\mathcal{I}_2^{(r)} &= \{(x_2 - x_1)^a (y_2 - y_1)^b (z_2 - z_1)^c \mid a + b + c = r - 1\}, \qquad \text{and} \\ \mathcal{B}\mathcal{A}_2^{(r)} &= \{(x_2 - x_1)^a (y_2 - y_1)^b (z_2 - z_1)^c \mid a + b + c = r\}. \end{aligned}$$

It follows that

$$\mathcal{H}_2^{(r)}(\mathbf{w}; \mathbf{q}) = s_{r-1} S_2(\mathbf{w}) + s_r S_{11}(\mathbf{w}),$$

from which we get the Hilbert series

$$\mathcal{H}_2^{(r)}(\mathbf{q}) = s_{r-1} + s_r, \quad \text{and} \quad \mathcal{A}_2^{(r)}(\mathbf{q}) = s_r.$$

In particular,

$$\dim \mathcal{H}_2^{(r)} = (r+1)^2, \quad \text{and} \quad \dim \mathcal{A}_2^{(r)} = \frac{(r+1)(r+2)}{2}.$$

which agree with formulas (3) and (5).

The above formulas, which clearly generalize (2) and (4), are both encompassed in the following generalization of a Conjecture of [15], in conjunction with a second conjecture below.

Conjecture 1. *The graded Frobenius characteristic of* $\mathcal{H}_n^{(r)}$ *is given by the formula*

$$\mathcal{H}_n^{(r)}(\mathbf{w}; q_1, q_2, 1) = \sum_{\beta \in \mathcal{D}_n^{(r)}} \sum_{f \in \mathcal{P}_\beta} i_\beta^{(r)}(q_1)\, q_2^{\mathrm{dinv}(f)} Q_{\mathbf{co}(f)}(\mathbf{w}), \tag{15}$$

where β *run over the set of* r*-Dyck paths, on which we consider the* r*-Tamari order* (see Section 5)*, and* $\mathcal{P}_\beta$ *denotes the set of "parking functions" of "shape"* β*.*

To finish parsing the right-hand side of (15), recall from [15] that $Q_{\mathbf{co}(f)}(\mathbf{w})$ stands for the fundamental quasi-symmetric polynomials indexed by the composition $\mathbf{co}(f)$ associated to a parking function f as described in Section 5. We also recall in Section 5 other relevant notions pertaining to Dyck paths, parking functions and the Tamari poset, including the definition of the "dinv" statistic. For a given r-Dyck path β, let us denote by $i_\beta^{(r)}(q)$ the polynomial

$$i_\beta^{(r)}(q) := \sum_{\alpha \leq \beta} q^{d(\alpha,\beta)},$$

that q-enumerates the r-Dyck paths α that are smaller than β in the r-Tamari order. Such paths are weighted by $q^{d(\alpha,\beta)}$, where $d(\alpha,\beta)$ stands for the of **length of the longest chain** going from α to β in $\mathcal{D}_n^{(r)}$. We write $i_\beta^{(r)}$ for $i_\beta^{(r)}(1)$. Since $d(\alpha,\beta) = 0$ if and only if $\alpha = \beta$, we have $i_\beta^{(r)}(0) = 1$. Thus Conjecture 1 agrees with the shuffle conjecture of [15] at $q_1 = 0$. It also follows from results of [15] that the specialization at $q_1 = q_2 = 1$ of formula (15) gives

$$\mathcal{H}_n^{(r)}(\mathbf{w}; 1, 1, 1) = \sum_{\beta \in \mathcal{D}_n^{(r)}} i_\beta^{(r)}\, e_{\mathbf{co}(\beta)}. \tag{16}$$

Here $e_{r_1 \cdots r_k} := e_{r_1} \cdots e_{r_k}$, where e_r stands for the usual r^{th} elementary symmetric function (see [22]). It may be worth underlying that the right hand side of this last equation is simply, up to twisting by the sign, the Frobenius characteristic of the action of the symmetric group on the set of pairs (f, α), where f is a r-parking function and α is a r-Dyck path lying below the shape

of f. See section 5 for more on this. Calculations suggest that we have the following explicit formula for this Frobenius characteristic.

Conjecture 2.

(17)
$$\mathcal{H}_n^{(r)}(\mathbf{w};1,1,1) = \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} (r\,n+1)^{\ell(\lambda)-2} \prod_{k\in\lambda} \binom{(r+1)\,k}{k} \frac{1}{z_\lambda} p_\lambda(\mathbf{w}).$$

Recall that, for a partition λ having d_i parts of size i, it is usual to denote by z_λ the integer

$$z_\lambda := 1^{d_1} d_1!\, 2^{d_2} d_2! \cdots n^{d_n} d_n!.$$

It is interesting to note that the special case $r = 1$ of this last formula has also been conjectured indepentely by Loktev (see [21]). It follows from (17) that we have an explicit formula for the enumeration of pairs (f, α), as above, that are fixed points for the action of a permutation having given cycle type. In generating function format, this may be stated as

(18)
$$\sum_{n\geq 0} \sum_{\beta\in\mathcal{D}_n^{(r)}} i_\beta^{(r)}\, h_{\mathbf{co}(\beta)} = \frac{1}{(rn+1)^2} \exp\left(\sum_{k\geq 1} (r\,n+1) \binom{(r+1)\,k}{k} p_k(\mathbf{w})/k \right),$$

where the e-basis has been turned into the h-basis by the removal of signs.

This also implies that the dimension of $\mathcal{H}_n^{(r)}$ may be described in two different ways, giving rise to the following elegant formula

$$\sum_{\beta\in\mathcal{D}_n^{(r)}} i_\beta^{(r)} \binom{n}{\mathbf{co}(\beta)} = (r+1)^n\,(rn+1)^{n-2}. \tag{19}$$

A first draft of this paper prompted a collaborative effort between Bousquet-Mélou, Chapuy, and the second author, to give a direct proof of this new combinatorial identity (see [6]).

On another note, we will recast formula (18) (hence also formula (17)) using the following calculations. We start with establishing the formal power series identity

$$\exp\left(\sum_{k\geq 1} a \binom{b\,k}{k} \frac{t^k}{k} \right) = 1 + \sum_{j\geq 1} ab\,(ab+jb)_{(k)}\, \frac{t^j}{j!}, \tag{20}$$

where we use the (shifted) Pochhammer symbol notation

$$(u)_{(k)} := (u-1)\,(u-2)\cdots(u-k+1).$$

To check that the above equality hold, we may proceed as follows. Let us denote by $Z = Z(t)$ the series corresponding to the right-hand side of (20), at $a = 1$. Hence,

$$Z(t) := \exp\left(\sum_{k\geq 1} \binom{b\,k}{k} \frac{t^k}{k}\right).$$

We first show that $Z(t)$ satisfies the equation

$$Z(t) = (1 + t\,Z(t))^b. \tag{21}$$

Equivalently, the logarithmic derivative of $Z(t)$ is such that

$$\frac{Z'(t)}{Z(t)} = \frac{b\,Z(t)}{1-(b-1)\,t\,Z(t)}, \tag{22}$$

and we want to check that this is equal to

$$Y(t) := \sum_{k\geq 1} \binom{b\,k}{k} t^k.$$

Now, it happens that $Y(t)$ satisfies the equation

$$Y(t) = \frac{(b\,(1+t\,Y(t)))^b}{(1+(b-1)(1+t\,Y(t)))^{b-1}}. \tag{23}$$

Replacing $Y(t)$ by the right-hand side of (22) in this last equality, after calculations, we see that $Z(t)$ satisfies (21). Using Lagrange inversion, it is then easy to check that the right-hand side of (20) is indeed equal to $Z(t)^a$.

This calculation allows us to reformulate (18) in terms of the monomial symmetric functions $m_\lambda(\mathbf{w})$, using the Cauchy identity (see [22, page 65]):

$$\sum_{\lambda\vdash n} p_\lambda(\mathbf{u})\,\frac{1}{z_\lambda}\,p_\lambda(\mathbf{w}) = \sum_{\lambda\vdash n} h_\lambda(\mathbf{u})\,m_\lambda(\mathbf{w}). \tag{24}$$

Setting $a = rn+1$ and $b = r+1$ in (20), we get

$$\exp\left(\sum_{k\geq 1}(r\,n+1)\binom{(r+1)\,k}{k}\frac{p_k(\mathbf{w})}{k}\right) = \sum_{\lambda}\prod_{k\in\lambda}(r+1)\,(rn+1)\,((r+1)(rn+k+1))_{(k)}\,m_\lambda(\mathbf{w}),$$

since it is straightforward that

$$(r+1)\,(rn+1)\,((r+1)(rn+j+1))_{(k)} = \frac{rn+1}{rn+k+1}\binom{(r+1)(rn+k+1)}{k}.$$

It follows that the right-hand side of (18) may be rewritten in the form

$$\frac{1}{(rn+1)^2}\sum_{\lambda\vdash n}\prod_{k\in\lambda}\frac{rn+1}{rn+k+1}\binom{(r+1)(rn+k+1)}{k}m_\lambda(\mathbf{w}). \tag{25}$$

The multiplicity of the alternating representation in $\mathcal{H}_n^{(r)}$ is the coefficient of $m_n(\mathbf{w})$ in this last expression. In view of (16), this also suggests that we have the following generalization of a formula of Chapoton (see [7], in the case $r=1$) for the number of intervals in the r-Tamari poset:

$$\sum_{\beta\in\mathcal{D}_n^{(r)}} i_\beta^{(r)} = \frac{(r+1)}{n\,(rn+1)}\binom{(r+1)^2\,n+r}{n-1}. \tag{26}$$

A few months before this writing, the first author conjectured that this last equality should hold. This new combinatorial identity has since been shown to be true in [5].

In view of Proposition 3, the above formulas and conjectures imply that the following combinatorial identity should also hold

$$\sum_{\beta\in\mathcal{D}_n^{(r)}} i_\beta^{(r)}\,\chi(\mathbf{co}(\beta)=11\cdots 1) \;=\; \sum_{\beta\in\mathcal{D}_n^{(r-1)}} i_\beta^{(r-1)}, \tag{27}$$

where $\chi(P)$ is equal to 1 if P is true, and 0 otherwise. A direct bijective combinatorial proof of this identity seems to follow from the approach in [5]. Moreover, it appears that (18) may be amenable to a Polya theory extension of the approach of [6] and [5]. In particular, this will reduce Conjecture 2 to Conjecture 1, as well as giving a joint proof of (19) and (26).

2. Trivariate diagonal invariants and alternants

Invariants and scalar product

For our purpose, some interesting properties of trivariate diagonal invariants and alternants need to be recalled. Much of what is discussed here is a straightforward generalization to the context of three sets of variables of results of [12] for the bivariate case. They are included here to help make our presentation understandable on its own. Omitted proofs are entirely similar to the bivariate case. A particular case of a result of Weyl, in [24], establishes that the subring of trivariate diagonal invariants is generated by the **polarized power sums**

$$P_\alpha(X) := \sum_{j=1}^{n} X_j^\alpha, \tag{28}$$

for $\alpha = (a, b, c)$ running thought the set of all $\mathbb{N}$-vectors such that $1 \leq |\alpha| \leq n$.

To make better sense out of Definition (12), we need to discuss a few notions concerning trivariate diagonal alternants. Let us first observe that definition (13) is equivalent to

$$\langle f(X), g(X) \rangle = f(\partial X) g(X)\big|_{X=0}, \tag{29}$$

where $p(X)\big|_{X=0} := p(0)$ is the **constant term** of $p(X)$.

It is a well known fact that, with respect to this scalar product, the orthogonal complement $I^\perp$, of an ideal $I = (f_1, \ldots, f_N)$, is simply the set of polynomial zeros of the differential operators $f_i(\partial X)$. Indeed, let $g(X)$ be in $I^\perp$, and consider the leading term (for any suitable term order) of

$$f_i(\partial X) g(X) = c\, X^A + \ldots,$$

if any. Since $X^A f_i(X)$ lies in I, we must have $\langle X^A f_i(X), g(X) \rangle = 0$, but this means precisely that $c = 0$. Hence we must have $f_i(\partial X) G(X) = 0$. In particular, Weyl's result implies that

$$\mathcal{I}_n^\perp = \mathcal{H}_n^{(1)} = \mathcal{H}_n.$$

To show that the spaces we consider are finite dimensional, the following is very useful.

Proposition 1. *All monomials X^A, of total degree larger than $\binom{n}{2}$ lie in $\mathcal{I}_n$.*

Alternants

A set of linear generators for trivariate alternants is easily obtained by applying the **antisymetrization operator**

$$R^{\pm}(X^A) := \sum_{\sigma \in \mathbb{S}_n} \mathrm{sign}(\sigma)\, \sigma \cdot X^A, \tag{30}$$

to any monomial X^A, with A a $3 \times n$ matrix of non-negative integers. Evidently the product of an alternant by an invariant is yet again an alternant. We get a linear basis $\Delta_A(X) := \{R^{\pm}(X^A)\}_A$, of the module of alternants over the ring of invariants, by choosing matrices A having all their columns different and appearing in decreasing lexicographic order (from left to right). Any symmetric operator in the ring $\mathbb{C}[X, \partial X]$ sends alternants to alternants. In particular, this is the case for the operators $E^{(k)}_{\mathbf{uv}}$, and $P_\alpha(\partial X)$. A degree argument shows that $P_\alpha(\partial X)\Delta_A(X) = 0$, whenever A is an order-ideal under **component** comparison of columns:

$$B \leq C, \qquad \text{iff} \qquad \forall i\ (b_i \leq c_i).$$

For instance, this is the case for the Vandermonde determinants. Moreover, a simple direct calculation of the commutator of the operators $E^{(k)}_{\mathbf{yx}}$ and $P_\alpha(\partial X)$, with $\alpha = (a, b, c)$, gives that

$$[P_\alpha(\partial X), E^{(k)}_{\mathbf{yx}}] = a\, P_\beta(\partial X), \tag{31}$$

where $\beta = \alpha + (k, -1, 0)$. It is evident that similar results holds for other possible choices of variable sets in $E^{(k)}_{\mathbf{uv}}$. Putting all this together, we deduce that

Proposition 2. *If $A \in \mathbb{N}^{3\times n}$ is an order-ideal, then*

$$(\partial X^A)\Omega(\Delta_A(X)) \in \mathcal{H}_n, \tag{32}$$

for any A, and any composition Ω of the operators $E^{(k)}_{\mathbf{uv}}$.

The "Operator Theorem" of [18] states that Proposition 2 entirely characterizes the space $\mathcal{H}_n$ in the bivariate case. More precisely, we have the following.

Theorem 1 (Haiman). *The space of diagonal harmonics for the bivariate case is the smallest vector space containing the Vandermonde determinant*

$\Delta_n(\mathbf{x})$, *which is closed under taking partial derivatives and applications of the operators* $E^{(k)}_{\mathbf{yx}}$.

For the trivariate case, experiments suggest that the analogous statement should hold. However, the methods employed by Haiman to settle the bivariate case do not seem to generalize to the trivariate case.

Another result along these lines is that, for the bivariate case, we can calculate the entire space $\mathcal{A}_n^{(r)}$ (resp. $\mathcal{I}_n^{(r)}$) by successive applications of the operators $E^{(k)}_{\mathbf{yx}}$, starting with the r^{th}-power (resp. $(r-1)^{\text{th}}$-power) of the Vandermonde determinant $\Delta_n(\mathbf{x})$. However, we do not know how to explicitly construct a basis in this manner. All, this also appears to hold in the trivariate case. In particular, we have that

Proposition 3. *For all* n, *and all* $r \geq 1$, *the spaces* $\mathcal{I}_n^{(r)}$ *and* $\mathcal{A}_n^{(r-1)}$ *coincide.*

3. Action of the general linear group

To make apparent a special feature of both the Hilbert series and the (graded) Frobenius characteristic of the spaces $\mathcal{H}_n^{(r)}$, we need to consider it as a polynomial GL_3-sub-representation of $\mathcal{R}_n = \mathbb{C}[X]$, for the action

$$f(X) \mapsto f(M\,X), \qquad \text{for} \qquad M \in GL_3. \tag{33}$$

Clearly this action commutes with the action of S_n. Recall that the **character** of a GL_3 representation is the symmetric function of the parameters $\mathbf{q} = q_1, q_2, q_3$ obtained by calculating the trace of the linear transformation

$$Q^*(f(X)) := f(Q\,X),$$

where Q is the diagonal matrix $Q = [q_1, q_2, q_3]$. Observe that a polynomial $f(X)$ is **homogeneous** of degree d if and only if

$$f(Q\,X) = \mathbf{q}^d\, f(X).$$

Since an homogenous subspace $\mathcal{V}$, of $\mathcal{R}_n$, is simply a subspace that affords a basis $\mathcal{B}$ of homogeneous polynomials, it is G_3-invariant and its character $\mathcal{V}(\mathbf{q})$ is

$$\mathcal{V}(\mathbf{q}) = \sum_{f \in \mathcal{B}} \mathbf{q}^{\deg(f)}.$$

This is precisely what we have called previously the Hilbert series of $\mathcal{V}$. It is well known that the characters, of irreducible polynomial GL_3-representations,

are the Schur polynomials (evaluated at q_1, q_2, q_3). Hence, $\mathcal{V}(\mathbf{q})$ expands as a nonnegative integer coefficient linear combination of the $s_\mu(\mathbf{q})$, since the coefficients correspond to multiplicities of irreducibles.

Since the two actions commute, we get a double decomposition into irreducibles, for both $\mathbb{S}_n$ and GL_3, of any subspace that is stable under the two actions. This is the reason why the graded Frobenius characteristic of such a space expands as a positive sum of products $s_\mu(\mathbf{q})S_\lambda(\mathbf{w})$.

4. The whole story for the case $n = 3$

Using observations and results of [2], we can calculate explicitly $\mathcal{H}_3^{(r)}(\mathbf{w};\mathbf{q})$ in full generality. The crucial observation is that the only Schur function $s_\mu(\mathbf{q})$ that can occur in the expansion of $\mathcal{H}_3^{(r)}(\mathbf{w};\mathbf{q})$ are those indexed by partitions having at most two parts. We can thus deduce the trivariate case[2] expansion from the known bivariate case.

Recall that, in the bivariate case we have $\mathcal{H}_3^{(r)}(\mathbf{w};q,t) = \nabla^r(S_{111}(\mathbf{w}))$, where ∇ is an operator characterized as follows. As usual (see [22]), we denote by $\lambda \preceq \mu$ the **dominance order** on partitions, and μ' stands for the **conjugate** partition of μ. The **integral form Macdonald polynomials** $H_\mu(\mathbf{w};q,t)$, with μ a partition of n form a linear basis of the ring Λ, of symmetric functions in the variables $\mathbf{w}$, and are characterized by the equations

$$\begin{aligned}
&\text{(i)}\ \langle S_\lambda(\mathbf{w}), H_\mu[(1-q)\,\mathbf{w};q,t]\rangle = 0, && \text{if} \quad \lambda \not\preceq \mu,\\
&\text{(ii)}\ \langle S_\lambda(\mathbf{w}), H_\mu[(1-t)\,\mathbf{w};q,t]\rangle = 0, && \text{if} \quad \lambda \not\preceq \mu', \text{ and}\\
&\text{(iii)}\ \langle S_n(\mathbf{w}), H_\mu(\mathbf{w};q,t)\rangle = 1,
\end{aligned} \tag{34}$$

involving the usual "Hall" scalar product on symmetric functions (for which the Schur functions are orthonormal), and some plethystic notation (see [1]). The linear operator ∇ is characterized by the fact that it affords the Macdonald polynomials as eigenfunctions:

$$\nabla(H_\mu(\mathbf{w};q,t)) = H_\mu(\mathbf{w};q,t) \prod_{(a,b)\in\mu} q^a t^b.$$

This presentation may seem a bit terse, but for our purpose we only need the case $n = 3$ for which an explicit description of ∇ is given below. Using this description, we will infer the graded Frobenius of $\mathcal{H}_3^{(r)}$ for the trivariate case,

[2] In fact, this particular calculation holds for any number of set of variables.

out of the simple knowledge of the bivariate case. To this end, we consider the linear operator corresponding to the matrix

$$\nabla = \begin{pmatrix} 0 & s_{22} & s_{32} \\ 0 & -s_{21} & -s_{31} \\ 1 & s_1+s_2 & s_{11}+s_3 \end{pmatrix}, \tag{35}$$

with respect to the basis $\{S_3(\mathbf{w}), S_{21}(\mathbf{w}), S_{111}(\mathbf{w})\}$. Then[3],

Proposition 4. *For the trivariate case, we have the explicit formula*

$$\mathcal{H}_3^{(r)}(\mathbf{w};\mathbf{q}) = \nabla^r(S_{111}(\mathbf{w})). \tag{36}$$

We may calculate from (35) that

Lemma 1. *The characteristic polynomial of the operator ∇, over the ring $\Lambda^{(2)}(\mathbf{q})$, is*

$$\nabla^3 - (s_3 - s_{21} + s_{11})\nabla^2 + (s_{41} + s_{33} - s_{32})\nabla - s_{44}. \tag{37}$$

One may observe that $s_{11}(\mathbf{q}) = q_1q_2+q_1q_3+q_2q_3$ is one of the eigenvalues of ∇. The associated eigenspace is spanned by the symmetric function

$$S_3(\mathbf{w}) + (q_1+q_2+q_3)\,S_{21}(\mathbf{w}) + (q_1q_2+q_1q_3+q_2q_3)\,S_{111}(\mathbf{w}),$$

which is a three parameter version of a Macdonald polynomial. Wether one can develop a theory of these polynomials for all n remains an open question. One would expect these to appear as eigenfunctions for a suitable generalization of ∇.

In view Proposition 4, Lemma 1 leads to a recursive approach for the calculation of the graded Frobenius characteristic of any of the spaces $\mathcal{H}_3^{(r)}$, based on the initial values

$$\begin{aligned}
\mathcal{H}_3^{(0)}(\mathbf{w};\mathbf{q}) &= S_{111}(\mathbf{w}),\\
\mathcal{H}_3^{(1)}(\mathbf{w};\mathbf{q}) &= S_3(\mathbf{w}) + (s_2+s_1)\,S_{21}(\mathbf{w}) + (s_3+s_{11})\,S_{111}(\mathbf{w}),\\
\mathcal{H}_3^{(2)}(\mathbf{w};\mathbf{q}) &= (s_{11}+s_3)S_3(\mathbf{w}) + (s_{21}+s_4+s_{31}+s_5)S_{21}(\mathbf{w})\\
&\quad +(s_{2,2}+s_{41}+s_6)S_{111}(\mathbf{w}).
\end{aligned}$$

[3]All calculations are to be done over the ring $\Lambda^{(2)}(\mathbf{q})$ of symmetric functions modulo the ideal generated by the s_μ for which μ has at least three parts.

Calculations are to be done in the ring $\Lambda^{(2)}(\mathbf{q})$ of symmetric functions spanned by Schur functions indexed by partitions that have at most two parts. In order to describe the resulting formulas, let us introduce the notation

$$T_{nk} := s_n + s_{n-2,1} + \ldots + s_{n-2\,k,k}. \tag{38}$$

After calculations, we get the following.

Proposition 5. *The space $\mathcal{H}_3^{(r)}$ affords the explicit graded Frobenius character formula*

$$\begin{aligned}\mathcal{H}_3^{(r)}(\mathbf{w};\mathbf{q}) \;=\; & T_{3\,(r-1),r-1}\, S_3(\mathbf{w}) \\ & +(T_{3\,r-2,r-1} + T_{3\,r-1,r-1})\, S_{21}(\mathbf{w}) \\ & +T_{3\,r,r}\, S_{111}(\mathbf{w}).\end{aligned}$$

In particular, the Hilbert series of the alternating part $\mathcal{A}_3^{(r)}$ of this space appears as the coefficient of $S_{111}(\mathbf{w})$ in this formula, i.e.:

$$\mathcal{A}_3^{(r)}(\mathbf{q}) = s_{3\,r} + s_{3\,r-2,1} + \ldots + s_{r,r}. \tag{39}$$

For any $a \geq b \geq 0$, the Schur function $s_{ab} = s_{ab}(q_1, q_2, q_3)$ specializes to

$$\frac{1}{2}(b+1)(a+2)(a-b+1),$$

when one sets $q_1 = q_2 = q_3 = 1$. From this, it follows that

$$T_{nk}(1,1,1) = \frac{1}{12}\,(k+1)\,(k+2)\left(k^2 - (13+10\,n)\,k + 3\,(n+1)\,(n+2)\right)$$

We may then check directly that formula of Proposition 5 specializes, at $q_1 = q_2 = q_3 = 1$, to the case $n = 3$ of (17).

5. Dyck paths, Tamari posets, and parking functions

Generalized Dyck paths and Tamari posets

Recall that a r-Dyck path of height n is a path in $\mathbb{N} \times \mathbb{N}$, that starts at $(0,0)$, ends at (rn, n), and stays above the line $x = ry$. For instance, we have the 2-Dyck path of Figure 1. Such a path may be entirely described by a sequence $\alpha = (a_1, a_2, \ldots, a_n)$, where (a_i, i) corresponds to the leftmost point lying on

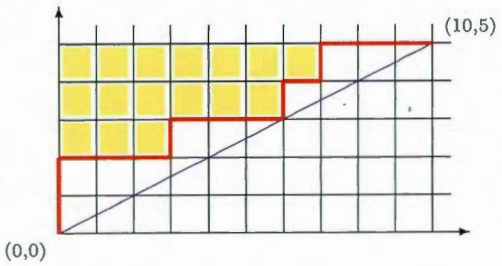

Figure 1: The 2-Dyck path encodes as 00367.

the path at height i. I other words, a_i is the number of "boxes" lying to the left of the path at height i. In this manner, α is a **r-Dyck path** if and only if

(1) $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n$,

(2) for each i, we have $a_i \leq r(i-1)$.

We denote by $\mathcal{D}_n^{(r)}$ the set of r-Dyck paths of height n. For example,

$$\mathcal{D}_3^{(2)} = \{000, 001, 002, 003, 004, 011, 012, 013, 014, 022, 023, 024\}$$

It is well known (see [9, 11]) that the number of r-Dyck paths of height n is given by the Fuss-Catalan number:

$$\#\mathcal{D}_n^{(r)} = \frac{1}{rn+1}\binom{(r+1)n}{n}.$$

We say that $(a_i, a_{i+1}, \ldots, a_k)$ is a **primitive subsequence**, of a r-Dyck path $(a_1, \ldots, a_n)$, if

(1) $a_j - a_i < r(j-i)$ for each $i < j \leq k$, and

(2) either $k = n$, or $a_{k+1} - a_i \geq r(k+1-i)$.

For each i, there is a unique such primitive subsequence. It corresponds to the portion of the r-Dyck path that starts at $(a_i, i-1)$ and ends at the "first return" of the path to the line of slope $1/r$ passing through the point $(a_i, i-1)$. Whenever i is such that $a_{i-1} < a_i$, we set $\alpha \leq \beta$, where

$$\beta := (a_1, \ldots, a_{i-1}, a_i - 1, \ldots, a_k - 1, a_{k+1}, \ldots, a_n),$$

with $(a_i, \ldots, a_k)$ equal to the primitive subsequence starting at a_i. The r-Tamari poset[4], on the underlying set $\mathcal{D}_n^{(r)}$, is the order obtained as the reflexive and transitive closure of this covering relation $\alpha \leq \beta$. Its largest element is the path $00\cdots0$, and its smallest is the path for which $a_i = r\,(i-1)$.

For a given r-Dyck path β, we have already considered the number $i_\beta^{(r)}$ of r-paths that smaller than β in $\mathcal{D}_n^{(r)}$:

$$i_\beta^{(r)} := \#\{\alpha \in \mathcal{D}_n^{(r)} \mid \alpha \leq \beta\},$$

and denoted by $d(\alpha, \beta)$ the length of the longest chain going from α to β in $\mathcal{D}_n^{(r)}$. Observe that, when α is the smallest element of $\mathcal{D}_n^{(r)}$, $d(\alpha, \beta)$ coincides with the usual "area" statistic,

$$\mathrm{area}(\beta) := r\binom{n}{2} - \sum_{i=1}^{n} a_i,$$

on r-Dyck path β. For $n = 4$ and $r = 1$, the Tamari poset is drawn up in Figure 2. To a r-Dyck path β we associate the composition $\mathrm{co}(\beta)$ whose

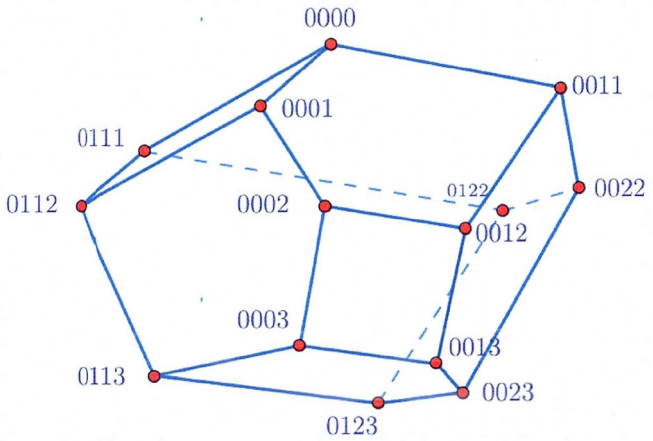

Figure 2: The 1-Tamari poset for $n = 4$.

parts are the length of runs of 1 in β. Thus $\mathrm{co}(00112) = 221$.

[4]As far as we know this poset has not been considered before.

Parking functions

For a sequence of positive integers $f = (f_1, f_2, \ldots, f_n)$ (we also write $f = f_1 f_2 \cdots f_n$), let

$$\binom{\alpha(f)}{\beta(f)} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} \tag{40}$$

be the **lex-increasing rearrangement** of

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ f_1 & f_2 & \ldots & f_n \end{pmatrix},$$

ordering first with respect to the bottom row. This is to say that $b_i \leq b_{i+1}$, for all $1 \leq i < n$, and that $a_i < a_{i+1}$ when $b_i = b_{i+1}$. One says that f is a **r-parking function** if $\beta(f)$ is such that $b_k \leq r(k-1)$, for all k. Perforce, $\beta(f)$ is thus a r-Dyck path. It is said to be the **shape** of f.

For example, with $r = 2$, we have the following 49 r-parking functions of length 3:

000	001	002	003	004	010	011
012	013	014	020	021	022	023
024	030	031	032	040	041	042
100	101	102	103	104	110	120
130	140	200	201	202	203	204
210	220	230	240	300	301	302
310	320	400	401	402	410	420.

The set of r-parking functions of length n, denoted by $\mathcal{P}_n^{(r)}$, has cardinality $(rn+1)^{n-1}$ (see [25]). For a r-Dyck path β, we further denote by $\mathcal{P}_\beta$ the set of all parking functions of shape β. Observe that these are automatically r-parking functions. Clearly we get a permutation action of symmetric group $\mathbb{S}_n$ on each of the $\mathcal{P}_\beta$ (by permutation of values). The corresponding Frobenius characteristic is easily seen to be equal to $h_{\mathbf{co}(\beta)}(\mathbf{w})$, where $h_{c_1 \cdots c_k}(\mathbf{w}) := h_{c_1}(\mathbf{w}) \cdots h_{c_k}(\mathbf{w})$ is the complete homogeneous symmetric function associated to the composition $\mathbf{co}(\beta) = c_1 \cdots c_k$. In particular, the number of parking functions of shape β is given by the multinomial coefficient

$$\#\mathcal{P}_\beta = \binom{n}{\mathbf{co}(\beta)} = \frac{n!}{c_1! c_2! \cdots c_k!}. \tag{41}$$

Formula (41) reflects the fact that any representation of $\mathbb{S}_n$, having $h_c(\mathbf{w})$ as its Frobenius characteristic, must be of dimension $\binom{n}{c}$. From this we get the identity

$$(rn+1)^{n-1} = \sum_{\beta \in \mathcal{D}_n^{(r)}} \binom{n}{\mathbf{co}(\beta)} \tag{42}$$

Recall that it has been shown in [18] that the (ungraded) Frobenius characteristic of the $\mathbf{z}$-free component of $\mathcal{H}_n$ is equal to the character of the $\mathbb{S}_n$-module $\mathcal{P}_n^{(r)}$, twisted by the sign representation. From the point of view of Frobenius characteristics, this twist turns $h_c(\mathbf{w})$ into $e_c(\mathbf{w})$. Hence, one consequence of the results in [18] may be stated as

$$\mathcal{H}_n^{(r)}(\mathbf{w}; 1, 1, 0) = \sum_{\beta \in \mathcal{D}_n^{(r)}} e_{\mathbf{co}(\beta)}(\mathbf{w}). \tag{43}$$

The composition $\mathbf{co}(f) = c_1 c_2 \cdots c_k$, associated to a r-parking function f, is simply the composition that encodes the descents of the permutation $\alpha(f)$.

For the purpose of stating Conjecture 1, and discussing potential extensions (see next section), let us recall (from [15]) how to calculate the "dinv" statistic of a parking function. Given a r-parking function f, let a_i and b_i be as in (40), and let $c_i := r\,i - b_i$. For $1 \leq i < j \leq n$ and $0 \leq d \leq r-1$, we say that a triple (i, j, d) is a d-inversion of f if either

- $c_i - c_j + d = 0$ and $a_i < a_j$, or
- $1 \leq c_i - c_j + d \leq r-1$, or
- $c_i - c_j + d = r$ and $a_i > a_j$.

Then $\mathrm{dinv}(f)$ is simply the number of d-inversions of f. Notice that this actually depends on the value of r.

6. Final remarks

A missing component in (15) is a "statistic" $\nu(f, \alpha)$ that would account for the behavior of the third parameter q_3, with respect to pairs (f, α), for $f \in \mathcal{P}_\beta$ and $\alpha \leq \beta$ in $\mathcal{D}_n^{(r)}$. Such a statistic would give a complete combinatorial description of $\mathcal{H}_n^{(r)}(\mathbf{w}; q_1, q_2, q_3)$ in the form

$$\mathcal{H}_n^{(r)}(\mathbf{w}; q_1, q_2, q_3) = \sum_{\beta \in \mathcal{D}_n^{(r)}} \sum_{f \in \mathcal{P}_\beta} \sum_{\alpha \leq \beta} q_1^{d(\alpha,\beta)} q_2^{\mathrm{dinv}(f)} q_3^{\nu(f,\alpha)} Q_{\mathbf{co}(f)}(\mathbf{w}). \tag{44}$$

We would expect ν to be such that $\nu(f,\alpha)=0$ if and only if α is the smallest element of $\mathcal{D}_n^{(r)}$. Then, at $q_3=0$, formula (44) would specialize precisely to the Conjecture of Haiman *et al.* in [15]. However, such a statistic still has to be found, even in the case $r=1$.

Much of what is discussed here seems to hold for generalized permutation groups $G(r,n)$. This will be explored in upcoming work.

Thanks

We thank Mark Haiman for insightful suggestions to the effect that we should extend our approach to the case of the r-spaces $\mathcal{H}_n^{(r)}$.

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CHAPITRE IV

DEUXIÈME ARTICLE : THE NUMBER OF INTERVALS IN THE m-TAMARI LATTICES

Cet article a été publié dans *The electronic journal of combinatorics*[1]. Notre résultat principal est l'énumération des intervalles du treillis de m-Tamari et nous résolvons ainsi la conjecture 9 de Bergeron. Voici les grandes lignes de cet article.

Nous donnons une nouvelle caractérisation de la relation d'ordre entre les chemins de Dyck dans le treillis de m-Tamari en terme d'une « fonction distance ». Plus précisément, soit $D \in \text{Dyck}_m(n)$. Pour chacun des n pas nord dans D, soit D_i le sous-chemin de D commencant par le $i^{\text{ème}}$ pas nord dans D et qui est un chemin de m-Dyck primitif. Pour $m = 1$, la *fonction distance* $\text{dist}_D : \{1, 2, .., n\} \to \{1, 2, .., n\}$ est définie en posant que $\text{dist}_D(i)$ est le nombre de pas nord dans D_i. Dans le cas du treillis de m-Tamari, chacun des pas nord d'un chemin de m-Dyck est grossi par un facteur m. Pour $D \in \text{Dyck}_m(n)$, on obtient donc une fonction $\text{dist}_D : \{1, .., mn\} \to \{1, .., mn\}$.

Nous avons obtenu le lemme suivant.

Lemme 1 *Soit $D_1, D_2 \in Dyck_m(n)$. Alors $D_1 \leq D_2$ dans le treillis de m-Tamari si et seulememt si $\forall i \in \{1, .., mn\}$: $dist_{D_1}(i) \leq dist_{D_2}(i)$.*

Ce lemme permet une description récursive des intervalles du treillis de m-Tamari.

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1. Je remercie *The electronic journal of combinatorics* de m'avoir autorisé d'insérer cet article dans cette thèse.

Soit $[D_1, D_2]$ un intervalle du treillis de m-Tamari. Tout comme dans le cas $m = 1$ de la section 1.10.2, la variable x « compte » le nombre de contacts de D_1 avec la droite de pente $1/m$ passant par l'origine. Nous ajoutons également une variable y dont la puissance compte le nombre de pas nord consécutifs (la hauteur) dans la première montée de D_2 (la première montée est celle qui commence à l'origine). Soit Δ l'opérateur défini par l'équation 1.12. Soit $F(t; x, y)$ la fonction génératrice (ordinaire) des intervalles du treillis de m-Tamari donnée explicitement par

$$F(t; x, y) := \sum_{n \geq 0} t^n \sum_{D_1 \in \mathrm{Dyck}_m(n)} x^{\mathrm{contacts}(D_1)} \sum_{D_1 \leq D_2} y^{\mathrm{montée}(D_2)},$$

où la somme de droite a lieu sur les chemins de Dyck plus grands ou égaux à D_1 dans le treillis de m-Tamari. Nous avons démontré que la série $F(t; x, y)$ satisfait l'équation fonctionnelle suivante :

$$F(t; x, y) = \frac{1}{1 - xyt\,(F(t; x, 1)\ \Delta)^m}\ (x), \tag{4.1}$$

qui est clairement équivalente à

$$F(t; x, y) = x + xyt\,(F(t; x, 1)\ \Delta)^m\,(F(t; x, y)). \tag{4.2}$$

Nous avons introduit une belle paramétrisation de $F(t; x, y)$ à l'aide des changements de variables suivants

$$t = z(1 - z)^{m^2+2m}, \quad x = \frac{1 + u}{(1 + zu)^{m+1}} \quad \text{et} \quad y = \frac{1 + v}{(1 + zv)^{m+1}}. \tag{4.3}$$

On obtient alors

$$\frac{F(t; x, y)}{x} = \frac{(1 + u)(1 + zu)(1 + v)(1 + zv)}{(u - v)(1 - zuv)(1 - z)^{m+2}} \left(\frac{(1 + zv)^{m+1}}{1 + v} - \frac{(1 + zu)^{m+1}}{1 + u} \right). \tag{4.4}$$

Nous avons démontré (après changements de variables) que cette paramétrisation satisfait bien l'équation 4.2. En spécialisant à $y = 1$, on trouve

$$F(t; x, 1) = \frac{(1 + u)(1 + zu)}{(1 + zu)^{m+1}(1 - z)^{m+2}} \left(\frac{1 + u - (1 + zu)^{m+1}}{u} \right), \tag{4.5}$$

Puis, pour $x = 1$, on a

$$F(t; 1, 1) = \frac{1 - (m + 1)z}{(1 - z)^{m+2}}. \tag{4.6}$$

En utilisant l'inversion de Lagrange, nous obtenons ainsi une preuve de la conjecture 9 proposée par Bergeron, à savoir que :

Corollaire 3 *Le nombre d'intervalles de hauteur n dans le treillis de m-Tamari est donné par*

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}. \tag{4.7}$$

Un intervalle est dit primitif si le chemin inférieur de l'intervalle possède uniquement deux contacts. Compter le nombre d'intervalles primitifs revient donc à calculer $[x^2]F(t;x,1)$. En utilisant l'inversion de Lagrange à partir de la formule 4.5, on obtient que le nombre d'intervalles primitifs de hauteur n est donné par

$$\frac{m}{n((m+1)n-1)}\binom{(m+1)^2n-m-1}{n-1}.$$

Soit $\breve{F}(t;x,y) = \frac{F(t;x;y)}{x}$. Il est clair d'après la paramétrisation 4.4 que $\breve{F}(t;x,y) = \breve{F}(t;y,x)$. Ceci implique le résultat suivant :

Corollaire 4 *La double distribution du nombre de retours du chemin inférieur et de la hauteur de la première montée du chemin supérieur des intervalles dans le treillis de m-Tamari est symétrique.*

Soit $F(t;x,y,q_1)$ la même série en t qu'auparavant à laquelle on ajoute la variable q_1 qui compte la longueur de la plus grande chaîne dans un intervalle. La généralisation suivante de l'équation 4.2 permet de calculer cette série :

$$F(t;x,y,q_1) = x + xyt\,(F(t;x,1,q_1)\;\Delta_{q_1})^m\,(F(t;x,y,q_1)), \tag{4.8}$$

où

$$\Delta_{q_1}(R(t;x,q_1)) := \frac{R(t;xq_1,q_1) - R(t;1,q_1)}{xq_1 - 1}.$$

The number of intervals in the m-Tamari lattices

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To Doron Zeilberger, on the occasion of his 60th birthday

Abstract

An m-ballot path of size n is a path on the square grid consisting of north and east steps, starting at $(0,0)$, ending at (mn,n), and never going below the line $\{x = my\}$. The set of these paths can be equipped with a lattice structure, called the m-Tamari lattice and denoted by $\mathcal{T}_n^{(m)}$, which generalizes the usual Tamari lattice $\mathcal{T}_n$ obtained when $m = 1$. We prove that the number of intervals in this lattice is

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$$

This formula was recently conjectured by Bergeron in connection with the study of diagonal coinvariant spaces. The case $m = 1$ was proved a few years ago by Chapoton. Our proof is based on a recursive description of intervals, which translates into a functional equation satisfied by the associated generating function. The solution of this equation is an algebraic series, obtained by a guess-and-check approach. Finding a bijective proof remains an open problem.

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1 Introduction

A *ballot path* of size n is a path on the square lattice, consisting of north and east steps, starting at $(0,0)$, ending at (n,n), and never going below the diagonal $\{x = y\}$. There are three standard ways, often named after Stanley, Kreweras and Tamari, to endow the set of ballot paths of size n with a lattice structure (see [15, 20, 22], and [4] or [21] for a survey). We focus here on the *Tamari lattice* $\mathcal{T}_n$, which, as detailed in the following proposition, is conveniently described by the associated covering relation. See Figure 1 for an illustration.

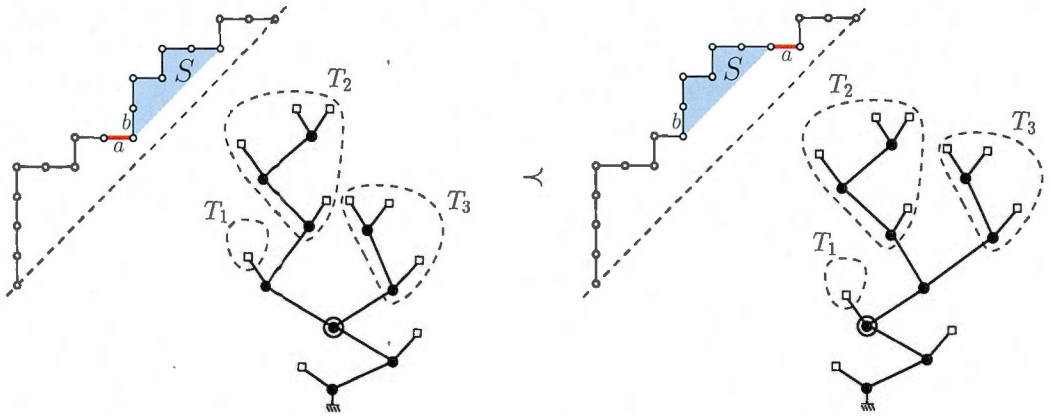

Figure 1: A covering relation in the Tamari lattice, shown on ballot paths and binary trees. The path encodes the postorder of the tree (apart from the first leaf).

Proposition 1. [4, Prop. 2.1] *Let P and Q be two ballot paths of size n. Then Q covers P in the Tamari lattice $\mathcal{T}_n$ if and only if there exists in P an east step a, followed by a north step b, such that Q is obtained from P by swapping a and S, where S is the shortest factor of P that begins with b and is a (translated) ballot path.*

Alternatively, the Tamari lattice $\mathcal{T}_n$ is often described in terms of rooted binary trees. The covering relation amounts to a re-organization of three subtrees, often called *rotation* (Figure 1). The equivalence between the two descriptions is obtained by reading the tree in postorder, and encoding each leaf (resp. inner node) by a north (resp. east) step (apart from the first leaf, which is not encoded). We refer to [4, Sec. 2] for details. The Hasse diagram of the lattice $\mathcal{T}_n$ is the 1-skeleton of the *associahedron*, or *Stasheff polytope* [11].

A few years ago, Chapoton [12] proved that the number of *intervals* in $\mathcal{T}_n$ (*i.e.*, pairs $P, Q \in \mathcal{T}_n$ such that $P \leq Q$) is

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$

He observed that this number is known to count 3-connected planar triangulations on $n+3$ vertices [30]. Motivated by this result, Bernardi and Bonichon found a beautiful bijection

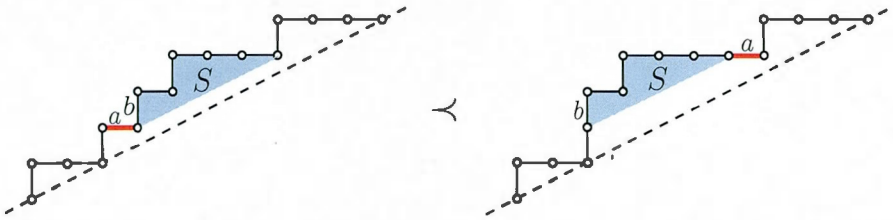

Figure 2: The relation $\prec$ between m-ballot paths ($m = 2$).

between Tamari intervals and triangulations [4]. This bijection is in fact a restriction of a more general bijection between intervals in the Stanley lattice and *Schnyder woods*. A further restriction leads to the enumeration of intervals of the Kreweras lattice.

In this paper, we study a generalization of the Tamari lattices to *m-ballot paths* due to Bergeron, and count the intervals of these lattices. Again, a remarkably simple formula holds (see (1)). As we explain below, this formula was first conjectured by F. Bergeron, in connection with the study of coinvariant spaces.

An *m-ballot path* of size n is a path on the square grid consisting of north and east steps, starting at $(0, 0)$, ending at (mn, n), and never going below the line $\{x = my\}$. It is a classical exercice to show that there are $\frac{1}{mn+1}\binom{(m+1)n}{n}$ such paths [14]. Consider the following relation $\prec$ on m-ballot paths, illustrated in Figure 2.

Definition 2. *Let P and Q be two m-ballot paths of size n. Then $P \prec Q$ if there exists in P an east step a, followed by a north step b, such that Q is obtained from P by swapping a and S, where S is the shortest factor of P that begins with b and is a (translated) m-ballot path.*

As we shall see, the transitive closure of $\prec$ defines a lattice on m-ballot paths of size n. We call it the m-Tamari lattice of size n, and denote it by $\mathcal{T}_n^{(m)}$. Of course, $\mathcal{T}_n^{(1)}$ coincides with $\mathcal{T}_n$. See Figure 3 for examples. The main result of this paper is a closed form expression for the number $f_n^{(m)}$ of intervals in $\mathcal{T}_n^{(m)}$:

$$f_n^{(m)} = \frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}. \tag{1}$$

The first step of our proof establishes that $\mathcal{T}_n^{(m)}$ is in fact isomorphic to a sublattice (and more precisely, an upper ideal) of $\mathcal{T}_{mn}$. We then proceed with a recursive description of the intervals of $\mathcal{T}_n^{(m)}$, which translates into a functional equation for the associated generating function (Section 2, Proposition 8). This generating function keeps track of the size of the paths, but also of a *catalytic parameter*[1] that is needed to write the equation. This parameter is the number of contacts of the lower path with the line $\{x = my\}$. A general theorem asserts that the solution of the equation is algebraic [7], and gives a systematic procedure to solve it for small values of m. However, for a generic value of

[1]This terminology is due to Zeilberger [32].

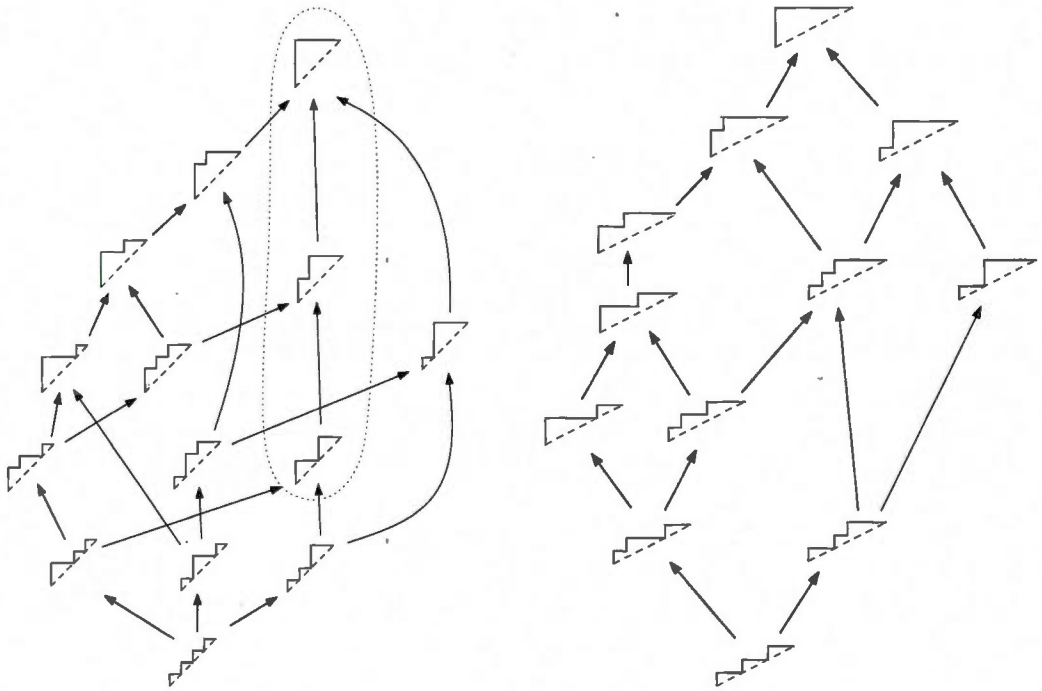

Figure 3: The m-Tamari lattice $\mathcal{T}_n^{(m)}$ for $m = 1$ and $n = 4$ (left) and for $m = 2$ and $n = 3$ (right). The three walks surrounded by a line in $\mathcal{T}_4^{(1)}$ form a lattice that is isomorphic to $\mathcal{T}_2^{(2)}$. This will be generalized in Section 2.

m, we have to resort to a guess-and-check approach to solve the equation (Section 3, Theorem 10). We enrich our enumeration by taking into account the *initial rise* of the upper path, that is, the length of its initial run of north steps. We obtain an unexpected symmetry result: the joint distribution of the number of contacts of the lower path (minus one) and the initial rise of the upper path is symmetric. Section 4 presents comments and questions.

To conclude this introduction, we describe the algebraic problem that led Bergeron to conjecture (1).

Let $X = (x_{i,j})_{\substack{1 \le i \le \ell \\ 1 \le j \le n}}$ be a matrix of variables, for some positive integers $\ell, n \ge 1$. We call each line of X a *set of variables*. Let $\mathbb{C}[X]$ be the ring of polynomials in the variables of X. The symmetric group $\mathfrak{S}_n$ acts as a representation on $\mathbb{C}[X]$ by permuting the columns of X. That is, if $\sigma \in \mathfrak{S}_n$ and $f(X) \in \mathbb{C}[X]$, then

$$\sigma(f(X)) = f(\sigma(X)) = f((x_{i,\sigma(j)})_{\substack{1 \le i \le \ell \\ 1 \le j \le n}}).$$

We consider the ideal I of $\mathbb{C}[X]$ generated by $\mathfrak{S}_n$-invariant polynomials having no constant term. The quotient ring $\mathbb{C}[X]/I$ is (multi-)graded because I is (multi-)homogeneous, and is a representation of $\mathfrak{S}_n$ because I is invariant under the action of $\mathfrak{S}_n$. We focus on the dimension of this quotient ring, and to the dimension of the sign subrepresentation. We denote by W^ε the sign subrepresentation of a representation W.

Let us begin with the classical case of a single set of variables. When $X = [x_1, \ldots, x_n]$, we consider the *coinvariant space* R_n, defined by

$$R_n = \mathbb{C}[X] \Big/ \Big\langle \Big\{ \sum_{i=1}^{n} x_i^r \Big| r \geq 1 \Big\} \Big\rangle,$$

where $\langle S \rangle$ denotes the ideal generated by the set S. It is known [1] that R_n is isomorphic to the regular representation of $\mathfrak{S}_n$. In particular, $\dim(R_n) = n!$ and $\dim(R_n^{\varepsilon}) = 1$. There exist explicit bases of R_n indexed by permutations.

Let us now move to two sets of variables. In the early nineties, Garsia and Haiman introduced an analogue of R_n for $X = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$, and called it the *diagonal coinvariant space* [19]:

$$\mathrm{DR}_{2,n} = \mathbb{C}[X] \Big/ \Big\langle \Big\{ \sum_{i=1}^{n} x_i^r y_i^t \Big| r + t \geq 1 \Big\} \Big\rangle.$$

About ten years later, using advanced algebraic geometry [18], Haiman settled several conjectures of [19] concerning this space, proving in particular that

$$\dim(\mathrm{DR}_{2,n}) = (n+1)^{n-1} \quad \text{and} \quad \dim(\mathrm{DR}_{2,n}^{\varepsilon}) = \frac{1}{n+1}\binom{2n}{n}. \tag{2}$$

He also studied an extension of $\mathrm{DR}_{2,n}$ involving an integer parameter m and the ideal $\mathcal{A}$ generated by *alternants* [16, 17]:

$$\mathcal{A} = \big\langle \{ f(x) | \sigma(f(X)) = (-1)^{\mathrm{inv}(\sigma)} f(X), \forall \sigma \in \mathfrak{S}_n \} \big\rangle.$$

There is a natural action of $\mathfrak{S}_n$ on the quotient space $\mathcal{A}^{m-1}/\mathcal{J}\mathcal{A}^{m-1}$. Let us twist this action by the $(m-1)^{\text{st}}$ power of the sign representation ε: this gives rise to spaces

$$\mathcal{DR}_{2,n}^{m} := \varepsilon^{m-1} \otimes \mathcal{A}^{m-1}/\mathcal{J}\mathcal{A}^{m-1},$$

so that $\mathcal{DR}_{2,n}^{1} = \mathcal{DR}_{2,n}$. Haiman [18, 17] generalized (2) by proving

$$\dim(\mathrm{DR}_{2,n}^{m}) = (mn+1)^{n-1} \quad \text{and} \quad \dim(\mathrm{DR}_{2,n}^{m\,\varepsilon}) = \frac{1}{mn+1}\binom{(m+1)n}{n}.$$

Both dimensions have simple combinatorial interpretations: we recognize in the latter the number of m-ballot paths of size n, and the former is the number of *m-parking functions* of size n (these functions can be described as m-ballot paths of size n in which the north steps are labelled from 1 to n in such a way the labels increase along each run of north steps; see *e.g.* [31]). However, it is still an open problem to find bases of $\mathrm{DR}_{2,n}^{m}$ or $\mathrm{DR}_{2,n}^{m\,\varepsilon}$ indexed by these simple combinatorial objects.

For $\ell \geq 3$, the spaces $\mathrm{DR}_{\ell,n}$ and their generalization $\mathrm{DR}_{\ell,n}^{m}$ can be defined similarly. Haiman explored the dimension of $\mathrm{DR}_{\ell,n}$ and $\mathrm{DR}_{\ell,n}^{\varepsilon}$. For $\ell = 3$, he observed in [19] that, for small values of n,

$$\dim(\mathrm{DR}_{3,n}) = 2^n (n+1)^{n-2} \quad \text{and} \quad \dim(\mathrm{DR}_{3,n}^{\varepsilon}) = \frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$

Following discussions with Haiman, Bergeron came up with conjectures that directly imply the following generalization (since $\mathrm{DR}^1_{3,n}$ coincides with $\mathrm{DR}_{3,n}$):

$$\dim(\mathrm{DR}^m_{3,n}) = (m+1)^n(mn+1)^{n-2} \quad \text{and} \quad \dim(\mathrm{DR}^{m\ \varepsilon}_{3,n}) = \frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$$

Both conjectures are still wide open.

A much simpler problem consists in asking whether these dimensions again have a simple combinatorial interpretation. Bergeron, starting from the sequence $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$, found in Sloane's Encyclopedia that this number counts, among others, certain ballot related objects, namely intervals in the Tamari lattice [12]. From this observation, and the role played by m-ballot paths for two sets of variables, he was led to introduce the m-Tamari lattice $\mathcal{T}_n^{(m)}$, and conjectured that $\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}$ is the number of intervals in this lattice. This is the conjecture we prove in this paper. Another of his conjectures is that $(m+1)^n(mn+1)^{n-2}$ is the number of Tamari intervals where the larger path is "decorated" by an m-parking function [3]. This is proved in [6, 5].

2 A functional equation for the generating function of intervals

The aim of this section is to describe a recursive decomposition of m-Tamari intervals, and to translate it into a functional equation satisfied by the associated generating function (Proposition 8). There are two main tools:

- we prove that $\mathcal{T}_n^{(m)}$ can be seen as an upper ideal of the usual Tamari lattice $\mathcal{T}_{mn}$,
- we give a simple criterion to decide when two paths of the Tamari lattice are comparable.

2.1 An alternative description of the m-Tamari lattices

Our first transformation is totally harmless: we apply a 45 degree rotation to 1-ballot paths to transform them into *Dyck paths*. A Dyck path of size n consists of steps $(1,1)$ (up steps) and steps $(1,-1)$ (down steps), starts at $(0,0)$, ends at $(0,2n)$ and never goes below the x-axis.

We now introduce some terminology, and use it to rephrase the description of the (usual) Tamari lattice $\mathcal{T}_n$. Given a Dyck path P, and an up step u of P, the shortest portion of P that starts with u and forms a (translated) Dyck path is called the *excursion of u in P*. We say that u and the final step of its excursion *match* each other. Finally, we say that u has *rank i* if it is the i^{th} up step of P.

Given two Dyck paths P and Q of size n, Q covers P in the Tamari lattice $\mathcal{T}_n$ if and only if there exists in P a down step d, followed by an up step u, such that Q is obtained from P by swapping d and S, where S is the excursion of u in P. This description implies the following property [4, Cor. 2.2].

Property 3. *If $P \leq Q$ in $\mathcal{T}_n$ then P is below Q. That is, for $i \in [0..2n]$, the ordinate of the vertex of P lying at abscissa i is at most the ordinate of the vertex of Q lying at abscissa i.*

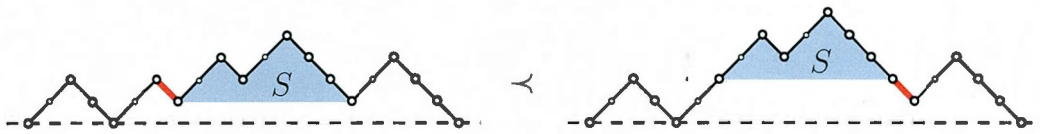

Figure 4: The relation $\prec$ of Figure 2 reformulated in terms of m-Dyck paths ($m = 2$).

Consider now an m-ballot path of size n, and replace each north step by a sequence of m north steps. This gives a 1-ballot path of size mn, and thus, after a rotation, a Dyck path. In this path, for each $i \in [0..n-1]$, the up steps of ranks $mi+1, \ldots, m(i+1)$ are consecutive. We call the Dyck paths satisfying this property *m-Dyck paths*. Clearly, m-Dyck paths of size mn are in one-to-one correspondence with m-ballot paths of size n. Consider now the relation $\prec$ of Definition 2: once reformulated in terms of Dyck paths, it becomes a covering relation in the (usual) Tamari lattice (Figure 4). Conversely, it is easy to check that, if P is an m-Dyck path and Q covers P in the usual Tamari lattice, then Q is also an m-Dyck path, and the m-ballot paths corresponding to P and Q are related by $\prec$. We have thus proved the following result.

Proposition 4. *The transitive closure of the relation $\prec$ defined in Definition 2 is a lattice on m-ballot paths of size n. This lattice is isomorphic to the sublattice of the Tamari lattice $\mathcal{T}_{mn}$ consisting of the elements that are larger than or equal to the Dyck path $u^m d^m \ldots u^m d^m$. The relation $\prec$ is the covering relation of this lattice.*

Notation. From now on, we only consider Dyck paths. We denote by $\mathcal{T}$ the set of Dyck paths, and by $\mathcal{T}_n$ the Tamari lattice of Dyck paths of length n. By $\mathcal{T}^{(m)}$ we mean the set of m-Dyck paths, and by $\mathcal{T}_n^{(m)}$ the Tamari lattice of m-Dyck paths of size mn. This lattice is a sublattice of $\mathcal{T}_{mn}$. Note that $\mathcal{T}^{(1)} = \mathcal{T}$ and $\mathcal{T}_n^{(1)} = \mathcal{T}_n$.

2.2 Distance functions

Let P be a Dyck path of size n. For an up step u of P, we denote by $\ell_P(u)$ the size of the excursion of u in P. The function $D_P : [1..n] \to [1..n]$ defined by $D_P(i) = \ell(u_i)$, where u_i is the i^{th} up step of P, is called the *distance function* of P. It will sometimes be convenient to see D_P as a vector $(\ell(u_1), \ldots, \ell(u_n))$ with n components. In particular, we will compare distance functions component-wise. The main result of this subsection is a description of the Tamari order in terms of distance functions. This simple characterization seems to be new.

Proposition 5. *Let P and Q be two paths in the Tamari lattice $\mathcal{T}_n$. Then $P \leq Q$ if and only if $D_P \leq D_Q$.*

In order to prove this, we first describe the relation between the distance functions of two paths related by a covering relation.

Lemma 6. *Let P be a Dyck path, and d a down step of P followed by an up step u. Let S be the excursion of u in P, and let Q be the path obtained from P by swapping d and S. Let u' be the up step matched with d in P, and i_0 the rank of u' in P. Then $D_Q(i) = D_P(i)$ for each $i \neq i_0$ and $D_Q(i_0) = D_P(i_0) + \ell_P(u)$.*

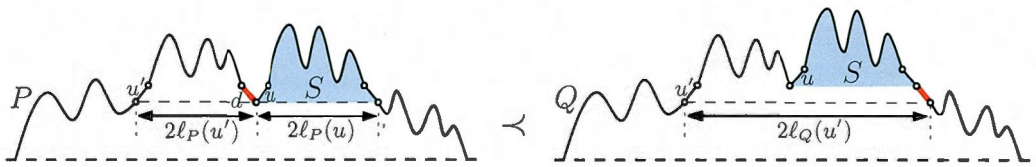

Figure 5: How the distance function changes in a covering relation.

This lemma is easily proved using Figure 5. It already implies that $D_P \leq D_Q$ if $P \leq Q$. The next lemma establishes the reverse implication, thus concluding the proof of Proposition 5.

Lemma 7. *Let P and Q be two Dyck paths of size n such that $D_P \leq D_Q$. Then $P \leq Q$ in the Tamari lattice $\mathcal{T}_n$.*

Proof. Let us first prove, by induction on the size, that P is below Q (in the sense of Property 3). This is clearly true if $n = 0$, so we assume $n > 0$.

Let u be the first up step (in P and Q). Note that $\ell_P(u) = D_P(1) \leq D_Q(1) = \ell_Q(u)$. Let P' (resp. Q') be the path obtained from P (resp. Q) by contracting u and the down step matched with u. Observe that $D_{P'}$ is obtained by deleting the first component of D_P, and similarly for $D_{Q'}$ and D_Q. Consequently $D_{P'} \leq D_{Q'}$, and hence by the induction hypothesis, P' is below Q'. Let us consider momentarily Dyck paths as functions, and write $P(i) = j$ if the vertex of P lying at abscissa i has ordinate j. Note that $P(i) = P'(i-1) + 1$ for $1 \leq i < 2\ell_P(u)$, and $P(i) = P'(i-2)$ for $2\ell_P(u) \leq i \leq 2n$. Similarly $Q(i) = Q'(i-1) + 1$ for $1 \leq i < 2\ell_Q(u)$, and $Q(i) = Q'(i-2)$ for $2\ell_Q(u) \leq i \leq 2n$. Since $\ell_P(u) \leq \ell_Q(u)$ and $P'(i) \leq Q'(i)$ for $0 \leq i \leq 2n-2$, one easily checks that $P(i) \leq Q(i)$ for $0 \leq i \leq 2n$, so that P is below Q.

In order to prove that $P \leq Q$, we proceed by induction on $||D_P - D_Q||$, where $||(x_1, \ldots, x_n)|| = |x_1| + \cdots + |x_n|$. If $D_P = D_Q$ then $P = Q$, because P is below Q and Q is below P. So let us assume that $D_P \neq D_Q$. Let i be minimal such that $D_P(i) < D_Q(i)$. We claim that P and Q coincide at least until their up step of rank i. Indeed, since P lies below Q, the paths P and Q coincide up to some abscissa, and then we find a down step δ in P but an up step in Q. Let j be the rank of the up step that matches δ in P. This up step belongs also to Q, and, since $\delta \notin Q$, we have $D_P(j) < D_Q(j)$. Hence $j \geq i$ by minimality of i, and P and Q coincide at least until their up step of rank i, which we denote by u. Let d be the down step matched with u in P (Figure 6). Since

$D_P(i) < D_Q(i)$, the step d is *not* a step of Q. The step of Q located at the same abscissa as d ends strictly higher than d, and in particular, at a positive ordinate. Hence d is not the final step of P. Let s be the step following d in P.

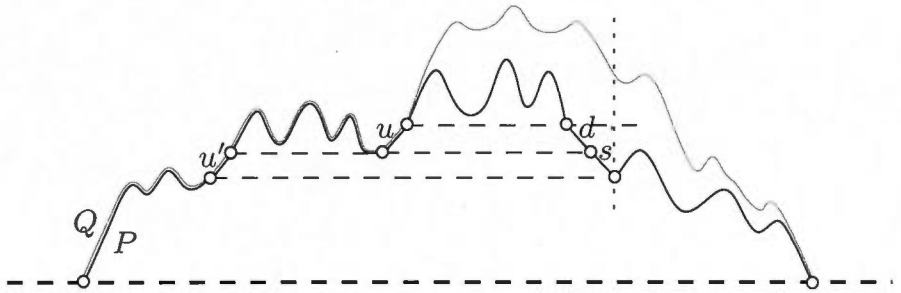

Figure 6: Why s cannot be descending.

Let us prove *ad absurdum* that s is an up step. Assume s is down. Then s is matched in P with an up step u' of rank $j < i$ (Figure 6). Hence u' belongs to Q and has rank j in Q. Since s cannot belong to Q, this implies that $D_P(j) < D_Q(j)$, which contradicts the minimality of i.

Hence s is an up step of P (Figure 7). Let S be the excursion of s in P. Since $\ell_Q(u) > \ell_P(u)$ and since Q is above P, we have $\ell_Q(u) \geq \ell_P(u) + \ell_P(s)$, *i.e.*, $D_Q(i) \geq D_P(i) + \ell_P(s)$. Let P' be the path obtained from P by swapping d and S. Then P' covers P in the Tamari lattice. By Lemma 6, $D_P = D_{P'}$ except at index i (the rank of u), where $D_{P'}(i) = D_P(i) + \ell_P(s)$. Since $D_P(i) + \ell_P(s) \leq D_Q(i)$, we have $D_{P'} \leq D_Q$. But $||D_{P'} - D_Q|| = ||D_P - D_Q|| - \ell_P(s)$ and by the induction hypothesis, $P' \leq Q$ in the Tamari lattice. Hence $P < P' \leq Q$, and the lemma is proved. ∎

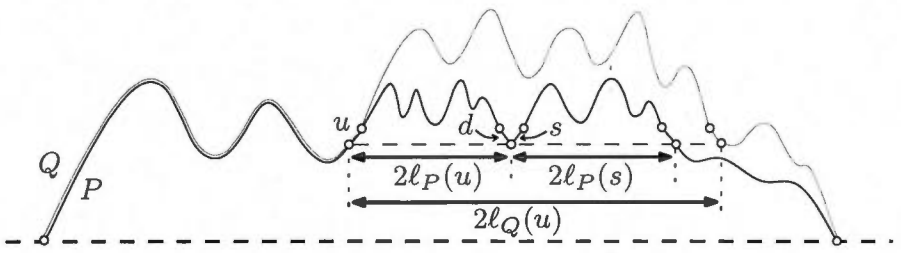

Figure 7: General form of P and Q.

2.3 Recursive decomposition of intervals and functional equation

A *contact* of a Dyck path P is a vertex of P lying on the x-axis. It is *initial* if it is $(0, 0)$. A *contact* of a Tamari interval $[P, Q]$ is a contact of the *lower* path P. The recursive decomposition of intervals that we use makes the number of contacts crucial, and we say

that this parameter is *catalytic*. We also consider another, non-catalytic parameter, which we find to be equidistributed with non-initial contacts (even more, the joint distribution of these two parameters is symmetric). Given an m-Dyck path Q, the length of the initial run of up steps is of the form mk; the integer k is called the *initial rise* of Q. The *initial rise* of an interval $[P, Q]$ is the initial rise of the *upper* path Q. The aim of this subsection is to establish the following functional equation.

Proposition 8. *For $m \geq 1$, let $F(x) \equiv F^{(m)}(t; x)$ be the generating function of m-Tamari intervals, where t counts the size (divided by m) and x the number of contacts. Then*

$$F(x) = x + xt\,(F(x) \cdot \Delta)^{(m)}\,(F(x)),$$

where Δ is the following divided difference operator

$$\Delta S(x) = \frac{S(x) - S(1)}{x - 1},$$

and the power m means that the operator $G(x) \mapsto F(x) \cdot \Delta G(x)$ is applied m times to $F(x)$.

More generally, if $F(x, y) \equiv F^{(m)}(t; x, y)$ keeps track in addition of the initial rise (via the variable y), we have the following functional equation:

$$F(x, y) = x + xyt\,(F(x, 1) \cdot \Delta)^{(m)}\,(F(x, y)). \tag{3}$$

Note that each of the above two equations defines a unique formal power series in t (think of extracting inductively the coefficient of t^n in $F^{(m)}(t; x)$ or $F^{(m)}(t; x, y)$).

Examples

1. When $m = 1$, the above equation reads

$$\begin{aligned} F(x, y) &= x + xytF(x, 1) \cdot \Delta(F(x, y)) \\ &= x + xytF(x, 1)\,\frac{F(x, y) - F(1, y)}{x - 1}. \end{aligned}$$

When $y = 1$, we obtain, in the terminology of [7], a quadratic equation with one catalytic variable:

$$F(x) = x + xtF(x)\,\frac{F(x) - F(1)}{x - 1}.$$

2. When $m = 2$,

$$\begin{aligned} F(x, y) &= x + xyt\ F(x, 1) \cdot \Delta(F(x, 1) \cdot \Delta(F(x, y))) \\ &= x + xyt\ F(x, 1) \cdot \Delta\left(F(x, 1)\,\frac{F(x, y) - F(1, y)}{x - 1}\right) \\ &= x + \frac{xyt}{x - 1}\ F(x, 1)\,\left(F(x, 1)\,\frac{F(x, y) - F(1, y)}{x - 1} - F(1, 1)F'(1, y)\right), \end{aligned}$$

where the derivative is taken with respect to the variable x. When $y = 1$, we obtain a cubic equation with one catalytic variable:

$$F(x) = x + \frac{xt}{x-1}F(x)\left(F(x)\,\frac{F(x)-F(1)}{x-1} - F(1)F'(1)\right).$$

The solution of (3) will be the topic of the next section. For the moment we focus on the proof of this equation.

We say that a vertex q lies to the right of a vertex p if the abscissa of q is greater than or equal to the abscissa of p. A *k-pointed Dyck path* is a tuple $(P; p_1, \ldots, p_k)$ where P is a Dyck path and $p_1, \ldots, p_k$ are contacts of P such that p_{i+1} lies to the right of p_i, for $1 \leq i < k$ (note that some p_i's may coincide). Given an m-Dyck path P of positive size, let $u_1, \ldots, u_m$ be the initial (consecutive) up steps of P, and let $d_1, \ldots, d_m$ be the down steps matched with $u_1, \ldots, u_m$, respectively. The *m-reduction* of P is the m-pointed Dyck path $(P'; p_1, \ldots, p_m)$ where P' is obtained from P by contracting all the steps $u_1, \ldots, u_m, d_1, \ldots, d_m$, and $p_1, \ldots, p_m$ are the vertices of P' resulting from the contraction of $d_1, \ldots, d_m$. It is easy to check that they are indeed contacts of P' (Figure 8).

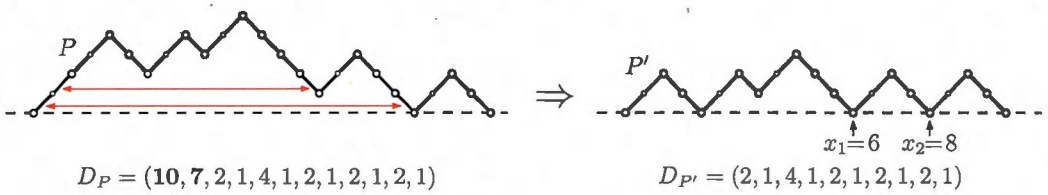

Figure 8: The m-reduction of an m-Dyck path ($m = 2$).

The map $P \mapsto (P'; p_1, \ldots, p_m)$ is clearly invertible, hence m-Dyck paths of size mn are in bijection with m-pointed m-Dyck paths of size $m(n-1)$. Note that the non-initial contacts of P correspond to the contacts of P' that lie to the right of p_m. Note also that the distance function $D_{P'}$ (seen as a vector with $m(n-1)$ components) is obtained by deleting the first m components of D_P. Conversely, denoting by $2x_i$ the abscissa of p_i, D_P is obtained by prepending to $D_{P'}$ the sequence $(x_m + m, x_{m-1} + m - 1, \ldots, x_1 + 1)$. In view of Proposition 5, this gives the following recursive characterization of intervals.

Lemma 9. *Let P and Q be two m-Dyck paths of size $mn > 0$. Let $(P'; p_1, \ldots, p_m)$ and $(Q'; q_1, \ldots, q_m)$ be the m-reductions of P and Q respectively. Then $P \leq Q$ in $\mathcal{T}_n^{(m)}$ if and only if $P' \leq Q'$ in $\mathcal{T}_{n-1}^{(m)}$ and for $i \in [1..m]$, the point q_i lies to the right of p_i.*

The non-initial contacts of P correspond to the contacts of P' located to the right of p_m.

Let us call *k-pointed interval* in $\mathcal{T}^{(m)}$ a pair consisting of two k-pointed m-Dyck paths $(P; p_1, \ldots, p_k)$ and $(Q; q_1, \ldots, q_k)$ such that $P \leq Q$ and for $i \in [1..k]$, the point q_i lies to the right of p_i. An *active contact* of such a pair is a contact of P lying to the right of p_k (if $k = 0$, all contacts are declared active). For $0 \leq k \leq m$, let us denote by

$G^{(m,k)}(t;x,y) \equiv G^{(k)}(x,y)$ the generating function of k-pointed m-Tamari intervals, where t counts the size (divided by m), x the number of active contacts, and y the initial rise (we drop the superscript m since it will not vary). In particular, the series we are interested in is

$$F(x,y) = G^{(0)}(x,y). \tag{4}$$

Moreover, Lemma 9 implies

$$F(x,y) = x + xyt\, G^{(m)}(x,y). \tag{5}$$

We will prove that, for $k \geq 0$,

$$G^{(k+1)}(x,y) = F(x,1) \cdot \Delta G^{(k)}(x,y). \tag{6}$$

The functional equation (3) then follows using (4) and (5).

For $k \geq 0$, let $I = [P^\bullet, Q^\bullet]$ be a $(k+1)$-pointed interval in $\mathcal{T}^{(m)}$, with $P^\bullet = (P; p_1, \ldots, p_{k+1})$ and $Q^\bullet = (Q; q_1, \ldots, q_{k+1})$ (see an illustration in Figure 9 when $k = 0$). Since P is below Q, the contact q_{k+1} of Q is also a contact of P. By definition of pointed intervals, q_{k+1} is to the right of $p_1, \ldots, p_{k+1}$. Decompose P as $P_\ell P_r$ where P_ℓ is the part of P to the left of q_{k+1} and P_r is the part of P to the right of q_{k+1}. Decompose similarly Q as $Q_\ell Q_r$, where the two factors meet at q_{k+1}. The distance function D_P (seen as a vector) is D_{P_ℓ} concatenated with D_{P_r}, and similarly for D_Q. In particular, $D_{P_\ell} \leq D_{Q_\ell}$ and $D_{P_r} \leq D_{Q_r}$. By Proposition 5, $I_r := [P_r, Q_r]$ is an interval, while $I_\ell := [P^\circ, Q^\circ]$, with $P^\circ = (P_\ell; p_1, \ldots, p_k)$ and $Q^\circ = (Q_\ell; q_1, \ldots, q_k)$, is a k-pointed interval. Its initial rise equals the initial rise of I. We denote by Φ the map that sends I to the pair of intervals (I_r, I_ℓ).

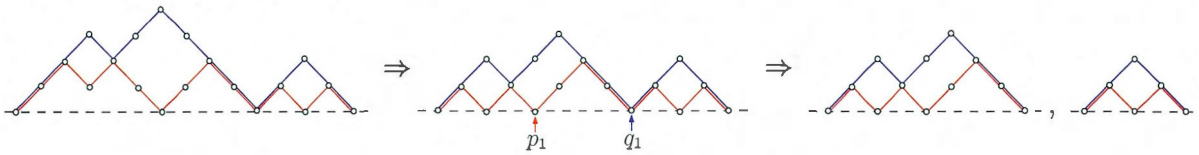

Figure 9: The recursive decomposition of intervals. Starting from an m-Tamari interval of size n (here, $m = 1$ and $n = 7$), one first obtains by reduction an m-pointed interval of size $n-1$ (Lemma 9). This interval is further decomposed into two intervals, the first one being $(m-1)$-pointed.

Conversely, take an interval $I_r = [P_r, Q_r]$ and a k-pointed interval $I_\ell = [P^\circ, Q^\circ]$, where $P^\circ = (P_\ell; p_1, \ldots, p_k)$ and $Q^\circ = (Q_\ell; q_1, \ldots, q_k)$. Let $P = P_\ell P_r$, $Q = Q_\ell Q_r$, and denote by q_{k+1} the point where Q_ℓ and Q_r (and P_r and P_ℓ) meet. This is a contact of P and Q. Then the preimages of (I_r, I_ℓ) by Φ are all the intervals $I = [P^\bullet, Q^\bullet]$ such that $P^\bullet = (P; p_1, \ldots, p_{k+1})$ and $Q^\bullet = (Q; q_1, \ldots, q_{k+1})$, where p_{k+1} is any *active* contact of P_ℓ. If P_ℓ has i active contacts and P_r has j contacts, then (I_r, I_ℓ) has i preimages, having respectively $j, 1+j, \ldots, i+j-1$ active contacts (j active contacts when $p_{k+1} = q_{k+1}$,

and $i+j-1$ active contacts when $p_{k+1}=p_k$). Let us write $G^{(k)}(x,y)=\sum_{i\geq 0} G_i^{(k)}(y)x^i$, so that $G_i^{(k)}(y)$ counts (by the size and the initial rise) k-pointed intervals with i active contacts. The above discussion gives

$$\begin{aligned} G^{(k+1)}(x,y) &= F(x,1)\sum_{i\geq 1} G_i^{(k)}(y)(1+x+\cdots+x^{i-1}) \\ &= F(x,1)\sum_{i\geq 1} G_i^{(k)}(y)\frac{x^i-1}{x-1} \\ &= F(x,1)\cdot \Delta G^{(k)}(x,y), \end{aligned}$$

as claimed in (6). The factor $F(x,1)$ accounts for the choice of I_r, and the term $\Delta G^{(k)}(x,y)$ for the choice of I_ℓ and p_{k+1}. This completes the proof of Proposition 8. ∎

3 Solution of the functional equation

In this section, we solve the functional equation of Proposition 8, and thus establish the main result of this paper. We obtain in particular an unexpected symmetry property: the series $yF^{(m)}(t;x,y)$ is symmetric in x and y. In other words, the joint distribution of the number of non-initial contacts (of the lower path) and the initial rise (of the upper path) is symmetric.

For any ring $\mathbb{A}$, we denote by $\mathbb{A}[x]$ the ring of polynomials in x with coefficients in $\mathbb{A}$, and by $\mathbb{A}[[x]]$ the ring of formal power series in x with coefficients in $\mathbb{A}$. This notation is extended to the case of polynomials and series in several indeterminates $x_1, x_2, \ldots$

Theorem 10. *For $m\geq 1$, let $F^{(m)}(t;x,y)$ be the generating function of m-Tamari intervals, where t counts the size (divided by m), x the number of contacts of the bottom path, and y the initial rise of the upper path. Let z, u and v be three indeterminates, and set*

$$t=z(1-z)^{m^2+2m}, \quad x=\frac{1+u}{(1+zu)^{m+1}}, \quad \text{and} \quad y=\frac{1+v}{(1+zv)^{m+1}}. \tag{7}$$

Then $F^{(m)}(t;x,y)$ becomes a formal power series in z with coefficients in $\mathbb{Q}[u,v]$, and this series is rational. More precisely,

$$yF^{(m)}(t;x,y)=\frac{(1+u)(1+zu)(1+v)(1+zv)}{(u-v)(1-zuv)(1-z)^{m+2}}\left(\frac{1+u}{(1+zu)^{m+1}}-\frac{1+v}{(1+zv)^{m+1}}\right). \tag{8}$$

In particular, $yF^{(m)}(t;x,y)$ is a symmetric series in x and y.

Remark. This result was first guessed for small values of m. More precisely, we first guessed the values of $\frac{\partial^i F}{\partial x}(1,1)$ for $0\leq i\leq m-1$, and then combined these conjectured values with the functional equation to obtain conjectures for $F(x,1)$ and $F(x,y)$. Let us

illustrate our guessing procedure on the case $m = 1$. We first consider the case $y = 1$, where the equation reads

$$F(x,1) = x + xtF(x,1)\frac{F(x,1) - F(1,1)}{x-1}. \tag{9}$$

Our first objective is to guess the value of $F(1,1)$. Using the above equation, we easily compute, say, the 20 first coefficients of $F(1,1)$. Using the MAPLE package **gfun** [27], we conjecture from this list of coefficients that $f \equiv F(1,1)$ satisfies

$$1 - 16\,t - (1 - 20\,t)\,f - \left(3\,t + 8\,t^2\right) f^2 - 3\,t^2 f^3 - t^3 f^4 = 0.$$

Using the package **algcurves**, we find that the above equation admits a rational parametrization, for instance

$$t = z(1-z)^3, \quad f = F(1,1) = \frac{1-2z}{(1-z)^3}.$$

This is the end of the "guessing" part[2]. Assume the above identity holds, and replace t and $F(1,1)$ in (9) by their expressions in terms of z. This gives an algebraic equation in $F(x,1)$, x and z. Again, the package **algcurves** reveals that this equation, seen as an equation in $F(x,1)$ and x, has a rational parametrization, for instance

$$x = \frac{1+u}{(1+zu)^2}, \quad F(x,1) = \frac{(1+u)\,(1-2\,z-z^2u)}{(1+zu)\,(1-z)^3}.$$

Let us finally return to the functional equation defining $F(x,y)$:

$$F(x,y) = x + xytF(x,1)\frac{F(x,y) - F(1,y)}{x-1}.$$

In this equation, replace t, x and $F(x,1)$ by their conjectured expressions in terms of z and u. This gives

$$\left(1 + zu - zy\frac{(1+u)^2}{u}\right) F(x,y) = \frac{1+u}{1+zu} - zy\frac{(1+u)^2}{u}F(1,y). \tag{10}$$

We conclude by applying to this equation the *kernel method* (see, e.g. [2, 8, 26]): let $U \equiv U(z;y)$ be the unique formal power series in z (with coefficients in $\mathbb{Q}[y]$) satisfying

$$U = zy(1+U)^2 - zU^2.$$

Equivalently,

$$U = z\frac{1+v}{1-2z-z^2v}, \quad \text{with} \quad y = \frac{1+v}{(1+zv)^2}.$$

[2]For a general value of m, one has to guess the series $\frac{\partial^i F}{\partial x}(1,1)$ for $0 \le i \le m-1$. All of them are found to be rational functions of z, when $t = z(1-z)^{m^2+2m}$.

Setting $u = U$ in (10) cancels the left-hand side, and thus the right-hand side, giving

$$yF(1,y) = \frac{(1+v)(1-2z-z^2v)}{(1+zv)(1-z)^3}.$$

A conjecture for the trivariate series $F(t;x,y)$ follows, using (10). This conjecture coincides with (8). □

Before we prove Theorem 10, let us give a closed form expression for the number of intervals in $\mathcal{T}_n^{(m)}$.

Corollary 11. *Let $m \geq 1$ and $n \geq 1$. The number of intervals in the Tamari lattice $\mathcal{T}_n^{(m)}$ is*

$$f_n^{(m)} = \frac{\bar{m}}{n(mn+1)}\binom{n\bar{m}^2+m}{n-1},$$

where we denote $\bar{m} = m+1$. For $2 \leq i \leq n+1$, the number of intervals in which the bottom path has i contacts with the x-axis is

$$f_{n,i}^{(m)} = \frac{(n\bar{m}^2 - i\bar{m} + m)!(i\bar{m}-m)!}{(n\bar{m}^2 - n - im + 2m)!(n-i+1)!(mi)!(i-2)!}P_m(n,i), \tag{11}$$

where $P_m(n,i)$ is a polynomial in n and i. In particular,

$$P_1(n,i) = 2, \quad P_2(n,i) = 6(33in - 9i^2 + 15i - 2n - 2).$$

More generally,

$$\begin{aligned} i(i-1)P_m(n,i) &= -\bar{m}!(m-1)!(n-i+1)\binom{i\bar{m}}{m}\binom{nm(m+2)-im+2m}{m-1} \\ &+ \sum_{k=1}^{m-2} kk!^2(m-k-2)!(m-k-1)!((i+1)m\bar{m}+2\bar{m}+k)(n-i)(n-i+1)\times \\ &\binom{i\bar{m}-k-1}{m-k-1}\binom{im}{k}\binom{n\bar{m}^2-i\bar{m}+m+k}{k}\binom{nm(m+2)-im+2m}{m-k-2} \\ &+ m!^2\binom{im}{m-1}\left(i\binom{n\bar{m}^2-i\bar{m}+2m}{m} \right. \\ &\left. -\frac{(m-1)(i\bar{m}+2)(n-i+1)}{m}\binom{n\bar{m}^2-i\bar{m}+2m-1}{m-1}\right). \end{aligned} \tag{12}$$

Remarks

1. The case $m = 1$ of (11) reads

$$f_{n,i}^{(1)} = \frac{(i-1)(4n-2i+1)!}{(3n-i+2)!(n-i+1)!}\binom{2i}{i}.$$

This result can also obtained using Bernardi and Bonichon's bijection between intervals of size n in the (usual) Tamari lattice and planar 3-connected triangulations having $n+3$

vertices [4]. Indeed, through this bijection, the number of contacts in the lower path of the interval becomes the degree of the root-vertex of the triangulation, minus one [4, Def. 3.2]. The above result is thus equivalent to a result of Brown counting triangulations by the number of vertices and the degree of the root-vertex [10, Eq. (4.7)].

2. Our expression of P_m is not illuminating, but we have given it to prove that P_m is indeed a polynomial. If we fix i rather than m, then, experimentally, $P_m(n,i)$ seems to be a sum of two hypergeometric terms in m and n. More precisely, it appears that

$$P_m(n,i) = \frac{m\bar{m}!(im)!}{(i\bar{m}-m)!\binom{n}{i-1}} \times \left(\bar{m}R_i(m,n)\binom{n\bar{m}^2-(i-2)\bar{m}-1}{\bar{m}} + Q_i(m,n)\binom{nm(m+2)-(i-2)m}{m}\right),$$

where R_i and Q_i are two polynomials in m and n. This holds at least for small values of i.

3. The coefficients of the trivariate series $F(t;x,y)$ do not seem to have small prime factors, even when $m=1$.

Proof of Theorem 10. The functional equation of Proposition 8 defines a unique formal power series in t (think of extracting inductively the coefficient of t^n in $F(t;x,y)$). The coefficients of this series are polynomials in x and y. The parametrized expression of $F(t;x,y)$ given in Theorem 10 also defines $F(t;x,y)$ uniquely as a power series in t, because (7) defines z, u and v uniquely as formal power series in t (with coefficients in $\mathbb{Q}$, $\mathbb{Q}[x]$ and $\mathbb{Q}[y]$ respectively). Thus it suffices to prove that the series $F(t;x,y)$ of Theorem 10 satisfies the equation of Proposition 8.

If $G(t;x,y) \equiv G(x,y)$ is any series in $\mathbb{Q}[x,y][[t]]$, then performing the change of variables (7) gives $G(t;x,y) = H(z;u,v)$, where

$$H(z;u,v) \equiv H(u,v) = G\left(z(1-z)^{m^2+2m}; \frac{1+u}{(1+zu)^{m+1}}, \frac{1+v}{(1+zv)^{m+1}}\right).$$

Moreover, if $F(x,y)$ is given by (8), then

$$F(x,1) = \frac{(1+u)(1+zu)}{u(1-z)^{m+2}}\left(\frac{1+u}{(1+zu)^{m+1}} - 1\right),$$

and

$$F(x,1)\Delta G(x,y) = \frac{(1+u)(1+zu)}{(1-z)^{m+2}}\frac{H(u,v)-H(0,v)}{u}.$$

Let us define an operator Λ as follows: for any series $H(z;u,v) \in \mathbb{Q}[u,v][[z]]$,

$$\Lambda H(z;u,v) := (1+u)(1+zu)\frac{H(z;u,v)-H(z;0,v)}{u}. \tag{13}$$

Then the series $F(t;x,y)$ of Theorem 10 satisfies the equation of Proposition 8 if and only if the series $H(u,v)$ obtained by performing the change of variables (7) in $y(1-z)^{m+2}F(x,y)$, that is,

$$H(u,v) = \frac{(1+u)(1+zu)(1+v)(1+zv)}{(u-v)(1-zuv)}\left(\frac{1+u}{(1+zu)^{m+1}} - \frac{1+v}{(1+zv)^{m+1}}\right). \tag{14}$$

satisfies

$$z\Lambda^{(m)}H(u,v) = \frac{(1+zu)^{m+1}(1+zv)^{m+1}}{(1+u)(1+v)}H(u,v) - (1-z)^{m+2}. \tag{15}$$

Hence we simply have to prove an identity on rational functions. Observe that both $H(u,v)$ and the conjectured expression of $\Lambda^{(m)}H(u,v)$ are symmetric in u and v. More generally, computing (with the help of MAPLE) the rational functions $\Lambda^{(k)}H(u,v)$ for a few values of m and k suggests that these fractions are *always* symmetric in u and v. This observation raises the following question: Given a symmetric function $H(u,v)$, when is $\Lambda H(u,v)$ also symmetric? This leads to the following lemma, which will reduce the proof of (15) to the case $v=0$.

Lemma 12. *Let $H(z;u,v) \equiv H(u,v)$ be a series of $\mathbb{Q}[u,v][[z]]$, symmetric in u and v. Let Λ be the operator defined by* (13)*, and denote $H_1(u,v) := \Lambda H(u,v)$. Then $H_1(u,v)$ is symmetric in u and v if and only if H satisfies*

$$H(u,v) = \frac{u(1+v)(1+zv)H(u,0) - v(1+u)(1+zu)H(v,0)}{(u-v)(1-zuv)}. \tag{16}$$

If this holds, then $H_1(u,v)$ also satisfies (16) *(with H replaced by H_1). By induction, the same holds for $H_k(u,v) := \Lambda^{(k)}H(u,v)$.*

The proof is a straightforward calculation. ∎

Note that a series H satisfying (16) is characterized by the value of $H(u,0)$. The series $H(u,v)$ given by (14) satisfies (16), with

$$H(u,0) = \frac{(1+u)(1+zu)}{u}\left(\frac{1+u}{(1+zu)^{m+1}} - 1\right) = \Lambda\left(\frac{1+u}{(1+zu)^{m+1}}\right).$$

Moreover, one easily checks that the right-hand side of (15) also satisfies (16), as expected from Lemma 12. Thus it suffices to prove the case $v=0$ of (15), namely

$$z\Lambda^{(m+1)}\left(\frac{1+u}{(1+zu)^{m+1}}\right) = \frac{(1+u)(1+zu)}{u}\left(1 - \frac{(1+zu)^{m+1}}{1+u}\right) - (1-z)^{m+2}. \tag{17}$$

This will be a simple consequence of the following lemma.

Lemma 13. *Let Λ be the operator defined by* (13)*. For $m \geq 1$,*

$$\Lambda^{(m)}\left(\frac{1}{(1+zu)^m}\right) = (1-z)^m - (1+zu)^m.$$

Proof. We will actually prove a more general identity. Let $1 \leq k \leq m$, and denote $w = 1 + zu$. Then

$$\Lambda^{(k)}\left(\frac{1}{(1+zu)^m}\right) = \frac{(1-z)^k}{w^{m-k}} - \sum_{i=k}^{m-1}\sum_{j=1}^{k} \frac{(-1)^{k+j} z^{k-j+1}}{w^{m-i-1}} \binom{k}{j-1}\binom{i-j+1}{k-j} + \sum_{i=1}^{k-1}\sum_{j=1}^{i} (-1)^{j-1} z^j w^{k-i} \binom{i-1}{j-1}\binom{m-k+j-1}{j} - w^k. \quad (18)$$

The case $k = m$ is the identity of Lemma 13. In order to prove (18), we need an expression of $\Lambda(w^p)$, for all $p \in \mathbb{Z}$. Using the definition (13) of Λ, one obtains, for $p \geq 1$,

$$\begin{cases} \Lambda\left(\dfrac{1}{w^p}\right) &= \dfrac{1-z}{w^{p-1}} - z\displaystyle\sum_{a=0}^{p-2} \frac{1}{w^a} - w, \\ \Lambda(1) &= 0, \\ \Lambda\left(w^p\right) &= (z-1)w + z\displaystyle\sum_{a=2}^{p} w^a + w^{p+1}. \end{cases} \quad (19)$$

We now prove (18), by induction on $k \geq 1$. For $k = 1$, (18) coincides with the expression of $\Lambda(1/w^p)$ given above (with p replaced by m). Now let $1 \leq k < m$. Apply Λ to (18), use (19) to express the terms $\Lambda(w^p)$ that appear, and then check that the coefficient of $w^a z^b$ is what it is expected to be, for all values of a and b. The details are a bit tedious, but elementary. One needs to apply a few times the following identity:

$$\sum_{r=r_1}^{r_2} \binom{r-a}{b} = \frac{(r_2+1-a-b)}{b+1}\binom{r_2+1-a}{b} - \frac{(r_1-a-b)}{b+1}\binom{r_1-a}{b}.$$

We give in the appendix a constructive proof of Lemma 13, which does not require to guess the more general identity (18). It is also possible to derive (18) combinatorially from (19) using one-dimensional lattice paths (in this setting, (19) describes what steps are allowed if one starts at position p, for any $p \in \mathbb{Z}$). ∎

Let us now return to the proof of (17). We write

$$z\frac{1+u}{(1+uz)^{m+1}} = \frac{1}{(1+uz)^m} + \frac{z-1}{(1+uz)^{m+1}}.$$

Thus

$$\begin{aligned} z\Lambda^{(m+1)}\left(\frac{1+u}{(1+uz)^{m+1}}\right) &= \Lambda\left(\Lambda^{(m)}\left(\frac{1}{(1+uz)^m}\right)\right) + (z-1)\Lambda^{(m+1)}\left(\frac{1}{(1+uz)^{m+1}}\right) \\ &= \Lambda\left((1-z)^m - (1+uz)^m\right) \\ &\quad + (z-1)\left((1-z)^{m+1} - (1+uz)^{m+1}\right) \end{aligned}$$

by Lemma 13. Eq. (17) follows, and Theorem 10 is proved. ∎

Proof of Corollary 11. Let us first determine the coefficients of $F(t;1,1)$. By letting u and v tend to 0 in the expression of $yF(t;x,y)$, we obtain

$$F(t;1,1) = \frac{1-(m+1)z}{(1-z)^{m+2}},$$

where $t = z(1-z)^{m^2+2m}$. The Lagrange inversion formula gives

$$[t^n]F(t;1,1) = \frac{1}{n}[t^{n-1}]\frac{1-(m+1)^2t}{(1-t)^{nm(m+2)+m+3}},$$

and the expression of $f_n^{(m)}$ follows after an elementary coefficient extraction.

We now wish to express the coefficient of $t^n x^i$ in

$$F(t;x,1) = \frac{(1+u)(1+zu)}{u(1-z)^{m+2}}\left(\frac{1+u}{(1+zu)^{m+1}} - 1\right).$$

We will expand this series, first in x, then in t, applying the Lagrange inversion formula first to u, then to z. We first expand $(1-z)^{m+2}F(t;x,1)$ in partial fractions of u:

$$(1-z)^{m+2}F(t;x,1) = -z\mathbb{1}_{m>1} - (1+zu) - \sum_{k=1}^{m-2}\frac{z}{(1+uz)^k} + \frac{1-z^2}{z(1+uz)^{m-1}} - \frac{(1-z)^2}{z(1+uz)^m}.$$

By the Lagrange inversion formula, applied to u, we have, for $i \geq 1$ and $p \geq -m$,

$$[x^i](1+zu)^p = \frac{p}{i}\binom{i\bar{m}+p-1}{i-1}z^i(1-z)^{im+p},$$

with $\bar{m} = m+1$. Hence, for $i \geq 1$,

$$\begin{aligned} i[x^i]F(t;x,1) = &-\binom{i\bar{m}}{i-1}z^i(1-z)^{(i-1)m-1} + \sum_{k=1}^{m-2}k\binom{i\bar{m}-k-1}{i-1}z^{i+1}(1-z)^{(i-1)m-k-2} \\ &- (m-1)\binom{i\bar{m}-m}{i-1}z^{i-1}(1+z)(1-z)^{(i-2)m} + m\binom{(i-1)\bar{m}}{i-1}z^{i-1}(1-z)^{(i-2)m}. \end{aligned}$$

We rewrite the above line as

$$\binom{i\bar{m}-m}{i-1}\left(\frac{i}{i\bar{m}-m}z^{i-1}(1-z)^{(i-2)m} - (m-1)z^i(1-z)^{(i-2)m}\right).$$

Recall that $z = \frac{t}{(1-z)^{m^2+2m}}$. Hence, for $i \geq 1$,

$$\begin{aligned} i[x^it^n]F(t;x,1) = &-\binom{i\bar{m}}{i-1}[t^{n-i}]\frac{1}{(1-z)^{\bar{m}(im+1)}} \\ &+ \sum_{k=1}^{m-2}k\binom{i\bar{m}-k-1}{i-1}[t^{n-i-1}]\frac{1}{(1-z)^{(i+1)m\bar{m}+2\bar{m}+k}} \\ &+ \binom{i\bar{m}-m}{i-1}\left(\frac{i}{i\bar{m}-m}[t^{n-i+1}]\frac{1}{(1-z)^{m(i\bar{m}-m)}} - (m-1)[t^{n-i}]\frac{1}{(1-z)^{m(i\bar{m}+2)}}\right). \end{aligned}$$

By the Lagrange inversion formula, applied to z, we have, for $p \geq 1$ and $n \geq 1$,

$$[t^n]\frac{1}{(1-z)^p} = \frac{p}{n}\binom{n\bar{m}^2+p-1}{n-1}.$$

This formula actually holds for $n = 0$ if we write it as

$$[t^n]\frac{1}{(1-z)^p} = \frac{p\,(n\bar{m}^2+p-1)!}{n!\,(n\bar{m}^2-n+p)!},$$

and actually for $n < 0$ as well with the convention $\binom{a}{n-1} = 0$ if $n < 0$. With this convention, we have, for $1 \leq i \leq n+1$,

$$\begin{aligned} i[x^i t^n]F(t;x,1) = &-\frac{\bar{m}(im+1)}{n-i}\binom{i\bar{m}}{i-1}\binom{n\bar{m}^2-i\bar{m}+m}{n-i-1} \\ &+\sum_{k=1}^{m-2} k\frac{(i+1)m\bar{m}+2\bar{m}+k}{n-i-1}\binom{i\bar{m}-k-1}{i-1}\binom{n\bar{m}^2-i\bar{m}+m+k}{n-i-2} \\ &+m\binom{i\bar{m}-m}{i-1}\left(\frac{i}{n-i+1}\binom{n\bar{m}^2-i\bar{m}+2m}{n-i}\right. \\ &\left.-(m-1)\frac{i\bar{m}+2}{n-i}\binom{n\bar{m}^2-i\bar{m}+2m-1}{n-i-1}\right). \end{aligned}$$

This gives the expression (11) of $f_{n,i}^{(m)}$, with $P_m(n,i)$ given by (12). Clearly, $i(i-1)P_m(n,i)$ is a polynomial in n and i, but we still have to prove that it is divisible by $i(i-1)$.

For $m \geq 1$ and $1 \leq k \leq m-2$, the polynomials $\binom{i\bar{m}}{m}$ and $\binom{im}{k}$ are divisible by i. The next-to-last term of (12) contains an explicit factor i. The last term vanishes if $m = 1$, and otherwise contains a factor $\binom{im}{m-1}$, which is a multiple of i. Hence each term of (12) is divisible by i.

Finally, the right-hand side of (12) is easily evaluated to be 0 when $i = 1$, using the `sum` function of MAPLE. ∎

4 Final comments

Bijective proofs? Given the simplicity of the numbers (1), it is natural to ask about a bijective enumeration of m-Tamari intervals. A related question would be to extend the bijection of [4] (which transforms 1-Tamari intervals into triangulations) into a bijection between m-Tamari intervals and certain maps (or related structures, like *balanced trees* or *mobiles* [28, 9]). Counting these structures in a bijective way (as is done in [25] for triangulations) would then provide a bijective proof of (1).

Symmetry. The fact that the joint distribution of the number of non-initial contacts of the lower path and the initial rise of the upper path is symmetric remains a combinatorial mystery to us, even when $m = 1$. What *is* easy to see is that the joint distribution of the

number of non-initial contacts of the lower path and the *final descent* of the upper path is symmetric. Indeed, there exists a simple involution on Dyck paths that reverses the Tamari order and exchanges these two parameters: If we consider Dyck paths as postorder encodings of binary trees, this involution amounts to a simple reflection of trees. Via the bijection of [4], these two parameters correspond to the degrees of two vertices of the root-face of the triangulation [4, Def. 3.2], so that the symmetry is also clear in this setting.

A q-analogue of the functional equation. As described in the introduction, the numbers $f_n^{(m)}$ are conjectured to give the dimension of certain polynomial rings $\mathrm{DR}_{3,n}^{m,\epsilon}$. These rings are tri-graded (with respect to the sets of variables $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$), and it is conjectured [3] that the dimension of the homogeneous component in the x_i's of degree k is the number of intervals $[P, Q]$ in $\mathcal{T}_n^{(m)}$ such that the longest chain from P to Q, in the Tamari order, has length k. One can recycle the recursive description of intervals described in Section 2.3 to generalize the functional equation of Proposition 8, taking into account (with a new variable q) this distance. Eq. (3) remains valid, upon defining the operator Δ by

$$\Delta S(x) = \frac{S(qx) - S(1)}{qx - 1}.$$

The coefficient of t^n in the series $F(t, q; x, y)$ does not seem to factor, even when $x = y = 1$. The coefficients of the bivariate series $F(t, q; 1, 1)$ have large prime factors.

More on m-Tamari lattices? We do not know of any simple description of the m-Tamari lattice in terms of rotations in $m+1$-ary trees (which are equinumerous with m-Dyck paths). A rotation for ternary trees is defined in [23], but does not give a lattice. However, as noted by the referee, if we interpret m-ballot paths as the *prefix* (rather than postfix) code of an $m+1$-ary tree, the covering relation can be described quite simply. One first chooses a leaf ℓ that is followed (in prefix order) by an internal node v. Then, denoting by $T_0, \ldots, T_m$ the $m+1$ subtrees attached to v, from left to right, we insert v and its first m subtrees in place of the leaf ℓ, which becomes the rightmost child of v. The rightmost subtree of v, T_m, finally takes the former place of v (Figure 10).

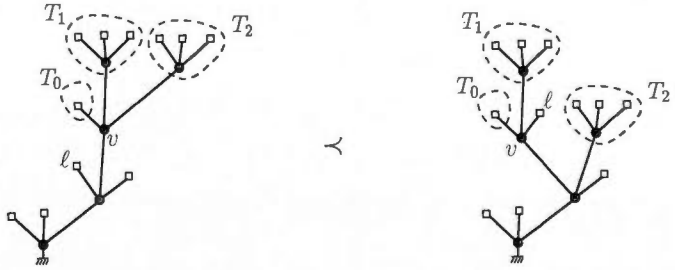

Figure 10: The covering relation of Figure 2 translated in terms of ternary trees.

More generally, it may be worth exploring analogues for the m-Tamari lattices of the numerous questions that have been studied for the usual Tamari lattice. To mention only

one, what is the *diameter* of the m-Tamari lattice, that is, the maximal distance between two m-Dyck paths in the Hasse diagram? When $m = 1$, it is known to be $2n - 6$ for n large enough, but the proof is as complicated as the formula is simple [13, 29].

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Appendix. A constructive approach to Lemma 13. In order to prove Lemma 13, we had to prove the more general identity (18). This identity was first guessed by expanding $\Lambda^{(k)}(1/w^m)$ in w and z, for several values of k and m. Fortunately, the coefficients in this expansion turned out to be simple products of binomial coefficients.

What if these coefficients had not been so simple? A constructive approach goes as follows. Introduce the following two formal power series in[3] t and s, with coefficients in $\mathbb{Q}[w, 1/w, z]$:

$$P(t;s) = \sum_{m\geq 1, k\geq 0} t^k s^{m-1} \Lambda^{(k)}(w^m) \quad \text{and} \quad N(t;s) = \sum_{m\geq 0, k\geq 0} t^k s^m \Lambda^{(k)}\left(\frac{1}{w^m}\right),$$

where we still denote $w = 1 + zu$. Observe that

$$P(t;0) = \sum_{k\geq 0} t^k \Lambda^{(k)}(w).$$

We want to compute the coefficient of $t^m s^m$ of $N(t;s)$, since this coefficient is $\Lambda^{(m)}(1/w^m)$.

Eq. (19) yield functional equations for the series P and N. For $P(t;s)$ first,

$$\begin{aligned} P(t;s) &= \sum_{m\geq 1} s^{m-1} w^m + t \sum_{m\geq 1, k\geq 1} t^{k-1} s^{m-1} \Lambda^{(k-1)}\left((z-1)w + z\sum_{a=2}^{m} w^a + w^{m+1}\right) \\ &= \frac{w}{1-sw} + \frac{t(z-1)}{1-s} P(t;0) + \frac{tz}{1-s}\left(P(t;s) - P(t;0)\right) + t\,\frac{P(t;s) - P(t;0)}{s}. \end{aligned}$$

Equivalently,

$$\left(1 - \frac{tz}{1-s} - \frac{t}{s}\right) P(t;s) = \frac{w}{1-sw} - \frac{tP(t;0)}{s(1-s)}. \tag{20}$$

Now for $N(t;s)$, we have

$$\begin{aligned} N(t;s) &= \sum_{m\geq 0} \frac{s^m}{w^m} + t \sum_{m\geq 1, k\geq 1} t^{k-1} s^m \Lambda^{(k-1)}\left(\frac{1-z}{w^{m-1}} - z\sum_{a=0}^{m-2} \frac{1}{w^a} - w\right) \\ &= \frac{1}{1-s/w} + ts(1-z)N(t;s) - \frac{tzs^2}{1-s} N(t;s) - \frac{ts}{1-s} P(t;0). \end{aligned}$$

[3]The variable t that we use here has nothing to do with the variable t that occurs in the generating function $F(t;x,y)$ of intervals.

Equivalently,

$$\left(1 - ts + \frac{tzs}{1-s}\right) N(t;s) = \frac{1}{1-s/w} - \frac{ts}{1-s} P(t;0). \tag{21}$$

Equation (20) can be solved using the kernel method (see e.g. [2, 8, 26]): let $S \equiv S(t,z)$ be the unique formal power series in t, with coefficients in $\mathbb{Q}[z]$, having constant term 0 and satisfying

$$1 - \frac{tz}{1-S} - \frac{t}{S} = 0.$$

That is,

$$S = \frac{1 + t - tz - \sqrt{1 - 2t(1+z) + t^2(1-z)^2}}{2}. \tag{22}$$

Then setting $s = S$ cancels the left-hand side of (20), giving

$$P(t;0) = \frac{wS(1-S)}{t(1-wS)}.$$

Combined with (21), this yields an explicit expression of $N(t;s)$:

$$N(t;s) = \frac{1}{1 - ts + \frac{tzs}{1-s}} \left(\frac{1}{1-s/w} - \frac{wsS(1-S)}{(1-s)(1-wS)} \right).$$

We want to extract from this series the coefficient of $t^m s^m$, and obtain the simple expression $(1-z)^m - w^m$ predicted by Lemma 13. Clearly, the first part of the above expression of $N(t;s)$ (with non-positive powers of w) contributes $(1-z)^m$, as expected. For $i \geq 1$, the coefficient of w^i in the second part of $N(t;s)$ is

$$R_i := -\frac{sS^i(1-S)}{(1-s)\left(1 - ts + \frac{tzs}{1-s}\right)}.$$

Recall that S, given by (22), depends on t and z, but not on s. Since $S = t + O(t^2)$, the coefficient of $t^m s^m$ in R_i is zero for $i > m$. When $i = m$, it is easily seen to be -1, as expected. In order to prove that the coefficient of $t^m s^m$ in R_i is zero when $0 < i < m$, we first perform a partial fraction expansion of R_i in s, using

$$(1-s)\left(1 - ts + \frac{tzs}{1-s}\right) = (1 - sS)(1 - st/S),$$

where S is defined by (22). This gives

$$R_i = -\frac{S^{i+1}(1-S)}{t - S^2}\left(\frac{1}{1 - ts/S} - \frac{1}{1 - sS}\right),$$

so that

$$[s^m]R_i = -\frac{S^{i+1-m}(1-S)}{t - S^2}\left(t^m - S^{2m}\right) = \sum_{j=0}^{m-1} t^{m-1-j} S^{2j+i-m+1}(S-1)$$

and

$$[s^m t^m]R_i = \sum_{j=0}^{m-1} [t^{j+1}]S^{2j+i-m+1}(S-1) = \sum_{j=0}^{m-i} [t^{j+1}]S^{2j+i-m+1}(S-1). \tag{23}$$

The Lagrange inversion gives, for $n \geq 1$ and $k \in \mathbb{Z}$,

$$[t^n]S^k(S-1) = \begin{cases} 0 & \text{if } n < k; \\ -1 & \text{if } n = k; \\ 1-kz & \text{if } n = k+1; \\ \dfrac{1}{n}\displaystyle\sum_{p=1}^{n-k} z^p \binom{n}{p}\binom{n-k-1}{p-1}\frac{n-p-kp}{n-k-1} & \text{otherwise.} \end{cases}$$

Returning to (23), this gives

$$[s^m t^m]R_i = -(m-i-1)z + \sum_{j=0}^{m-i-2}\sum_{p=1}^{m-i-j} \frac{z^p}{j+1}\binom{j+1}{p}\binom{m-i-j-1}{p-1}\frac{j+1-p(2j+i-m+2)}{m-i-j-1}.$$

Proving that this is zero boils down to proving, that, for $1 \leq p \leq m-i$,

$$\sum_{j=0}^{m-i-2} \frac{1}{j+1}\binom{j+1}{p}\binom{m-i-j-1}{p-1}\frac{j+1-p(2j+i-m+2)}{m-i-j-1} = (m-i-1)\mathbb{1}_{p=1}.$$

This is easily proved using Zeilberger's algorithm [24, Chap. 6], via the MAPLE package EKHAD (command `zeil`), or directly using the MAPLE command `sum`.

CHAPITRE V

TROISIÈME ARTICLE : THE REPRESENTATION OF THE SYMMETRIC GROUP ON m-TAMARI INTERVALS

L'article présenté dans ce chapitre constitue le point culminant de cette thèse. Nous donnons un peu plus de détails dans ce chapitre que dans les deux précédents. Nous devons cependant mentionner que les résultats présentés dans le chapitre précédent se sont avérés indispensables pour l'obtention des résultats dans le présent chapitre. Nous y reviendrons.

Nous avions soumis une version préliminaire sur arXiv (arXiv :1109.2398) que nous avions décidé de ne pas soumettre à un journal parce que les deux articles s'intersectent beaucoup. Nous discutons des deux versions de l'article ici en terminant par celui qui se situe à la fin de ce chapitre. Voici les contributions du premier de ces deux articles (que nous n'incluons pas dans cette thèse). Il s'intitule *Tamari lattices and parking functions : proof of a conjecture of F. Bergeron.*

Dans cet article, on prouve la conjecture 10 due à Bergeron. Soit $G(t; x, y)$ la série exponentielle (en t) des intervalles étiquetés du treillis de m-Tamari où les variables x et y ont le même rôle que pour les intervalles sans étiquettes. Plus précisemment, soit

$$G(t;x,y) := \sum_{n\geq 0} \frac{t^n}{n!} \sum_{D_1 \in \mathrm{Dyck}_m(n)} x^{\mathrm{contacts}(D_1)} \sum_{D_1 \leq D_2} y^{\mathrm{montée}(D_2)} \sum_{P \in \mathrm{Park_m(D_2)}} 1.$$

Nous avons premièrement démontré que la série $G(t;x;y)$ satisfait l'équation suivante [1]

$$G(t;x,y) = \exp\left(xyt(G(t,x,1)\Delta)^m\right)(x). \tag{5.1}$$

En dérivant par rapport à y, on obtient l'équation équivalente (avec la condition initiale)

$$\frac{\partial G}{\partial y}(t;x,y) = tx\left(G(t;x,1)\cdot\Delta\right)^{(m)}\left(G(t;x,y)\right) \quad , \quad G(t;x,0) = x. \tag{5.2}$$

Elle est dite différentielle-catalytique car x est une variable catalytique et différentielle, étant donné qu'on dérive par rapport à y. Bien qu'il existe plusieurs méthodes qui aident à résoudre des équations algébriques avec variable catalytique [2], il n'en existe que très peu pour les équations différentielles algébriques (avec variable catalytique) comme l'équation 5.2.

La clé de notre démarche consiste à exploiter une analogie [3] entre séries ordinaires et séries exponentielles, qui transforme $\frac{1}{1-z}$ en e^z. Grâce à cette analogie, nous avons été capables de résoudre l'équation 5.2. En effet, par analogie, les changements de variables que nous avons obtenus dans le cas non-étiqueté :

$$t = z(1-z)^{m^2+2m} \quad \text{et} \quad x = \frac{1+u}{(1+zu)^{m+1}}$$

se tranforment, en version exponentielle, en :

$$t = z'e^{-m(m+1)z'} \quad \text{et} \quad x = (1+u')e^{-mz'u'}. \tag{5.3}$$

De même pour la version paramétrique de $F(t;x,1)$:

$$F(t;x,1) = \frac{(1+u)(1+zu)}{(1+zu)^{m+1}(1-z)^{m+2}}\left(\frac{1+u-(1+zu)^{m+1}}{u}\right)$$

---

1. La démonstration de cette équation est en fait la même que le cas sans étiquettes.

2. Plusieurs de ces équations apparaissent dans le domaine de l'énumération de familles de cartes planaires enracinées

3. Cette analogie n'est pas sans rappeler la transformée de Laplace-Carson, qui transforme $\frac{1}{1-z}$ en e^z.

se transforme, en version exponentielle, en :

$$H_1(z';u',1) = \frac{(1+u')\,e^{z'u'}}{e^{mz'u'}\,e^{-(m+1)z'}} \left(\frac{1+u'-e^{mz'u'}}{u'} \right). \tag{5.4}$$

Le but est de démontrer que $G(t;x,1) = H_1(z';u',1)$ à partir de l'équation 5.2. Soit $H(z';u',y) \equiv G(t;x,y)$. En effectuant les changements de variables en 5.3, l'équation 5.2 se réécrit sous la forme :

$$\frac{\partial H}{\partial y}(z';u',y) = z'(1+u')e^{-mz'u'-m(m+1)z'} \left(\frac{u'H(z';u',1)}{(1+u')e^{-mz'u'}-1}\ \Delta'_u \right)^{(m)} H(z';u',y), \tag{5.5}$$

avec $\Delta'_u R(u') = \frac{R(u')-R(0)}{u'}$, et la condition initiale

$$H(z';u',0) = (1+u')e^{-mz'u'}. \tag{5.6}$$

Remplaçant $H(z';u',1)$ par $H_1(z';u',1)$ dans la récurrence 5.5, et dénotant $\tilde{H}(z';u',y)$ la solution de la nouvelle équation, on a

$$\frac{\partial \tilde{H}}{\partial y}(z';u',y) = z'(1+u')e^{-mz'u'-m(m+1)z'} \left(\frac{u'H_1(z';u',1)}{(1+u')e^{-mz'u'}-1}\ \Delta'_u \right)^{(m)} \tilde{H}(z';u',y), \tag{5.7}$$

avec la condition initiale

$$\tilde{H}(z';u',0) = (1+u')e^{-mz'u'}. \tag{5.8}$$

Dans le cas $m = 1$, nous avons résolu directement l'équation 5.7 (avec la condition initiale 5.8) grâce à un joli argument de symétrie. Nous avons ainsi obtenu une paramétrisation assez simple de la série $G(t,x,y)$ pour ce cas[4].

Dans le cas m général, la preuve est plus compliquée. Nous avons démontré que la solution de l'équation 5.7 est solution d'une nouvelle équation que l'on a « devinée » grâce à système de calcul formel. Nous avons ensuite démontré que la solution de cette nouvelle équation est unique, et donc que la solution des deux équations doit être la même. Ces deux étapes sont loin d'être triviales. Nous avons dû utiliser, entre autres, le

4. Ceci revient en quelque sorte à montrer directement l'équation 5.1 lorsque y=1 en controllant toute la somme dans l'exponentielle.

théorème de Puiseux, l'interpolation lagrangienne, la théorie des fonctions symétriques, et le calcul de singularités de séries. Notre nouvelle équation a l'avantage de permettre de fixer $y = 1$ directement, contrairement à l'équation 5.7. Notre théorème d'unicité s'applique alors directement, même lorsque $y = 1$. Avec les outils développés jusque-là, il n'est pas trop difficile de démontrer que $H_1(z'; u', 1)$ satisfait bien cette nouvelle équation lorsque $y = 1$. Ceci démontre que $\tilde{H}(z'; u', 1) = H_1(z'; u', 1)$ et donc que $G(t; x, y) = \tilde{H}(z'; u', y)$. En résumé, on obtient

$$G(t; x, 1) = \frac{(1+u')\, e^{z'u'+(m+1)z'}}{e^{mz'u'}} \left(\frac{1+u'-e^{mz'u'}}{u'} \right), \tag{5.9}$$

et donc que

$$G(t; 1, 1) = (1 - mz)e^{(m+1)z}. \tag{5.10}$$

En utilisant l'inversion de Lagrange, nous obtenons comme corollaire la preuve de la conjecture 10 due à Bergeron.

Corollaire 5 *Les intervalles étiquetés dans le treillis de m-Tamari sont comptés par*

$$(m+1)^n (mn+1)^{n-2}. \tag{5.11}$$

Soit $G(t; x, y, q_1)$ le q_1-analogue de la fonction génératrice des intervalles étiquetés où la variable q_1 compte la longueur de la plus grande chaîne dans chacun des intervalles. La série $G(t; x, y, q_1)$ satisfait l'équation

$$G(t; x, y, q_1) = exp\ (xyt(G(t, x, 1, q_1)\Delta_{q_1})^m)(x), \tag{5.12}$$

où

$$\Delta_{q_1}(R(t; x, q_1)) := \frac{R(t; xq_1, q_1) - R(t; 1, q_1)}{xq_1 - 1}.$$

Dans l'article qui suit (celui qui a été soumis à un journal), nous avons été très surpris de constater que l'on pouvait généraliser le résultat précédent en calculant toute la série $\mathcal{F}(R_{\mathrm{Tam}_m(n)})$ et ainsi démontrer la conjecture 15 de Bergeron et P.-R. . Soit $B(t, p; x, y)$ la série de Frobenius raffinée de $R_{\mathrm{Tam}_m(n)}$ définie par

$$B(t,p;x,y)=\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{I\in\mathrm{Fix}(\sigma)}x^{\mathrm{contacts(I)}}\,y^{\mathrm{montée(I)}}p_{\lambda(\sigma)}, \tag{5.13}$$

où $\mathrm{Fix}(\sigma)$ est l'ensemble des intervalle étiquetés dans le treillis de m-Tamari fixés sous l'action de σ, $\mathrm{contacts}(I)$ est le nombre de contacts de l'extrémité inférieure de I, $\mathrm{montée}(I)$ est la hauteur de la première montée de l'extrémité supérieure de I et $\lambda(\sigma)$ est le partage formé des longeurs des cycles de σ.

L'équation 5.1 se généralise facilement en

$$B(t,p;x,y)=\exp\left(y\sum_{k\geq 1}\frac{p_k}{k}\left(tx(B(t,p;x,1)\Delta)^{(m)}\right)^{(k)}\right)(x). \tag{5.14}$$

En dérivant par rapport à y, nous obtenons l'équation équivalente

$$\frac{\partial B}{\partial y}(t,p;x,y)=\sum_{k\geq 1}\frac{p_k}{k}\left(tx(B(t,p;x,1)\Delta)^{(m)}\right)^{(k)}(B(t,p;x,y)), \tag{5.15}$$

avec la condition initiale

$$B(t,p;x,0)=x. \tag{5.16}$$

Par analogie, les changements de variables 5.3 deviennent

$$t=z''e^{-mL(z'',p)}\quad\text{and}\quad x=(1+u'')e^{-mK(z'',p;u'')}, \tag{5.17}$$

où $L(z'',p)$ et $K(z'',p;u'')$ sont les séries en z'' suivantes :

$$\begin{aligned}L(z'',p)&=\sum_{k\geq 1}\frac{p_k}{k}\binom{(m+1)k}{k}(z'')^k,\\ K(z'',p;u'')&=\sum_{k\geq 1}\frac{p_k}{k}(z'')^k\sum_{i=1}^{k}\binom{(m+1)k}{k-i}(u'')^i,\end{aligned}$$

et la paramétrisation 5.10 devient

$$B(t,p;x,1)=(1+u'')\,e^{K(z'',p;u'')+L(z'',p)}\left(\frac{(1+u'')e^{-mK(z'',p;u'')}-1}{u''}\right). \tag{5.18}$$

La preuve présentée précédemment se transpose à peu près à ce cas plus général, modulo l'exploitation de liens entre la combinatoire des chemins et des identités d'opérateurs

sur les séries formelles. La paramétrisation 5.18 implique que

$$B(t,p;1,1) = e^{L(z'',p)}\left(1 - m\sum_{k\geq 1}\frac{p_k}{k}(z'')^k\binom{(m+1)k}{k-1}.\right) \tag{5.19}$$

Via l'inversion de Lagrange, on obtient la preuve de la conjecture 15, à savoir que :

Corollaire 6

$$\mathcal{F}(R_{\mathrm{Tam}_m(n)}) = \sum_{\lambda=(\lambda_1,..,\lambda_\ell)\vdash n}(mn+1)^{\ell-2}\prod_{1\leq i\leq \ell}\binom{(m+1)\lambda_i}{\lambda_i}\frac{p_\lambda(w)}{z_\lambda}. \tag{5.20}$$

Prenant le coefficient de x^2 dans la série $B(t,p;x,1)$ donne que

Corollaire 7 *La série de Frobenius de la sous-représentation de* $R_{\mathrm{Tam_m(n)}}$ *ayant comme base les intervalles étiquetés primitifs du treillis de m-Tamari est donnée par*

$$\sum_{\lambda=(\lambda_1,..,\lambda_\ell)\vdash n}((m+1)n-1)^{\ell-2}\prod_{1\leq i\leq \ell}\binom{(m+1)\lambda_i-1}{\lambda_i}\frac{p_\lambda(w)}{z_\lambda}. \tag{5.21}$$

La preuve de ces résultats est très technique pour le cas $m > 1$. Une preuve plus simple serait grandement appréciée. Nous en reparlerons dans le prochain chapitre.

THE REPRESENTATION OF THE SYMMETRIC GROUP ON m-TAMARI INTERVALS

MIREILLE BOUSQUET-MÉLOU, GUILLAUME CHAPUY, AND LOUIS-FRANÇOIS PRÉVILLE-RATELLE

ABSTRACT. An m-ballot path of size n is a path on the square grid consisting of north and east unit steps, starting at $(0,0)$, ending at (mn,n), and never going below the line $\{x = my\}$. The set of these paths can be equipped with a lattice structure, called the m-Tamari lattice and denoted by $\mathcal{T}_n^{(m)}$, which generalizes the usual Tamari lattice $\mathcal{T}_n$ obtained when $m = 1$. This lattice was introduced by F. Bergeron in connection with the study of diagonal coinvariant spaces in three sets of n variables. The representation of the symmetric group $\mathfrak{S}_n$ on these spaces is conjectured to be closely related to the natural representation of $\mathfrak{S}_n$ on (labelled) intervals of the m-Tamari lattice, which we study in this paper.

An interval $[P, Q]$ of $\mathcal{T}_n^{(m)}$ is *labelled* if the north steps of Q are labelled from 1 to n in such a way the labels increase along any sequence of consecutive north steps. The symmetric group $\mathfrak{S}_n$ acts on labelled intervals of $\mathcal{T}_n^{(m)}$ by permutation of the labels. We prove an explicit formula, conjectured by F. Bergeron and the third author, for the character of the associated representation of $\mathfrak{S}_n$. In particular, the dimension of the representation, that is, the number of labelled m-Tamari intervals of size n, is found to be

$$(m+1)^n(mn+1)^{n-2}.$$

These results are new, even when $m = 1$.

The form of these numbers suggests a connection with parking functions, but our proof is not bijective. The starting point is a recursive description of m-Tamari intervals. It yields an equation for an associated generating function, which is a refined version of the Frobenius series of the representation. The form of this equation is highly non-standard: it involves two additional variables x and y, a derivative with respect to y and iterated divided differences with respect to x. The hardest part of the proof consists in solving it, and we develop original techniques to do so.

1. INTRODUCTION AND MAIN RESULT

An *m-ballot path* of size n is a path on the square grid consisting of north and east unit steps, starting at $(0,0)$, ending at (mn,n), and never going below the line $\{x = my\}$. It is well-known that there are

$$\frac{1}{mn+1}\binom{(m+1)n}{n},$$

such paths [8], and that they are in bijection with $(m+1)$-ary trees with n inner nodes.

François Bergeron recently defined on the set $\mathcal{T}_n^{(m)}$ of m-ballot paths of size n an order relation. It is convenient to describe it via the associated covering relation, exemplified in Figure 1.

Definition 1. *Let P and Q be two m-ballot paths of size n. Then Q covers P if there exists in P an east step a, followed by a north step b, such that Q is obtained from P by swapping a and S, where S is the shortest factor of P that begins with b and is a (translated) m-ballot path.*

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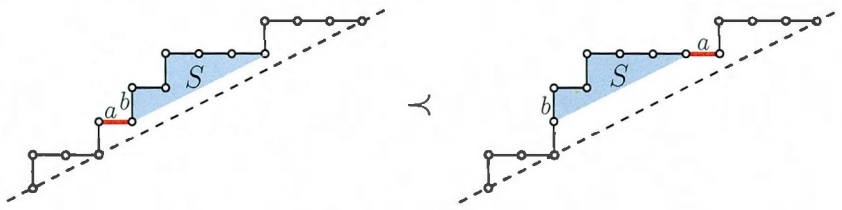

FIGURE 1. The covering relation between m-ballot paths ($m = 2$).

It was shown in [7] that this order endows $\mathcal{T}_n^{(m)}$ with a lattice structure, which is called the *m-Tamari lattice of size n*. When $m = 1$, it coincides with the classical Tamari lattice $\mathcal{T}_n$ [5, 9, 23, 24]. Figure 2 shows two of the lattices $\mathcal{T}_n^{(m)}$.

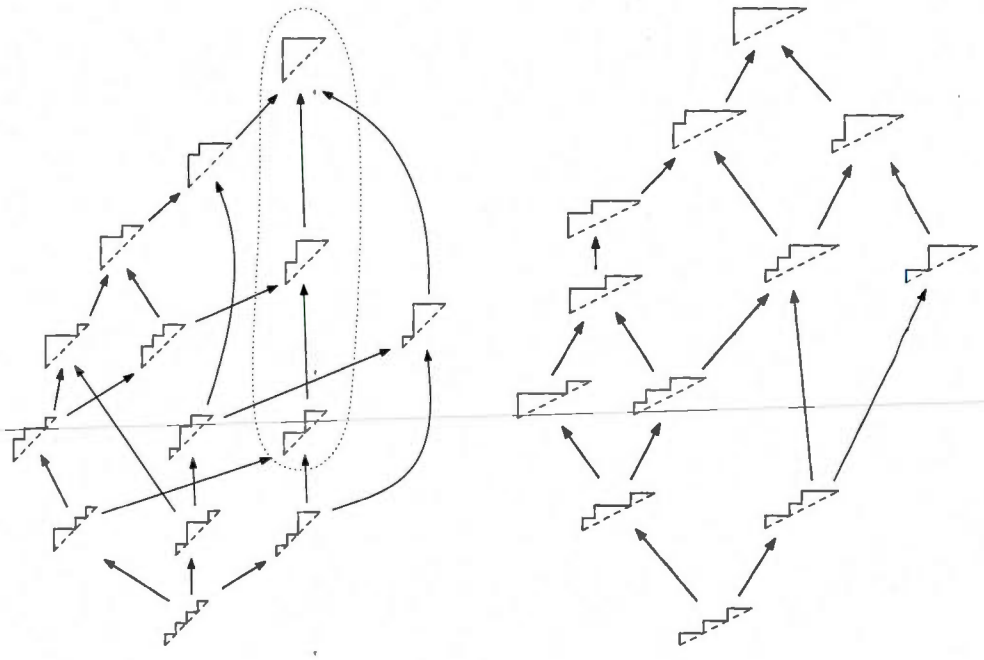

FIGURE 2. The m-Tamari lattice $\mathcal{T}_n^{(m)}$ for $m = 1$ and $n = 4$ (left) and for $m = 2$ and $n = 3$ (right). The three walks surrounded by a line in $\mathcal{T}_4^{(1)}$ form a lattice that is isomorphic to $\mathcal{T}_2^{(2)}$ (see Proposition 6).

The interest in these lattices is motivated by their — still conjectural — connections with trivariate diagonal coinvariant spaces [4, 7]. Some of these connections are detailed at the end of this introduction. In particular, it is believed that the representation of the symmetric group on these spaces is closely related to the representation of the symmetric group on *labelled* m-Tamari intervals. The aim of this paper is to characterize the latter representation, by describing explicitly its character.

So let us define this representation and state our main result. Let us call *ascent* of a path a maximal sequence of consecutive north steps. An m-ballot path of size n is *labelled* if the north steps are labelled from 1 to n, in such a way the labels increase along ascents (see the upper paths in Figure 3). These paths are in bijection with $(1, m, \dots, m)$-*parking functions* of size n, in the sense of [30, 31]: the function f associated with a path Q satisfies $f(i) = k$ if the north step

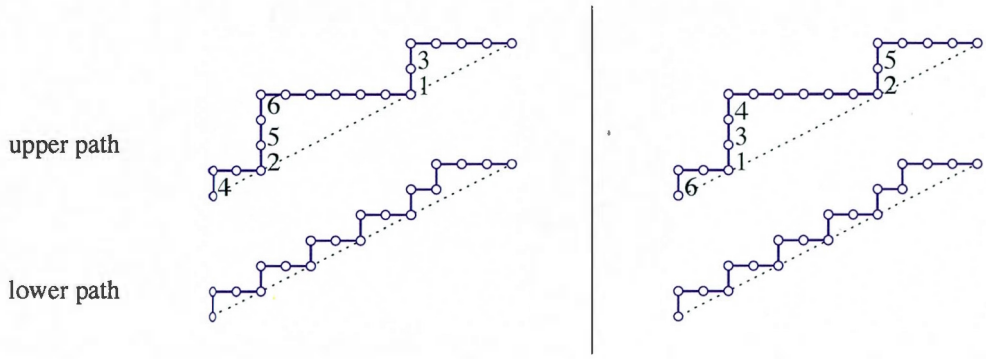

FIGURE 3. A labelled 2-Tamari interval, and its image under the action of $\sigma = 2\,3\,5\,6\,1\,4$.

of Q labelled i lies at abscissa $k-1$. The symmetric group $\mathfrak{S}_n$ acts on labelled m-ballot paths of size n by permuting labels, and then reordering them in each ascent (Figure 3, top paths). The character of this representation, evaluated at a permutation of cycle type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, is

$$(mn+1)^{\ell-1}.$$

This formula is easily proved using the cycle lemma [27]. As recalled further down, this representation is closely related to the representation of $\mathfrak{S}_n$ on diagonal coinvariant spaces in two sets of variables.

Now an m-Tamari interval $[P, Q]$ is *labelled* if the upper path Q is labelled. The symmetric group $\mathfrak{S}_n$ acts on labelled intervals of $\mathcal{T}_n^{(m)}$ by rearranging the labels of Q as described above (Figure 3). We call this representation the *m-Tamari* representation of $\mathfrak{S}_n$. Our main result is an explicit expression for its character χ_m, which was conjectured by Bergeron and the third author [4].

Theorem 2. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of n and σ a permutation of $\mathfrak{S}_n$ having cycle type λ. Then for the m-Tamari representation of $\mathfrak{S}_n$,*

$$\chi_m(\sigma) = (mn+1)^{\ell-2} \prod_{1 \le i \le \ell} \binom{(m+1)\lambda_i}{\lambda_i}. \tag{1}$$

Since $\mathfrak{S}_n$ acts by permuting labelled intervals, this is also the number of labelled m-Tamari intervals left unchanged under the action of σ. The value of the character only depends on the cycle type λ, and will sometimes be denoted $\chi_m(\lambda)$.

In particular, the dimension of the representation, that is, the number of labelled m-Tamari intervals of size n, is

$$\chi_m(\mathrm{id}) = (mn+1)^{n-2}(m+1)^n. \tag{2}$$

We were unable to find a bijective proof of these amazingly simple formulas. Instead, our proof uses generating functions and goes as follows. We introduce a generating function $F^{(m)}(t, p; x, y)$ counting labelled intervals according to multiple parameters, where p is an infinite sequence of variables $p_1, p_2, \ldots$, which can be thought of as power sums. We call this series the *refined Frobenius series* of the m-Tamari representation (Section 2). Then, we describe a recursive construction of intervals and translate it into a functional equation defining $F^{(m)}(t, p; x, y)$ (Proposition 5, Section 3). The form of this equation, which involves both continuous and discrete derivatives (a.k.a. divided differences) is new to us, and its solution is the most difficult and original part of the paper. The principles of our approach are explained in Section 4, and exemplified with the case $m = 1$. The general case is solved in Section 5. The techniques we use are borrowed from a former prepublication[1] by the same authors, which only proves (2). Since going from (2)

[1] which will not be submitted, as it is subsumed by the present paper

to (1) implies a further complexification, the prepublication [6] may be a good introduction to our techniques. However, the present paper is completely self-contained. Section 6 gathers a few final comments. In particular, we reprove a result of [7] giving the number of *unlabelled* intervals of $\mathcal{T}_n^{(m)}$.

In the remainder of this section, we recall some of the conjectured connections between Tamari intervals and trivariate diagonal coinvariant spaces. They seem to parallel the (now largely proved) connections between ballot paths and *bivariate* diagonal coinvariant spaces, which have attracted considerable attention in the past 20 years [12, 14, 15, 18, 21, 20, 26] and are still a very active area of research today [1, 2, 10, 11, 19, 16, 22, 25].

Let $X = (x_{i,j})_{\substack{1\le i\le k\\ 1\le j\le n}}$ be a matrix of variables. The diagonal coinvariant space $\mathcal{DR}_{k,n}$ is defined as the quotient of the ring $\mathbb{C}[X]$ of polynomials in the coefficients of X by the ideal $\mathcal{J}$ generated by constant-term free polynomials that are invariant under permuting the columns of X. For example, when $k = 2$, denoting $x_{1,j} = x_j$ and $x_{2,j} = y_j$, the ideal $\mathcal{J}$ is generated by constant-term free polynomials f such that for all $\sigma \in \mathfrak{S}_n$,

$$f(X) = \sigma(f(X)), \quad \text{where} \quad \sigma(f(X)) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, y_{\sigma(1)}, \dots, y_{\sigma(n)}).$$

An *m-extension* of the spaces $\mathcal{DR}_{k,n}$ is of great importance here [13, p. 230]. Let $\mathcal{A}$ be the ideal of $\mathbb{C}[X]$ generated by *alternants* under the diagonal action described above; that is, by polynomials f such that $\sigma(f(X)) = \varepsilon(\sigma)f(X)$. There is a natural action of $\mathfrak{S}_n$ on the quotient space $\mathcal{A}^{m-1}/\mathcal{J}\mathcal{A}^{m-1}$. Let us twist this action by the $(m-1)^{\text{st}}$ power of the sign representation ε: this gives rise to spaces

$$\mathcal{DR}_{k,n}^m := \varepsilon^{m-1} \otimes \mathcal{A}^{m-1}/\mathcal{J}\mathcal{A}^{m-1},$$

so that $\mathcal{DR}_{k,n}^1 = \mathcal{DR}_{k,n}$. It is now a famous theorem of Haiman [20, 17] that, as representations of $\mathfrak{S}_n$,

$$\mathcal{DR}_{2,n}^m \cong \varepsilon \otimes \mathrm{Park}_m(n)$$

where $\mathrm{Park}_m(n)$ is the m-parking representation of $\mathfrak{S}_n$, that is, the representation on m-ballot paths of size n defined above.

In the case of three sets of variables, Bergeron and Préville-Ratelle [4] conjecture that, as representations of $\mathfrak{S}_n$,

$$\mathcal{DR}_{3,n}^m \cong \varepsilon \otimes \mathrm{Tam}_m(n),$$

where $\mathrm{Tam}_m(n)$ is the m-Tamari representation of $\mathfrak{S}_n$. The fact that the dimension of this space seems to be given by (2) is an earlier conjecture due to F. Bergeron. This was also observed earlier for small values of n by Haiman [21] in the case $m = 1$.

2. The refined Frobenius series

2.1. Definitions and notation

Let $\mathbb{L}$ be a commutative ring and t an indeterminate. We denote by $\mathbb{L}[t]$ (resp. $\mathbb{L}[[t]]$) the ring of polynomials (resp. formal power series) in t with coefficients in $\mathbb{L}$. If $\mathbb{L}$ is a field, then $\mathbb{L}(t)$ denotes the field of rational functions in t. This notation is generalized to polynomials, fractions and series in several indeterminates. We denote by bars the reciprocals of variables: for instance, $\bar{u} = 1/u$, so that $\mathbb{L}[u, \bar{u}]$ is the ring of Laurent polynomials in u with coefficients in $\mathbb{L}$. The coefficient of u^n in a Laurent polynomial $P(u)$ is denoted by $[u^n]P(u)$.

We use classical notation relative to integer partitions, which we recall briefly. A *partition* λ of n is a non-increasing sequence of integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell > 0$ summing to n. We write $\lambda \vdash n$ to mean that λ is a partition of n. Each component λ_i is called a *part*. The number of parts or *length* of the partition is denoted by $\ell(\lambda)$. The *cycle type* of a permutation $\sigma \in \mathfrak{S}_n$ is the partition of n whose parts are the lengths of the cycles of σ. This partition is denoted by $\lambda(\sigma)$. The number of permutations $\sigma \in \mathfrak{S}_n$ having cycle type $\lambda \vdash n$ equals $\frac{n!}{z_\lambda}$ where $z_\lambda := \prod_{i\ge 1} i^{\alpha_i}\alpha_i!$, where α_i is the number of parts equal to i in λ.

We let $p = (p_1, p_2, \dots)$ be an infinite list of independent variables, and for λ a partition of n, we let $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}$. The reader may view the p_λ's as power sums in some ground set of variables (see e.g. [28]). This point of view is not really needed in this paper, but it explains why we call *refined Frobenius series* our main generating function. Throughout the paper, we denote by $\mathbb{K} = \mathbb{Q}(p_1, p_2, \dots)$ the field of rational fractions in the p_i's with rational coefficients.

Given a Laurent polynomial $P(u)$ in a variable u, we denote by $[u^{\geq}]P(u)$ the *non-negative part of* $P(u)$ *in* u, defined by

$$[u^{\geq}]P(u) = \sum_{i \geq 0} u^i [u^i]P(u).$$

The definition is then extended by linearity to power series whose coefficients are Laurent polynomials in u. We define similarly the positive part of $P(u)$, denoted by $[u^{>}]P(u)$.

We now introduce several series and polynomials which play an important role in this paper. They depend on two independent variables u and z. First, we let $v \equiv v(u)$ be the following Laurent polynomial in u:

$$v = (1+u)^{m+1} u^{-m}.$$

We now consider the following series:

$$V(v) = \sum_{k \geq 1} \frac{p_k}{k} v^k z^k. \tag{3}$$

It is is a formal power series in z whose coefficients are Laurent polynomials in u over the field $\mathbb{K}$. Finally we define the two following formal power series in z:

$$L \equiv L(z,p) \quad := \quad [u^0]V(v) = \sum_{k \geq 1} \frac{p_k}{k} \binom{(m+1)k}{k} z^k, \tag{4}$$

$$K(u) \equiv K(z,p;u) \quad := \quad [u^{>}]V(v) = \sum_{k \geq 1} \frac{p_k}{k} z^k \sum_{i=1}^{k} \binom{(m+1)k}{k-i} u^i. \tag{5}$$

As shown with these series, we often omit to denote the dependence of our series in certain variables (like z and p above). This is indicated by the symbol $\equiv$.

2.2. A REFINED THEOREM

As stated in Theorem 2, the value of the character $\chi_m(\sigma)$ is the number of labelled intervals fixed under the action of σ, and one may see (1) as an enumerative result. Our main result is a refinement of (1) where we take into account two more parameters, which we now define. The first parameter is the number of *contacts* of the interval: A *contact* of a ballot path P is a vertex of P lying on the line $\{x = my\}$, and a *contact* of a Tamari interval $[P, Q]$ is a contact of the *lower* path P. We denote by $c(P)$ the number of contacts of P.

By definition of the action of $\mathfrak{S}_n$ on m-Tamari intervals, a labelled interval $I = [P, Q]$ is fixed by a permutation $\sigma \in \mathfrak{S}_n$ if and only if σ stabilizes the set of labels of each ascent of Q. Equivalently, each cycle of σ is contained in the label set of an ascent of Q. If this holds, we let $a_\sigma(Q)$ be the number of cycles of σ occurring in the *first* ascent of Q: this is our second parameter.

The main object we handle in this paper is a generating function for pairs (σ, I), where σ is a permutation and $I = [P, Q]$ is a labelled m-Tamari interval fixed by σ. In this series $F^{(m)}(t, p; x, y)$, pairs (σ, I) are counted by the size $|I|$ of I (variable t), the number $c(P)$ of contacts (variable x), the parameter $a_\sigma(Q)$ (variable y), and the cycle type of σ (one variable p_i for each cycle of size i in σ). Moreover, $F^{(m)}(t, p; x, y)$ is an exponential series in t. That is,

$$F^{(m)}(t, p; x, y) = \sum_{I=[P,Q],\ \text{labelled}} \frac{t^{|I|}}{|I|!} x^{c(P)} \sum_{\sigma \in \mathrm{Stab}(I)} y^{a_\sigma(Q)} p_{\lambda(\sigma)}, \tag{6}$$

where the first and second sums are taken respectively over all labelled m-Tamari intervals I, and over all permutations σ fixing I.

Note that when $(x,y)=(1,1)$, we have:

$$F^{(m)}(t,p;1,1)=\sum_{I=[P,Q]}\frac{t^{|I|}}{|I|!}\sum_{\sigma\in\mathrm{Stab}(I)}p_{\lambda(\sigma)}=\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathfrak{S}_n}\chi_m(\sigma)p_{\lambda(\sigma)}=\sum_{n\geq 0}t^n\sum_{\lambda\vdash n}\chi_m(\lambda)\frac{p_\lambda}{z_\lambda},$$

since the number of intervals fixed by a permutation depends only on its cycle type, and since $\frac{n!}{z_\lambda}$ is the number of permutations of cycle type λ. Hence, in representation theoretic terms, $[t^n]F^{(m)}(t,p;1,1)$ is the *Frobenius characteristic* of the m-Tamari representation of $\mathfrak{S}_n$, also equal to

$$\sum_{\lambda\vdash n}c(\lambda)s_\lambda,$$

where s_λ is the Schur function of shape λ and $c(\lambda)$ is the multiplicity of the irreducible representation associated with λ in the m-Tamari representation [28, Chap. 4]. For this reason, we call $F^{(m)}(t,p;x,y)$ a *refined Frobenius series.*

The most general result of this paper is a (complicated) parametric expression of $F^{(m)}(t,p;x,y)$ which becomes simpler when $y=1$. We state here the result for $y=1$.

Theorem 3. *Let $F^{(m)}(t,p;x,y)\equiv F(t,p;x,y)$ be the refined Frobenius series of the m-Tamari representation, defined by (6). Let z and u be two indeterminates, and write*

$$t=ze^{-mL}\quad\text{and}\quad x=(1+u)e^{-mK(u)},\tag{7}$$

where $L\equiv L(z,p)$ and $K(u)\equiv K(z,p;u)$ are defined by (4) and (5). Then $F(t,p;x,1)$ becomes a series in z with polynomial coefficients in u and the p_i, and this series has a simple expression:

$$F(t,p;x,1)=(1+\bar{u})e^{K(u)+L}\left((1+u)e^{-mK(u)}-1\right)\tag{8}$$

with $\bar{u}=1/u$. In particular, in the limit $u\to 0$, we obtain

$$F(t,p;1,1)=e^L\left(1-m\sum_{k\geq 1}\frac{p_k}{k}z^k\binom{(m+1)k}{k-1}\right).\tag{9}$$

Theorem 2 will follow by extracting from $F(t,p;1,1)$ the coefficient of p_λ/z_λ (via Lagrange's inversion). Our expression of $F^{(m)}(t,p;x,y)$ is given in Theorem 21. When $m=1$, it takes a reasonably simple form, which we now present. The case $m=2$ is also detailed at the end of Section 5 (Corollary 22).

Theorem 4. *Let $F^{(1)}(t,p;x,y)\equiv F(t,p;x,y)$ be the refined Frobenius series of the 1-Tamari representation, defined by (6). Define the series $V(v)$, L and $K(u)$ by (3–5), with $m=1$, and perform the change of variables (7), still with $m=1$. Then $F(t,p;x,y)$ becomes a formal power series in z with polynomial coefficients in u and y, which is given by*

$$F(t,p;x,y)=(1+u)\,[u^{\geq}]\left(e^{yV(v)-K(u)}-\bar{u}e^{yV(v)-K(\bar{u})}\right),\tag{10}$$

with $\bar{u}=1/u$.

Remarks

1. It is easily seen that the case $y=1$ of (10) reduces to the case $m=1$ of (8) (the proof relies on the fact that L and $K(u)$ are respectively the constant term and the positive part of $V(v)$ in u, and that $v=(1+u)(1+\bar{u})$ is symmetric in u and $\bar{u}$).

2. When $p_1=1$ and $p_i=0$ for $i>1$, the only permutation that contributes in (6) is the identity. We are thus simply counting labelled 1-Tamari intervals, by their size (variable t), the number of contacts (variable x) and the size of the first ascent (variable y). Still taking $m=1$, we have

$V(v) = zv = z(1+u)(1+\bar{u})$, $K(u) = zu$ and the extraction of the positive part in u in (10) can be performed explicitly:

$$\begin{aligned} F(t,p;x,y) &= (1+u)[u^{\geq}]\left(e^{yzv-zu} - \bar{u}e^{yzv-z\bar{u}}\right) \\ &= (1+u)e^{2yz}\left(\sum_{0\leq i\leq j} u^{j-i}\frac{z^{i+j}y^i(y-1)^j}{i!j!} - \sum_{0\leq j<i} u^{i-j-1}\frac{z^{i+j}y^i(y-1)^j}{i!j!}\right). \end{aligned}$$

When $x = 1$, that is, $u = 0$, the double sums in this expression reduce to simple sums, and the generating function of labelled Tamari intervals, counted by the size and the height of the first ascent, is expressed in terms of Bessel functions:

$$\frac{F(t,p;1,y)}{e^{2yz}} = \sum_{i\geq 0}\frac{z^{2i}y^i(y-1)^i}{i!^2} - \sum_{j\geq 0}\frac{z^{2j+1}y^{j+1}(y-1)^j}{(j+1)!j!}.$$

3. A FUNCTIONAL EQUATION

The aim of this section is to establish a functional equation satisfied by the series $F^{(m)}(t,p;x,y)$.

Proposition 5. *For $m \geq 1$, let $F^{(m)}(t,p;x,y) \equiv F(x,y)$ be the refined Frobenius series of the m-Tamari representation, defined by (6). Then*

$$\begin{aligned} F(x,y) &= \sum_{k\geq 0}\tilde{h}_k(y)\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x) \\ &= \exp\left(y\sum_{k\geq 1}\frac{p_k}{k}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}\right)(x), \end{aligned}$$

where

$$\tilde{h}_k(y) = \sum_{\lambda\vdash k}\frac{p_\lambda}{z_\lambda}y^{\ell(\lambda)}, \tag{11}$$

Δ is the following divided difference operator

$$\Delta S(x) = \frac{S(x) - S(1)}{x-1},$$

and the powers (m) and (k) mean that the operators are applied respectively m times and k times.

Equivalently, $F(x,0) = x$ and

$$\frac{\partial F}{\partial y}(x,y) = \sum_{k\geq 1}\frac{p_k}{k}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(F(x,y)). \tag{12}$$

The above equations rely on a recursive construction of labelled m-Tamari intervals. Our description of the construction is self-contained, but we refer to [7] for several proofs and details.

3.1. RECURSIVE CONSTRUCTION OF TAMARI INTERVALS

We start by modifying the appearance of 1-ballot paths. We apply a 45 degree rotation to transform them into *Dyck paths*. A Dyck path of size n consists of steps $u = (1,1)$ (up steps) and steps $d = (1,-1)$ (down steps), starts at $(0,0)$, ends at $(2n,0)$ and never goes below the x-axis. We say that an up step has *rank* i if it is the i^{th} up step of the path. We often represent Dyck paths by words on the alphabet $\{u,d\}$. An ascent is thus now a maximal sequence of u steps.

Consider an m-ballot path of size n, and replace each north step by a sequence of m north steps. This gives a 1-ballot path of size mn, and thus, after a rotation, a Dyck path. In this path, for each $i \in [\![0, n-1]\!]$, the up steps of ranks $mi+1,\ldots,m(i+1)$ are consecutive. We call the Dyck paths satisfying this property *m-Dyck paths*, and say that the up steps of ranks $mi+1,\ldots,m(i+1)$ form a *block*. Clearly, m-Dyck paths of size mn (*i.e.*, having n blocks) are in one-to-one correspondence with m-ballot paths of size n.

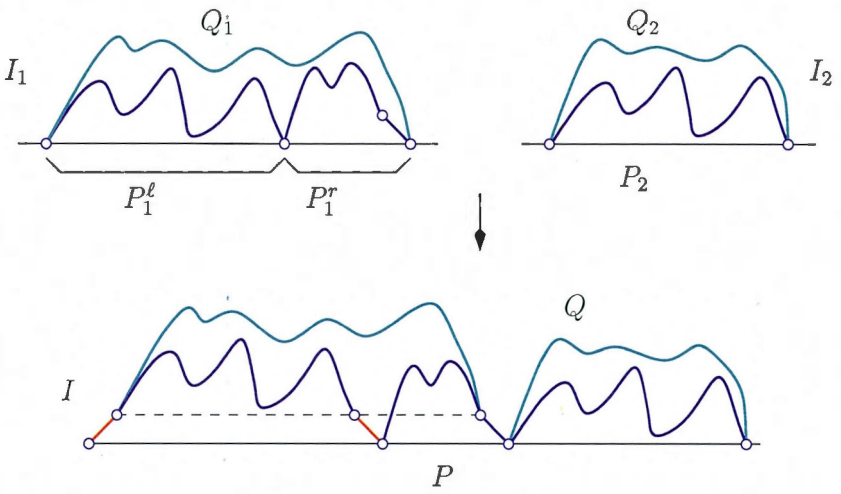

FIGURE 4. The recursive construction of Tamari intervals.

We often denote by $\mathcal{T}_n$, rather than $\mathcal{T}_n^{(1)}$, the usual Tamari lattice of size n. Similarly, the intervals of this lattice are called Tamari intervals rather than 1-Tamari intervals. As proved in [7], the transformation of m-ballot paths into m-Dyck paths maps $\mathcal{T}_n^{(m)}$ on a sublattice of $\mathcal{T}_{mn}$.

Proposition 6 ([7, Prop. 4]). *The set of m-Dyck paths with n blocks is the sublattice of $\mathcal{T}_{nm}$ consisting of the paths that are larger than or equal to $u^m d^m \dots u^m d^m$. It is order isomorphic to $\mathcal{T}_n^{(m)}$.*

We now describe a recursive construction of (unlabelled) Tamari intervals, again borrowed from [7]. Thanks to the embedding of $\mathcal{T}_n^{(m)}$ into $\mathcal{T}_{nm}$, it will also enable us to describe recursively m-Tamari intervals, for any value of m, in the next subsection.

A Tamari interval $I = [P, Q]$ is *pointed* if its lower path P has a distinguished contact. Such a contact splits P into two Dyck paths P^ℓ and P^r, respectively located to the left and to the right of the contact. The pointed interval I is *proper* is P^ℓ is not empty, *i.e.*, if the distinguished contact is not $(0,0)$. We often use the notation $I = [P^\ell P^r, Q]$ to denote a pointed Tamari interval. The contact $(0,0)$ is called the *initial* contact.

Proposition 7. *Let $I_1 = [P_1^\ell P_1^r, Q_1]$ be a pointed Tamari interval, and let $I_2 = [P_2, Q_2]$ be a Tamari interval. Construct the Dyck paths*

$$P = uP_1^\ell dP_1^r P_2 \quad \text{and} \quad Q = uQ_1 dQ_2$$

as shown in Figure 4. Then $I = [P, Q]$ is a Tamari interval. Moreover, the mapping $(I_1, I_2) \mapsto I$ is a bijection between pairs (I_1, I_2) formed of a pointed Tamari interval and a Tamari interval, and Tamari intervals I of positive size. Note that I_1 is proper if and only if the first ascent of P has height larger than 1.

Remarks

1. To recover P_1^ℓ, P_1^r, Q_1, P_2 and Q_2 from P and Q, one proceeds as follows: Q_2 is the part of Q that follows the first return of Q to the x-axis; this also defines Q_1 unambiguously. The path P_2 is the suffix of P having the same size as Q_2. This also defines $P_1 := uP_1^\ell dP_1^r$ unambiguously. Finally, P_1^r is the part of P_1 that follows the first return of P_1 to the x-axis, and this also defines P_1^ℓ unambiguously.
2. This proposition is obtained by combining Proposition 5 in [7] and the case $m = 1$ of

Lemma 9 in [7]. With the notation $(P';p_1)$ and (Q',q_1) used therein, the above paths P_2 and Q_2 are respectively the parts of P' and Q' that lie to the right of q_1, while $P_1^\ell P_1^r$ and Q_1 are the parts of P' and Q' that lie to the left of q_1. The pointed vertex p_1 is the endpoint of P_1^ℓ. Proposition 5 in [7] guarantees that, if $P \preceq Q$ in the Tamari order, then $P_1^\ell P_1^r \preceq Q_1$ and $P_2 \preceq Q_2$.
3. One can keep track of several parameters in the construction of Proposition 7. For instance, the number of non-initial contacts of P is

$$c(P) - 1 = (c(P_1^r) - 1) + c(P_2). \tag{13}$$

3.2. From the construction to the functional equation

We now prove Proposition 5 through a sequence of lemmas. The first one describes $F^{(m)}(t,p;x,y)$ in terms of homogeneous symmetric functions rather than power sums.

Lemma 8. *Let $\tilde{h}_k(y)$ be defined by* (11)*, and set*

$$h_k = \tilde{h}_k(1) = \sum_{\lambda \vdash k} \frac{p_\lambda}{z_\lambda}.$$

Hence h_k is the k^{th} homogenous symmetric function if p_k is the k^{th} power sum. Then the refined Frobenius series $F^{(m)}(t,p;x,y)$, defined by (6)*, can also be written as the following* ordinary *generating function:*

$$F^{(m)}(t,p;x,y) = \sum_{I=[P,Q],\ \text{unlabelled}} t^{|I|} x^{c(P)} \tilde{h}_{a_1}(y) \prod_{i\geq 2} h_{a_i}, \tag{14}$$

where the sum runs over unlabelled *m-Tamari intervals I, and a_i is the height of the i^{th} ascent of the upper path Q. In particular, $F^{(m)}(t,p;x,1) \equiv F^{(m)}(x,1)$ is the ordinary generating function of m-Tamari intervals, counted by the size (t), the number of contacts (x), and the distribution of ascents (h_i for each ascent of height i in the upper path).*

Proof. Let $I = [P,Q]$ be an unlabelled Tamari interval, and let a_i be the height of the i^{th} ascent of Q. Denote $n = |I|$. An *I-partitioned permutation* is a permutation $\sigma \in \mathfrak{S}_n$, together with a partition of the set of cycles of σ into labelled subsets $A_1, A_2, \ldots$, such that the sum of the lengths of the cycles of A_i is a_i. In the expression (6) of $F^{(m)}$, the contribution of labelled intervals $\bar{I} = [P,\bar{Q}]$ obtained by labelling Q in all possible ways is $x^{c(P)}\phi(I)$, where

$$\phi(I) = \frac{t^{|I|}}{|I|!} \sum_{\bar{I}=[P,\bar{Q}]} \sum_{\sigma \in \mathrm{Stab}(\bar{I})} y^{a_\sigma(\bar{Q})} p_{\lambda(\sigma)}.$$

In other words, this is the exponential generating function of I-partitioned permutations, counted by the size (variable t), the number of cycles in the block A_1 (variable y), and the number of cycles of length j (variable p_j), for all j. By elementary results on exponential generating functions, this series factors over ascents of Q. The contribution of the i^{th} ascent is the exponential generating function of permutations of $\mathfrak{S}_{a_i}$, counted by the size, the number of cycles of length j for all j, and also by the number of cycles if $i = 1$. But this is exactly $t^{a_i} h_{a_i}$ (or $t^{a_1}\tilde{h}_{a_1}(y)$ if $i = 1$), since

$$t^a \tilde{h}_a(y) = t^a \sum_{\lambda \vdash a} \frac{p_\lambda}{z_\lambda} y^{\ell(\lambda)} = \frac{t^a}{a!} \sum_{\sigma \in \mathfrak{S}_a} p_\lambda(\sigma)\, y^{\ell(\lambda(\sigma))}.$$

Hence

$$\phi(I) = t^{|I|}\tilde{h}_{a_1}(y) \prod_{i\geq 2} h_{a_i},$$

and the proof is complete[2]. ∎

[2] An analogous result was used without proof in the study of the parking representation of the symmetric group [21, p. 28].

Lemma 9. *In the expression* (14) *of* $F^{(m)}(t,p;x,y) \equiv F(x,y)$, *the contribution of intervals* $I=[P,Q]$ *such that the first ascent of* Q *has height* k *is*

$$\tilde{h}_k(y)\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x).$$

This proves the first equation satisfied by $F^{(m)}(x,y)$ *in Proposition* 5.

Proof. We constantly use in this proof the inclusion $\mathcal{T}_n^{(m)} \subset \mathcal{T}_{nm}$ given by Proposition 6. That is, we identify elements of $\mathcal{T}_n^{(m)}$ with m-Dyck paths having n blocks. The size of an interval is thus now the number of blocks, and the height of the first ascent becomes the number of blocks in the first ascent.

Lemma 9 relies on the recursive description of Tamari intervals given in Proposition 7. We actually apply this construction to a slight generalization of m-Tamari intervals. For $\ell \geq 0$, an *ℓ-augmented m-Dyck path* is a Dyck path Q of size $\ell + mn$ for some integer n, where the first ℓ steps are up steps, and all the other up steps can be partitioned into *blocks* of m consecutive up steps. The ℓ first steps of Q are not considered to be part of a block, even if ℓ is a multiple of m. We denote by $a(Q)$ the number of blocks contained in the first ascent[3] of Q. A Tamari interval $I=[P,Q]$ is an *ℓ-augmented m-Tamari interval* if both P and Q are ℓ-augmented m-Dyck paths.

For $k,\ell \geq 0$ let $F_{k,\ell}(x)^{(m)} \equiv F_{k,\ell}(x)$ be the generating function of ℓ-augmented m-Tamari intervals $I=[P,Q]$ such that $a(Q)=k$, counted by the number of blocks (variable t), the number of *non-initial* contacts (variable x) and the number of non-initial ascents of Q having i blocks (one variable h_i for each $i \geq 1$, as before). We are going to prove that for all $k,\ell \geq 0$,

$$F_{k,\ell}(x) = \begin{cases} \frac{1}{x}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x) & \text{if } \ell = 0, \\ (F(x,1)\Delta)^{(\ell)}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x) & \text{if } \ell > 0. \end{cases} \tag{15}$$

We claim that (15) implies Lemma 9. Indeed, m-Tamari intervals coincide with 0-augmented m-Tamari intervals. Since the initial contact and the first ascent are not counted ~~in $F_{k,0}(x)$,~~ but ~~are counted in $F^{(m)}(x,y)$,~~ the contribution in $F^{(m)}(x,y)$ of intervals such that $a(Q)=k$ is $x\tilde{h}_k(y)F_{k,0}(x)$, as stated in the lemma.

We now prove (15), by lexicographic induction on (k,ℓ). For $(k,\ell)=(0,0)$, the unique interval involved in $F_{k,\ell}(x)$ is $\{\bullet\}$, where $\bullet$ is the path of length 0. Its contribution is 1 (since the initial and only contact is not counted). Therefore $F_{0,0}(x)=1$ and (15) holds. Let $(k,\ell) \neq (0,0)$ and assume that (15) holds for all lexicographically smaller pairs $(k',\ell') < (k,\ell)$. We are going to show that (15) holds at rank (k,ℓ).

If $k>0$ and $\ell=0$, then we are considering 0-augmented m-Tamari intervals, that is, usual m-Tamari intervals. But an m-Tamari interval $I=[P,Q]$ having n blocks and k blocks in the first ascent can be seen as an m-augmented m-Tamari interval having $n-1$ blocks and $k-1$ blocks in the first ascent, by considering that the first m up steps of P and Q are not part of a block. This implies that:

$$F_{k,0}(x) = tF_{k-1,m}(x) = \frac{1}{x}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x)$$

by the induction hypothesis (15) applied at rank $(k-1,m)$. This is exactly (15) at rank $(k,\ell=0)$.

Now assume $\ell \neq 0$. The series $F_{k,\ell}(x)$ counts ℓ-augmented m-Tamari intervals $I=[P,Q]$ such that $a(Q)=k$. By Proposition 7, such an interval can be decomposed into a pointed interval $I_1=[P_1^\ell P_1', Q_1]$ and an interval $I_2=[P_2,Q_2]$ (the "ℓ" in the notation P_1^ℓ is a bit unfortunate here; we hope it will not raise any difficulty). Note that I_2 is an m-Tamari interval, while I_1 is an $(\ell-1)$-augmented pointed m-Tamari interval. Conversely, starting from such a pair (I_1,I_2),

[3]Since the number of blocks does not depend on Q only, but also on ℓ, it should in principle be denoted $a^{(\ell)}(Q)$. We hope that our choice of a lighter notation will not cause any confusion.

the construction of Proposition 7 produces an ℓ-augmented m-Tamari interval, unless I_1 is not proper and $\ell > 1$. Moreover, $a(Q_1) = a(Q)$. Using (13), we obtain:

$$F_{k,\ell}(x) = F(x,1)\left(F^{\bullet}_{k,\ell-1}(x) + \mathbb{1}_{\ell=1}F^{\circ}_{k,\ell-1}(x)\right) \tag{16}$$

where $F^{\bullet}_{k,\ell-1}(x)$ (resp. $F^{\circ}_{k,\ell-1}(x)$) is the generating function of proper (resp. non-proper) pointed $(\ell-1)$-augmented m-Tamari intervals $I_1 = [P_1^{\ell}P_1^{r}, Q_1]$ such that $a(Q_1) = k$, counted by the number of blocks (variable t), the number of non-initial ascents of Q_1 of height i (variable h_i) for each $i \geq 1$, and the number of non-initial contacts of P_1^r (variable x). The factor $F(x,1)$ in (16) is the contribution of the interval I_2.

To determine the series $F^{\bullet}_{k,\ell-1}(x)$, expand the series $F_{k,\ell-1}(x)$ as

$$F_{k,\ell-1}(x) = \sum_{i\geq 1} F_{k,\ell-1,i}x^i,$$

where $F_{k,\ell-1,i} = [x^i]F_{k,\ell-1}(x)$ is the generating function of $(\ell-1)$-augmented m-Tamari intervals $[P_1, Q_1]$ such that $r(Q_1) = k$, and having i non-initial contacts. Each such interval can be pointed in i different ways to give rise to i different proper pointed intervals $[P_1^{\ell}P_1^{r}, Q_1]$, having respectively $0, 1, \ldots, i-1$ non-initial contacts. Therefore,

$$\begin{aligned} F^{\bullet}_{k,\ell-1}(x) &= \sum_i F_{k,\ell-1,i}(1 + x + \cdots + x^{i-1}) \\ &= \sum_i F_{k,\ell-1,i}\frac{x^i - 1}{x-1} \\ &= \frac{1}{x-1}\left(F_{k,\ell-1}(x) - F_{k,\ell-1}(1)\right) \\ &= \Delta F_{k,\ell-1}(x). \end{aligned} \tag{17}$$

This, together with (16), already allows us to prove (15) when $\ell > 1$. Indeed, one then has:

$$F_{k,\ell}(x) = F(x,1)\Delta F_{k,\ell-1}(x) = (F(x,1)\Delta)^{(\ell)}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x),$$

by the induction hypothesis. This is (15) at rank (k,ℓ).

It remains to treat the case $\ell = 1$. To this end we need to determine the series $F^{\circ}_{k,0}(x)$. By definition, a pointed interval $I_1 = [P_1^{\ell}P_1^{r}, Q_1]$ is non-proper if P_1^{ℓ} is empty, in which case I_1 can be identified with the (non-pointed) interval $[P_1^r, Q_1]$. This implies that $F^{\circ}_{k,0}(x) = F_{k,0}(x)$. Therefore (16) and (17) give:

$$\begin{aligned} F_{k,1}(x) &= F(x,1)\left(\Delta F_{k,0}(x) + F_{k,0}(x)\right) \\ &= F(x,1)\Delta\big(xF_{k,0}(x)\big). \end{aligned}$$

By the induction hypothesis, $F_{k,0}(x) = \frac{1}{x}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x)$, so that

$$F_{k,1}(x) = F(x,1)\Delta\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}.$$

We recognise (15) at rank $(k, \ell = 1)$, and this settles the last case of the induction. ∎

Proof of Proposition 5. By Lemmas 8 and 9, and the definition (11) of $\tilde{h}_k(y)$, we have:

$$F(x,y) = \sum_{k\geq 0}\tilde{h}_k(y)\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x) = \sum_{\lambda} y^{\ell(\lambda)}\frac{p_\lambda}{z_\lambda}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(|\lambda|)}(x).$$

Letting α_i be the number of parts equal to i in the partition λ, and summing on the α_i's rather than on λ, we can rewrite this sum as:

$$\begin{aligned} F(x,y) &= \sum_{\alpha_1,\alpha_2,\dots} \prod_i \left(\frac{y^{\alpha_i}}{\alpha_i!}\left(\frac{p_i}{i}\right)^{\alpha_i}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(i\alpha_i)}\right)(x) \\ &= \prod_{i\geq 1} \exp\left(y\,\frac{p_i}{i}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(i)}\right)(x) \\ &= \exp\left(y\sum_i \frac{p_i}{i}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(i)}\right)(x). \end{aligned}$$

We have used the fact that the operator Δ commutes with the multiplication by y and by p_i. This is the second functional equation satisfied by $F(x,y)$ given in Proposition 5. The third one, (12), follows by differentiating with respect to y. ∎

4. Principle of the proof, and the case $m = 1$

4.1. Principle of the proof

Let us consider the functional equation (12), together with the initial condition $F(t,p;x,0) = x$. Perform the change of variables (7), and denote $G(z,p;u,y) \equiv G(u,y) = F(t,p;x,y)$. Then $G(u,y)$ is a series in z with coefficients in $\mathbb{K}[u,y]$ (where $\mathbb{K} = \mathbb{Q}(p_1,p_2,\dots)$) satisfying

$$\frac{\partial G}{\partial y}(u,y) = \sum_{k\geq 1}\frac{p_k}{k}\left(z(1+u)e^{-m(K(u)+L)}\left(\frac{uG(u,1)}{(1+u)e^{-mK(u)}-1}\,\Delta_u\right)^{(m)}\right)^{(k)} G(u,y), \qquad (18)$$

with $\Delta_u H(u) = \frac{H(u)-H(0)}{u}$, and the initial condition

$$G(u,0) = (1+u)e^{-mK(u)}. \qquad (19)$$

Observe that this pair of equations defines $G(u,y) \equiv G(z,p;u,y)$ uniquely as a formal power series in z. Indeed, the coefficient of z^n in G can be computed inductively from these equations: one first determines the coefficient of z^n in $\frac{\partial G}{\partial y}$, which can be expressed, thanks to (18), in terms of the coefficients of z^i in G for $i < n$; then the coefficient of z^n in G is obtained by integration with respect to y, using the initial condition (19). Hence, if we exhibit a series $\tilde{G}(z,p;u,y)$ that satisfies both equations, then $\tilde{G}(z,p;u,y) = G(z,p;u,y)$. We are going to construct such a series.

Let

$$G_1(z,p;u) \equiv G_1(u) = (1+\bar{u})e^{K(u)+L}\left((1+u)e^{-mK(u)}-1\right). \qquad (20)$$

Then $G_1(u)$ is a series in z with coefficients in $\mathbb{K}[u]$, which, as we will see, coincides with $G(u,1)$. Consider now the following equation, obtained from (18) by replacing $G(u,1)$ by its conjectured value $G_1(u)$:

$$\begin{aligned} \frac{\partial \tilde{G}}{\partial y}(z,p;u,y) &= \sum_{k\geq 1}\frac{p_k}{k}\left(z(1+u)e^{-m(L+K(u))}\left(\frac{uG_1(u)}{(1+u)e^{-mK(u)}-1}\,\Delta_u\right)^{(m)}\right)^{(k)}\tilde{G}(z,p;u,y) \\ &= \sum_{k\geq 1}\frac{p_k}{k}\left(z(1+u)e^{-m(L+K(u))}\left((1+u)e^{K(u)+L}\,\Delta_u\right)^{(m)}\right)^{(k)}\tilde{G}(z,p;u,y), \qquad (21) \end{aligned}$$

with the initial condition

$$\tilde{G}(z,p;u,0) = (1+u)e^{-mK(u)}. \qquad (22)$$

Eq. (21) can be rewritten as

$$\frac{\partial \tilde{G}}{\partial y}(z,p;u,y) = \sum_{k\geq 1}\frac{p_k}{k}\left(zvA(u)^m\Lambda^{(m)}\right)^{(k)}\tilde{G}(z,p;u,y), \qquad (23)$$

where

$$A(u) = \frac{u}{1+u} e^{-K(u)}, \tag{24}$$

Λ is the operator defined by

$$\Lambda(H)(u) = \frac{H(u) - H(0)}{A(u)}, \tag{25}$$

and $v = (1+u)^{m+1}u^{-m}$ as before. Again, it is not hard to see that (23) and the initial condition (22) define a unique series in z, denoted $\tilde{G}(z,p;u,y) \equiv \tilde{G}(u,y)$. The coefficients of this series lie in $\mathbb{K}[u,y]$. The principle of our proof can be described as follows.

> *If we prove that $\tilde{G}(u,1) = G_1(u)$, then the equation (21) satisfied by $\tilde{G}$ coincides with the equation (18) that defines G, and thus $\tilde{G}(u,y) = G(u,y)$. In particular, $G_1(z,p;u) = \tilde{G}(z,p;u,1) = G(z,p;u,1) = F(t,p;x,1)$, and Theorem 3 is proved.*

Remark. Our proof relies on the fact that we have been able to *guess* the value of $G(u,1)$, given by (20). This was a difficult task, which we discuss in greater details in Section 6.1.

4.2. The case $m = 1$

Take $m = 1$. In this subsection, we describe the three steps that, starting from (23), prove that $\tilde{G}(u,1) = G_1(u)$. In passing, we establish the expression (10) of $F(t,p;x,1)$ (equivalently, of $\tilde{G}(z,p;u,1)$) given in Theorem 4. The case of general m is difficult, and we hope that studying in details the case $m = 1$ will make the ideas of the proof more transparent. Should this specialization not suffice, we invite the reader to set further $p_i = \mathbb{1}_{i=0}$, in which case we are simply counting labelled Tamari intervals (see also [6]).

4.2.1. ***A homogeneous differential equation and its solution.*** When $m = 1$, the equation (23) defining $\tilde{G}(z;u,y) \equiv \tilde{G}(u,y)$ reads

$$\frac{\partial \tilde{G}}{\partial y}(u,y) = \sum_{k\geq 1} \frac{p_k}{k} z^k \Big((1+u)(1+\bar{u})\,\Omega\Big)^{(k)} \tilde{G}(u,y), \tag{26}$$

where $\bar{u} = 1/u$ and the operator Ω is defined by $\Omega H(u) = H(u) - H(0)$, with the initial condition

$$\tilde{G}(u,0) = (1+u)e^{-K(u)}. \tag{27}$$

These equations imply that $\tilde{G}(-1,y) = 0$. The following lemma provides us with a symmetry property which is crucial in our approach.

Lemma 10. *For $k \geq 0$ one has:*

$$\Big((1+u)(1+\bar{u})\Omega\Big)^{(k)} \tilde{G}(u,y) = (1+u)^k(1+\bar{u})^k \tilde{G}(u,y) - P_k(v),$$

where $P_k \in \mathbb{K}[y][[z]][v]$ and $v = (1+u)(1+\bar{u})$.

Proof. This is easily proved by induction on k. Alternatively, readers well acquainted with lattice path enumeration may view this lemma as a form of André's reflection principle. ∎

Observe that the quantity $P_k(v)$, being a function of $v = (1+u)(1+\bar{u})$, is left invariant by the substitution $u \mapsto \bar{u}$. This symmetry is the keystone of our approach, as it enables us to eliminate some *a priori* intractable terms in (26). Replacing u by $\bar{u}$ in (26) gives

$$\frac{\partial \tilde{G}}{\partial y}(\bar{u},y) = \sum_{k} \frac{p_k}{k} z^k \Big((1+u)(1+\bar{u})\Omega\Big)^{(k)} \tilde{G}(\bar{u},y),$$

so that, applying Lemma 10 and using $v(u) = v(\bar{u})$ we obtain:

$$\frac{\partial}{\partial y}\Big(\tilde{G}(u,y) - \tilde{G}(\bar{u},y)\Big) = \sum_{k\geq 1} \frac{p_k}{k} z^k (1+u)^k(1+\bar{u})^k \Big(\tilde{G}(u,y) - \tilde{G}(\bar{u},y)\Big) = V(v)\Big(\tilde{G}(u,y) - \tilde{G}(\bar{u},y)\Big),$$

where $V(v)$ is given by (3). This is a *homogeneous* linear differential equation satisfied by $\tilde{G}(u,y) - \tilde{G}(\bar{u},y)$. It is readily solved, and the initial condition (27) yields

$$\tilde{G}(u,y) - \tilde{G}(\bar{u},y) = (1+u)\left(e^{-K(u)} - \bar{u}e^{-K(\bar{u})}\right)e^{yV(v)}. \tag{28}$$

4.2.2. ***Reconstruction of*** $\tilde{G}(u,y)$. Recall that $\tilde{G}(u,y) \equiv \tilde{G}(z,p;u,y)$ is a series in z with coefficients in $\mathbb{K}[u,y]$. Hence, by extracting from the above equation the positive part in u (as defined in Section 2.1), we obtain

$$\tilde{G}(u,y) - \tilde{G}(0,y) = [u^>]\left((1+u)\left(e^{-K(u)} - \bar{u}e^{-K(\bar{u})}\right)e^{yV(v)}\right).$$

For any Laurent polynomial P, we have

$$[u^>]\left((1+u)P(u)\right) = (1+u)[u^>]P(u) + u[u^0]P(u). \tag{29}$$

Hence

$$\tilde{G}(u,y) - \tilde{G}(0,y) = (1+u)[u^>]\left(e^{yV(v)}\left(e^{-K(u)} - \bar{u}e^{-K(\bar{u})}\right)\right) + u[u^0]\left(e^{yV(v)}\left(e^{-K(u)} - \bar{u}e^{-K(\bar{u})}\right)\right).$$

Setting $u=-1$ in this equation gives, since $\tilde{G}(-1,y)=0$,

$$-\tilde{G}(0,y) = -[u^0]\left(e^{yV(v)}\left(e^{-K(u)} - \bar{u}e^{-K(\bar{u})}\right)\right),$$

so that finally,

$$\begin{aligned}\tilde{G}(u,y) &= (1+u)[u^>]\left(e^{yV(v)}\left(e^{-K(u)} - \bar{u}e^{-K(\bar{u})}\right)\right) \\ &\qquad +(1+u)[u^0]\left(e^{yV(v)}\left(e^{-K(u)} - \bar{u}e^{-K(\bar{u})}\right)\right) \\ &= (1+u)[u^{\geq}]\left(e^{yV(v)}\left(e^{-K(u)} - \bar{u}e^{-K(\bar{u})}\right)\right). \end{aligned} \tag{30}$$

As explained in Section 4.1, $\tilde{G}(u,y) = G(u,y)$ will be proved if we establish that $\tilde{G}(u,1)$ coincides with the series $G_1(u)$ given by (20). This is the final step of our proof.

4.2.3. ***The case*** $y=1$. Equation (30) completely describes the solution of (26). It remains to check that $\tilde{G}(u,1) = G_1(u)$, that is

$$\tilde{G}(u,1) = (1+\bar{u})e^{K(u)+L}\left((1+u)e^{-K(u)} - 1\right). \tag{31}$$

Let us set $y=1$ in (30). We find, using $V(v) = K(\bar{u}) + L + K(u)$:

$$\begin{aligned}\tilde{G}(u,1) &= (1+u)[u^{\geq}]\left(e^{L+K(\bar{u})} - \bar{u}e^{L+K(u)}\right) \\ &= (1+u)e^{L}\left(1 - \bar{u}e^{K(u)} + \bar{u}\right),\end{aligned}$$

which coincides with (31). Hence $\tilde{G}(z,p;u,y) = G(z,p;u,y) = F(t,p;x,y)$ (with the change of variables (7)), and Theorem 4 is proved, using (30).

5. Solution of the functional equation: the general case

We now adapt to the general case the solution described for $m=1$ in Section 4.2. Recall from Section 4.1 that we start from (23), and want to prove that $\tilde{G}(u,1) = G_1(u)$. We first obtain in Section 5.1 the counterpart of (28), that is, an explicit expression for a linear combination of the series $\tilde{G}(u_i,y)$, where $u_0 = u, u_1, \ldots, u_m$ are the $m+1$ roots of the equation $v(u) = v(u_i)$, with $v(u) = (1+u)^{m+1}\bar{u}^m$. In Section 5.2, we reconstruct from this expression the series $\tilde{G}(u,y)$, by taking iterated positive parts. This generalizes (30). The third part of the proof differs from Section 4.2.3, because we are not able to derive from our expression of $\tilde{G}(u,y)$ that

$\tilde{G}(u,1) = G_1(u)$. Instead, the arguments of Section 5.2 imply that the counterpart of (28) has also a unique solution when $y = 1$, and we check that $G_1(u)$ is a solution.

5.1. A HOMOGENEOUS DIFFERENTIAL EQUATION AND ITS SOLUTION

Let us return to the equation (23) satisfied by $\tilde{G}(u,y)$. This equations involves the quantity

$$v \equiv v(u) = (1+u)^{m+1}\bar{u}^m$$

In the case $m = 1$, this (Laurent) polynomial was $(1+u)(1+\bar{u})$, and took the same value for u and $\bar{u}$. We are again interested in the series u_i such that $v(u_i) = v(u)$.

Lemma 11. *Denote $v = (1+u)^{m+1}u^{-m}$, and consider the following polynomial equation in U:*

$$(1+U)^{m+1} = U^m v.$$

This equation has no double root. We denote its $m+1$ roots by $u_0 = u, u_1, \ldots, u_m$.

Proof. A double root would also satisfy

$$(m+1)(1+U)^m = mU^{m-1}v,$$

and this is easily shown to be impossible. ∎

Remark. One can express the u_i's as Puiseux series in u (see [29, Ch. 6]), but this will not be needed here, and we will think of them as abstract elements of an algebraic extension of $\mathbb{Q}(u)$. In fact, in this paper, the u_i's always occur in *symmetric rational functions* of the u_i's, which are thus rational functions of v. At some point, we will have to prove that a symmetric polynomial in the u_i's (and thus a polynomial in v) vanishes at $v = 0$, that is, at $u = -1$, and we will then consider series expansions of the u_i's around $u = -1$.

The following proposition generalizes (28).

Proposition 12. *Denote $v = (1+u)^{m+1}u^{-m}$, and let the series u_i be defined as above. Denote $A_i = A(u_i)$, where $A(u)$ is given by (24). Then*

$$\sum_{i=0}^{m} \frac{\tilde{G}(u_i,y)}{\prod_{j\neq i}(A_i - A_j)} = ve^{yV(v)}. \tag{32}$$

By $\prod_{j\neq i}(A_i - A_j)$ we mean $\prod_{0\leq j\leq m, j\neq i}(A_i - A_j)$ but we prefer the shorter notation when the bounds on j are clear. Observe that the A_i's are distinct since the u_i's are distinct (the coefficient of z^0 in $A(u)$ is $u/(1+u)$). Note also that when $m = 1$, then $u_0 = u$, $u_1 = \bar{u}$, and (32) coincides with (28). In order to prove the proposition, we need the following two lemmas.

Lemma 13. *Let $x_0, x_1, \ldots, x_m$ be $m+1$ variables. Then*

$$\sum_{i=0}^{m} \frac{{x_i}^m}{\prod_{j\neq i}(x_i - x_j)} = 1 \tag{33}$$

and

$$\sum_{i=0}^{m} \frac{1/x_i}{\prod_{j\neq i}(x_i - x_j)} = (-1)^m \prod_{i=0}^{m} \frac{1}{x_i}. \tag{34}$$

Moreover, for any polynomial Q of degree less than m,

$$\sum_{i=0}^{m} \frac{Q(x_i)}{\prod_{j\neq i}(x_i - x_j)} = 0. \tag{35}$$

Proof. By Lagrange interpolation, any polynomial R of degree at most m satisfies:

$$R(X) = \sum_{i=0}^{m} R(x_i) \prod_{j \neq i} \frac{X - x_j}{x_i - x_j}.$$

Equations (34) and (35) follow by evaluating this equation at $X = 0$, respectively with $R(X) = 1$ and $R(X) = XQ(X)$. Equation (33) is obtained by taking $R(X) = X^m$ and extracting the leading coefficient. ∎

Our second lemma replaces Lemma 10 for general values of m.

Lemma 14. *Let $k \geq 0$, and let Λ be the operator defined by (25). Let $H(z;u) \equiv H(u)$ be a formal power series in z, having coefficients in $\mathbb{L}(u)$, with $\mathbb{L} = \mathbb{K}(y)$. Assume that these coefficients have no pole at $u = 0$. Then there exists a polynomial $P_k(X, Y) \in \mathbb{L}[[z]][X, Y]$ of degree less than m in X, such that*

$$\left(zvA(u)^m \Lambda^{(m)}\right)^{(k)} H(u) = (zv)^k H(u) - P_k(A(u), v). \tag{36}$$

Proof. We denote by $\mathcal{L}$ the subring of $\mathbb{L}(u)[[z]]$ formed by formal power series whose coefficients have no pole at $u = 0$. By assumption, $H(u) \in \mathcal{L}$. We use the notation $O(u^k)$ to denote an element of $\mathbb{L}(u)[[z]]$ of the form $u^k J(z;u)$ with $J(z;u) \in \mathcal{L}$.

First, note that $A(u) = ue^{-K(u)}/(1+u)$ belongs to $\mathcal{L}$. Moreover,

$$A(u) = u + O(u^2). \tag{37}$$

We first prove that for all series $I(u) \in \mathcal{L}$, there exists a sequence of formal power series $(g_j^I)_{j \geq 0} \in \mathbb{L}[[z]]^{\mathbb{N}}$ such that for all $\ell \geq 0$,

$$I(u) = \sum_{j=0}^{\ell-1} g_j^I A(u)^j + O(u^\ell). \tag{38}$$

We prove (38) by induction on $\ell \geq 0$. The identity holds for $\ell = 0$ since $I(u) \in \mathcal{L}$. Assume it holds for some $\ell \geq 0$: there exists series $g_0^I, \ldots, g_{\ell-1}^I$ in $\mathbb{L}[[z]]$ and $J(u) \in \mathcal{L}$ such that

$$I(u) = \sum_{j=0}^{\ell-1} g_j^I A(u)^j + u^\ell J(u).$$

By (37) and by induction on r, we have $u^r = A(u)^r + O(u^{r+1})$ for all $r \geq 0$. Using this identity with $r = \ell$, and rewriting $J(u) = J(0) + O(u)$, we obtain $u^\ell J(u) = J(0)A(u)^\ell + O(u^{\ell+1})$, so that:

$$I(u) = \sum_{j=0}^{\ell} g_j^I A(u)^j + O(u^{\ell+1}),$$

with $g_\ell^I := J(0) \in \mathbb{L}[[z]]$. Thus (38) holds for $\ell + 1$.

We now prove that for all $q \geq 0$, one has:

$$\Lambda^{(q)} I(u) = \frac{1}{A(u)^q} \left(I(u) - \sum_{j=0}^{q-1} g_j^I A(u)^j \right), \tag{39}$$

where the series g_j^I are those that satisfy (38). Again, we proceed by induction on $q \geq 0$. The identity clearly holds for $q = 0$. Assume it holds for some $q \geq 0$. In (39), replace $I(u)$ by its expression (38) obtained with $\ell = q+1$, and let u tend to 0: this shows that g_q^I is in fact $\Lambda^{(q)} I(0)$. From the definition of Λ one then obtains

$$\Lambda^{(q+1)} I(u) = \frac{\Lambda^{(q)} I(u) - g_q^I}{A(u)} = \frac{1}{A(u)^{q+1}} \left(I(u) - \sum_{j=0}^{q} g_j^I A(u)^j \right).$$

Thus (39) holds for $q+1$.

We finally prove, by induction on $k \geq 0$, that (36) holds and that the left-hand side of (36) is an element of $\mathcal{L}$. For $k=0$, these results are clear, with $P_0 = 0$. Assume they hold for some $k \geq 0$, for any $H(u) \in \mathcal{L}$. Let $H(u) \in \mathcal{L}$ and let $M(u)$ be the left-hand side of (36). By the induction hypothesis, $M(u) \in \mathcal{L}$, so that applying (39) with $I(u) = M(u)$ and $q = m$ gives:

$$\begin{aligned} zvA(u)^m \Lambda^{(m)} M(u) &= zv\left(M(u) - \sum_{j=0}^{m-1} g_j^M A(u)^y\right). \qquad (40) \\ &= \left(zvA(u)^m \Lambda^{(m)}\right)^{(k+1)} H(u) \qquad \text{by definition of } M. \end{aligned}$$

By the induction hypothesis (36), we have $M(u) = (zv)^k H(u) - P_k(A(u), v)$ with $P_k(X,Y)$ of degree less than m in X, so that the above equation gives:

$$\left(zvA(u)^m \Lambda^{(m)}\right)^{(k+1)} H(u) = (zv)^{k+1} H(u) - P_{k+1}(A(u), v),$$

with

$$P_{k+1}(X,Y) := zY\left(P_k(X,Y) + \sum_{j=0}^{m-1} g_j^M X^j\right).$$

Note that $P_{k+1}(X,Y)$ has still degree less than m in X.

It remains to prove that $\left(zvA(u)^m \Lambda^{(m)}\right)^{(k+1)} H(u) \in \mathcal{L}$. Applying (38) with $I(u) = M(u)$ and $\ell = m+1$, and substituting in (40), we obtain:

$$\begin{aligned} \left(zvA(u)^m \Lambda^{(m)}\right)^{(k+1)} H(u) &= zv\left(g_m^M A(u)^m + O(u^{m+1})\right) \\ &= zvu^m\left(g_m^M + O(u)\right), \end{aligned}$$

since $A(u)^m = u^m + O(u^{m+1})$. Since $v = (1+u)^{m+1}u^{-m}$, this shows that $\left(zvA(u)^m \Lambda^{(m)}\right)^{(k+1)} H(u)$ belongs to $\mathcal{L}$, which completes the proof. ∎

Proof of Proposition 12. Thanks to Lemma 14, we can rewrite (23) as

$$\frac{\partial \tilde{G}}{\partial y}(u,y) = \sum_{k \geq 1} \frac{p_k}{k}\Big((zv)^k \tilde{G}(u,y) - P_k(A(u), v)\Big), \qquad (41)$$

where $v \equiv v(u) = (1+u)^{m+1}\bar{u}^m$, and for all $k \geq 1$, $P_k(X,Y)$ is a polynomial of degree less than m in X with coefficients in $\mathbb{K}(y)[[z]]$.

As was done in Section 4.2.1, we are going to use the fact that $v(u_i) = v$ for all $i \in [\![0, m]\!]$ to eliminate the (infinitely many) unknown polynomials $P_k(A(u), v)$. For $0 \leq i \leq m$, the substitution $u \mapsto u_i$ in (41) gives:

$$\frac{\partial \tilde{G}}{\partial y}(u_i,y) = \sum_{k \geq 1} \frac{p_k}{k}\Big((zv)^k \tilde{G}(u_i,y) - P_k(A_i, v)\Big), \qquad (42)$$

with $A_i = A(u_i)$. Consider the linear combination

$$R(u,y) := \sum_{i=0}^{m} \frac{\tilde{G}(u_i,y)}{\prod_{j \neq i}(A_i - A_j)}. \qquad (43)$$

Recall that A_i is independent of y. Thus by (42),

$$\begin{aligned}\frac{\partial R}{\partial y}(u,y) &= \sum_{k\geq 1}\frac{p_k}{k}\left((zv)^k R(u,y) - \sum_{i=0}^{m}\frac{P_k(A_i,v)\ .}{\prod_{j\neq i}(A_i-A_j)}\right),\\ &= \sum_{k\geq 1}\frac{p_k}{k}(zv)^k R(u,y), \qquad \text{by (35)},\\ &= V(v)R(u,y),\end{aligned}$$

where $V(v)$ is defined by (3). This homogeneous linear differential equation is readily solved:

$$R(u,y) = R(u,0)e^{yV(v)}.$$

Recall the expression (43) of R in terms of $\tilde{G}$. The initial condition (22) can be rewritten $\tilde{G}(u,0) = vA(u)^m$, which yields

$$\begin{aligned}R(u,0) &= v\sum_{i=0}^{m}\frac{{A_i}^m}{\prod_{j\neq i}(A_i-A_j)}\\ &= v\end{aligned}$$

by (33). Hence $R(u,y) = ve^{yV(v)}$, and the proposition is proved. ∎

5.2. Reconstruction of $\tilde{G}(u,y)$

We are now going to prove that (32), together with the condition $\tilde{G}(-1,y) = 0$ derived from (21–22), characterizes the series $\tilde{G}(u,y)$. We will actually obtain a (complicated) expression for this series, generalizing (30).

We first introduce some notation. Consider a formal power series in z, denoted $H(z;u) \equiv H(u)$, having coefficients in $\mathbb{L}[u]$, where $\mathbb{L} = \mathbb{K}(y)$. We define a series H_k in z whose coefficients are rational symmetric functions of $k+1$ variables $x_0,\dots,x_k$:

$$H_k(x_0,\dots,x_k) = \sum_{i=0}^{k}\frac{H(x_i)}{\prod_{0\leq j\leq k, j\neq i}(A(x_i)-A(x_j))}, \tag{44}$$

where, as above, A is defined by (24).

Lemma 15. *The series $H_k(x_0,\dots,x_k)$ has coefficients in $\mathbb{L}[x_0,\dots,x_k]$. If, moreover, $H(-1) = 0$, then the coefficients of H_k are multiples of $(1+x_0)\cdots(1+x_k)$.*

Proof. Using the fact that $e^{-K(u)} = 1 + O(z)$, it is not hard to prove that

$$\frac{1}{A(x_i)-A(x_j)} = \frac{1}{x_i-x_j}B(x_i,x_j), \tag{45}$$

where $B(x_i,x_j)$ is a series in z with *polynomial* coefficients in x_i and x_j. Hence

$$H_k(x_0,\dots,x_k)\prod_{0\leq i<j\leq k}(x_i-x_j)$$

has polynomial coefficients in the x_i's. But these polynomials are anti-symmetric in the x_i's (since H_k is symmetric), hence they must be multiples of the Vandermonde $\prod_{i<j}(x_i-x_j)$. Hence $H_k(x_0,\dots,x_k)$ has polynomial coefficients.

A stronger property than (45) actually holds, namely:

$$\frac{1}{A(x_i)-A(x_j)} = \frac{(1+x_i)(1+x_j)}{x_i-x_j}C(x_i,x_j),$$

where $C(x_i, x_j)$ is a series in z with polynomial coefficients in x_i and x_j. Hence, if $H(-1) = 0$, that is, if $H(x) = (1+x)K(x)$ where $K(x)$ has polynomial coefficients in x,

$$H_k(x_0, \dots, x_k) = \sum_{i=0}^{k} K(x_i)(1+x_i)^{k+1} \prod_{j \neq i} \frac{(1+x_j)C(x_i, x_j)}{x_i - x_j}.$$

Setting $x_0 = -1$ shows that $H_k(-1, x_1, \dots, x_k) = 0$, so that $H_k(x_0, \dots, x_k)$ is a multiple of $(1+x_0)$. By symmetry, it is also a multiple of all $(1+x_i)$, for $1 \leq i \leq k$. ∎

Our treatment of (32) actually applies to equations with an arbitrary right-hand side. We consider a formal power series $H(z;u) \equiv H(u)$ with coefficients in $\mathbb{L}[u]$, satisfying $H(-1) = 0$ and

$$\sum_{i=0}^{m} \frac{H(u_i)}{\prod_{j \neq i}(A_i - A_j)} = \Phi_m(v),$$

for some series $\Phi_m(v) \equiv \Phi_m(z;v)$ with coefficients in $v\mathbb{L}[v]$, where $v = (1+u)^{m+1}\bar{u}^m$. Using the notation (44), this equation can be rewritten as

$$H_m(u_0, \dots, u_m) = \Phi_m(v).$$

We will give an explicit expression of $H(u)$ involving two standard families of symmetric functions, namely the homogeneous functions h_λ and the monomial functions m_λ.

> **Caveat.** *These symmetric functions will be evaluated at $(u_0, u_1, \dots u_m)$ or $(A(u_0), \dots, A(u_m))$. They have nothing to do with the variables p_k involved in the generating function $F^{(m)}(t, p; x, y)$.*

We also use the following notation: For any subset $V = \{i_1, \dots, i_k\}$ of $[\![0, m]\!]$, of cardinality k, and any sequence $(x_0, \dots, x_m)$, we denote $x_V = \{x_{i_1}, \dots, x_{i_k}\}$.

Proposition 16. *Let $H(z;u) \equiv H(u)$ be a power series in z with coefficients in $\mathbb{L}[u]$, satisfying $H(-1) = 0$ and*

$$H_m(u_0, \dots, u_m) = \Phi_m(v), \tag{46}$$

where $\Phi_m(v) \equiv \Phi_m(z;v)$ is a series in z with coefficients in $v\mathbb{L}[v]$.

There exists a sequence $\Phi_0, \dots, \Phi_m$ of series in z with coefficients in $v\mathbb{L}[v]$, which depend only on Φ_m, such that for $0 \leq k \leq m$, and for all subset V of $[\![0, m]\!]$ of cardinality $k+1$,

$$H_k(u_V) = \sum_{j=k}^{m} \Phi_j(v) h_{j-k}(A_V). \tag{47}$$

In particular, $H(u) \equiv H_0(u)$ is completely determined if Φ_m is known:

$$H(u) = \sum_{j=0}^{m} \Phi_j(v) A(u)^j.$$

The series $\Phi_k(v) \equiv \Phi_k(z;v)$ can be computed by a descending induction on k as follows. Let us denote by $\Phi_{k-1}^{>}(u)$ the positive part in u of $\Phi_{k-1}(v)$, that is

$$\Phi_{k-1}^{>}(u) := [u^{>}]\Phi_{k-1}(\bar{u}^m(1+u)^{m+1}).$$

Then for $1 \leq k \leq m$, this series can be expressed in terms of $\Phi_k, \dots, \Phi_m$:

$$\Phi_{k-1}^{>}(u) = -\frac{1}{\binom{m}{k}}[u^{>}] \left(\sum_{j=k}^{m} \Phi_j(v) \sum_{\lambda \vdash j-k+1} \binom{m-\ell(\lambda)}{k-\ell(\lambda)} m_\lambda(A_1, \dots, A_m) \right). \tag{48}$$

The extraction makes sense since, as will be seen, $vm_\lambda(A_1, \dots, A_m)$ belongs to $\mathbb{K}[u, \bar{u}][[z]]$. Finally, $\Phi_{k-1}(v)$ can be expressed in terms of $\Phi_{k-1}^{>}$:

$$\Phi_{k-1}(v) = \sum_{i=0}^{m} \left(\Phi_{k-1}^{>}(u_i) - \Phi_{k-1}^{>}(-1) \right). \tag{49}$$

We first establish three lemmas dealing with symmetric functions of the series u_i defined in Lemma 11.

Lemma 17. *The elementary symmetric functions of $u_0 = u, u_1, \dots, u_m$ are*

$$e_j(u_0, u_1, \dots, u_m) = (-1)^j \binom{m+1}{j} + v\mathbb{1}_{j=1}$$

with $v = u^{-m}(1+u)^m$.

The elementary symmetric functions of $u_1, \dots, u_m$ are

$$e_{m-j}(u_1, \dots, u_m) = \begin{cases} 1 & \text{if } j = m, \\ (-1)^{m-j-1} \sum_{p=0}^{j} \binom{m+1}{p} u^{p-j-1} & \text{otherwise.} \end{cases}$$

In particular, they are polynomials in $1/u$, and so is any symmetric polynomial in $u_1, \dots, u_m$.

Finally,

$$\prod_{i=0}^{m} (1+u_i) = v.$$

Proof. The symmetric functions of the roots of a polynomial can be read from the coefficients of this polynomial. Hence the first result follows directly from the equation satisfied by the u_i's, for $0 \le i \le m$, namely

$$(1+u_i)^{m+1} = v u_i^m.$$

For the second one, we need to find the equation satisfied by $u_1, \dots, u_m$, which is

$$0 = \frac{(1+u_i)^{m+1}u^m - (1+u)^{m+1}u_i^m}{u_i - u} = u^m u_i^m - \sum_{j=0}^{m-1} u_i^j u^{m-j-1} \sum_{p=0}^{j} \binom{m+1}{p} u^p.$$

The second result follows.

The third one is obtained by evaluating at $U = -1$ the identity

$$\prod_{i=0}^{m} (U - u_i) = (1+U)^{m+1} - vU^m.$$

■

Lemma 18. *Denote $v = \bar{u}^m(1+u)^{m+1}$. Let P be a polynomial. Then $P(v)$ is a Laurent polynomial in u. Let $P^>(u) := [u^>]P(v)$ denote its positive part. Then*

$$P(v) = P(0) + \sum_{i=0}^{m} (P^>(u_i) - P^>(-1)). \tag{50}$$

Proof. The right-hand side of (50) is a symmetric polynomial of $u_0, \dots, u_m$, and thus, by the first part of Lemma 17, a polynomial in v. Denote it by $\tilde{P}(v)$. The second part of Lemma 17 implies that the positive part of $\tilde{P}(v)$ in u is $P^>(u_0) = P^>(u)$. That is, $P(v)$ and $\tilde{P}(v)$ have the same positive part in u. In other words, the polynomial $Q := P - \tilde{P}$ is such that $Q(v)$ is a Laurent polynomial in u of non-positive degree. But since $v = (1+u)^{m+1}\bar{u}^m$, the degree in u of $Q(v)$ coincides with the degree of Q, and so Q must be a constant. Finally, by setting $u = -1$ in $\tilde{P}(v)$, we see that $\tilde{P}(0) = P(0)$ (because $u_i = -1$ for all i when $u = -1$, as follows for instance from Lemma 17). Hence $Q = 0$ and the lemma is proved. ■

Lemma 19. *Let $0 \le k \le m$, and let $R(x_0, \dots, x_k)$ be a symmetric rational function of $k+1$ variables $x_0, \dots, x_k$, such that for any subset V of $[\![0, k]\!]$ of cardinality $k+1$,*

$$R(u_V) = R(u_0, \dots, u_k).$$

Then there exists a rational fraction in v equal to $R(u_0, \dots, u_k)$.

Proof. Let $\tilde{R}$ be the following rational function in $x_0, \dots, x_m$:

$$\tilde{R}(x_0, \dots, x_m) = \frac{1}{\binom{m+1}{k+1}} \sum_{V \subset [\![0,m]\!],\ |V|=k+1} R(x_V).$$

Then $\tilde{R}$ is a symmetric function of $x_0, \dots, x_m$, and hence a rational function in the elementary symmetric functions $e_j(x_0, \dots, x_m)$, say $S(e_1(x_0, \dots, x_m), \dots, e_{m+1}(x_0, \dots, x_m))$. By assumption,

$$\tilde{R}(u_0, \dots, u_m) = S(e_1(u_0, \dots, u_m), \dots, e_{m+1}(u_0, \dots, u_m)) = R(u_0, \dots, u_k).$$

Since S is a rational function, it follows from the first part of Lemma 17 that $R(u_0, \dots, u_k)$ can be written as a rational function in v. ∎

Proof of Proposition 16. We prove (47) by descending induction on k. For $k = m$, it holds by assumption. Let us assume that (47) holds for some $k > 0$, and prove it for $k - 1$.

Observe that

$$(A(x_{k-1}) - A(x_k))H_k(x_0, \dots, x_k) = H_{k-1}(x_0, \dots, x_{k-2}, x_{k-1}) - H_{k-1}(x_0, \dots, x_{k-2}, x_k).$$

This is easily proved by collecting the coefficient of $H(x_i)$, for all $i \in [\![0, k]\!]$, in both sides of the equation. We also have, for any indeterminates $a_0, \dots, a_m$,

$$(a_{k-1} - a_k)h_{j-k}(a_0, \dots, a_k) = h_{j-k+1}(a_0, \dots, a_{k-2}, a_{k-1}) - h_{j-k+1}(a_0, \dots, a_{k-2}, a_k).$$

Let V be a subset of $[\![0, m]\!]$ of cardinality $k - 1$, and let p and q be two elements of $[\![0, m]\!] \setminus V$. Multiplying (47) by $A_p - A_q$, and using the two equations above gives

$$H_{k-1}(u_V, u_p) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A_V, A_p) = H_{k-1}(u_V, u_q) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A_V, A_q).$$

This implies that the series

$$H_{k-1}(x_0, \dots, x_{k-1}) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A(x_0), \dots, A(x_{k-1}))$$

takes the same value at all points u_V, for $V \subset [\![0, m]\!]$ of cardinality k. Hence Lemma 19, applied to the coefficients of this series, implies that there exists a series in z with *rational* coefficients in v, denoted $\Phi_{k-1}(v)$, such that for all $V \subset [\![0, m]\!]$ with $|V| = k$:

$$H_{k-1}(u_V) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A_V) = \Phi_{k-1}(v). \tag{51}$$

This is exactly (47) with k replaced by $k - 1$.

The next point we will prove is that the coefficients of Φ_{k-1} belong to $v\mathbb{L}[v]$. In order to do so, we symmetrize (51) over $u_0, \dots, u_m$. By (51),

$$\binom{m+1}{k} \Phi_{k-1}(v) = \sum_{V \subset [\![0,m]\!], |V|=k} H_{k-1}(u_V) - \sum_{j=k}^{m} \left(\Phi_j(v) \sum_{V \subset [\![0,m]\!], |V|=k} h_{j-k+1}(A_V) \right). \tag{52}$$

We will prove that both sums in the right-hand side of this equation are series in z with coefficients in $v\mathbb{L}[v]$.

By Lemma 15,

$$\sum_{V \subset [\![0,m]\!], |V|=k} H_{k-1}(x_V)$$

is a series in z with polynomial coefficients in $x_0, \dots, x_m$, which is symmetric in these variables. By Lemma 17, the first sum in (52) is thus a series in z with *polynomial* coefficients in v. We still need to prove that this series vanishes at $v = 0$, that is, at $u = -1$. But this follows from the second part of Lemma 15, since $u_i = -1$ for all i when $u = -1$.

Let us now consider the second sum in (52), and more specifically the term

$$\Phi_j(v) \sum_{V\subset[\![0,m]\!],|V|=k} h_{j-k+1}(A_V). \tag{53}$$

Recall that

$$A_i = \frac{u_i}{1+u_i}\, e^{-K(u_i)}.$$

But by Lemma 17,

$$\frac{1}{1+u_i} = \frac{1}{v} \prod_{0\le j\neq i\le m} (1+u_j).$$

Hence (53) can be written as a series in z with coefficients in $\mathbb{L}[1/v, u_0, \dots, u_m]$, symmetric in $u_0, \dots, u_m$. By the first part of Lemma 17, these coefficients belong to $\mathbb{L}[v, 1/v]$. We want to prove that they actually belong to $v\mathbb{L}[v]$, that is, that they are not singular at $v = 0$ (equivalently, at $u = -1$) and even vanish at this point. From the equation $(1+u_i)^{m+1} = vu_i^m$, it follows that we can label $u_1, \dots, u_m$ in such a way that

$$1 + u_i = \xi^i(1+u) + o(1+u),$$

where ξ is a primitive $(m+1)^{\text{st}}$ root of unity. Since $\Phi_j(v)$ is a multiple of $v = \bar{u}^m(1+u)^{m+1}$, and the symmetric function h_{j-k+1} has degree $j-k+1 \le m$, it follows that the series (53) is not singular at $u = -1$, and even vanishes at this point. Hence its coefficients belong to $v\mathbb{L}[v]$.

So far, $\Phi_{k-1}(v)$ has been expressed in terms of H (and the series Φ_j), and we now want to obtain an expression in terms of the Φ_j only. Lemma 18, together with $\Phi_{k-1}(0) = 0$, establishes (49). To express $\Phi^{>}_{k-1}(u)$, we now symmetrize (51) over $u_1, \dots, u_m$. With the above notation,

$$\binom{m}{k}\Phi_{k-1}(v) = \sum_{V\subset[\![1,m]\!],|V|=k} H_{k-1}(u_V) - \sum_{j=k}^{m} \left(\Phi_j(v) \sum_{V\subset[\![1,m]\!],|V|=k} h_{j-k+1}(A_V) \right). \tag{54}$$

As above,

$$\sum_{V\subset[\![1,m]\!],|V|=k} H_{k-1}(x_V)$$

is a series in z with polynomial coefficients in $x_1, \dots, x_m$, which is symmetric in these variables. By the second part of Lemma 17, the first sum in (54) is thus a series in z with polynomial coefficients in $1/u$. Since $\Phi_{k-1}(v)$ has coefficients in $\mathbb{L}[v]$, and hence in $\mathbb{L}[u, 1/u]$, the second sum in (54) is also a zeries in z with coefficients in $\mathbb{L}[u, 1/u]$. We can now extract from (54) the positive part in u, and this gives

$$\binom{m}{k}\Phi^{>}_{k-1}(u) = -[u^{>}] \left(\sum_{j=k}^{m} \left(\Phi_j(v) \sum_{V\subset[\![1,m]\!],|V|=k} h_{j-k+1}(A_V) \right) \right).$$

One easily checks that, for indeterminates $a_1, \dots, a_m$,

$$\sum_{V\subset[\![1,m]\!],|V|=k} h_{j-k+1}(a_V) = \sum_{\lambda\vdash j-k+1} \binom{m-\ell(\lambda)}{k-\ell(\lambda)} m_\lambda(a_1, \dots, a_m),$$

so that the above expression of $\Phi^{>}_{k-1}(u)$ coincides with (48). ∎

5.3. THE CASE $y = 1$

As explained in Section 4.1, Theorem 3 will be proved if we establish $\tilde{G}(u,1) = G_1(u)$, where

$$G_1(u) = (1+\bar{u})e^{K(u)+L}\left((1+u)e^{-mK(u)} - 1\right).$$

A natural attempt would be to set $y = 1$ in the expression of $\tilde{G}(u,y)$ that can be derived from Proposition 16, as we did when $m = 1$ in Section 4.2.3. However, we have not been able to do so, and will proceed differently.

We have proved in Proposition 12 that the series $\tilde{G}(u,y)$ satisfies (46) with $\Phi_m(v) = ve^{yV(v)}$. In particular, $\tilde{G}(u,1)$ satisfies (46) with $\Phi_m(v) = ve^{V(v)}$. By Proposition 16, this equation, together with the initial condition $\tilde{G}(-1,1) = 0$, characterizes $\tilde{G}(u,1)$. It is clear that $G_1(-1) = 0$. Hence it suffices to prove the following proposition.

Proposition 20. *The series $G_1(u)$ satisfies (46) with $\Phi_m(v) = ve^{V(v)}$.*

Proof. First observe that

$$G_1(u) = e^L\left(vA(u)^{m-1} - \frac{1}{A(u)}\right).$$

Using Lemma 13 with $x_i = A_i$, it follows that

$$\begin{aligned}\sum_{i=0}^{m} \frac{G_1(u_i)}{\prod_{j\neq i}(A_i - A_j)} &= 0 + (-1)^{m+1}e^L \prod_{i=0}^{m} \frac{1}{A_i} \qquad \text{(by (34) and (35))} \\ &= (-1)^{m+1}e^{L+\sum_i K(u_i)} \prod_{i=0}^{m} \frac{(1+u_i)}{u_i}.\end{aligned}$$

By Lemma 17 one has $\prod_i(1+u_i) = v$ and $\prod_i u_i = (-1)^{m+1}$, so it only remains to show that $L + \sum_{i=0}^{m} K(u_i) = V(v)$.

Recall that $V(v)$ belongs to $v\mathbb{K}[v][[z]]$ and that $K(u) = [u^>]V(v)$. Therefore Lemma 18 gives:

$$V(v) = 0 + \sum_{i=0}^{m} \big(K(u_i) - K(-1)\big).$$

But it follows from (5) that

$$K(-1) = \sum_{k\geq 1} \frac{p_k}{k} z^k \sum_{i=1}^{k} \binom{(m+1)k}{k-i}(-1)^i = -\sum_{k\geq 1} \frac{p_k}{k} z^k \frac{k}{(m+1)k}\binom{(m+1)k}{k} = \frac{-L}{m+1},$$

where we have used the identity

$$\sum_{i=1}^{a} \binom{b}{a-i}(-1)^i = -\binom{b-1}{a-1} = -\frac{a}{b}\binom{b}{a},$$

valid for $b \geq a \geq 0$, which is easily proved by induction on a. Therefore $V(v) = L + \sum_{i=0}^{m} K(u_i)$, and the proof is complete. ∎

We have finally proved that $\tilde{G}(u,1) = G_1(u)$. As explained in Section 4.1, this implies that $F^{(m)}(x,y) = \tilde{G}(u,y)$ after the change of variables (7). In particular, $F^{(m)}(x,1) = G_1(u)$, and (8) is proved. One then obtains (9) in the limit $u \to 0$, using

$$[u]K(u) = \sum_{k\geq 1} \frac{p_k}{k}\binom{(m+1)k}{k-1} z^k.$$

5.4. From series to numbers

We now derive from (9) the expression of the character given in Theorem 2. We will extract from $F^{(m)}(t,p;1,1)$ the coefficient of t^n. We find convenient to rewrite the factor e^L occurring in this series as $\tilde{z}/s$, where $s^m = t$ and $\tilde{z} = se^L$ (so that $\tilde{z}^m = z$).

Hence

$$\begin{aligned}[t^n]F(t,p;1,1) &= [s^{mn+1}]\left(\tilde{z} - m\sum_{k\geq 1}\frac{p_k}{k}\tilde{z}^{km+1}\binom{(m+1)k}{k-1}\right)\\ &= \frac{1}{mn+1}[\tilde{z}^{mn}]\left(1 - m\sum_{k\geq 1}\frac{p_k}{k}(km+1)\tilde{z}^{km}\binom{(m+1)k}{k-1}\right)e^{(mn+1)L}\end{aligned}$$

by the Lagrange inversion formula. This can be rewritten in terms of $z = \tilde{z}^m$:

$$[t^n]F(t,p;1,1) = \frac{1}{mn+1}[z^n]\left(1 - m\sum_{k\geq 1}p_k z^k\binom{(m+1)k}{k}\right)e^{(mn+1)L}.$$

The sum inside the brackets is closely related to the derivative of L with respect to z:

$$\begin{aligned}[t^n]F(t,p;1,1) &= \frac{1}{mn+1}[z^n]\left(1 - mz\frac{\partial L}{\partial z}\right)e^{(mn+1)L}\\ &= \frac{1}{mn+1}[z^n]\left(1 - \frac{mz}{mn+1}\frac{\partial}{\partial z}\right)e^{(mn+1)L}\\ &= \frac{1}{mn+1}\left(1 - \frac{mn}{mn+1}\right)[z^n]e^{(mn+1)L}\\ &= \frac{1}{(mn+1)^2}[z^n]\prod_{k\geq 1}\exp\left((mn+1)\frac{p_k}{k}z^k\binom{(m+1)k}{k}\right)\\ &= \frac{1}{(mn+1)^2}\sum_{\alpha_1+2\alpha_2+\cdots=n}(mn+1)^{\sum\alpha_k}\prod_k\frac{p_k^{\alpha_k}}{k^{\alpha_k}\alpha_k!}\binom{(m+1)k}{k}^{\alpha_k}\\ &= \frac{1}{(mn+1)^2}\sum_{\lambda=(\lambda_1,\ldots)\vdash n}(mn+1)^{\ell(\lambda)}\frac{p_\lambda}{z_\lambda}\prod_{i\geq 1}\binom{(m+1)\lambda_i}{\lambda_i}.\end{aligned}$$

The final equation is precisely Theorem 2.

5.5. The complete series $F(t,p;x,y)$

We finally give an explicit expression of the complete series $F(x,y) \equiv F^{(m)}(t,p;x,y)$. Recall that $F(x,y) = \tilde{G}(u,y)$ after the change of variables (7), and that the series $\tilde{G}(u,y)$ satisfies (46) with $\Phi_m(v) = ve^{yV(v)}$ (Proposition 12). Hence Proposition 16 gives an explicit, although complicated, expression of the complete series $F(t,p;x,y)$.

Theorem 21. *Let $F^{(m)}(t,p;x,y) \equiv F(t,p;x,y)$ be the refined Frobenius series of the m-Tamari representation, defined by (6). Let z and u be two indeterminates, and write*

$$t = ze^{-mL} \quad \text{and} \quad x = (1+u)e^{-mK(u)},$$

where L and $K(u)$ are defined by (4–5). Then $F(t,p;x,y)$ becomes a series in z with polynomial coefficients in u, y and the p_i, and this series can be computed by an iterative extraction of positive parts. More precisely,

$$F(t,p;x,y) = \sum_{k=0}^{m}\Phi_k(v)A(u)^k, \tag{55}$$

where $v = u^{-m}(1+u)^{m+1}$, $A(u)$ *is defined by* (24), *and* $\Phi_k(v) \equiv \Phi_k(z;v)$ *is a series in* z *with polynomial coefficients in* v, y *and the* p_i*'s. This series can be computed by a descending induction on* k *as follows. First,* $\Phi_m(v) = ve^{yV(v)}$ *where* $V(v)$ *is defined by* (3). *Then for* $1 \leq k \leq m$,

$$\Phi_{k-1}(v) = \sum_{i=0}^{m} \left(\Phi_{k-1}^{>}(u_i) - \Phi_{k-1}^{>}(-1)\right)$$

where

$$\begin{aligned}\Phi_{k-1}^{>}(u) &= [u^{>}]\Phi_{k-1}(v) \\ &= -\frac{1}{\binom{m}{k}}[u^{>}]\left(\sum_{j=k}^{m}\Phi_j(v)\sum_{\lambda\vdash j-k+1}\binom{m-\ell(\lambda)}{k-\ell(\lambda)}m_\lambda(A(u_1),\ldots,A(u_m))\right), \quad (56)\end{aligned}$$

and $u_0 = u, u_1, \ldots, u_m$ *are the* $m+1$ *roots of the equation* $(1+u_i)^{m+1} = u_i^m v$.

We can rewrite (55) in a slightly different form, which gives directly (10) when $m = 1$. This rewriting combines (55) with the expression of $[u^{>}]\Phi_0(v)$ derived from (56). The case $k = 1$ of (56) reads

$$[u^{>}]\Phi_0(v) = -\frac{1}{m}[u^{>}]\left(\sum_{j=1}^{m}\Phi_j(v)\sum_{i=1}^{m}A(u_i)^j\right). \quad (57)$$

Recall that $F(t,p;x,y) = \tilde{G}(z,p;u,y)$ has polynomial coefficients in u, and that $x = 1$ when $u = 0$. Hence, returning to (55):

$$\begin{aligned}F(t,p;x,y) &= F(t,p;1,y) + [u^{>}]\left(\sum_{k=0}^{m}\Phi_k(v)A(u)^k\right) \\ &= F(t,p;1,y) + [u^{>}]\left(\sum_{k=1}^{m}\Phi_k(v)\left(A(u)^k - \frac{1}{m}\sum_{i=1}^{m}A(u_i)^k\right)\right) \qquad \text{(by (57))} \\ &= (1+u)[u^{\geq}]\left(\sum_{k=1}^{m}\frac{\Phi_k(v)}{1+u}\left(A(u)^k - \frac{1}{m}\sum_{i=1}^{m}A(u_i)^k\right)\right) \quad (58)\end{aligned}$$

by (29), and given that $F(t,p;x,y) = 0$ when $u = -1$. The proof that $\Phi_k(v)\sum_{i=1}^{m}A(u_i)^k$ has coefficients in $(1+u)\mathbb{K}[u,\bar{u}]$ (which is needed to apply (29)) is similar to the proof that (53) has coefficients in $v\mathbb{K}[v]$.

Examples. We now specialize (58) to $m = 1$ and $m = 2$. When $m = 1$, (58) coincides with (10) (recall that $\Phi_m = ve^{yV(v)}$). When $m = 2$, we obtain the following expression for $F^{(2)}$.

Corollary 22. *Let* $V(v), L$ *and* $K(u)$ *be the series given by* (3–5), *with* $m = 2$. *Perform the change of variables* (7), *still with* $m = 2$. *Then the weighted Frobenius series of the* 2*-Tamari representation satisfies*

$$\frac{F^{(2)}(t,p;x,y)}{1+u} = [u^{\geq}]\left(\frac{\Phi_1(v)}{1+u}\left(A(u) - \frac{A(u_1)}{2} - \frac{A(u_2)}{2}\right) + (1+\bar{u})^2 e^{yV(v)}\left(A(u)^2 - \frac{A(u_1)^2}{2} - \frac{A(u_2)^2}{2}\right)\right),$$

where

$$u_{1,2} = \frac{1+3u \pm (1+u)\sqrt{1+4u}}{2u^2}, \qquad A(u) = \frac{u}{1+u}e^{-K(u)},$$

and

$$\Phi_1(v) = \Phi_1^{>}(u) + \Phi_1^{>}(u_1) + \Phi_1^{>}(u_2) - 3\Phi_1^{>}(-1),$$

with

$$\Phi_1^{>}(u) = -[u^{>}]\left((1+u)^3\bar{u}^2 e^{yV(v)}\left(A(u_1)+A(u_2)\right)\right).$$

This expression has been checked with MAPLE, after computing the first coefficients of $F^{(2)}(t,p;x,y)$ from the functional equation (12).

6. FINAL COMMENTS

6.1. A CONSTRUCTIVE PROOF?

Our proof would not have been possible without a preliminary task consisting in *guessing* the expression (8) of $F(t,p;x,1)$. This turned out to be difficult, in particular because the standard guessing tools, like the MAPLE package GFUN, can only guess D-finite generating functions, while the generating function of the numbers (1), or even (2), is not D-finite. The expression of $F(t,p;x,1)$ actually *becomes* D-finite in z (at least when only finitely many p_i's are non-zero) after the change of variables (7). The correct parametrization of the variable t by z was not hard to obtain using the (former) conjecture (1) and the Lagrange inversion formula, but we had no indication on the correct parametrization of x. Our discovery of it only came after a long study of special cases (for instance $m=1$ and $p_i = \mathbb{1}_{i=0}$), and an analogy with the enumeration of unlabelled Tamari intervals [7]. Obviously, a constructive proof of our result would be most welcome, not to mention a bijective one.

6.2. THE ACTION OF $\mathfrak{S}_n$ ON PRIME m-TAMARI INTERVALS

Other remarkable formulas, as simple as (1) and (2), can be derived from our expression (8) of the series $F^{(m)}(t,p;x,1)$. Let us for instance focus on the action of $\mathfrak{S}_n$ on *prime* intervals, that is, intervals $[P,Q]$ such that P has only two contacts with the line $\{x=my\}$. The character $\tilde{\chi}_m$ of this representation is obtained by extracting the coefficient of x^2 from $F^{(m)}(t,p;x,1)$, and the Lagrange inversion formula gives, for a partition λ of length ℓ:

$$\tilde{\chi}_m(\lambda) = ((m+1)n-1)^{\ell-2}\prod_{1\le i\le \ell}\binom{(m+1)\lambda_i - 1}{\lambda_i}.$$

In particular, the number of prime labelled m-Tamari intervals of size n is

$$((m+1)n-1)^{n-2}m^n.$$

For *unlabelled* intervals, it follows from [7, Coro. 11] that the corresponding numbers are

$$\frac{m}{n((m+1)n-1)}\binom{(m+1)^2n-m-1}{n-1}.$$

6.3. THE NUMBER OF UNLABELLED m-TAMARI INTERVALS

Recall from Lemma 8 that the series $F^{(m)}(t,p;x,y)$ can also be understood as the generating function of (weighted) unlabelled m-Tamari intervals. In particular, when $p=\mathbf{1}=(1,1,\ldots)$ and $y=1$, we have

$$h_k = \sum_{\lambda\vdash k}\frac{1}{z_\lambda} = 1,$$

(because $k!/z_\lambda$ counts permutations of cycle type λ), so that

$$F^{(m)}(t,\mathbf{1};x,1) = \sum_{I=[P,Q] \text{ unlabelled}} t^{|I|}x^{c(P)}.$$

By specializing Theorem 3 to the case $y=1, p=\mathbf{1}$, we recover the following result, already proved in [7]. The result of [7] also keeps track of the size of the first ascent, but we have not been able to recover it in this generality.

Proposition 23 ([7]). *Let z' and u' be two indeterminates, and write*

$$t = z'(1-z')^{m^2+2m} \quad \text{and} \quad x = \frac{1+u'}{(1+z'u')^{m+1}}. \tag{59}$$

Then the ordinary generating function of unlabelled m-Tamari intervals, counted by the size and the number of contacts, becomes a series in z' with polynomial coefficients in u', and admits the following closed form expression:

$$F^{(m)}(t, \mathbf{1}; x, 1) = \frac{(1+u')(1+z'u')}{u'(1-z')^{m+2}} \left(\frac{1+u'}{(1+z'u')^{m+1}} - 1 \right). \tag{60}$$

As shown in [7], this implies that the number of unlabelled m-Tamari intervals of size n is

$$\frac{m+1}{n(mn+1)} \binom{(m+1)^2 n + m}{n-1}.$$

Proof. We need to relate the parametrizations (7) and (59), and then the expressions (8) and (60). Let $M \equiv M(z)$ be the unique formal power series in z satisfying

$$M = 1 + zM^{m+1}. \tag{61}$$

We claim that, when $p = \mathbf{1}$,

$$e^L = M^{m+1} \quad \text{and} \quad e^{K(u)} = \frac{1}{1 - zuM^{m+1}} = \frac{1}{1 - u(M-1)}. \tag{62}$$

This establishes the equivalence between the parametrizations (7) and (59), with

$$M = \frac{1}{1-z'} \quad \text{and} \quad u = \frac{u'(1-z')}{1+u'z'}.$$

The equivalence between the two expressions of $F^{(m)}$, namely (8) and (60), also follows.

We will prove (62) using combinatorial interpretations of the series K, L, M in terms of lattice paths on the square grid, starting at $(0,0)$ and formed of north and east steps. First, note that M counts m-ballot paths (defined in the introduction) by the size. Also,

$$B(z) := z\frac{d}{dz} L(z, \mathbf{1}) = \sum_{k \geq 1} \binom{(m+1)k}{k} z^k$$

counts, by the number of north steps, non-empty paths ending on the line $\{x = my\}$ (often called *bridges*, hence the notation B). We have $M = 1/(1-P)$, where P counts *prime* ballot paths (those that only have two contacts). By a variant of the cycle lemma [3, Section 4.1], there exists a size preserving bijection between non-empty bridges and pairs formed of a prime excursion with a marked step, and an excursion. Since a bridge having n north steps has $(m+1)n$ steps in total, this gives:

$$z\frac{d}{dz} L(z, \mathbf{1}) = B(z) = \frac{z(m+1)P'(z)}{1-P(z)} = z\frac{d}{dz} \left(\ln M(z)^{m+1} \right). \tag{63}$$

Integrating over z and then exponentiating gives the first part of (62).

Let us now consider the series $K(z, \mathbf{1}; u)$. We will interpret it in terms of paths of length $k(m+1)$ for some k (to generalize the terminology used for ballot paths, we say that such paths have *size* k). The *depth* of path ending at (x, y) is $x - my$. Observe that

$$z\frac{d}{dz} K(z, \mathbf{1}; u) = \sum_{k \geq 1} z^k \sum_{i=1}^{k} \binom{(m+1)k}{k-i} u^i,$$

counts paths of length multiple of $(m+1)$ having a positive depth (z accounts for the size, divided by $(m+1)$, and u for the depth, also divided by $(m+1)$). Let w be such a path, and look at the shortest prefixes of w of depth 1, then depth 2, and so on up to depth $(m+1)i$.

This factors w into a sequence $(M_1, e, M_2, e, \ldots, M_{(m+1)i}, e, B)$, where the M_i are ballot paths, e stands for an east step and B is a bridge. Accordingly,

$$z\frac{d}{dz}K(z,\mathbf{1};u) = (1+B(z))\left(\frac{1}{1-zuM(z)^{m+1}}-1\right) = z\frac{d}{dz}\left(\ln\frac{1}{1-zuM(z)^{m+1}}\right),$$

by (63). Integrating and exponentiating gives the second part of (62). ∎

6.4. A q-ANALOGUE OF THE FUNCTIONAL EQUATION

As described in the introduction, the numbers (2) are conjectured to give the dimension of certain polynomial rings generalizing $\mathrm{DR}_{3,n}$. These rings are tri-graded (with respect to the sets of variables $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$), and it is conjectured [4] that the dimension of the homogeneous component in the x_i's of degree k is the number of labelled intervals $[P,Q]$ in $\mathcal{T}_n^{(m)}$ such that the longest chain from P to Q, in the Tamari order, has length k. One can recycle the recursive description of intervals described in Section 3 to generalize the functional equation of Proposition 5 (taken when $p_i = \mathbb{1}_{i=0}$), by taking into account (with a new variable q) this distance. Eq. (12) becomes

$$\frac{\partial F}{\partial y}(x,y) = tx(F(x,1)\Delta)^{(m)}(F(x,y)),$$

where now

$$\Delta S(x) = \frac{S(qx)-S(1)}{qx-1}.$$

Here $F(1,1)$ counts labelled m-Tamari intervals by the size and the above defined distance. But we have not been able to conjecture any simple q-analogue of (2).

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CHAPITRE VI

PROBLÈMES OUVERTS

Dans ce dernier chapitre, nous énonçons certains problèmes qui ont retenu notre attention durant cette thèse et qui restent ouverts.

On commence avec une conjecture qui ne semble pas difficile. Elle est la conjecture la plus simple d'un série plutôt empirique de conjectures.

Conjecture 16 ($m > 1$)

$$\sum_{D \in \mathrm{Dyck}_m(n)} q_1^{d(\widehat{0},D)} = \sum_{D \in \mathrm{Dyck}_m(n)} q_1^{d(D,\widehat{1})}. \tag{6.1}$$

Une jolie preuve bijective serait la bienvenue (ceci est peut-être facile, voir la conjecture 21). Notons que l'identité dans la conjecture précédente est triviale pour $m = 1$ car le treillis de Tamari est auto-dual.

Soit $D \in \mathrm{Dyck}_m(n)$ et $a_1, a_2, ..., a_n$ la séquence des distances horizontales des n pas nord dans D par rapport à la droite de pente $1/m$ passant par l'origine (comme dans la section 2.3.2). On partitionne les n pas nord de D dans des ensembles de la manière suivante. Les nombres i et j sont dans le même ensemble dans cette partition si et seulement si $a_i = a_j$ et $a_r \geq a_i$ pour tous les $i < r < j$. Une *fonction de m-stationnement diagonale* sur D est un étiquetage des pas nord de D croissant sur chacun des ensembles de pas nord dans la partition précédente (avec les valeurs $\{1, 2, ..., n\}$ comme étiquettes). La figure 6.1 donne un exemple d'une fonction de stationnement diagonale.

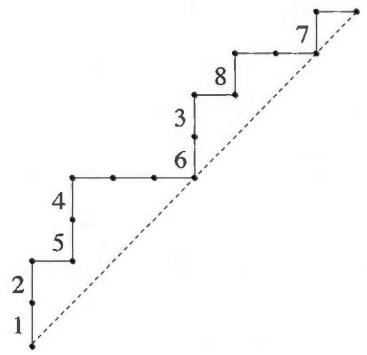

Figure 6.1 Une fonction de stationnement diagonale, car $1<6<7, 2<5, 3<8$.

On dénote cet ensemble de fonctions de m-stationnement diagonales par $\mathrm{DPark}_m(n)$ (sur tous les chemins de m-Dyck de hauteur n). Il est facile de voir que

$$|\mathrm{DPark}_m(n)| = (mn+1)^{n-1}.$$

Le conjecture 16 suggère le problème suivant :

Problème 3 *Trouver une statistique stat sur les éléments de* $\mathrm{Dyck}_m(n)$ *et de* $\mathrm{DPark}_m(n)$ *telle que*

$$\sum_{D\in \mathrm{Dyck_m(n)}} q_1^{\mathrm{area}(D)} q_2^{\mathrm{dinv}(D)} = \sum_{D\in \mathrm{Dyck_m(n)}} q_1^{d(D,\widehat{1})} q_2^{\mathrm{stat(D)}}, \tag{6.2}$$

$$\sum_{P\in \mathrm{Park_m(n)}} q_1^{\mathrm{area}(P)} q_2^{\mathrm{dinv}(D)} = \sum_{P\in \mathrm{DPark_m(n)}} q_1^{d(P,\widehat{1})} q_2^{\mathrm{stat(P)}}, \tag{6.3}$$

Notons que les membres de gauche de 6.2 et de 6.3 sont les valeurs conjecturales de $\mathcal{H}(\mathcal{DR}_{2,n}^{m\ \varepsilon}; q_1, q_2)$ et de $\mathcal{H}(\mathcal{DR}_{2,n}^{m}; q_1, q_2)$, respectivement.

Soit $D \in \mathrm{Dyck}_m(n)$. Soit $c_1, c_2, ..., c_t$ les cardinalités des ensembles dans la partition des n pas nord utilisés dans la définition des fonctions de m-stationnement diagonales. Soit $e(D) = e_{c_1} e_{c_2} ... e_{c_t}$, où les $e_1, e_2, ...$ sont des variables. Soit b_i la hauteur de la $i^{\text{ème}}$ montée (non-nulle) de D. On obtient donc une séquence $b_1, b_2, ..., b_h$ pour un certain $1 \leq h \leq n$ telle que $\sum b_i = n$. Soit $r(D) = r_{b_1} r_{b_2} ... r_{b_h}$, où les $r_1, r_2, ...$ sont des variables. Soit retours(D) le nombre de retours de D avec la doite de pente $1/m$ passant par

l'origine. Notons que retours(D) = contacts$(D) - 1$. Soit montée(D) la hauteur de la première montée de D. Soit

$$M(t;x,y,e,r,q_1) = \sum_{[D_1,D_2]} t^n x^{\text{retours}(D_1)} y^{\text{montée}(D_2)} e(D_1) r(D_2) q_1^{d(D_1,D_2)},$$

où la première somme est indexée par les intervalles de hauteur n du treillis de m-Tamari. Cette série est très similaire à la série $F(t;x,y)$ dans le chapitre 4 qui compte les intervalles du treillis de m-Tamari.

Conjecture 17

$$M(t;x,y,e,r,q_1) = M(t;y,x,r,e,q_1). \tag{6.4}$$

Dans le cas $m = 1$, il est possible de démontrer la conjecture 17 ainsi. Soit $M'(t;x,y,e,r,q_1)$ une distribution analogue à $M(t;x,y,e,r,q_1)$, où y compte la longueur de la dernière séquence de pas est dans le chemin supérieur d'un intervalle, les variables r comptent les longueurs des séquences de pas est dans le chemin supérieur, et les variables t, x, e, q_1 comptent les mêmes choses que dans la série M. Alors ces deux séries satisfont la même équation, donc $M(t;x,y,e,r,q_1) = M'(t;x,y,e,r,q_1)$. Par une simple réflexion sur les arbres binaires, il est facile de voir que $M'(t;x,y,e,r,q_1) = M'(t;y,x,r,e,q_1)$. Ceci démontre la conjecture 17 dans le cas $m = 1$. Gilles Schaeffer a introduit des objets combinatoires, que nous ne montrerons pas ici, mais qui permettent de démontrer par une très jolie involution la conjecture 17 lorsque $m = 1$ et $q_1 = 1$. Notons que la conjecture 17 (avec $q_1 = 1$) implique la conjecture suivante, dont le cas $m = 1$ est prouvé par les arguments précédents, qui ne semble pas difficile à démontrer à première vue.

Conjecture 18

$$\sum_{[D_1,D_2]} \sum_{P \in \text{Park}_m(D_2)} 1 = \sum_{[D_1,D_2]} \sum_{P \in \text{DPark}_m(D_1)} 1. \tag{6.5}$$

Le membre de gauche de 6.5 est égal à $(m+1)^n(mn+1)^{n-2}$ par le corollaire 5. Notons également qu'un cas particulier de la conjecture 17 généralise la conjecture 16 en gardant trace de la statistique $d(,)$.

L'équation 4.2 et la paramétrisation 4.4 impliquent que la série $\ddot{F}(t;x,y) = \frac{yF(t;x,y)}{x}$ satisfait l'équation suivante :

$$\ddot{F}(t;x,y) = y + xyt\left(\ddot{F}(t;1,y)\ \Delta_y\right)^m (\ddot{F}(t;x,y)), \tag{6.6}$$

où $\Delta_y(R(y)) = \frac{R(y)-R(1)}{y-1}$. Une démonstration de cette équation directement à partir des intervalles permettrait probablement de démontrer la conjecture 18. Si cela s'avère difficile, il serait intéressant de trouver une (autre) équation fonctionnelle satisfaite par la série génératrice des structures combinatoires comptées par le membre de droite de la conjecture 18.

Les q_1-analogues donnés dans les équations 4.8 et 5.12 comptent la longueur de la plus grande chaîne pour chacun des intervalles non-étiquetés et étiquetés, respectivement. Il est probable que nous nous attaquerons prochainement au problème suivant :

Problème 4 *Trouver des identités, des égalités, des asymptotiques ou des distributions limites liées à ces q_1-analogues.*

Les conjectures 11, 12 et 13 suggèrent plusieurs des conjectures et des problèmes suivants[1].

Conjecture 19

$$\sum_{[D_1,D_2]} q_1^{d(D_1,D_2)} = \sum_{[D_1,D_2]} q_1^{\mathrm{dinv(D_2)}}.$$

Conjecture 20

$$\sum_{[D_1,D_2]} \sum_{P\in \mathrm{Park_m(D_2)}} q_1^{d(D_1,D_2)} = \sum_{[D_1,D_2]} \sum_{P\in \mathrm{Park_m(D_2)}} q_1^{\mathrm{dinv(P)}}.$$

Problème 5 *Trouver la troisième statistique thirdstat sur les intervalles telle que*

$$\mathcal{H}(\mathcal{DR}_{3,n}^{m\ \varepsilon}; q_1, q_2, q_3) = \sum_{[D_1,D_2]} q_1^{d(D_1,D_2)} q_2^{\mathrm{dinv(D_2)}} q_3^{\mathrm{thirdstat(D_1,D_2)}}.$$

---

1. Toutes ces sommes sont indexées par les intervalles $[D_1, D_2]$ de hauteur n du treillis de m-Tamari.

Nous avons remarqué avec un logiciel de calcul formel qu'une telle statistique ne dépend pas juste de D_1 (l'extrémité inférieure de l'intervalle). Il faudrait ensuite étendre cette statistique aux intervalles étiquetés. L'approche qui nous semble la plus raisonnable pour trouver cette troisième statistique est de considérer l'ensemble des intervalles étiquetés du treillis de m-Tamari dont la statistique dinv est égale à 0. Il est très probable que nous réattaquerons prochainement à ce problème :

Problème 6 *Trouver une statistique cstat, sur les intervalles étiquetés dans le treillis de m-Tamari dont la statistique dinv est égale à 0, telle que*

$$\sum_{P\in \mathrm{Park_m}(n)} q_1^{\mathrm{area}(P)} q_2^{\mathrm{dinv}(P)} = \sum_{[D_1\leq D_2]} q_1^{d(D_1,D_2)} \sum_{P\in \mathrm{Park}_m(D_2),\, \mathrm{dinv}(P)=0} q_2^{\mathrm{cstat}(D_1,P)}. \quad (6.7)$$

Notons que le membre de gauche de 6.7 est l'expression conjecturale de $\mathcal{H}(\mathcal{DR}^m_{2,n}; q_1, q_2)$ d'après la conjecture 3. Une façon d'attaquer le problème précédent serait d'abord de résoudre la conjecture 16 et au moins un des deux problèmes suivants.

Problème 7 *Trouver une bijection entre l'ensemble des fonctions de m-stationnement et l'ensemble des intervalles de stationnement du treillis de m-Tamari dont la statistique dinv est égale à 0, qui envoie la statistique dinv sur la statistique $d(,)$.*

Problème 8 *Trouver une bijection entre l'ensemble des fonctions de m-stationnement et l'ensemble des intervalles de stationnement du treillis de m-Tamari dont la statistique dinv est égale à 0, qui envoie la statistique area sur la statistique $d(,)$.*

Cette troisième statistique sur les intervalles permettrait peut-être d'attaquer le problème suivant.

Problème 9 *Trouver un modèle combinatoire pour les $\mathcal{DR}^m_{4,n}$.*

Pour des petites valeurs de n, il semblerait, en utilisant un logiciel de calcul formel, que les multichaînes de longueur 2 du treillis de m-Tamari donnent un ensemble à peine plus grand que $|\mathcal{DR}^{m\ \varepsilon}_{4,n}|$.

Le reste de ce chapitre est constitué de plusieurs problèmes dans un ordre plutôt arbitraire.

Problème 10 *Serait-il possible d'utiliser l'ordre de Tamari pour se guider pour tenter de démontrer les conjectures 1, 3 et 5 dans le cas de 2 jeux de variables ?*

Les conjectures 1, 3 et 5 sont difficiles d'entrée car il faut bien savoir manipuler les fonctions symétriques (comme les fonctions de Macdonald) et les opérateurs. Il faut ensuite trouver des récurrences qui fonctionnent aussi bien sur les fonctions symétriques (avec opérateurs) que sur les objets combinatoires.

Les changements de variables et la paramétrisation utilisés pour compter les intervalles étiquetés (et pour la série raffinée de Frobenius de ces intervalles étiquetés) restent mystérieux. Le problème suivant est probablement très difficile.

Problème 11 *Est-it possible de construire les changements de variables 5.3, 5.17 et les paramétrisations 5.9, 5.18 ?*

Plus généralement, étendre les méthodes pour les équations algébriques à variable catalytique aux équations différentielles-catalytiques semble pour l'instant relever de l'impossible. Néanmoins, les fortes similitudes entre le cas non-étiqueté et le cas étiqueté nous suggèrent la question (difficile) suivante.

Problème 12 *Pourrait-on obtenir les changements de variables et la paramétrisation pour le cas étiqueté à partir d'une certaine transformation agissant sur les changements de variables et la paramétrisation du cas non-étiqueté ?*

Les preuves des formules d'énumérations 5.11 et 5.20 restent difficiles pour le cas $m > 1$ même lorsque l'on assume les paramétrisations.

Problème 13 *Est-il possible de simplifier les preuves (en assumant les paramétrisations) des formules d'énumérations 5.11 et 5.20 pour le cas $m > 1$?*

Une façon d'attaquer le problème précédent serait d'avoir plus de contrôle sur la variable y comme dans le cas $m = 1$ (ce qui revient à contrôler toute la somme dans 5.1 et 5.14).

Nous voudrions mentionner que l'on peut généraliser les équations 4.2, 5.1 et 5.14 que nous avons résolues en ajoutant un paramètre $r \in \mathbb{P}$. Les coefficients de t (lorsque $x = y = 1$) semblent être des produits de petits facteurs premiers et d'une polynomialité de degré prévisible. Nous n'avons pas essayé de démontrer ces généralisations car la combinatoire ne semble pas très agréable. Cependant, après quelques essais, il semblerait qu'elles aient de jolis changements de variables et paramétrisations. Ces équations sont données par :

$$F(t;x,y) = x + xyt\,(F(t;x,1)\;\Delta^r)^m\,(F(t;x,y)),$$

$$G(t;x,y) = exp\;(xyt(G(t,x,1)\Delta^r)^m)(x),$$

$$B(t,p;x,y) = \exp\left(y\sum_{k\geq 1}\frac{p_k}{k}\left(tx(B(t,p;x,1)\Delta^r)^{(m)}\right)^{(k)}\right)(x).$$

Voici un problème qui a occupé une très grande partie de notre temps et qui reste sans réponse. Il semble difficile.

Problème 14 *Trouver une preuve bijective uniforme des formules d'énumération 4.7 et 5.20.*

Dans une tout autre direction, nous étudierons des structures de poset, définis sur d'autres familles de chemins combinatoires dotés d'une double distribution symétrique, en imposant des relations d'ordre entre les chemins. Des famille de chemins intéressantes à regarder apparaissent dans (Haglund, 2008) et dans (Loehr, 2003), comme par exemple les chemins trapézoïdaux. Une autre famille de chemins à considérer est la suivante (voir

(Armstong, 2012)). Soient $a, b \in \mathbb{P}$. Soit $\mathrm{Dyck}_{\frac{b}{a}}(a)$ l'ensemble des chemins constitués de pas nord et de pas est, commençant en $(0,0)$, terminant en (b,a) et ne descendant jamais sous la diagonale de pente $\frac{a}{b}$ passant par l'origine[2].

Voici quelques questions que nous nous poserons sur ces posets mentionnés dans le paragraphe précédent.

Problème 15 *Ces posets sont-ils des treillis ?*

Problème 16 *La double distribution* $\{(d(,), \mathrm{dinv})\}$ *sur les intervalles (non-étiquetés et étiquetés) semble-t-elle symétrique ?*

Problème 17 *Les intervalles (non-étiquetés et étiquetés) dans les posets semblent-ils être comptés par de jolies formules ?*

Notons que nous avons brièvement regardé un tel poset sur les chemins dans $\mathrm{Dyck}_{\frac{b}{a}}(a)$, qui pour des valeurs $a, b \leq 10$ est un treillis, et que les intervalles dans ce cas ne semblent pas être comptés par de jolis nombres. Cependant, ce poset semble avoir de belles propriétés combinatoires. Par exemple, soit $\mathrm{Tam}(a,b)$ le poset défini sur les chemins dans l'ensemble $\mathrm{Dyck}_{\frac{b}{a}}(a)$ mentionné auparavant. Il semblerait alors que[3]

Conjecture 21

$$\mathrm{Tam}(a,b) \cong (\mathrm{Tam}(b,a))^*.$$

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2. Dans (Armstong, 2012), il est mentionné la découverte d'une statistique dinv sur les chemins appartenant à $\mathrm{Dyck}_{\frac{b}{a}}(a)$. Sur ces derniers, la double distribution (area, dinv) semble également symétrique.

3. La conjecture 21 implique la conjecture 16.

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