PSEUDOVALUATION DOMAINS WITH VAPNIK-CHERVONENKIS CLASSES OF DEFINABLE SETS

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Abstract

We use model-theoretic methods to give examples of pseudovaluation domains with Vapnik-Chervonenkis classes of definable sets.

1 Introduction

Pseudovaluation domains were introduced by Hedstrom and Houston in [9] and have been extensively studied. A domain A is called a *pseudovaluation domain*, if every prime ideal P has the property that for all $x, y \in Q(A)$, the field of fractions of A, $xy \in P$ implies $x \in P$ or $y \in P$; or equivalently, if and only if A is local and $Q(A)$ has a (unique) valuation ring containing A and with the same maximal ideal as A. Given a set X and a finite subset F, a collection $\mathcal{C}$ of subsets of X is said to *shatter* F if for every subset F' of F there is $C \in \mathcal{C}$ with $F' = F \cap C$. The collection $\mathcal{C}$ is called a Vapnik-Chervonenkis class (or VC class) if there is an n such that no subset of X of size n is shattered by $\mathcal{C}$. We will be concerned with collections which are *definable* parametrized family in a pseudovaluation domain A, i.e. of the form $\mathcal{C}_\phi = \{\phi(A^m, b) : b \in A^k\}$, where $\phi(A^m, b) = \{c \in A^m : \phi(c, b) \text{ holds}\}$, $\phi(\overrightarrow{x}, \overrightarrow{y})$ being a formula in the sense of first-order logic built from the algebraic operations and constants $+, -, \cdot, 0, 1$, and $\overrightarrow{x}, \overrightarrow{y}$ denoting variables $x_1, \ldots, x_m$ and $y_1, \ldots y_k$. Such Vapnik-Chervonenkis classes are connected with a standard property in model theory. The formula $\phi(\overrightarrow{x}, \overrightarrow{y})$ has the *independence property* with respect to A if for every n there is a sequence $(a_i)_{i<n}$ of elements from A^k so that for every subset w of $\{0, \ldots, n-1\}$ there is a $c_w \in A^m$ such that $\phi(c_w, a_i)$ holds iff $i \in w$. Laskowski [12] observed that $\mathcal{C}_\phi$ is a VC class if and only if $\phi(\overrightarrow{x}, \overrightarrow{y})$ does not have the independence property. We will use this equivalence to give examples of pseudovaluation

domains where every definable family is a VC class (corollary 4.6 and section 5, below). We refer to [12] for background on VC classes and the connection with model theory, and [5] for background on model theory.

All rings are commutative with 1. Let A be a pseudovaluation domain. We will denote by V_A the valuation ring associated to A and by k_A the residue field of A. We say that a valued field or a valuation ring is unramified if the base field is of characteristic 0 and, either the characteristic of the residue field is 0, or it is $p > 0$ and p generates the maximal ideal of the valuation ring. We say that a pseudovaluation domain is unramified if the associated valuation ring is.

Given a structure for some first-order language, we say that this structure has the VC property if every definable family is a VC class. We will denote by L the standard first-order language of rings, and by $\mathcal{L}$ the first-order language of valued fields with two sorts, one for the base field and another one for the residue field, a predicate for the valuation ring, a function symbol for the residue map and the standard language of rings for each of the two sorts. Let L_0 be a first-order language and M_1, M_2 two L_0-structures. We recall that $M_1 \equiv M_2$ (M_1 is elementary equivalent to M_2) means that M_1, M_2 satisfy the same first-order L_0-statements, and $M_1 \preceq M_2$ (M_1 is an elementary substructure of M_2) that M_1 is a L_0-substructure of M_2 and they satisfy the same first-order L_0-statements *with parameters from* M_1.

2 Pseudovaluation domains

It is straightforward to check from the definition that a valuation ring is a pseudovaluation domain. Now let V be a valuation ring, k its residue field, $k_0 \subset k$ a proper subfield of k, and A the pullback of k_0 along the residue map $V \to k$. Then A is a pseudovaluation domain. All pseudovaluation domains are of this form (thm 2.1 below). To get a pseudovaluation domain which is not a valuation ring, make the above construction with a valuation ring of the form $k + M$ ([9], example 2.1). Here is a noetherian pseudovaluation domain which is not a valuation ring ([9], example 3.6): let m be a square-free positive integer congruent to $5 \; mod \; 8$, and $D = Z[\sqrt{m}]$; $N = (2, 1+\sqrt{m})$ is a maximal ideal of D and D_N is the example.

Being a pseudovaluation domain is equivalent to many properties (see [9]). We isolate the characterization by the above construction, which is the key fact for our purpose.

Theorem 2.1 *([7]) Let A be a domain. Then A is a pseudovaluation domain if and only if there exists a valuation ring V, a subfield $k \subseteq k_V$ of* *the residue field of V, a surjective map $\overline{\nu} : A \to k$ and an injective map $\overline{u} : A \to V$, such that $(A, \overline{\nu}, \overline{u})$ is the pullback of the inclusion $u : k \hookrightarrow k_V$ along the canonical surjection $\nu : V \to k_V$. In this situation, after identifying A with $\overline{u}(A)$,and if M is the maximal ideal of V, we have:*

1. *$M \cap A = M = \{x \in V : xA \subseteq V\}$.*
2. *The canonical map $Spec(V) \overset{\overline{u}^\star}{\to} Spec(A)$ is the identity homeomorpism; it is a scheme isomorphism outside $\{M\}$.*
3. *$Q(A) = Q(V)$.*
4. *If A is not a valuation ring, then the valuation ring V is unique and equal to $M^{-1} = \{x \in Q(A) : xM \subseteq A\} = \{x \in Q(A) : xM \subseteq M\}$.*

We also need a characterization formulated in terms of the elements of A, which will ensure that pseudovaluation domains are axiomatized by a set of sentences in first-order logic. Here is one ([9], thm 1.5): for every $x \in Q(A) \setminus A$ and every non unit $a \in A$, $x^{-1}a \in A$. It translates into a L-statement in first-order logic: $\forall a, b, c \in A$ (a divides b or c is invertible or b divides ac).

It is a key fact for the model theory of pseudovaluation domains that they are henselian at the same time as their associated valuation rings.

Lemma 2.2 *Let A be a pseudovaluation domain and V_A its associated valuation ring. Then A is henselian if and only if V_A is henselian.*

Proof. Let M be the common maximal ideal of A and V_A. Suppose A is henselian. It suffices to see that every polynomial of the form $1 + X + a_2X + \ldots + a_nx^n$ has a root in V_A, where $a_i \in M$. But since A is henselian, it already has a root in $A \subseteq V_A$. Now, suppose V_A is henselian. Then, it follows directly from the description of A as a pullback along the residue map of V_A and the question of lifting simple roots, that A is also henselian.□

3 Bi-interpretability with enriched valued fields

If A is a pseudovaluation domain and M its maximal ideal, then the attached valuation ring $V_A = M^{-1} = \{x \in Q(A) : xM \subseteq A\}$ is L-interpretable in A, so that the structure $(Q(A), V_A, k_A \subseteq k_{V_A})$ is also L-interpretable in A. In other words, these constructions from A can be described by first-order L-formulas. For example, one describes the elements of $Q(A)$ using couples of elements of A with the usual equivalence relation, M is the set of non-invertible elements of A etc. We can go the other way around: let $\mathcal{L}_{\mathcal{E}}$ be the language $\mathcal{L}$ expanded by a unary symbol $\mathcal{E}$ to denote k_A, then A is $\mathcal{L}_{\mathcal{E}}$-interpretable in $(Q(A), V_A, k_A \subseteq k_{V_A})$ as the pullback of k_A in V_A via the natural residue map $V_A \to k_{V_A}$. We note that these interpretations are *uniform* in both classes of pseudovaluation domains and structures $(K, V, k \subseteq k_V)$ consisting of a valued field (K, V, k_V) together with a fixed subfield $k \subseteq k_V$ of the residue field. Furthermore, consider the category of pseudovaluation domains with *local* embeddings as morphisms, and the category of structures $(K, V, k \subseteq k_V)$ with $\mathcal{L}_{\mathcal{E}}$-embeddings as morphisms. A local embedding of pseudovaluation domains will induce in a natural way an embedding of the corresponding valued field structures and vice versa. We see that these two categories are isomorphic. This immediately yields the following correspondance between the first-order theories in the two classes.

Theorem 3.1 *Let A, B be two pseudovaluation domains.*

1. $A \equiv B$ *if and only if* $(Q(A), V_A, k_A \subseteq k_{V_A}) \equiv (Q(B), V_B, k_B \subseteq k_{V_B})$.
2. *Suppose* $A \subseteq B$ and the inclusion is local. *Then* $A \preceq B$ *if and only if* $(Q(A), V_A, k_A \subseteq k_{V_A}) \preceq (Q(B), V_B, k_B \subseteq k_{V_B})$.

Proof. The two are similar. (1) Sufficiency follows because of the uniform interpretability of $(Q(A), V_A, k_A \subseteq k_{V_A})$ in A . For necessity, suppose

$$(Q(A), V_A, k_A \subseteq k_{V_A}) \equiv (Q(B), V_B, k_B \subseteq k_{V_B})$$

By taking suitable ultrapowers we can assume that

$$(Q(A), V_A, k_A \subseteq k_{V_A}) \simeq (Q(B), V_B, k_B \subseteq k_{V_B})$$

But then this isomorphism carries over to the pullback diagrams yielding A and B, and thus A,B are isomorphic. For (2) transpose the above discusssion using the elementary diagram of A and $(Q(A), V_A, k_A \subseteq k_{V_A})$. □

A first-order structure is said to have the independence property if there is a formula $\phi(x, \overrightarrow{y})$, where x is a variable denoting a *single* element and not a tuple, having the independence property. It turns out (see [12]) that a first-order structure has the VC property if and only if it does not have the independence property. Also, the independence property carries from one first-order structure to another by first-order interpretability. From all this, it follows that a pseudovaluation domain A has the VC property if and only if $(Q(A), V_A, k_A \subseteq k_{V_A})$ does. We are thus reduced to consider the structures $(K, V, k \subseteq k_V)$.

4 Transfer theorems

Our model-theoretic results about pseudovaluation domains are transfer theorems à la Ax-Kochen-Ershov, which reduce the study of a pseudovaluation domain to the associated value group and pair of residue fields. We will deduce them from relative quantifier elimination results for the $\mathcal{L}_{\mathcal{E}}$-structures $(K, V, k \subseteq k_V)$. We need to introduce extra functions. A coefficient map is a homomorphism from the multiplicative group of K into that of k_V, which extends the residue map on units of V. If the characteristic of k_V is $p > 0$, a coefficient map of order n is a homomorphism from the multiplicative group of K into the multiplicative group of the residue ring $V/(p^{n+1})$, which extends the natural residue map on units of V.

Theorem 4.1 *Let $\mathcal{L}_{co,\mathcal{E}}$ be our language of valued fields $\mathcal{L}$ augmented by a symbol co for a coefficient map and a predicate $\mathcal{E}$ for a subfield of the residue field. The theory of henselian valued fields with a residue field of characteristic 0 has elimination of base field quantifiers in the language $\mathcal{L}_{co,\mathcal{E}}$.*

Theorem 4.2 *Let $\mathcal{L}_{co_\omega,\mathcal{E}_\omega}$ be our language of valued fields $\mathcal{L}$ augmented by symbols co_n for a coefficient map of order n, $n \geq 0$, a predicate $\mathcal{E}_0$ for a subfield of the residue field, and predicates $\mathcal{E}_n$ for a subring of the the valuation ring mod p^{n+1}, with the axioms that $\mathcal{E}_n$ is the inverse image of $\mathcal{E}_0$ under the natural map and the appropriate compatibility of co_n's and $\mathcal{E}_n$'s with the canonical inverse system of maps. The theory of henselian unramified valued fields with a residue field of characteristic $p > 0$ has elimination of base field quantifiers in the language $\mathcal{L}_{co_\omega,\mathcal{E}_\omega}$.*

In [3] we gave proofs of analog results but without the predicates $\mathcal{E}, \mathcal{E}_n$. It is straighforward to check that these predicates do not raise obstructions, and that the same proofs carry over (one only needs to keep track of the new predicates). We immediately get Ax-Kochen-Ershov principles in the respective languages $\mathcal{L}_{co,\mathcal{E}}$, $\mathcal{L}_{co_\omega,\mathcal{E}_\omega}$. Now, since any unramified valued field has an elementary extension with a coefficient map or a compatible system of coefficient maps of order n (see [3]), we get the Ax-Kochen-Ershov principles in the languages $\mathcal{L}_{\mathcal{E}}, \mathcal{L}_{\mathcal{E}_\omega}$ as well. Since the residue rings $V/(p^{n+1})$ are isomorphic to the ring of Witt vectors of length $n+1$ over k_V, and these are uniformly L-interpretable in k_V, we get the results in the language $\mathcal{L}_{\mathcal{E}}$ in both cases. The language for the pairs $(k_A \subseteq k_{V_A})$ is L enriched by a unary predicate symbol to denote the smaller field.

Corollary 4.3 *Let $(K_i, V_i, k_i \subseteq k_{V_i})$, $i = 1, 2$, be two henselian unramified valued fields with a distinguished subfield of the residue field. Then*

1. $(K_1, V_1, k_1 \subseteq k_{V_1}) \equiv (K_2, V_2, k_2 \subseteq k_{V_2})$ *if and only if* $vK_1 \equiv vK_2$ *and* $(k_1 \subseteq k_{V_1}) \equiv (k_2 \subseteq k_{V_2})$.

2. *Suppose* $(K_1, V_1, k_1 \subseteq k_{V_1})$ *is a substructure of* $(K_2, V_2, k_2 \subseteq k_{V_2})$. *Then* $(K_1, V_1, k_1 \subseteq k_{V_1}) \preceq (K_2, V_2, k_2 \subseteq k_{V_2})$ *if and only if* $vK_1 \preceq vK_2$ *and* $(k_1 \subseteq k_{V_1}) \preceq (k_2 \subseteq k_{V_2})$.

This corollary can also be proved by established methods (cf. [11]) or deduced e.g. from theorem 4.3 in [14]. We need the elimination theorems to deal with the independence property. By the preceding section and lemma 2.2 we get

Theorem 4.4 *Let* A, B *be two henselian unramified pseudovaluation domains. Then*

1. $A \equiv B$ *if and only if* $v(Q(A)) \equiv v(Q(B))$ *and* $(k_A \subseteq k_{V_A}) \equiv (k_B \subseteq k_{V_B})$.

2. *Suppose* $A \subseteq B$ and the inclusion is local, *then* $A \preceq B$ *if and only if* $v(Q(A)) \preceq v(Q(B))$ *and* $(k_A \subseteq k_{V_A}) \preceq (k_B \subseteq k_{V_B})$.

As we have remarked already, the VC property, or equivalently the independence property, for a pseudovaluation domain A reduces to the corresponding enriched valued field structure $(Q(A), V_A, k_A \subseteq k_{V_A})$. For valued fields (K, V, k_V), it is known ([6], [2], [3]) that a henselian unramified valued field (K, V, k_V) has the independence property if and only if k_V does. The arguments rely on the description à la Delon of types and coheirs, Poizat's criterion for the independence property by counting coheirs, and the fact that no abelian ordered group has the independence property ([8]). Here, the extra predicates for the subrings of the residue rings will not interact with the base field. Given the elimination theorems 4.1 and 4.2, the arguments used in [2] and [3] carry through, and the known transfer theorems extend in a natural way. We get the corresponding result for pseudovaluation domains.

Theorem 4.5 *Let* $(K, V, k \subseteq k_V)$ *be a valued field with a fixed subfield of the residue field, such that* (K, V, k_V) *is henselian unramified. Then* $(K, V, k \subseteq k_V)$ *has the independence property if and only if the structure* $(k \subseteq k_V)$ *does.*

Corollary 4.6 *A henselian unramified pseudovaluation domain* A *has the independence property if and only if the pair* $(k_A \subseteq k_{V_A})$ *does. Hence,* A *has the VC property if and only if the structure* $(k_A \subseteq k_{V_A})$ *does.*

5 The examples

We thus get examples of pseudovaluation domains with the VC property by taking henselian unramified valuation rings V, subfields $k_0 \subset k_V$ such that the pair $(k_0 \subset k_V)$ has the VC property, and lifting k_0 along the residue map. Now, the independence property for a structure is, in fact, a property of its complete first-order theory, i.e. the set of first-order sentences true in it. By theorem 4.4 we obtain complete theories of henselian unramified pseudovaluation domains by combining complete theories of ordered abelian groups and complete theories of pairs of fields. The complete theories of ordered abelian groups are known (e.g. see [8]). Less is known about theories of fields, and much less about theories of *pairs* of fields. We list the main examples $(k \subseteq k')$ where the theory of the pair is determined by the L-theory of each of its constituents: k, k' are finite; k' is algebraically closed and k is any subfield, but specify the degree of the extension appropriately $(1, 2, \infty)$ (see [10]); k', k are separably closed and $k \preceq k'$ (Delon, see [4]); k, k' are real closed with k dense in k' (A. Robinson, see [10]); k, k' are real closed and the extension is a *separated extension* with respect to the smallest convex valuation ([1]); k, k' are p-adically closed and k is dense in k' ([13]).

The following fields are known not to have the independence property: because they are stable, any finite field, algebraically closed field, separably closed field (see [12]); any real closed field (see [12]); any p-adically closed field (L. Matthews, see [3]). The following pairs of fields are known not to have the independence property: pairs of algebraically closed fields, pairs of separably closed fields where the inclusion is elementary (see [4], section 2). Given the preceding remarks, it is straightforward to see that we also have the following examples: a finite subfield inside an algebraically closed field, or inside any other field known not to have the independence property, a real closed field inside its algebraic closure.

So then, take k any algebraically closed field of characteristic 0 and k_0 a proper algebraically closed subfield or a real closed subfield of codimension 2. The ring $k[[T]]$ of power series in the single variable T is a henselian unramified valuation ring. Let A be the pseudovaluation domain obtained using k_0, i.e. by taking all power series with the first term belonging to k_0. Then A has the VC property, and it is not a valuation ring (see section 2). The same construction will work with the valuation ring of Puiseux series over k or with generalized power series rings $k[[T^G]]$, where G is any ordered abelian group. Take k any algebraically closed or finite field of characteristic $p > 0$ and k_0 a proper algebraically closed or finite subfield, or k, k_0 separably closed of characteristic $p > 0$ such that $k_0 \prec k$. The ring $W[k]$ of Witt vectors over k is a henselian unramified valuation ring. Let A be the pseudovaluation domain obtained using k_0, i.e by taking all Witt vectors with the first component belonging to k_0. Then A has the VC property, and is also not a valuation ring. A particular case is the completion of the noetherian pseudovaluation domain $Z[\sqrt{m}]_{(2,1+\sqrt{m})}$ already encountered: here k is the field with 4 elements, $W[k] = Z_2[\frac{1+\sqrt{m}}{2}]$, Z_2 the 2-adic integers, and k_0 is the field with 2 elements.

The model theory of power series rings over finite fields is still an open problem, so our methods fail in that context.

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