UNIVERSITÉ DU QUÉBEC À MONTRÉAL
DÉPARTEMENT DES SCIENCES ÉCONOMIQUES

ESSAIS SUR L'APPOCHE DES OPTIONS RÉELLES APPLIQUÉE À LA GESTION ET L'EXPLOITATION DES RESSOURCES NATURELLES

THÈSE PRÉSENTÉE
COMME EXIGENCE PARTIELLE DU
DOCTORAT EN ÉCONOMIE

PAR
SKANDER BEN ABDALLAH

OCTOBRE 2009

UNIVERSITY OF QUEBEC AT MONTREAL
DEPARTMENT OF ECONOMICS

ESSAYS ON THE REAL OPTIONS APPROACH APPLIED TO NATURAL RESOURCES MANAGEMENT AND EXPLOITATION

A THESIS PRESENTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE
DOCTOR OF PHILOSOPHY
IN ECONOMICS

BY
SKANDER BEN ABDALLAH

OCTOBER 2009

Je dédie ce travail à mes chers parents Zakia et Ahmed, ma chère épouse Imen et mon adorable fils Ahmed, en signe de ma gratitude pour toute la patience et les sacrifices qu'ils m'ont consenti.

REMERCIEMENTS

Alhamdoulilallah,

Je tiens tout d'abord à remercier mes parents Zakia et Ahmed et mon épouse Imen de m'avoir soutenu pour réussir mon doctorat.

Je remercie également mon directeur de thèse, Professeur Pierre Lasserre, de m'avoir financé, encadré et encouragé tout au long de ces années de doctorat. En particulier, je suis très reconnaissant au Professeur Lasserre d'avoir contribué à l'amélioration de mes connaissances. Je demeure pour toujours plein de gratitude envers mon Professeur qui, durant les longues heures de travail que nous avons endurées, m'a marqué par sa patience, son intelligence et sa faculté d'innovation. Je tiens à lui faire part de mon profond respect.

Mes remerciements s'étendent au Président du jury, Professeur Claude-Denys Fluet (UQAM) ainsi qu'aux membres du jury, professeur Margaret Insley (University of Waterloo) et professeur Nicolas Marceau (UQAM), de leur patience et de leurs commentaires fort utiles.

Je remercie tous ceux qui m'ont aidé dans l'accomplissement de mon doctorat dont en particulier Professeur Gérard Gaudet de l'Université de Montréal et Professeur Hassan Benchekroun de McGill.

Je ne saurai oublier mes amis à l'Université du Québec à Montréal dont en particulier Ajmi Ben Khalifa, Nicolas Petrosky-Nadeau, Lin Zhang, Louise Lavoie et Jérémy Chaudourne, ainsi que le personnel administratif dont je cite en particulier Jacinthe Lalonde, Martine Boisselle, Francine Germain et Lorraine Brisson.

Je tiens aussi à remercier les institutions québécoises et canadiennes qui ont contribué au financement de ma thèse : l'Université du Québec à Montréal, "The Sustainable Forest Management Network of Canada", "The Social Science Research Council of Canada" et le Fonds québécois de la recherche sur la société et la culture.

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RÉSUMÉ

Cette thèse se compose de trois chapitres ayant trait à l'économie des ressources naturelles et de l'environnement. Elle porte sur l'application de l'approche des options réelles à la gestion et l'exploitation des ressources naturelles. Son principal objectif est d'étudier les décisions d'investissements indéfiniment répétés sous l'incertitude.

Dans les deux premiers chapitres, on utilise l'approche des options réelles pour déterminer comment extraire optimalement le bois d'une forêt qui constitue un habitat stochastique pour une espèce menacée d'extinction.

Dans le premier chapitre qui s'intitule "A Real options Approach to Forest-Management Decision Making to Protect Caribou under the Threat of Extinction", la politique optimale visant la protection de l'espèce menacée d'extinction consiste à banir définitivement à un moment optimal la coupe forestière sans aucun coût. Le modèle, comportant des aspects économiques, écologiques et sociaux, est calibré afin de déterminer une politique optimale arbitrant entre les revenus économiques de l'exploitation forestière et le risque d'extinction du *caribou Rangifer tarandus*, une espèce menacée au centre du Labrador, Canada.

Le deuxième chapitre fait suite au premier et s'intitule "A Real Options Approach to the Protection of a Habitat Dependent Endangered Species". Dans ce deuxième chapitre, nous utilisons la méthode des options réelles pour déterminer quand un planificateur social doit interrompre ou reprendre l'exploitation d'une telle forêt. Le processus d'interruption et reprise peut cette fois se reproduire indéfiniment. Pour des paramètres correspondant au cas du *caribou Rangifer tarandus* du Labrador, la politique optimale d'interruption et de reprise de l'exploitation forestière s'avère attractive ; elle n'exige pas de longs arrêts de l'exploitation alors qu'elle réduit significativement le risque d'extinction tout en augmentant la valeur de la forêt.

Le troisième chapitre s'intitule "Alternative and Indefinitely Repeated Investments : Species Choice and Harvest Age in Forestry". Dans ce chapitre, on considère le problème de choisir la date optimale pour couper un arbre comme un problème archétypique d'investissement ; en effet, ce problème évoque le temps, l'incertitude et l'irréversibilité de décisions ayant des conséquences dans le futur. L'exercice de l'option de couper donne naissance à l'option de planter un autre arbre ou bien d'utiliser la terre à d'autres fins. On enrichit ce problème en mettant l'accent sur la décision de planter et en supposant l'existence de deux essences alternatives d'arbres. Pour chaque essence, on suppose que la fonction de croissance est déterministe alors que le prix unitaire de la récolte est stochastique. Dans le cas d'une seule rotation (le problème de Wicksel), le manager forestier doit planter immédiatement l'essence dont le prix (actuel) est suffisament élevé par rapport au prix de l'autre essence. Cependant, si les prix des deux essences sont relativement proches, il doit attendre jusqu'à ce ces prix se distinguent suffisamment pour éviter de choisir la mauvaise essence. En revanche, quand le nombre de rotations est arbitrairement grand (le problème de Faustman), on montre qu'il est toujours optimal de planter l'une des deux essences immédiatement. L'age optimal à la coupe dépend du prix relatif des deux essences d'une façon non monotone ; en effet, il varie autour de l'age de Faustman correspondant à l'essence plantée et ce tant que le prix relatif est inférieur à un certain niveau seuil indiquant la nécessité de planter l'essence alternative. La valeur du terrain est une fonction croissante du prix relatif de l'essence plantée et croît avec l'incertitude liée aux prix des deux essences. La pente de la fonction est continue même au voisinage du seuil signalant la nécessité de changer l'essence à planter. Par contre, en absence d'incertitude, la pente est généralement discontinue au voisinage de ce seuil. La valeur du terrain est similaire à la valeur d'une option américaine avec une frontière libre, une date d'expiration infinie et une valeur endogène à l'exercice. La détermination de la valeur du terrain est basée sur la méthode de pénalité appliquée simultanément aux valeurs de la forêt constituée par chacune des deux essences d'une part, et d'autre part, par un processus newtonien itératif appliqué à la valeur du terrain.

ABSTRACT

This thesis consists of three chapters related to the economics of natural resources and the environment. We focus on the application of the real options approach to the management and exploitation of natural resources. Our main objective is studying indefinitely repeated investment decisions under uncertainty.

In the first two chapters, we use the real options approach to determine optimally how to manage timber harvesting in situations where an endangered species relies on forest habitat for its survival, and that habitat evolves stochasticly.

In the first chapter, entitled "A Real Options Approach to Forest-Management Decision Making to Protect Caribou under the Threat of Extinction", the policy rule designed to protect the endangered species consists in banning logging at no cost and forever. The model incorporates economic, ecological and social features, and is calibrated to generate an optimal forest management rule that balances the benefits from commercial forest exploitation with the risks of extinction facing *Rangifer tarandus caribou*, an endangered species in central Labrador, Canada.

The second chapter, entitled "A Real Options Approach to the Protection of a Habitat Dependent Endangered Species", builds on the first chapter. We use the real options approach to determine optimally when a social planner has to stop or resume logging at some cost. The process of interruption and resume can be repeated indefinitely. For the reasonable parameters used in our application to the *Rangifer tarandus caribou*, the policy of banning logging temporarily is quite attractive as it does not require long banning periods while it drastically reduces the extinction risk and increases forest value.

The third chapter is entitled "Alternative and Indefinitely Repeated Investments : Species Choice and Harvest Age in Forestry". In this chapter, we consider the tree cutting problem of forest management as an archetypal investment problem ; it involves time, uncertainty, and irreversible actions with consequences in the future. The exercise of the option to cut a tree opens the option of planting a new one or of using the land for alternative purposes. We enrich the tree cutting problem by focusing on the planting decision. Two alternative tree species are available; for each species, the corresponding growth function is assumed deterministic but the unity price of its harvest is stochastic. In the case of a single rotation (Wicksell's problem), the forest manager should plant one species immediately if its price is sufficiently high relative to the price of the other species. However, if prices are close to each other, the manager should wait in order to avoid the mistake of planting the wrong species. In contrast, when the number of rotations is arbitrarily high (Faustman's problem), it is optimal to plant immediately one of the two species once a harvest has taken place. We show that the optimum cutting age depends on the relative timber price of the species currently planted. It varies around its Faustman's age when the relative price is below some threshold value signalling the necessity to switch to the alternative species. The land value is an increasing function of the relative price and increases with the uncertainty related to timber prices. As long as uncertainty is present, the slope of that function is continuous even around the switching threshold. On the contrary, under certainty, the slope of the land value function generally changes at the switching threshold. The stand value is similar to the value of an American option with a free boundary, an infinite expiry date, and an endogenous payoff. The computation is based on the penalty method applied simultaneously to the stand value function of each species, and on a Newton iterative process applied to the land value.

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Introduction

L'objectif de cette thèse est d'étudier les décisions d'investissements indéfiniment répétés sous l'incertitude en adoptant l'approche des options réelles appliquée à l'économie des ressources naturelles et de l'environnement. Dans ce premier chapitre, on introduit l'approche des options réelles ainsi que ses plus importantes applications dans la littérature économique. On mettra l'accent sur celles ayant trait à la gestion des ressources naturelles et de l'environnement, dont en particulier la coupe forestière, considérée comme un archétype des problèmes d'investissement.

La théorie élémentaire de l'investissement basée sur la valeur présente nette suppose implicitement que la décision d'investissement est réversible ou que c'est une décision à prendre "maintenant ou jamais". Cependant, en pratique, la plupart des décisions d'investissement sont partiellement ou complètement irréversibles, indivisibles et donnent lieu à des revenus incertains dans le futur. Dans le but d'éviter des pertes irréversibles, les firmes préfèrent reporter les investissements jusqu'à ce que l'incertitude liée aux revenus futurs soit suffisamment réduite et que la valeur présente nette de l'investissement devienne suffisamment grande. Plus précisément, la règle optimale est d'investir dés que la valeur présente nette excède le coût d'investissement augmenté d'un coût d'opportunité exprimant l'engagement ou la perte de la flexibilité. En d'autres termes, la valeur présente nette doit dépasser une valeur strictement positive qui dépend des niveaux de l'irréversibilité, de l'indivisibilité et de l'incertitude liés à l'investissement.

Conséquemment, quand une firme dispose d'une opportunité d'investissement qui est (même partiellement) irréversible et dont les revenus futurs associés sont incertains, la firme se comporte comme si elle détenait une option financière d'achat (elle a le droit mais pas l'obligation d'acheter cet actif). Quand cette firme investit la règle optimale est d'investir dés que la valeur présente nette excède le coût d'investissement augmenté d'un coût d'opportunité exprimant l'engagement ou la perte de la flexibilité. En d'autres termes, la valeur présente nette doit dépasser une valeur strictement positive qui dépend des niveaux de l'irréversibilité, de l'indivisibilité et de l'incertitude liés à l'investissement.

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Grâce à cette analogie, les origines de l'approche des options réelles remontent à l'économie financière avec Black et Scholes (1973) et Merton (1973). Cette approche a été d'abord appliquée à des projets individuels discrets où l'incertitude couvre une ou deux variables d'état qui suivent généralement un processus géométrique Brownien. Parmi les premiers articles pionniers figurent McDonald et Siegel (1985) qui déterminent la date optimale pour investir dans un projet dont la valeur est stochastique. Pindyck (1988) montre que la capacité d'une firme est à son optimum quand le coût d'une unité marginale est égal à la valeur présente nette des revenus anticipés correspondants. En concordance avec l'approche des options réelles, le coût de la capacité marginale unitaire doit inclure le coût reflétant l'exercice de l'option d'acheter une unité additionnelle de capacité. L'approche des options réelles a été par la suite largement appliquée dans d'autres domaines ayant trait à la gestion corporative moderne tels que le développement de marchés, la R&D, l'adoption de nouvelles technologies, ainsi que la gestion et l'exploitation des ressources naturelles et la protection de l'environnement. A titre d'exemples, Pindyck (2000) analyse la date optimale pour l'adoption d'une politique environnementale qui est considérée comme un investissement irréversible ayant des coûts et des bénéfices incertains pour la société. Il considère le réchauffement planétaire comme exemple d'application. Brennan et Schwartz (1985) examinent la date optimale pour exploiter ou abandonner une mine de cuivre. Motivé par l'exploitation de l'énergie éolienne, Murto (2003) évalue l'investissement dans un projet caractérisé par des revenus et un progrès technologique incertains. D'autres articles sont dédiés plus particulièrement à l'exploitation forestière comme Clarke et Reed (1989) qui analysent la rotation forestière quand la croissance des arbres et le prix du bois sont incertains. Thomson (1992) considère le cas où le prix du bois suit un processus de diffusion stochastique et généralise la règle de Faustmann exprimant la maturité financière d'un arbre, applicable dans le cas où le prix du bois est constant dans le temps.

Initialement, la littérature des options réelles se concentre sur les opportunités d'investissement uniques ou séquentielles. Récemment, on commence à traiter des situations où l'investisseur est appelé à choisir entre plusieurs opportunités alternatives. On s'aperçoit alors que chaque fois que les opportunités alternatives d'investissement ont des valeurs anticipées très proches, le décideur préfère attendre et ne rien faire durant un certain temps pour éviter de prendre une décision qui peut s'avérer regrettable plus tard. Cette inaction pourrait être optimale alors que chaque opportunité d'investissement, considérée seule, satisfait toutes les conditions pour un investissement immédiat.

On introduit dans les chapitres 2 et 3 un modèle d'opportunités alternatives d'usage où l'exercice d'une opportunité réduit la viabilité de l'autre. En fait, on considère un manager (un planificateur social) qui maximise les revenus provenant de deux usages alternatifs de son actif (une forêt) : L'exploitation forestière (coupe de bois) et la protection d'une forme de biodiversité. Plus précisément, on détermine optimalement quand le manager doit arrêter ou reprendre l'exploitation de la forêt constituant un habitat stochastique d'une espèce menacée d'extinction. L'exploitation forestière augmente le risque d'extinction de l'espèce alors que la suspension de l'exploitation ne réduit que partiellement ce risque puisque l'espèce demeure naturellement menacée. Le manager prend des décisions successives d'exploitation et de suspension aussi longtemps que l'espèce est en existence ; cependant, le manager exploite définitivement la forêt si l'espèce subit l'extinction. Ainsi, ce modèle illustre un processus de prise de décisions répétées qui peut s'interrompre définitivement. Il est calibré et appliqué au cas du Caribou Rangifer tarandus, une espèce du centre du Labrador (Canada) menacée d'extinction.

En vue d'étudier davantage les opportunités d'investissement indéfiniment répétés, on introduit dans le chapitre 4 un modèle où la prise de décision donne naissance à plusieurs nouvelles opportunités d'investissement. Bien qu'une telle situation constitue une pratique courante (ex. le remplacement d'équipements), elle se heurte à des difficultés de résolution qui explique pourquoi elle n'a pas été investiguée théoriquement dans le passé. Pour ce faire, on reconsidère le problème de la coupe forestière en introduisant deux essences d'arbres dont les prix du bois suivent deux processus stochastiques (géométriques browniens) différents. Le manager forestier doit alors évaluer à chaque instant la valeur de son actif (la forêt) et décider de sa gestion optimale : À quel moment il faut procéder à la récolte ? Faut-il attendre après la récolte ? Faut-il replanter la même essence ou planter l'essence alternative ?

The objective of this thesis is studying indefinitely repeated investment decisions under uncertainty using the real options approach applied to the economics of natural resources and the environment. In this chapter, we first introduce the real options approach as well as its main applications in the economic literature. We focus on applications related to natural resources and the environment and particularly to the tree cutting problem as an archetypal investment problem.

The elementary investment theory based on net present value supposes implicitly that the investment decision is reversible and that it is a "now-or-never" decision. However in practice, most investment decisions are partially or completely irreversible, indivisible, and yield uncertain future rewards. In order to avoid irreversible loss, firms would like to delay investments as long as this helps reduce the uncertainty surrounding future rewards while increasing its net present value. More precisely, the rule to invest is whether the project present value exceeds or not the investment cost augmented by an opportunity cost expressing the "engagement" or loss of flexibility. In other words, the net present value should exceed a strictly positive amount that depends on irreversibility, indivisibility and uncertainty levels of the investment.

Hence, when a firm holds an investment opportunity which is (even partly) irreversible and whose future rewards are affected by uncertainty, the firm is acting as if it held a financial call option (it has the right but not the obligation to buy an asset). When the firm invests and buys the asset, it exercises its financial option. In this sense, holding an investment opportunity is analogous to holding a financial option and the corresponding area of research has been named "real options".

Due to the analogy with financial options, the real options approach has its foundations in financial economics with Black and Scholes (1973) and Merton (1973). Leading articles are those of McDonald and Siegel (1985) who determine the optimal timing to invest in return for a project whose value evolves stochastically. Pindyck (1988) shows that the firm's capacity choice is optimal when the cost of the marginal capacity unit equals the present value of the corresponding expected cash flow. According to the real options approach, the cost of the marginal capacity should include the cost of exercising the option of buying an additional capacity unit. The real options approach has then been largely applied to many areas relevant to modern corporations such as market development, R&D, technology adoption, as well as in natural resource management and exploitation and the environment protection. For instance, Pindyck (2000) analyzes the optimal timing to adopt an environmental policy, seen as an irreversible investment with uncertain costs and benefits on the society and considers Global Warming as an example of applications. Brennan and Schwartz (1985) examine the optimal time of invest in a copper mine and to abandon it. Motivated by wind energy features, Murto (2003) evaluates the investment in a project with uncertainty covering both the revenue stream and future technological progress. Some articles are more closely related to the forest rotation. This is the case of Clarke and Reed (1989) who consider the forest rotation problem tree growth and timber price are stochastic. Thomson (1992) considers the optimal forest rotation timber prices follow a diffusion Process as the Faustmann rule for financial maturity is applicable when timer prices are constant over time.

While the real option literature initially focused on unique investment opportunities or sequential investment opportunities, the recent literature has begun to treat situations where the decision maker has more than one opportunity to choose from. An interesting result in such situations is that a new reason for postponing action arises. Whenever the alternatives are too close to each other and uncertain, the decision maker will wait in order to avoid making a decision that might prove erroneous in the future. This inaction may be optimal although each project, taken in isolation, would satisfy the requirements for immediate investment under conditions of irreversibility and uncertainty.

In chapters 2 and 3, we introduce a model with alternative usages of the same asset : The forest provides timber and a habitat for an endangered species. More precisely, we consider a forest manager (a social planner) who is maximizing its revenues from two alternative usages of her asset (the forest) and to determine when the harvest must take place, whether or not it is optimal harvest timber or protect a form of biodiversity. More precisely, we determine optimally when a forest manager has to ban temporarily harvesting a forest that provides a stochastic habitat to an endangered species. The forest harvest increases the extinction risk of the endangered species while the ban on harvesting reduces but only partially that risk as the species is naturally threatened. The manager decides when to ban harvesting and when to resume it as long as the species is in existence. This is a situation where the social planner can make repeated investment decisions but the decision process may stop definitely in the future. The model is calibrated and applied to the to the Rangifer tarandus caribou, an endangered species in Central Labrador (Canada).

In order to further analyze indefinitely repeated investment decisions, we introduce in chapter 4 a model where exercising an option opens up new options indefinitely. Although this is a most common practical situation (e.g. equipment retirement) it raises difficulties that explain why it has never been investigated theoretically before. To do so, we reconsider the tree cutting problem and introduce two alternative tree species whose timber prices follow two stochastic processes (geometric Brown). The social planner has to evaluate at each instant her asset (the forest) value and to determine when the harvest must take place, whether or not it is optimal to wait before replanting, and which species has to be planted.

Chapitre 1

A Real Options Approach to Forest-Management Decision Making to Protect Caribou under the Threat of Extinction

1.1 Introduction

Managing for ecosystem services, such as timber and wildlife, is fraught with various forms of uncertainty. Sources of uncertainty include : uncertainty of external disturbances, such as wildfire, uncertainty of how systems will respond to change, such as ecological change from timber extraction, and uncertainty about the underlying structure of systems given that there is incomplete knowledge of how ecosystems respond to forest management (Walters 1986). The irreversibility of forestry and wildlife decisions complicates the management of forest ecosystems and wildlife populations. For example, once the decision is made to cut old growth forests the option to preserve the forest is lost (Conrad 2000). Alternatively, a decision to stop harvesting would preclude the socio-economic opportunity of timber extraction that may benefit a resource dependent community's well being. Trading off the socio-economic risks of preservation and ecological risks of timber harvesting is a fundamental challenge for resource management decision making. There is a need for methods to deal with the risk, uncertainty and irreversibility of resource management decision making. To investigate this need we adapted an approach used in economics to evaluate financial decision making called real options (Dixit and Pindyck 1994).

The risk and uncertainty associated with management decisions are included in a real options problem formulation (Dixit and Pindyck 1994). A real option is defined as "the value of being able to choose some characteristic (e.g. the timing) of a decision with irreversible consequences, which affects a real asset (as opposed to a financial asset)" (Saphores and Carr 2000). Under real options, problems are structured so that they can be solved by numerical methods. It has been applied in resource management related decision making, such as species re-introduction (Bakshi and Saphores 2004), biodiversity (Kassar and Lasserre 2004), forest harvesting (Insley 2002), land use decisions (Marwah and Zhao 2002) and climate change (Boyer et al 2003). The real options approach uses a flexible approach to uncertainty by identifying its sources, developing future scenarios and by the construction of decision rules (Boyer et al 2003). It attempts to reduce risk by monitoring the implementation of its decisions and requiring decision making to be adaptive throughout the lifecycle of a project. To gain the most benefit from a venture, be it a business venture or the management of a natural resource, real options values future opportunities. For example, real options treats the loss of a wildlife species like losing a future economic opportunity, be it monetary or otherwise, by recognizing that the species may have some future value (Kassar and Lasserre 2004).

Real options explicitly accounts for uncertainty in the determination of an optimal decision given the stochasticity of an asset's value. For example, Brennan and Schwartz (1985) looked at the decision to open, close and abandon a mine where the price and hence the value of the resource is stochastic. Pindyck (2000) evaluated the timing of adopting an environmental policy when, due to climate change, there are uncertain costs and benefits to society. Irreversibility of resource management decisions are also considered by real options, such as the evaluation of harvesting old growth forests given their amenity value (Forsyth 2000), growth, and timber price (Clarke and Reed 1989 and Reed and Clarke 1990). Marwah and Zhao (2002) examined the problem of irreversibility and uncertainty in land acquisition for wildlife conservation. Conversion of land from other uses, such as agriculture or forestry, may be irreversible, with costs invested in preserving the land being lost if that area becomes unsuitable for wildlife. Irreversibility infers unintended change, where something of value is lost and must be considered when making risky decisions. Thus the decision maker prefers to delay making a decision in order to get better information about its consequences. However, by waiting, opportunities may deteriorate or be lost, such as vanishing chances to gain financial revenues or the decline in the population of a valued wildlife species. In the case of Marwah and Zhao (2002), they used real options methods, in their determination of the optimal timing and amount of land to purchase for conservation, to evaluate the effect of different decision maker strategies for managing uncertainty. They found that the timing of land investments was highly dependent on how a decision maker synthesized existing and new information, where a decision maker that actively preserves land and thereby learns about its costs and benefits to conservation objectives produces a more optimal solution than a passive decision maker that waits for better information.

In this analysis, we demonstrate how a decision maker would make use of the real options method to evaluate the trade-off between harvesting timber and maintaining sufficient old forest to support an endangered woodland caribou (*Rangifer tarandus caribou*) population in central Labrador, Canada. We focus on the interaction of natural wild fire and timber exploitation increasing the probability of local caribou extinction, and a community forgoing timber revenues when forest harvesting is banned. To apply this methodology the decision maker tracks the amount of caribou habitat and determines the optimal point to stop timber harvesting, given the uncertainty of future natural disturbances and the amount of habitat required to support a viable caribou population. The optimal stopping time is also informed by the social trade-off of maintaining the economic benefits to the community of timber harvesting and the conservation benefits of woodland caribou. The methodology presented in this paper is a component of a broader sustainable forest management project focused on central Labrador (Sturtevant et al 2007).

Central Labrador's Red Wine Mountains caribou population declined significantly from over 700 animals in the 1980s to 151 by 1997 (Schaefer 1999). In 2002, the population was listed as "threatened" under both the Endangered Species Act of Newfoundland and Labrador and under the Canadian federal Species at Risk Act (Shmelzer et al 2004). Caribou are an integral part of the communities of central Labrador. Caribou meat has historically constituted a large portion of the diet of the central Labrador people. The Innu, the local first nation, have a strong spiritual and cultural connection with the caribou (Schmelzer et al 2004, Armitage 1992). As a conservation measure, the non-subsistence hunting of the Red Wine Mountains herd stopped in 1972, and subsistence hunting stopped in 2002 (Schmelzer et al 2004). The George River herd, one of the worlds largest at $600,000$ to $800,000$ animals (Couturier et al 1996), is considered healthy and is legally hunted. However, the ranges of the George River and Red Wine Mountains herds overlap and the animals are nearly indistinguishable. When the two herds mix the Red Wine Mountains caribou are frequently mistaken for the George River animals and shot, contributing to their decline (Schaefer 1999, Schaefer et al 2001, Schmelzer et al 2004).

Currently, there is a proposal to substantially expand forest harvesting, some of which is planned within the Red Wine Mountains caribou herd's historic range (Labrador Department of Forest Resources and Agrifoods 2003), and it is anticipated that it will increase the livelihood of local communities. Past timber exploitation has not been implicated in the decline of the Red Wine Mountains caribou herd, it has been recent and small-scale. Historical hunting of the caribou has been identified as the main agent in the reduction of the herd's population (Schmelzer et al 2004). Notwithstanding, commercial forestry has negatively affected caribou population dynamics and behavior across the boreal forest (Chubbs et al 1993, Seip and Cichowski 1996, James and Stuart-Smith 2000, Schaefer 2003) and there is a concern that the expanded commercial forestry and associated human activity in the Red Wine Mountains caribou's range may further compromise their viability. As a measure to protect the endangered Red Wine Mountains herd the local resource management plan has set aside large areas free from forest harvesting. However, it is uncertain if this is enough to ensure the caribou's survival.

Sedentary caribou (Bergerud 1988), such as the Red Wine Mountains herd, exist in low numbers. Lichen is a key food source for caribou, however its availability is not considered to be the limiting factor in the persistence of caribou populations, instead it is the distribution of winter and summer habitat (Seip and Cichowski 1996). Caribou need to be spatially separated from their predators and require large tracks of undisturbed forests. If caribou are confined to small areas they are easier to find by predators (Seip 1991). Smith et al (2000) found that as forest harvesting progressed in a landscape occupied by caribou, their daily movement rates and winter range size decreased and they avoided recently fragmented areas. Conservation research has shown that a species decline is typically associated with a degradation of its range (Channell and Lomolino 2000). Forest harvesting also causes a shift in the forest age structure to a higher occurrence of young forest, which is more favorable to other ungulates, such as moose (Alces alces; Rempel et al 1997). With a larger prey base the wolf (Canis lupus) population expands, preying on moose and the resident caribou (Bergerud and Elliot 1986, Seip 1992, Seip and Cichowski 1996). Wolves are the main non-human source of mortality for the adult Red Wine Mountains caribou (Schaefer 1999). Recently, it has been observed that central Labrador's moose population has increased while the number of Red Wine Mountains caribou has declined (Schaefer 1999). An additional concern of expanded commercial forestry is the increased road density. The efficiency of predators in a harvested landscape is facilitated by the increase in the number of roads and trails that result from forest harvesting activities (James and Stuart-Smith 2000). Further, access, facilitated by increased road densities, may increase poaching and the legal hunting of the George River herd and the overlapping Red Wine Mountains caribou.

There are conflicting social and ecological risks for the Labrador study area. There is the ecological risk of the Red Wine Mountains caribou becoming extinct with expanded commercial forestry. As well, there is a risk that socio-economic opportunity will be lost if the decision to stop timber harvesting is made too soon. The timing of the decision to stop harvesting, before a critical minimum amount of caribou habitat is lost, is also dependent on the amount of certainty that the decision maker has about the viability of the caribou population under various levels of forestry activity. The decision maker needs to evaluate the certainty associated with how natural ecological dynamics may interact with timber harvesting and how it may undermine the amount of caribou habitat. In the social domain, the decision maker needs to consider what level of risk society is willing to accept given their interest in both a viable caribou population and the economic benefits that forestry brings to their community.

In this study we describe a real options methodology and how it deals with risk, uncertainty and irreversibility. Also presented is background on data requirements and how parameters were calculated for the real options model. Caribou habitat is captured by using a coarse habitat indicator. The real options methodology, assumptions used and model are presented, followed by a discussion on the utility of the approach in resource management decision making and specifically in the Labrador study area.

1.2 Methods

1.2.1 Study Area

Our study area is defined as Labrador's District 19A. The Red Wine Mountains caribou herd overlaps this area (Figure 1). The local forest management plan outlines strategies to expand commercial forestry and to protect the resident woodland caribou habitat (Labrador Department of Forest Resources and Agrifoods 2003).

The study area is approximately 2 million hectares, and is located in the lower section of the Churchill River Valley and the coastal plain surrounding Lake Melville (Figure 1). Human impact, primarily roads, historic low levels of timber harvesting and human caused fires, is mainly confined to the area on the north side of the Churchill River (Forsyth et al 2003). The area has cool summers and cold winters and is the most heavily forested area of Labrador. Fire is infrequent and patchy and the area's disturbance regime is dominated by individual tree mortality. This results in a mix of age and cover types, and a multi-layered canopy. The landscape is dominated by lakes, rivers and wetlands, with forests of black spruce (Picea mariana) and balsam fir (Abies balsamea), open sphagnum forests, lichen woodlands, black spruce bogs and birch (Betual papyrifera) trembling aspen (Populus tremuloides) and balsam poplar (Populus balsamifera) hardwoods (Wilton 1965).

The resident woodland caribou prefer mature forests that have high lichen abundance. Their main non-human predators are wolf and black bears (Brown 1986, Schaefer et al 2001). There are specific caribou reserves that exclude timber harvesting in the current Labrador District 19A forest management plans. With very little commercial development in the area, the people, the land and the caribou still reflect historic patterns and interactions.

1.2.2 Model Overview

We used a mean-reverting process (Dixit and Pindyck 1994) to describe the expected amount and temporal variability of caribou habitat. Mean-reverting processes incorporate the volatility and the speed of reversion to the mean of the system being assessed. In our application, the volatility and speed of reversion are dictated by the forest dynamics from natural and human sources. The stopping rule used in our real options problem formulation specifies the timing of closing down harvesting when the amount of caribou habitat approaches a critical minimum habitat threshold. The timing of the decision to end harvesting reflects the social trade-off between the loss of the socioeconomic opportunity from timber harvesting and the risk of having insufficient habitat to maintain the caribou population. The timing is sensitive to the valuation by society of both the existence of the caribou herd and the benefits associated with timber exploitation; it is also sensitive to the uncertainty associated with the system. Decision makers may be more cautious if their understanding of the system is limited, for example, if they lack confidence about the extent and frequency of forest disturbance, or if they have reservations about how a caribou population will respond to timber harvesting activities.

1.2.3 Ecological Context

To construct the real options model we first examined the ecological dynamics of the central Labrador study area and we investigated the habitat requirements of the Red Wine Mountains Caribou herd. To model the trade-off between caribou habitat and timber harvesting, the real options methodology needs to capture the inherent ecological and forest management processes and ecological boundaries of the system. Tracking the amount and variability of habitat through time is central to the model. As well, the model requires some approximation of a minimum amount of habitat below which the caribou's survival becomes questionable.

Caribou Habitat

To maintain a species a minimum viable population is required. This is characterized as a population that can exist without facing extinction from natural disasters or demographic, environmental or genetic stochasticity (Shaffer 1981). The model used in this study focused on using the amount of habitat as a surrogate for maintaining a viable population. As commercial harvesting progresses across the landscape the forest would become increasingly fragmented and roaded increasing caribou mortality by compromising their ability to spread out across the landscape to avoid predators and exposing them to incidental hunting. We assume that other measures, such as hunting restrictions, required to protect the caribou would be maintained. Caribou habitat typically includes large contiguous areas of old forest with terrestrial lichen, peat land and bog complexes with a minimum of human disturbance (Schmelzer et al 2004, Johnson et al 2002, Seip 1998). To capture their old forest requirement we characterized forest stands greater than 160 years old as woodland caribou habitat. This habitat requirement happens to be the one most directly affected by forest management.

It is widely considered that caribou require some minimum amount of habitat to survive in landscapes with commercial forestry, although the exact level is hard to identify (Seip and Cichowski 1996, Smith et al 2000, Schaefer 2003). Attempts have been made to identify the critical amount of habitat required for a species to persist (Fahrig 2001). However, in field studies a minimum habitat threshold is often not known until crossed (Carpenter et al 2001). Given the challenges of explicitly identifying an appropriate habitat threshold, we relied on theoretical limits. The conservation literature suggests that, in general, a landscape becomes fragmented to an organism when less than 30% of its habitat is intact (Andrén 1994, Fahrig 2002). We chose 30% of the expected amount of old forest as our critical threshold due to the overwhelming impact that fragmentation has on caribou population viability. We recognize that the actual minimum threshold for the Red Wine Mountains caribou may be different due to landscape and species differences when compared to the theoretical literature (Andrén 1999).

Habitat Dynamics

The expected amount of old forest, and therefore habitat, can be estimated based on the frequency and extent of stand replacing disturbance, fire and timber harvesting, and forest growth. We used a landscape level disturbance simulation model (Fall et al 2004) to estimate the expected amount and variability of habitat. In overview, the simulation model captures both landscape disturbance and forest management and generates indicators of forest structure through time. It was implemented using the SELES (*Spatially Explicit Landscape Event Simulator*) modelling system (Fall and Fall 2001). This software is a flexible tool for building and processing grid-based, spatiotemporal models. Fire was modelled based on its historic frequency and extent (Sturtevant, unpublished data) and the forestry regime was modelled based on the rate of harvest as described in current forest management planning documents (Forsyth et al 2003). The simulation model was then used to conduct a set of Monte Carlo simulations, based on the fire and harvesting regimes, generating 100 data sets containing the amount and variability of caribou habitat through a $1,000$ year period. This data was then used to generate the real options model parameters.

Fire is the main natural disturbance agent in the Boreal Forest (Johnson 1992, Payette 1992). To estimate the fire regime we used Labrador provincial forest fire data for the study area (Sturtevant, unpublished data). Based on the size and frequency of fires in the past thirty five years a fire rotation - the amount of time required to burn an area equivalent to a study area - of 343 years was calculated. This fire rotation is consistent with the longer 500 year rotations reported for the wetter east coast of Labrador (Foster 1983), and is longer than the $123 - 273$ year rotations in the drier areas to the west in central Quebec (Bergeron 2001). To aid in parameterization of the landscape simulation model the theoretical mean expected amount of forest greater than 160 years, our caribou habitat, can be calculated using a negative exponential distribution (Van Wagner 1978, Johnson et al 1995). The equation has the form $A(t) = e^{-t/b}$, where t is the time since last disturbance (in years) and b is the disturbance cycle (343 years). Solving the equation $A(160) = 1,117,327e^{-160/343}$ of forested area, gives the estimate of $700,800$ hectares of forest older than 160 years. Given that we have defined the critical minimum habitat threshold required to maintain the Red Wine Mountains caribou population as 30% of the expected amount of old forest, we can calculate the minimum amount of habitat as 30% of $700,800$ hectares ($H_e = 210,240\ ha$).

To parameterize the simulation model estimates of the fire cycle and number of fires per year are needed (Fall et al 2004). The number of fires per year is equal to the forested area divided by the mean fire size and by the fire cycle. Using the 343 year fire cycle and a $1,003\ ha$ average fire size, calculated based on the size of fires in the study area over the past 35 years (Sturtevant, unpublished data), would imply a mean of 3.25 fires per year. Using the expected mean number of fires and mean fire size the simulation model selects the fire size and number of fires per simulated year from an exponential distribution (Fall et al 2004). It was found that simulated fires did not burn as large as expected due to fire starts on islands and other various barriers to fires reaching their full extent. To force the model to match the extent of burning predicted by theory the simulation model was adjusted to burn more frequently by using a 295 year return interval and a mean of 3.78 fires per year, which leads to an age class structure consistent with a 343 year return interval.

The simulation model uses forest growth and harvesting sub-models to characterize the study area's timber management regime and spatially captures timber harvesting blocks and road networks. With each simulated time step the forest is aged and timber volume is calculated based on growth and yield projections (Newfoundland and Labrador Department Natural Resources, unpublished data). Planned harvesting activities are modeled by specifying an annual volume of timber to be harvested. An "oldest first" harvest rule is used to prioritize forest stands for harvest.

1.2.4 The Real Options Model

Formulation of the model

While the current amount of habitat (H) is known at any point in time, uncertainty exists about future projection of H. Its behavior is not totally random, as it tends towards some average long-run level, which differs depending on the presence of timber harvesting. Due to the many uncertainties that affect the future amount of caribou habitat, such as wildfire, it becomes impossible to know at what speed the long-run equilibrium level is being approached. Further, the amount of habitat may fluctuate through time, such that it oscillates around an expected level.

The expected amount of old forest, and therefore habitat, was estimated using the results of the landscape level forest disturbance simulation modeling. The habitat time series generated by the Monte Carlo simulations was analyzed to determine the most appropriate stochastic process that describes the habitat dynamics. Using the econometrics software EVIEWS, one can show that the Neperian logarithm h of the habitat amount H ($h = Log(H)$) follows approximately a mean-reverting process. The mean-reverting process is characterized by the speed of reversion α, the volatility σ^2 and the equilibrium long-run level h, such that

$$dh = \alpha(\underline{h} - h)dt + \sigma dz \tag{2.1}$$

where dz is the increment of a Wiener process, a stochastic term with a normal distribution, and dt is the change in time. Thus dh represents the change in h over a time interval dt. The component $\alpha(\underline{h} - h)dt$ reflects the deterministic behavior of the amount of habitat, what would happen absent of any stochasticity. The deterministic component is null when h is equal to its equilibrium long-run level ($h = \underline{h}$). When h is higher than its equilibrium long-run level $\alpha(\underline{h} - h)dt$ it is negative. This implies a reduction in h; when h is lower than h the opposite is true. In both cases the speed of adjustment is proportional to the gap $\underline{h} - h$ and to α the speed of reversion parameter.

The stochastic component

$$\sigma dz = \sigma \varepsilon \sqrt{dt}$$

reflects the unpredictable natural variation due to external effects, such as fire, where $\varepsilon\,\tilde{}\,N(0,1)$.

If the log of the amount of habitat is h_0 at time zero, the expected amount of habitat at any time t is

$$E_0(h_t) = \underline{h} + (h_0 - \underline{h})e^{-\alpha t}$$

and its variance is

$$VAR_0(h_t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

Over the long-run, the amount of habitat is expected to fluctuate on average around h, a level that will differ based on the presence or absence of timber harvesting.

To address the impact of forestry (harvesting being allowed or banned) on habitat, we assume that the forestry regime affects only the level of h. Depending on whether timber harvesting is prohibited or not the long-run amount of habitat h may take two possible values, h_a or h_b. h_a is the long-run amount of habitat when timber cutting is allowed whereas hb is the long-run amount of habitat when timber harvesting is banned. That is

$$\Delta h \equiv h_a - h_b < 0$$

represents the long-run effect of timber harvesting on the amount of caribou habitat.

Whether forestry is allowed or not, the amount of habitat may decline due to forest dynamics, such as wildfire, to a critical amount h_e at which time the caribou herd may become extinct. However timber harvesting, by reducing the long-run equilibrium level of habitat, will cause h to be on average lower than if timber exploitation was not allowed, thereby increasing the probability of the resident caribou becoming extinct. Consequently, the decision maker has to balance the benefits of allowing timber harvesting against the costs in terms of an increased probability of caribou extinction.

Parameterization

The critical minimum habitat threshold, to maintain a viable population of caribou, has been estimated based on the ecological literature as $h_e = 12.26$ ($H_e =$

210, 240 *ha*). The stochastic process parameters were estimated using the Maximum Likelihood Approach (Gourieuroux and Jasiak 2001). Using this process when harvesting is banned $h_b = 13.45$, $\alpha = 0.0532$ and $\sigma = 0.0529$. Similarly when harvesting is allowed $h_a = 12.70$, $\alpha = 0.0575$ and $\sigma = 0.0533$. We can assume that $\alpha = 0.05$ and $\sigma = 0.05$ and that they are almost independent of the harvesting regime.

The Objective Functions

We assume that the local community is concerned about the extinction of the Red Wine Mountains caribou herd, but they are less concerned about the number of caribou. Caribou provide a constant instantaneous utility, denoted by s to the community as long as they exist and zero if they become extinct. Further, we denote an instantaneous utility, denoted by ρ, to the sale of timber. We assume that the decision maker espouses the objectives of the community. Then, the Net Present Value of the forest when harvesting is banned is the expected utility flow due to the caribou's existence, or implied by their extinction, and is

$$V(h) = E \int_0^{+\infty} s_t e^{-rt} dt \tag{2.2}$$

Where s_t is defined as follows

$$\begin{aligned} If\ \forall s &\in [0,t],\ h_s > h_e\ then\ s_t = s \\ If\ \exists\forall s &\in [0,t],\ h_s = h_e\ then\ s_t = 0 \end{aligned} \tag{2.3}$$

This means that the utility flow associated with the caribou herd is s as long as h remains higher than the critical habitat threshold value h_e, and becomes null forever as soon as that level is reached, even if there is a recovery in the amount of habitat. As the Net Present Value $V(h)$ depends on the moment when the instantaneous utility st shifts from s to zero, then $V(h)$ depends on the stochastic process followed by h which is itself determined by the forestry policy regime. Prohibiting timber harvesting decreases the probability of caribou extinction but implies foregoing future timber revenues as long as the prohibition is effective. We assume that, once decided, the prohibition of timber harvesting is irreversible, costless and applies forever. We assume that under the harvesting regime, the forest provides a constant revenue flow of ρ by unit of time.

We assume that timber harvesting will be allowed at the beginning of the model. Further, we are investigating a forest which has not been extensively commercially exploited before, so we assume that h is initially high and it is possible to start harvesting timber without threatening the caribou with extinction for some time into the future. However, as forestry proceeds h will diminish according to the process discussed above. At some stage habitat may become dangerously close to the threshold h_e and timber harvesting may have to be banned to reduce the risk that the level of habitat falls below h_e.

When harvesting is still allowed, the decision maker has an option to prohibit harvesting. Once in place the ban is irreversible. Let F be the value of the forest when harvesting is allowed and the caribou still exist. The value of F includes the option to prohibit harvesting once and for all and stems its value from utility provided by caribou existence and timber revenues. F is entirely anticipated by the decision maker based on the expected future amount of habitat according to the stochastic process governing h. F is then a function of h. This value is enhanced because of the flexibility of the decision maker to improve caribou protection by banning harvesting. The decision maker must choose a decision rule that will yield the optimal future time to prohibit harvesting. To achieve this goal the following maximization problem needs to be solved by choosing the banning date

$$F(h) = \max_{T} E\left\{e^{-rT}V(h_T)\right\} \tag{2.4}$$

The solution must satisfy the Value Matching (VM) and Smooth Pasting (SP) conditions

$$(VM) \quad : \quad F(h^*) = V(h^*) \tag{2.5a}$$

$$(SP) \quad : \quad F'(h^*) = V'(h^*) \tag{2.5b}$$

The decision maker decides to prohibit timber harvesting at the time T when the amount of caribou habitat hits, for the first time, a stopping threshold h^*. This threshold is "sufficiently" above the critical minimum habitat threshold h_e. h^* should depend negatively on α but positively on h_a, h_b and h_e. To decide when to ban harvesting, the decision maker has to monitor the current amount of caribou habitat and ban harvesting forever as soon as h hits the critical value h^*. In order to characterize the function $F(h)$, we first need to determine the expression of $V(h)$ where the log of the habitat amount h is governed by the differential equation

$dh = \alpha(h_b - h)dt + \sigma dz$

We focus on the case of interest $h > h_e$; then,

$V(h) = E\left(\int_0^{dt} se^{-rt}dt + \int_{dt}^{+\infty} se^{-rt}dt\right)$

When neglecting terms of smaller order than dt, we obtain

$V(h) = sdt + e^{-rt}EV(h + dh)$

$\Rightarrow V(h) = sdt + (1 - rdt)E\left[V(h) + V'(h)dh + \frac{1}{2}V''(h)dh^2\right]$

$\Rightarrow \frac{E(dh)^2}{dt}\frac{V''(h)}{2} + \frac{Edh}{dt}V'(h) - rV(h) + s = 0$

$\Rightarrow$

$$\frac{\sigma^2}{2}V''(h) + \alpha(h_b - h)V'(h) - rV(h) + s = 0, \forall h > h_e$$

along with the boundary condition

$$V(h_e) = 0$$

This boundary condition is said to be "absorbing", when h hits h_e for the first time. In this case, the caribou becomes extinct, due to a lack of habitat, and the stochastic process becomes irrelevant (one says it is terminated or "killed").

To compute $F(h)$, consider an initial value h_0 of h such that $h_0 > h^* > h_e$. The habitat h will then evolves until it hits h^* at time $T \in [0, +\infty[$. At any time before T i.e. for any $h > h^*$, $F(h)$ satisfies the Bellmann's Equation

$F(h) = (\rho + s)dt + e^{-rt}EF(h + dh)$

where h becomes $h + dh$ during the infinitesimal time period dt. Thus, $F(h)$ is governed by the following partial differential equation, that is

$$\frac{\sigma^2}{2}F''(h) + \alpha(h_a - h)F'(h) - rF(h) + \rho + s = 0, \forall h > h^*$$

along with the following value matching and smooth pasting boundary conditions

$$\begin{aligned} F(h^*) &= V(h^*) \\ F'(h^*) &= V'(h^*) \end{aligned}$$

The first conditions says that, when forestry is costly banned, the value of the forest stems from the existence of the caribou only, as described above. The second condition requires this to happen "smoothly", as it can be shown that it would not be optimal to ban forestry at that time if that condition was not met.

The complete resolution consists in computing h^*, once $V(h)$ and $F(h)$ have been determined. The decision maker will then be able to ban harvesting optimally at some time in the future as follows : they would monitor the amount of caribou habitat over time and ban timber exploitation the first time h hits the threshold value h^*. Hence, the Decision maker can not predict the time banning will be applied as it depends on the future realizations of h. It is worth noting that banning may never happen if over time h diverges from h^*.

1.3 Results

The real options model was applied to the Labrador study area by computing numerically $V(h)$, $F(h)$ and the stopping threshold h^*. The real options analysis involves generating a series of model runs that are then interpreted by a forest manager. The runs vary, as reflected in the value of current habitat h, due to the stochastic nature of the model, which captures the combined dynamics of the forest management regime and natural disturbance. A decision maker evaluates model output and gains insights into how the amount of caribou habitat fluctuates in time according to various random future events.

To illustrate the real options methodology we present three examples of our real option model output. The examples show the critical threshold h_e's importance and sensitivity. It balances the benefit from continuing harvesting while keeping caribou under an acceptable, but perhaps higher, probability of extinction on the one hand with the economic loss from prohibiting harvesting in order to reduce this probability on the other hand. Table 1 shows the parameters used in the real options model for the Labrador study area. We have assigned an instantaneous value from timber (ρ)

to 1, the instantaneous value of caribou (s) has been assigned 2 to demonstrate the real options methodology. The values of s and ρ reflect how risk averse the Labrador decision maker is, a higher value of s will stop harvesting sooner, and lesser values will stop timber extraction later. Finally, the interest rate is taken to be equal to 5%. The numerical resolution of the model leads to $h^* = 12.2670$, which reflects the Net Present Value of the instantaneous utility value of s and ρ.

Figures 2 through 4 illustrate three possible realizations of the habitat amount over a period of 1000 years. Each realization is obtained by generating h using the differential equation of dh and starting from the current habitat amount (h_0). Among many possible realizations generated this way, we have chosen examples that demonstrate how the model works and how decisions can be made.

As previously discussed, depending on the future realization of h, starting from its present value $h_0 > h^*$, harvesting is irreversibly banned as soon as h hits h^*. The examples (Figures 2 to 4) illustrate how the time to ban harvesting is a stochastic variable taking value for the interval $[0, +\infty[$. Initially harvesting is allowed and the current value of h is h_0. This is sufficiently larger than h_c to justify harvesting, at least for a while.

Figure 2 illustrates an outcome that maintains caribou habitat above h^* and thus allows for timber harvesting. According to this future realization of h, with h above h^*, harvesting will be allowed for at least 1000 years and there will be enough habitat, theoretically, to sustain the caribou population.

In Figure 3, the 2^{nd} future realization of h, h diminishes to h^* in approximately 300 years. This implies that harvesting must be prohibited at that time because it is getting close to the critical minimum habitat threshold h_c. Prohibiting harvesting when h hits h^* will let h be governed by the new stochastic differential equation

(1.6a) instead of (1.6b) where

$$dh = \alpha(h_a - h)dt + \sigma dz \tag{1.6a}$$

$$dh = \alpha(h_b - h)dt + \sigma dz \tag{1.6b}$$

As $h_b > h_a$, prohibiting harvesting will give a better chance for h to rise and stay away from h_e. This does not guarantee that the caribou herd will not become extinct over the long run but makes that outcome less probable.

In Figure 4, the 3^{rd} possible future realization of h shows that prohibiting harvesting after almost 800 years does not succeed in keeping h above h_e as the amount of habitat declines to the critical minimum habitat amount threshold (h_e) at which time the caribou herd hypothetically goes extinct despite harvesting being banned. In the graph, the habitat amount is maintained equal to h_e after the caribou have gone extinct to underline the fact that extinction is an irreversible event.

1.4 Conclusion

The real options model that we have described focuses on uncertainty and irreversibility in a dynamic context. The irreversibility refers to events such as extinction or decisions such as banning timber harvesting. The uncertainty pertains to how the amount of caribou habitat will change : the current level of habitat is observable ; its future level is unknown. The model specifies the objective of the decision maker as that of maximizing future benefits from the forest, given uncertain benefits derived from caribou and from timber exploitation. For this model, the sole instrument that the decision maker can use to achieve the objective of maximizing future benefits is to ban timber harvesting forever, a decision to be taken at some future time. Given the uncertainty surrounding the future, the decision maker would be mistaken to specify a definite future date at which harvesting should be banned : the habitat level might be more than adequate at that date, hence no ban is required; or it might become dangerously low before that date, in which case the ban should be introduced earlier. Consequently, the decision maker should not choose a date but a decision rule to be applied at all future dates. That rule consists in observing future levels of habitat and banning timber exploitation the first time the threshold value H^* is reached. While the date at which this may happen is uncertain, the threshold itself is not random. It is computed according to the various ecological, biological, and economical parameters of the model and according to the stochasticity of the process governing H.

Analyses of real options models indicate some interesting properties of the decision rule and the threshold value. It can be shown that, by applying the stopping rule, the decision maker attempts to give their community exposure to favorable outcomes (cases when H grows more than expected), while seeking protection from unfavorable outcomes (low growth), thus maximizing future benefit. In practice, this is achieved by banning harvesting when H is still sufficiently above the critical minimum habitat threshold. This does not guarantee that the caribou will not be extirpated from the study area, but will increase their chance of survival. In that respect the model is a rigorous application of the precautionary principle. It does not prohibit risk taking, but agrees with the intuitive conventional wisdom that decisions should bend the distribution of risk a community is exposed to in a way that reduces the probability of irreversible catastrophes and thereby maximizing the future benefits of an active forest industry and the existence of woodland caribou.

One key parameter in the model is the critical minimum habitat threshold (He) below which the caribou, according to the underlying biological theory, becomes extinct. This value is exogenous to the model as it is determined by ecologists. Although certain according to the theory, it is not known accurately in practice. Underestimating He may lead to accidental extinction as the model would wait too long before prohibiting harvesting. Too high an estimate may appear wise but implies foregoing timber revenues unnecessarily. In fact the choice of the stopping threshold (H^*) as determined by the model, incorporates the precautionary principle, but only to the extent that risk arises from uncertainty in the evolution of H. The scientific risk of an error in He is not taken into account.

Socioeconomic issues can and must also be addressed. It is, at least conceptually, easy to evaluate the benefits from timber harvesting leading to parameter ρ, the economic revenue from timber extraction, and it is certainly useful to do sensitivity analyses around it. More difficult is the issue of the value associated with caribou existence (s). Some will argue that caribou cannot be valued. Does this mean that they are valueless in which case the forest has no value when harvesting is banned and the Net Present Value is 0 ($V(h) = 0$)? Does it mean that their value is infinite? A positive answer to any of these two questions implies that there is no timber harvesting issue : allow harvesting forever in the first instance ; prohibit harvesting forever in the second instance. Thus the harvesting decision is not a trivial one and the caribou must have a finite value. The model does not determine that value. But the model can help investigate the consequences of the value, in terms of allowing or banning timber harvesting. Given the complexity of the problem what is required is fundamentally a social decision on the value of the caribou herd and the value of timber harvesting, in order to balance the uncertainties, risks and irreversibility issues involved in the forestry-caribou conflict.

The real options methodology provides several advantages over traditional approaches to managing forestry-wildlife conflicts, such as our Labrador case study. Unlike conventional forest management approaches, which depend on certainty and consistency of the future supply of the resource (Gunderson 2000), the uncertainty about the complex dynamics of natural systems is central to the real options decision making process. Real options does not provide a deterministic solution, when timber harvesting should be stopped, but instead provides a decision rule and process that allow future decisions to take new information into account as it arises. By adapting to new developments this process has the potential to maximize the future supply, and therefore the benefit, of socially valued ecosystem services, such as timber and wildlife. A decision maker is not presumed to have complete knowledge of the current system and its future, but only the capacity to respond to change.

Implementing forest dynamics numerically required a simplification, using only a mean and deviation. More complex behaviours, such as longer term oscillations resulting from climate cycling, are far more challenging to include. Similarly, it is a challenge to include, in this numerical method, wildlife population lag effects to changes in habitat. In this application a simple threshold was used, caribou population dynamics were not incorporated explicitly. As well, caribou habitat was characterized simplistically, as forests greater than 160 years old and other aspects of caribou habitat, such as habitat connectivity, were not included.

Some improvements and extensions might be considered in further work. For example the ban on timber harvesting might be reversible. In such case, the decision maker would consider two threshold levels. One lower lever at which harvesting would be temporarily banned ; one higher level at which harvesting would be reinstated temporarily if H recovered well enough. Finally there may be actions and decisions other than timber harvesting that affect caribou population and survival probability. These should ideally be considered in the timber harvesting caribou preservation debate. However, as explained in this paper, it is when some degree of irreversibility combines with uncertainty that the real options approach is most useful and telling.

Parameter description	Value	lognormal value
Amount of habitat at time 0	$H_0 = 467,185\ ha$	$h_0 = 13.0545$
Critical minimum habitat threshold	$H_e = 210,240\ ha$	$h_e = 12.2560$
Speed of reversion	$\alpha = 0.05$	
Volatility	$\sigma = 0.05$	
Discount rate	$r = 5\%$	
Flow of timber revenue per time unit	$\rho = 1$	
Flow of instantaneous utility per time unit	$s = 1$	

Tab. 2.1: Descriptions and values of parameters for the real options model

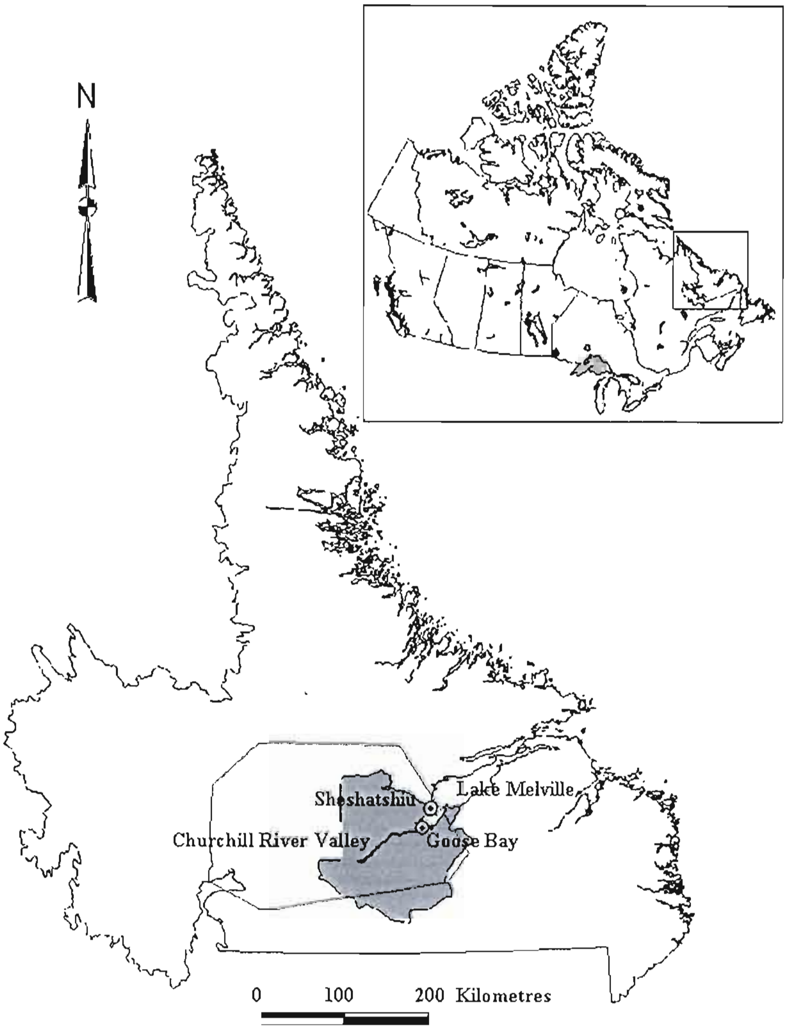

Fig. 2.1: District 19*A* study area and an outline of the historic range of the Red Wine Mountains caribou in central Labrador, Canada.

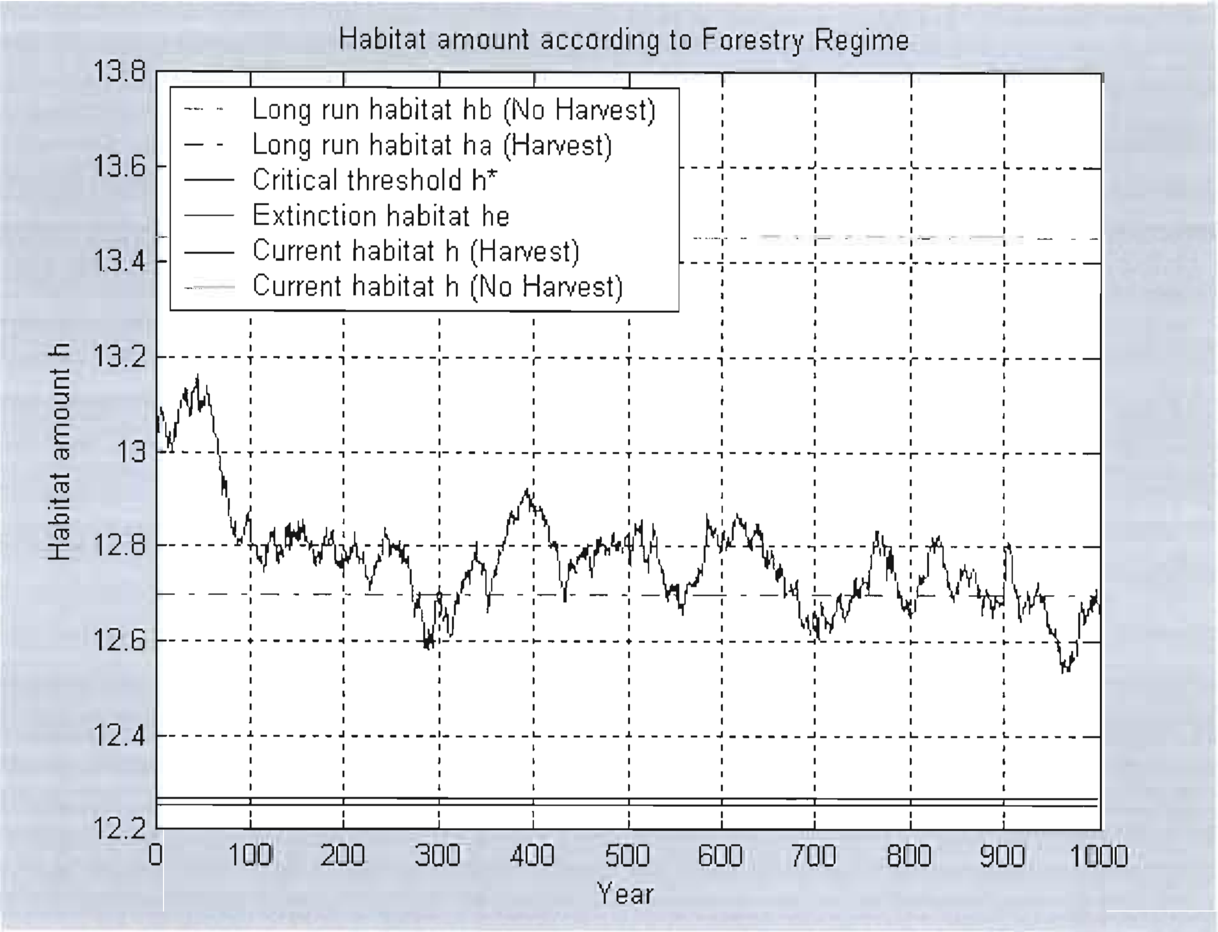

Fig. 2.2: Monte Carlo simulation of the habitat amount over a period of 1000 years.

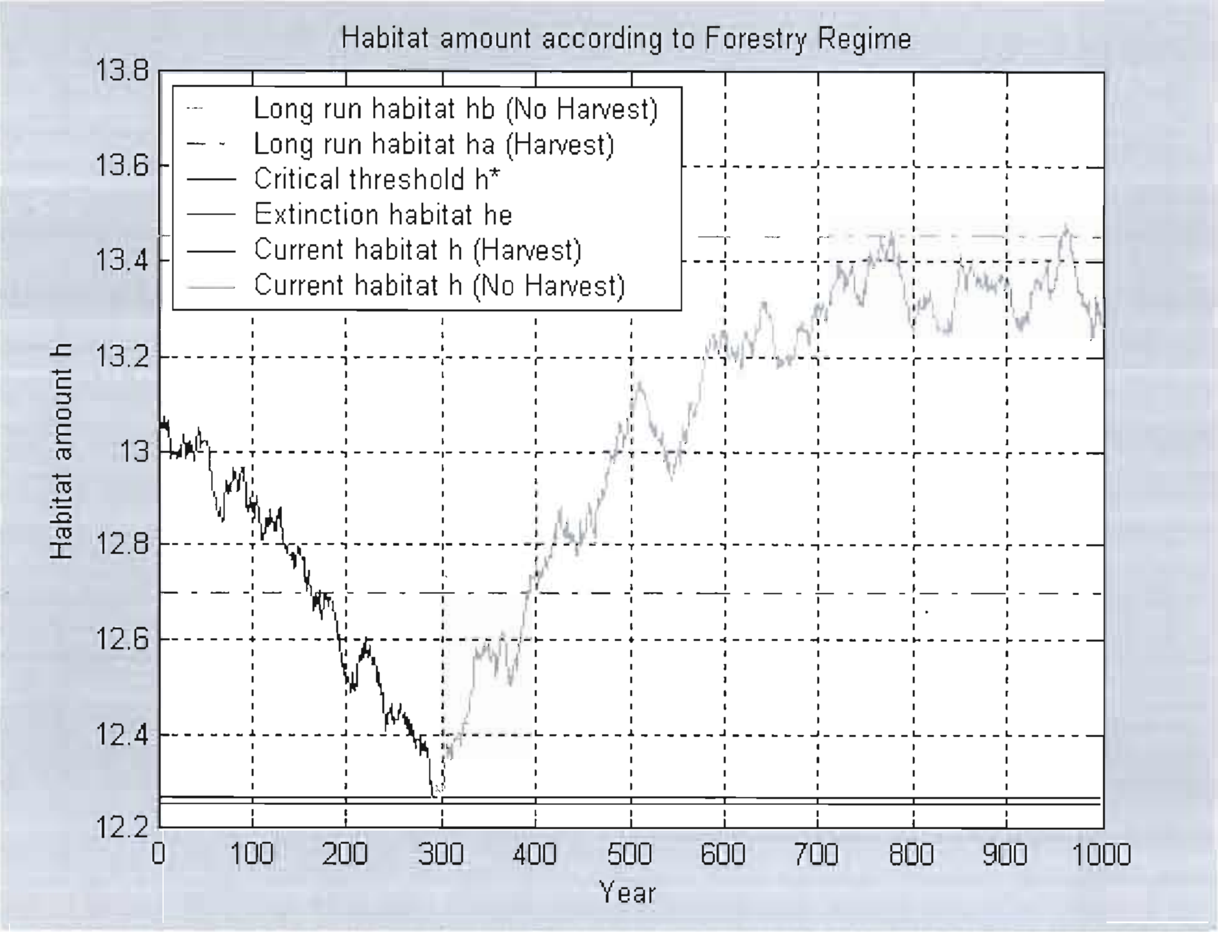

Fig. 2.3: Monte Carlo simulation of the habitat amount over a period of 1000 years.

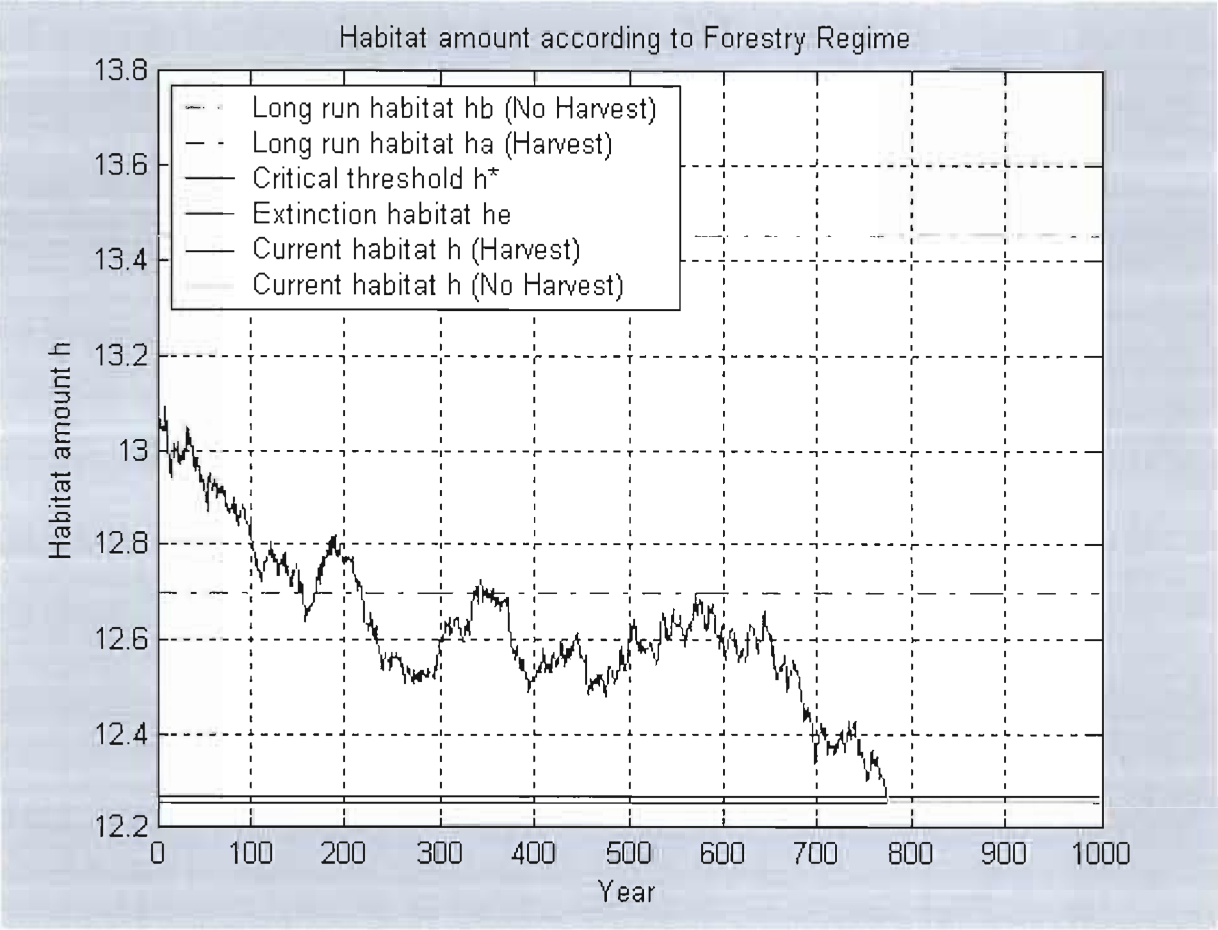

Fig. 2.4: Monte Carlo simulation of the habitat amount over a period of 1000 years.

Chapitre 2

A Real Options Approach to the Protection of a Habitat Dependent Endangered Species

2.1 Introduction

Commercially exploiting a forest which is home to some endangered species increases the latter's probability of extinction. A temporary ban on logging may prevent or delay extinction but implies foregone timber revenues. This paper uses real option theory to address these issues, with an application to the *Rangifer tarandus* caribou, an endangered species in Labrador, Canada. Caribous are a crucial component of the social, economic, and cultural lives of central Labrador's Innus. As conservation measures, non-subsistence and subsistence hunting were prohibited respectively in 1972 and in 2002. Despite these protection efforts, the caribou population kept declining and was listed as threatened under both the Endangered

Species Act of Newfoundland and Labrador, and under the Canadian Federal Species at Risk Act (Schmelzer et al 2004). Nevertheless, it is currently proposed to substantially develop logging in the region, with a view to improve the livelihood of local communities (Labrador Department of Forest Resources and Agrifoods 2003).

Besides being fraught with uncertain and irreversible consequences, decisions to alter current forest management are costly. The decision to initiate timber exploitation implies investments such as road construction and equipment acquisition. Similarly, a decision to interrupt logging implies a major disruption in economic activities and may require various actions such as site rehabilitation and the relocation of some communities. As a result, trading socioeconomic costs of preservation such as the loss of jobs and timber revenues for ecological risks from logging is a fundamental challenge for the decision maker.

Several papers use real options to deal with biodiversity protection or forest management. Among others, Pindyck (2000) applied the real option theory to environmental policy adoption ; Conrad (2000) studied land use decisions ; Insley (2002) treated forest logging as the exercise of a real option ; Kassar and Lasserre (2004) analyzed biodiversity preservation decisions ; Saphores (2003) studied the exploitation of an endangered species ; Saphores and Shogren (2005) showed how to optimize the use of pesticides.

To our knowledge, this paper is the first one modelling a situation where exercise decisions, rather than just changing or controlling some underlying state variable, affects the diffusion process of that variable. The crucial state variable in this case is the forest habitat of the *Rangifer tarandus* caribou. While there is little ecological data that could be used to characterize the stochastic behavior of caribou populations directly, much more is known about their habitat because of forestry research.

The *Rangifer tarandus* caribou lives in mature boreal forests. We use a landscape level disturbance simulation model (Fall and Fall, 2001) to generate Monte Carlo habitat series reflecting the local conditions of the Labrador forest. These habitat series are then used to estimate the habitat diffusion process when logging is going on or when logging is banned.

The theoretical model establishes an optimal rule for interrupting or resuming logging, as many times as desired to help prevent extinction, based on the current level of habitat. Depending on the value that the community attributes to the caribous' existence, some levels of extinction risk are deemed acceptable. Logging is managed in such a way as to optimize the trade off between the extinction risk and economic profits from timber production as, realistically, no human policy can protect the caribous from extinction entirely. In our model, human actions only affect the extinction risk by affecting the diffusion process of the caribous' habitat.

Besides establishing optimal habitat thresholds for interrupting or resuming logging, we describe the optimum policy in terms of the expected duration of each forestry regime (logging or no logging) and in terms of the impact of regime changes on the caribous' short-run extinction risk. Such information can help anticipate the impact of the optimal policy on important components of social life such as the duration of forestry contracts or the type of communities that logging may be expected to sustain in terms of duration and continuity. We discuss how this optimum policy is affected by the caribous' existence value, the discount rate, and the costs to ban and to resume logging. We show that a logging ban may be optimal only if the existence value of caribous not only matches timber revenues but exceeds them by a certain premium. The reason is that caribous constitute an asset whose existence is naturally threatened. A premium is then required to cover the risk of substituting timber revenues that are certain for the existence of caribous over an uncertain time period.

When a ban is introduced it reduces the chosen extinction risk less significantly when the discount rate is high than when it is low. Indeed, extinction being a future occurrence, it weighs less against the current cost of giving up logging when the discount rate is high. Similarly if logging is banned the decision maker will accept a higher level of risk when resuming logging if the discount rate is high.

Despite the effectiveness of the optimal policy in reducing the extinction risk, the bans are on average of short duration relative to logging periods. For the reasonable parameters used in our computations, a logging ban introduced optimally decreases the extinction risk by 80% ; meanwhile, the expected duration of the ban is about 8 years whereas the mean duration of a logging regime is around 86 years.

The rest of the paper is organized as follows. We continue in Section 2 with a brief general analysis of habitat dependent endangered species and we present the dynamics of habitat. Then in Section 3 we introduce the real option model and solve the decision problem. We compute the mean duration of each forestry regime and define the notion of short-run extinction risk which is used further in the paper as a quantitative measure of policy effectiveness. As already mentioned, this policy consists in banning logging if the habitat is dangerously low ; and possibly resuming logging afterwards if habitat has recovered sufficiently, as many times as required. Section 4 describes the empirical application to the *Rangifer tarandus* caribous of Central Labrador (Canada) and Section 5 concludes.

2.2 The habitat of the endangered species

We consider a species that relies on its habitat for survival. In our application to the *Rangifer tarandus* caribou this habitat is mature boreal forest. Various characteristics of the habitat make its existence and integrity crucial for the survival of the species. It may provide food and shelter, but also protection from predators and conditions favorable to reproduction as well as protection against disease or parasites. In such circumstances biologists often find that a minimum habitat size is required if the species is to survive. This critical value of the habitat level is referred to as the extinction threshold. Indeed, habitat loss and fragmentation has been recognized to be the main threat to many species' survival (Debinski and Holt, 2000). The extinction threshold depends mainly on the species reproductive potential, mortality and emigration rates, but also on landscape characteristics. There is no extinction threshold common to all species; it may vary from less than 1% to over 99% of the habitat (Fahrig 2001). Generally, the extinction threshold can be considered constant except in the two following cases.

The first case is related to metapopulations, that is to say to groups of spatially separated populations of the same species. Recent studies have shown that the extinction threshold may be low in such populations because of the rescue effect induced by the possibility for some populations to migrate towards the threatened populations (Brown and Kodric-Brown 1977) thus postponing extinction. This may be a momentary effect though. Although the *Rangifer tarandus* caribous of Central Labrador are sometimes in contact with their northern brothers, we assume that the rescue effect can be considered negligible.

The second case is when the habitat is highly fragmented. Habitat fragmentation affects the extinction threshold via the Allee effect (Allee et al 1949), under which the extinction threshold is relatively higher when the habitat is highly fragmented. Fragmentation has little impact on population persistence as long as the proportion of intact habitat remains above 20% of habitat (Andrén 1994 and Fahrig 2001). The Allee effect is negligible in our application as the proportion of intact habitat is expected to remain well above 20%. In fact, according to the ecological conservation literature discussed in Morgan et al (2008), a landscape becomes inadequate as caribou habitat when it contains less than 30% of intact old growth habitat. Under such circumstances, the caribous would already be extinct when the habitat reached the 20% fragmentation threshold if this was to happen. Thus the extinction habitat level can be considered as a known constant.

We measure habitat as the logarithm of the area (in hectares) covered with mature trees, within the relevant forest district in Labrador. As other aspects of the forest, the surface occupied by mature trees evolves over time. It has a deterministic component as it is subject to predictable biological changes, and a stochastic component because biological growth is subject to random circumstances such as weather and other environmental factors, and because of unpredictable natural disasters such as wildlife fires and diseases.

The forest may be exploited for timber. Whether logging is allowed or not, the level of habitat may decline to the extinction level because of the stochastic component in its evolution. However, logging affects habitat negatively, not only because mature areas may be logged, but also because some trees may be cut when they reach financial maturity even if they do not fall into the category of biologically mature trees. In other words logging in areas other than caribou habitat affect the flow of trees entering the habitat category over time. Such commercial activity reduces the average habitat level and increases its probability to reach the extinction threshold.

Let h_t denote the habitat level at time t. In order to model these characteristics, we assume that h_t follows an Ornstein-Uhlenbeck mean-reverting diffusion process, whose parameters depend on the forestry regime (logging allowed, $i = a$; or banned, $i = b$) :

$$dh_t = \lambda_i \left(\mu_i - h_t\right) dt + \sigma_i dz_{it},\ i = a, b \tag{3.1}$$

where dh_t represents the change in h_t over a time interval dt and dz_{it} is the increment of a Wiener process. The long-run expected habitat level and the instantaneous variance of the process are respectively μ_i and σ_i^2 with $E(h_t) = \mu_i + (h_0 - \mu_i)e^{-\lambda_i t}$ and $var(h_t) = \frac{\sigma_i^2}{2\lambda_i}(1 - e^{-2\lambda_i t})$, where $\lambda_i > 0$ is the speed of reversion (Dixit and Pindyck, 1994).

The stochastic component $\sigma_i dz_{it}$ accounts for unpredictable natural variations due to stochastic events such as wild fires, diseases, or temperature. Although occurrences such as fires are discontinuous events at the micro level, and as such would be better represented, e.g., by a Poisson process, the process that we are modelling pertains to a large area where several occurrences of these discontinuous events are to be expected over any relevant period. Consequently, the law of large numbers justifies the use of a Gaussian white noise as an approximation.

The deterministic component $\lambda_i \left(\mu_i - h_t\right) dt$ of the process describes what would happen in the absence of uncertainty. It vanishes when h_t is equal to μ_i, which is the unique and stable equilibrium in the absence of uncertainty. When h_t is higher than the long-run mean μ_i then $\lambda_i \left(\mu_i - h_t\right) dt$ is negative, implying a reduction in habitat which may be offset or reinforced by the stochastic component ; when h_t is lower than μ_i the opposite is true. In both cases the deterministic speed of adjustment is proportional to the gap $\mu_i - h_t$ and to the speed of reversion λ_i. Since logging affects habitat negatively on average, μ_a must be lower than μ_b ; the difference $\mu_a - \mu_b$ thus represents the effect of logging on the long-run mean level of the species habitat.

Clearly, the values of the parameters in the two alternative diffusion processes defined by (3.1) depend on empirical circumstances. However it should be noted that values of μ_b below the extinction threshold h_e would be highly unlikely as the sole non stochastic equilibrium of the process would then be extinction. Current existence of the species would arise as a statistical aberration in a forest that had never been logged. In a forest where logging were introduced recently, μ_a could be either lower or higher than h_e. In the theoretical treatment of the model, we make the following assumption to rule out irrelevant configurations.

Assumption 1 *The long-run habitat level in the absence of logging μ_b strictly exceeds the extinction habitat level h_e.*

In the empirical application to *Rangifer tarandus* Caribous, we find that $\mu_a = 12.7034$ and $\mu_b = 13.4504$. This means that the long-run expected habitat level under a permanent logging regime, and the long-run expected habitat level under a permanent "no logging" regime respectively represent 29% and 62% of the total forest area of 1 117 327 hectares. Meanwhile, the extinction habitat level required to maintain the Red Wine Mountains caribou population is estimated to be 30% of the expected area of old growth forest (Morgan et al, 2008), which is 694 110 hectares ($e^{\mu_b} = 694\ 110$). The extinction habitat level is thus 208 230 hectares or 19% of the total forest area ; that is $h_e = 12.2464$. Consequently, even when logging is going on, the forest in our empirical application tends to revert to a situation where the caribou habitat level exceeds the extinction level. This needs not be true in other applications of the model.

2.3 The real option model

2.3.1 Objective function, decisions, and costs

We assume that the decision maker shares the objectives of the local community. She is interested in revenues from logging but is concerned about the survival of the endangered species. Logging provides a constant flow of benefits ω except at times when it is banned. This flow is equal to the constant price of timber times the long-term sustained yield of the forest. We assume that caribous provide a constant instantaneous utility flow s as long as they are in existence. This utility flow vanishes if the species goes extinct[1].

The decision maker has control over whether or not logging is allowed to take place at any given time. As implied by the discussion on habitat dynamics she has no control over logging intensity, which must be either zero, implying that the habitat follows process (2.1) for $i = b$, or some exogenous positive level implying (2.1) for $i = a$[2].

When logging is resumed after some interruption, a cost of $I_a \geq 0$ is incurred; this corresponds to equipment and infrastructure expenditures, but also to social, information, and consensus building costs. Similarly, when logging is interrupted, workers and the community experience costs that extend from site rehabilitation

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[1] It is possible to define the instantaneous utility function as a concave and increasing function of the habitat level. for instance $u(h) = s(1 - e^{-\beta(h-h_e)})$ where β is a positive constant. However, in that case, the decision to ban logging does not only reflect the extinction risk but also current and future marginal and average utility effects. This unduly complicates the analysis without bringing any important intuition to light.

[2] Without difficulty, the complete logging ban regime can be replaced with a low intensity logging regime.

to the necessity for a significant proportion of the logging community to find new jobs or to move to other locations. We assume that the ban implies a lump cost $I_b \geq 0$. For simplicity, it is assumed that neither I_a nor I_b depend on the length of any previous logging or ban periods.

2.3.2 The extinction risk

The habitat diffusion process defined by (2.1) is not only regular in the sense that the probability for habitat to reach any other finite level within the process interval over a period of infinite length is strictly positive (Karlin and Taylor, 1981 p. 158) but habitat will reach any finite level over an infinite period with probability one. In particular, extinction is certain under both forestry regimes in the very long run. However it must be true that the probability of extinction over any finite period is lower when logging is banned ($i = b$) than when logging is going on ($i = a$). In order to evaluate the impact of any rule governing logging on the risk of extinction it is thus necessary to do so over a finite period; in other words what is needed is a notion of short-run risk.

Consider a logging ban introduced at some relatively low level $\underline{h} < \mu_b$. If h reaches its natural long-run mean level μ_b at least once after the introduction of the ban, extinction remains certain in the very long run, but it has to be considered a natural occurrence. Similarly, if at the same habitat level $\underline{h}$ the ban is not introduced and logging continues, and if nonetheless habitat recovers to reach its long-run mean level μ_a at least once, it can be argued that the failure to ban logging when h was at $\underline{h}$ did not cause extinction in the short run. Thus, we define the short run as the period until the long-run habitat level is reached under either a logging regime ($i = a$) or under a logging ban ($i = b$).

Let $P_i^{h_e,\mu_i}(h)$ denote the probability for the habitat to reach the habitat level μ_i before it reaches h_e given that the current habitat level is h and assuming that the current logging regime (3.1) will be maintained forever for either $i = a$ or $i = b$; that is $P_i^{h_e,\mu_i}(x) = \Pr[T_i^{\mu_i}(x) < T_i^{h_e}(x)]$, where $T_i^y(x)$ is the date at which habitat level reaches y for the first time under forestry regime i given an initial habitat level of x at time zero. If $h > h_e \geq \mu_i$ then $P_i^{h_e,\mu_i}(h)$ is zero because h cannot reach μ_i before reaching h_e; if $h \geq \mu_i > h_e$ then $P_i^{h_e,\mu_i}(h)$ is unity because there is no way for h to reach h_e without going through μ_i, and this is certain to happen. However, for $h \in [h_e, \mu_i]$, h may diminish and reach the extinction level h_e without reaching μ_i despite the fact that it tends to revert to that long-run mean level. Such an outcome may be defined as short-run extinction under regime i. Its probability is $R_i(h) = 1 - P_i^{h_e,\mu_i}(h)$. The following lemma allows one to compute the short-run extinction risk under regimes $i = a, b$.

Lemma 1 *Let $\mu_i > h_e$; then $P_i^{h_e,\mu_i}(h)$ is an increasing function of h over $[h_e, \mu_i]$ given by*

$$P_i^{h_e,\mu_i}(h) = [S_i(h)]\,[S_i(\mu_i)]^{-1} \tag{3.2}$$

where $S_i(h) = \int_{h_e}^{h} e^{\frac{\lambda_i}{\sigma_i^2}(x-\mu_i)^2} dx$ is the scale function of the diffusion process (3.1).

Proof. For short, let $P_i(h)$ denote $P_i^{h_e,\mu_i}(h)$. Karlin and Taylor (1981) show that if $h_e < \mu_i$ then for any $h \in]h_e, \mu_i[$ where h follows the diffusion process (3.1) and for a sufficiently small time interval dt, $P_i(h) = E\left[P_i(h + dh)\right] + o(dt)$ or $E\left[dP_i(h)\right] = o(dt)$ where the levels of h at time t and at time $t + dt$ are respectively h and $h + dh$. By applying Ito's lemma to $P_i(h)$ and the expectation operator to $dP_i(h_t)$, we obtain $EdP_i\,(h) = \frac{\sigma_i^2}{2} P_i''(h)dt + \lambda_i\,(\mu_i - h)\,P_i'(h)dt$, $\forall h \in]h_e, \mu_i[$ where terms of smaller order than dt are neglected. Combining these two results implies that $P_i(h)$ satisfies the differential equation

$$\frac{\sigma_i^2}{2} P_i''(h) \; + \; \lambda_i (\mu_i - h) P_i'(h) \; = \; 0$$

over $]h_e, \mu_i[$. Two boundary conditions also apply : $P_i(h_e) = 0$ and $P_i(\mu_i) = 1$. Integrating the differential equation gives

$$P_i(h) \; = \; \int^h e^{\int^x - \frac{2\lambda_i(\mu_i - \xi)}{\sigma_i^2} d\xi} dx$$

Let $S_i(h) = \int_{h_e}^h e^{-\int_{\mu_i}^x \frac{\lambda_i(\mu_i - \zeta)}{\sigma_i^2/2} d\zeta} dx$, or equivalently $S_i(h) = \int_{h_e}^h e^{\frac{\lambda_i}{\sigma_i^2}(x-\mu_i)^2} dx$. This function is known as the scale function of process (3.1)[3]. The two boundary conditions imply that $P_i(h) = \frac{S_i(h)}{S_i(\mu_i)}$. The scale function increases in h implying that $P_i(h)$ has the same property. ■

The notion of short-run risk just introduced has two major advantages. First it avoids arbitrariness by relying on the concept of long-run equilibrium represented by μ_i in the Ornstein-Uhlenbeck process : the short run ends the first time the process reaches its long-run level. Second, it focuses on the down-side risk : in both regimes, the short-run risk is appropriately zero when habitat is higher than its long-run level and is higher the closer the extinction threshold.

For the purpose of comparing situations with logging and without logging, however, this notion of short-run risk has the drawback of being regime specific. The risk does culminate when h tends toward h_e under both regimes, which is a desirable property. However, by definition, short-run risk is zero for $h \geq \mu_i$; since $\mu_a < \mu_b$, this leads to the paradox that $R_a(h) = 0 < R_b(h)$ when $\mu_a \leq h < \mu_b$. A ban introduced

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[3] The process $x_t = S(h_t)$ is said to be natural or canonical as its probability to hit x_2 before x_1 is equal to $\frac{x_2 - x}{x_2 - x_1}$ when its current level is $x \in [x_1, x_2]$ with $x_1 < x_2$. This justifies the name given to the function $S(h)$ as it allows to convert its corresponding process into a natural one.

at a relatively high value of h might then result in a negative value of $R_a(h) - R_b(h)$ and be falsely interpreted as a deterioration in the risk situation. In that sense the measure understates the risk improvement associated with the introduction of a ban at habitat values lower than μ_b but too close to, or higher than, μ_a. In contrast, a drop in short-run extinction at the introduction of a logging ban, that is to say a positive value of $R_a(h) - R_b(h)$, definitely indicates an improvement in the short-run risk situation. This is typically what happens at habitat values sufficiently lower than μ_a. Since bans turn out to be introduced at low habitat values as will become clear next when we establish the optimal rule, the change in short-run extinction risk provides a proper measure of the impact of the optimal policy.

2.3.3 Interrupting and resuming logging optimally

The solution of the general problem is an optimum rule consisting in interrupting logging, and resuming logging, temporarily, as many times as desired according to the level of caribou habitat. Suppose that logging is currently allowed ; such logging may be interrupted if habitat decreases to some low threshold level. If habitat recovers correctly, then logging may be allowed again when it increases to some high threshold level. This alternating pattern will go on for as long as the species is not extinct. Since the date is of no relevance to these decisions, which obviously depend on the state variable only, the problem is time autonomous.

Clearly it would not make economic sense to spend I_a to allow logging if the maximum cumulative present value revenues from such a decision did not at least match the cost. Thus we make this assumption to rule out irrelevant situations where logging would never be resumed or introduced in the first place :

Assumption 2 $I_a \leq \omega / r$.

Let $V_i(h)$ be the forest value function when the forestry regime is $i = a, b$ and the species is not extinct. Precisely, during a ban on logging, habitat follows the diffusion process (3.1) for $i = b$, whose deterministic component pulls it toward μ_b, a higher long-run mean than μ_a. The decision maker earns the flow s of utility associated with the existence of the endangered species, and holds the option to resume logging. That option is to be exercised at cost I_a if and when habitat reaches a threshold h_b whose optimal value must be determined. Since, given enough time, the habitat is certain to reach any positive finite level, however high, and since at such high habitat level the gain from maintaining a logging ban tends toward zero, it is certain by Assumption 2 that there exists such a threshold h_b at which the option to end the ban on logging should be exercised. It is the smallest solution to the following problem which defines the forest value function during a logging ban $V_b(h)$, for any $h \in]h_e, h_b]$:

$$V_b(h) = \max_{x \geq h_e} E\left[\int_0^{T_b^x(h)} se^{-r\tau} d\tau + e^{-rT_b^x(h)}\left(V_a(x) - I_a\right) | h_0 = h\right] \tag{3.3}$$

The following proposition characterizes the forest value function when the species is in existence while logging is temporarily banned.

Proposition 1 *When the species is not extinct and logging is banned temporarily :* *(1) There exists a finite habitat level above which logging should be allowed. If the ban is not to be lifted immediately, then the forest value function $V_b(h)$ satisfies Bellman equation*

$$\frac{\sigma_b^2}{2} V_b''(h) + \lambda_b(\mu_b - h) V_b'(h) - rV_b(h) + s = 0, \ \forall h \in]h_e, h_b[\tag{3.4a}$$

where $h_b > h_e$ is the finite threshold value of h at which the ban is lifted. At h_b, V_b satisfies the value-matching condition

$$V_b(h_b) = V_a(h_b) - I_a \tag{3.4b}$$

the smooth-pasting condition

$$V_b'(h_b) = V_a'(h_b) \tag{3.4c}$$

and the boundary condition

$$V_b(h_e) = \frac{\omega}{r} - I_a \tag{3.4d}$$

(2) If the ban is to be lifted immediately, then

$$V_b(h) = V_a(h) - I_a$$

Proof. 1. The existence of the finite threshold h_b follows from the argument just preceding the proposition. Bellman equation, the smooth-pasting condition, and the value-matching condition are obtained by standard methods. We sketch their derivation. Using the law of iterative expectations, $V_b(h)$ satisfies $V_b(h) = E\left[E_{dt}\left[\int_0^{dt} se^{-r\tau}d\tau + \int_{dt}^{T_b^{h_b}(h)} se^{-r\tau}d\tau + e^{-rT_b^{h_b}(h)}\left(V_a(h_b) - I_a\right)\right]\right]$ for any $h \in]h_e, h_b[$ and for any sufficiently small time interval dt; the habitat level is $h_0 = h$ at time zero and $h_{dt} = h + dh$ at time dt. As $T_b^{h_b}(h) = T_b^{h_b}(h + dh) + dt$, neglecting terms of smaller order than dt leads to $V_b(h) = sdt + e^{-rdt}E\left[E_{dt}\left[\int_0^{T_b^{h_b}(h+dh)} se^{-r\tau}d\tau + e^{-rT_b^{h_b}(h+dh)}\left(V_a(h_b) - I_a\right)\right]\right]$, that is $V_b(h) = sdt + e^{-rdt}E\left[V_b(h + dh)\right]$. Applying Ito's lemma to $V_b(h + dh)$ gives $V_b(h) = sdt + e^{-rdt}E\left[V_b(h) + V_b'(h)dh + \frac{1}{2}V_b''(h)dh^2\right]$. Using (3.1), one obtains equation (3.4a) by letting dt go to zero. The restriction to $h \in]h_e, h_b[$ is obvious since h_e is absorbing and h_b is the exercise trigger. For the value-matching and smooth-pasting conditions see, e.g. Dixit (1993a). If h_e is reached despite the logging ban, the species goes extinct and it is useless to continue the ban. Logging will be resumed if the cost of doing so does not exceed the revenues; since no further ban will be forthcoming revenues are certain to be $\frac{\omega}{r}$ so that $V_b(h_e) = \max(\frac{\omega}{r} - I_a, 0)$ which equals $\frac{\omega}{r} - I_a$ by Assumption 2.

2. The result stems directly from (3.3). ■

When logging is going on, the habitat diffusion process (3.1) for $i = a$ is pulling the habitat level toward a lower long-run level than when logging is banned. The current benefit flow stems from logging and from the existence value of the caribous. In addition, the decision maker holds the option to ban logging. This option is not exercised as long as the habitat level remains above some threshold denoted h_a to be chosen optimally. Suppose that such a threshold exists so that the option to ban is valuable, and consider values of h higher than h_a, implying that the option has not been exercised yet. Then the forest value $V_a(h)$ is the sum of the expected value of the entitlement to the flow $s + \omega$, and the value of the option to ban logging, defined for any $h \in [h_a, +\infty[$. Consequently, h_a is the highest solution (the first to be reached) to the following problem :

$$V_a(h) = \max_{x \geq h_e} E\left[\int_0^{T_a^x(h)} (s+\omega)\, e^{-r\tau} d\tau + e^{-rT_a^x(h)} \left(V_b(x) - I_b\right) | h_0 = h\right] \tag{3.5}$$

It would not make economic sense to spend I_b to ban logging in an effort to improve the chance of survival if the maximum cumulative present value benefits from permanent survival did not at least match that cost. In that case no finite value of h_a would solve the maximization problem defined by (3.5). In that configuration the extinction threshold may be reached at some date $T_a^{h_e}(h)$ without any ban being imposed before extinction ; the forest value is then the sum of the expected value of the entitlement to the flow $s + \omega$ until $T_a^{h_e}(h)$, and thereafter the proceeds from logging alone :

$$\tilde{V}_a(h) = E\left[\int_0^{T_a^{h_e}(h)} (s+\omega)\, e^{-r\tau} d\tau + \int_{T_a^{h_e}(h)}^{+\infty} \omega e^{-r\tau} d\tau \quad |h_0 = h\right], \ \forall h \in [h_e, +\infty[\tag{3.6}$$

As implied by the foregoing discussion, the following assumption is necessary but not sufficient for logging ever to be banned :

Assumption 3 $I_b \leq s/r$.

Proposition 2 characterizes the value function when the species is in existence while logging is temporarily allowed.

Proposition 2 *Suppose that logging is allowed and that the species is not extinct : (1) If a finite habitat threshold h_a exists at which it is optimal to ban logging as soon as h falls to that level then $h_e < h_a \leq h_b$. Furthermore, the forest value function $V_a(h)$ is defined by* (3.5) *and satisfies Bellman equation*

$$\frac{\sigma_a^2}{2} V_a''(h) + \lambda_a (\mu_a - h) V_a'(h) - rV_a(h) + s + \omega = 0, \ \forall h \in]h_a, +\infty[\tag{3.7a}$$

along with the value-matching condition

$$V_a(h_a) = V_b(h_a) - I_b \tag{3.7b}$$

the smooth-pasting condition

$$V_a'(h_a) = V_b'(h_a) \tag{3.7c}$$

and the boundary condition

$$\lim_{h \to +\infty} V_a(h) = \frac{s+\omega}{r} \tag{3.7d}$$

(2) If there exists no threshold value $h_a > h_e$ at which it is optimal to ban logging then the forest value function is $\tilde{V}_a(h)$ defined by (3.6) and satisfies the following differential equation

$$\frac{\sigma_a^2}{2}\tilde{V}_a''(h) + \lambda_a\left(\mu_a - h\right)\tilde{V}_a'(h) - r\tilde{V}_a(h) + s + \omega = 0,\ \forall h \in]h_e, +\infty[$$

with the two boundary conditions

$$\tilde{V}_a(h_e) = \frac{\omega}{r}$$

$$\lim_{h \to +\infty} \tilde{V}_a(h) = \frac{s+\omega}{r}$$

(3) For any set of parameter values ($\omega > 0,\ I_a \geq 0,\ I_b \geq 0$) satisfying Assumption 2, there exists a value $\underline{s}$ of s satisfying Assumption 3 such that h_a exists for $s \geq \underline{s}$ and does not exist for $s < \underline{s}$. (4) If h_a exists then $h_a = h_b$ if and only if $I_a = I_b = 0$.

Proof. (1) To obtain the Bellman equation as well as the value-matching, smooth-pasting, and boundary conditions, adapt the proof of Proposition 1. Note that $h_e \leq h_a$ by definition and $V_b\left(h_e\right) = \frac{\omega}{r} - I_a$ by (3.4d). A ban on harvesting when habitat reaches h_e cannot bring any benefit as the caribous go into extinction ; consequently, if it exists, h_a must be strictly higher than h_e. The property $h_a \leq h_b$ is a logical necessity. As shown in Proposition 1 h_b exists and defines a set $[h_b, +\infty[$ of values at which it is optimal to allow logging. Being the habitat level below which it is optimal to ban logging, h_a cannot exist unless $h_a \leq h_b$.

(2) Again adapt the proof of Proposition 1 with the following difference. Since there is no optimization problem in the present configuration, there is no Bellman equation. However by definition $\tilde{V}_a(h) = E\left[\int_0^{T_a^{h_e}(h)}\left(s+\omega\right)e^{-r\tau}d\tau + \int_{T_a^{h_e}(h)}^{+\infty}\omega e^{-r\tau}d\tau\right]$, $\forall h \in [h_e, +\infty[$;then for any $h \in]h_e, +\infty[$ and for a sufficiently small time interval dt, $\tilde{V}_a(h) = E\left[E_{dt}\left[\int_0^{T_a^{h_e}(h)}\left(s+\omega\right)e^{-r\tau}d\tau + \int_{T_a^{h_e}(h)}^{+\infty}\omega e^{-r\tau}d\tau\right]\right]$, given that the

habitat level is $h_0 = h$ at time zero and $h_{dt} = h + dh$ at time dt. $\tilde{V}_a(h) = E\left[E_{dt}\left[\int_0^{T_a^{h_e}(h+dh)+dt}(s+\omega)\,e^{-r\tau}d\tau + \int_{T_a^{h_e}(h+dh)+dt}^{+\infty}\omega e^{-r\tau}d\tau\right]\right]$ or $\tilde{V}_a(h) = (s+\omega)\,dt + e^{-rdt}E\tilde{V}_a(h+dh)$. Thus the proof proceeds as in Proposition 1, though the implied differential equation is not a Bellman equation. Also, since expression (3.6) is not an optimization, there are no value-matching or smooth-pasting conditions associated with it. The boundary conditions are obvious given the discussion just preceding the proposition.

(3) For any level of h, imposing a ban of any arbitrary duration increases the expected extinction date and yields a benefit that is proportional to s. The cost from the ban is I_b plus the present value of foregone logging revenues; neither depend on s. Consequently for a ban of any arbitrary duration, including a ban whose duration would be determined by applying an optimal decision rule, benefits exceed costs at high levels of s and *vice versa*. The existence of $\underline{s}$ follows by continuity.

(4) Suppose h_a exists and $h_a = h_b$; then (3.4b) and (3.7b) imply that $I_a + I_b = 0$; consequently, $I_a = I_b = 0$, as I_a and I_b are non negative. Suppose now that h_a exists and that $I_a = I_b = 0$; then we show by contradiction that $h_a = h_b$. If $h_a \neq h_b$ then by Proposition 2.1, $h_a < h_b$. By (3.5), for any $h \in [h_a, h_b]$, $V_a(h) = E\left[\int_0^{T_a^{h_a}(h)}(s+\omega)\,e^{-r\tau}d\tau + e^{-rT_a^{h_a}(h)}V_b(h_a)\right]$, or $V_a(h) = \frac{s+\omega}{r}\left(1 - Ee^{-rT_a^{h_a}(h)}\right) + V_b(h_a)\,Ee^{-rT_a^{h_a}(h)}$. As V_a is strictly increasing on $[h_a, +\infty[$ and tends to $\frac{s+\omega}{r}$ at infinity, then for any finite habitat level $\widetilde{h} \in [h_a, h_b[$, $V_a(\widetilde{h}) < \frac{s+\omega}{r}$ so that $V_a(h) > V_a(\widetilde{h})\left(1 - Ee^{-rT_a^{h_a}(h)}\right) + V_b(h_a)\,Ee^{-rT_a^{h_a}(h)}$, or $V_a(h) > V_a(\widetilde{h}) + \left(V_b(h_a) - V_a(\widetilde{h})\right)Ee^{-rT_a^{h_a}(h)}$. As h_a is finite, the last inequality must hold for $h = h_a$ where $T_a^{h_a}(h_a) = 0$ so that it implies $V_a(h_a) > V_b(h_a)$, a contradiction as $V_b(h_a) = V_a(h_a)$ by (3.7b). ■

In the configuration of Proposition 2.1, there exists a threshold value h_a such that, if it is reached from above when logging is allowed, logging is banned. In the configuration of Proposition 2.2, there exists no such threshold, so that the logging regime lasts until h reaches h_e and extinction occurs. Logging continues thereafter, with the forest value remaining constant at $\frac{\omega}{r}$ forever. Consequently there is one fewer variable to be determined which explains why the number of equations characterizing the solution is lower by one in Proposition 2.2 than in Proposition 2.1.

The above two propositions fully characterize the forest value functions and the habitat thresholds. Precisely, Proposition 1 together with Proposition 2.1 characterize V_a and V_b together with the thresholds h_a (end of logging) and h_b (end of ban) in configurations where logging bans are optimal if h becomes dangerously low. Meanwhile Proposition 1 together with Proposition 2.2 characterize V_a and V_b together with the threshold h_b in configurations where logging is never to be prohibited once allowed.

The first instance is perhaps more interesting as logging may be banned and resumed several times. Proposition 2.3 indicates that it occurs at high values of the existence value of the species, that is to say at values of s exceeding $\underline{s}$. Intuitively, this is because, for a ban to make sense, the benefits should outweigh the costs. The benefits result from an increase in the duration of caribou survival, which is more valuable the higher s. The costs consist of the cost of starting the ban I_b to which one must add the cost of foregone logging revenues, which is more significant, the higher ω, and the cost of reintroducing logging if and when the ban is lifted. Thus the configuration involving a possible ban on logging is more likely to arise the higher s and the lower ω.

Meanwhile, since extinction is possible, the contribution of caribou to cumulated expected value is strictly lower than $\frac{s}{r}$ while logging's contribution may reach $\frac{\omega}{r}$ if no ban is ever introduced. However, the possibility to manage logging regimes and bans reduces the opportunity cost of a logging ban to less than $\frac{\omega}{r}$ so that s should not necessarily exceed ω in order for the configuration involving a possible ban on logging to arise. In our application to *Rangifer tarandus* caribous though, it is found numerically that s should exceed ω by more than 10% in order that configuration to arise.

The optimal decision rule is illustrated in Figure 3.1. If the species is in existence and logging is allowed, which is compatible with past applications of the optimal rule if $h \in [h_a, +\infty[$, then logging should continue as long as the current habitat level, driven by (3.1) for $i = a$, remains above h_a (upper arrow) ; the forest value is then $V_a(h)$.

As soon as h falls down to h_a, logging is banned at cost I_b and the forest value becomes $V_b(h)$; habitat is then driven by (3.1) with $i = b$. Logging remains banned as long as the current habitat level is between h_e and h_b (lower arrow). Although this ban is favorable to habitat recovery, extinction may occur as it remains possible for h to fall to h_e. If and when this happens, logging is resumed forever at cost I_a yielding a forest value $\frac{\omega}{r}$; the net present value of a forest reaching caribou extinction during a logging ban is thus $V_b(h_e) = \frac{\omega}{r} - I_a$ as the cost of allowing logging must be incurred prior to drawing benefits. Although habitat may recover and reach levels above h_e later on, extinction is irreversible so that forest value remains at $\frac{\omega}{r}$ even at values of h exceeding h_e once extinction has occurred[4]. However, when logging is banned, the probability that the habitat level remains above h_e is higher than when logging is allowed, as h is driven towards $\mu_b > \mu_a$ by the non stochastic component of (3.1) for $i = b$. During a ban, habitat is likely to increase and so does $V_b(h)$ until

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[4] If $\frac{\omega}{r} - I_a < 0$, in violation of Assumption 2, logging is never resumed (in fact it was never worth undertaking) and the forest has no value anymore after extinction.

h_b is reached. At that habitat level the risk of extinction no longer justifies foregoing logging income and the option to end the ban is exercised.

The introduction of a logging ban reduces the short-run risk of extinction. Just before the ban, the short-run risk of extinction is $R_a(h_a)$; once logging is banned, the risk becomes $R_b(h_a)$. Intuition suggests, and our numerical results confirm, that h_a increases with the existence value of caribous : when s is higher, logging is interrupted farther above the extinction level because the cost of extinction, which is the expected value of being deprived from caribous, is higher. Raising h_a, by reducing the probability of extinction, reduces that cost. For the opposite reason, a rise in the cost of interrupting logging justifies accepting a higher risk of extinction in the hope of avoiding or postponing that cost : h_a diminishes when I_b increases. A rise in the cost of resuming has the same effect, although weakened because that cost is incurred in the future. As indicated in Proposition 2.2 though, adjustments to changes in s, I_a or I_b can reach a limit : for any given value of I_a and I_b, there is a value of s, $\underline{s}$, below which h_a no longer exists.

2.3.4 The mean durations of the forestry regimes

Besides the direct costs I_a and I_b of interrupting or resuming logging and besides effects on timber revenues, switching between alternative forestry regimes implies various social disruptions that are likely to be more acute, the more frequent the switches. Longer logging periods are probably better than short logging spikes for forest communities. In any case the expected duration of logging regimes and bans is a crucial characteristic of the optimal rule. The mean duration of a ban $\widetilde{T_b}$ is the expected time for h to hit either h_b or h_e when its current level is h_a and it follows the diffusion process (2.1), with $i = b$. Similarly, the expected duration of a logging period $\widetilde{T}_a$ is the expected time for the diffusion process (3.1) to reach h_a for the first time when its current level is h_b and when $i = a$.

Proposition 4 gives explicit expressions for the mean durations of each forestry regime. In order to establish that proposition, we use the following lemma adapted from Karlin and Taylor (1981). Let $T_i^{\underline{h},\overline{h}}(h)$ denote the date at which the diffusion process (3.1) hits either $\underline{h}$ or $\overline{h}$ ($\underline{h} < \overline{h}$) for the first time given that its current level $h \in [\underline{h}, \overline{h}]$; that is $T_i^{\underline{h},\overline{h}}(h) = \min(T_i^{\underline{h}}(h), T_i^{\overline{h}}(h))$.

Lemma 3 *The expected time for the regular diffusion process* (3.1) *to reach either* $\underline{h}$ *or* $\overline{h}$ *given its value* $h \in [\underline{h}, \overline{h}]$ *at time zero is* $ET_i^{\underline{h},\overline{h}}(h)$

$$ET_i^{\underline{h},\overline{h}}(h) = 2\left\{P_i^{\underline{h},\overline{h}}(h)\int_h^{\overline{h}}\left(S_i(\overline{h}) - S_i(\xi)\right)m_i(\xi)d\xi + \left(1 - P_i^{\underline{h},\overline{h}}(h)\right)\int_{\underline{h}}^{h}\left(S_i(\xi) - S_i(\underline{h})\right)m_i(\xi)d\xi\right\}$$

where $m_i(h) = \frac{1}{\sigma_i^2 S_i'(h)}$ *denotes the speed density function of process* (3.1) *and* $P_i^{\underline{h},\overline{h}}(h) = \left[\int_{\underline{h}}^{h} e^{\frac{\lambda_i}{\sigma_i^2}(x-\mu_i)^2} dx\right]\left[\int_{\underline{h}}^{\overline{h}} e^{\frac{\lambda_i}{\sigma_i^2}(x-\mu_i)^2} dx\right]^{-1}$ *is the probability for the process to reach* $\overline{h}$ *before* $\underline{h}$ *given its current level* h.

Proof. Let $\widetilde{T}_i(h)$ denote $ET_i^{\underline{h},\overline{h}}(h)$. For $h \in]\underline{h}, \overline{h}[$ and for a sufficiently small time interval dt, the diffusion process (3.1) does not hit either $\underline{h}$ or $\overline{h}$ during a time interval dt over which h evolves to $h + dh$; that is $\widetilde{T}_i(h) = dt + \widetilde{T}_i(h + dh)$. By applying Ito's lemma to $\widetilde{T}_i(h)$, one shows that $\widetilde{T}_i(h)$ satisfies the differential equation $\frac{\sigma_i^2}{2}T_i''(h) + \lambda_i(\mu_i - h)T_i'(h) + 1 = 0$. To solve that equation, use the canonical representation of the differential infinitesimal operator associated with the diffusion process (3.1), that is express $\frac{\sigma_i^2}{2}T_i''(h) + \lambda_i(\mu_i - h)T_i'(h)$ as $\frac{1}{2}\frac{d}{dM_i}\left[\frac{dT_i(h)}{dS_i(h)}\right]$ with $M_i(h) = \int^h m_i(\zeta)d\zeta$ where $m_i(h) = \frac{1}{\sigma_i^2 S_i'(h)}$ is the speed density function of the diffusion process (3.1)[5]. To establish this canonical representation, note that the scale

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[5] The name of 'speed density' can be justified as follows : for a natural and regular diffusion

function $S_i(h)$ of (3.1) satisfies by definition $\frac{\lambda_i(\mu_i - h)}{\sigma_i^2/2} = -\frac{S_i''(h)}{S_i'(h)}$. As the differential equation at hand becomes $\frac{1}{2}\frac{d}{dM_i}\left[\frac{dT_i(h)}{dS_i(h)}\right] = -1$, it can be integrated successively. Clearly, two boundary conditions apply : $\widetilde{T}_i(\underline{h}) = \widetilde{T}_i(\overline{h}) = 0$. For more details, see Karlin and Taylor (1981 *p.* 197). To compute $m_i(h) = \frac{1}{\sigma_i^2 S_i'(h)}$, note that $S_i(h) = \int_{h_e}^{h} e^{\frac{\lambda_i}{\sigma_i^2}(x-\mu_i)^2} dx$ by Proposition 1, so that $m_i(h) = \frac{1}{\sigma_i^2} e^{-\frac{\lambda_i}{\sigma_i^2}(h-\mu_i)^2}$. ■

Proposition 4 follows from Lemma 3 by noting that $\widetilde{T}_a = ET_a^{h_a,+\infty}(h_b)$, $\widetilde{T}_b = ET_b^{h_e,h_b}(h_a)$, and that $\lim_{\overline{h}\to+\infty} P_a^{h_a,\overline{h}}(h_b) = 0$.

Proposition 4 *Suppose that the species is not extinct and that h_a exists : (1) The mean duration of a logging ban is*

$$\widetilde{T}_b = 2\left\{P_b^{h_e,h_b}(h_a)\int_{h_a}^{h_b}\left(S_b(h_b) - S_b(\xi)\right)m_b(\xi)d\xi \right. \tag{3.8a}$$
$$\left. + \left(1 - P_b^{h_e,h_b}(h_a)\right)\int_{h_e}^{h_a}\left(S_b(\xi) - S_b(h_e)\right)m_b(\xi)\right\} d\xi$$

(2) The mean duration of a logging period is

$$\widetilde{T}_a = 2\int_{h_a}^{h_b}\left(S_a(\xi) - S_a(h_a)\right)m_a(\xi)d\xi \tag{3.8b}$$

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process x_t, $\varepsilon^2 m(x)$ is of the same order as the expected time for the process to leave the interval $]x-\varepsilon, x+\varepsilon[$ where x is its state at time zero and ε is positive and small relative to x.

2.4 Optimal forest management rule and sensitivity analysis

2.4.1 Estimation of the stochastic process governing caribou habitat

The herd of Central Labrador's *Rangifer tarandus* Caribous, also known as Red Wine Mountains caribous, occupies an area of about two million hectares corresponding approximately to Labrador's Forest District $19A$, near Goose Bay, Newfoundland and Labrador, Canada. Although non-subsistence hunting was prohibited in 1972, its population declined significantly from over 700 animals in the 1980's to 151 by 1997 (Schaefer et al 2001). The species was listed as threatened in 2002 and subsistence hunting was banned. With little other threats to their livelihood, it is believed that the caribous hold a good prospect for survival. The main risk that they face is encroachments by human activities and development on their habitat. Caribou habitat typically consists of contiguous areas of old forest with minimal human disturbance (Schmelzer et al, 2004).

The town Happy Valley-Goose Bay has about $8,000$ aboriginal and non-aboriginal inhabitants. The nearby Innu community of Sheshatshiu has a population of about $1,600$ people and lives on Forest District $19A$. In the past, the Canadian Forces Base in Goose Bay was a major source of local employment and socio-economic development although benefits to the Innu Nation remain a contentious issue. Currently, the base operates on a reduced scale. Caribou may not be as critical to the Innu community as they once were but they are still central to their culture and highly valued. Labrador's Forest District $19A$ has never been commercially exploited in any significant way. It is currently proposed to substantially develop logging in the region, a project jointly conducted by the Innu Nation government and the Labrador Department of Forest Resources and Agrifoods. The main impact on the caribou herd would be habitat destruction or fragmentation.

Using a landscape level disturbance simulation model, the Spatially Explicit Landscape Event Simulator (SELES) due to Fall and Fall (2001), we generated caribou habitat series simulating forest management with logging and without logging alternatively. The SELES model is a grid-based, spatiotemporal model, that incorporates both landscape characteristics, plausible disturbances such as wild fires or infestations, and forest management practices, to generate indicators of forest structure as it evolves over time.

For the purpose of this application, the logging regime is based on the rate of harvest described in the forest management planning documents (Forsyth et al, 2003) for the period 2003 – 2023. The logging rate is assumed constant at 62 000 m^3 per year, which corresponds to the maximum sustainable yield. Being the main natural perturbation affecting caribou habitat, wild fires are the sole source of stochastic disturbance explicitly introduced in the model. Based on the size and frequency of fires in the past thirty five years (Morgan et al, 2008), the model is set to assume a fire rotation[6] of 343 years.

Considering the slow rate of growth of the boreal forest (Fahrig 2001), we generate Monte-Carlo habitat series that extend over one thousand years on ten-year intervals, starting with current period 2003 – 2008 conditions. One such series for each forestry regime is illustrated in Figure 3.2. One notes that extinction occurs after about six hundred years in the sample curve with logging. No such event hap-

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[6] The fire rotation is the time required to burn a surface equivalent to the reference area.

pens in the case where logging is prohibited; however, while less probable in the short run, extinction occurs with probability one over an infinite period.

The analysis of the habitat series generated by SELES indicates that their autoregressive correlation coefficients are decreasing while their partial autoregressive correlation coefficients are negligible except for the first-order coefficient. Consequently, both the habitat series with logging ($i = a$) and without logging ($i = b$) can be assumed to be Gaussian autoregressive processes of order 1 ($AR1$). We write them as discrete versions of (3.1) :

$$h_{it} = \mu_i(1 - \rho_i) + \rho_i h_{t-1} + \eta_i \varepsilon_{it}, \; i = a, b \tag{3.9}$$

with $\lambda_i > 0$, $\rho_i = e^{-\lambda_i} < 1$, $\eta_i^2 = \sigma_i^2 \frac{1-e^{-2\lambda_i}}{2\lambda_i}$, and where ε_{it} is a standardized Gaussian white noise (for more details, see Gouriéroux and Jasiak, 2001, chapter 11). Box-Pierce and Breusch-Godfrey Lagrange multiplier tests validate this representation as they exhibit uncorrelated error terms. The sample was used to estimate (3.9). The Maximum Likelihood estimators are[7] :

Forestry Regime :	logging allowed : $i = a$		logging banned : $i = b$	
Estimated values :	mean	st. dev.	mean	st. dev.
Long-run expected habitat : $\widehat{\mu}_i$	12.7034	0.00497	13.4504	0.00513
Speed of reversion (per decade) : $\widehat{\lambda}_i$	0.0575	0.011	0.0532	0.0106
Variance (per decade) : $\widehat{\sigma}_i$	0.0028		0.0028	

According to the estimated parameters, logging affects the long-run level of habitat ($\widehat{\mu_a} < \widehat{\mu}_b$) as expected but has little effect on the speed of reversion and on

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[7] Let $T = 100$ be the number of observations. For $i = a, b$, $\widehat{\mu}_i = \overline{h}_{iT}$, $\widehat{\lambda}_i = -\log\left[\frac{\sum_{t=1}^{T}(h_{it}-\overline{h}_{iT})(h_{i,t-1}-\overline{h}_{iT})}{\sum_{t=1}^{T}(h_{it}-\overline{h}_{iT})^2}\right]$; and $\widehat{\sigma}_i^{\,2} = \frac{2\widehat{\lambda}_i}{1-e^{-2\widehat{\lambda}_i}}\widehat{\eta}_i^{\,2}$ where $\overline{h}_{iT} = \frac{\sum_{t=1}^{T} h_{it}}{T}$, $\widehat{\eta}_i^{\,2} = \frac{\sum_{t=1}^{T}\widehat{\varepsilon}_{it}^2}{T}$, and $\widehat{\varepsilon}_{it}^2 = h_{it} - \overline{h}_{iT} - e^{-\widehat{\lambda}_i}\left(h_{i,t-1} - \overline{h}_{iT}\right)$.

the volatility. Introducing logging causes the long-run level of caribou habitat to decrease from 694 110 ha to 328 860 ha (from 62% to 29% of the forest area), a drop of more than 50%.

As discussed in Section 2, a landscape becomes inadequate as caribou habitat when less than 30% of old growth habitat is intact. In this context old growth habitat must be interpreted as the expected long-run habitat level when logging is banned. Then the level of habitat corresponding to caribou's extinction can be estimated to be around 30% of e^{μ_b} or 208 230 hectares, which is significantly lower than the long-run habitat level when logging is taking place; in logarithm, this translates to $h_e = 12.2464$, or 18% of the forest area.

2.4.2 Application of the real option model

Assuming constant timber prices, the flow of logging revenues is constant. We set ω to unity, so that all monetary values in the model have to be multiplied by the actual logging revenue flow to give nominal dollar figures. We use a discount rate of 5%. As it is difficult to get accurate estimations of the costs to ban and to resume logging, we assume I_a and I_b to represent alternative percentages of logging revenues and conduct an analysis of the decision rule's sensitivity to that percentage. Unless otherwise mentioned, I_a and I_b are both set to 10% of an hypothetical perpetuity $\frac{\omega}{r}$ from logging revenues, that is $I_a = I_b = 2$ for $\omega = 1$ and $r = 5\%$.

The existence value of the caribous is a very important and controversial issue. While some Innus consider the caribous to be priceless and their existence beyond evaluation, some also consider that the very existence of the Innus as a distinct culture is compromised by economic hardship. Logging and learning how to control that activity might be one way toward survival and adaptation. While these views appear in contradiction with each other, the model clarifies at least two points about the relative values of ω and s. The first one is that, by Proposition 1, there exists some upper threshold h_b above which logging should be allowed however high the existence value of the species. The second point is that, by Proposition 2, it is possible for the existence value of the caribous to be low enough with respect to logging revenues for a logging ban never to be optimal. This relative value depends on all other parameters and can be computed ; in the current application, s must be higher than ω by more than 10% if logging is ever to be banned. Consequently we describe the optimal decision rule for a range of s/ω values strictly above unity ; as a base case and unless otherwise mentioned, s equals 10 times ω.

The forest value functions V_b and V_a are computed numerically as solutions to $(3.4a)-(3.4d)$ and $(3.7a)-(3.7d)$ respectively by adapting the box method described in Zwillinger (1998). Figure 3.3 represents the forest value function $V_a(h)$ for values of h above h_a. The threshold h_a applies when it is crossed from the right to the left ; it is equal to 12.2972 which implies that logging is to be interrupted if habitat gets down to 219 082 hectares, which is only 5% above the extinction level and dangerously close to it. The figure also illustrates the value-matching and smooth-pasting conditions that link $V_a(h)$ and $V_b(h)$ at h_a. Since the value matching condition is $V_a(h_a) = V_b(h_a) - I_b$ and the relevant function to the left of h_a is $V_b(h)$, the plain curve to the left of h_a represents the forest value function net of the cost of banning logging, $V_b(h) - I_b$.

Once logging has been banned, the forest value function is $V_b(h)$ and this situation continues as long as habitat does not cross, from the right, the extinction threshold h_e or, from the left, the threshold h_b at which logging should be resumed. That threshold is equal to 12.3567 or 232 513 hectares, which amounts to 12%

more than the extinction threshold. When h_b is reached, logging is resumed at cost I_a and the forest value function becomes $V_a(h)$ as long as habitat remains above h_a. Figure 3.4 illustrates the value-matching and smooth-pasting conditions between $V_b(h)$ and $V_a(h) - I_a$ at h_b.

As implied by Proposition 2.3, even if Assumptions 2 and 3 are satisfied, there may not exist any value of h compatible with current caribou existence at which a ban on logging is optimal. However, while h_a may not exist, h_b always does. Both thresholds, as well as the short-run extinction risks $R_a(h_a)$ and $R_b(h_a)$, vary with the existence value of the caribous as represented by s. In the base case, the short-run extinction risk decreases from approximately $R_a(h_a) = 50\%$ to $R_b(h_a) = 10\%$ when logging is banned at $h = h_a$. This is a reduction of about 80%.

Figure 3.5 illustrates the sensitivity to s of the thresholds h_a and h_b, and of the short-run extinction risks at the time logging is interrupted. The extinction risk curves are decreasing as implied by Proposition 1. The curves describing h_a and h_b as functions of s are rising. That is to say, at higher values of s, a logging regime is interrupted earlier (h_a is higher) and a ban on logging is ended later (h_b is higher), than at low values of s. When s is low, habitat is allowed to come closer to the extinction level before logging is banned ; however the banning threshold is strictly higher than h_e. As implied by Proposition 2.3, h_a does not exist for values of s below $\underline{s}$; this critical species value turns out to be 10% higher than ω and is associated with a threshold $\underline{h_a}$ which exceeds h_e by only 0.5% ; indeed the risk of extinction is allowed to become as high as 82% before logging is interrupted.

Figure 3.6 applies Proposition **4** to compare the mean duration of a ban with the mean duration of a logging regime at various levels of s. In the base case, when $s = 10\omega$, the former is about 8 years while the latter is around 86 years ; furthermore this difference in favor of logging increases as s increases. Meanwhile, as the figure also indicates, the relative change in extinction risk when a ban is introduced is quite drastic, at about 80% in the base case, and rising with s. This means that a policy optimally designed to protect the caribous by banning logging temporarily is quite attractive as it does not require long banning periods while it drastically reduces the risk of extinction[8].

The longer expected duration of logging regimes stems from the fact that such regimes apply at higher habitat values : $h \in [h_a, +\infty[$. Since $\mu_a \in [h_a, +\infty[$, this means that the interval includes values where the deterministic component of the habitat's motion is zero or small ; at higher values, the motion may be fast but no regime change may occur. Thus h tends to stay for extended periods in that regime. In contrast, bans involve values of h in $[h_e, h_b]$; as h_b turns out to be lower than μ_b, this means that bans are associated with relatively fast changes in habitat, so that they do not last on average as long as logging regimes do.

It is also surprising at first to see that the mean duration of logging regimes increases with s. Indeed, since h_a increases with s it takes less time, the higher s, for habitat to reach h_a from any level $h > h_a$; this factor reduces the expected duration of logging at higher values of s. However, since h_b also increases with s, logging is introduced at higher levels of h, the higher s ; this means that it takes longer for h to reach from h_b any given level below h_b ; this factor increases the expected duration. Furthermore h_b being closer to μ_b, the higher s, logging starts in a zone where h is not moving fast, and more so, the higher s. Together these last two factors dominate the first one and explain why the mean duration of logging regimes increases with

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[8] This property emerges from the numerical analysis and appears to be robust to the range of parameters that we have considered.

the existence value of the species.

The next two figures illustrate the impact of the discount rate. Figure 3.7 demonstrates its impact on h_a, h_b, and on the short-run extinction risk at the time logging is interrupted. Higher discount rates require logging to be resumed earlier and banning to occur later as both h_a and h_b decrease when the discount rate increases. In both cases, a higher discount rate reduces the weight assigned to future extinction so that it promotes risk taking.

Figure 3.8 illustrates the sensitivity to the discount rate of the expected logging and ban durations. The expected duration of a logging period decreases as h_a and h_b become closer to each other as a result of the increase in r ; similarly the expected duration of a ban decreases as h_b becomes closer to h_e.

The costs of interrupting or reintroducing logging are other factors affecting thresholds and optimal short-run risks of extinction. Figure 3.9 depicts how the habitat thresholds and extinction risks h_b and h_a vary according to I_a and I_b. When I_b increases, the decision to ban is delayed so that h_a becomes closer to the extinction level h_e. Similarly, when I_a increases, the decision to resume logging is delayed and h_b increases. The costs I_a or I_b are incurred at the time the change in logging regime is implemented and their effect is intuitively obvious. However another, indirect, cost of a change in logging regime is the cost that will be incurred in the future when a new regime change is called for. If the cost of reverting in the future to the logging regime being abandoned is high, the decision will be postponed. This means, *e.g.*, that an increase in I_b not only reduces h_a but also raises h_b. The figure does not separate such secondary effect from the direct effect as I_a and I_b are taken to be equal and vary together as a percentage of ω/r.

Finally, Figure 3.10 illustrates the extent to which adopting the optimal management rule described in this paper increases forest value relative to the value achieved under two alternative, sub-optimal, policies. The first alternative consists in allowing logging forever while the second alternative consists in banning logging forever as long as the species is in existence, and introduce logging if it becomes extinct. In order for the alternative value curves to be comparable, we assume that the initial situation is such that no logging is going on. Introducing logging consequently requires the initial outlet I_a whether this is a permanent change or a temporary one. On the contrary, the permanent ban policy does not involve any expenditure.

Under the optimal policy, the initial logging ban continues until either habitat falls to the extinction threshold (then logging is resumed forever), or habitat increases to h_b (then logging is allowed and goes on as long as habitat remains above h_a). The continuous curves represent the corresponding value function.

Allowing logging forever amounts to giving up the option to protect the species; this option is less valuable when habitat is abundant and the species is not immediately threatened. The forest value function corresponding to that policy is $\tilde{V}_a(h)$, defined by (3.6) and characterized in Proposition 2.2. It is lower than the optimum value function and approaches it when h tends to infinity. At finite habitat values, the difference is strict; for example, the optimum policy increases forest value by 11% over the permanent logging policy when $h = h_a$.

The policy of banning logging until extinction is a lesser mistake when habitat is low because the optimal policy then requests to prohibit logging anyway, at least temporarily. In fact, when h tends toward the extinction threshold, the value achieved by a permanent ban tends toward that achieved by the optimal policy. This is because, in case of extinction, logging is introduced or reintroduced in both cases.

The forest value function arising from a permanent ban policy is :

$$\tilde{V}_b(h) = E\left[\int_0^{T_b^{h_e}(h)} se^{-r\tau}d\tau + e^{-rT_b^{h_e}(h)}\left(\frac{\omega}{r} - I_a\right) |h_0 = h\right], \; \forall h \in [h_e, +\infty[$$

One can show that it must solve

$$\frac{\sigma_b^2}{2}\tilde{V}_b''(h) + \lambda_b\left(\mu_b - h\right)\tilde{V}_b'(h) - r\tilde{V}_b(h) + s = 0, \;\forall h \in]h_e, +\infty[$$

with the two boundary conditions :

$$\tilde{V}_b(h_e) = \frac{\omega}{r} - I_a$$

$$\lim_{h \to +\infty} \tilde{V}_b(h) = \frac{s}{r}$$

When $h = h_b$, the optimal policy yields 7% more than a permanent ban; when h tends toward infinity, it yields 9% more.

2.5 Conclusion

The real options model that we have presented applies when human activity may affect a species or a natural site adversely. It optimizes the trade-off between the benefits associated with this activity and the risks involved for the natural environment. Partial or total irreversibility is present not only as extinction is final but also as policy changes are costly to introduce and to undo. Uncertainty affects the evolution of the species or site.

In the empirical application that we have presented, the habitat of the species is a stochastic variable which is currently observable but whose future level is unknown. The objective of the decision maker is to maximize expected future benefits from the forest, that is to say benefits derived from the existence of the endangered species and benefits derived from timber exploitation.

The sole instrument to achieve this objective is the discretion to ban logging if habitat becomes dangerously low, and to resume logging if habitat recovers sufficiently. Besides the obvious effect on wood harvest, such changes in the logging regime affect the stochastic process governing habitat. In the empirical application presented in the paper the stochastic process was estimated by Monte Carlo methods for each regime.

Given the uncertainty surrounding the future, the decisions to authorize or to ban logging are not specified in advance; what is specified by solving the optimization problem is a rule to be followed in the future and according to which the current logging regime is maintained or interrupted depending on the current habitat level. As usual in real options models, the optimal rule takes advantage of uncertainty in such a way as to increase exposure to favorable outcomes (when the habitat grows more than expected), while seeking protection from unfavorable outcomes (low habitat levels). Interrupting logging when habitat is dangerously low does not guarantee that extinction will not occur but reduces its probability, thus providing (partial) protection against that unfavorable outcome. In that respect the model just presented is a rigorous application of the precautionary principle. It does not rule out risk taking, but agrees with the conventional wisdom that decisions should bend the distribution of risk a community is exposed to in such a way as to reduce the probability of irreversible catastrophes.

Whether the current regime allows or prohibits logging, there is a well defined probability of extinction over any finite horizon, and that probability is higher, the closer habitat is to the extinction level. The decision rule established in the paper optimizes the trade-off between the risk of extinction and the benefits derived from logging. We have described it in terms of the habitat levels that trigger logging bans or resumptions, but also in terms of the risks of extinction, and in terms of the value derived by the community from the forest. We have also compared the forest value achieved by using the optimal policy to the values implied by less sophisticated policies; the gain is of the order of 10% at empirically relevant habitat levels. Such magnitudes provide alternative, perhaps more intuitive, descriptions of the optimal policy, as also do the mean durations of each regime. Whatever the way the optimal policy is examined, it appears to provide an attractive and simple solution to the problem of protecting a species while not giving up other benefits altogether.

The existence value of the endangered species is central to the model and determines the risk of extinction implied by the optimum rule. While the existence value is always difficult to estimate and controversial, to the point that stakeholders frequently deny that it is even amenable to any form of estimation, the same actors are often willing to consider risks of extinction and to evaluate the effect of policy decisions on such risks. By making explicit the relationship between existence value and the willingness to increase the extinction risk under the optimal policy, our model can also be viewed, and used, as a way to infer a species valuation from the willingness to take risks with respect to its survival.

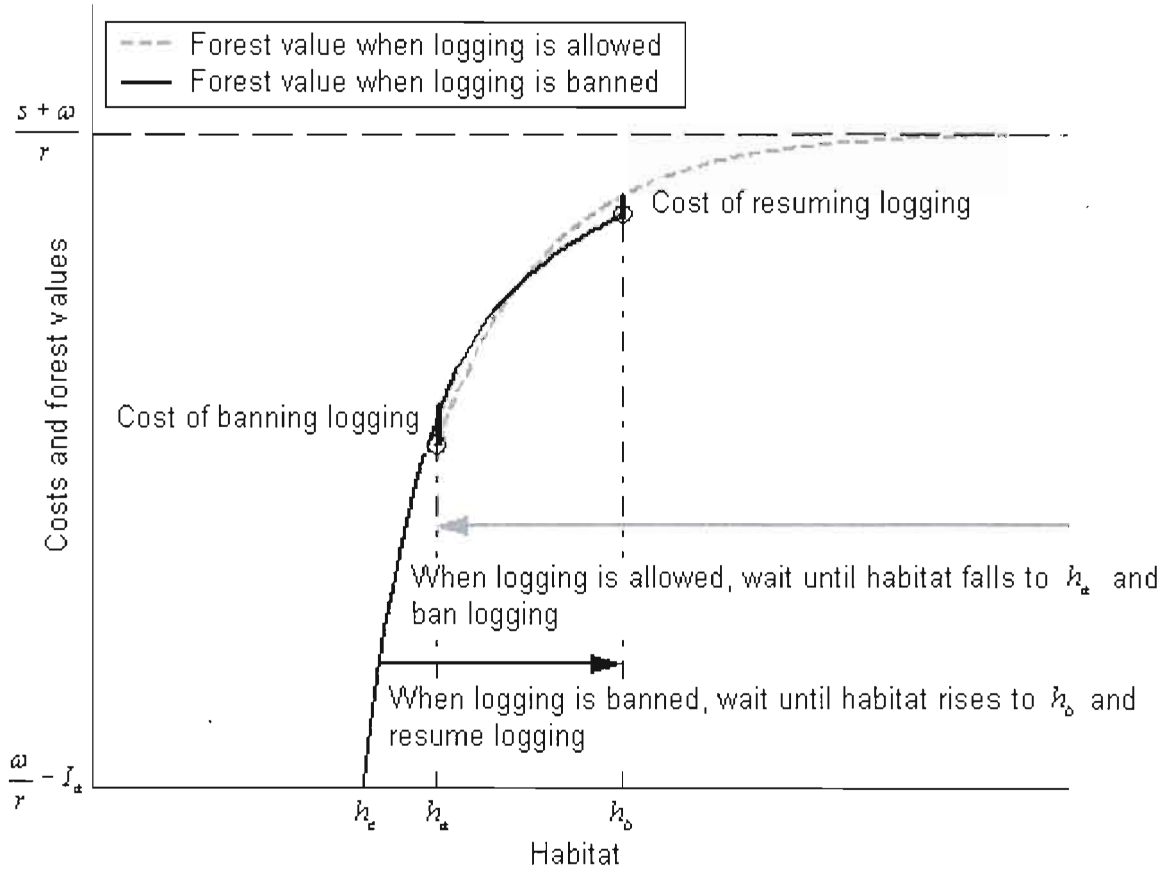

Fig. 3.1: The decision rule and the value functions at alternative habitat levels when logging is allowed or prohibited

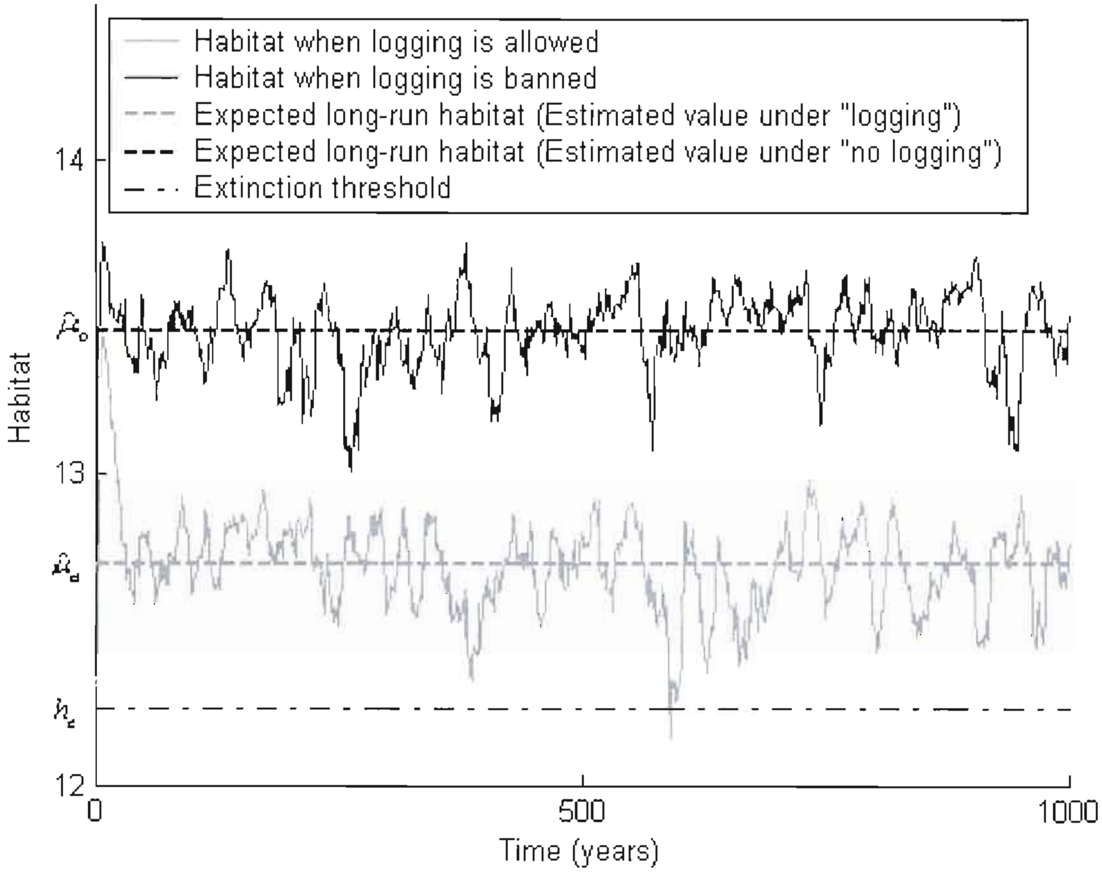

Fig. 3.2: Two realizations of caribou habitat from the SELES landscape model (logging and "no logging" regimes)

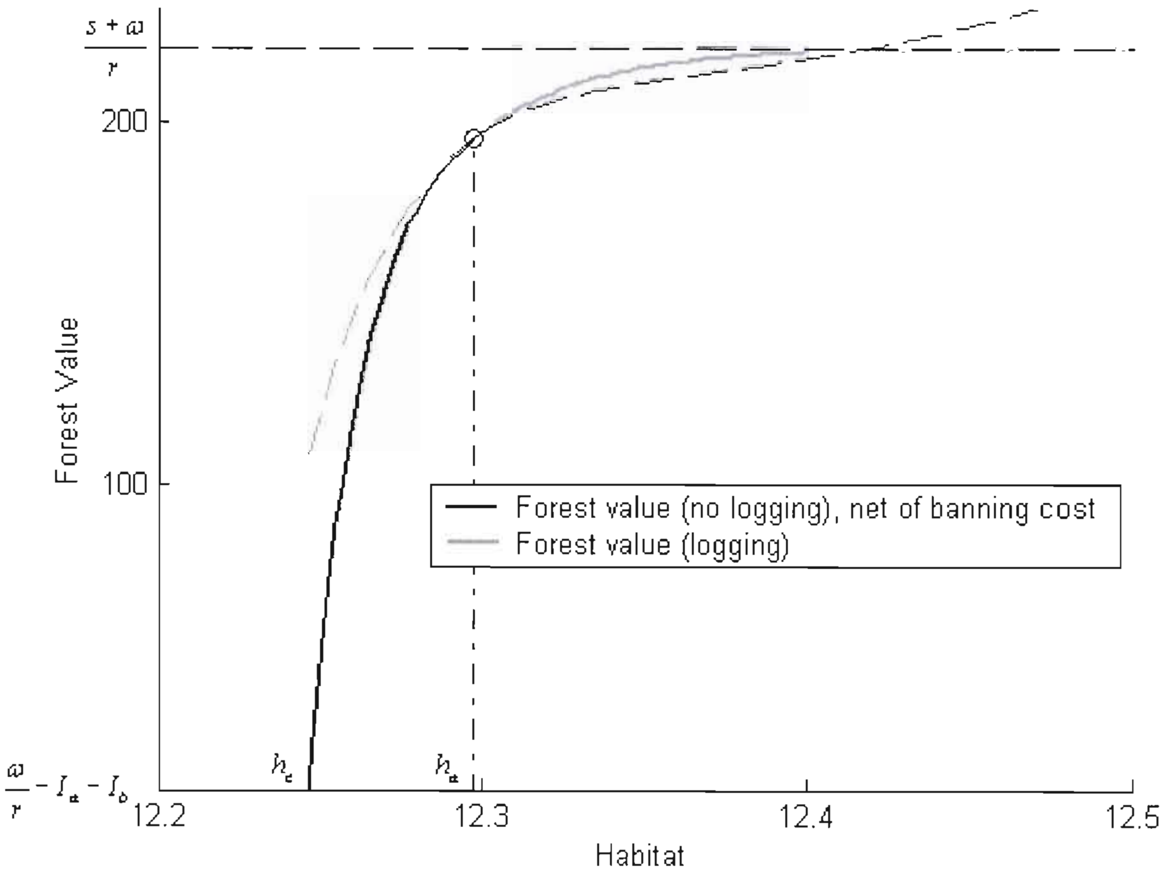

Fig. 3.3: Forest value and habitat threshold during a logging regime

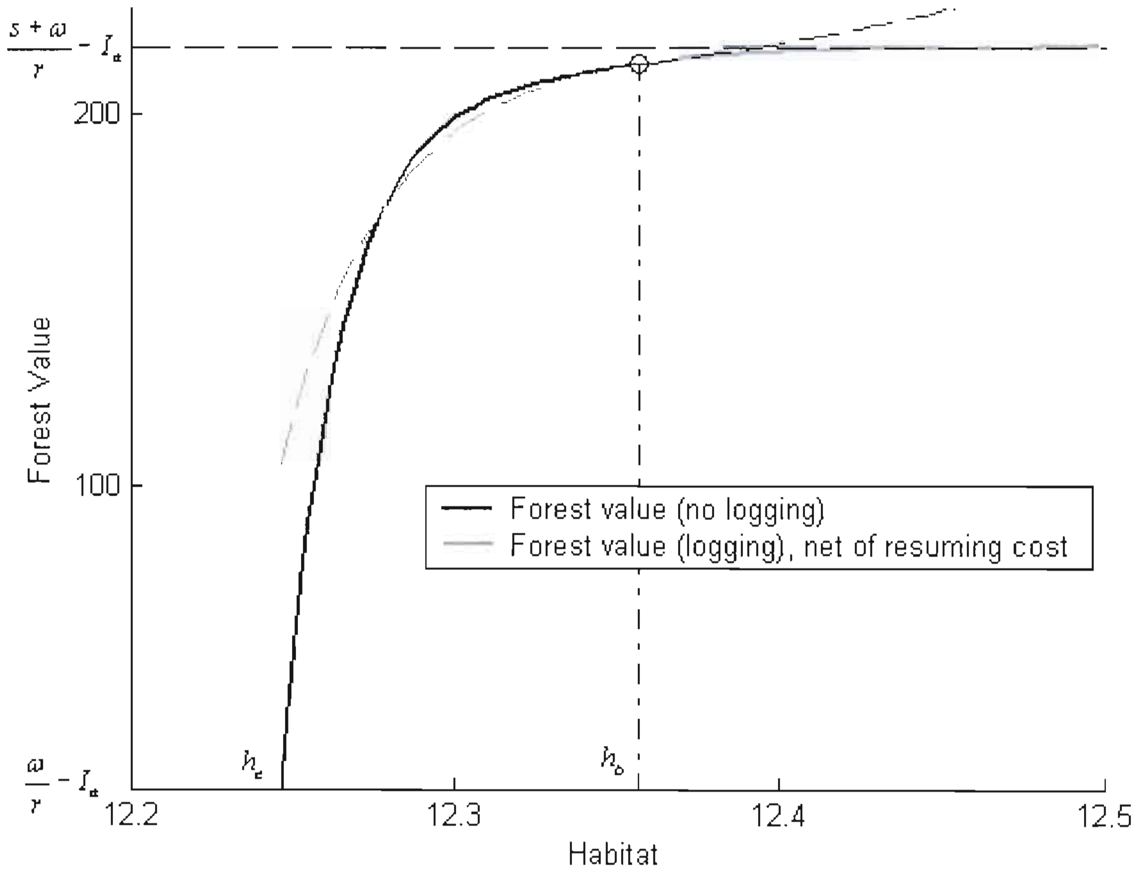

Fig. 3.4: Forest value and habitat threshold during a logging ban

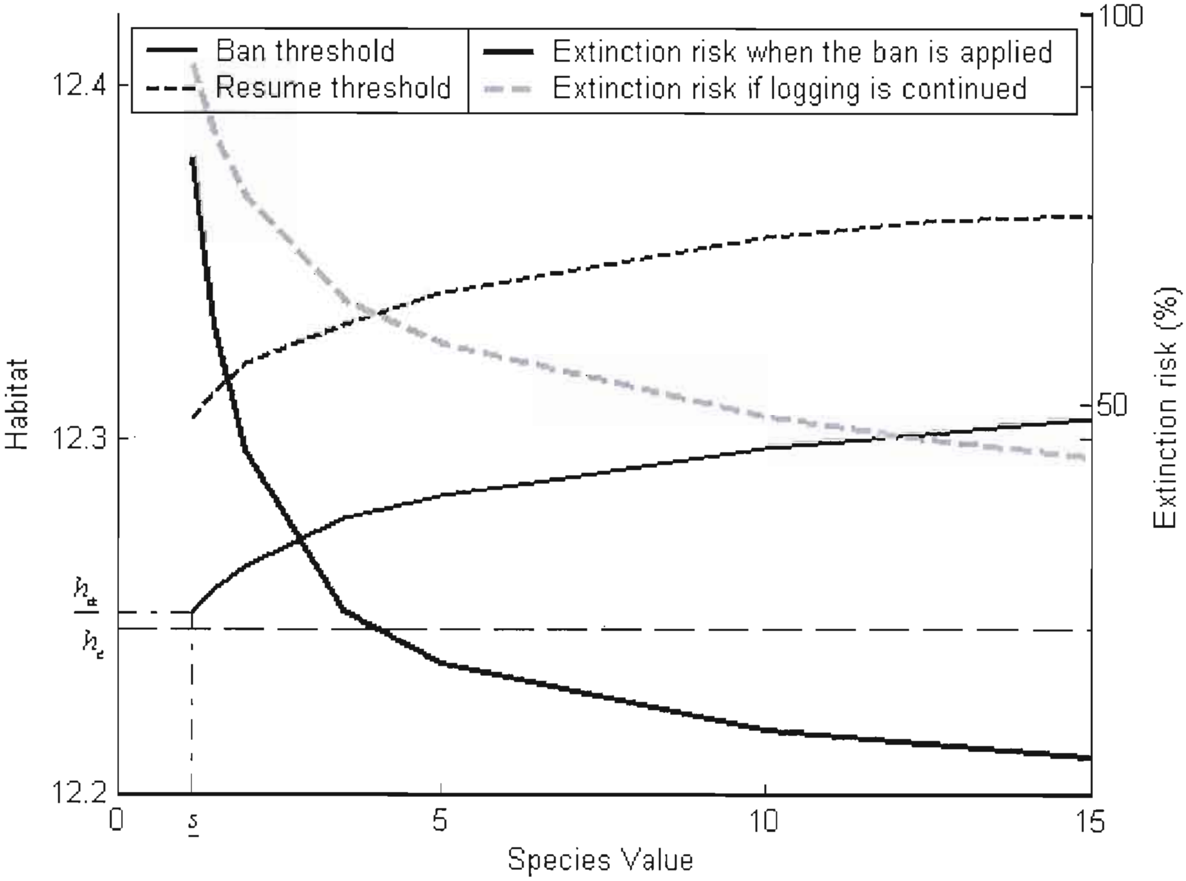

Fig. 3.5: Habitat triggers (left-hand scale) and short-run extinction risk (right-hand scale) according to species value

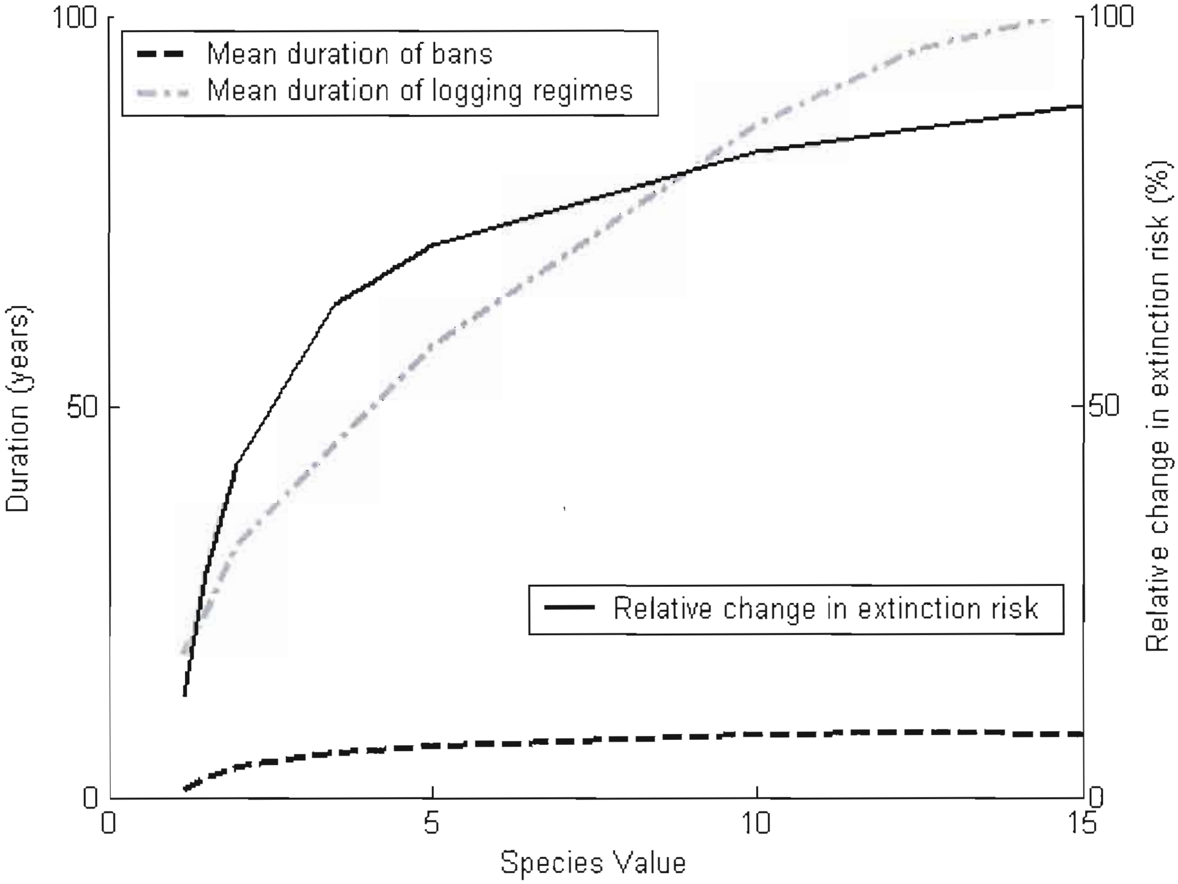

Fig. 3.6: Mean duration of forestry regimes (left-hand scale) and impact of a ban on short-run extinction risk (right-hand scale) according to species value

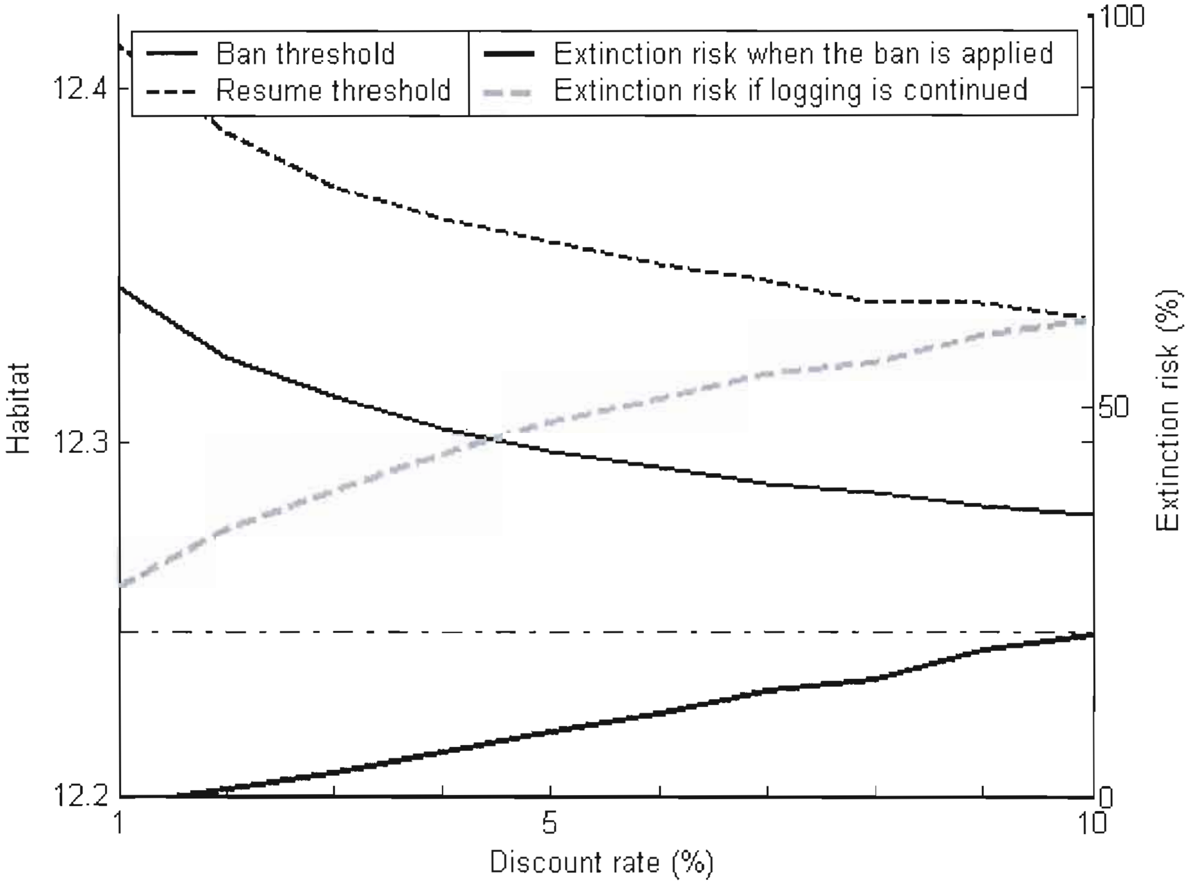

Fig. 3.7: Habitat thresholds (left-hand scale) and short-run extinction risk (right-hand scale) according to the discount rate

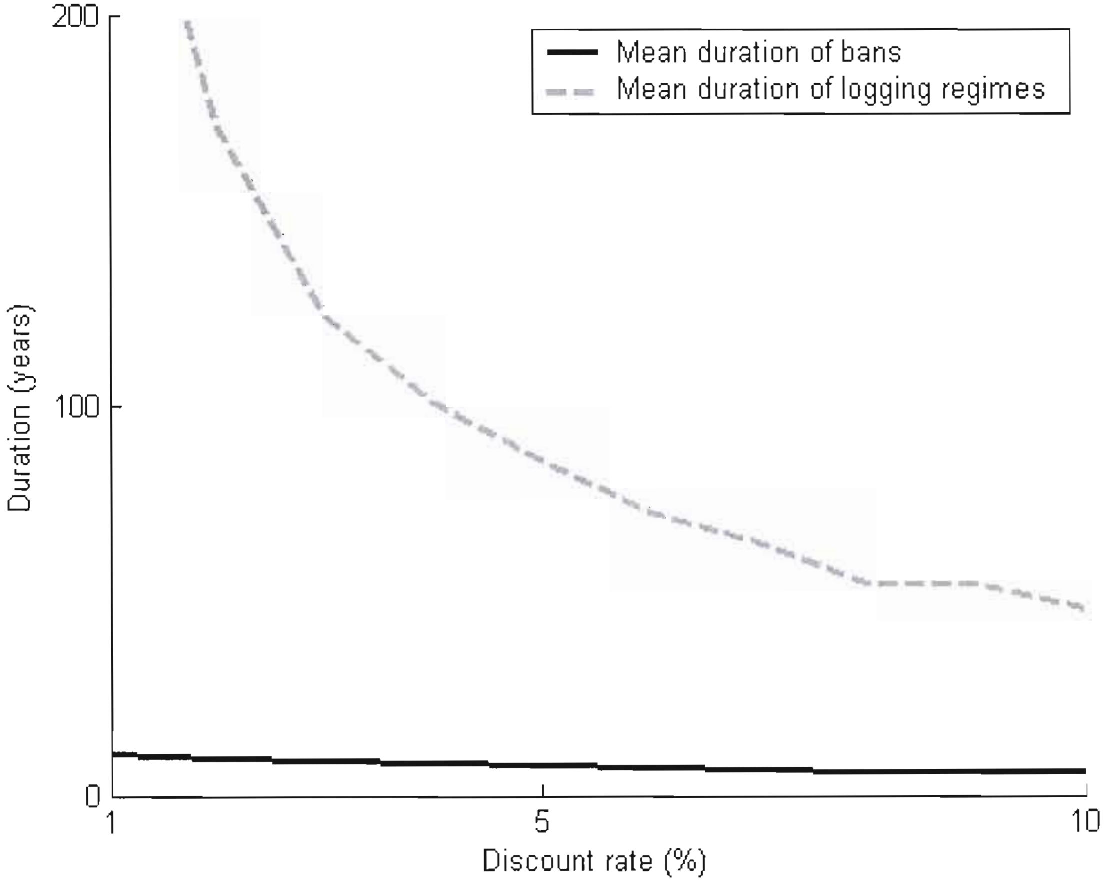

Fig. 3.8: Mean duration of each forestry regime according to discount rate

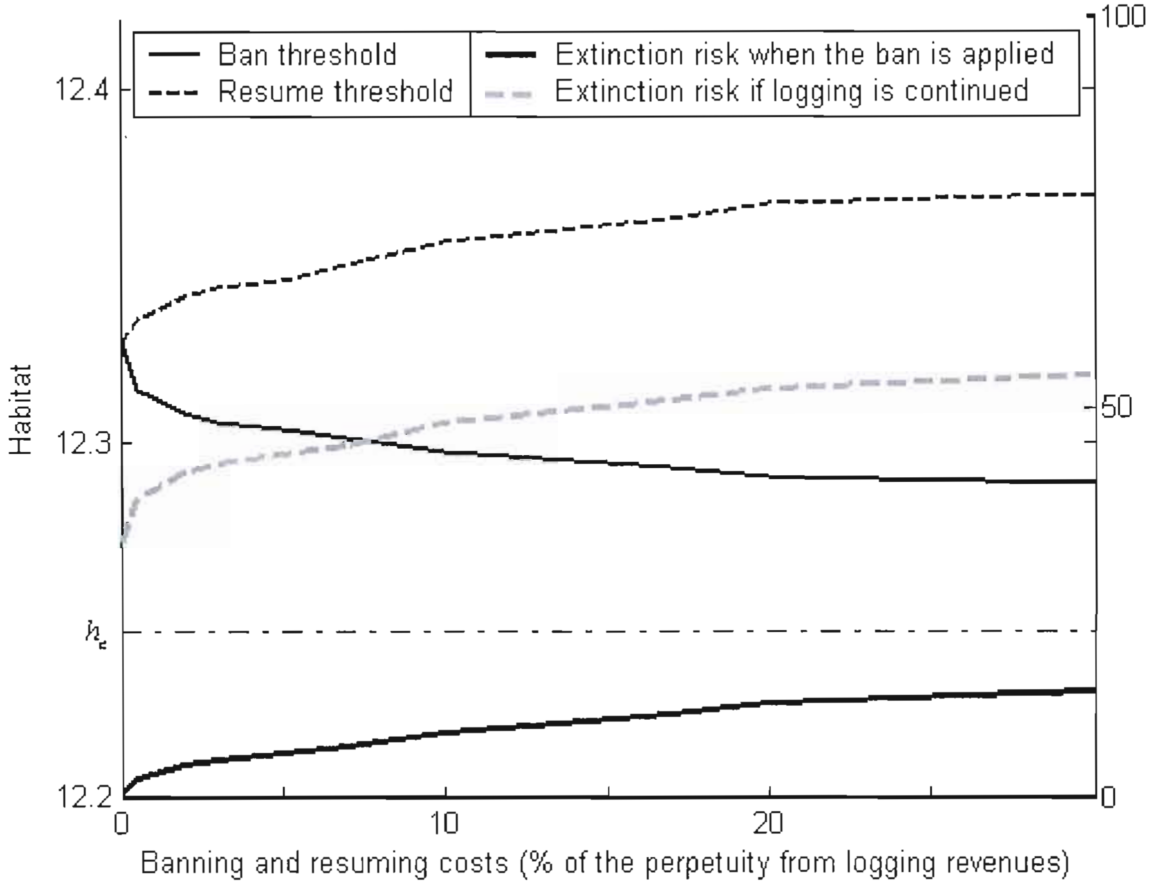

Fig. 3.9: Habitat triggers (left-hand scale) and short-run extinction risk (right-hand scale) according to the costs of interrupting or resuming logging

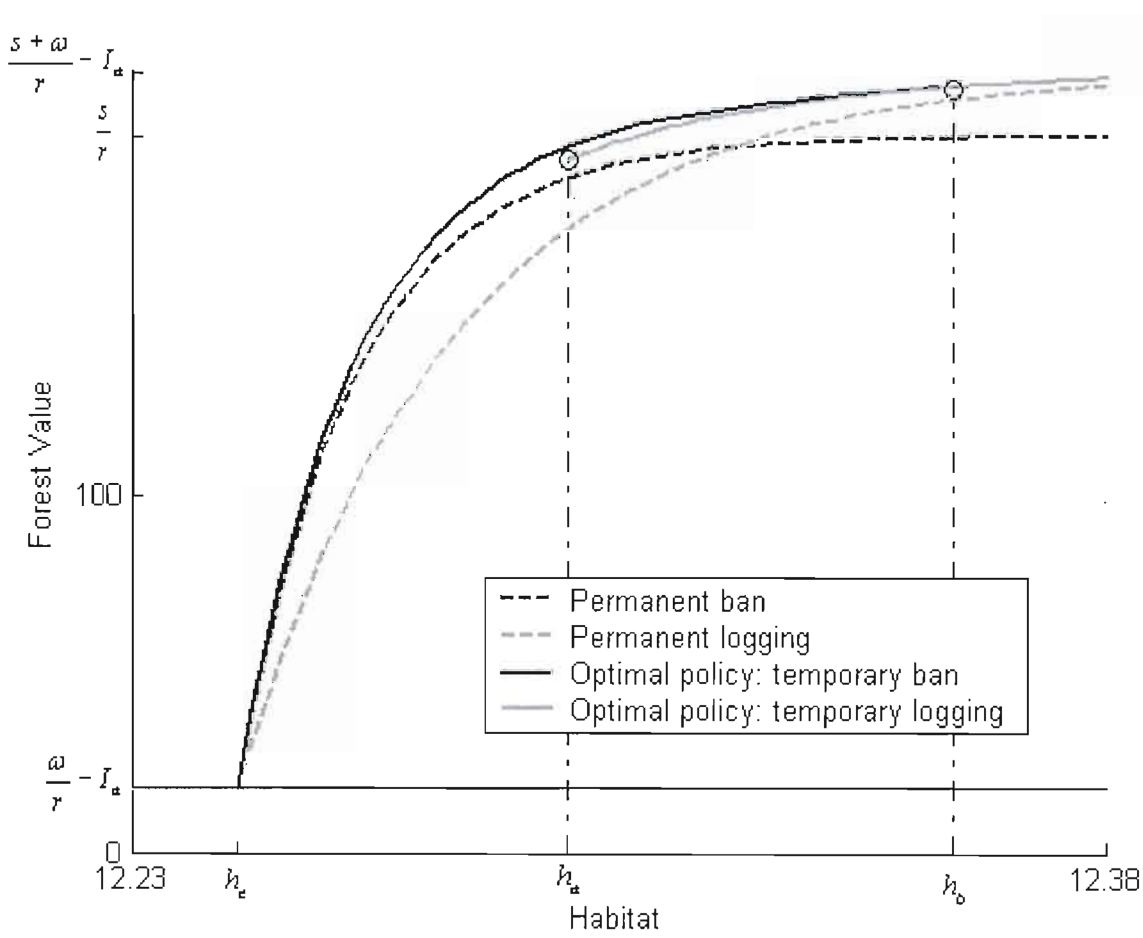

Fig. 3.10: Comparing forest values under the optimal policy, under a permanent ban, and under permanent logging

Chapitre 3

Alternative and Indefinitely Repeated Investments : Species Choice and Harvest Age in Forestry

3.1 Introduction

Forest management is an archetypal investment problem ; it involves time ; it involves uncertainty ; it involves irreversible actions with consequences in the future. It also exemplifies investments that open up new options : cutting a tree opens the option of planting a new one or using the land for different purposes. Faustman (1849) gave forestry economics its foundations by addressing the question : at what age should a stand of even-aged trees be harvested ? He did so under the assumption of constant timber prices by comparing the net marginal benefits from letting timber grow further, to the opportunity cost of land, itself a function of timber management decisions.

The real option approach has its foundations in financial economics and a much shorter history. It has been widely applied in natural resources exploitation and management. Although real options precisely focus on issues of concern in forestry, such as irreversibility and uncertainty, applications are not straightforward. In the real option framework, a typical investment involves an optimum stopping rule, the choice of the date at which the decision maker considers that conditions have become favorable enough to justify committing resources irreversibly. A typical result is that more uncertainty postpones investment although it increases project value.

Although cutting trees involves the permanent loss of the possibility to choose the cutting age, it also involves the option of establishing a new harvest, which may be a reason for cutting earlier or later. Using a numerical approach and simplifying assumptions about the value of bare land, Thomson (1992) considers the optimal forest rotation when timber prices follow a diffusion Process. His binomial option pricing model shows that the rotation is then generally longer than the rotation that would be optimal in a Faustman model with a price equal to the expected price in the binomial model. Clarke and Reed (1989) and Reed and Clarke (1990) use stochastic continuous-time dynamic programming to consider the forest rotation problem when tree growth and timber prices follow Brownian processes. They stress the sensitivity of optimum rules to stochastic processes; they also show that when the only source of stochasticity is price, the optimum rotation is independent of the price if there is no cost to plant or harvest. Willassen (1998) dismisses the optimal stopping methodology and uses impulse control. He studies the optimal harvesting age in the case where the forest value is governed by a generalized Brownian process and further generalizes Faustman's rule. However he does not provide any additional unambiguous result about rotation.

While the real option literature initially focused on unique investment opportunities or sequential investment opportunities, the recent literature has begun to treat situations where the decision maker has more than one irreversible investment opportunity to choose from. These alternative opportunities may differ with respect to investment costs and output amount as in Dixit (1993*b*) or Decamps et al (2006) or benefit trajectories as in Kassar and Lasserre (2004). An interesting result in such situations is that a new reason for postponing action arises. Whenever the alternatives are too close to each other and uncertain, the decision maker will wait in order to avoid making a decision that might prove erroneous in the future. This inaction may be optimal although each project, taken in isolation, would satisfy the requirements for immediate investment under conditions of irreversibility and uncertainty.

The decisions investigated in the literature vein just alluded to are never repeated. In this paper we introduce alternative projects in situations where decisions open up new options in a process that repeats itself indefinitely, and must be reevaluated at each instant. Although this is a most common practical situation it raises difficulties that explain why it has never been investigated theoretically before. For example, when they replace machinery, managers choose between alternative technologies as well as they choose the timing of their decision. The decision to cut trees and to choose, after each harvest, which species should be established on the bare land, exemplifies such situations. Since the tree cutting problem is such a well known one, it is natural to use it as a benchmark for the introduction of alternative opportunities in repeated decisions contexts. We do so by introducing a second tree species to choose from after each harvest in an otherwise standard tree harvesting model. Each species is characterized by a different, stochastic, timber price process and by a distinct, although not stochastic, growth function.

In Section 2, We first introduce and solve a basic model where the decision maker chooses once and for all between two mutually exclusive projects under price uncertainty. Then we investigate, in Section 3, the case of a single rotation, also known as Wicksell's tree cutting problem. Stochastic versions of that problem for a single species have been discussed by Willassen (1998). An important difference brought about by the introduction of an alternative species is that we do not only consider the decision to cut trees, but also, and first, we consider the decision to plant one, or the alternative species. We do not assume that a young crop has already been established as was previously done in the literature. An important result in that model is the existence of hysteresis, conditions about relative species prices under which the decision maker waits rather than establishing a crop, despite the fact that each species would be worth planting in isolation. In Section 4 we extend the analysis to multiple rotations. The decision maker must decide at what age the trees of the current stand must be cut ; after harvest she must decide whether she should establish a new crop or wait ; when she makes such decision, she must choose one, or the other, species. This process is repeated indefinitely. We find that the hysteresis of the Wicksellian problem disappears.

The qualitative properties of the decision rules and value functions are described in a number of propositions. In particular, we show that, under uncertainty, the stand value is similar to the value of an American option with a free boundary and an expiry date equals to infinity but with endogenous payoff. The analysis is completed with a numerical resolution based on the penalty method (Zvan et al

1998) applied simultaneously to the stand value function of each species, and on a Newton iterative process applied to the land value. Most importantly, We show that the optimum cutting age increases and overpasses Faustman' age when the relative price approaches some threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species; letting existing trees grow older is a way to postpone the choice.

Land value increases with the uncertainty of timber prices. It is an increasing function of the relative price. As long as uncertainty is present, the slope of that function is continuous even around the switching threshold; this prevents any arbitrage. On the contrary, under certainty, the slope of the land value function generally changes at the switching threshold. Finally, Section 5 concludes.

3.2 Choosing between two alternative projects

Consider the problem faced by a risk-neutral decision maker when choosing between two alternative projects (P) and (P') under output price uncertainty. Investment and operational costs are nil. The project (P) (resp. (P')) produces one unit of output forever at a price p (resp. p') that follows a geometric Brownian motion (GBM) with drift μ (resp. μ') and volatility σ (resp. σ') :

$$dp = \mu p dt + \sigma p dz \tag{3.1a}$$

$$dp' = \mu' p' dt + \sigma' p' dz \tag{3.1b}$$

where $dz = \varepsilon\sqrt{dt}$ and $dz' = \varepsilon'\sqrt{dt}$ are the increments of a Wiener process while ε and ε' are standardized Gaussian white noises whose correlation is ρ. In the rest of the paper variable that depend on time will be indexed unless they are considered at the current date, in which case the time index is omitted.

In order to make a decision, the manager must compare the projects' expected net present values at investment time t, given the information available at this time. The expected net present values of investing in project (P) or (P') at time t, when prices are p and p', are respectively $E\int_t^{+\infty} e^{-r(s-t)}p_s ds = \frac{p}{\delta}$ and $E\int_t^{+\infty} e^{-r(s-t)}p'_s ds = \frac{p'}{\delta'}$ where $\delta \equiv r-\mu$, $\delta' \equiv r-\mu'$, r is the constant discount rate and E is the expectation operator conditional on information available at time t. We assume that $\delta > 0$ and $\delta' > 0$; otherwise it would be optimal to delay the investment forever.

The current option value function of investing in either project is then

$$F(P,P') = \max_t E\left[e^{-rt}\max\left(P_t, P'_t\right)\right] \tag{4.2}$$

where $P \equiv \frac{p}{\delta}$, $P' \equiv \frac{p'}{\delta'}$, and t is the date of the investment. As $F(P,P')$ is homogenous in (P,P') it can be written $F(P,P') = Pf(\theta)$ where $\theta \equiv \frac{P'}{P}$.

As long as the investment has not occurred, in the continuation region, $F(P,P')$ should satisfy Bellman's Equation $EdF = rFdt$. Itô's lemma implies that $f(\theta)$ satisfies :

$$\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_\theta(\theta) - \delta f(\theta) = 0 \tag{4.3}$$

with $\overline{\sigma}^2 \equiv \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$ and $\overline{\mu} \equiv \mu' - \mu = \delta - \delta'$.

This differential equation has solutions when its characteristic equation $Q(\beta) = 0$ has real roots, where $Q(\beta) = \frac{\overline{\sigma}^2}{2}\beta(\beta-1)+\overline{\mu}\beta-\delta$. Its determinant $\Delta \equiv (\overline{\mu}-\frac{\overline{\sigma}^2}{2})^2+2\overline{\sigma}^2\delta$ is zero if and only if ($\mu = \mu'$, $\sigma = \sigma'$, $\rho = 1$) or ($\mu = \mu'$, $\sigma = \sigma' = 0$). In such cases, prices follow parallel trajectories, so that the higher price will remain so forever ; it is then optimal to invest immediately in the project with the highest price. To rule out these trivial cases, we assume from now on that $(\mu,\sigma) \neq (\mu',\sigma')$, so that the determinant of $Q(\beta)$ is strictly positive and the characteristic equation has two roots $\beta_1 > 1$ and $\beta_2 < 0$.

In this case, $f(\theta) = b_1\theta^{\beta_1} + b_2\theta^{\beta_2}$ where b_1 and b_2 are two constants to be determined. The option value function is then $F(P, P') = Pf(\theta)$ in the continuation region. This means that, at some levels of θ, it is optimal to wait rather than invest immediately.

For a current price level P, sufficiently high relative to P', that is for θ smaller than a critical threshold θ^*, it is optimal to invest immediately in (P); then $F(P, P') = P$ and $f(\theta) = 1$. On the boundary $\theta = \theta^*$, where the manager is just indifferent between waiting for an infinitesimal time or investing immediately, the option value function $F(P, P')$ and the value function P meet smoothly. This leads to value-matching and smooth-pasting conditions (Dixit 1993*a*) $f(\theta^*) = 1$ and $f_\theta(\theta'^*) = 0$, implying $F(P^*, P'^*) = P^*$, $F_p(P^*, P'^*) = 1$, and $F_{p'}(P^*, P'^*) = 0$.

Similarly, at levels P' sufficiently high relatively to P, that is for θ bigger than a critical threshold θ'^*, it is optimal to invest immediately in (P'); then $F(P, P') = P'$ or $f(\theta) = \theta$, and the value-matching and smooth-pasting conditions are respectively $f(\theta'^*) = \theta'^*$ and $f_\theta(\theta'^*) = 1$, implying $F(P^*, P'^*) = P'^*$, $F_P(P^*, P'^*) = 0$, and $F_{P'}(P^*, P'^*) = 1$.

Equations $f(\theta^*) = 1$, $f_\theta(\theta'^*) = 0$, $f(\theta'^*) = \theta'^*$, and $f_\theta(\theta'^*) = 1$ determine b_1, b_2,

θ^* and θ'^* as follows :

$$b_1 \equiv \frac{1-\beta_2}{\beta_1-\beta_2}\left(\frac{[\beta_1/(\beta_1-1)]^{\beta_1}}{[\beta_2/(\beta_2-1)]^{\beta_2}}\right)^{(1-\beta_1)/(\beta_1-\beta_2)} \tag{4.4a}$$

$$b_2 \equiv \frac{\beta_1-1}{\beta_1-\beta_2}\left(\frac{[\beta_1/(\beta_1-1)]^{\beta_1}}{[\beta_2/(\beta_2-1)]^{\beta_2}}\right)^{(1-\beta_2)/(\beta_1-\beta_2)} \tag{4.4b}$$

$$\theta^* \equiv \left(\frac{[\beta_1/(\beta_1-1)]^{\beta_1-1}}{[\beta_2/(\beta_2-1)]^{\beta_2-1}}\right)^{1/(\beta_1-\beta_2)} \tag{4.4c}$$

$$\theta'^* \equiv \left(\frac{[\beta_1/(\beta_1-1)]^{\beta_1}}{[\beta_2/(\beta_2-1)]^{\beta_2}}\right)^{1/(\beta_1-\beta_2)} \tag{4.4d}$$

The thresholds θ^* and θ'^* are independent of P and P'. The ratio of the slopes of the two linear boundaries $\theta = \theta^*$ and $\theta = \theta'^*$ is $\left(\frac{\beta_1/(\beta_1-1)}{\beta_2/(\beta_2-1)}\right)^{1/(\beta_1-\beta_2)}$, a function of r, $\overline{\mu}$, and $\bar{\sigma}$. This ratio measures the size of the continuation region, that is the combination of prices for which it is optimal to wait rather than invest in either project. The results are gathered in the following proposition.

Proposition 1 *The optimal decision involves choosing the time of the investment, and selecting one of the two alternative projects : (1) When the relative value* θ *of the projects lies between* θ^* *and* θ'^**, the optimum decision is to wait; when* θ *exceeds* θ'^**, the optimum decision is to invest in project* (P')*; when* θ *is lower than* θ*, the optimum decision is to invest in project* (P)*. (2) The size of the continuation region, decreases in* r *and increases in* $\bar{\sigma}^2$*. (3) The continuation region is symmetric if and only if* $\mu = \mu'$*, whatever the volatilities* σ *and* σ' *and whatever* $\bar{\sigma}^2$*.*

Proof. 1. is proven in the text.

2. We study the ratio of the boundaries' slopes $\left(\frac{\beta_1/(\beta_1-1)}{\beta_2/(\beta_2-1)}\right)^{1/(\beta_1-\beta_2)}$. Replacing β_1 and β_2 by their expressions, one can show that $\frac{1}{\beta_1-\beta_2}$ decreases in r and that

the derivative of $\frac{\beta_1/(\beta_1-1)}{\beta_2/(\beta_2-1)}$ with respect to r has the same sign as $-\overline{\mu}^2 - \frac{\overline{\sigma}^2}{4}$ which is negative. So $\frac{1}{\beta_1-\beta_2}$ and $\frac{\beta_1/(\beta_1-1)}{\beta_2/(\beta_2-1)}$ decrease in r and so does the continuation region size. Similarly, $\frac{1}{\beta_1-\beta_2}$ increases in $\overline{\sigma}^2$ and that the derivative of $\frac{\beta_1/(\beta_1-1)}{\beta_2/(\beta_2-1)}$ with respect to $\overline{\sigma}^2$ has the sign of $-2\overline{\mu}^2 - (\delta + \delta')\sigma^2$ which is negative. Thus, $\frac{\beta_1/(\beta_1-1)}{\beta_2/(\beta_2-1)}$ and $\frac{1}{\beta_1-\beta_2}$ decreases in $\overline{\sigma}^2$ and so does the size continuation region.

3. The continuation region is symmetric when both boundaries have complementary angles with the axis $P' = 0$, that is when $\theta^*\theta'^{*^{-1}} = 1$, which is equivalent to $(2\beta_1 - 1)\ln\frac{\beta_1}{\beta_1-1} = (2\beta_2 - 1)\ln\frac{\beta_2}{\beta_1-1}$ or $S(\beta_1) = S(\beta_2)$ where $S(x) = (2x-1)\ln\frac{x}{x-1}$. S is defined on $]-\infty, 0[$ and $]1, +\infty[$, is reversible on each of these intervals, and satisfies $S(x) = S(1-x)$. As $\beta_1 \neq \beta_2$, then $S(\beta_1) = S(\beta_2)$ implies $\beta_1 = 1 - \beta_2$, which is equivalent to $\mu = \mu'$. ■

As the proposition makes clear, the decision to make the investment is postponed as long as the projects have not differentiated themselves clearly enough from one another. The current price ratio must be sufficiently in favor of one, or the other, project in order for one project to be undertaken. This reduces the probability of choosing a project whose returns might later on be overtaken by the returns of the alternative project. As such occurrence is more probable in the distant future, the implied *ex post* regrets weight more in the investment decision, the lower the discount rate so that the continuation region is then larger. Higher volatility in the projects returns also requires the projects to be more clearly differentiated before investment takes place. However individual project volatility does not matter; what matters is the sole combined volatility $\overline{\sigma}^2$.

3.3 Choosing between two forestry projects : an extension of Wicksell's problem

"The Wicksellian tree cutting problem" refers to the problem of choosing the age at which a stand of even-aged trees will be harvested. One single harvest is considered. The optimal harvesting age is determined by the well-known Wicksellian rule under which the optimal age is chosen in such a way that the marginal value growth of the trees is equal to the opportunity cost of holding them. We assume that the volume growth functions have the following properties :

Assumption 4 *It exists* $\underline{a} > 0$ *and* $\underline{a}' > 0$*, such that the timber volume functions* $V(a)$ *and* $V'(a)$ *are continuous over* $[0, +\infty[$ *but nil over* $[0. \underline{a}]$ *and* $[0, \underline{a}']$ *respectively, positive, continuous, differentiable and concave over* $[\underline{a}, +\infty[$ *and* $[\underline{a}', +\infty[$ *respectively. In addition,* $\lim\limits_{a \to +\infty} V_a(a) = 0$ *and* $\lim\limits_{a \to +\infty} V_a'(a) = 0$.

Volume growth functions usually have a convex initial part and become concave once the trees have reached some strictly positive age. This implies that it is never optimum to harvest at an arbitrarily low age. Assumption 4 ensures that this stylized property is satisfied while avoiding some of the complications associated with non convexities[1].

We modify Wicksell's problem in two essential ways. First we start with bare land and consider the tree planting decision. Second we study a situation where two tree species, not simply one, are available. The reason why these new features are

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[1] For illustrations in the next sections, we use the same volume growth function for both species ; this better isolates the role of the stochastic price processes defined by (3.1a) and (3.1b). The volume growth function used is $V(a) = V_\infty \left(1 - e^{-\alpha(a-\underline{a})}\right)$ where $V_\infty = 100$ is the timber volume when the age tends to infinity, $\alpha = 0.01$, and $\underline{a} = 10$ is the minimum age for postive growth.

important are the following. First, since the timber value of each species evolves stochastically, following a GBM, the revenues derived upon harvest depend on the species initially planted and on the path followed by its price between planting and harvesting. The price of one species may overtake the other one, implying that the decision maker may regret the initial choice. Second, the availability of more than one species opens up the possibility of diversification, taking the form of a proportion λ of the land being devoted to one species, and a proportion $1 - \lambda$ being devoted to the second species.

Thus consider a decision maker who chooses the proportion λ of a bare piece of land devoted to either one of the species while simultaneously choosing the planting date. The rest of the land is devoted to the other species and planted simultaneously or later. Planting costs are assumed to be absent, which allows complete focus on opportunity costs. Harvests take place later on, at times to be chosen optimally for each species. Assuming without loss of generality that species (P) is planted at T, no later than species (P') which is established at T', the value of the project is then

$$\begin{aligned} F(p,p') &= E \max_{T',\lambda\in[0,1]} \Big(\lambda e^{-rT} \max_{s} E_T e^{-rs} V(s) p_{T+s} \\ &\quad + (1-\lambda)\, e^{-rT'} \max_{s'} E_{T'} e^{-rs'} V'(s') p'_{T+s'} \Big) \end{aligned} \tag{4.5}$$

where p and p' are the current prices of a unit of timber (P) or (P'), E is the expected value operator conditional on current information, and E_T (respectively $E_{T'}$) is the expected value operator conditional on information at T (respectively T'). Functions V and V' give the commercial volumes of timber from species (P) and (P') that can be harvested as a functions of the stand age when $\lambda = 1$ or $\lambda = 0$ respectively; finally s and s' are the respective ages at which species (P) and (P')

are to be optimally harvested.

Since λ and $1-\lambda$ do not affect the choices of s or s' in problem (4.5), and since they enter the objective function multiplicatively, they do not affect the choices of T or T'. Consequently λ can be determined given the optimal rules for choosing T and T' :

$$\begin{aligned} F(p,p') &= E\left\{\max_{\lambda\in[0,1]}\left[\lambda \max_{T} E\left(e^{-rT}\max_{s} E_T e^{-rs}V(s)p_{T+s}\right)\right.\right. \\ &\quad \left.\left. +(1-\lambda)\max_{T'} E\left(e^{-rT'}\max_{s'} E_{T'} e^{-rs'}V'(s')p'_{T+s'}\right)\right]\right\} \end{aligned}$$

The above problem is linear in λ. Consequently the solution is either $\lambda = 1$ or $\lambda = 0$. This establishes that diversification is not optimum, as stated in the following proposition.

Proposition 2 *When two tree species may be grown simultaneously on a forest land, it is optimal to specialize into one, or the other, species rather than diversify.*

Let $G(p,a)$ ($G'(p',a)$) be the stand value function when species (P) (respectively species (P')) is in place, the age of the trees is a, and the current price of the species in place is p (respectively a' and p') :

$$G(p,a) = \max_{s} E\left[e^{-rs}V(a+s)p_s\right] \tag{4.6a}$$

$$G'(p',a) = \max_{s} E\left[e^{-rs}V'(a+s)p'_s\right] \tag{4.6b}$$

Consider $G(p,a)$. As $G(p,a)$ is homogenous in p, $G(p,a) = pg(a)$ where $g(a) \equiv G(1,a)$. $G(p,a)$ must satisfy Bellman's equation $EdG = rGda$ which implies, by Itô's lemma, that $g(a)$ satisfies $\delta g(a) - g_a(a) = 0$. Then $g(a) = be^{\delta a}$, where b is a constant to be determined using the value-matching and smooth-pasting conditions as follows.

At cutting age a_w and for any price p, $G(p, a_w) = V(a_w)p$, $G_p(p, a_w) = V(a_w)$, and $G_a(p, a_w) = V_a(a_w)p$, implying $g(a_w) = V(a_w)$ and $g_a(a_w) = V_a(a_w)$. Consequently,

$$\begin{aligned} G(p,a) &= be^{\delta a}p && (4.7a) \\ b &= V(a_w)e^{-\delta a_w} && (4.7b) \\ \frac{V_a(a_w)}{V(a_w)} &= \delta && (4.7c) \end{aligned}$$

where a_w is determined implicitly by the last equation.

In particular when the expected value of timber price is constant ($\mu = 0$), the optimum harvest age is given by condition $\frac{V_a(a_w)}{V(a_w)} = r$, which is Wicksell's rule in the absence of uncertainty (Willassen, 1998).

To find out about the effect of the existence of the second tree species, note similarly that

$$\begin{aligned} G'(p',a) &= b'e^{\delta' a}p' && (4.8a) \\ b' &= V'(a'_w)e^{-\delta' a'_w} && (4.8b) \\ \frac{V'_a(a'_w)}{V'(a'_w)} &= \delta' && (4.8c) \end{aligned}$$

Then, the value function in presence of two tree species is

$$F(p,p') = \max_T E e^{-rT}\left\{\max\left[P_T, P'_T\right]\right\}$$

where $P \equiv bp$ and $P' \equiv b'p'$.

Thus the value function has the same form as in the basic alternative projects model analyzed in the previous section. It may then be optimal to delay planting for as long as the prices of the two alternative species are too close to each other. This is stated in the following proposition.

For short, $F(p,p')$ will be called the the land value while $G(p,a)$ (resp. $G(p',a)$) will be called the (P) stand value (resp. ($P\prime$) stand value).

Proposition 5 *When a bare forest is to be planted with either one of two possible tree species for a single harvest, then it is optimal to delay planting until prices are sufficiently far apart. The continuation period lasts as long as current price levels satisfy*

$$\theta^* \frac{b}{b'} < \frac{p'}{p} < \theta'^* \frac{b}{b'} \tag{3.9}$$

where θ^*, θ'^*, b, *and* b' *are given by* (3.4*c*), (3.4*d*), (3.7*b*), *and* (3.8*b*) *respectively.*

3.4 Choosing between alternative forestry projects repeatedly : an extension of Faustman's problem

When a unique species (P) is available, the expected value of harvesting at age $a + s$ a forest stand whose current age is a, and then replanting and harvesting the same species for an arbitrary number of rotations, is

$$G(p, a) = \max_s E e^{-rs} \left[p_s V(a+s) + F(p_s)\right] \tag{3.10}$$

where V is the timber growth function satisfying Assumption 4 ; $G(p, a)$ is the (P) stand value at age a when timber price is p subject to (3.1a) and $F(p)$ is the land value. Note that the land value $F(p)$ can be expressed as follows

$$F(p) = \max_s E e^{-rs} \left[p_s V(s) + F(p_s)\right] \tag{3.11}$$

Thus, the land value coincides with the stand value at age zero as there are no planting or other fixed management cost, that is $F(p) = G(p, 0)$.

When the timber price is constant, the optimal harvesting age a_f is independent of timber price[2] and determined by Faustman's rule (1849),

$$\frac{V_a(a_f)}{V(a_f)} = \frac{r}{1 - e^{-ra_f}} \tag{4.12}$$

In the case where the timber price is governed by (4.1a), and since the investment (planting) cost is nil, the land value function $F(p)$ and the stand value functions $G(p, a)$ are homogenous in p as stated in the following lemma.

Lemma 2 *When the timber price p follows the process* (4.1a), *the land value* $F(p)$ *and the value of the stand at age a,* $G(p, a)$, *are homogenous of degree one in p, and the optimal cutting age does not depend on p.*

Proof. $F(p)$ may be written as $F(p) = \max\limits_{\{a_k\}} E \sum_{k=1}^{\infty} e^{-r\tau_k} p_{\tau_k} V(a_k)$ where current time is $t = 0$ and $\tau_k = \sum_{s=1}^{k} a_s$ is the sum of all cutting ages from the next harvest to the k^{th} harvest. Since p is a GBM, multiplying p by λ implies that p_s is multiplied by λ for any $s > t$. Then $F(\lambda p) = \max\limits_{\{a_k\}} E \sum_{k=1}^{\infty} e^{-r\tau_k} (\lambda p_{\tau_k}) V(a_k)$
$= \lambda \max\limits_{\{a_k\}} E \sum_{k=1}^{\infty} e^{-r\tau_k} p_{\tau_k} V(a_k)$

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[2] However, when there is a positive investment (or planting) cost, Faustman's rotation not only is affected by that cost but it is also affected by the (parametric) price. There is a substantial literature with implications on the price-harvest age relationship. When the price is stochastic, Thomson (1992) shows in a discrete model with positive investment cost that the option to harvest is exercised when the stand reaches a critical age that depends on timber price. Willasen (1998) studies Faustman's tree-cutting problem in the case of stochastic growth. His approach, which makes use of impulse control theory, can handle harvest value functions generated by diffusions based on the logistic equation. Clarke and Reed (1989) and Reed and Clarke (1990) earlier had studied a problem that was more general in the sense that it involved two sources of uncertainty : stochastic biological growth and stochastic price, but less general in that the diffusions were Brownian motions.

$= \lambda F(p)$

Consequently, (4.11) can be written as

$pF(1) = \max_s E e^{-rs}\left[p_s V(s) + p_s F(1)\right]$ or

$pF(1) = \max_s \left[pe^{-\delta s} V(s) + pe^{-\mu s} F(1)\right]$, that is

$F(1) = \max_s \left[e^{-\delta s} V(s) + e^{-\mu s} F(1)\right]$. This means that the maximization which defines harvest age is independent of the current price level ■.

Let $G(p,a) = pg(a)$ where $g(a)$ is a function of the stand age to be determined. $G(p,a)$ satisfies Bellman's equation $E\left(dG(p,a)\right) = rG(p,a)da$ where $EdG = G_a da + G_p \mu p da + \frac{\sigma^2}{2} p^2 G_{pp} da + o(da)$, $G_a = pg_a(a)$, $G_p = g(a)$, and $G_{pp} = 0$. Consequently $EdG = pg_a(a) + \mu g(a) p da + o(da)$, and Bellman's equation implies that $g(a)$ must satisfy $\delta g(a) = g_a(a)$ for $a \in [0, a_f]$, where $\delta \equiv r - \mu$. Furthermore, $G(p,a)$ should satisfy the value-matching and smooth-pasting conditions linking $G(p,a_f)$ and $V(a_f)p + F(p)$ at harvest. Dividing both conditions by p yields $g(a_f) = V(a_f) + g(0)$ and $g_a(a_f) = V_a(a_f)$.

Solving in the usual fashion one finds that the forest and land value functions are determined as follows

$$G(p,a) = be^{\delta a_f} p \tag{4.13a}$$

$$F(p) = bp \tag{4.13b}$$

$$b = \frac{V(a_f)}{e^{\delta a_f} - 1} \tag{4.13c}$$

$$\frac{V_a(a_f)}{V(a_f)} = \frac{\delta}{1 - e^{-\delta a_f}} \tag{4.13d}$$

Both $F(p)$ and $G(p,a)$ depend linearly on the current price of the unique tree species.

The optimal harvest age is defined by (4.13d); it depends on the drift of the stochastic process but it is independent of the current price level. It is smaller than the optimal harvest age corresponding to just one rotation given by (4.7c) as $\frac{V_a(a)}{V(a)}$ is decreasing in a.

We consider now the case where two species (P) and $(P\prime)$ are available and assume that the manager has the option to harvest and to replant the land with either species (P) or (P'), immediately after the harvest or after any delay, and to harvest again. The process goes on forever. As in the case of a single harvest, it can be shown that, to an expected value maximizer, any forest diversification strategy involving growing two species simultaneously would be dominated by one consisting in establishing one single species if at all. The value of the land, including the option, is denoted $G(p, p', a)$. This is the (P) forest value function where p and p' are the current prices of each species and a is the current age of the trees. The manager may choose (i) to wait and observe the stand growing up, or (ii) to harvest the stand (P) and replant the same species immediately, or (iii) to harvest the stand (P) and plant the alternative species (P'), or finally (iv) to harvest the stand and wait before planting one or the other species. As before we assume that there is no fixed cost associated with harvesting and establishing a new crop.

Consider the last possibility. After harvesting, the forest manager may wait before establishing a new crop. Then the land remains bare and we assume that this does not involve any cost (e.g. weeding). By doing so the manager is able to wait until the prices of each species evolve in such a way that she is reasonably convinced to choose the right one. No gain can be achieved by using that strategy, though. Indeed suppose the manager decides to establish one species at some date t strictly posterior to the harvest. Had she planted that species immediately after harvest, she would then be better off at t because the trees would have grown already. Had she planted the other species, she could cut the trees and plant the preferred species at no cost.

In that case she would either be as well off, if cutting the existing trees produced no income, or she would be better off if cutting the existing trees produced some income. Hence the following lemma, which implies that only the first three possible decisions outlined in the past paragraph need be considered.

Lemma 3 *It is optimal to establish a new crop immediately after harvest.*

This result contrasts with Proposition 5 indicating that it is optimal, when there is only one harvest, to delay planting when the projects are not clearly differentiated. There are two explanations. First, in the case of multiple rotations, it is possible to postpone the harvest rather than cut and wait, if the best planting decision is not clear. Second and more important, a decision turning out to be undesirable *ex post* has a lower opportunity cost in case of multiple rotations. The undesirable trees may be cut and replaced with the desirable species. In the single rotation case, the possibility of a more profitable harvest is lost.

Consider now the first three management possibilities. Let $F(p,p')$ denote the value of the options to plant and harvest repeatedly one unit of a bare forest land when two tree species (P) and (P') are available. Let $G(p,p',a)$ and $G'(p,p',a)$ denote respectively the value of a forest land planted with trees of species (P) (resp. (P')) and age a when the price for the currently planted species is p (resp. p') and the price for the other species is p' (resp. p). Precisely,

$$G(p,p',a) = \max_s E e^{-rs}\left[p_s V(s+a) + F(p_s,p'_s)\right] \tag{4.14a}$$

$$G'(p,p',a) = \max_s E e^{-rs}\left[p'_s V'(s+a) + F(p_s,p'_s)\right] \tag{4.14b}$$

For short F will be called the land value function while G and G' will be called the forest value functions.

Lemma 4 *When p and p' follow the processes (4.1a) and (4.1b) respectively, $F(p,p')$, $G(p,p',a)$, and $G'(p,p',a)$ are homogenous of degree one in (p,p') and the optimal cutting age depends on the current timber relative price θ only.*

Proof. First, we will prove that $F(p,p')$ is homogenous of degree one in (p,p'). Define α_k as a dichotomous variable taking the value 1 if species (P) is planted after the $(k-1)^{th}$ harvest or the value 0 otherwise. Similarly, α'_k is a dichotomous variable taking the value 1 if species (P') is planted after the $(k-1)^{th}$ harvest or 0 if species (P) is planted at that time. Thus, α_k and α'_k satisfy $\alpha_k \in \{0,1\}$, $\alpha'_k \in \{0,1\}$, and $\alpha_k\alpha'_k = 0$. $F(p,p')$ can be written as $F(p,p') = \max\limits_{\{\alpha_k,\alpha'_k,a_k\}} E\sum_{k=1}^{\infty} e^{-r\tau_k}\left(\alpha_k p_{\tau_k} V(a_k) + \alpha'_k p'_{\tau_k} V'(a_k)\right)$. Since p is a GBM, multiplying p_t by λ implies that $p_{t'}$ is multiplied by λ for any $t' > t$. Then $F(p,p')$ is homogenous of degree one in (p,p'). Considering equations (4.14a) and (4.14b), the functions $G(p,p',a)$ and $G'(p,p',a)$ are homogenous of degree one in (p,p') as well. Now, suppose that species (P) is currently planted; we want to prove that the optimal cutting age depends on θ_s only, where θ_s is the value of θ when the stand is optimally cut. As $G(p,p',a) = \max_{s\geqslant 0} Ee^{-rs}\left[p_sV(s+a) + F(p_s,p'_s)\right]$, then $G(1,\theta,a) = \max_{s\geqslant 0}\left[e^{-\delta s}V(s+a) + Ee^{-rs}\frac{p_s}{p}F(1,\theta_s)\right]$. Note that $\frac{p_s}{p}$ is the value at time s of a GBM with drift μ and volatility σ whose value is 1 at time 0; thus $\frac{p_s}{p}$ is independent of p (as $\frac{p_s}{p}$ is a GBM with initial value 1). Since it is optimal to cut when

$argmax_s Ee^{-rs}\left[p_sV(s+a) + F(p_s,p'_s)\right] = 0$, the optimal cutting age depends on θ only. ■

Considering Lemma 4 define the 'reduced' functions $g(\theta,a)$, $g'(\theta,a)$, and $f(\theta)$ as follows :

$g(\theta,a) = \frac{1}{p}G(p,p',a)$

$g'(\theta,a) = \frac{1}{p}G'(p,p',a)$

$f(\theta) = \frac{1}{p}F(p,p')$

The species currently established on the forest land is (P). We are interested in characterizing the continuation region, the set of points (θ, a) in $\mathbb{R}^+ \times \mathbb{R}^+$ where it is optimum to wait rather than exercise the option of harvesting. Let $a(\theta)$ denote the minimum age at which it is optimum to harvest given θ. The set $\{(\theta, a(\theta))\}$ defines the boundary of the continuation region. Outside the continuation region it is optimal to harvest and immediately reestablish one or the other species.

As long as it is optimal to wait rather than harvest, the forest value function $G(p,p',a)$ should satisfy Bellman's equation $E(dG(p,p',a)) = rG(p,p',a)da$. On the boundary of the continuation region, value matching and smooth pasting conditions apply, as stated in the following proposition.

Proposition 3 *Suppose that species* (P) *is currently in place. The reduced forest value function* $g(\theta,a)$ *satisfies the partial differential equation* (4.15*a*) *in the continuation region; on the boundary of the continuation region, it satisfies the value-matching condition* (4.15*b*) *as well as the smooth-pasting condition* (4.15*c*), *where*

$$\begin{aligned}
\frac{\overline{\sigma}^2}{2}\theta^2 g_{\theta\theta} + \overline{\mu}\theta g_\theta - \delta g + g_a &= 0 && (4.15a)\\
g(\theta, a(\theta)) &= V(a(\theta)) + f(\theta) && (4.15b)\\
g_\theta(\theta, a(\theta)) &= f_\theta(\theta) \text{ or } g_a(\theta, a(\theta)) = V_a(a(\theta)) && (4.15c)
\end{aligned}$$

Proof. The value matching condition stands as the stand value function $g(\theta,a)$ is continuous. See Appendix for the smooth pasting condition. The smooth pasting condition can be written in two forms as in (4.15*c*) that, given the value-matching condition, are equivalent. ■

Proposition 4 *The reduced land value function* $f(\theta)$ *is non decreasing in* θ *with* $\lim\limits_{\theta\to 0} f(\theta) = b$*, the reduced land value when only species* (P) *is available,* $\lim\limits_{\theta\to 0} f_\theta(\theta) = 0$*,* $f(\theta) \underset{\theta\to+\infty}{\sim} b'\theta$*, the reduced land value when only species* (P') *is available, and* $\lim\limits_{\theta\to+\infty} f_\theta(\theta) = b'$ *where* $b = \frac{V(a_f)}{e^{\delta a_f}-1}$ *and* $b' = \frac{V'(a'_f)}{e^{\delta' a'_f}-1}$*. Furthermore, the reduced land value is equal to the stand value* (P) *or* $(P\prime)$ *at age zero, that is* $f(\theta) = g(\theta, 0) = g'(\theta, 0)$.

Proof. An increase in θ can be analyzed as an increase in p' at constant p; then all future values of p' are raised in the same proportion. This cannot decrease land value $pf(\theta)$, which implies that the reduced land value function $f(\theta)$ is non decreasing in θ. Suppose that p is strictly positive and that p' is zero; as (4.1*b*) is absorbing at zero, (P) will remain the preferred species forever. Therefore, using (4.13*b*), $\lim\limits_{\theta\to 0} pf(\theta) = pb$ or $\lim\limits_{\theta\to 0} f(\theta) = b$. When p is a positive constant and p' varies in the vicinity of zero, the land value does not vary as the preferred species will remain (P) forever; consequently $\lim\limits_{\theta\to 0} f_\theta(\theta) = 0$. A similar proof shows that $\lim\limits_{\theta\to+\infty} f(\theta) = b'\theta$ and $\lim\limits_{\theta\to+\infty} f_\theta(\theta) = b'$. Suppose that it is optimal to plant species (P) on a bare unit of land; then its value is $F(p,p') = \max\limits_{s} Ee^{-rs}\left[p_s V(s) + F(p_s, p'_s)\right]$, i.e. $G(p, p', 0)$ by (4.14*a*). When it is not optimal to plant (P), it is then optimal to cut any stand (P) of age zero immediately and plant the alternative species, that is $G(p,p',0) = V(0) + F(p,p')$ where $V(0) = 0$ by Assumption 4. Consequently, $G(p,p',0) = F(p,p')$ or $f(\theta) = g(\theta, 0)$ for any $\theta \geq 0$. A similar proof shows that $f(\theta) = g'(\theta, 0)$ for any $\theta \geq 0$. ■

Other properties of the repeated decision with alternative species can be identified by characterizing the boundary of the continuation region. Clearly, at values of θ for which the continuation region exists, the latter has an upper boundary as trees mature physically. As a consequence of Lemma 3, it is certain that the trees being harvested are replaced immediately. At relative prices below some switching value θ_0, species (P) is planted again; above θ_0, there is a switch to species (P'). While on bare land it is clear that species (P') must be planted when $\theta > \theta_0$, if the stand has a strictly positive age, trees of species (P) might be worth keeping and harvesting at some later age. Indeed such trees are either already older than $\underline{a}$ and experiencing strictly positive growth, or they are closer to $\underline{a}$ than if their age was zero. This suggests that, for some values of the relative price above θ_0, it is optimal to keep trees that are older than some minimum age; the continuation region then has a lower boundary, as well as an upper boundary. This is characterized in Proposition 5.

Proposition 5 *Suppose that species (P) is currently in place. The continuation region has the following properties : (1) There exists $\theta_0 > 0$ such that, if $\theta < \theta_0$ and the land is bare, it is optimal to plant species (P) immediately. If $\theta > \theta_0$ and the land is bare, it is optimal to plant species (P') immediately. For $\theta = \theta_0$, indifferently planting (P) or (P') is optimal. (2) For $0 \leq \theta \leq \theta_0$, the continuation region includes all $(\theta e^{\overline{\mu}a}, a)$ pairs such that $0 \leq a \leq \arg\max_s Ee^{-rs}\left[V(s) + f\left(\theta e^{\overline{\mu}s}\right)\right]$. Its upper boundary $a(\theta)$ includes $(0, a_f)$. (3) For $\theta_0 < \theta \leq \bar{\theta}$, the continuation region includes all (θ, a) pairs such that $a \geq 0$ and $V(a) + f(\theta) \leq \max_{s \geq 0} Ee^{-rs}\left[V(s) + f\left(\theta e^{\overline{\mu}s}\right)\right]$. Its upper boundary $a^+(\theta)$ is a decreasing curve starting from $(\theta e^{\overline{\mu}\underline{a}_0}, \underline{a}_0)$ where $\underline{a}_0 = \arg\max_s Ee^{-rs}\left[V(s) + f\left(\theta_0 e^{\overline{\mu}s}\right)\right]$. Its lower boundary $a^-(\theta)$ is an increasing curve starting from $(\theta_0, 0)$. The upper and lower boundaries end together at $(\bar{\theta}, \underline{a})$ where $\bar{\theta}$ is defined by the condition*

$$argmax_s Ee^{-rs}\left[V(\underline{a} + s) + f\left(\bar{\theta} e^{\overline{\mu}s}\right)\right] = 0$$

Proof. 1. The set of $\theta > 0$ for which it is optimal to plant (P) is not empty as for θ

positive and sufficiently small it is optimal to plant (P). This set is bounded as for θ sufficiently high it is optimal to plant (P'). Being not empty and bounded, the set of $\theta > 0$ for which it is optimal to plant (P) has a finite maximum θ_0 that belongs to it.

2. The first sentence is a definition. The boundary $a(\theta)$ corresponds to pairs satisfying the definition with equality. When θ equals zero, the cutting age is the Faustman's age a_f given by (4.12).

3. The condition defines all (θ, a) pairs where it is not yet worth harvesting an inherited (P) stand that was not optimal to plant at date zero. By definition, $(\theta e^{\overline{\mu}\underline{a}_0}, \underline{a}_0)$ and $(\theta_0, 0)$ belong to respectively the upper and the lower boundaries of this continuation region.

For a given stand age that is positive but smaller than $\underline{a}$, the decision maker prefers waiting when price p is sufficiently high with respect to p', that is for $\theta = \frac{p'}{p}$ sufficiently low but higher than θ_0. Otherwise, she prefers harvesting immediately and switching to species ($P\prime$). The corresponding threshold of θ increases with $a \in [0, \underline{a}]$ because a positive crop is closer in the future.

Similarly, for a given stand age that is positive but higher than $\underline{a}$, the decision maker prefers waiting when price p is sufficiently high with respect to p', that is for $\theta = \frac{p'}{p}$ sufficiently low but higher than θ_0. However, for higher θ, the decision maker would prefer harvesting and switching to the alternative species earlier as the opportunity cost of keeping the current species increases with θ. Consequently, the corresponding threshold of θ decreases with age a or equivalently $a^+(\theta)$ decreases with θ.

As it is not worth keeping an inherited (P) stand for ever, $a^-(\theta)$ and $a^+(\theta)$ necessarily meet at some point $(\bar{\theta}, a(\bar{\theta}))$ with $\bar{\theta} > \theta_0$.

If $a(\bar{\theta}) < \underline{a}$ then it would be possible to inherit a (P) stand for which the decision maker decides to wait for a while and then harvest and switch to the alternative species later on before age $\underline{a}$, without earning any positive harvest value. This is not optimal as the decision maker is better off switching earlier.

If $a(\bar{\theta}) > \underline{a}$ then it exists θ such that $\theta_0 < \theta < \bar{\theta}$. Assume that the manager inherits a (P) stand characterized by (θ, a_1) with $a_1 > \underline{a}$ and (θ, a_1) belongs to the lower switching boundary. She decides then to harvest immediately and plant the other species. Assume that the manager inherits a (P) stand characterized by (θ, a_2) in the continuation region with $a_2 > a_1 > \underline{a}$. She decides then to wait for a strictly positive period before harvesting and switching to the alternative species. This is contradictory as the marginal gain from waiting is higher in the first case because of the decreasing marginal growth for stand ages higher than $\underline{a}$ whereas the opportunity cost is the same in both cases.

We conclude than that the upper and lower boundaries meet at $(\bar{\theta}, \underline{a})$. Therefore, $\bar{\theta}$ can be characterized by the condition

$$argmax_s E e^{-rs}\left[V(\underline{a}+s) + f\left(\bar{\theta} e^{\bar{\mu} s}\right)\right] = 0$$

as at $(\bar{\theta}, \underline{a})$ its optimal to harvest immediately.

■

3.4.1 Analytical resolution under certainty

We now assume that $\sigma = \sigma' = 0$. Prices are certain, although not necessarily constant.

If $\mu = \mu'$ then the relative price θ remains constant. If a species is planted optimally then it will continue to be planted and harvested successively forever.

Switching from one species to the alternative one can not occur unless the problem starts with an inherited stand that should not have been planted in the first place.

When $\mu \neq \mu'$ switches are possible. Without loss of generality, assume that $\overline{\mu} = \mu' - \mu > 0$; the drift of the price of the currently planted species (P) is smaller than the drift of the price of the alternative species (P'). Consider a stand of species (P) at age a when its price is p and the price of the alternative species p' is still relatively low. The decision maker will cut this stand and replant the same species and so on until the price of the alternative species (P') has sufficiently increased with respect to the price of the species currently planted. Suppose that species (P) is replanted n times until the switch to (P') occurs. At that moment, species (P) will be cut for the last time and species (P') will be planted thereafter forever. Therefore, by analogy with (4.13b), the land value at that moment will be $b'p'$. It will be convenient to index cutting ages and the corresponding prices in reverse chronological order. Thus a_0 is the age at which species (P) is cut for the last time. Cutting ages a_n, a_{n-1},..,a_0 will be chosen to maximize the stand value $G(p, p', a) = pg(\theta, a)$, that is

$$\begin{aligned} G(p,p',a) &= \max_{a_0,\dots a_n} \Big[p_n V(a_n) e^{-r(a_n - a)} + \dots \\ &\quad + p_0 V(a_0) e^{-r(a_n + \dots + a_0 - a)} + b' p'_0 e^{-r(a_n + \dots + a_0 - a)}\Big] \end{aligned}$$

where $p_i = pe^{\mu(a_n + \dots + a_i - a)}$ for $i = 0, \dots n$ and $p'_0 = p' e^{\mu'(a_n + \dots + a_0 - a)}$. This implies that

$$\begin{aligned} G(p,p',a) &= p \max_{a_0,\dots a_n} \Big[V(a_n) e^{-\delta(a_n - a)} + \dots + V(a_0) e^{-\delta(a_n + \dots + a_0 - a)} \\ &\quad + b' \frac{p'}{p} e^{-\delta'(a_n + \dots + a_0 - a)}\Big] \end{aligned}$$

so that

$$g(\theta,a) = \max_{a_0,\ldots a_n} \Big[\big(V(a_n)e^{-\delta a_n} + \ldots + V(a_0)e^{-\delta(a_n+\ldots+a_0)}\big) e^{\delta a} + b'\theta e^{-\delta'(a_n+\ldots+a_0)} e^{\delta' a}\Big] \tag{4.16}$$

Proposition 6 *It exists* $\theta_0 > 0$ *such that the manager is indifferent between planting on a bare land species* (P') *forever or species* (P) *for one rotation of length* $\underline{a}_0$ *and then species* (P') *forever. Both* θ_0 *and* $\underline{a}_0$ *are determined by*

$$\frac{V_a(\underline{a}_0)}{V(\underline{a}_0)} = \delta + \frac{\delta'}{e^{\delta' \underline{a}_0} - 1} \tag{4.17a}$$

$$K(\underline{a}_0) = \frac{\delta'}{\delta} b'\theta_0 e^{\overline{\mu}\underline{a}_0} \tag{4.17b}$$

Besides, $\underline{a}_0$ *satisfies* $\underline{a} < \underline{a}_0 < a_f$.

Proof. The existence of θ_0 and $\underline{a}_0$ stems from Proposition 5. As by definition the manager is indifferent between planting (P') forever or planting (P) for just one rotation of length $\underline{a}_0$ then $b'\theta_0 = \max_s \left[V(s)e^{-\delta s} + b'\theta_0 e^{-\delta' s}\right]$ where $\underline{a}_0 = \arg\max_s \left[V(s)e^{-\delta s} + b'\theta_0 e^{-\delta' s}\right]$. The first order condition of this maximization problem is equivalent to equation (4.17b). The second order condition is $K_a(\underline{a}_0) - \overline{\mu}K(\underline{a}_0) < 0$, satisfied for $\underline{a}_0 \in]\underline{a}, a_w[$ as $V(a)$ is decreasing and concave on this interval and $\overline{\mu} > 0$. The first order condition, together with $b'\theta_0 = V(\underline{a}_0)e^{-\delta a_0} + b'\theta_0 e^{-\delta' \underline{a}_0}$, gives Equation (4.17$a$) that allows to determine $\underline{a}_0$ unambiguously whereas (4.17b) determines θ_0. As $\frac{V_a(a)}{V(a)}$ is decreasing on $[\underline{a}, +\infty[$ and $\lim_{a\to\underline{a}} \frac{V_a(a)}{V(a)} = +\infty$ then $\underline{a}_0 > \underline{a}$. Faustman's age a_f is determined by Equation (4.13d) that is equivalent to $\frac{V_a(a_f)}{V(a_f)} = \delta + \frac{\delta}{e^{\delta a_f} - 1}$. One can prove that $\frac{\delta}{e^{\delta a}-1} < \frac{\delta'}{e^{\delta' a}-1}$ for any $a > 0$ as $\delta' < \delta$. Consequently, $\frac{V_a(a_f)}{V(a_f)} < \frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$, thus $\underline{a} < \underline{a}_0 < a_f$. ■

When $\theta > \theta_0$, it is conceivable that a stand of age a of species (P) may be inherited by the decision maker while not resulting from a rational decision to establish it. Depending on the stand age and on the value of θ, the decision maker may be better off harvesting immediately and planting the alternative species, or letting the stand reach maturity before switching.

Proposition 6 *For $\theta \geq \theta_0$, the upper switching boundary $a^+(\theta)$ is defined analytically as*

$$\left\{(\theta, a)/\ \theta_0 \leq \theta \leq \bar{\theta},\ \underline{a} \leq a \leq \overline{a},\ \text{and } K(a) = b'\frac{\delta'}{\delta}\theta\right\} \tag{4.18}$$

where $\bar{\theta}$ and $\overline{a}$ are respectively the unique solutions to $V_a(\underline{a}) = \delta' b' \bar{\theta} e^{\overline{\mu}\underline{a}}$ and $K(\overline{a}) = b'\frac{\delta'}{\delta}\theta_0$. It is a decreasing curve in the plan (θ, a) stretching between $(\theta_0, \overline{a})$ and $(\bar{\theta}, \underline{a})$.

Proof. Consider an inherited stand (P) characterized by (θ, a) such that with $\theta \geq \theta_0$ and $a \geq \underline{a}$. The maximization problem is

$$\max_s \left[V(s)e^{-\delta s} + b'\theta e^{-\delta' s}\right]$$

whose first order condition is equivalent to $K(a) = b'\frac{\delta'}{\delta}\theta$ whereas the second order condition is $K_a(a) - \overline{\mu}K(a) < 0$, satisfied for $a \in]\underline{a}, \overline{a}[$ as $V(a)$ is decreasing and concave on this interval and $\overline{\mu} > 0$. The first order condition defines a decreasing curve $a(\theta)$. The proof of monotonicity is immediate since $K(a) = \frac{V_a(a)}{\delta} - V(a)$ is decreasing in a. The highest value of θ compatible with $a \geq \underline{a}$ defines $\bar{\theta}$ with $K(\underline{a}) = \frac{V_a(\underline{a})}{\delta} = \frac{\delta'}{\delta} b' \bar{\theta} e^{\overline{\mu}\underline{a}}$ or $V_a(\underline{a}) = \delta' b' \bar{\theta} e^{\overline{\mu}\underline{a}}$. The highest value of a is $\overline{a}$ compatible with θ_0 with $\overline{a} < a_w$ as $K(\overline{a}) = b'\frac{\delta'}{\delta}\theta_0 > K(a_w) = 0$. ■

Proposition 7 *For $\theta \in [\theta_0, \bar{\theta}]$, the lower switching boundary of the waiting region $a^-(\theta)$ is defined analytically as*

$$\begin{gathered}\{(\theta, a)/\ \theta_0 \leq \theta \leq \bar{\theta},\ 0 \leq a \leq \underline{a}, \exists s \geq 0/ \\ \left(1 - e^{-\delta' s}\right) b'\theta = e^{-\delta s} V(a + s)\ \text{and } K(a + s) = \frac{\delta'}{\delta} b'\theta e^{\overline{\mu} s}\right\}\end{gathered} \tag{4.19}$$

It is an increasing curve in the plan (θ, a) stretching between $(\theta_0, 0)$ and $(\bar{\theta}, \underline{a})$.

Proof. For $a \in [0, \underline{a}]$, the set of points (θ, a) for which the decision maker is indifferent between harvesting immediately to earn $p\,[b'\theta + V(a)]$ or harvesting after a time period s maximizing $e^{-rs}[p_s V(a+s) + b'p'_s]$ defines the lower switching boundary of the waiting region. It is defined as the set of points (θ, a) solution to $b'\theta + V(a) = e^{-\delta s} V(a+s) + e^{-\delta' s} b'\theta$ and $K(a+s) = \frac{\delta'}{\delta} b'\theta e^{\overline{\mu} s}$ where $V(a) = 0$, $\underline{a} - a \leq s \leq \underline{a}_0 - a$, and $a + s \in [\underline{a}, \underline{a}_0]$ is the age at which the trees will be cut and a is their age on the lower boundary. Pairs (θ, a) below or to the right of the lower boundary command immediate cutting; while pairs above the lower boundary but below the upper boundary belong to the continuation region. The two equations in Proposition 7 lead to $\frac{V_a(a+s)}{V(a+s)} = \delta + \frac{\delta'}{e^{\delta' s} - 1}$. For a given $a \in [0, \underline{a}]$, $\frac{V_a(a+s)}{V(a+s)}$ decreases in $s \in [\underline{a} - a, \underline{a}_0 - a]$ from $+\infty$ to $\frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$ whereas $\delta + \frac{\delta'}{e^{\delta' s} - 1}$ decreases from a finite positive value $\delta + \frac{\delta'}{e^{\delta'(\underline{a} - a)} - 1}$ to $\delta + \frac{\delta'}{e^{\delta'(\underline{a}_0 - a)} - 1} > \delta + \frac{\delta'}{e^{\delta' \underline{a}_0} - 1} = \frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$ by (4.17a). Therefore, for any $a \in [0, \underline{a}]$, there exists a unique $s \in [\underline{a} - a, \underline{a}_0 - a]$ such that $\frac{V_a(a+s)}{V(a+s)} = \delta + \frac{\delta'}{e^{\delta' s} - 1}$ that we denote $s(a)$. Furthermore, as $\frac{V_a(a+s)}{V(a+s)}$ decreases in a, then $s(a)$ decreases in a, too. The second equation $K(a+s) = \frac{\delta'}{\delta} b'\theta e^{\overline{\mu} s}$ shows that for any $a \in [0, \underline{a}]$, there exists a unique $\theta = \theta(a)$ satisfying the condition in Proposition 7. To show that $\theta_a(a) < 0$, derive $\left(1 - e^{-\delta' s}\right) b'\theta = e^{-\delta s} V(a+s)$ with respect to a to obtain $\left(1 - e^{-\delta' s}\right) b'\theta_a = -\delta e^{-\delta s} V(a+s) s_a + e^{-\delta s}(1 + s_a) V_a(a+s) - \delta' e^{-\delta' s} b'\theta s_a$ or

$$\left(1 - e^{-\delta' s}\right) b'\theta_a = -\delta e^{-\delta s} V(a+s) s_a + e^{-\delta s}(1+s_a) V_a(a+s) - \delta e^{-\delta s} s_a \left(\frac{V_a(a+s)}{\delta} - V(a+s)\right)$$

that is

$\left(1 - e^{-\delta' s}\right) b'\theta_a = e^{-\delta s} s_a V_a(a+s)$. As $s_a(a) < 0$ then $\theta_a(a) < 0$.

The point $(\theta_0, 0)$ satisfies the condition in Proposition 7 with $s = \underline{a}_0$ and $a = 0$. The point $(\bar{\theta}, \underline{a})$ satisfies this condition with $s = 0$ and $a = \underline{a}$. The lower switching boundary $a^-(\theta)$ is then an increasing curve in the plan (θ, a) stretching between $(\theta_0, 0)$ and $(\bar{\theta}, \underline{a})$. ■

An implication of the two last propositions is that there does not exist any continuation zone for values of θ above $\overline{\theta}$. If a stand (P) exists for such values, it must be harvested immediately and the alternative tree species must be established.

Proposition 7 *Assume that* $\mu < \mu'$ *and the land is occupied by the stand* (P) *characterized by the pair* (θ, a) *with* $\theta < \theta_0$. *If the manager has not to harvest the stand immediately then she must optimally harvest it and replant the land with the same species* (P) *at pairs* $(\theta e^{\mu(a_n+...+a_k-a)}, a_k)$, $k = 1, ..., n$, *and finally harvest for the last time the stand* (P) *at pair* $(\theta e^{\overline{\mu}(a_n+...+a_0-a)}, a_0)$ *and immediately switch once and forever to species* $(P\prime)$. *The cutting ages* $(a_n, ..., a_{1,}a_0)$ *can be determined by the following* $n+1$ *equations.*

$$K(a_k) = L(a_{k-1}), \textit{ for } k = 1, ..., n \tag{4.20a}$$

$$K(a_0) = \frac{\delta'}{\delta} b'\theta e^{\overline{\mu}(a_n+...+a_0-a)} \tag{4.20b}$$

where the number of rotations $n+1$ *is determined endogenously to satisfy* $\theta e^{\overline{\mu}(a_n+...+a_0-a)} \geq \theta_0$ *and* $\theta e^{\overline{\mu}(a_n+...+a_1-a)} \leq \theta_0$.

Proof. Note that the solution $(a_n, ..., a_0)$ to the maximization problem corresponding to $g(\theta, a)$ is the same as the one corresponding to $g(\theta_0, 0)$ where $\theta_0 e^{(\mu'-\mu)a} = \theta$. As so, it is possible to set a to zero in the maximization problem. Consider then the maximization problem $g(\theta, 0) = \max_{a_0,...a_n} W(\theta, a_n, ..., a_0)$ where $W(\theta, a_n, ..., a_0)$ denotes $V(a_n)e^{-\delta a_n} + ... + V(a_0)e^{-\delta(a_n+...+a_0)} + b'\theta e^{-\delta'(a_n+...+a_0)}$. The $n+1$ first order conditions of this maximization problem are $V_a(a_k)e^{-\delta(a_k+...+a_n)} - \delta\sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)} - \delta' b'\theta e^{-\overline{\mu}a}e^{-\delta'(a_0+...+a_n)} = 0$ for $k = 0, ..., n$. This set of $n+1$ equations can be equivalently simplified into another set of $n+1$ equations obtained by keeping the 1^{st} order condition for $k = 0$, and for $k = 1, ..., n$, substituting the k^{th} first order condition for the equation obtained by subtracting the $(k-1)^{th}$ first order condition from the k^{th} one. This transformation of the $n+1$ first order conditions of the maximization problem leads to the following equivalent set of $n+1$ equations $V_a(a_0) - \delta V(a_0) - \delta' b' \theta e^{\overline{\mu}(a_0+\ldots+a_n-a)} = 0$ or $K(a_0) = \frac{\delta'}{\delta} b' \theta_0$, and $V_a(a_k)e^{-\delta(a_k+\ldots+a_n)} - \delta V(a_k)e^{-\delta(a_k+\ldots+a_n)} = V_a(a_{k-1})e^{-\delta(a_{k-1}+\ldots+a_n)}$ or $K(a_k) = L(a_{k-1})$ for $k = 1, \ldots, n$. We show in the appendix that the second order condition is satisfied. ■

Conditions $(4.20a)$ apply at harvests where it is optimal to replant the same species ; they can be called replanting conditions. The last condition $(4.20b)$ applies when switching to the alternative species is optimal, upon the last harvest of species (P). A consequence of Proposition 7 is the existence of a strictly decreasing sequence $(\theta_k)_{k\in\mathbb{N}}$ with first term θ_0 and $\lim_{n\to+\infty}\theta_n = 0$ such that if the land is bare and relative timber price is $\theta \in [\theta_{n+1}, \theta_n]$ than it is optimal to plant it with species (P) exactly n times before switching definitely to species $(P\prime)$. As illustrated in Figure 4.2, when θ varies in the interval $[\theta_{n+1}, \theta_n]$, let $[\underline{a}_n, \overline{a}_n]$ denote the interval of the corresponding cutting age.

For the purpose of next propositions, let $R(a)$ denotes the unique positive number in $]\underline{a}, +\infty[$ associated to $a \in]\underline{a}, +\infty[$ such that $K(R(a)) = L(a)$. As illustrated in Figure 4.1, on the interval $]\underline{a}, +\infty[$, the function R has the following proprieties

$$\begin{aligned}
\forall a &\in]\underline{a}, +\infty[, \ R(a) = a \Longleftrightarrow a = a_f \\
\forall a &\in]\underline{a}, +\infty[, \ R(a) < a \Longleftrightarrow a > a_f \\
\forall a &\in]\underline{a}, +\infty[, \ R(a) > a \Longleftrightarrow a < a_f
\end{aligned}$$

Proposition 8 *The boundary $a(\theta)$ of the continuation region for $\theta \leq \theta_0$ is the union of infinity of decreasing and discontinuous curves $a_n(\theta)$ stretching between $(\theta_{n+1}e^{\overline{\mu}\ \overline{a}_n}, \overline{a}_n)$ and $(\theta_n e^{\overline{\mu}\ \underline{a}_n}, \underline{a}_n)$ with $\underline{a}_{n+1} < \overline{a}_n$, $a_f \in [\underline{a}_{n+1}, \overline{a}_{n+1}] \subset [\underline{a}_n, \overline{a}_n]$, for*

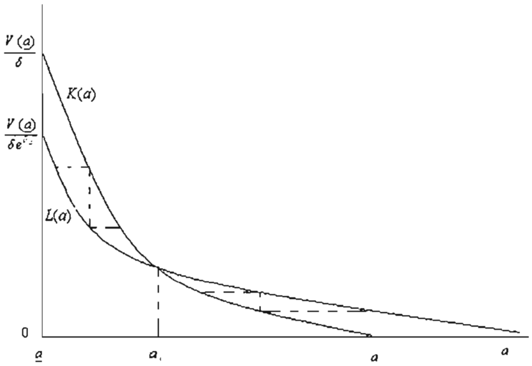

Fig. 4.1: Functions $K(a)$ and $L(a)$.

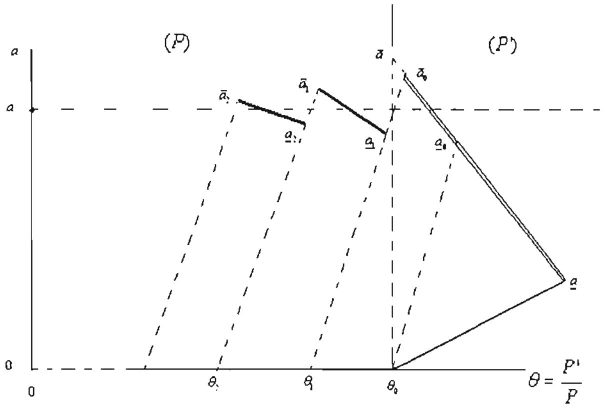

Fig. 4.2: Boundary of the continuation region under certainty

$n = 0, ..., +\infty$. *Precisely, sequences* $(\underline{a}_n)_{n\in\mathbb{N}}$ *and* $(\overline{a}_n)_{n\in\mathbb{N}}$ *are respectively increasing and decreasing with* $\lim_{n\to+\infty} \underline{a}_n = \lim_{n\to+\infty} \overline{a}_n = a_f$.

Proof. Assume that $\mu < \mu'$. When the relative timber price θ varies in $[\theta_1, \theta_0]$, by definition of θ_0 and θ_1, it is optimal to plant a bare land with species (P) and cut the stand at age $a(\theta)$ such that $K(a(\theta)) = \frac{\delta'}{\delta} b'\theta e^{\overline{\mu}a(\theta)}$ where $\theta e^{\overline{\mu}a(\theta)} > \theta_0$ and switch immediately to species $(P\prime)$. Recall that, by Proposition 6, $a^+(\theta)$ is a decreasing curve in the plan (θ, a) stretching between $(\theta_0, \overline{a})$ and $(\overline{\theta}, \underline{a})$. By definition of θ_1, when the relative timber price is equal to θ_1, the manager is indifferent between planting species (P) and harvest later on at age $\overline{a}_0$ and switching to species $(P\prime)$, or harvesting and replanting the same species for the last time at age $\underline{a}_1$. Note that $\overline{a}_0 \geq \underline{a}_1$, by definition of $\overline{a}_0$, $\underline{a}_1$, and θ_0. Assume that $\overline{a}_0 = \underline{a}_1$, that is the harvest will take place at some time in the future at which the manager is indifferent between species (P) or $(P\prime)$ to be planted. Then, necessarily $\overline{a}_0 = \underline{a}_1 = \overline{a}$, $\theta_0 = \theta_1$, and $K(\overline{a}) = L(\underline{a}_0)$ and $K(\overline{a}) = \frac{\delta'}{\delta} b'\theta_0$ by Proposition 7. As $K(\underline{a}_0) = \frac{\delta'}{\delta} b'\theta_0 e^{\overline{\mu}\underline{a}_0}$ then, $L(\underline{a}_0) = K(\underline{a}_0)e^{\overline{\mu}\underline{a}_0}$ implying $\frac{V_a(\underline{a}_0)}{V(\underline{a}_0)} = \delta + \frac{\delta}{e^{\delta'\underline{a}_0 - 1}}$ that contradicts equation (4.17a) as long as $\delta \neq \delta'$. We conclude that $\overline{a}_0 > \underline{a}_1$.

Consider now the cutting boundary $a_1(\theta)$ corresponding to a stand that has been planted with species (P) at date 0 when timber relative price was $\theta \in [\theta_1, \theta_0]$. Consider two pairs $(xe^{\overline{\mu}a(x)}, a(x))$ and $(ye^{\overline{\mu}a(y)}, a(y))$ of this curve such that $x < y$. There exist only two pairs $(x'e^{\overline{\mu}a(x')}, a(x'))$ and $(y'e^{\overline{\mu}a(y')}, a(y'))$ on the upper switching curve $a^+(\theta)$ such that $R(a(x)) = a(x')$, $R(a(y)) = a(y')$, $x' < y'$, and $a(x') > a(y')$ as it is decreasing in the plan (θ, a), as depicted in Figure 4.1. One can show that the function R conserves the order relationship and therefore necessarily $a(x') < a(y')$. We conclude that $a_1(\theta)$ is a decreasing curve in the plan (θ, a). The proof can be recursively repeated to show that all curves $a_n(\theta)$, $n = 2, ... + \infty$, are decreasing in the plan (θ, a).

For $n = 0, ..., +\infty$, by definition of $a_{n+1}(\theta)$, for any pair $(x, a(x)) \in a_{n+1}(\theta)$, there exists a unique pair $(x', a(x')) \in a_n(\theta)$ such that $R(a(x)) = a(x')$. Given the proprieties of the function R, this implies that $[\underline{a}_{n+1}, \overline{a}_{n+1}] \subset [\underline{a}_n, \overline{a}_n]$

and $\lim_{n\to+\infty} \underline{a}_n = \lim_{n\to+\infty} \overline{a}_n = a_f$.

■

Proposition 9 *When $\mu < \mu'$, the reduced land value function $f(\theta)$ and the corresponding cutting age can be computed recursively using*

$$f(\theta) = \max_{\theta' \geq \theta} \left(\frac{\theta'}{\theta}\right)^{-\frac{\delta}{\delta-\delta'}} \left[V\left(\frac{\log(\frac{\theta'}{\theta})}{\mu' - \mu}\right) + f(\theta')\right] \tag{4.21}$$

for $\theta \leq \theta_0$ and using $f(\theta) = b'\theta$ for $\theta \geq \theta_0$.

Proof. When the current prices are given by p and $\theta \leq \theta_0$, it is optimal to plant species (P) and the land value function is defined as $pf(\theta) = \max_{s\geqslant 0} e^{-rs}\left[p_s V(s) + p_s f(\theta_s)\right]$. Under certainty, $p_s = pe^{\mu s}$ and $\theta_s = \theta e^{(\mu'-\mu)s} = \theta e^{(\delta-\delta')s}$. Consequently, $f(\theta) = \max_{s\geqslant 0} e^{-\delta s}\left[V(s) + f(\theta_s)\right]$ where s is replaced by $\frac{1}{\mu'-\mu}\log(\frac{\theta_s}{\theta})$, to obtain the expression of $f(\theta)$ in Proposition 9. ■

Note that the cutting age as a function of θ is a discontinuous curve. The discontinuous part corresponds to $\theta \leq \theta_0$, that is the replanting part of the cutting age. The discontinuity is present to distinguish between a stand that has to be replanted for n times with species (P) before switching definitely to species (P') and another who has to be replanted exactly $n - 1$ times before switching to species (P').

Proposition 9 is used to compute the land value function illustrated in Figure 4.4. At relative prices equal to or higher than the switching threshold θ_0, land value equals $b'\theta$, the land value function that applies when the sole species is (P').

Note that when $\mu > \mu'$, one can show that

$$f(\theta) = \max_{\theta' \leq \theta} \left(\frac{\theta'}{\theta}\right)^{-\frac{\delta}{\delta-\delta'}} \left[V(\frac{\log(\frac{\theta'}{\theta})}{\mu' - \mu}) + f(\theta')\right]$$

with $f(\theta) = b$ for $\theta \leq \theta_0$, in which case (θ_0, a_0) is the unique solution to $K(a_0) = \frac{\delta'}{\delta} b' \theta_0 e^{(\mu'-\mu)a_0}$ and $\frac{V_a'(a_0)}{V'(a_0)} = \delta' + \frac{\delta}{e^{\delta a_0} - 1}$. In this case, for relative price equal or lower than the switching threshold θ_0, the land value coincides with b, the land value function when species (P) is solely available. When both timber prices have equal drifts, that is $\mu = \mu'$, the relative price is a constant and $f(\theta) = \max(b, b'\theta)$. In this case, $\theta_0 = 1$ and $a_0 = a_f$.

3.4.2 Numerical resolution with uncertainty

In order to conclude the description of the solution of the model with uncertainty, in particular in order to describe the boundary of the continuation region, it is necessary to use numerical methods. The reduced forest value function $g(\theta, a)$ must satisfy the partial differential equation and the value-matching and smooth pasting conditions stated in Proposition 3. The partial differential equation governing $g(\theta, a)$ can be simplified by performing the change of variable $x = \log(\theta)$. Let $h(x, a) = g(\theta, a)$ and $l(x) = f(\theta)$, then the partial differential equation governing $h(x, a)$ is

$$\frac{\overline{\sigma}^2}{2} h_{xx} + \left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right) h_x - \delta h + h_a = 0 \tag{3.22}$$

The optimal stopping problem of valuing the forest value when species (P) is planted is similar to the problem of valuing an American-type option with free boundary. Because the free boundary location is not known in advance, the value-matching and smooth pasting conditions cannot be of immediate help. These conditions can be used to localize the free cutting boundary once the stand value is determined. We do so iteratively as explained later in this section. To compute the forest value function $h(x,a)$, it is helpful to specify the corresponding optimal stopping problem as a linear complementarity one (Zvan et al, 1998). Let $\mathcal{L}$ be the linear operator defined as

$$\mathcal{L}.h = \frac{\overline{\sigma}^2}{2} h_{xx} + \left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right) h_x - \delta h + h_a$$

Then, the linear complementarity problem is

$$\begin{aligned} \mathcal{L}.h(x,a) &\geq 0 \\ h(x,a) - (l(x) + V(a)) &\geq 0 \\ \mathcal{L}.h(x,a)\,[h(x,a) - (l(x) + V(a))] &= 0 \end{aligned}$$

Note that this formulation does not imply any explicit use of the free cutting boundary $a(\theta)$. It shows that the value function $h(x,a)$ can be considered as the value of an American option with expiry date equals to infinity, an underlying asset which is a geometric diffusion process with drift $\overline{\mu}$ and volatility $\overline{\sigma}^2$, and a discount factor equal to δ. Indeed, in the continuation region where it is optimal to continue holding the option to harvest, the required return δh is equal to the actual return or equivalently $\mathcal{L}.h(x,a) = 0$ and the option value is higher than the payoff, that is $h(x,a) - (l(x) + V(a)) > 0$. Consequently, it is not yet optimal to exercise. In the stopping region, it is no more optimal to continue holding the option to harvest but it is optimal to harvest immediately because the required return δh is less than the actual return or equivalently $\mathcal{L}.h(x,a) > 0$ and the option value must equal the payoff that is $h(x,a) - (l(x) + V(a)) = 0$. The free cutting boundary is just where the decision maker is indifferent between harvesting immediately or continuing to hold this option, that is when $\mathcal{L}.h(x,a) = 0$ and $h(x,a) - (l(x) + V(a)) = 0$.

When valuing an American option, the use of the complementarity formulation is straightforward. The option value is then computed numerically by performing a discretization of the linear complementarity problem. The main difference between valuing an American option and the forest value is that the payoff in the former case is known as a function of the underlying asset when exercising the option whereas it should be endogenously determined in the latter as it is the sum of the timber crop value and the land value $l(x) = h(x, 0)$. For this reason, valuing the forest value function $h(x, a)$ as a complementarity problem as specified above is seemingly not possible. Instead, we need to consider simultaneously the second problem consisting in valuing the forest value function $g'(\theta, a) = h'(x, a)$ as follows :

$$\mathcal{L}.h(x,a) \geq 0 \tag{4.23a}$$

$$h(x,a) - (l(x) + V(a)) \geq 0 \tag{4.23b}$$

$$\mathcal{L}.h(x,a)\,[h(x,a) - (l(x) + V(a))] = 0 \tag{4.23c}$$

$$\mathcal{L}.h'(x,a) \geq 0 \tag{4.24a}$$

$$h'(x,a) - (l(x) + e^{x}V(a)) \geq 0 \tag{4.24b}$$

$$\mathcal{L}.h'(x,a)\,[h'(x,a) - (l(x) + e^{x}V(a))] = 0 \tag{4.24c}$$

$$l(x) = h(x,0) = h'(x,0) \tag{4.25}$$

Equations $(4.23a) - (4.24c)$ represent the complementarity problem respectively for stand (P) and stand (P') whereas equation (4.25) says that each forest value at age zero must equal the land value by Proposition 4. As specified by equations $(4.23a) - (4.24c)$, and equation (4.25), the problem of valuing simultaneously both stands can then be solved numerically. The value-matching and smooth-pasting conditions will be used to localize the age cutting boundary. The forest value functions $h(x,a)$ and $h'(x,a)$ are computed iteratively as specified by the following pseudo-code. First, (i) assume that after the $n-1^{th}$ iteration, the land value function is $l^{(n)}$; then (ii) compute forest value function $h^{(n)}$ as solution to $(4.23a)-(4.23c)$

and forest value function $h'^{(n)}$ as solution to $(4.24a)-(4.24c)$; (iii) deduce the new land value function as $l^{(n+1)}(x)=\max(h^{(n)}(x,0),h'^{(n)}(x,0))$; (iv) continue iteratively until convergence is reached when $h(x,0)$ and $h'(x,0)$ are approximately equal.

The initial value ascribed to the land value can be arbitrarily chosen. We find that the algorithm converges when the land value has initially any positive values. However, the convergence is generally faster when the initial land value function is equal to $\max(b,b'\theta)$ or $\max(b,b'e^{x})$. Indeed, this is the minimal land value when both tree species (P) and (P') are available.

The numerical discretization to compute $h^{(n)}$ and $h'^{(n)}$ at step (ii) is based on a fully implicit finite difference method. With respect to a fully explicit finite difference method, the implicit method is unconditionally stable and more robust (Brennan and Schwartz 1978). Other numerical methods to solve option valuation problems are discussed in Wilmott et al (1993). We use the penalty method (Zvan et al, 1998) to solve the linear complementarity problem as in Insley (2002).

Figure **4.3** illustrates the cutting age boundary in the case where $\mu<\mu'$. The relative price threshold θ^* for which the decision maker is indifferent between planting any of the two species on a bare land (the switching threshold) corresponds to the relative price for which it is optimal to cut the currently planted species at age zero. It is the same relative price at which the replanting cutting boundary meets the switching one. It is higher under uncertainty than its corresponding level under certainty.

Figure 3.4 shows the land value function in the case where $\mu < \mu'$. The land has a greater value under uncertainty than under certainty and it is more valuable when the uncertainty level, measured by $\overline{\sigma}^2$, increases. Unlike the case under certainty, the land value function has a continuous slope around the relative price switching threshold under uncertainty to prevent any arbitrage. When the relative price tends respectively to zero or to infinity, that is when respectively species (P') or (P) is worthless, the reduced land value functions tend toward their single species level b or $b'\theta$ respectively.

3.5 Conclusion

We have examined the decision to undertake projects that differ in that they generate different future income flows. The focus has been extensions of the conventional forestry economics model. In our extended forestry model, two alternative species may be planted, so that outputs as well as timber prices differ across projects. This entails more sophisticated planting and harvesting decisions than had been considered before.

When choosing between two alternative species for just one rotation, the decision maker plants immediately if the price of one species is sufficiently high compared to the price of the other species. However, the decision maker prefers to wait if both prices are sufficiently close. This is so even while the decision does not involve any direct cost. Indeed, even in the absence of explicit investment costs, there is the potential cost of establishing the wrong species. This mistake is irreversible

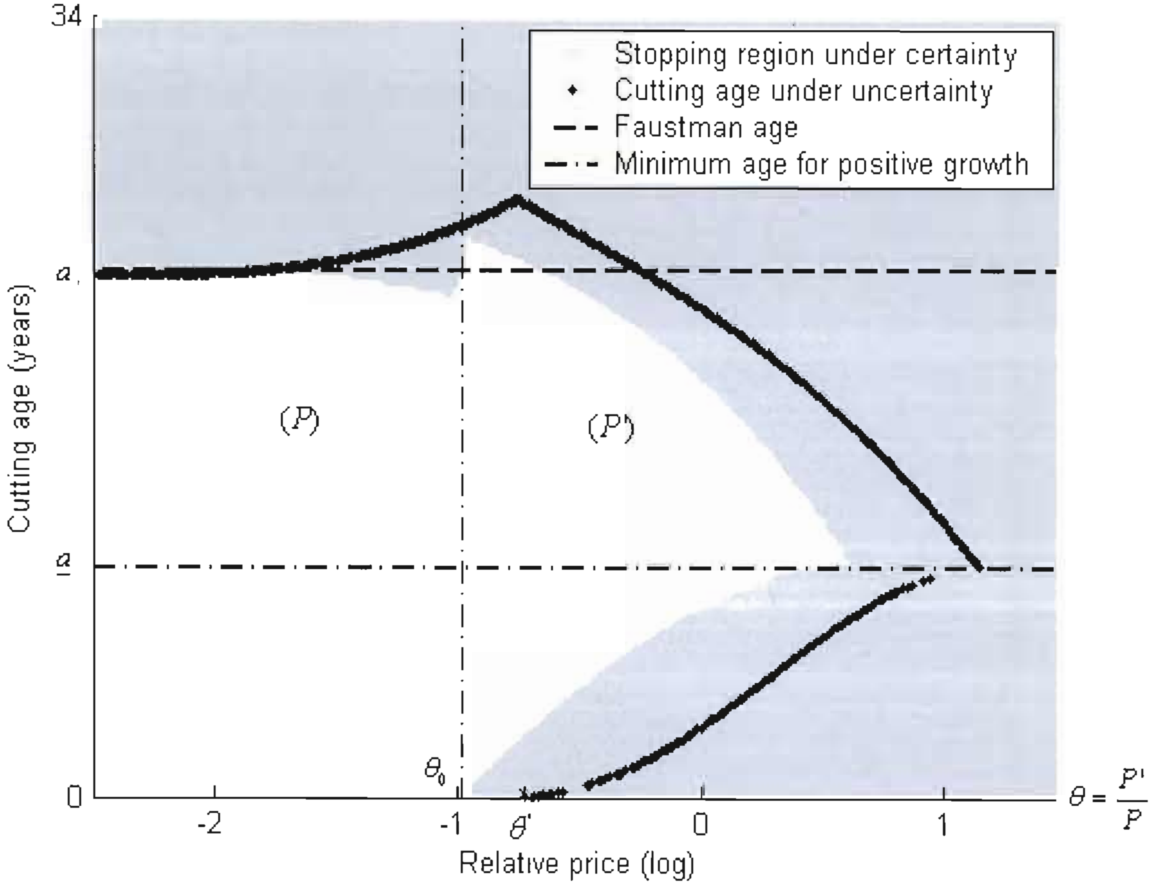

Fig. 4.3: Cutting age as function of the relative price ($\mu < \mu'$)

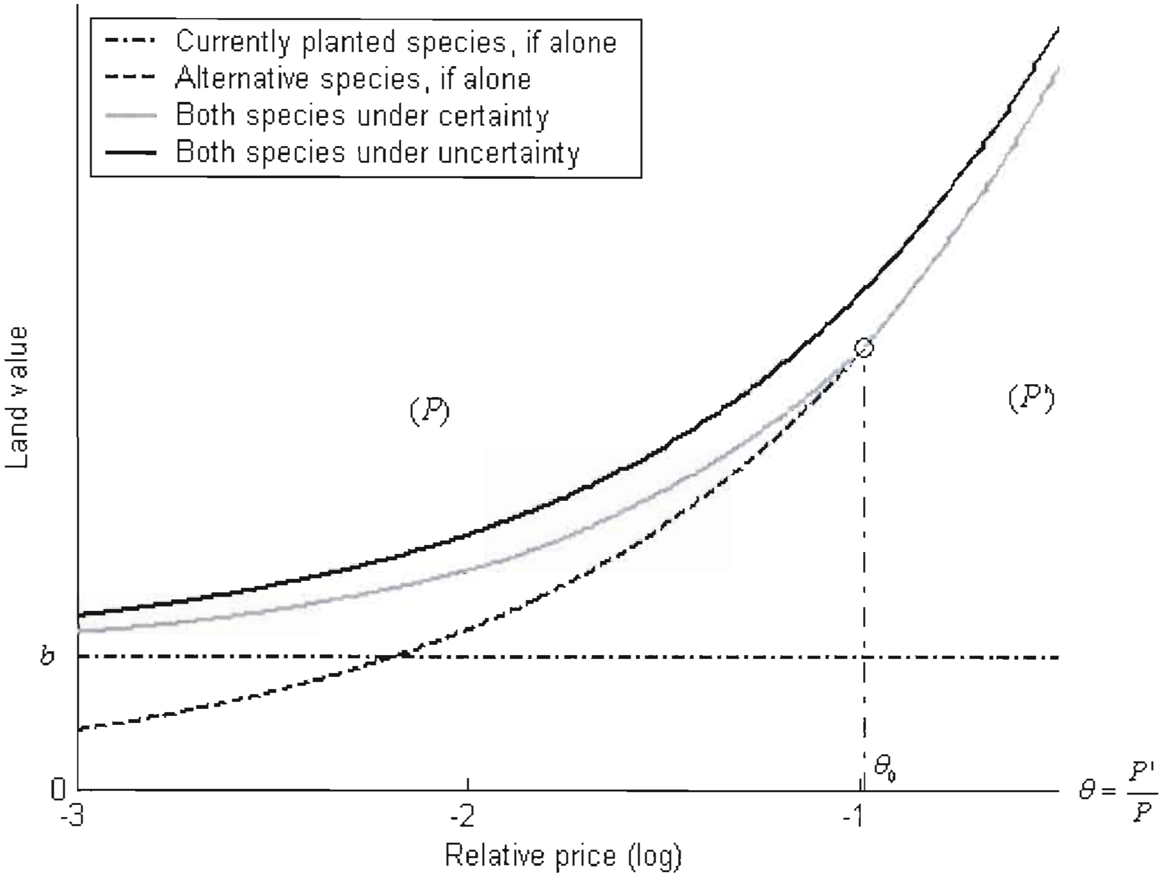

Fig. 4.4: Land value as function of the relative price ($\mu < \mu'$)

because the project involves one harvest only. Consequently, the decision maker prefers waiting until the prices of the alternative species are sufficiently far away from each other to make the probability of a price reversal acceptably low.

We then considered situations where the decision maker has the opportunity to exploit the forest land for an indefinite number of rotations. Here the decision maker must decide at what age the current stand should be harvested and whether the same species, or the alternative species, should be replanted, immediately or after some delay.

We showed that, absent any planting costs, the investor immediately establishes the species whose relative price exceeds some threshold; otherwise the alternative species is selected. This rule differs from the one rotation case because introducing many rotations has the effect of diluting the irreversibility effect present in the one rotation case. The decision maker plants immediately because it is possible to switch to the alternative species whenever desirable. This is not an easy decision, however, as trees may in that case have to be harvested while they have little or no value. We have shown that, although waiting without planting may help choose the best species, nothing can be gained and some loss may be made, by using that strategy.

We have characterized the value functions and the optimal management strategy in this stochastic repeated rotations context, although they were not provided in explicit form. In the space of relative species prices and wood stand ages, we characterized the set of points where the manager is willing to wait or to exercise the option of harvesting. The exercise frontier divides itself into a zone where the same species is reestablished immediately, and, at higher relative prices a zone where the other species is chosen. The relative price that separates these two zones is independent of stand age. At still higher relative prices, the continuation zone and exercise frontier disappear altogether : a stand of the "wrong" species should then be harvested immediately.

Land value is higher than when one species only is available. It converges to the limiting cases of one species, or the other, when their relative price tends to zero or to infinity. When both species are available, the optimum cutting age is non monotonous but oscillates around Faustman's' age when the relative price is below some threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species and adjust the number of times the same species has to be planted. The land value increases with the uncertainty related to timber prices. It is an increasing function of the relative price with a continuous slope even around the switching threshold to prevent any arbitrage as long as the uncertainty is present. On the contrary, under certainty, the slope of the land value function generally changes at the switching threshold. For relative prices higher than the switching threshold, the optimal harvesting age decreases until it hits the minimum age at which timber volumes become positive. Similar results are established explicitly in the certainty case.

Annexe A Proof of the Second Order Condition

We will show that the Hessian matrix $\left[\frac{\partial^2 W}{\partial a_i \partial a_j}\right]_{1\leq i,j\leq n+1}$ associated to $W(\theta, a_n, ..., a_0)$ twice continuously differentiable with respect to $(a_n, ..., a_0)$ is negative definite on $]\underline{a}, +\infty[^{n+1}$, where

$$g(\theta, 0) = \max_{a_n,...,a_0} W(\theta, a_n, ..., a_0)$$

To do so, we will show that its leading principal minors of order k has the sign of $(-1)^k$ for $k = 1, ..., n+1$. First determine $W_{kk} = \frac{\partial^2 W}{\partial a_k \partial a_k}$ for $1 \leq k \leq n+1$, and $W_{kl} = \frac{\partial^2 W}{\partial a_l \partial a_k}$ for l such that $k < l \leq n+1$ as the Hessian matrix is symmetric :

$W_{kl} = -\delta V_a(a_k)e^{-\delta(a_k+...+a_n)} + \delta^2 \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)} + \delta'^2 b'\theta e^{-\delta'(a_0+...+a_n)}$

$W_{kk} = V_{aa}(a_k)e^{-\delta(a_k+...+a_n)} - \delta V_a(a_k)e^{-\delta(a_k+...+a_n)} - \delta V_a(a_k)e^{-\delta(a_k+...+a_n)}$

$+\delta^2 \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)} + \delta'^2 b'\theta e^{-\delta'(a_0+...+a_n)}$. Using the first order conditions, we show that W_{kl} is independent of k and l, and that $W_{kl} < 0 : W_{kl} = -\delta V_a(a_k)e^{-\delta(a_k+...+a_n)} + \delta^2 \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)}$

$+\delta' \left[V_a(a_k)e^{-\delta(a_k+...+a_n)} - \delta \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)}\right]$

$W_{kl} = \overline{\mu} V_a(a_k)e^{-\delta(a_k+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)}$

$W_{kl} = \delta\overline{\mu} K(a_k)e^{-\delta(a_k+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k-1} V(a_i)e^{-\delta(a_i+...+a_n)}$

$W_{kl} = \delta\overline{\mu} L(a_{k-1})e^{-\delta(a_k+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k-1} V(a_i)e^{-\delta(a_i+...+a_n)}$

$W_{kl} = \overline{\mu} V_a(a_{k-1})e^{-\delta(a_{k-1}+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k-1} V(a_i)e^{-\delta(a_i+...+a_n)}$

$W_{kl} = \delta\overline{\mu} K(a_{k-1})e^{-\delta(a_{k-1}+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k-2} V(a_i)e^{-\delta(a_i+...+a_n)}$

We continue so and show that,

$W_{kl} = \delta\overline{\mu} K(a_2)e^{-\delta(a_2+...+a_n)} - \delta\overline{\mu} \left[V(a_0)e^{-\delta(a_0+...+a_n)} + V(a_1)e^{-\delta(a_1+...+a_n)}\right]$

$W_{kl} = \delta\overline{\mu} L(a_1)e^{-\delta(a_2+...+a_n)} - \delta\overline{\mu} \left[V(a_0)e^{-\delta(a_0+...+a_n)} + V(a_1)e^{-\delta(a_1+...+a_n)}\right]$

$W_{kl} = \overline{\mu} V_a(a_1)e^{-\delta(a_1+...+a_n)} - \delta\overline{\mu} \left[V(a_0)e^{-\delta(a_0+...+a_n)} + V(a_1)e^{-\delta(a_1+...+a_n)}\right]$

$W_{kl} = \delta\overline{\mu} K(a_1)e^{-\delta(a_1+...+a_n)} - \delta\overline{\mu} V(a_0)e^{-\delta(a_0+...+a_n)}$

$W_{kl} = (\delta' - \delta)\, V_a(a_0)e^{-\delta(a_0+...+a_n)} - \delta\overline{\mu}V(a_0)e^{-\delta(a_0+...+a_n)}$
$W_{kl} = \delta\overline{\mu}e^{-\delta(a_0+...+a_n)}\left[\frac{V_a(a_0)}{\delta} - V(a_0)\right]$
$W_{kl} = \delta\overline{\mu}e^{-\delta(a_0+...+a_n)}K(a_0) < 0$ as $\overline{\mu} = \delta' - \delta < 0$ and $K(a_0) > 0$.

Denote $W_{kl} = \beta < 0$ and $\alpha_k = \delta e^{-\delta(a_k+...+a_n)}K_a(a_k) < 0$ then $W_{kk} = \alpha_k + \beta$. The k^{th} leading principal minor for $k = 1, ..., n+1$, is therefore

$$\begin{vmatrix} \alpha_1+\beta & \beta & \ldots & \beta & \beta \\ \beta & \alpha_2+\beta & \ldots & . & . \\ . & . & \ldots & . & . \\ . & . & \ldots & \beta & . \\ . & . & \ldots & \alpha_k+\beta & \beta \\ \beta & \beta & \ldots & \beta & \alpha_k+\beta \end{vmatrix}$$

that is denoted $H(\alpha_1, ..., \alpha_k, \beta)$. By subtracting the second line from the first one and then develop the determinant according the first line, we obtain

$$H(\alpha_1, ..., \alpha_k, \beta) = \begin{vmatrix} \alpha_1 & -\alpha_2 & \ldots & 0 & 0 \\ \beta & \alpha_2+\beta & \ldots & \beta & \beta \\ . & \beta & \ldots & . & . \\ . & . & \ldots & \beta & . \\ . & . & \ldots & \alpha_{k-1}+\beta & \beta \\ \beta & \beta & \ldots & \beta & \alpha_k+\beta \end{vmatrix}$$

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \alpha_2 \begin{vmatrix} \beta & \beta & \ldots & . & \beta \\ \beta & \alpha_3+\beta & \ldots & . & . \\ . & \beta & \ldots & . & . \\ . & . & \ldots & \alpha_{k-1}+\beta & \beta \\ \beta & \beta & \ldots & \beta & \alpha_k+\beta \end{vmatrix} .$$

By subtracting the last line from the first one to compute the last determinant and continue to do so, we obtain

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \alpha_2 \begin{vmatrix} 0 & -\alpha_3 & 0 & 0 & 0 \\ \beta & \alpha_3 + \beta & \dots & \beta & \beta \\ \beta & \beta & \dots & . & . \\ . & . & \dots & \alpha_{k-1} + \beta & \beta \\ \beta & . & \dots & \beta & \alpha_k + \beta \end{vmatrix}$$

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \alpha_2\alpha_3 \begin{vmatrix} \beta & \beta & \dots & \beta \\ . & \alpha_3 + \beta & \dots & . \\ . & . & \dots & . \\ \beta & \beta & \dots & \alpha_k + \beta \end{vmatrix}$$

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \alpha_2\alpha_3 ... \alpha_{k-1} \begin{vmatrix} \beta & \beta \\ \beta & \alpha_k + \beta \end{vmatrix}$$

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \beta\alpha_2\alpha_3 ... \alpha_k.$$

Now, it is possible to show recursively that the k^{th} leading principal minor has the sign of $(-1)^k$. Indeed, one can check that $H(\alpha_1, \beta) = \alpha_1 + \beta < 0$, and that $H(\alpha_1, \alpha_2, \beta) = (\alpha_1 + \beta)(\alpha_2 + \beta) - \beta^2 = \alpha_1\alpha_2 + (\alpha_1 + \alpha_2)\beta > 0$. Assume now that the leading principal minor of order $k - 1$, $2 \leqslant k \leqslant n + 1$, has the sign of $(-1)^{k-1}$, then $H(\alpha_2, ..., \alpha_k, \beta)$ has the sign of $(-1)^{k-1}$. Consequently, in the expression above, $\alpha_1 H(\alpha_2, ..., \alpha_k, \beta)$ and $\beta\alpha_2\alpha_3 ... \alpha_k$ have both the sign of $(-1)^k$, therefore the k^{th} leading principal minor $H(\alpha_1, ..., \alpha_k, \beta)$ has the sign of $(-1)^k$.

Annexe B The partial differential equation of the stand value function

$G(p, p', a) = pg(\theta, a)$ must satisfy Bellman's equation $EdG(p, p', a) = rG(p, p', a)da$.

As $dG = G_a da + G_p dp + \frac{1}{2} G_{pp} dp^2 + G_{p'} dp' + \frac{1}{2} G_{p'p'} dp'^2 + G_{p'p} dp dp' + o(da)$ where

$G_a(p, p', a) = pg_a(\theta, a)$,

$G_p(p, p', a) = g(\theta, a) - \theta g_\theta(\theta, a)$,

$G_{pp}(p, p', a) = \frac{\theta^2}{p} g_{\theta\theta}(\theta, a)$,

$G_{p'}(p, p', a) = g_\theta(\theta, a)$,

$G_{p'p'}(a, p, p') = \frac{1}{p} g_{\theta\theta}(\theta, a)$, and

$G_{pp'}(p, p', a) = -\frac{\theta}{p} g_{\theta\theta}(\theta, a)$.

Then $EdG(p, p', a)$

$= pg_a da + \mu pg da + p\theta g_\theta \left(\mu' - \mu\right) da + \theta^2 g_{\theta\theta} \left(\frac{\sigma^2}{2} - \rho\sigma\sigma' + \frac{\sigma'^2}{2}\right) da + o(da)$

and so $\frac{\overline{\sigma}^2}{2} \theta^2 g_{\theta\theta} + \overline{\mu} \theta g_\theta - \delta g + g_a = 0$ where $\overline{\sigma}^2 = \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$ and $\overline{\mu} = \mu' - \mu$.

Note that θ is a GBM with drift $(\mu' - \mu) + \sigma^2 - \rho\sigma\sigma'$ and volatility $\overline{\sigma}^2$.

Annexe C Smooth Pasting Conditions

The harvesting boundary between the continuation region and the stopping region is a locus of points (a, θ) on which the decision maker is just indifferent between waiting for a infinitesimal time period da and harvesting immediately. At any point on the boundary, the gain from harvesting immediately, $pV(a) + pf(\theta)$, should be equal to the expected gain from waiting for da, i.e. letting the trees reach age $a + da$ and θ change by $d\theta$.

We will consider small increments da in such a way as to neglect terms of order higher than 1/2 in the processes governing p, p', and θ. Then

$$d\theta = \sigma\theta dz + o\left(da^{1/2}\right)$$

and the sign of $d\theta$ is the sign of dz, irrespective of da.

Let π_1 denote the probability that $d\theta$ is positive ; and π_2 the probability that $d\theta \leq 0$. Starting from (θ, a) on the boundary a change $d\theta > 0$ implies that the stand should be harvested. Let $d\theta_1 = E\left(d\theta \middle| \, d\theta > 0\right)$ and $d\theta_2 = E\left(d\theta \middle| \, d\theta \leq 0\right)$; correspondingly let $dp_1 = E\left(dp \middle| \, d\theta > 0\right)$ and $dp_2 = E\left(dp \middle| \, d\theta \leq 0\right)$.With probability π_1 the stand should be harvested immediately in which case the gain is $(p + dp_1)\, V(a + da) + (p + dp_1)\, f(\theta + d\theta_1)$; with probability π_2, the manager should wait further in which case the gain is $(p + dp_2)\, g(a + da, \theta + d\theta_2)$. The non-arbitrage condition on the cutting boundary then stipulates that

$$\begin{aligned} pV(a) + pf(\theta) &= e^{-rda}\left\{\pi_1 (p + dp_1)\left(V(a + da) + f(\theta + d\theta_1)\right)\right. \\ &\quad \left. + \pi_2 (p + dp_2)\, g(a + da, \theta + d\theta_2)\right\} \end{aligned}$$

where terms in da are negligible.

This condition may be interpreted as the Smooth-Pasting condition with respect to θ. Indeed, taking Taylor expansions of V around a, of f around θ, and of g around (θ, a); letting da tend to zero; and dividing by p, leads to

$$\begin{aligned} V(a) + f(\theta) &= \pi_1 (V(a) + f(\theta)) + \pi_1 f_\theta(\theta) d\theta_1 + \pi_1 \frac{dp_1}{p} (V(a) + f(\theta)) \\ &\quad + \pi_2 g(\theta, a) + \pi_2 g_\theta(\theta, a) d\theta_2 + \pi_2 g(\theta, a) \frac{dp_2}{p} + o(da^{1/2}) \end{aligned}$$

Using the Value-Matching condition $V(a) + f(\theta) = g(\theta, a)$; noting that $\pi_1 + \pi_2 = 1$, $Edp = \pi_1 dp_1 + \pi_2 dp_2 = \mu p da$

$$\pi_1 f_\theta(\theta) d\theta_1 + \pi_2 g_\theta(\theta, a) d\theta_2 + \mu p g(\theta, a) da + o(da^{1/2}) = 0$$

Since $Ed\theta = \pi_1 d\theta_1 + \pi_2 d\theta_2 = \gamma \theta da$ where γ is the drift of θ, $\pi_2 d\theta_2 = \gamma \theta da - \pi_1 d\theta_1$,

$$\pi_1 f_\theta(\theta) d\theta_1 + (\gamma \theta da - \pi_1 d\theta_1) g_\theta(\theta, a) + \mu p g(\theta, a) da + o(da^{1/2}) = 0$$

The terms in da are negligible and can be dropped. Consequently,

$$\pi_1 d\theta_1 (f_\theta(\theta) - g_\theta(\theta, a)) + o(da^{1/2}) = 0$$

Since $\pi_1 = \frac{1}{2} + O\left(da^{\frac{1}{2}}\right)$ (Dixit and Pindyck, 1994, p. 69) and $\pi_1 d\theta_1$ is of order $da^{1/2}$, then necessarily,

$$f_\theta(\theta) = g_\theta(\theta, a)$$

which is the Smooth-Pasting condition.

Differentiating the Value-Matching condition

$$g(\theta, a(\theta)) = V(a(\theta)) + f(\theta)$$

with respect to θ leads to

$$f_\theta(\theta) + V_a(a(\theta)) a_\theta(\theta) = g_a(\theta, a(\theta)) a_\theta(\theta) + g_\theta(\theta, a(\theta))$$

Consequently,

$$V_a(a(\theta)) = g_a(\theta, a(\theta))$$

The smooth-pasting condition can be equivalently expressed as $f_\theta(\theta) = g_\theta(\theta, a)$ or as $V_a(a(\theta)) = g_a(\theta, a(\theta))$.

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