UNIVERSITÉ DU QUÉBEC À MONTRÉAL

# ASPECTS COMBINATOIRES DES POLYNÔMES DE MACDONALD

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# Combinatorial Aspects of Macdonald Polynomials

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ADOLFO RODRÍGUEZ

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CONTENTS

# RÉSUMÉ

La théorie sur les polynômes de Macdonald a fait l'objet d'une quantité importante de recherches au cours des dernières années. Définis originellement par Macdonald comme une généralisation de quelques-unes des bases les plus importantes de l'anneau des fonctions symétriques, ces polynômes ont des applications dans des domaines tels que la théorie des représentations des groupes quantiques et physique des particules. Ce travail présente quelques-uns des résultats combinatoires les plus importants entourant ces polynômes, en mettant particulièrement l'accent sur la formule combinatoire prouvée récemment par Haglund, Haiman et Loehr pour les polynômes de Macdonald.



Mots-clés : Polynômes de Macdonald, fonctions symétriques, combinatoire algébrique, combinatoire enumerative, théorie des représentations, géométrie algébrique.

## ABSTRACT

The theory of Macdonald polynomials has been subject to a substantial amount of research in recent years. Originally defined by Macdonald as a common generalization to some of the most important known basis of the ring of symmetric polynomials, they have shown to have striking applications in a wide variety of non elementary topics such as representation theory of quantum groups and particle physics. This work outlines some of the most important combinatorial results surrounding this theory, making special emphasis on the combinatorial formula for Macdonald polynomials proven recently by Haglund, Haiman and Loehr.

Keywords: Macdonald polynomials, symmetric functions, algebraic combinatorics, enumerative combinatorics, representation theory, algebraic geometry.

# INTRODUCTION

A great number of beautiful results involving Macdonald polynomials have been established since these were first introduced in (Macdonald, 1988), and a large number of conjectures have been raised, making the theory of Macdonald symmetric polynomials one of the most active subjects in Algebraic Combinatorics. A recent conjecture by Haglund (Haglund, 2004), giving a combinatorial expression for Macdonald polynomials, almost immediately proven in (Haglund, Haiman, Loehr, 2005), raised hopes of finding a combinatorial formula for the $q, t$-Kostka coefficients that appear upon expressing Macdonald polynomials in terms of Schur symmetric functions. A combinatorial proof of the positivity of $q, t$-Kostka coefficients was finally achieved this year in (Assaf, 2007) by giving an interpretation of the coefficients of LLT polynomials in terms of Schur polynomials, and using Haglund's expression of Macdonald polynomials in terms of LLT. Most of the body of this work is devoted to developing the concepts that are needed to understand and prove Haglund's formula for Macdonald polynomials.

The first chapter of this monograph introduces the basic combinatorial structures (namely; diagrams, partitions, compositions, fillings, super fillings and tableaux) that index most of the formulas in the following chapters. The second chapter is meant to be a short introduction to the theory of symmetric functions or symmetric polynomials in an infinite number of variables. Some classic results on this theory are proven while others are conveniently cited from the references in order to keep a light but still self-contained approach. The third chapter is the main of this work, showing (Haglund, Haiman, Loehr, 2005)'s proof of Haglund's formula for Macdonald polynomials, along with a few of the most important combinatorial aspects of this theory. Finally, the last chapter is an appendix of some interesting conjectures and open problems involving the topics studied in the first three chapters.

# CHAPTER I

# BASIC NOTIONS

The aim of this chapter is to define the combinatorial structures that will index most of the definitions and relations in the following ones. Symmetric polynomials are normally indexed by *Young diagrams* or *skew partitions*, and in some cases by *tuples of skew partitions*. Quasisymmetric polynomials are indexed by *compositions*. The definitions of these polynomials are often weighed sums over a convenient set of *fillings* or *super fillings*, where the weights depend on parameters that are referred to as *statistics* throughout this work.

## 1.1 Diagrams and Partitions

**Definition 1.1.1.** For the purpose of this work, a *diagram* is defined as a subset of $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \mathbb{Z}_+ \cup \{0\} = \{0, 1, 2, \ldots\}$.

A diagram can be pictorially represented as a set of cells on the first quadrant of a Cartesian coordinate system. In such representation, the cell corresponding to $(i, j)$ is the unitary square with vertices $(i, j), (i, j+1), (i+1, j), (i+1, j+1)$ (see figure 1.1).

In what follows, a diagram and its pictorial representation will be referred to without distinction. The elements of a diagram are commonly called *cells*.

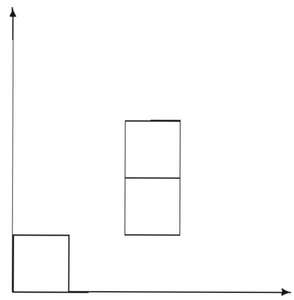


**Figure 1.1** Representation of the diagram $\{(0,0),(2,1),(2,2)\}$.

## Young Diagrams

**Definition 1.1.2.** A *Young diagram* is a diagram $\mu \in \mathbb{N} \times \mathbb{N}$ such that if $(i_0, j_0) \in \mu$, then $(i,j) \in \mu$ for all $i,j \in \mathbb{N}$ with $i \leq i_0$ and $j \leq j_0$ (see figure 1.2). The $(j+1)$*th row* of a *Young diagram* is the one consisting of all the cells with second coordinate equal to $j$, this is, the $(j+1)$th row from bottom to top. Similarly, the $(i+1)$*th column* is the one consisting of all the cells with first coordinate equal to $i$. The *size* of $\mu$, denoted $|\mu|$, is the number of cells in $\mu$.

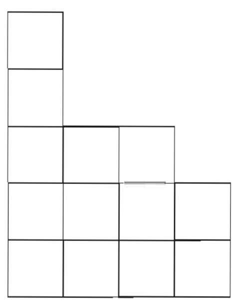


**Figure 1.2** A Young diagram. When drawing a Young diagram, coordinate axes are conveniently omitted.

**Definition 1.1.3.** A cell $(i,j)$ of a Young diagram $\mu$ is said to be a *corner* if $(i+1,j),(i,j+1)$ are not in $\mu$. The corners of the Young diagram in figure 1.2 are $(0,4)$, $(2,2)$ and $(3,1)$.

**Definition 1.1.4.** Let $\mu$ be a Young diagram. The *conjugate* of $\mu$, denoted $\mu'$, is the Young diagram which results of reflecting each cell of $\mu$ through the line $y = x$ in the coordinate plane. Notice that if $\eta = \mu'$ then $\mu = \eta'$. In such case, it is said that "$\mu$ and $\eta$ are conjugates".

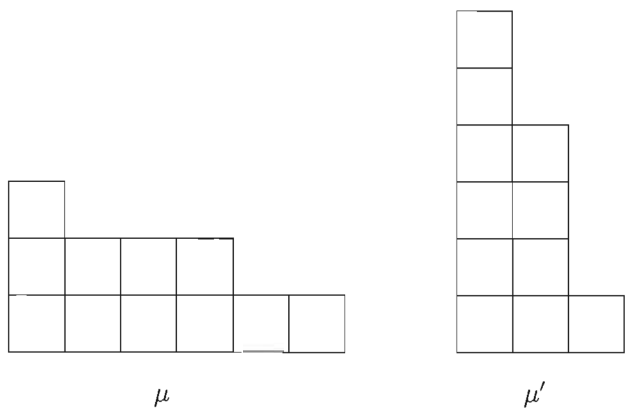



**Figure 1.3** A Young diagram and its conjugate.

The concept of Young diagrams is linked to that of partitions of integers:

**Definition 1.1.5.** Let $n$ be a non-negative integer. A *partition* $\lambda$ of $n$ is a (possibly empty) tuple $(\lambda_1, \ldots, \lambda_k)$ of integers satisfying the two conditions

$$\lambda_1 \geq \cdots \geq \lambda_k > 0$$

$$\lambda_1 + \cdots + \lambda_k = n$$

The integers $\lambda_1, \ldots, \lambda_k$ are called the *parts* of $\lambda$. The number of parts of $\lambda$ is denoted $l(\lambda)$ and is called the *length* of $\lambda$. The statement "$\lambda$ is a partition of $n$" is abbreviated $\lambda \vdash n$. The integer $n$ is called the *size* of $\lambda$. The empty partition is denoted by the number 0. In order to simplify some formulas, the parts $\lambda_i$ are set to be zero for $i > l(\lambda)$.

**Remark.** Partitions of a non-negative integer $n$ are in natural bijection with Young diagrams of size $n$. In fact, if $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of $n$, the Young diagram with row lengths $\lambda_r$ ($r = 1, \dots, k$) is in unique correspondence with $\lambda$. The Young diagram in figure 1.2 corresponds to the partition $(4, 4, 3, 1, 1) \vdash 13$.

This said, partitions and Young diagrams will be referred to without distinction, identifying every partition $(\mu_1, \dots, \mu_r)$ with the unique Young diagram of row lengths $\mu_1, \dots, \mu_r$. Notice that if $\mu = (\mu_1, \dots, \mu_r)$ and $\mu' = (\mu'_1, \dots, \mu'_c)$ are conjugates as in definition 1.1.4, then $\mu'_1, \dots, \mu'_c$ are the heights of the columns of $\mu$. Also $\mu'_i$ is the number of parts of $\mu$ that are greater than or equal to $i$. As a consequence

$$l(\mu) = \mu'_1 \tag{1.1}$$

$$l(\mu') = \mu_1 \tag{1.2}$$

With this same notation, the *corners* of $\mu$ can be equivalently defined as the cells $(i-1, \mu_i - 1)$ in $\mu$ such that $\mu_{i+1} < \mu_i$.

The number of partitions of a nonnegative integer $n$ is commonly denoted $p(n)$. There is no known simple formula for $p(n)$, however, its generating function can be deduced by conveniently rewriting partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ as $(j^{m_j}, \dots, 1^{m_1})$, where $m_i$ ($1 \leq i \leq j$) denotes the number of parts of $\lambda$ that are equal to $i$. Thus $p(n)$ is simply the number of ways of expressing $n$ as a sum $m_1 1 + m_2 2 + \cdots$. This is summarized in the formula;

$$\sum_{n \geq 0} p(n) x^n = \prod_{i \geq 1} \sum_{m \geq 0} x^{mi} = \prod_{i \geq 1} \frac{1}{1 - x^i} \tag{1.3}$$

A few important definitions concerning Young diagrams follow.

**Definition 1.1.6.** Let $\mu$ be a Young diagram. The *arm* of a cell $(i, j) \in \mu$ is the subset of cells of $\mu$ that are in the same row and to the right of $(i, j)$. More formally, it is the set

$$\{(i', j) \in \mu : i' > i\}$$

The number of cells in the arm of $(i, j)$ is denoted $\mathrm{arm}(i, j)$ (see figure 1.4).

**Definition 1.1.7.** The *leg* of a cell $(i, j)$ in a Young diagram $\mu$, is the subset of cells of $\mu$ that are in the same column and above $(i, j)$. More formally, it is the set

$$\{(i, j') \in \mu : j' > j\}$$

. The number of cells in the leg of $(i, j)$ is denoted $\text{leg}(i, j)$ (see figure 1.4).

**Definition 1.1.8.** The *hook* of a cell $(i, j)$ in a Young diagram $\mu$, is the union of the arm, the leg, and the cell $(i, j)$ itself. More formally, it is the set

$$\{(i', j) \in \mu : i' \geq i\} \cup \{(i, j') \in \mu : j' > j\}$$

. The number of cells in the hook of $(i, j)$ will be denoted $\text{hook}(i, j)$ (see figure 1.4).

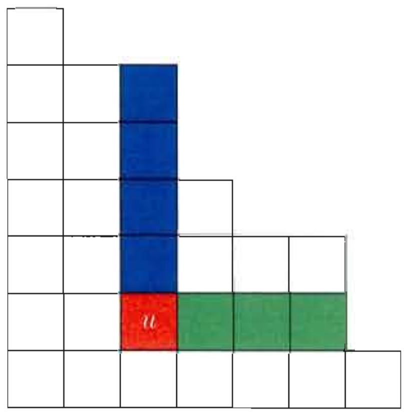



**Figure 1.4** A cell $u = (2, 1)$ in a Young diagram, with $\text{arm}(u) = 3$, $\text{leg}(u) = 4$, and $\text{hook}(u) = 8$.

Let $\mu = (\mu_1, \ldots, \mu_r)$ and $\mu' = (\mu'_1, \ldots, \mu'_c)$ be conjugate partitions. Recall that $\mu_{j+1}$ is simply the length of the $(j+1)$th row of $\mu$, and $\mu'_{i+1}$ is the length of the $(i+1)$th column of $\mu$. The three equations below follow from these observations:

$$\text{arm}(i, j) = \mu_{j+1} - i - 1 \tag{1.4}$$

$$\text{leg}(i, j) = \mu'_{i+1} - j - 1 \tag{1.5}$$

$$\text{hook}(i, j) = \text{arm}(i, j) + \text{leg}(i, j) + 1 \tag{1.6}$$

**Definition 1.1.9.** A *skew partition* is a diagram of the form $\mu \backslash \lambda$, where $\mu$ and $\lambda$ are Young diagrams. Young diagrams are themselves skew partitions. The definitions of *arm*, *leg* and *hook* of a cell can be naturally extended to skew partitions. In fact, if

$u \in \nu = \mu\backslash\lambda$, then the arm, leg, and hook of $u$ in $\nu$, are the same as those in $\mu$. The notation $\mu/\lambda = \mu\backslash\lambda$ is frequently used in the literature.

**Definition 1.1.10.** A skew partition $\nu$ is a *horizontal strip* (respectively a *vertical strip*) if it does not have two cells on the same column (respectively on the same row). Figure 1.5 shows a horizontal and a vertical strip.

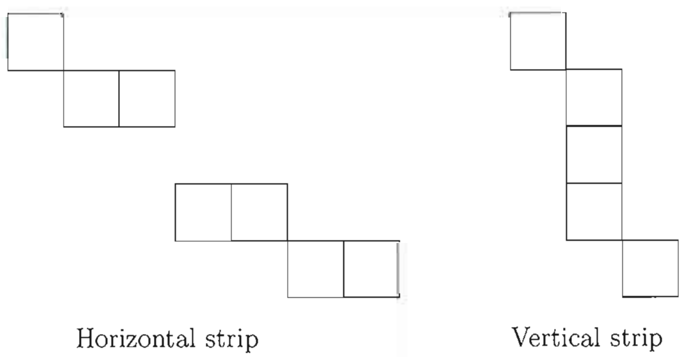



**Figure 1.5** A horizontal strip and a vertical strip.

**Definition 1.1.11.** The *content* of a cell $(i, j)$ of a diagram is defined by

$$c(i, j) = j - i$$

| | | | | |
|---|---|---|---|---|
| 4 | 3 | 2 | 1 | 0 |
| 3 | 2 | 1 | 0 | -1 |
| 2 | 1 | 0 | -1 | -2 |
| 1 | 0 | -1 | -2 | -3 |
| 0 | -1 | -2 | -3 | -4 |

**Figure 1.6** Content function.

**Definition 1.1.12.** Let $\mu \in \mathbb{N} \times \mathbb{N}$ be any diagram (not necessarily a Young diagram). The parameter $n(\mu)$ is given by the sum

$$n(\mu) = \sum_{(i,j)\in\mu} i$$

These are a few of the most important properties of the parameter $n(\mu)$:

1. $(n(\mu), n(\mu')) = \sum_{(i,j)\in\mu} (i,j)$
2. If $\mu = (\mu_1, \dots, \mu_r)$ is a partition and $\mu' = (\mu'_1, \dots, \mu'_c)$ its conjugate, then
$$n(\mu) = \mu'_2 + 2\mu'_3 + \cdots + (c-1)\mu'_c = \sum_{0\le i\le c-1} i\mu_{i+1} \tag{1.7}$$
3. If $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_l)$ are partitions with $\lambda \subseteq \mu$ (where $\subseteq$ denotes the usual contention relation, viewing $\lambda$ and $\mu$ as Young diagrams), and $\nu = \mu\backslash\lambda$ is a skew partition, then
$$\begin{aligned} n(\nu) = n(\mu) - n(\lambda) &= (\mu'_2 - \lambda'_2) + 2(\mu'_3 - \lambda'_3) + \cdots + (r-1)(\mu'_r - \lambda'_r) \\ &= \sum_{0\le i\le r-1} i(\mu'_{i+1} - \lambda'_{i+1}) \end{aligned} \tag{1.8}$$
where $r = \max\{l(\mu'), l(\lambda')\} = \max\{\mu_1, \lambda_1\}$.
4. $n(\mu') - n(\mu) = \sum_{u\in\mu} c(u)$.

**Definition 1.1.13.** A skew partition $\nu$ is a *ribbon* if it satisfies the following conditions (see figure 1.7):

1. It is connected. This is, you can go from any cell to any other through a series of length 1 horizontal or vertical steps without falling out of the diagram.
2. It does not contain any $2 \times 2$ square.
3. The lower-right cell has content 1 (see definition 1.1.11).

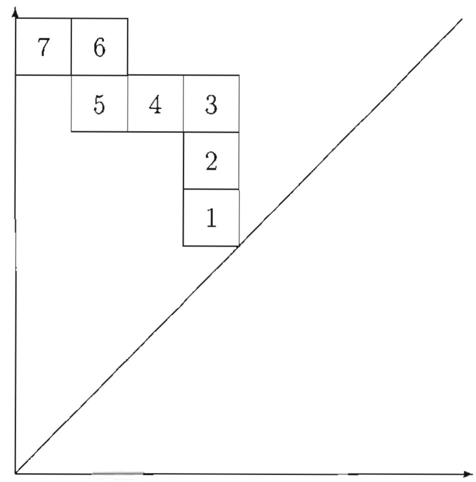



**Figure 1.7** A ribbon and its content function $c$.

Notice that the contents of the cells of a ribbon of size $n$ are always the consecutive numbers $1, 2, \ldots, n$ (see figure 1.7).

**Definition 1.1.14.** The *descent set* of a ribbon $\nu$ is the set of contents $c(u)$ of the cells $u = (i, j)$ such that $(i, j-1) \in \nu$. This set is denoted $\mathrm{d}(\nu)$. The descent set of the ribbon in figure 1.7 is $\{2, 3, 6\}$. Clearly for $|\nu| = n$, the set $\mathrm{d}(\nu)$ can be any subset of $\{2, 3, \ldots, n\}$. Also the set $\mathrm{d}(\nu)$ defines the ribbon $\nu$ uniquely.

## Orders on Partitions

It is appropriate at this point to introduce some partial and total orders on partitions that will be useful in the future.

**Contention order (partial).** Two partitions $\lambda$ and $\mu$ are said to satisfy $\lambda \subseteq \mu$ (reads "$\lambda$ is contained in $\mu$" or "$\mu$ contains $\lambda$") if their associated Young diagrams satisfy the usual set contention relation $\lambda \subseteq \mu$. In other words, $\lambda \subseteq \mu$ if they satisfy the following two properties:

1. $l(\mu) \geq l(\lambda)$

2. $\mu_i \geq \lambda_i$ $(1 \leq i \leq l(\lambda))$.

For example $(2,2,1) \subset (4,3,1,1,1)$. See figure 1.8 for a visual example.

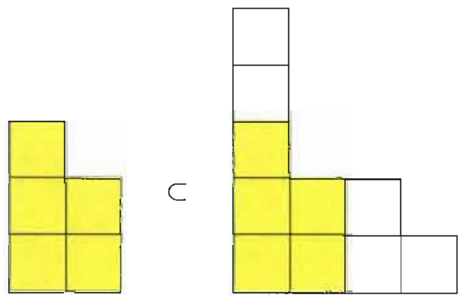


**Figure 1.8** Contention order on partitions.

**Dominance order (partial).** A partition $\mu$ is said to *dominate* another partition $\lambda$, if $\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$ for all $k \geq 0$. This relation is written $\lambda \preceq \mu$. For example $(2,2,2,1,1) \prec (3,3,2)$. Notice that unlike the contention order, the dominance order is still interesting when restricted to partitions of a given positive integer $n$. In fact, when the relation $\lambda \preceq \mu$ is written, it is frequently assumed that $|\lambda| = |\mu|$ unless stated otherwise.

**Lexicographic order (total).** Two partitions are said to satisfy $\lambda <_{\text{lex}} \mu$ if the first non-zero difference $\mu_i - \lambda_i$ is positive. For example $(5,4,2,2,2) < (5,4,3,1)$ since $5-5 = 4-4 = 0$ and $3-2 = 1 > 0$. See figure 1.9 for a visual example.

Notice that if two partitions $\mu$ and $\lambda$ satisfy $\lambda \prec \mu$, and $i$ is the first positive integer for which $\mu_i \neq \lambda_i$, then $\mu_i = (\mu_1 + \cdots + \mu_i) - (\lambda_1 + \cdots + \lambda_i) + \lambda_i \geq \lambda_i$, and so $\lambda <_{\text{lex}} \mu$. In other words, the lexicographic order is a linear extension of the dominance order. Also if $\lambda \prec \mu$ for $\mu$ and $\lambda$ partitions of the same positive integer $n$, then adding up the inequalities

$$\lambda_1 + \ldots + \lambda_i \leq \mu_1 + \cdots + \mu_i$$

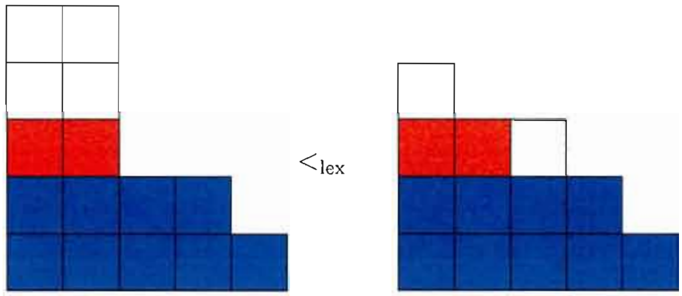



**Figure 1.9** Lexicographic order on partitions.

for $1 \leq i \leq m = \max\{l(\lambda), l(\mu)\}$, knowing that at least one of them is strict, one obtains

$$\sum_{i=0}^{m-1}(m-i)\lambda_{i+1} < \sum_{i=0}^{m-1}(m-i)\mu_{i+1},$$

which substracted to the equality $\sum_{i=0}^{m-1} m\lambda_{i+1} = \sum_{i=0}^{m-1} m\mu_{i+1} = mn$, yields

$$n(\lambda) > n(\mu)$$

Another interesting feature of the dominance order is the fact that it is symmetric with respect to conjugation when restricted to partitions of a single positive integer $n$, this is;

**Proposition 1.1.1** (Classic result). *Let $\lambda$ and $\mu$ be two partitions of a positive integer $n$. Then:*

$$\mu \prec \lambda \Leftrightarrow \mu' \succ \lambda'$$

*Proof.* Suppose that $\mu'$ does not dominate $\lambda'$ and let $r$ be the first positive integer for which $\mu'_1 + \ldots + \mu'_r < \lambda'_1 + \ldots + \lambda'_r$. Clearly $\mu'_r < \lambda'_r$. Thus, considering that $\mu'_r$ and $\lambda'_r$ are the number of $\mu_i$'s and $\lambda_i$'s respectively that are greater than or equal to $r$;

$$0 < (\lambda'_1 + \ldots + \lambda'_r) - (\mu'_1 + \ldots + \mu'_r)$$

$$= (\mu'_{r+1} + \ldots + \mu'_{l(\mu')}) - (\lambda'_{r+1} + \ldots + \lambda'_{l(\lambda')})$$

$$= |\{(i,j) \in \mu : i \geq r\}| - |\{(i,j) \in \lambda : i \geq r\}|$$

$$= \left(\sum_{i=1}^{\mu'_r} \mu_i - r\right) - \left(\sum_{i=1}^{\lambda'_r} \lambda_i - r\right)$$

$$\leq \left(\sum_{i=1}^{\mu'_r} \mu_i - r\right) - \left(\sum_{i=1}^{\mu'_r} \lambda_i - r\right) = \left(\sum_{i=1}^{\mu'_r} \mu_i\right) - \left(\sum_{i=1}^{\mu'_r} \lambda_i\right)$$

Therefore $\lambda$ does not dominate $\mu$. $\square$

### Orders on the Cells

Some of the development in this work requires to assign a partial or total order to the cells of a diagram. Some general results depend on a given such ordering. These orders are often restrictions of total or partial orders in $\mathbb{Z} \times \mathbb{Z}$. These are some of the orders on $\mathbb{Z} \times \mathbb{Z}$ that will be useful in further sections:

**Reading order (total).** This order is defined by reading each row starting from the one on the top to the one on the bottom, reading the cells of each row from left to right (see figure 1.10). This is the usual "reading" order of western languages. More formally:

$$(i, j) \leq_{\text{read}} (i', j') \text{ if } \begin{cases} j' < j, \text{ or} \\ j' = j \text{ and } i' \geq i \end{cases}$$

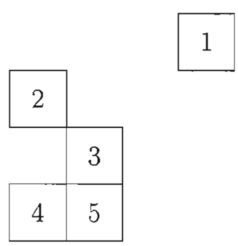


**Figure 1.10** Cells labeled in the reading order.

**Natural order (partial).** In this order, $u \leq v$ if the relation $\leq$ is satisfied coordinate-wise. This is:

$$(i, j) \leq (i', j') \text{ if } i' - i \geq 0 \text{ and } j' - j \geq 0$$

**Lexicographic order (total).** In this order, $u <_{\text{lex}} v$ if the first nonzero coordinate of $v - u$ is positive. This is:

$$(i,j) \leq_{\text{lex}} (i',j') \text{ if } \begin{cases} i' > i, \text{ or} \\ i' = i \text{ and } j' \geq j \end{cases}$$

An immediate relation between the reading and lexicographic orders is that

$$(i,j) \leq_{\text{lex}} (i',j') \Leftrightarrow (j,-i) \geq_{\text{read}} (j',-i')$$

The natural order on cells allows for a simpler definition of Young diagrams as those sets $\mu \subseteq \mathbb{N} \times \mathbb{N}$ such that $(i,j) \in \mu$ if $(i',j') \in \mu$ and $(i,j) \leq (i',j')$.

## 1.2 Fillings, Super Fillings and Tableaux

For the purpose of this and future sections, an *alphabet* $A$ will be a totally ordered set of "letters".

**Definition 1.2.1.** Let $\mu \in \mathbb{N} \times \mathbb{N}$ be a diagram. A *filling* $\varphi$ of $\mu$ with values in $A$ is simply a function $\varphi : \mu \to A$. A filling $\varphi : \mu \to A$ can be represented by writing the values $\varphi(u)$ inside each cell $u \in \mu$. The diagram $\mu$ is called the *shape* of $\varphi$. The values $\varphi(u)$ $(u \in \mu)$ are called the *entries* of $\varphi$.

The total order $\leq$ on the alphabet $A$ allows for the definition of *semi standard* fillings;

**Definition 1.2.2.** Let $\mu \in \mathbb{N} \times \mathbb{N}$ be a diagram. A filling $\varphi : \mu \to A$ is said to be *semi standard* if its entries are strictly increasing upwards on every column of $\mu$ , and weakly increasing from left to right on every row of $\mu$ (See figure 1.11). This is, if and only if;

1. $\varphi(i_1,j) \leq \varphi(i_2,j)$ for all $(i_1,j),(i_2,j) \in \mu$ with $i_1 < i_2$; and,

2. $\varphi(i,j_1) < \varphi(i,j_2)$ for all $(i,j_1),(i,j_2) \in \mu$ with $j_1 < j_2$.

**Definition 1.2.3.** A *standard filling* of a diagram $\mu$ is a bijective semi standard filling with entries in $\{1,2,\ldots,|\mu|\}$ (See figure 1.12).

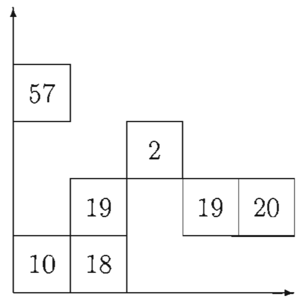



**Figure 1.11** A semi standard filling.

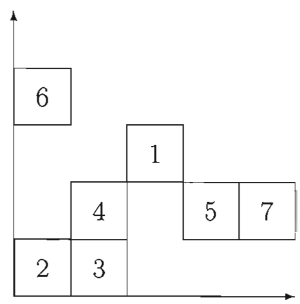



**Figure 1.12** A standard filling.

**Definition 1.2.4.** A semi standard filling of a skew partition $\nu$ (possibly a Young diagram) with entries in $\mathbb{Z}_+$, is called a *semi standard Young tableau of shape* $\nu$. The set of semi standard Young tableaux of shape $\nu$ is denoted $\mathrm{SSYT}(\nu)$.

**Definition 1.2.5.** A standard filling of a skew partition $\nu$ (possibly a Young diagram) is called a *standard Young tableau of shape* $\nu$. The entries of standard Young tableaux are strictly increasing on every row from left to right and upwards on every column. The set of standard Young tableaux of shape $\nu$ is denoted $\mathrm{SYT}(\nu)$. This is clearly always a finite set of at most $|\nu|!$ elements. figure 1.13 shows all standard Young tableaux of shape $(2, 2, 1)$.

A standard Young tableau $\tau : \mu \to \{1, 2, \ldots, n\}$, where $\mu$ is a partition and $n = |\mu|$, can be identified with a maximal chain $\mu^{(0)} \subset \mu^{(1)} \subset \ldots \subset \mu^{(n)}$ in the contention order

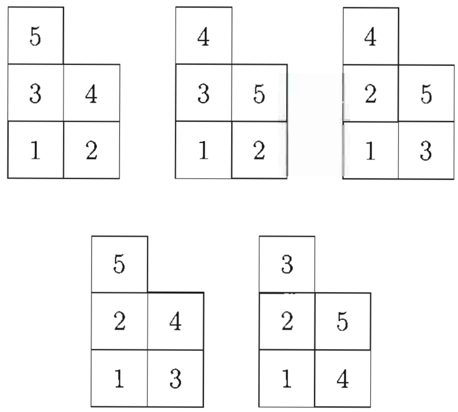



**Figure 1.13** Standard Young tableaux of shape $(2, 2, 1)$.

on partitions, where $\mu^{(0)} = 0$ (the empty partition) and $\mu^{(n)} = \mu$. This maximal chain is obtained by setting $\mu^{(k)}$ to be the subset of cells $(i, j) \in \mu$ that satisfy $\tau(i, j) \leq k$. The maximality is clear from the fact that $|\mu^{(k+1)}| - |\mu^{(k)}| = 1$. Also the $\mu^{(k)}$'s are all Young diagrams, since $\tau$ is increasing in columns and rows. This correspondence is clearly bijective. figure 1.14 shows a standard Young tableau and its corresponding maximal chain.

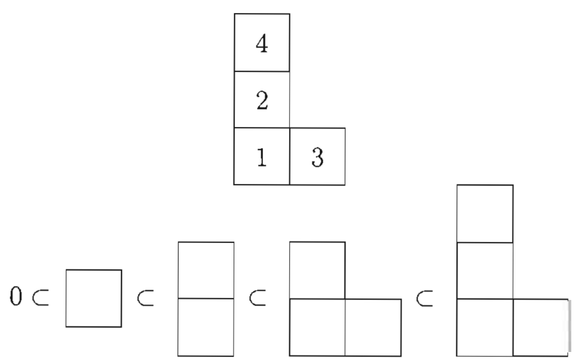



**Figure 1.14** Standard Young tableau of shape $(2, 1, 1)$ and its corresponding maximal chain.

When $\mu$ is a partition, $f^{\mu} = |\mathrm{SYT}(\mu)|$ denotes the number of standard Young tableaux of shape $\mu$. This number can be expressed in terms of the hook lengths hook$(i,j)$ of the cells of $\mu$. If $|\mu| = n$, then

$$f^{\mu} = \frac{n!}{\prod_{(i,j)\in\mu} \text{hook}(i,j)} \tag{1.9}$$

This is known as the "hook length formula" and its proof can be found in (Bergeron, 2008) among others. The hook lengths of $\mu = (2,2,1)$ are shown in figure 1.15. As a result;

$$f^{(2,2,1)} = \frac{5!}{1\times 1\times 2\times 3\times 4} = 5$$

which confirms the number of fillings in figure 1.13.

| | |
|---|---|
| 1 | |
| 3 | 1 |
| 4 | 2 |

**Figure 1.15** Hook lengths of $\mu = (2,2,1)$.

Semi standard Young tableaux can be counted when their entries are restricted:

**Definition 1.2.6.** Let $\lambda$ and $\mu$ be two partitions of the same positive integer $n$. The *Kostka number* $K_{\lambda\mu}$ is the number of semi standard Young tableaux $\tau : \lambda \to \mathbb{Z}_+$ such that $|\tau^{-1}(i)| = \mu_i$ for $i = 1,\ldots,l(\mu)$.

The following are a some of the most important basic properties of Kostka numbers:

1. If $\mu = (1^n) = (\underbrace{1,\ldots,1}_{n\,\text{times}})$, then $K_{\lambda\mu} = f^{\lambda}$. This is a direct result of the definition above.

2. $K_{\lambda\lambda} = 1$. Indeed, if a semi standard filling of $\lambda$ has exactly $\lambda_1$ 1's, then all of them must be in the first row. Similarly, all 2's must be in the second row, and so on, leaving only one possible semi standard filling $\tau : \lambda \to \mathbb{Z}_+$ satisfying $|\tau^{-1}(i)| = \lambda_i$ for $i = 1,\ldots,l(\lambda)$.

3. $K_{\lambda\mu} > 0 \Rightarrow \mu \preceq \lambda$. Indeed, suppose that $K_{\lambda\mu} > 0$ and let $\tau : \lambda \to \mathbb{Z}_+$ be a semi standard filling $\tau : \lambda \to \mathbb{Z}_+$ satisfying the condition in definition 1.2.6. Suppose that there exists $k \geq 1$ such that $\mu_1 + \cdots + \mu_k > \lambda_1 + \cdots + \lambda_k$. This is, $\tau^{-1}\{1, \ldots, k\} > \lambda_1 + \cdots + \lambda_k$. Thus there is at least one number in $\{1, \ldots, k\}$ placed above the $k$th row of $\lambda$. In other words, there is at least one cell $(i, j) \in \lambda$ such that $j \geq k \geq \tau(i, j)$. Therefore, since $\tau$ is semi standard, $\tau(i, 0) \leq \tau(i, j) - j \leq \tau(i, j) - k \leq 0$ and so $\tau(i, 0) \notin \mathbb{Z}_+$, which is a contradiction. As a result, $\lambda$ must necessarily dominate $\mu$.

4. $\mu \preceq \lambda \Rightarrow K_{\lambda\mu} > 0$. This fact can be seen as a consequence of the symmetry of Schur polynomials (see expression 2.25 in section 2.2 for its proof).

### Statistics on Fillings

As in previous sections, assume the alphabet $A$ to be a totally ordered set.

**Definition 1.2.7.** Let $\nu \in \mathbb{N} \times \mathbb{N}$ be a skew partition (possibly a Young diagram) and let $\varphi : \nu \to A$ be a filling of $\nu$. A cell $(i, j) \in \nu$ is said to be a *descent* of $\varphi$ if $j > 0$ and $\varphi(i, j) > \varphi(i, j-1)$. The set of descents of a filling $\varphi$ is denoted $\mathrm{Des}(\varphi)$ (see figure 1.16).

Recall from basic combinatorics that the "descents" of a word $a_1 \cdots a_n \in A^n$ are the $i$'s $(1 \leq i \leq n-2)$ for which $a_i > a_{i+1}$. Notice that in the case $\nu = (1^n)$, the number $|\mathrm{Des}(\varphi)|$ counts the number of descents of the word $\varphi(0, n-1)\varphi(0, n-2) \cdots \varphi(0, 0)$.

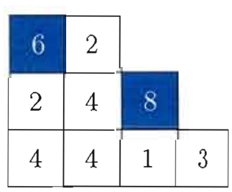



**Figure 1.16** Filling $\varphi$ with $\mathrm{Des}(\varphi) = \{(0, 2), (2, 1)\}$.

The following definition will be necessary to introduce another important set;

**Definition 1.2.8.** Two cells $(i, j) <_{\text{read}} (i', j')$ of a skew partition $\nu$ are said to *attack* each other if either;

$$j' = j - 1 \text{ and } i' < i, \text{ or,}$$

$$j' = j \text{ and } i' > i$$

Figure 1.17 shows three examples of pairs of cells that attack each other in a Young diagram.

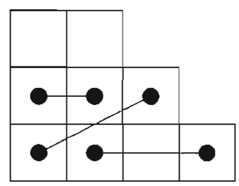


**Figure 1.17** Examples of pairs of cells that attack each other in a Young diagram.

**Definition 1.2.9.** Let $\nu \in \mathbb{N} \times \mathbb{N}$ be a skew partition (possibly a Young diagram) and let $\varphi : \nu \to A$ be a filling of $\nu$. A pair of cells $(u, v)$ of $\nu$, with $u <_{\text{read}} v$, is said to be an *inversion* of $\varphi$, if $u$ and $v$ attack each other and $\varphi(u) > \varphi(v)$. The set of inversions of a filling $\varphi$ is denoted $\text{Inv}(\varphi)$ (see figure 1.18).

Recall from basic combinatorics that the "inversions" of a word $a_1 \cdots a_n \in A^n$ are the pairs $(i, j)$ with $1 \leq i < j \leq n$, for which $a_i > a_j$. In the case $\nu = (n)$, the number $|\text{Inv}(\varphi)|$ counts the number of inversions of the word $\varphi(0,0)\varphi(1,0) \cdots \varphi(n-1,0)$.

**Definition 1.2.10.** Let $\nu \in \mathbb{N} \times \mathbb{N}$ be a skew partition (possibly a Young diagram) and let $\varphi : \mu \to A$ be a filling of $\nu$. The statistics maj and inv are defined as follows:

$$\text{maj}(\varphi) = \sum_{u \in \text{Des}(\varphi)} (\text{leg}(u) + 1)$$

$$\text{inv}(\varphi) = |\text{Inv}(\varphi)| - \sum_{u \in \text{Des}(\varphi)} \text{arm}(u)$$

A simple calculation yields $\text{maj}(\varphi) = 5$ and $\text{inv}(\varphi) = 3$ for the filling in figure 1.18.

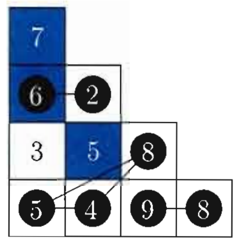
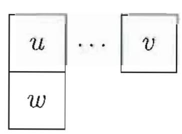



**Figure 1.18** Inversions and descents of a filling of shape $(4,3,2,1)$ (descents are in blue).

In the case $\nu = (1^n)$, the number $\operatorname{maj}(\varphi)$ is the sum of the positions of the descents of the word $\varphi(0, n-1)\varphi(0, n-2)\cdots\varphi(0,0)$, which is the usual maj statistic on words. In the case $\nu = (n)$, the number $\operatorname{inv}(\varphi)$ is simply $|\operatorname{Inv}(\varphi)|$, which, as mentioned before, counts the number of inversions of the word $\varphi(0,0)\varphi(1,0)\cdots\varphi(n-1,0)$.

**Proposition 1.2.1.** (Haglund, Haiman, Loehr, 2005) *The parameters* inv *and* maj *as described in the previous definition are always nonnegative.*

*Proof.* Let $\nu$ be a skew partition and let $\varphi : \nu \to A$ be a filling of $\nu$. The number $\operatorname{maj}(\varphi)$ is clearly nonnegative. In order to prove that $\operatorname{inv}(\varphi)$ is nonnegative, consider all triplets $(u, v, w)$ of cells of $\nu$ such that $u <_{\text{read}} v <_{\text{read}} w$, with $u$ and $v$ in the same row, and $w$ is directly below $u$. More visually, $u$, $v$ and $w$ must be placed in the following way;




Clearly each descent of $\varphi$ appears as $u$ in a triplet $(u, v, w)$ exactly $\operatorname{arm}(u)$ times and each inversion that is not of the form $((i,j),(i',j))$ with $(i, j-1) \notin \nu$, appears as $(u, v)$ or $(v, w)$ exactly once. As a result, if $T$ is the set of all such triplets and $m$ is the number of inversions of the form $((i,j),(i',j))$ with $(i, j-1) \notin \nu$, then;

$$\operatorname{inv}(\varphi) = m + \sum_{(u,v,w)\in T} i_\varphi(u,v) + i_\varphi(v,w) - i_\varphi(u,w)$$

where $i_\varphi(a,b)$ is defined by

$$i_\varphi(a,b) = \begin{cases} 1 \text{ if } \varphi(a) > \varphi(b) \\ 0 \text{ otherwise} \end{cases} \tag{1.10}$$

for $a, b \in \mu$. The number $i_\varphi(u,v) + i_\varphi(v,w) - i_\varphi(u,w)$ is always 0 or 1, thus $\mathrm{inv}(\varphi) \geq 0$. □

In (Haglund, Haiman, Loehr, 2005), the authors define a *reading descent set* for bijective fillings $\xi : \biguplus \boldsymbol{\nu} \to \{1, 2, \ldots, |\nu|\}$ which is useful to relate quasisymmetric functions to Macdonald polynomials. In order to avoid confusing this set with the one in definition 1.2.7, the notation $\mathrm{DR}(\xi)$ will be used here.

**Definition 1.2.11.** Let $\nu$ be a skew partition and $\xi : \nu \to \{1, 2, \ldots, n = |\nu|\}$ a bijective filling. Define the *reading descent set* of $\xi$ as follows:

$$\mathrm{DR}(\xi) = \{i \in \{1, \ldots, n-1\} : \xi^{-1}(i+1) <_{\text{read}} \xi^{-1}(i)\}$$

### Super Fillings

*Super fillings* can be regarded as an extension of fillings in which the entries are not in an alphabet $A$ but in a *super alphabet* $\mathcal{A} = A \uplus A_-$. The sets $A$ and $A_-$ are respectively referred to as the sets of *positive* and *negative* letters of $\mathcal{A}$. Furthermore, the total order $\leq$ in $\mathcal{A}$ must be an extension to the one in the alphabet $A$. The most common setting is $A = \mathbb{Z}_+$ (with the order $1 < 2 < 3 < \ldots$), $A_- = \mathbb{Z}_-$ and $\mathcal{A} = \mathbb{Z}_\pm = \mathbb{Z}_+ \uplus \mathbb{Z}_-$ (with some total order that satisfies $1 < 2 < 3 < \ldots$). This will be the setting in all the examples. Super fillings will be useful in the next chapters to prove combinatorial results involving quasisymmetric polynomials.

**Definition 1.2.12.** Let $\nu$ be a skew partition (possibly a Young diagram) and $\mathcal{A} = A \uplus A_-$ a super alphabet. An *$\mathcal{A}$-valued super filling* of $\mu$ is simply a filling with entries in $\mathcal{A}$. As stated before, it is advisable to regard this definition as a "generalization" of fillings (see figure 1.19).

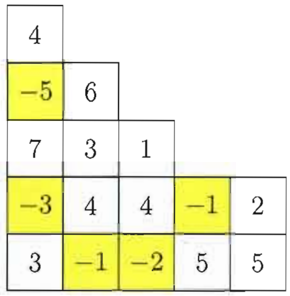



**Figure 1.19** A $(\mathbb{Z}_+ \uplus \mathbb{Z}_-)$-valued super filling.

Since fillings are themselves super fillings, most general theorems on super fillings are also true for fillings. On the other hand, the parameters defined for fillings in the previous section, can be extended to super fillings.

**Definition 1.2.13.** Let $\nu$ be a skew partition (possibly a Young diagram) and let $\varphi$ be an $\mathcal{A}$-valued super filling of $\nu$, for $\mathcal{A} = A \uplus A_-$. The parameter $I_\varphi(u, v)$ is defined for $u, v \in \nu$ as follows:

$$I_\varphi(u,v) = \begin{cases} 1 \text{ if } \varphi(u) > \varphi(v) \text{ or } \varphi(u) = \varphi(v) \in A_- \\ 0 \text{ if } \varphi(u) < \varphi(v) \text{ or } \varphi(u) = \varphi(v) \in A \end{cases} \tag{1.11}$$

This is an extension of the parameter $i_\varphi(u, v)$ defined in equation 1.10. In fact, if $\varphi(u), \varphi(v) \in A$, then $I_\varphi(u, v) = i_\varphi(u, v)$, since the total order in $A$ is a restriction to that in $\mathcal{A}$.

**Definition 1.2.14.** Let $\nu$ be a skew partition (possibly a Young diagram) and let $\varphi$ be an $(A \uplus A_-)$-valued super filling of $\nu$. A cell $(i, j) \in \nu$ is said to be a *descent* of $\varphi$ if $j > 0$ and $I_\varphi((i, j), (i, j - 1)) = 1$. The set of descents of $\varphi$ is denoted $\mathrm{Des}(\varphi)$.

**Definition 1.2.15.** Let $\nu$ be a skew partition (possibly a Young diagram) and let $\varphi$ be an $(A \uplus A_-)$-valued super filling of $\nu$. A pair of cells $(u, v)$ of $\nu$, with $u <_{\text{read}} v$, is said to be an *inversion* of $\varphi$ if $u$ and $v$ attack each other and $I_\varphi(u, v) = 1$. The set of inversions of $\varphi$ is denoted $\mathrm{Inv}(\varphi)$.

It is clear that in the case $\varphi(\nu) \subseteq A$, the two previous definitions are compatible with definitions 1.2.7 and 1.2.9.

**Definition 1.2.16.** Let $\nu$ be a skew partition (possibly a Young diagram) and let $\varphi$ be a super-filling of $\nu$. The statistics maj and inv are defined exactly as in definition 1.2.10:

$$\mathrm{maj}(\varphi) = \sum_{u \in \mathrm{Des}(\varphi)} (\mathrm{leg}(u) + 1)$$

$$\mathrm{inv}(\varphi) = |\mathrm{Inv}(\varphi)| - \sum_{u \in \mathrm{Des}(\varphi)} \mathrm{arm}(u)$$

Using the total order on $\mathbb{Z}_{\pm}$ defined by $1 < -1 < 2 < -2 < \cdots$, a simple calculation yields $\mathrm{maj}(\varphi) = 16$ and $\mathrm{inv}(\varphi) = 6$ for the super filling on figure 1.19.

**Proposition 1.2.2.** (Haglund, Haiman, Loehr, 2005) *The parameters* inv *and* maj *on super fillings are always nonnegative.*

*Proof.* The proof is exactly the same as the one of proposition 1.2.1, using the parameter $I_\varphi$ in place of $i_\varphi$. $\square$

**Definition 1.2.17.** Let $\nu$ be a skew partition and let $\varphi : \nu \to \mathcal{A}$ be a super filling with $\mathcal{A} = A \uplus A_-$. The *standardization* of $\varphi$ is the unique bijective filling $\mathrm{st}(\varphi) : \nu \to \{1, 2, \ldots, |\nu|\}$ such that $\varphi \circ \mathrm{st}(\varphi)^{-1}$ is weakly increasing, and each restriction of the form

$$\mathrm{st}(\varphi)|_{\varphi^{-1}(\{x\})} : \varphi^{-1}(\{x\}) \to \{1, 2, \ldots, |\nu|\} \quad (x \in \varphi(\nu))$$

is strictly increasing with respect to the reading order if $x \in A$ and strictly decreasing with respect to the reading order if $x \in A_-$.

In the cases $\nu = (1^n)$ and $\nu = (n)$, for which $\varphi(\nu) \subseteq A$, the standardization of $\varphi$ is equivalent to the usual standardization of words, when the entries of $\varphi$ and $\mathrm{st}(\varphi)$ are written in the reading order.

**Proposition 1.2.3.** (Haglund, Haiman, Loehr, 2005) *Let $\nu$ be a skew partition (possibly a Young diagram) and let $\varphi : \nu \to A \uplus A_-$ be a super filling. Then:*

$$\mathrm{Des}(\mathrm{st}(\varphi)) = \mathrm{Des}(\varphi),$$

$$\mathrm{Inv}(\mathrm{st}(\varphi)) = \mathrm{Inv}(\varphi),$$

$$\mathrm{maj}(\mathrm{st}(\varphi)) = \mathrm{maj}(\varphi), \textit{ and,}$$

$$\mathrm{inv}(\mathrm{st}(\varphi)) = \mathrm{inv}(\varphi)$$

*Proof.* Set $\xi = \mathrm{st}(\varphi)$. Suppose that $\xi(u) > \xi(v)$ (or $\xi(u) < \xi(v)$) for some $u, v \in \nu$ with $u <_{\text{read}} v$. Then:

$$\varphi(u) = (\varphi \circ \xi^{-1})(\xi(u)) \geq (\varphi \circ \xi^{-1})(\xi(v)) = \varphi(v)$$

$$(\text{respectively } \varphi(u) = (\varphi \circ \xi^{-1})(\xi(u)) \leq (\varphi \circ \xi^{-1})(\xi(v)) = \varphi(v))$$

If $\varphi(u) = \varphi(v) = x$, then $u, v \in \varphi^{-1}(x)$. Since $u <_{\text{read}} v$ and $\xi(u) > \xi(v)$ (respectively $\xi(u) < \xi(v)$); $x$ must be in $A_-$ (respectively $A$). Thus $I_\varphi(u,v) = 1$ (respectively $I_\varphi(u,v) = 0$). Otherwise, if $\varphi(u) > \varphi(v)$ (respectively $\varphi(u) < \varphi(v)$), then clearly $I_\varphi(u,v) = 1$ (respectively $I_\varphi(u,v) = 0$). As a result, if $u <_{\text{read}} v$, then $i_\xi(u,v) = I_\varphi(u,v)$. This clearly proves the first two equalities. The last two are a direct result of the first two. □

**Definition 1.2.18.** Let $\nu$ be a skew partition. A super filling $\varphi : \nu \to A \uplus A_-$ is said to be *semi standard* if it is weakly increasing on its rows from left to right and upwards on its columns, and such that the set $\tau^{-1}(i)$ is a horizontal strip for each $i \in A$ and a vertical strip for each $i \in A_-$. When all the entries of $\varphi$ are in $A$, $\varphi$ is simply a semi standard tableau as in definition 1.2.2.

*Super tableaux* are the super filling extensions of semi standard tableaux:

**Definition 1.2.19.** Let $\nu$ be a skew partition. Dote the set $\mathbb{Z}_\pm = \mathbb{Z}_+ \uplus \mathbb{Z}_-$ with a total order $\leq$ whose restriction to $\mathbb{Z}_+$ is the usual order on integers. A *super tableau* of shape $\nu$ is a semi standard super filling with entries in $\mathbb{Z}_\pm$. The set of super tableaux of shape $\nu$ is denoted $\mathrm{SSYT}_\pm(\nu)$.

A clear consequence of this definition is the following:

$$\mathrm{SSYT}(\nu) = \{\tau \in \mathrm{SSYT}_\pm(\nu) : \tau(\nu) \subseteq \mathbb{Z}_+\} \tag{1.12}$$

**Proposition 1.2.4.** (Haglund, Haiman, Loehr, 2005) *Let $\tau : \nu \to \mathbb{Z}_+ \uplus \mathbb{Z}_-$ be a super filling. Then* $\mathrm{st}(\tau)$ *is a standard Young tableau if and only if $\tau$ is semi standard.*

*Proof.* Set $\xi = \mathrm{st}(\tau)$. The fact that $\tau \circ \xi^{-1}$ is weakly increasing shows that $\tau(u) < \tau(v) \Rightarrow \xi(u) < \xi(v) \Rightarrow \tau(u) \leq \tau(v)$. On the other hand, since $\xi$ is increasing with respect to the reading order when restricted to $\tau^{-1}(i)$ for $i \in A$ and decreasing for $i \in A_-$, then the entries of $\xi$ are increasing in columns upwards and in rows from left to right, if and only if $\tau^{-1}(i)$ is a horizontal strip for $i \in A$ and a vertical strip for $i \in A_-$. □

The following result, last in this section, will be useful to relate quasisymmetric polynomials with Macdonald symmetric polynomials.

**Proposition 1.2.5.** (Haglund, Haiman, Loehr, 2005) *Let $\nu$ be a skew partition, $\xi : \nu \to \{1, 2, \ldots, n = |\nu|\}$ a bijective filling of shape $\nu$, and $a : \{1, 2, \ldots, n\} \to \mathcal{A}$ a weakly increasing function onto a super alphabet $\mathcal{A} = A \uplus A_-$. Then* $\mathrm{st}(a \circ \xi) = \xi$ *if and only if*

$$a(i) = a(i+1) \in A \Rightarrow i \notin \mathrm{DR}(\xi)$$

*and*

$$a(i) = a(i+1) \in A_- \Rightarrow i \in \mathrm{DR}(\xi)$$

*Proof.* Suppose that $\mathrm{st}(a \circ \xi) = \xi$. If $a(i) = a(i+1) \in A$, then clearly $\xi^{-1}(i), \xi^{-1}(i+1) \in (a \circ \xi)^{-1}(x)$ for some $x \in A$. Since $\xi$ must be increasing with respect to the reading order when restricted to $(a \circ \xi)^{-1}(x)$, then $\xi^{-1}(i) <_{\text{read}} \xi^{-1}(i+1)$, and so $i \notin \mathrm{DR}(\xi)$. On the other hand, if $a(i) = a(i+1) \in A_-$, then $\xi^{-1}(i), \xi^{-1}(i+1) \in (a \circ \xi)^{-1}(x)$ for some $x \in A_-$, and so $i \in \mathrm{DR}(\xi)$, since $\xi$ is decreasing with respect to the reading order when restricted to $(a \circ \xi)^{-1}(x)$. Now suppose that the weakly increasing function $a : \{1, 2, \ldots, n\} \to \mathcal{A}$ satisfies

$$a(i) = a(i+1) \in A \Rightarrow i \notin \mathrm{DR}(\xi)$$

and

$$a(i) = a(i+1) \in A_- \Rightarrow i \in \mathrm{DR}(\xi)$$

Let $u <_{\text{cread}} v$ be two cells of $\nu$ such that $u, v \in (a \circ \xi)^{-1}(x) = \xi^{-1}(a^{-1}(x))$ for some $x \in A$. Since $a$ is weakly increasing, the set $a^{-1}(x)$ is an interval of the form $\{m+1, m+2, \ldots, m+j\}$ for some $j > 1$ (because $\xi$ is bijective and $\xi^{-1}(a^{-1}(x))$ has at least two elements $u$ and $v$). Since $a(m+1) = \cdots = a(m+j) = x \in A$, then $m+1, \ldots, m+j-1 \notin \mathrm{DR}(\xi)$, and so $\xi^{-1}(m+1) <_{\text{read}} \cdots <_{\text{read}} \xi^{-1}(m+j)$. As a result $\xi(u) < \xi(v)$. This proves that $\xi$ is increasing with respect to the reading order when restricted to a set of the form $(a \circ \xi)^{-1}(x)$ for some $x \in A$. A similar argument shows that $\xi$ is decreasing with respect to the reading order when restricted to a set of the form $(a \circ \xi)^{-1}(x)$ for $x \in A_-$. Thus $\xi$ is the standardization of $a \circ \xi$. □

**Remark.** The previous proposition shows that for a given bijective filling $\xi : \nu \to \{1, 2, \ldots, n = |\nu|\}$, there is a one to one correspondence between super fillings $\varphi : \nu \to \mathcal{A}$ such that $\mathrm{st}(\varphi) = \xi$, and weakly increasing sequences $a(1), a(2), \ldots, a(n = |\nu|) \in \mathcal{A}$, satisfying the two conditions above. In this correspondence, the entries of $\varphi$ are the same as the images of $a$. In the case that $\xi$ is a standard Young tableau, the correspondence is with semi standard super fillings. And in the case the output of $a$ is restricted to the alphabet $A$, the bijective correspondence holds with fillings when $\xi$ is any bijective filling, and with semi standard Young tableaux when $\xi$ is a standard Young tableau.

## 1.3 Tuples of Skew Partitions

This section introduces some notation on tuples of skew partitions and deals with some of the most important parameters and properties surrounding such tuples. The study of tuples of skew partitions will be essential later to relate LLT polynomials, quasisymmetric polynomials and Macdonald polynomials. An important note to readers is the fact that most of the parameters and properties studied in this section for super fillings of tuples of skew partitions, are not direct extensions of the parameters found in previous sections for super fillings of skew partitions. For instance, when restricted to tuples with just one component, the concept of "standardization" of fillings of tuples of skew partitions is not equivalent to the concept of standardization of fillings of skew partitions introduced in definition 1.2.17.

**Definition 1.3.1.** Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ be a tuple of skew partitions. A *filling* of $\boldsymbol{\nu}$ with values in an alphabet $A$, is simply a tuple $\boldsymbol{\varphi} = (\varphi^{(1)}, \dots, \varphi^{(k)})$ such that $\varphi^{(i)}$ is a filling of $\nu^{(i)}$ with values in $A$ for $i = 1, \dots, k$. This can be written simply as

$$\boldsymbol{\varphi} : \uplus\boldsymbol{\nu} \to A$$

The disjoint union symbol $\uplus$ is necessary to make clear that two fillings $\varphi^{(i)}, \varphi^{(j)}$ are allowed to return different values for the same cell $u \in \mathbb{N} \times \mathbb{N}$.

**Definition 1.3.2.** Let $\mathcal{A} = A \uplus A_-$ be a super alphabet. An $\mathcal{A}$-valued *super filling* of $\boldsymbol{\nu}$ is simply a filling $\boldsymbol{\varphi} : \uplus\boldsymbol{\nu} \to \mathcal{A}$.

If $\boldsymbol{\varphi} = (\varphi^{(1)}, \dots, \varphi^{(k)})$ is a filling (or a super filling) of a tuple of skew partitions $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$, then define the notation

$$\boldsymbol{\varphi}(u) = \varphi^{(i)}(u)$$

for $u \in \uplus\boldsymbol{\nu} = \uplus_{i=1}^{k} \nu^{(i)}$.

**Definition 1.3.3.** Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ be a tuple of skew partitions and $\boldsymbol{\varphi} : \boldsymbol{\nu} \to \mathcal{A} = A \uplus A_-$ a super filling of $\boldsymbol{\nu}$. The parameter $I_{\boldsymbol{\varphi}}(u, v)$ is defined as follows for $u, v \in \uplus\boldsymbol{\nu}$:

$$I_{\boldsymbol{\varphi}}(u,v) = \begin{cases} 1 \text{ if } \boldsymbol{\varphi}(u) > \boldsymbol{\varphi}(v) \text{ or } \boldsymbol{\varphi}(u) = \boldsymbol{\varphi}(v) \in A_- \\ 0 \text{ if } \boldsymbol{\varphi}(u) < \boldsymbol{\varphi}(v) \text{ or } \boldsymbol{\varphi}(u) = \boldsymbol{\varphi}(v) \in A \end{cases}$$

**Definition 1.3.4.** Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ be a tuple of skew partitions and $\boldsymbol{\varphi} : \boldsymbol{\nu} \to \mathcal{A}$ a super filling of $\boldsymbol{\nu}$. A pair of cells $(u, v) \in \nu^{(i)} \times \nu^{(j)}$ such that $I_{\boldsymbol{\varphi}}(u, v) = 1$, is said to be an *inversion* of $\boldsymbol{\varphi}$ if either

$$i < j \text{ and } c(u) = c(v), \text{ or,}$$

$$i > j \text{ and } c(u) = c(v) + 1$$

where $c$ is the content function as in definition 1.1.11. This definition is not a direct extension of the concept of inversions of fillings and super fillings. The set of inversions of $\varphi$ is denoted $\mathrm{Inv}(\varphi)$ and the number of inversions is denoted $\mathrm{inv}(\varphi) = |\mathrm{Inv}(\varphi)|$.

**Definition 1.3.5.** Let $\mu = (\mu_1, \ldots, \mu_r)$ be a partition, and let $D$ be a subset of $\{(i,j) \in \mu : j > 0\}$. Define the tuple of ribbons

$$\mathbf{rbb}(\mu, D) = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\mu_1)})$$

as the only one satisfying $|\nu^{(i+1)}| = \mu'_{i+1}$ and $\mathrm{d}(\nu^{(i)}) = \{j+1 : (i,j) \in D\}$ for $i = 0, 1, \ldots, \mu_1 - 1$. This is clearly a bijective correspondence between pairs $(\mu, D)$ and tuples of ribbons of weakly decreasing sizes, which allows for the convenient notation

$$\mathbf{rbb}^{-1}(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\mu_1)}) = (\mu, D)$$

Figure 1.20 shows an example of this correspondence. Notice also that this gives a natural bijection between the cells of $\mu$ and those of $\mathbf{rbb}(\mu, D)$, by assigning to each cell $(i-1, j-1) \in \mu$, the cell of $\nu^{(i)}$ that has content $j$.

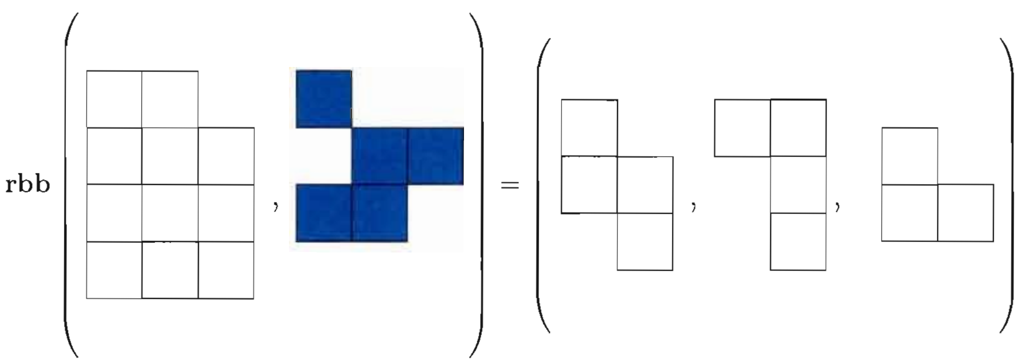



**Figure 1.20** $\mathbf{rbb}(\mu, D)$

The following proposition relates definitions 1.2.15 and 1.3.4.

**Proposition 1.3.1.** (Haglund, Haiman, Loehr, 2005) *Let $\mu = (\mu_1, \ldots, \mu_r)$ be a partition and $\varphi : \mu \to \mathcal{A}$ a super filling of $\mu$. Let $D$ be any subset of $\{(i,j) \in \mu : j > 0\}$ and set $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(\mu_1)}) = \mathbf{rbb}(\mu, D)$. Define $\varphi$ as the super filling of $\boldsymbol{\nu}$ given by:*

$$\varphi(u) = \varphi(i-1, j-1) \text{ for } u \in \nu^{(i)} \text{ with } c(u) = j$$

*Then* $|\mathrm{Inv}(\varphi)| = |\mathrm{Inv}(\varphi)|$.

*Proof.* Let $\nu_j^{(i)} \in \uplus\boldsymbol{\nu}$ denote the only cell of $\nu^{(i)}$ such that $c\left(\nu_j^{(i)}\right) = j$. Then

$$\varphi\left(\nu_j^{(i)}\right) = \varphi(i-1, j-1), \text{ and,}$$

$$I_\varphi\left(\nu_{j_1}^{(i_1)}, \nu_{j_2}^{(i_2)}\right) = I_\varphi((i_1-1, j_1-1), (i_2-1, j_2-1))$$

Thus the pair of cells $\left(\nu_{j_1}^{(i_1)}, \nu_{j_2}^{(i_2)}\right) \in \uplus\boldsymbol{\nu} \times \uplus\boldsymbol{\nu}$ is an inversion of $\varphi$ if and only if $I_\varphi((i_1 - 1, j_1-1), (i_2-1, j_2-1)) = 1$ and either

$$i_1 < i_2 \text{ and } j_1 = j_2, \text{ or,}$$

$$i_1 > i_2 \text{ and } j_1 = j_2 + 1$$

This is, if and only if the pair of cells $((i_1-1, j_1-1), (i_2-1, j_2-1)) \in \mu \times \mu$ is an inversion of $\mu$. □

**Definition 1.3.6.** Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ be a tuple of skew partitions. A super filling $\varphi : \uplus\boldsymbol{\nu} \to A \uplus A_-$ is said to be *semi standard*, if it is semi standard on each of the skew partitions $\nu^{(i)}$ $(1 \leq i \leq k)$.

**Definition 1.3.7.** For a tuple $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ of skew partitions, set

$$\mathrm{SSYT}(\boldsymbol{\nu}) = \mathrm{SSYT}(\nu^{(1)}) \times \cdots \times \mathrm{SSYT}(\nu^{(k)}), \text{ and,}$$

$$\mathrm{SSYT}_\pm(\boldsymbol{\nu}) = \mathrm{SSYT}_\pm(\nu^{(1)}) \times \cdots \times \mathrm{SSYT}_\pm(\nu^{(k)})$$

As expected, the elements of $\mathrm{SSYT}(\boldsymbol{\nu})$ are called *semi standard Young tableaux of shape* $\boldsymbol{\nu}$ and the elements of $\mathrm{SSYT}_\pm(\boldsymbol{\nu})$ are called *super tableaux of shape* $\boldsymbol{\nu}$.

**Definition 1.3.8.** Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ be a tuple of skew partitions. A *standard Young tableau* of shape $\boldsymbol{\nu}$ is a bijective semi standard Young tableau $\boldsymbol{\tau} : \uplus\boldsymbol{\nu} \to \{1, \dots, n\}$, where $n = |\boldsymbol{\nu}| = \sum_{i=1}^k |\nu_i|$. $\mathrm{SYT}(\boldsymbol{\nu})$ denotes the set of standard Young tableaux of shape $\boldsymbol{\nu}$.

Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ be a tuple of skew partitions. Define the *content reading order* $\leq_{\mathrm{cread}}$ as the one resulting of reading the cells of $\boldsymbol{\nu}$ in decreasing order of their contents. The cells of equal content are read from $\nu^{(1)}$ to $\nu^{(k)}$, and the cells of equal content in the same skew partition are read diagonally from bottom left to top right (see figure 1.21). In the case $\boldsymbol{\nu} = \mathbf{rbb}(\mu, D)$, this order corresponds to the reading order on cells of $\mu$, through the bijection between cells of $\mu$ and $\boldsymbol{\nu}$ mentioned at the end of definition 1.3.5.

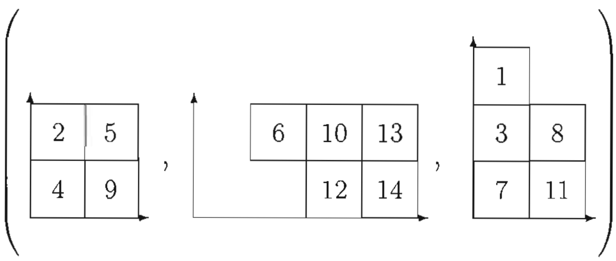



**Figure 1.21** Cells labeled in the content reading order on a tuple of skew partitions.

**Definition 1.3.9.** Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ be a tuple of skew partitions and $\boldsymbol{\varphi} : \uplus\boldsymbol{\nu} \to A \uplus A_-$ a super filling of $\boldsymbol{\nu}$. The *standardization* of $\boldsymbol{\nu}$, denoted $\mathbf{st}(\boldsymbol{\varphi})$ is the only bijective filling $\boldsymbol{\xi} : \uplus\boldsymbol{\nu} \to \{1, 2, \dots, |\boldsymbol{\nu}| = \sum_{l=1}^{k} \nu_l\}$ such that $\boldsymbol{\varphi} \circ \boldsymbol{\xi}^{-1}$ is weakly increasing and the restriction $\boldsymbol{\xi}|_{\boldsymbol{\varphi}^{-1}(i)} : \boldsymbol{\varphi}^{-1}(i) \to A \uplus A_-$ is increasing with respect to the content reading order for $i \in A$ and decreasing for $i \in A_-$.

**Remark.** When $\boldsymbol{\nu}$ has only one component, the previous definition is not the usual definition of standardization on fillings and super fillings. They only match when the super filling $\boldsymbol{\varphi}$ is semi standard, as stated in the next proposition.

**Proposition 1.3.2.** (Haglund, Haiman, Loehr, 2005) *Let $\nu$ be a skew partition and $\varphi : \nu \to A \uplus A_-$ a semi standard super filling of $\nu$. Let $\boldsymbol{\nu} = (\nu)$ be the "tuple" of skew partitions with only one component $\nu$, and let $\boldsymbol{\varphi} : \uplus\boldsymbol{\nu} \to A \uplus A_-$ be the super filling of $\boldsymbol{\nu}$ that corresponds to $\varphi$. Then the standardizations $\mathrm{st}(\varphi)$ and $\mathbf{st}(\boldsymbol{\varphi})$ (from definitions 1.2.17 and 1.3.9 respectively) are coincident on each entry of $\nu$.*

*Proof.* If $\varphi$ is semistandard, then the set $\varphi^{-1}(i)$ is a horizontal strip for $i \in A$ and a vertical strip for $i \in A_-$. In vertical and horizontal strips, the content reading order and the reading order are the same. The result follows from that observation. $\square$

**Proposition 1.3.3.** (Haglund, Haiman, Loehr, 2005) *If $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ is a tuple* *of skew partitions, and* $\varphi : \uplus\boldsymbol{\nu} \to A \uplus A_-$ *is a super filling of* $\boldsymbol{\nu}$*, then*

$$\mathrm{Inv}(\mathbf{st}(\varphi)) = \mathrm{Inv}(\varphi)$$

*Proof.* Recall that a pair $(u, v) \in \nu^{(i)} \times \nu^{(j)}$ of cells of $\boldsymbol{\nu}$ is an inversion of $\varphi$ if $I_\varphi(u, v) = 1$ and either

$$i < j \text{ and } c(u) = c(v), \text{ or,}$$

$$i > j \text{ and } c(u) = c(v) + 1$$

In both cases $u <_{\text{cread}} v$, so it is enough to prove that $I_\varphi(u, v) = I_{\mathbf{st}(\varphi)}(u, v)$ whenever $u <_{\text{cread}} v$. If $I_\varphi(u, v) = 1$, then either $\varphi(u) > \varphi(v)$ or $\varphi(u) = \varphi(v) \in A_-$. If $\varphi(u) > \varphi(v)$, then $\varphi(\boldsymbol{\xi}^{-1}(\boldsymbol{\xi}(u))) = \varphi(u) > \varphi(v) = \varphi(\boldsymbol{\xi}^{-1}(\boldsymbol{\xi}(v)))$ for $\boldsymbol{\xi} = \mathbf{st}(\varphi)$. Since $\varphi \circ \boldsymbol{\xi}^{-1}$ is weakly increasing, then $\boldsymbol{\xi}(u) > \boldsymbol{\xi}(v)$ and so $I_{\boldsymbol{\xi}}(u, v) = I_\varphi(u, v) = 1$. If $\varphi(u) = \varphi(v) \in A_-$, then $u, v \in \varphi^{-1}(i)$ for $i \in A_-$. Since $\boldsymbol{\xi}$ is decreasing in $\varphi^{-1}(i)$ with respect to the reading order, then $\boldsymbol{\xi}(u) > \boldsymbol{\xi}(v)$ and so $I_{\boldsymbol{\xi}}(u, v) = I_\varphi(u, v) = 1$ as wanted. On the other hand, if $I_\varphi(u, v) = 0$, then either $\varphi(u) < \varphi(v)$ or $\varphi(u) = \varphi(v) \in A$. If $\varphi(u) < \varphi(v)$, then $\varphi(\boldsymbol{\xi}^{-1}(\boldsymbol{\xi}(u))) = \varphi(u) < \varphi(v) = \varphi(\boldsymbol{\xi}^{-1}(\boldsymbol{\xi}(v)))$. Since $\varphi \circ \boldsymbol{\xi}^{-1}$ is weakly increasing, then $\boldsymbol{\xi}(u) < \boldsymbol{\xi}(v)$ and so $I_{\boldsymbol{\xi}}(u, v) = I_\varphi(u, v) = 0$. If $\varphi(u) = \varphi(v) \in A$, then $u, v \in \varphi^{-1}(i)$ for $i \in A$. Since $\boldsymbol{\xi}$ is increasing in $\varphi^{-1}(i)$ with respect to the reading order, then $\boldsymbol{\xi}(u) < \boldsymbol{\xi}(v)$ and so $I_{\boldsymbol{\xi}}(u, v) = I_\varphi(u, v) = 0$ as wanted. $\square$

**Proposition 1.3.4.** (Haglund, Haiman, Loehr, 2005) *Let* $\boldsymbol{\tau} : \uplus\boldsymbol{\nu} \to A \uplus A_-$ *be a super filling of the tuple of skew partitions* $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$*. Then* $\mathbf{st}(\boldsymbol{\tau})$ *is a standard Young tableau if and only if* $\boldsymbol{\tau}$ *is semi standard.*

*Proof.* Set $\boldsymbol{\xi} = \mathbf{st}(\boldsymbol{\tau})$. The fact that $\boldsymbol{\tau} \circ \boldsymbol{\xi}^{-1}$ is weakly increasing shows that $\boldsymbol{\tau}(u) < \boldsymbol{\tau}(v) \Rightarrow \boldsymbol{\xi}(u) < \boldsymbol{\xi}(v) \Rightarrow \boldsymbol{\tau}(u) \leq \boldsymbol{\tau}(v)$. Now let $\nu$ be a component of $\boldsymbol{\nu}$. The reading and content reading orders in a given column or row of $\nu$ are the same. Consequently, since $\boldsymbol{\xi}$ is increasing with respect to the content reading order in sets of the form $\boldsymbol{\tau}^{-1}(i) \cap \nu$ for $i \in A$ and decreasing for $i \in A_-$, the entries of $\boldsymbol{\xi}$ are increasing in columns upwards and in rows from left to right if and only if $\boldsymbol{\tau}^{-1}(i) \cap \nu$ is a horizontal strip for $i \in A$ and a vertical strip for $i \in A_-$. $\square$

**Definition 1.3.10.** Let $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$ be a tuple of skew partitions and $\boldsymbol{\xi} : \uplus\boldsymbol{\nu} \to \{1, 2, \ldots, |\boldsymbol{\nu}|\}$ a bijective filling. The *content reading descent set* of $\boldsymbol{\xi}$ is given by:

$$\mathrm{DC}(\boldsymbol{\xi}) = \{i \in \{1, 2, \ldots, |\boldsymbol{\nu}| - 1\} : \boldsymbol{\xi}^{-1}(i+1) <_{\text{cread}} \boldsymbol{\xi}^{-1}(i)\} \tag{1.13}$$

This definition can be specialized to skew partitions by setting $\boldsymbol{\nu}$ to have exactly one component $\nu$, and so the set $\mathrm{DC}(\xi)$ can be naturally defined for bijective fillings $\xi : \nu \to \{1, 2, \ldots, |\nu|\}$. An important observation is the fact that when $\xi$ is a standard Young tableau of shape $\nu$, the relations $\boldsymbol{\xi}^{-1}(i+1) <_{\text{cread}} \boldsymbol{\xi}^{-1}(i)$ and $\boldsymbol{\xi}^{-1}(i+1) <_{\text{read}} \boldsymbol{\xi}^{-1}(i)$ are equivalent. Indeed, if $\boldsymbol{\xi}^{-1}(i+1) <_{\text{cread}} \boldsymbol{\xi}^{-1}(i) <_{\text{read}} \boldsymbol{\xi}^{-1}(i+1)$ then clearly $\boldsymbol{\xi}^{-1}(i+1) < \boldsymbol{\xi}^{-1}(i)$ (see natural order), which is not allowed in standard Young tableaux. On the other hand, if $\boldsymbol{\xi}^{-1}(i+1) <_{\text{read}} \boldsymbol{\xi}^{-1}(i) <_{\text{cread}} \boldsymbol{\xi}^{-1}(i+1)$, then $\boldsymbol{\xi}^{-1}(i+1)$ must be strictly above and strictly to the right of $\boldsymbol{\xi}^{-1}(i)$, which would imply $i < \boldsymbol{\xi}(u) < i+1$ for the cell $u$ on the position directly above $\boldsymbol{\xi}^{-1}(i)$. These contradictions confirm the wanted equivalence, and prove the equality;

$$\mathrm{DC}(\xi) = \mathrm{DR}(\xi) \quad \text{for} \quad \xi \in \mathrm{SYT}(\nu) \tag{1.14}$$

**Proposition 1.3.5.** (Haglund, Haiman, Loehr, 2005) *Let* $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$ *be a tuple of skew partitions,* $\boldsymbol{\xi} : \uplus\boldsymbol{\nu} \to \{1, 2, \ldots, n = |\boldsymbol{\nu}|\}$ *a bijective filling of shape* $\boldsymbol{\nu}$*, and* $a : \{1, 2, \ldots, n\} \to \mathcal{A}$ *a weakly increasing function on a super alphabet* $\mathcal{A} = A \uplus A_-$*. Then* $\mathbf{st}(a \circ \boldsymbol{\xi}) = \boldsymbol{\xi}$ *if and only if*

$$a(i) = a(i+1) \in A \Rightarrow i \notin \mathrm{DC}(\boldsymbol{\xi})$$

*and*

$$a(i) = a(i+1) \in A_- \Rightarrow i \in \mathrm{DC}(\boldsymbol{\xi})$$

*Proof.* The proof is exactly the same as the proof of proposition 1.2.5, using the content reading order instead of the reading order. $\square$

**Remark.** The previous proposition shows that for a given bijective filling $\boldsymbol{\xi} : \uplus\boldsymbol{\nu} \to \{1, 2, \ldots, n = |\boldsymbol{\nu}|\}$, there is a bijective correspondence between super fillings $\boldsymbol{\varphi} : \uplus\boldsymbol{\nu} \to \mathcal{A}$ such that $\mathbf{st}(\boldsymbol{\varphi}) = \boldsymbol{\xi}$, and weakly increasing sequences $a(1), a(2), \ldots, a(n = |\boldsymbol{\nu}|) \in \mathcal{A}$, satisfying the two conditions above. In this correspondence, the entries of $\boldsymbol{\varphi}$ are the same as the images of $a$. In the case that $\boldsymbol{\xi}$ is a standard Young tableau, the correspondence is with semi standard super fillings. And in the case the output of $a$ is restricted to the alphabet $A$, the bijective correspondence holds with regular fillings when $\boldsymbol{\xi}$ is any bijective filling, and with semi standard Young tableaux when $\boldsymbol{\xi}$ is a standard Young tableau.

## 1.4 RSK Algorithm

Given two totally ordered alphabets $A$ and $B$ (generally subsets of $\mathbb{Z}_+$). A *lexicographic word*, sometimes called *generalized permutation* or simply *two-line array* is a matrix of the form

$$\begin{pmatrix} \mathrm{a} \\ \mathrm{b} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

with each $a_i$ in $A$ and each $b_i$ in $B$, such that $(a_i, b_i) \leq_{\text{lex}} (a_{i+1}, b_{i+1})$ for $i = 1, \ldots, n-1$. This is

$$a_i < a_{i+1}$$

or

$$a_i = a_{i+1} \text{ and } b_i \leq b_{i+1}$$

for $i = 1, \ldots, n-1$. The *RSK algorithm* gives an bijective correspondence between lexicographic words $\begin{pmatrix} \mathrm{a} \\ \mathrm{b} \end{pmatrix}$ and pairs of semi standard fillings $(\tau, \rho)$ of the same shape, such that $\tau$ has entries in $B$ and $\rho$ has entries in $A$. In this bijection, the entries of $\tau$ are $b_1, \ldots, b_n$ and the entries of $\rho$ are $a_1, \ldots, a_n$. In particular, when $a_i = i$ for $i = 1, \ldots, n$ and $b_i = \sigma(i)$ for a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, the RSK algorithm gives a correspondence between pairs of standard Young tableaux of the same shape $\mu \vdash n$ and permutations of $\{1, \ldots, n\}$. A result of this particular case is the formula

$$\sum_{\mu \vdash n} (f^\mu)^2 = n! \tag{1.15}$$

Before introducing the RSK algorithm, define the *row insertion* $(\tau \longleftarrow b)$ for a semi standard filling $\tau : \mu \to B$ and $b \in B$, as follows:

1. Start with $i = 0$ and $b^{(0)} = b$.
2. Let $j$ be the smallest nonnegative integer for which $\tau(i,j) > b^{(i)}$.
   (a) If $j$ does not exist then place $b^{(i)}$ immediately to the right of the $(i+1)$th row of $\mu$ (i.e., set $\tau(i, \mu_i) := b^{(i)}$), return $\tau$ and stop.
   (b) If $j$ does exist, set $b^{(i+1)} := \tau(i,j)$, $\tau(i,j) := b^{(i)}$, $i := i+1$ (in that order) and go to step 2.

It is clear that the resulting filling $\tau'$ is still a semi standard filling, and the corresponding diagram $(\tau')^{-1}(B)$ is still a partition. Also the difference between the new partition $(\tau')^{-1}(B)$ and $\mu$ (i.e., the diagram shape$(\tau')\backslash$shape$(\tau)$) consists of only one cell. Given a lexicographic word

$$\binom{\mathbf{a}}{\mathbf{b}} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

The RSK algorithm works as follows:

1. Start with $\tau$ and $\rho$ being empty fillings.
2. Do the following steps for $i = 1, \ldots, n$.
   (a) Define $\tau' := (\tau \longleftarrow b_i)$.
   (b) Set $\rho(\text{shape}(\tau')\backslash\text{shape}(\tau)) := a_i$
   (c) Set $\tau := \tau'$.

This produces two semi standard fillings $\tau$, $\rho$, and the resulting correspondence between lexicographic words and pairs of semi standard fillings is bijective.

The *Dual RSK algorithm*, or simply *RSK* algorithm*, is similar to the RSK algorithm, except that it uses a different *row insertion* $\tau \overset{*}{\leftarrow} b$. This row insertion is given by the following steps:

1. Start with $i = 0$ and $b^{(0)} = b$.

2. Let $j$ be the smallest nonnegative integer for which $\tau(i, j) \geq b^{(i)}$ (notice the symbol $\geq$ instead of $>$).

   (a) If $j$ does not exist then place $b^{(i)}$ immediately to the right of the $(i + 1)$th row of $\mu$ (i.e., set $\tau(i, \mu_i) := b^{(i)}$), return $\tau$ and stop.

   (b) If $j$ does exist, set $b^{(i+1)} := \tau(i,j)$, $\tau(i,j) := b^{(i)}$, $i := i + 1$ (in that order) and go to step 2.

The RSK* algorithm provides a bijection between lexicographic words $\binom{\mathbf{a}}{\mathbf{b}}$ with no repetitions (this is, $(a_i, b_i) \neq (a_{i+1}, b_{i+1})$ for $i = 1, \ldots, n$) and pairs of fillings $\tau, \rho$ of the same shape with entries in $B$ and $A$ respectively, such that $\tau^t$ (the traspose of $\tau$) and $\rho$ are both semi standard. In this bijection, as with the RSK algorithm, $\tau$ has entries $b_1, b_2, \ldots$ and $\rho$ has entries $a_1, a_2, \ldots$ respectively. Refer to (Stanley, 1999) for a deeper insight on both algorithms.

## 1.5 Compositions

Before going ahead with the definition of symmetric polynomials, it is necessary to introduce the concept of *compositions* of positive integers.

**Definition 1.5.1.** Let $n$ be a nonnegative integer. A *composition* $\mathbf{c}$ of $n$ is a (possibly empty) finite tuple $(c_1, \cdots, c_k)$ of nonnegative integers satisfying

$$c_1 + \cdots + c_k = n$$

For instance $(0,1,5,0,0,1,0,0,0)$ is a composition of 7. As with partitions, the number $n$ is called the *size* of $\mathbf{c}$, denoted $|\mathbf{c}|$, and the integers $c_1, \ldots, c_k$ are called the *parts* of $\mathbf{c}$. The *length* of $\mathbf{c}$ is the number of parts of $\mathbf{c}$ and is denoted $l(\mathbf{c})$. The composition of $n = 0$ that has no parts is called the *empty composition.* Many of the references define compositions as those with only positive parts. However, this definition is more convenient for this work. In what follows, a composition with only positive parts will be called a *positive composition.*

Although the number of compositions of a positive integer $n$ is infinite, the number of positive compositions is clearly finite. For example, the 16 compositions of 5 with only positive parts are given by the sums $1+1+1+1+1 = 1+1+1+2 = 1+1+2+1$ $= 1+2+1+1 = 2+1+1+1 = 1+1+3 = 1+3+1 = 3+1+1 = 1+2+2 = 2+1+2$ $= 2+2+1 = 1+4 = 4+1 = 2+3 = 3+2 = 5$.

Decreasingly ordering the parts of a composition $\mathbf{c}$ of $n$, produces a partition of $n$. This partition is denoted $\overset{\frown}{\mathbf{c}}$. For example, if $\mathbf{c} = (0,1,5,0,0,1,0,0,0)$, then $\overset{\frown}{\mathbf{c}} = (5,1,1) \vdash 7$. The following are a few observations involving compositions of integers:

1. There is a natural bijective correspondence between compositions of $n$ with $k$ parts and ordered sums of $(k-1)$ 0's and $n$ 1's. In one such sum, a 0 indicates the begining of a new part. For example the composition $(3,0,1,5,0,0,1,0,0,0)$ is associated with the sum $1+1+1+0+0+1+0+1+1+1+1+1+0+0+0+1+0+0+0$. As a result, the number of compositions of $n$ with $k$ parts is equal to the binomial coefficient $\binom{n+k-1}{n}$.

2. The generating function of the number of positive compositions of $n > 0$ is given by
$$\sum_{k\geq 1}(x+x^2+x^3+\cdots)^k = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{x}{1-2x} = \sum_{n\geq 1} 2^{n-1}x^n$$

3. Compositions of $n$ with $k$ positive parts, are in bijective correspondence with subsets of $k-1$ elements of $\{1,\ldots,n-1\}$. Indeed, for a composition $\mathbf{c} = (c_1, \cdots, c_k)$

of $n$ with positive parts, define

$$\text{set}(\mathbf{c}) = \{c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{k-1}\} \tag{1.16}$$

Conversely, for a subset $S = \{i_1, \ldots, i_{k-1}\} \subseteq \{1, \ldots, n-1\}$ such that $i_1 < \cdots < i_{k-1}$, one can define

$$\text{co}(S) = (i_1, i_2 - i_1, \ldots, i_{k-1} - i_{k-2}, n - i_{k-1}) \tag{1.17}$$

Notice that the number $n$ must be implicit for $c(S)$ to make sense. For example, for $n = 6$; $\text{set}(2,3,1) = \{2,5\}$, $\text{set}(1,1,4) = \{1,2\}$, $\text{co}(\{1,5\}) = (1,4,1)$ and $\text{co}(\{2,3\}) = (2,1,3)$. This shows that there are exactly $\binom{n-1}{k-1}$ positive compositions of $n$ with $k$ parts, and gives another proof of the generating function above.

4. Let $\lambda = (r^{m_r}, \ldots, 1^{m_1})$ be the partition of size $n$ with $m_i$ parts $i$ for $i = 1, \ldots, r > 0$ and zero parts $i$ for $i > r$ (where $n = 1m_1 + \cdots + rm_r$). Then the number of positive compositions $\mathbf{c}$ such that $\overset{\frown}{\mathbf{c}} = \lambda$, is equal to the multinomial coefficient:
$$\binom{l(\lambda) = \sum_{i=1}^{r} m_i}{m_1, \ldots, m_r} = \frac{l(\lambda)!}{m_1! \cdots m_r!} \tag{1.18}$$

5. In general, the number of compositions $\mathbf{c}$ with $l(\mathbf{c}) = k$ and $\overset{\frown}{\mathbf{c}} = \lambda$, is equal to the multinomial coefficient
$$\binom{k}{m_1, \ldots, m_r} = \frac{k!}{m_1! \cdots m_r! (k - l(\lambda))!} \tag{1.19}$$

**Definition 1.5.2.** Define the *refinement* order for compositions as follows: $(c_1, \ldots, c_k) \leq_{\text{ref}} (d_1, \ldots, d_l)$ if and only if there are integers $0 = i_0 < i_1 < \cdots < i_k = l$ such that

$$c_j = \sum_{i_{j-1} < i \leq i_j} d_i$$

for $j = 1, \ldots, k$.

Notice that $\mathbf{c} \leq_{\text{ref}} \mathbf{d} \Leftrightarrow \text{set}(\mathbf{c}) \subseteq \text{set}(\mathbf{d})$ for $|\mathbf{c}| = |\mathbf{d}|$. This observation will be useful to understand the relation between the two possible ways of indexing quasisymmetric polynomials.

# CHAPTER II

# SYMMETRIC AND QUASISYMMETRIC POLYNOMIALS

This chapter consists of a short exposition of some of the most important concepts in the theory of symmetric polynomials. The main goal here is to introduce the necessary ideas to be able to define Macdonald polynomials and outline the proof of its combinatorial formula. The first section defines the classic families of symmetric functions and studies some relations among them. The second section studies the Schur symmetric polynomials, which play an important role in this work. The third section introduces the concept of plethistic substitution, which will be necessary to understand the renormalization that defines Macdonald polynomials. Lastly, the fourth and fifth sections give a short introduction to quasisymmetric polynomials and LLT symmetric polynomials, both of which are essential to be able to interpret Macdonald polynomials in a combinatorial way.

For the definitions of the next section it is necessary to understand some basic notation on monomials. In what follows the bold letter $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ will denote the infinite vectors of commuting variables $(x_1, x_2, \ldots)$, $(y_1, y_2, \ldots)$ and $(z_1, z_2, \ldots)$ respectively, unless specified otherwise.

Given any function $\varphi : S \to \mathbb{Z}_+$ from any set $S$ onto the set of positive integers ($\varphi$ could be a filling for instance), define:

$$\mathbf{x}^{\varphi} = \prod_{u \in S} x_{\varphi(u)}$$

Also, for a finite vector $(x_1, \ldots, x_m)$, define the monomial

$$(x_1, \ldots, x_m)^\varphi = \mathbf{x}^\varphi|_{x_{m+1}=x_{m+2}=x_{m+3}=\cdots=0}$$

For example, if $\varphi$ is the filling in figure 1.11, then $\mathbf{x}^\varphi = (x_1, \ldots, x_m)^\varphi = x_2x_{10}x_{18}x_{19}^2x_{20}x_{57}$ for $m \geq 57$ and $(x_1, \ldots, x_m)^\varphi = 0$ for $m < 57$.

Another important notation is $\mathbf{x}^\mathbf{c}$ for a composition $\mathbf{c} = (c_1, \ldots, c_k)$, defined by

$$\mathbf{x}^\mathbf{c} = x_1^{c_1} \cdots x_k^{c_k}$$

And as above:

$$(x_1, \ldots, x_m)^\mathbf{c} = \mathbf{x}^\mathbf{c}|_{x_{m+1}=x_{m+2}=x_{m+3}=\cdots=0}$$

This can be extended to partitions by simply seeing them as weakly decreasing compositions.

The *degree* of a monomial $\mathbf{x}^\mathbf{c}$ is the size of the composition $\mathbf{c}$. For instance, the monomial $x_1^3x_3^4$ has degree 7. This and future chapters are concerned with functions of the form

$$f(\mathbf{x}) = \sum_{|\mathbf{c}| \leq d} a_\mathbf{c} x^\mathbf{c}$$

where the indices $\mathbf{c}$ are compositions, and the coefficients $a_\mathbf{c}$ are elements of a field. Assume this field to be $\mathbb{Q}$ unless stated otherwise. The term *polynomial* is used for these functions, even though the number of terms is allowed to (and will often) be infinite. The ring of polynomials in the variables $x_1, x_2, \ldots$ is denoted $\mathbb{Q}[\mathbf{x}]$. The *degree* of $f$ is the maximum value of $|\mathbf{c}|$ for which $a_\mathbf{c} \neq 0$. A polynomial is said to be homogeneous of degree $d$ if it is of the form

$$f(\mathbf{x}) = \sum_{|\mathbf{c}| = d} a_\mathbf{c} x^\mathbf{c}$$

with at least one nonzero $a_\mathbf{c}$. Every polynomial is clearly a finite sum of homogeneous polynomials.

If a polynomial $f$ is defined in a infinite number of variables $x_1, x_2, \ldots$, it can be naturally defined for a finite set of variables $x_1, \ldots, x_m$ by setting $x_{m+1} = x_{m+2} = \cdots = 0$, so it is a common practice to omit the $\mathbf{x}$ and simply write $f$.

## 2.1 Classic Definitions and Identities

This section is concerned with a special class of polynomials that are invariant with respect to transpositions of variables. More formally:

**Definition 2.1.1.** Let $\mathfrak{S}$ be the symmetric group on $\mathbb{Z}_+$ (i.e., the group of permutations $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$) and define an action $\mathfrak{S} \times \mathbb{Q}[\mathbf{x}] \to \mathbb{Q}[\mathbf{x}]$ by linearly extending the relations

$$\sigma x_1^{c_1} \cdots x_k^{c_k} = x_{\sigma(1)}^{c_1} \cdots x_{\sigma(k)}^{c_k} \tag{2.1}$$

A polynomial $f \in \mathbb{Q}[\mathbf{x}]$ is said to be a *symmetric polynomial* (or a *symmetric function*) if

$$\sigma f = f$$

for all $\sigma \in \mathfrak{S}$.

The product and the sum of two symmetric polynomials is clearly a symmetric polynomial. Also symmetric polynomials are still symmetric (with respect to the group $\mathfrak{S}_n$ of permutations $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$) when restricted to a finite set of variables $x_1, \ldots, x_n$. Another important observation is the fact that if

$$f(\mathbf{x}) = \sum_{\mathbf{c}} a_{\mathbf{c}} \mathbf{x}^{\mathbf{c}}$$

is symmetric, then

$$\overset{\frown}{\mathbf{c}_1} = \overset{\frown}{\mathbf{c}_2} \Rightarrow a_{\mathbf{c}_1} = a_{\mathbf{c}_2}$$

Thus any symmetric polynomial can be indexed by partitions as follows:

$$f(\mathbf{x}) = \sum_{\lambda} a_\lambda \sum_{\overset{\frown}{\mathbf{c}} = \lambda} x^{\mathbf{c}}$$

And any homogeneous symmetric polynomial of degree $n$ can be written as:

$$f(\mathbf{x}) = \sum_{\lambda \vdash n} a_\lambda \sum_{\overset{\frown}{\mathbf{c}} = \lambda} x^{\mathbf{c}}$$

Hence for $\lambda \vdash n$, the polynomials $\sum_{\overset{\frown}{\mathbf{c}} = \lambda} x^{\mathbf{c}}$ are a basis of the space of homogeneous symmetric polynomials of degree $n$.

**Definition 2.1.2.** Let $\lambda$ be a partition. The *monomial symmetric polynomial* indexed by $\lambda$ is given by

$$m_\lambda(\mathbf{x}) = \sum_{\widehat{\mathbf{c}}=\lambda} \mathbf{x}^{\mathbf{c}} \tag{2.2}$$

For instance;

$$m_{(5,2)}(x_1, x_2, x_3) = x_1^5 x_2^2 + x_1^5 x_3^2 + x_2^5 x_1^2 + x_2^5 x_3^2 + x_3^5 x_1^2 + x_3^5 x_2^2$$

Any symmetric polynomial can be expressed as a linear combinations of monomial symmetric polynomials. Furthermore, a symmetric polynomial has only positive integer coefficients if an only if this linear combination has also positive integer coefficients. Other important families of symmetric functions are the *complete homogeneous*, *power sum* and *elementary* symmetric functions. Unlike monomial symmetric functions, these are indexed by positive integers.

**Definition 2.1.3.** Let $k$ be a positive integer. The *complete homogeneous*, *power sum* and *elementary* symmetric functions indexed by $k$, are given in terms of monomial symmetric functions, by the formulas:

$$h_k = \sum_{\lambda \vdash k} m_\lambda \tag{2.3}$$

$$p_k = m_{(k)} = x_1^k + x_2^k + \cdots \tag{2.4}$$

$$e_k = m_{(1^k)} = \sum_{0<i_1<\cdots<i_k} x_{i_1} \cdots x_{i_k} \tag{2.5}$$

where $(1^k) = (\underbrace{1, \ldots, 1}_{k\,\text{times}})$. The polynomials $h_k$, $p_k$ and $e_k$ are all homogeneous of degree $k$. Define a more general form of these symmetric functions, indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$, as follows

$$h_\lambda = \prod_{i=1}^{r} h_{\lambda_i} \tag{2.6}$$

$$p_\lambda = \prod_{i=1}^{r} p_{\lambda_i} \tag{2.7}$$

$$e_\lambda = \prod_{i=1}^{r} e_{\lambda_i} \tag{2.8}$$

In order to find relations between these families of symmetric polynomials, it is convenient to consider their generating functions. For this define:

$$H(t) = \sum_{k\geq 0} h_k(\mathbf{x})t^k \tag{2.9}$$

$$P(t) = \sum_{k\geq 1} \frac{p_k(\mathbf{x})}{k}t^k \tag{2.10}$$

$$E(t) = \sum_{k\geq 0} e_k(\mathbf{x})t^k \tag{2.11}$$

A simple calculation yields

$$H(t) = \prod_{i\geq 1}(1 + x_it + x_i^2t^2 + \cdots) = \prod_{i\geq 1}\frac{1}{1-x_it} \tag{2.12}$$

$$P(t) = \sum_{i\geq 1}\left(x_it + \frac{x_i^2t^2}{2} + \frac{x_i^3t^3}{3} + \cdots\right) = \sum_{i\geq 1}\log\left(\frac{1}{1-x_it}\right)$$

$$= \log\left(\prod_{i\geq 1}\frac{1}{1-x_it}\right) = \log(H(t)) \tag{2.13}$$

$$E(t) = \prod_{i\geq 1}(1 + x_it) = \frac{1}{H(-t)} \tag{2.14}$$

From 2.14, $E(-t)H(t) = 1$, and so

$$\sum_{k=0}^{n}(-1)^k e_k h_{n-k} = 0 \text{ for } n > 0 \tag{2.15}$$

On the other hand, from 2.13;

$$H(t) = e^{P(t)} = \sum_{r\geq 0}\frac{P(t)^r}{r!} = \sum_{k\geq 0}\sum_{c_1+\cdots+c_r=k}\frac{p_{c_1}(\mathbf{x})\cdots p_{c_r}(\mathbf{x})}{c_1\cdots c_r r!}t^k$$

where the numbers $c_1, \ldots, c_r$ are positive integers. Indexing by partitions, one gets:

$$H(t) = \sum_{k\geq 0}\sum_{\lambda=(j^{m_j},\ldots,2^{m_2},1^{m_1})\vdash k}\binom{l(\lambda)}{m_1,\ldots,m_j}\frac{p_\lambda(\mathbf{x})}{1^{m_1}\cdots j^{m_j}l(\lambda)!}t^k$$

where $i^{m_i}$ in the sum index is just another way of writing $\underbrace{i,\cdots,i}_{m_i \text{ times}}$. And so

$$H(t) = \sum_{k\geq 0}\sum_{\lambda\vdash k}\frac{p_\lambda(\mathbf{x})}{z_\lambda}t^k$$

This is:

$$h_k = \sum_{\lambda \vdash k} \frac{p_\lambda}{z_\lambda} \tag{2.16}$$

where

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots j^{m_j} m_j! \tag{2.17}$$

for any partition $\lambda = (j^{m_j}, \dots, 2^{m_2}, 1^{m_1})$. A similar manipulation using that $E(t) = H(-t)^{-1} = e^{-P(-t)}$, produces:

$$e_k = \sum_{\lambda \vdash k} (-1)^{k-l(\lambda)} \frac{p_\lambda}{z_\lambda} \tag{2.18}$$

The generating function relations 2.12, 2.13 and 2.14 show that any homogeneous, power sum or elementary symmetric polynomial can be expressed as a linear combination of any of the other two families. Thus those three families generate the same space. The *fundamental theorem of symmetric functions* states that they actually generate the whole space of symmetric polynomials:

**Theorem 2.1.1** (Fundamental theorem of symmetric functions). *Every symmetric polynomial can be written uniquely as a polynomial on the elementary symmetric functions* $e_k$, $k = 1, 2, 3, \dots$

Refer to (Stanley, 1999) for a proof of this theorem. More useful relations between symmetric polynomials $m_\lambda$, $h_\lambda$, $p_\lambda$ and $e_\lambda$ can be found in (Bergeron, 2008; Macdonald, 1988; Stanley, 1999).

Define the multiplicative and linear involution $\omega$ on symmetric polynomials by the relation:

$$\omega(p_k) = (-1)^{k-1} p_k \tag{2.19}$$

It is immediate from this that

$$\omega(p_\lambda) = (-1)^{|\lambda| - l(\lambda)} p_\lambda \tag{2.20}$$

for any partition $\lambda$. Also from equations 2.16 and 2.18, it is clear that

$$\omega(h_k) = e_k \tag{2.21}$$

and

$$\omega(e_k) = h_k \tag{2.22}$$

The operator $\omega$ is uniquely defined by any of the four relations above.

## 2.2 Schur Polynomials

The family of Schur polynomials is one of the most important known basis of the space of symmetric polynomials. It plays an essential role in several areas outside combinatorics such as representation theory. Though such applications are closely related to the study of Macdonald polynomials, they will not be presented here in order to maintain a purely combinatorial approach. Refer to (Bergeron, 2008; Haglund, 2008; Macdonald, 1995; Sagan 2001) for some of its applications to this area.

**Definition 2.2.1.** Let $\nu$ be a skew partition. The *Schur polynomial* indexed by $\nu$ is defined by the equality:

$$s_\nu(\mathbf{x}) = \sum_{\tau \in \mathrm{SSYT}(\nu)} \mathbf{x}^\tau \tag{2.23}$$

The symmetry of Schur polynomials is not evident from this definition. To prove that $s_\nu(\mathbf{x})$ is symmetric, it suffices to show that it is invariant upon switching two variables $x_m, x_{m+1}$. For this it is enough to construct an involution $t : \mathrm{SSYT}(\nu) \to \mathrm{SSYT}(\nu)$ that switches the number of $m$'s with the number of $(m+1)$'s on each semi standard Young tableau $\tau : \nu \to \mathbb{Z}_+$.

Let $\alpha_m \subseteq \tau^{-1}(m)$ be the set of all the cells $(i,j) \in \nu$ such that $\tau(i,j) = m$ and $\tau(i,j+1) = m+1$. Let $\alpha_{m+1} \subseteq \tau^{-1}(m+1)$ be the set of all the cells $(i,j) \in \nu$ such that $\tau(i,j) = m+1$ and $\tau(i,j-1) = m$. Since $\tau$ is strictly increasing on columns, it is clear that $|\alpha_m| = |\alpha_{m+1}|$.

If $(i,j) \in \alpha_m$ and $\tau(i-1,j) = m$, then $m = \tau(i-1,j) < \tau(i-1,j+1) \leq \tau(i,j+1) = m+1$, so $\tau(i-1,j+1) = m+1$ and $(i-1,j) \in \alpha_m$. On the other hand, if $(i,j) \in \alpha_{m+1}$ and $\tau(i+1,j) = m+1$, then $m+1 = \tau(i+1,j) > \tau(i+1,j-1) \geq \tau(i,j-1) = m$, so $\tau(i+1,j-1) = m$ and $(i+1,j) \in \alpha_{m+1}$.

As a result, if $\beta \subseteq \boldsymbol{\nu}$ is any row of $\boldsymbol{\nu}$, then the set $\tau^{-1}(\{m, m+1\})\backslash(\alpha_m \cup \alpha_{m+1})$ is a sequence of consecutive cells in $\beta$. If $u_1 <_{\text{read}} \cdots <_{\text{read}} u_a <_{\text{read}} u_{a+1} <_{\text{read}} \cdots <_{\text{read}} u_{a+b}$ are these consecutive cells with $\tau(u_1) = \cdots = \tau(u_a) = m$ and $\tau(u_{a+1}) = \cdots = \tau(u_{a+b}) = m+1$, then change these values to $\tau(u_1) = \cdots = \tau(u_b) = m$ and $\tau(u_{b+1}) = \cdots = \tau(u_{b+a}) = m+1$. This transformation is clearly an involution when applied on every row of $\nu$, and it switches the number of $m$'s with the number of $(m+1)$'s appearing in the entries of $\tau$. Also the new filling is still a semi standard Young tableau. This proves the symmetry of Schur polynomials.

An interesting result of this symmetry is the following: If $\mathbf{c}^{(1)} = (c_1^{(1)}, \ldots, c_k^{(1)})$ and $\mathbf{c}^{(2)} = (c_1^{(2)}, \ldots, c_k^{(2)})$ are two positive compositions with $\overset{\curvearrowright}{\mathbf{c}^{(1)}} = \overset{\curvearrowright}{\mathbf{c}^{(2)}}$, and $\nu$ is a skew partition with $|\nu| = |\mathbf{c}^{(1)}| = |\mathbf{c}^{(2)}|$, then the number of semi standard Young tableaux of shape $\nu$ and exactly $c_i^{(1)}$ entries equal to $i$ $(1 \leq i \leq k)$ is the same as the number of semi standard Young tableaux of shape $\nu$ and exactly $c_i^{(2)}$ entries equal to $i$ $(1 \leq i \leq k)$. In particular, if $\mathbf{c} = (c_1, \ldots, c_r)$ is a composition with $\mu = \overset{\curvearrowright}{\mathbf{c}}$, then:

$$K_{\lambda \mathbf{c}} = K_{\lambda \mu} \tag{2.24}$$

where

$$K_{\lambda \mathbf{c}} = |\{\tau \in \mathrm{SSYT}(\lambda) : |\tau^{-1}(i)| = c_i \text{ for } i = 1, \ldots, r\}|$$

This observation is essential in the proof of the following result:

$$\mu \preceq \lambda \Rightarrow K_{\lambda \mu} > 0 \tag{2.25}$$

*Proof of (2.25).* Suppose that $|\mu| = |\lambda| > 0$ and $\mu = (\mu_1, \ldots, \mu_r) \preceq \lambda = (\lambda_1, \ldots, \lambda_k)$. Proceed by induction on $\lambda_1$: If $\lambda_1 = \cdots = \lambda_k = 1$ then clearly $r = k$ and $\mu_1 = \cdots = \mu_k$. In this case define $\tau : \lambda \to \mathbb{Z}_+$ by $\tau(0, j) = j + 1$ for $j = 1, \ldots, k-1$, which is evidently a semi standard Young tableau and satisfies $|\tau^{-1}(i)| = 1 = \mu_i$ for $i = 1, \ldots, r$. Thus $K_{\lambda\mu} > 0$. Now consider the case $\lambda_1 > 1$. Define $\lambda^{(1)} = (\lambda_1 - 1, \ldots, \lambda_k - 1)$ and $\mu^{(1)} = (\mu_1^{(1)}, \ldots, \mu_r^{(1)}) = (\mu_1 - 1, \ldots, \mu_k - 1, \mu_{k+1}, \ldots, \mu_r)$ (recall that $\mu \preceq \lambda \Rightarrow r \leq k$). The following two properties can be easily verified:

- $\left|\mu^{(1)}\right| = \left|\lambda^{(1)}\right|$.
- $\overset{\curvearrowright}{\mu^{(1)}} \preceq \lambda^{(1)}$.

As a result, from the inductive hypothesis:

$$K_{\lambda^{(1)}\mu^{(1)}} = K_{\lambda^{(1)}\overset{\curvearrowright}{\mu^{(1)}}} > 0$$

Thus there is a semi standard Young tableau $\tau^{(1)} : \lambda^{(1)} \longrightarrow \mathbb{Z}_+$ satisfying

$$\left|\left(\tau^{(1)}\right)^{-1}(i)\right| = \mu_i^{(1)} = \begin{cases} \mu_i - 1 & \text{for } i = 1, \ldots, k \\ \mu_i & \text{for } i = k+1, \ldots, r \end{cases}$$

A filling $\tau : \lambda \longrightarrow \mathbb{Z}_+$ is then defined in terms of $\tau^{(1)}$ as follows:

$$\tau(i,j) = \begin{cases} j+1 & \text{if } i = 0 \\ \tau^{(1)}(i-1,j) & \text{if } i \geq 1 \end{cases}$$

In other words, $\tau$ is constructed by placing numbers $1, \ldots, k$ on the first column of $\lambda$ and using a copy of $\tau^{(1)}$ to fill up the rest of the cells. The filling $\tau$ satisfies:

$$\begin{aligned} \left|\tau^{-1}(i)\right| &= \begin{cases} \left|\left(\tau^{(1)}\right)^{-1}(i)\right| + 1 & \text{for } i = 1, \ldots, k \\ \left|\left(\tau^{(1)}\right)^{-1}(i)\right| & \text{for } i = k+1, \ldots, r \end{cases} \\ &= \begin{cases} \mu_i - 1 + 1 & \text{for } i = 1, \ldots, k \\ \mu_i & \text{for } i = k+1, \ldots, r \end{cases} \\ &= \mu_i \text{ for } i = 1, \ldots, r \end{aligned}$$

and

$$\tau(1,j) = \tau^{(1)}(0,j) \geq j + \tau^{(1)}(0,0) \geq j + 1 = \tau(0,j)$$

Therefore $\tau$ is a semi standard Young tableau and $K_{\lambda\mu} > 0$.

□

Some important properties of Schur polynomials are the following:

$$s_{(1^k)} = e_k \tag{2.26}$$

$$s_{(k)} = h_k \tag{2.27}$$

$$s_\lambda = \sum_{\mu \preceq \lambda} K_{\lambda,\mu} m_\mu \text{ for any partition } \lambda \tag{2.28}$$

$$\prod_{i,j \in \mathbb{Z}_+} \frac{1}{1 - x_i y_j} = \sum_{\lambda \text{ partition}} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) \tag{2.29}$$

$$\prod_{i,j \in \mathbb{Z}_+} (1 + x_i y_j) = \sum_{\lambda \text{ partition}} s_\lambda(\mathbf{x}) s_{\lambda'}(\mathbf{y}) \tag{2.30}$$

$$\omega(s_\lambda) = s_{\lambda'} \text{ for any partition } \lambda \tag{2.31}$$

Equations 2.26, 2.27 and 2.28 are immediate from the definitions above. Equations 2.29 and 2.30 are known respectively as *Cauchy identity* and *dual Cauchy identity.* They are direct results of the application of the bijective correspondences given by the RSK and RSK* algorithm respectively. Equation 2.31 results from the following observations:

$$\prod_{i,j \in \mathbb{Z}_+} (1 + x_i y_j) = \prod_{i \geq 1} \prod_{j \geq 1} (1 + x_i y_j) = \prod_{i \geq 1} \sum_{k \geq 0} x_i^k e_k(\mathbf{y}) = \sum_{\lambda \text{ partition}} m_\lambda(\mathbf{x}) e_\lambda(\mathbf{y}) \tag{2.32}$$

$$\prod_{i,j \in \mathbb{Z}_+} \frac{1}{1 - x_i y_j} = \prod_{i \geq 1} \prod_{j \geq 1} \sum_{l \geq 0} x_i^l y_j^l = \prod_{i \geq 1} \sum_{k \geq 0} x_i^k h_k(\mathbf{y}) = \sum_{\lambda \text{ partition}} m_\lambda(\mathbf{x}) h_\lambda(\mathbf{y}) \tag{2.33}$$

From these two equations and the Cauchy identities above, it results;

$$\begin{aligned} \sum_{\lambda \text{ partition}} s_\lambda(\mathbf{x}) \omega_{\mathbf{y}}(s_\lambda(\mathbf{y})) &= \omega_{\mathbf{y}} \left( \sum_{\lambda \text{ partition}} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) \right) = \omega_{\mathbf{y}} \left( \sum_{\lambda \text{ partition}} m_\lambda(\mathbf{x}) h_\lambda(\mathbf{y}) \right) \\ &= \sum_{\lambda \text{ partition}} m_\lambda(\mathbf{x}) h_\lambda(\mathbf{y}) = \sum_{\lambda \text{ partition}} s_\lambda(\mathbf{x}) s_{\lambda'}(\mathbf{y}) \end{aligned} \tag{2.34}$$

where $\omega_{\mathbf{y}}$ is the operator $\omega$ on symmetric functions in the variables $\mathbf{y}$. Comparing coefficients of $s_\lambda(\mathbf{x})$ on both sides of equation 2.34, one obtains equation 2.31. A more general result, whose proof can also be found in (Stanley, 1999), is the following:

$$\omega(s_\nu) = s_{\nu'} \text{ for any skew partition } \nu \tag{2.35}$$

**Remark.** When restricted to a finite set of variables $(x_1, \ldots, x_n)$, Schur polynomials keep their combinatorial definition:

$$s_\nu(x_1, \ldots, x_n) = s_\nu(\mathbf{x})|_{x_{n+1} = x_{n+1} = \cdots = 0} = \sum_{\substack{\tau \in \mathrm{SSYT}(\nu) \\ \tau(\nu) \subseteq \{1, \ldots, n\}}} \mathbf{x}^\tau \tag{2.36}$$

Furthermore, fixing any possible total ordering $\leq'$ of $\mathbb{Z}_+$, the symmetry of Schur polynomials implies that the sum

$$\sum_{\substack{\tau:\nu\to\mathbb{Z}_+ \\ \tau\text{ semi standard} \\ \text{with respect to } \leq'}} \mathbf{x}^\tau$$

coincides with $s_\nu(\mathbf{x})$ when restricted to any finite set of variables $(x_1, \ldots, x_n)$, and so it must be equal to $s_\nu(\mathbf{x})$.

Another essential fact is that Schur polynomials indexed by partitions $\lambda$ of a given positive integer $n$, generate the space of homogeneous symmetric polynomials of degree $n$. A proof of this can be found in (Stanley, 2003).

## 2.3 Plethystic Substitution

Let $A$ be any rational expression on any number of variables. For a positive integer $k$, define the *plethystic* substitution $p_k[A]$ as the one that replaces every variable $a$ appearing in $A$ by its $k$th power $a^k$. For example:

$$p_k[x] = x^k$$

$$p_k\left[\frac{x+y}{z}\right] = \frac{x^k+y^k}{z^k}$$

$$p_k[x_1+x_2+x_3+\cdots] = x_1^k+x_2^k+x_3^k+\cdots$$

Extend multiplicatively this substitution by setting

$$p_{\lambda=(\lambda_1,\ldots,\lambda_k)}[A] = \prod_{i=1}^{k} p_{\lambda_i}[A] \tag{2.37}$$

Extend it now linearly as follows: If $f(\mathbf{x})$ is a symmetric polynomial, expressed in terms of power sum symmetric polynomials as follows:

$$f(\mathbf{x}) = \sum_{\lambda\text{ partition}} a_\lambda p_\lambda(\mathbf{x})$$

where the $a_\lambda$ are numbers. Then the plethystic substitution $f[A]$ is given by

$$f[A] = \sum_{\lambda\text{ partition}} a_\lambda p_\lambda[A] \tag{2.38}$$

These are some important properties of plethystic substitutions:

1. If $c$ is a constant, then $p_k[cA] = cp_k[A]$ and $p_\lambda[cA] = c^{l(\lambda)}p_\lambda[A]$.

2. $p_k[A+B] = p_k[A] + p_k[B]$.

3. $(f+g)[A] = f[A] + g[A]$.

4. $(fg)[A] = f[A]g[A]$.

5. If $X = x_1 + x_2 + \cdots$, then $f[X] = f(x_1, x_2, \ldots) = f(\mathbf{x})$.

6. $\omega(p_k)[A] = (-1)^{k-1}p_k[A]$ and so $\omega(p_\lambda)[A] = (-1)^{|\lambda|-l(\lambda)}p_\lambda[A]$.

7. $p_\lambda[-A] = (-1)^{l(\lambda)}p_\lambda[A] = (-1)^{|\lambda|}\omega(p_\lambda)[A]$.

8. In general, if $f$ is homogeneous of degree $n$, then $f[-A] = (-1)^n\omega(f)[A]$.

It will be useful later to have an expression for the plethystic substitution $s_\lambda[X+Y]$ ($\lambda$ being a partition) where $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$. For this notice that for any semi standard filling $\tau : \nu \to \{x_1, x_2, \ldots, y_1, y_2, \ldots\}$ (with the total ordering $x_1 < x_2 < \cdots < y_1 < y_2 < \cdots$), the set $\tau^{-1}(\{x_1, x_2, \ldots\})$ is a Young diagram $\mu \subseteq \lambda$, and so

$$s_\lambda[X+Y] = s_\lambda(\mathbf{x}, \mathbf{y}) = s_\lambda(x_1, x_2, \ldots, y_1, y_2, \ldots) = \sum_{\mu \subseteq \lambda} s_\mu(\mathbf{x}) s_{\lambda \backslash \mu}(\mathbf{y}) \tag{2.39}$$

Particular cases $\lambda = (1^k)$ and $\lambda = (k)$ yield

$$e_k(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{k} e_i(\mathbf{x}) e_{k-i}(\mathbf{y}) \tag{2.40}$$

$$h_k(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{k} h_i(\mathbf{x}) h_{k-i}(\mathbf{y}) \tag{2.41}$$

## 2.4 Quasisymmetric Polynomials

Quasisymmetric polynomials are a more general form of polynomials which include all symmetric polynomials. Their role in this theory is as a link between a particular plethystic substitution called *superization* and the combinatorial structures from the first chapter. This link will allow for a combinatorial interpretation of the axioms that define Macdonald polynomials.

### Monomials

**Definition 2.4.1.** Let $n$ be a positive integer and $D$ a subset of $\{1, 2, \ldots, n-1\}$. The degree $n$ *monomial quasisymmetric polynomial* indexed by $D$ is given by

$$M_{n,D}(\mathbf{x}) = \sum_{\substack{0<a_1\leq\cdots\leq a_n \\ a_i=a_{i+1}\Leftrightarrow i\notin D}} x_{a_1}\cdots x_{a_n} \tag{2.42}$$

where the $a_i$'s are integers. For example;

$$\begin{aligned} M_{5,\{2,4\}} &= x_1^2x_2^2x_3 + x_1^2x_2^2x_4 + \cdots + x_1^2x_3^2x_4 + x_1^2x_3^2x_5 + \cdots + x_2^2x_3^2x_4 + \cdots \\ &= \sum_{0<a<b<c} x_a^2x_b^2x_c \end{aligned}$$

As seen in section 1.5, subsets of $\{1, 2, \ldots, n-1\}$ are in bijective correspondence with compositions of $n$. Define the alternate notation

$$M_{\mathbf{c}} = M_{|\mathbf{c}|,\mathsf{set}(\mathbf{c})} \tag{2.43}$$

The example above is $M_{5,\{2,4\}} = M_{(2,4-2,5-4)} = M_{(2,2,1)}$. This notation allows for a somewhat simpler definition of the monomial quasisymmetric polynomials:

$$M_{\mathbf{c}}(\mathbf{x}) = \sum_{\substack{\mathbf{y}\subseteq\mathbf{x} \\ |\mathbf{y}|=l(\mathbf{c})}} \mathbf{y}^{\mathbf{c}} = \sum_{0<i_1<\cdots<i_k} x_{i_1}^{c_1}\cdots x_{i_k}^{c_k} \tag{2.44}$$

where $\mathbf{c} = (c_1, \ldots, c_k)$. Also using this same notation it is clear that

$$m_\mu = \sum_{\widehat{\mathbf{c}}=\mu} M_{\mathbf{c}} \tag{2.45}$$

### Fundamental basis

**Definition 2.4.2.** Let $n$ be a positive integer and $D$ a subset of $\{1, 2, \ldots, n-1\}$. The degree $n$ *Gessel's quasisymmetric polynomial* indexed by $D$ is given by

$$Q_{n,D}(\mathbf{x}) = \sum_{\substack{0<a_1\leq\cdots\leq a_n \\ a_i=a_{i+1}\Rightarrow i\notin D}} x_{a_1}\cdots x_{a_n} \tag{2.46}$$

where the $a_i$'s are integers. In other words;

$$Q_{n,D} = \sum_{D\subseteq D'} M_{n,D'} \tag{2.47}$$

For example;

$$Q_{5,\{2,4\}} = M_{5,\{2,4\}} + M_{5,\{1,2,4\}} + M_{5,\{2,3,4\}} + M_{5,\{1,2,3,4\}}$$

As with the monomials, define the alternate notation

$$Q_{\mathbf{c}} = Q_{|\mathbf{c}|,\mathrm{set}(\mathbf{c})} \tag{2.48}$$

which allows for the equivalent definition

$$Q_{\mathbf{c}} = \sum_{\mathbf{a}<_{\mathrm{ref}}\mathbf{c}} M_{\mathbf{a}} \tag{2.49}$$

The application of Möbius inversion to equation 2.47, yields a formula for the monomials in terms of Gessel's quasisymmetric polynomials:

$$M_{n,D} = \sum_{D\subseteq D'} (-1)^{|D'|-|D|} Q_{n,D'} \tag{2.50}$$

For example;

$$M_{5,\{2,4\}} = Q_{5,\{2,4\}} - Q_{5,\{1,2,4\}} - Q_{5,\{2,3,4\}} + Q_{5,\{1,2,3,4\}}$$

The space of quasisymmetric polynomials is the one generated by any of the two families defined above.

### Super Quasisymmetric Polynomials

**Definition 2.4.3.** Consider the super alphabet $\mathbb{Z}_+ \uplus \mathbb{Z}_-$ with a total order $\leq$ whose restriction to $\mathbb{Z}_+$ is the usual order on integers. Let $n$ be a positive integer and $D$ a subset of $\{1, 2, \ldots, n-1\}$. The degree $n$ *super quasisymmetric polynomial* indexed by $D$ is given by

$$\tilde{Q}_{n,D}(\mathbf{x}, \mathbf{y}) = \sum_{\substack{a_1 \leq \cdots \leq a_n \\ a_i = a_{i+1} \in \mathbb{Z}_+ \Rightarrow i \notin D \\ a_i = a_{i+1} \in \mathbb{Z}_- \Rightarrow i \in D}} z_{a_1} \cdots z_{a_n} \tag{2.51}$$

where the $a_i$'s are elements of $\mathbb{Z} \uplus \mathbb{Z}_-$ and

$$z_a = \begin{cases} x_a & \text{for } a \in \mathbb{Z}_+ \\ y_{-a} & \text{for } a \in \mathbb{Z}_- \end{cases}$$

An immediate property of super quasisymmetric polynomials is the following:

$$Q_{n,D}(\mathbf{x}) = \tilde{Q}_{n,D}(\mathbf{x}, 0) \tag{2.52}$$

Proposition 1.3.5 (or more specifically the remark right after its proof), along with equation 2.51, implies a formula of $\tilde{Q}_{n,D}(\mathbf{x}, 0)$ in terms of any bijective filling $\boldsymbol{\xi} : \uplus\boldsymbol{\nu} \to \{1, 2, \ldots, n\}$ of a tuple of skew partitions $\boldsymbol{\nu}$ of size $n$, satisfying $\mathrm{DC}(\boldsymbol{\xi}) = D$, namely;

$$\tilde{Q}_{n,\mathrm{DC}(\boldsymbol{\xi})}(\mathbf{x}, \mathbf{y}) = \sum_{\substack{\varphi : \uplus\boldsymbol{\nu} \to \mathbb{Z}_+ \uplus \mathbb{Z}_- \\ \mathsf{st}(\varphi) = \boldsymbol{\xi}}} \mathbf{z}^{\varphi} \tag{2.53}$$

If $\boldsymbol{\xi}$ is a standard Young tableau, then the equality becomes

$$\tilde{Q}_{n,\mathrm{DC}(\boldsymbol{\xi})}(\mathbf{x}, \mathbf{y}) = \sum_{\substack{\boldsymbol{\tau} \in \mathrm{SSYT}_{\pm}(\boldsymbol{\nu}) \\ \mathsf{st}(\boldsymbol{\tau}) = \boldsymbol{\xi}}} \mathbf{z}^{\boldsymbol{\tau}} \tag{2.54}$$

This formula will be useful to relate LLT polynomials to super quasisymmetric polynomials. Setting $\mathbf{y} = 0$ yields

$$Q_{n,\mathrm{DC}(\boldsymbol{\xi})}(\mathbf{x}) = \sum_{\substack{\boldsymbol{\tau} \in \mathrm{SSYT}(\boldsymbol{\nu}) \\ \mathsf{st}(\boldsymbol{\tau}) = \boldsymbol{\xi}}} \mathbf{x}^{\boldsymbol{\tau}} \tag{2.55}$$

From this equation and proposition 1.3.2, it can be deduced that

$$s_\nu(\mathbf{x}) = \sum_{\xi \in \mathrm{SYT}(\nu)} Q_{n,\mathrm{DC}(\xi)}(\mathbf{x}) = \sum_{\xi \in \mathrm{SYT}(\nu)} Q_{n,\mathrm{DR}(\xi)}(\mathbf{x}) \tag{2.56}$$

for any skew partition $\nu$, as a result of the combinatorial definition of Schur polynomials. This suggests the definition of *super Schur polynomials* as follows:

**Definition 2.4.4.** Let $\nu$ be a skew partition. The *super Schur polynomial* indexed by $\nu$ is given in terms of quasisymmetric polynomials by

$$\tilde{s}_\nu(\mathbf{x}, \mathbf{y}) = \sum_{\xi \in \mathrm{SYT}(\nu)} \tilde{Q}_{n,\mathrm{DR}(\xi)}(\mathbf{x}, \mathbf{y}) = \sum_{\tau \in \mathrm{SSYT}_\pm(\nu)} \mathbf{z}^\tau \tag{2.57}$$

The super Schur polynomials will be proven later to be independent of the total ordering assigned to the set $\mathbb{Z}_\pm = \mathbb{Z}_+ \uplus \mathbb{Z}_-$.

## Superization

**Definition 2.4.5.** The *superization* of a symmetric function $f$, denoted $\tilde{f}$, is defined by the plethystic substitution:

$$\tilde{f}(\mathbf{x}, \mathbf{y}) = \omega_Y f[X + Y] \tag{2.58}$$

where $X = x_1 + x_2 + \cdots$, $Y = y_1 + y_2 + \cdots$, and the operator $\omega_Y$ is the usual operator $\omega$ acting on $f$ as a symmetric function on the variables $y_1, y_2, \ldots$.

Equation 2.45 shows that any symmetric polynomial is itself quasisymmetric, thus it can be written in an unique way as a linear combination of the elements of any given linear basis of the ring of quasisymmetric polynomials. If $f(\mathbf{x})$ is a symmetric polynomial, then there are unique coefficients $c_{n,D}$ ($n \geq 1$, $D \subseteq \{1, \ldots, n-1\}$) such that

$$f(\mathbf{x}) = \sum_{\substack{n \geq 1 \\ D \subseteq \{1,\ldots,n-1\}}} c_{n,D} Q_{n,D}(\mathbf{x})$$

This unique expression is intimately related to the superization of symmetric polynomials as follows;

**Proposition 2.4.1.** (Haglund, Haiman, Loehr, 2005) *If $f$ is a symmetric function, given in terms of quasisymmetric polynomials as:*

$$f(\mathbf{x}) = \sum_{\substack{n\geq 1 \\ D\subseteq\{1,\ldots,n-1\}}} c_{n,D} Q_{n,D}(\mathbf{x}),$$

*then its superization $\tilde{f}(\mathbf{x},\mathbf{y}) = \omega_Y f[X+Y]$ is given by*

$$\tilde{f}(\mathbf{x},\mathbf{y}) = \sum_{\substack{n\geq 1 \\ D\subseteq\{1,\ldots,n-1\}}} c_{n,D} \tilde{Q}_{n,D}(\mathbf{x},\mathbf{y}),$$

*Proof.* Since Schur polynomials generate all symmetric polynomials, it is enough to prove that $\tilde{s}_\lambda(\mathbf{x},\mathbf{y})$ is the superization of $s_\lambda(\mathbf{x})$ where $\lambda$ is a partition. For a total ordering of $\mathbb{Z}_\pm = \mathbb{Z}_+ \uplus \mathbb{Z}_-$, and a positive integer $n$, define the indices $i_1, i_2, \ldots, i_{2n}$ as a reordering of the numbers $1,\ldots,n,-1,\ldots,-n$ such that $i_1 < i_2 < \ldots < i_{2n}$, and set $z_i = x_i$ for $i \in \mathbb{Z}_+$ and $z_i = y_{-i}$ for $i \in \mathbb{Z}_-$. Consider the restriction

$$\tilde{s}_\lambda(z_1,\ldots,z_n,z_{-1},\ldots,z_{-n}) = \sum_{\substack{\tau\in \mathrm{SSYT}_\pm(\lambda) \\ \tau(\lambda)\subseteq\{1,\ldots,n,-1,\ldots,-n\}}} \mathbf{z}^\tau$$

It is clear that for any $\tau$ as in a the previous sum, and any positive number $k$ with $1 \leq k \leq 2n$, the set $\tau^{-1}(\{i_1,\ldots,i_k\})$ is a partition $\mu^{(k)} \subseteq \lambda$, so the previous sum can be rewriten as follows:

$$\tilde{s}_\lambda(z_1,\ldots,z_n,z_{-1},\ldots,z_{-n}) = \sum_{0=\mu^{(0)}\subseteq\mu^{(1)}\subseteq\cdots\subseteq\mu^{(2n)}=\lambda} \prod_{k=1}^{2n} \tilde{s}_{\mu^{(k)}\backslash\mu^{(k-1)}}(z_{i_k})$$

It is clear that

$$\tilde{s}_{\mu^{(k)}\backslash\mu^{(k-1)}}(z_{i_k}) = \begin{cases} z_{i_k}^{|\mu^{(k)}\backslash\mu^{(k-1)}|} & \text{if } i_k \in \mathbb{Z}_+ \text{ and } \mu^{(k)}\backslash\mu^{(k-1)} \text{ is a horizontal stripe} \\ z_{i_k}^{|\mu^{(k)}\backslash\mu^{(k-1)}|} & \text{if } i_k \in \mathbb{Z}_- \text{ and } \mu^{(k)}\backslash\mu^{(k-1)} \text{ is a vertical stripe} \\ 0 & \text{otherwise.} \end{cases}$$

and so

$$\tilde{s}_{\mu^{(k)}\backslash\mu^{(k-1)}}(z_{i_k}) = \begin{cases} s_{\mu^{(k)}\backslash\mu^{(k-1)}}(z_{i_k}) & \text{if } i_k \in \mathbb{Z}_+ \\ s_{(\mu^{(k)}\backslash\mu^{(k-1)})'}(z_{i_k}) = \omega_Y(s_{\mu^{(k)}\backslash\mu^{(k-1)}}(z_{i_k})) & \text{if } i_k \in \mathbb{Z}_- \end{cases}$$

As a consequence:

$$\tilde{s}_\lambda(z_1,\ldots,z_n,z_{-1},\ldots,z_{-n}) = \omega_Y(s_\lambda(z_1,\ldots,z_n,z_{-1},\ldots,z_{-n}))$$

which implies the general result

$$\tilde{s}_\lambda(\mathbf{x},\mathbf{y}) = \omega_Y s_\lambda(\mathbf{x},\mathbf{y}) = \omega_Y s_\lambda[X+Y] \tag{2.59}$$

□

The previous proof implies the otherwise striking result that $\tilde{s}_\nu(\mathbf{x},\mathbf{y})$ does not depend on the total ordering assigned to $\mathbb{Z}_\pm = \mathbb{Z}_+ \uplus \mathbb{Z}_-$.

## 2.5 LLT Polynomials

In (Lascoux, Leclerc, Thibon, 1997), the authors introduced new families of symmetric functions on the variables $z_1, z_2, \ldots$ with coefficients in the field of rational functions $\mathbb{Q}(q)$. These were defined combinatorially in terms of the so called ribbon tableaux. The definition of LLT polynomials given here is a variant of one of these families which was introduced in (Haglund, Haiman, Loehr, Remmel, Ulyanov, 2005).

**Definition 2.5.1.** Let $\boldsymbol{\nu}$ be a tuple of skew partitions. The LLT polynomial indexed by $\boldsymbol{\nu}$ is given by:

$$G_{\boldsymbol{\nu}}(\mathbf{x};q) = \sum_{\boldsymbol{\tau}\in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(\boldsymbol{\tau})}\mathbf{x}^{\boldsymbol{\tau}}$$

### Expression in terms of Quasisymmetric Polynomials

As in the case of Schur polynomials, the symmetry of these functions on the variables $z_1, z_2, \ldots$ is not evident from their definition. An entirely combinatorial proof of the symmetry of LLT polynomials was done in (Haglund, Haiman, Loehr, 2005). This proof starts by expressing $G_{\boldsymbol{\nu}}(\mathbf{x};q)$ in terms of quasisymmetric polynomials as follows:

Consider the polynomial

$$\tilde{G}_{\boldsymbol{\nu}}(\mathbf{x},\mathbf{y};q) = \sum_{\boldsymbol{\tau}\in\mathrm{SSYT}_{\pm}(\boldsymbol{\nu})} q^{\mathrm{inv}(\boldsymbol{\tau})}\mathbf{z}^{\boldsymbol{\tau}} \tag{2.60}$$

where $z_i = x_i$ if $i \in \mathbb{Z}_+$ and $z_i = y_{-i}$ if $i \in \mathbb{Z}_-$. A direct consequence of equation 1.12 is the following:

$$\tilde{G}_{\boldsymbol{\nu}}(\mathbf{x},0;q) = G_{\boldsymbol{\nu}}(\mathbf{x};q) \tag{2.61}$$

Grouping the sum in equation 2.60 by the standardization of $\boldsymbol{\tau}$, one obtains

$$\tilde{G}_{\boldsymbol{\nu}}(\mathbf{x},\mathbf{y};q) = \sum_{\boldsymbol{\xi}\in\mathrm{SYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(\boldsymbol{\xi})} \sum_{\substack{\boldsymbol{\tau}\in\mathrm{SSYT}_{\pm}(\boldsymbol{\nu})\\ \mathbf{st}(\boldsymbol{\tau})=\boldsymbol{\xi}}} \mathbf{z}^{\boldsymbol{\tau}}$$

which combined with equation 2.55 yields

$$\tilde{G}_{\boldsymbol{\nu}}(\mathbf{x},\mathbf{y};q) = \sum_{\boldsymbol{\xi}\in\mathrm{SYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(\boldsymbol{\xi})}\tilde{Q}_{|\boldsymbol{\nu}|,\mathrm{DC}(\boldsymbol{\xi})}(\mathbf{x},\mathbf{y}) \tag{2.62}$$

And as a special case, from equations 2.52 and 2.61;

$$G_{\boldsymbol{\nu}}(\mathbf{x};q) = \sum_{\boldsymbol{\xi}\in\mathrm{SYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(\boldsymbol{\xi})}Q_{|\boldsymbol{\nu}|,\mathrm{DC}(\boldsymbol{\xi})}(\mathbf{x}) \tag{2.63}$$

This shows that $\tilde{G}_{\boldsymbol{\nu}}(\mathbf{x},\mathbf{y};q)$ is the superization of $G_{\boldsymbol{\nu}}(\mathbf{x};q)$. This superization is essential in Haglund's proof of the symmetry of $G_{\boldsymbol{\nu}}(\mathbf{x};q)$.

### Schur Positivity

The coefficients of LLT polynomials when expressed as a linear combination of Schur polynomials are all polynomials in $q$ with nonnegative integer coefficients. This fact has been proven by Grojnowski and Haiman in (Grojnowski, Haiman, 2007) and by S. H. Assaf (Assaf, 2007) using two very different approaches. In (Grojnowski, Haiman, 2007), the result follows as a particular case of a purely algebraic positivity result. The proof in (Assaf, 2007) is entirely combinatorial; it gives an interpretation of these coefficients in terms of a new combinatorial construction called *dual equivalence graph*. As will be made clear later, this result gives a combinatorial proof for the positivity of the coefficients of Macdonald polynomials when expressed as linear combinations of Schur polynomials.

# CHAPTER III

# MACDONALD POLYNOMIALS

Other important families of symmetric polynomials, not mentioned in the previous chapter, include the Zonal, Jack and Hall-Littlewood symmetric polynomials, all of them in the ring of symmetric polynomials in the variables $x_1, x_2, \ldots$ with coefficients in the field of rational functions on a subset of $\{q, t\}$. These three families, along with Schur polynomials, all indexed by partitions, satisfy an orthogonality relation $< f_\lambda, f_\mu >= 0$ when $\lambda \neq \mu$, for some convenient scalar products $<,>$ defined on the ring of symmetric polynomials with coefficients that are rational functions on $\{q, t\}$. These orthogonality relations allow these families to be uniquely obtained through an orthogonalization process, when some other conditions (such as triangularity relations) are given. For instance, Schur polynomials are uniquely defined by the axioms:

1. $s_\lambda = m_\lambda + \sum_{\mu \prec \lambda} c_{\lambda\mu} m_\mu$.
2. $\langle s_\lambda, s_\mu \rangle = 0$ whenever $\lambda \neq \mu$.

where the *Hall scalar product* $\langle\, ,\rangle$ is given by

$$\langle p_\lambda, p_\mu \rangle = \begin{cases} z_\lambda & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

and $z_\lambda$ is the coefficient defined in equation 2.17. It can be proven that

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} \tag{3.2}$$

for all partitions $\lambda, \mu$, using the general result that two basis $u_\lambda$ and $v_\lambda$ satisfy $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$ if and only if they satisfy $\sum_\lambda u_\lambda(\mathbf{x})v_\lambda(\mathbf{y}) = \prod_{i,j}(1-x_iy_j)^{-1}$. This result is obtained by observing that

$$\prod_{i,j}(1-x_iy_j)^{-1} = e^{\sum_{i,j} -\log(1-x_iy_j)} = e^{\sum_{i,j}\sum_{k\geq 1} x_i^k y_i^k/k}$$

$$= e^{\sum_{k\geq 1} p_k(\mathbf{x})p_k(\mathbf{y})/k} = \sum_{\lambda \text{ partition}} p_\lambda(\mathbf{x})\frac{p_\lambda(\mathbf{y})}{z_\lambda}$$

and it implies the relation

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} \tag{3.3}$$

Each of the three relations $\langle s_\lambda, s_\mu \rangle = \langle m_\lambda, h_\mu \rangle = \left\langle p_\lambda, \frac{p_\mu}{z_\mu} \right\rangle = \delta_{\lambda\mu}$ defines uniquely the Hall scalar product.

In (Macdonald, 1988), Macdonald introduced a common generalization to Schur, Zonal, Jack and Hall-Littlewood symmetric polynomials, as the unique family of homogeneous polynomials $P_\lambda(\mathbf{x}; q, t)$ satisfying the two axioms

1. $P_\lambda = m_\lambda + \sum_{\mu \prec \lambda} c_{\lambda\mu}(q,t)m_\lambda$ for some coefficients $c_{\lambda\mu} \in \mathbb{C}(q,t)$.

2. $< P_\lambda, P_\mu > = 0$ whenever $\lambda \neq \mu$.

where the scalar product $<,>$ is given by

$$< p_\lambda, p_\mu > = \begin{cases} Z_\lambda(q,t) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Refer to (Macdonald, 1988) or (Macdonald, 1995) for the original proof of the existence and uniqueness of this family of symmetric polynomials. The first axiom is called a *triangularity axiom* as it basically says that the matrix of the coefficients $c_{\lambda\mu}$ is upper triangular with respect to any total order that extends the inverse dominance order. This first axiom clearly shows that $\{P_\lambda\}_{\lambda \vdash n}$ generates the space of homogeneous symmetric polynomials of degree $n$ with coefficients in $\mathbb{C}(q,t)$.

The first problem that one encounters when working with these polynomials is the fact the coefficients are not polynomials, but rational functions on $\{q,t\}$, which makes tasks such as finding patterns or conjecturing combinatorial formulas very difficult. In order to simplify the expansions, Macdonald defined a natural renormalization of the polynomials $P_\lambda$ as follows:

**Definition 3.0.2.** Let $\mu$ be a partition. The *Macdonald polynomial* indexed by $\mu$ is given by the formula

$$H_\mu(\mathbf{x};q,t) = P_\mu\left[\frac{X}{1-t}, q, t^{-1}\right]\prod_{u\in\mu}(q^{\mathrm{arm}(u)} - t^{\mathrm{leg}(u)+1})$$

where $X = x_1 + x_2 + \cdots$.

The coefficients of $H_\mu$ are all polynomials in the variables $q,t$. Haiman (Haiman 1999) proved that the orthogonality and triangularity axioms that define the polynomials $P_\mu(\mathbf{x};q,t)$ are equivalent to the following axioms on the polynomials $H_\mu(\mathbf{x};q,t)$

1. $H_\mu[X(1-q);q,t] = \sum_{\lambda\succeq\mu} a_{\lambda\mu}(q,t)s_\lambda(\mathbf{x})$
2. $H_\mu[X(1-t);q,t] = \sum_{\lambda\succeq\mu'} b_{\lambda\mu}(q,t)s_\lambda(\mathbf{x})$
3. $\langle H_\mu(\mathbf{x};q,t), s_{(n)}(\mathbf{x})\rangle = 1$ for $n = |\mu|$

where $\langle\, ,\rangle$ is the Hall scalar product. The observations

- $f[-A] = (-1)^n\omega(f)[A]$ for any homogeneous symmetric function $f$ of degree $n$,
- $\omega(s_\lambda(\mathbf{x})) = s_{\lambda'}(\mathbf{x})$ for any partition $\lambda$,
- $\lambda \succeq \mu \Leftrightarrow \lambda' \preceq \mu'$ for any partitions $\lambda$ and $\mu$, and,
- the basis $\{s_\lambda\}_{\lambda\vdash n}$ and $\{m_\lambda\}_{\lambda\vdash n}$ are mutually lower triangular with respect to the dominance order,

imply that the three axioms above are equivalent to the following axioms:

1. $H_\mu[X(q-1);q,t] = \sum_{\lambda \preceq \mu'} c_{\lambda\mu}(q,t) m_\lambda(\mathbf{x})$

2. $H_\mu[X(t-1);q,t] = \sum_{\lambda \preceq \mu} d_{\lambda\mu}(q,t) m_\lambda(\mathbf{x})$

3. $\langle H_\mu(\mathbf{x};q,t), s_{(n)}(\mathbf{x})\rangle = 1$ for $n = |\mu|$

## 3.1 Combinatorial Formula

In this section, a combinatorial formula for Macdonald polynomials first conjectured by Haglund (Haglund, 2004) is presented. The proof outlined here is based on the one in (Haglund, Haiman, Loehr, 2005). This proof shows that the formula conjectured by Haglund satisfies the three axioms at the end of the previous section.

**Theorem 3.1.1.** (Haglund, Haiman, Loehr, 2005) *The Macdonald polynomial $H_\mu(\mathbf{x};q,t)$ admits the following combinatorial formula*

$$H_\mu(\mathbf{x};q,t) = \sum_{\varphi:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\varphi)} t^{\mathrm{maj}(\varphi)} \mathbf{x}^\varphi \tag{3.4}$$

*where* inv *and* maj *are the statistics on fillings studied in the first chapter.*

Among the most important immediate results of this theorem, is the fact that the coefficients of Macdonald polynomials are all in $\mathbb{Z}_+[q,t]$, a result that was first proven six years after Macdonald introduced the polynomials $H_\mu(\mathbf{x};q,t)$.

In order to prove equation 3.4, define the expression $C_\mu(\mathbf{x};q,t)$ as its right hand side:

$$C_\mu(\mathbf{x};q,t) = \sum_{\varphi:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\varphi)} t^{\mathrm{maj}(\varphi)} \mathbf{x}^\varphi \tag{3.5}$$

Before proving the equality $H_\mu(\mathbf{x};q,t) = C_\mu(\mathbf{x};q,t)$, a deeper study of $C_\mu(\mathbf{x};q,t)$ is necessary. Notice first that the symmetry of $C_\mu(\mathbf{x};q,t)$ is not evident from its definition. A proof of its symmetry follows.

*Proof that $C_\mu(\mathbf{x};q,t)$ is a symmetric function.* Group equation 3.5 by descent sets as follows:

$$C_\mu(\mathbf{x};q,t) = \sum_{D\subseteq\mu} \sum_{\substack{\varphi:\mu\to\mathbb{Z}_+\\ \mathrm{Des}(\varphi)=D}} q^{\mathrm{inv}(\varphi)} t^{\mathrm{maj}(\varphi)} \mathbf{x}^{\varphi}$$

$$= \sum_{D\subseteq\mu} q^{-\sum \mathrm{arm}(u)}\, t^{\sum(\mathrm{leg}(u)+1)} \sum_{\substack{\varphi:\mu\to\mathbb{Z}_+\\ \mathrm{Des}(\varphi)=D}} q^{|\mathrm{Inv}(\varphi)|}\mathbf{x}^{\varphi} \tag{3.6}$$

where the sums on the exponents of $q$ and $t$ are over all the cells $u \in \mu$.

For $D \subseteq \mu$, let $\boldsymbol{\nu} = \mathbf{rbb}(\mu, D) = (\nu^{(1)}, \ldots, \nu^{k})$ be the tuple of ribbons as in definition 1.3.5. Given a filling $\varphi : \mu \to \mathbb{Z}$ define the filling $\boldsymbol{\varphi} : \uplus\boldsymbol{\nu} \to \mathbb{Z}_+$ as the one corresponding entry by entry to $\varphi$ as in proposition 1.3.1. From the definition of $\mathbf{rbb}(\mu, D)$ it is clear that $\boldsymbol{\varphi}$ is a semi standard Young tableau if and only if $\mathrm{Des}(\varphi) = D$. Also from proposition 1.3.1; $|\mathrm{Inv}(\varphi)| = |\mathrm{Inv}(\boldsymbol{\varphi})|$. Therefore;

$$C_\mu(\mathbf{x};q,t) = \sum_{D\subseteq\mu} q^{-\sum \mathrm{arm}(u)} t^{\sum(\mathrm{leg}(u)+1)} \sum_{\boldsymbol{\varphi}\in\mathrm{SSYT}(\mathbf{rbb}(\mu,D))} q^{|\mathrm{Inv}(\boldsymbol{\varphi})|}\mathbf{x}^{\boldsymbol{\varphi}} \tag{3.7}$$

and from definition 2.5.1;

$$C_\mu(\mathbf{x};q,t) = \sum_{D\subseteq\mu} q^{-\sum \mathrm{arm}(u)} t^{\sum(\mathrm{leg}(u)+1)} G_{\mathbf{rbb}(\mu,D)}(\mathbf{x}, q) \tag{3.8}$$

The symmetry of LLT polynomials concludes the proof. □

Now that $C_\mu(\mathbf{x};q,t)$ has been proven to be symmetric, it makes sense to compute its superization, for which it is necessary to express it as a linear combination of quasisymmetric functions. Grouping equation 3.5 by the standardization of $\varphi$, produces:

$$C_\mu(\mathbf{x};q,t) = \sum_{\substack{\xi:\mu\to\{1,2,\ldots,|\mu|\}\\ \xi\ \text{bijective}}} q^{\mathrm{inv}(\xi)} t^{\mathrm{maj}(\xi)} \sum_{\substack{\varphi:\mu\to\mathbb{Z}_+\\ \mathrm{st}(\varphi)=\xi}} \mathbf{x}^{\varphi} \tag{3.9}$$

From proposition 1.2.5 (more specifically the remark right after its proof), it results that for any bijective filling $\xi : \mu \to \{1, 2, \ldots, |\mu|\}$, the following formulas hold for the quasisymmetric function $Q_{|\mu|,\mathrm{DR}(\xi)}(\mathbf{x})$ and the super quasisymmetric function $\tilde{Q}_{|\mu|,\mathrm{DR}(\xi)}(\mathbf{x}, \mathbf{y})$:

$$Q_{|\mu|,\mathrm{DR}(\xi)}(\mathbf{x}) = \sum_{\substack{\varphi:\mu\to\mathbb{Z}_+\\ \mathrm{st}(\varphi)=\xi}} \mathbf{x}^{\varphi} \tag{3.10}$$

$$\tilde{Q}_{|\mu|,\mathrm{DR}(\xi)}(\mathbf{x},\mathbf{y}) = \sum_{\substack{\varphi:\mu\to\mathbb{Z}_+\uplus\mathbb{Z}_-\\ \mathrm{st}(\varphi)=\xi}} \mathbf{z}^{\varphi} \tag{3.11}$$

with $z_i = x_i$ for $i \in \mathbb{Z}_+$ and $z_i = y_{-i}$ for $i \in \mathbb{Z}_-$. Thus combining equations 3.9 and 3.10 one can express $C_\mu(\mathbf{x}; q, t)$ in terms of quasisymmetric polynomials;

$$C_\mu(\mathbf{x}; q, t) = \sum_{\substack{\xi:\mu\to\{1,2,\ldots,|\mu|\}\\ \xi \text{ bijective}}} q^{\mathrm{inv}(\xi)} t^{\mathrm{maj}(\xi)} Q_{|\mu|,\mathrm{DR}(\xi)}(\mathbf{x}) \tag{3.12}$$

As a result, the superization of $C_\mu(\mathbf{x}; q, t)$ is given by

$$\begin{aligned}\tilde{C}_\mu(\mathbf{x},\mathbf{y}; q, t) &= \sum_{\substack{\xi:\mu\to\{1,2,\ldots,|\mu|\}\\ \xi \text{ bijective}}} q^{\mathrm{inv}(\xi)} t^{\mathrm{maj}(\xi)} \tilde{Q}_{|\mu|,\mathrm{DR}(\xi)}(\mathbf{x},\mathbf{y}) \\ &= \sum_{\varphi:\mu\to\mathbb{Z}_+\uplus\mathbb{Z}_-} q^{\mathrm{inv}(\varphi)} t^{\mathrm{maj}(\varphi)} \mathbf{z}^{\varphi} \qquad (3.13)\end{aligned}$$

In order to prove the equality $H_\mu(\mathbf{x}; q, t) = C_\mu(\mathbf{x}; q, t)$, it will be shown that $C_\mu$ satisfies the three axioms of the characterization of Macdonald polynomials presented at the beginning of this chapter, namely,

$$\textit{(T1)}\quad C_\mu[X(q-1); q, t] = \sum_{\lambda \preceq \mu'} c_{\lambda\mu}(q,t) m_\lambda(\mathbf{x})$$

$$\textit{(T2)}\quad C_\mu[X(t-1); q, t] = \sum_{\lambda \preceq \mu} d_{\lambda\mu}(q,t) m_\lambda(\mathbf{x})$$

$$\textit{(N)}\quad \langle C_\mu(\mathbf{x}; q, t), s_{(n)}(\mathbf{x}) \rangle = 1 \text{ for } n = |\mu|$$

*Proof that $C_\mu(\mathbf{x}; q, t)$ satisfies (N).* Since $s_{(n)} = h_n = h_{(n)}$, and $\langle m_\lambda, h_{(n)} \rangle = \delta_{\lambda,(n)}$ (from equation 3.3); $\langle C_\mu(\mathbf{x}; q, t), s_{(n)}(\mathbf{x}) \rangle$ is the coefficient of $m_{(n)}$ in the expansion of $C_\mu$ in terms of monomial symmetric polynomials. But $m_{(n)} = p_n = \sum_{k\geq 1} x_k^n$, thus it is enough to show that the coefficient of $x_k^n$ in $C_\mu(\mathbf{x}; q, t)$ is 1. This is evident since $\mathrm{inv}(\varphi) = \mathrm{maj}(\varphi) = 0$ for the constant filling $\varphi = k$. □

For *(T1)* and *(T2)* it is necessary to find a combinatorial formula for the expressions $C_\mu[X(q-1);q,t]$ and $C_\mu[X(t-1);q,t]$. The identity

$$\tilde{p}_k(\mathbf{x},-\mathbf{y}) = \left(p_k[X] + (-1)^{k-1}p_k[Y]\right)\Big|_{\mathbf{y}\leftarrow(-\mathbf{y})} = p_k[X-Y]$$

shows that in general

$$f[X-Y] = \tilde{f}(\mathbf{x},-\mathbf{y})$$

Which yields the two combinatorial formulas

$$C_\mu[X(q-1);q,t] = \tilde{C}_\mu(q\mathbf{x},-\mathbf{x};q,t)$$

$$= \sum_{\varphi:\mu\to\mathbb{Z}_+\uplus\mathbb{Z}_-} (-1)^{m(\varphi)} q^{p(\varphi)+\mathrm{inv}(\varphi)} t^{\mathrm{maj}(\varphi)} \mathbf{x}^{|\varphi|} \tag{3.14}$$

$$C_\mu[X(t-1);q,t] = \tilde{C}_\mu(t\mathbf{x},-\mathbf{x};q,t)$$

$$= \sum_{\varphi:\mu\to\mathbb{Z}_+\uplus\mathbb{Z}_-} (-1)^{m(\varphi)} q^{\mathrm{inv}(\varphi)} t^{p(\varphi)+\mathrm{maj}(\varphi)} \mathbf{x}^{|\varphi|} \tag{3.15}$$

where $m(\varphi)$ is the number of positive entries of $\varphi$, $p(\varphi)$ is the number of negative entries of $\varphi$, and $|\varphi|$ is the filling resulting from replacing each entry $i$ of $\varphi$ by $|i|$.

Recall that different total orderings of $\mathbb{Z}_\pm = \mathbb{Z}_+ \uplus \mathbb{Z}_+$ yield different definitions of inv and maj on super fillings, and these identities are true for any of these definitions.

For convenience, the following notation will be used;

$$\alpha(\varphi) = (-1)^{m(\varphi)} q^{p(\varphi)+\mathrm{inv}(\varphi)} t^{\mathrm{maj}(\varphi)} \mathbf{z}_{|\varphi|}$$

$$\beta(\varphi) = (-1)^{m(\varphi)} q^{\mathrm{inv}(\varphi)} t^{p(\varphi)+\mathrm{maj}(\varphi)} \mathbf{z}_{|\varphi|}$$

With this notation:

$$C_\mu[X(q-1);q,t] = \sum_{\varphi:\mu\to\mathbb{Z}_\pm} \alpha(\varphi) \tag{3.16}$$

$$C_\mu[X(t-1);q,t] = \sum_{\varphi:\mu\to\mathbb{Z}_\pm} \beta(\varphi) \tag{3.17}$$

*Proof that $C_\mu(\mathbf{x};q,t)$ satisfies (T1).* Consider the order $<_1$ in $\mathbb{Z}_\pm$ satisfying $1 <_1 -1 <_1 2 <_1 -2 \cdots$. Let $\varphi:\mu\to\mathcal{A}$ be a super filling on $\mu$. Let $\Psi$ be the involution on super fillings defined by the following two properties:

1. If there is no pair of attacking cells $u$, $v$ in $\mu$ such that $|\varphi(u)| = |\varphi(v)|$, then $\Psi\varphi = \varphi$.

2. Otherwise, let $a$ be the smallest integer satisfying $a = |\varphi(u)| = |\varphi(v)|$ for some pair of attacking cells $u$, $v$. Fix $v_0$ to be the last cell in the reading order that is part of an attacking pair $u$, $v_0$ with $|\varphi(u)| = |\varphi(v_0)| = a$ and fix $u_0$ to be the last cell in the reading order that attacks $v_0$ and satisfies $|\varphi(u_0)| = a$. Define $\Psi\varphi(u_0) = -\varphi(u_0)$ and $\Psi\varphi(w) = \varphi w$ for all $w \neq u_0$.

It is clear that $|\Psi\varphi| = |\varphi|$, and so the equality

$$\mathbf{x}^{|\Psi\varphi|} = \mathbf{x}^{|\varphi|} \tag{3.18}$$

holds for every super filling $\varphi$. Consider now the cases in which such pair of attacking cells $u$, $v$ satisfying $|\varphi(u)| = |\varphi(v)|$ exists. The following relation is evident from the definition of $\Psi$:

$$(-1)^{m(\Psi\varphi)} = -(-1)^{m(\varphi)} \tag{3.19}$$

Define the indicator $J(a,b)$ for $a, b \in \mathbb{Z}_\pm$ as the one satisfying $J(\varphi(u), \varphi(v)) = I_\varphi(u,v)$ for every super filling $\varphi$. In other words:

$$J(a,b) = \begin{cases} 1 & \text{if } a > b \text{ or } a = b \in \mathbb{Z}_- \\ 0 & \text{if } a < b \text{ or } a = b \in \mathbb{Z}_+ \end{cases}$$

Now notice that $J_\varphi(a,b) = J(a,-b)$ for all $a, b \in \mathbb{Z}_\pm$. Let $u_1$ be the cell immediately above $u_0$, then this relation implies

$$J(\Psi\varphi(u_1), \Psi\varphi(u_0)) = J(\varphi(u_1), -\varphi(u_0)) = J(\varphi(u_1), \varphi(u_0))$$

This is; $u_1$ is descent in $\Psi\varphi$ if and only if it is a descent in $\varphi$. Now let $u_2$ be the cell immediately below $u_0$. It is clear that $v_0$ and $u_2$ attack each other, and $v_0$ precedes $u_2$ in the reading order. Consequently, from the maximality of $v_0$ (in the reading order), the integer $|\varphi(u_2)|$ must be necessarily different from $|\varphi(v_0)| = |\varphi(u_0)|$. This implies the relation

$$J(\Psi\varphi(u_0), \Psi\varphi(u_2)) = J(-\varphi(u_0), \varphi(u_2)) = J(\varphi(u_0), \varphi(u_2))$$

This is; $u_0$ is a descent in $\Psi\varphi$ if and only if it is a descent in $\varphi$. These observations show that $\mathrm{Des}(\Psi\varphi) = \mathrm{Des}(\varphi)$, and consequently;

$$\mathrm{maj}(\Psi\varphi) = \mathrm{maj}(\varphi) \tag{3.20}$$

$$\mathrm{inv}(\Psi\varphi) - \mathrm{inv}(\varphi) = |\mathrm{Inv}(\Psi\varphi)| - |\mathrm{Inv}(\varphi)| \tag{3.21}$$

Let $u$ be any cell such that $u$ and $u_0$ attack each other and $u$ precedes $u_0$ in the reading order. Using again the relation $J(a,b) = J(a,-b)$, it is clear that $u$, $u_0$ form an inversion in $\Psi\varphi$ if and only if they form an inversion in $\varphi$.

Now let $u \neq v_0$ be any cell such that $u$ and $u_0$ attack each other and $u_0$ precedes $u$ in the reading order. It is clear that $u$ and $v_0$ attack each other, so from the maximality of $u_0$ (in the reading order); $|\varphi(u)|$ is necessarily different from $|\varphi(u_0)| = |\varphi(v_0)|$, thus, as in previous observations,

$$J(\Psi\varphi(u_0), \Psi\varphi(u)) = J(-\varphi(u_0), \varphi(u)) = J(\varphi(u_0), \varphi(u))$$

As a result; $u_0$, $u$ form an inversion in $\Psi\varphi$ if and only if they form an inversion in $\varphi$.

Since $|\varphi(u_0)| = |\varphi(v_0)|$ and $u_0$, $v_0$ attack each other; then $u_0$, $v_0$ form an inversion in any filling $\rho$ if and only if $\rho(u_0) \in \mathbb{Z}_-$, thus,

$$|\mathrm{Inv}(\Psi\varphi)| - |\mathrm{Inv}(\varphi)| = m(\Psi\varphi) - m(\varphi)$$

And since $m(\Psi\varphi) - m(\varphi) = p(\varphi) - p(\Psi\varphi)$;

$$|\mathrm{Inv}(\Psi\varphi)| - |\mathrm{Inv}(\varphi)| = p(\varphi) - p(\Psi\varphi) \tag{3.22}$$

Combining equations 3.21 and 3.22 one obtains;

$$p(\Psi\varphi) + \mathrm{inv}(\Psi\varphi) = p(\varphi) + \mathrm{inv}(\varphi) \tag{3.23}$$

and combining equations 3.18, 3.19, 3.20 and 3.23, one concludes that the relation

$$\alpha(\Psi\varphi) = -\alpha(\varphi)$$

holds for every super filling satisfying $\Psi\varphi \neq \varphi$. As a consequence, since $\Psi$ is an involution, one has the formula

$$C_\mu[X(q-1);q,t] = \sum_{\Psi\varphi=\varphi} (-1)^{m(\varphi)} q^{p(\varphi)+\mathrm{inv}(\varphi)} t^{\mathrm{maj}(\varphi)} \mathbf{x}^{|\varphi|} \tag{3.24}$$

Let $\lambda$ be a partition such that $\mathbf{x}^{|\varphi|} = \mathbf{x}^\lambda = x_1^{\lambda_1} \cdots x_l^{\lambda_l}$ for some super filling $\varphi : \mu \to \mathbb{Z}_\pm$ satisfying $\Psi\varphi = \varphi$. It only remains to prove that $\lambda \preceq \mu'$, this is, $\lambda_1 + \cdots + \lambda_j \leq \mu'_1 + \cdots + \mu'_j$ for all $j \geq 1$. $\lambda_1 + \cdots + \lambda_j$ is the number of entries in $\varphi$ with absolute value at most $j$. The super fillings $\varphi$ satisfying $\Psi\varphi = \varphi$ are those such that any pair $u, v$ of attacking cells satisfies $|\varphi(u)| \neq |\varphi(v)|$. In particular, all the entries in a given row must be different. As a result;

$$\lambda_1 + \cdots + \lambda_j \leq \sum_i \min\{m_i, j\} = \mu'_1 + \cdots + \mu'_j$$

$\square$

*Proof that $C_\mu(\mathbf{x};q,t)$ satisfies (T2).* As in the previous proof, the idea is to find an involution $\Phi$ on super fillings, that cancels out all terms in the sum of equation 3.17, except for those in the expansion of $m_\lambda$ for $\lambda \preceq \mu$. For this proof, the most convenient total ordering of $\mathbb{Z}_\pm$ is the one satisfying $1 <_2 2 <_2 3 <_2 \cdots <_2 -3 <_2 -2 <_2 -1$. Define the involution $\Phi$ on super fillings as the one satisfying the following two conditions for all super fillings $\varphi : \mu \to \mathbb{Z}_\pm$:

1. If each cell $u = (i,j) \in \mu$ satisfies $|\varphi(u)| > j$, define $\Phi\varphi = \varphi$.

2. Otherwise, let $a$ be the smallest integer for which there exists a cell $u = (i,j) \in \mu$ such that $a = |\varphi(u)| \leq j$, and let $u_0 = (i_0, j_0)$ be the first cell in the reading order such that $a = |\varphi(u)|$ (notice that this cell satisfies $a \leq j_0$). Define $\Phi\varphi(u_0) = -\varphi(u_0)$ and $\Phi\varphi(w) = \varphi(w)$ for all $w \neq u_0$.

It is clear from this definition that $|\Phi\varphi| = |\varphi|$, and so

$$\mathbf{x}^{|\Phi\varphi|} = \mathbf{x}^{|\varphi|} \tag{3.25}$$

Consider now a super filling $\varphi$ corresponding to the second case enumerated above. The following relation is clear from the definition of $\Phi$:

$$(-1)^{m(\Phi\varphi)} = -(-1)^{m(\varphi)} \tag{3.26}$$

Since $j_0 \geq a \geq 1$, $u_0$ is not in the bottom row. Let $u_1 = (i_0, j_0 - 1)$ be the cell immediately below $u_0$. If $|\sigma(u_1)| < |\varphi(u_0)| = a$, then from the minimality of $a$; $|\varphi(u_1)| > j_0 - 1$, which is a contradiction, since it implies $a > j_0$. Hence $|\varphi(u_1)|$ is necessarily greater than or equal to $|\varphi(u_0)|$. If $|\varphi(u_1)| > |\varphi(u_0)|$ and $\varphi(u_0) \in \mathbb{Z}_-$ then $J(\varphi(u_0), \varphi(u_1)) = 1$ and $J(\Phi\varphi(u_0), \Phi\varphi(u_1)) = 0$. If $|\varphi(u_1)| > |\varphi(u_0)|$ and $\varphi(u_0) \in \mathbb{Z}_+$ then $J(\varphi(u_0), \varphi(u_1)) = 0$ and $J(\Phi\varphi(u_0), \Phi\varphi(u_1)) = 1$. If $|\varphi(u_1)| = |\varphi(u_0)|$ and $\varphi(u_0) \in \mathbb{Z}_-$ then $J(\varphi(u_0), \varphi(u_1)) = 1$ and $J(\Phi\varphi(u_0), \Phi\varphi(u_1)) = 0$. Finally, if $|\varphi(u_1)| = |\varphi(u_0)|$ and $\varphi(u_0) \in \mathbb{Z}_+$ then $J(\varphi(u_0), \varphi(u_1)) = 0$ and $J(\Phi\varphi(u_0), \Phi\varphi(u_1)) = 1$. In general;

$$J(\Phi\varphi(u_0), \Phi\varphi(u_1)) - J(\varphi(u_0), \varphi(u_1)) = p(\varphi) - p(\Phi\varphi) \tag{3.27}$$

Suppose now that $u_0$ is not in the top row. Let $u_2 = (i_0, j_0 + 1)$ be the cell immediately above $u_0$. Since $a \leq j_0 < j_0 + 1$, then $|\varphi(u_2)|$ must be necessarily greater than $a = |\varphi(u_0)|$ (otherwise it would contradict either the minimality of $a$ or the minimality of $u_0$ in the reading order). If $\varphi(u_0) \in \mathbb{Z}_-$ then $J(\varphi(u_2), \varphi(u_0)) = 0$ and $J(\Phi\varphi(u_2), \Phi\varphi(u_0)) = 1$. Otherwise, if $\varphi(u_0) \in \mathbb{Z}_+$, then $J(\varphi(u_2), \varphi(u_0)) = 1$ and $J(\Phi\varphi(u_2), \Phi\varphi(u_0)) = 0$. In general;

$$J(\Phi\varphi(u_2), \Phi\varphi(u_0)) - J(\varphi(u_2), \varphi(u_0)) = p(\Phi\varphi) - p(\varphi) \tag{3.28}$$

Multiplying equation 3.27 times $(\mathrm{leg}(u_0) + 1)$ and equation 3.28 times $(\mathrm{leg}(u_2) + 1) = \mathrm{leg}(u_0)$, adding up the resulting equalities, and using the fact that all other cells besides $u_0$ have the same entries in $\varphi$ and $\Phi\varphi$, one obtains

$$\sum_{u \in \mu} (\mathrm{leg}(u) + 1)(J(\Phi\varphi(u), \Phi\varphi(u')) - J(\varphi(u), \varphi(u'))) = p(\varphi) - p(\Phi\varphi)$$

where $u'$ denotes the cell immediately below $u$. This equality is equivalent to

$$\mathrm{maj}(\Phi\varphi) + p(\Phi\varphi) = \mathrm{maj}(\varphi) + p(\varphi) \tag{3.29}$$

Notice that this equality holds even if $u_0$ is in the top row. In that case $(\mathrm{leg}(u_0)+1) = 1$ and equation 3.28 is not used.

Multiplying equation 3.27 times $\mathrm{arm}(u_0)$ and equation 3.28 times $\mathrm{arm}(u_2)$, adding up the resulting equalities, and using the fact that all other cells besides $u_0$ are have the same entries in $\varphi$ and $\Phi\varphi$, one obtains

$$\sum_{u\in\mu} \mathrm{arm}(u)(J(\Phi\varphi(u), \Phi\varphi(u')) - J(\varphi(u), \varphi(u'))) = k(p(\Phi\varphi) - p(\varphi)) \tag{3.30}$$

where $k = \mathrm{arm}(u_2) - \mathrm{arm}(u_0)$. In case $u_0$ is in the top row, this result holds for $k = -\mathrm{arm}(u_0)$. To avoid losing generality, define $\mathrm{arm}(u_2) = 0$ when $u_2 \notin \mu$.

Let $u \in \mu$ be any cell such that $u$, $u_0$ attack each other, and $u$ precedes $u_0$ in the reading order. Notice that there are exactly $\mathrm{arm}(u_2) + i_0$ of these cells. Using the same arguments as before, $|\varphi(u)| > |\varphi(u_0)|$, and so

$$J(\Phi\varphi(u), \Phi\varphi(u_0)) - J(\varphi(u), \varphi(u_0)) = p(\Phi\varphi) - p(\varphi) \tag{3.31}$$

Now let $u \in \mu$ be any cell such that $u$, $u_0$ attack each other, and $u_0$ precedes $u$ in the reading order. Notice that there are exactly $\mathrm{arm}(u_0) + i_0$ of these cells. From the same arguments above, $|\varphi(u)| \geq |\varphi(u_0)|$, and so

$$J(\Phi\varphi(u_0), \Phi\varphi(u)) - J(\varphi(u_0), \varphi(u)) = p(\varphi) - p(\Phi\varphi) \tag{3.32}$$

Summing up equations 3.31 and 3.32 over all their corresponding cells $u$, and considering the invariance of all other entries under $\Phi$, one obtains

$$|\mathrm{Inv}(\Phi\varphi)| - |\mathrm{Inv}(\varphi)| = k(p(\Phi\varphi) - p(\varphi)) \tag{3.33}$$

Combining equations 3.30 and 3.33, it results

$$\mathrm{inv}(\Phi\varphi) = \mathrm{inv}(\varphi) \tag{3.34}$$

And combining equations 3.25, 3.26, 3.29 and 3.34, one concludes

$$\alpha(\Phi\varphi) = -\alpha(\varphi)$$

for all super fillings $\varphi$ with $\Phi\varphi \neq \varphi$, which yields

$$C_\mu[X(t-1);q,t] = \sum_{\Phi\varphi=\varphi} (-1)^{m(\varphi)} q^{\mathrm{inv}(\varphi)} t^{p(\varphi)+\mathrm{maj}(\varphi)} \mathbf{x}^{|\varphi|}$$

Now let $\lambda$ be a partition such that $\mathbf{x}^{|\varphi|} = \mathbf{x}^\lambda = x_i^{\lambda_1} \cdots x_l^{\lambda_l}$ for some super filling $\varphi : \mu \to \mathbb{Z}_\pm$ satisfying $\Phi\varphi = \varphi$. $\lambda_1 + \cdots + \lambda_k$ is the number of entries in the filling $|\varphi|$ that are not greater than $k$. Since $|\varphi(i,j)| \leq j$ for all $(i,j) \in \mu$, these entries must be in the first $k$ rows of $\mu$, and consequently

$$\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + m_k$$

□

This concludes the proof of theorem 3.1.1. The following are some interesting specializations of Macdonald polynomials

$$H_\mu(\mathbf{x};0,0) = h_n = s_{(n)} \quad (n = |\mu|) \tag{3.35}$$

$$H_\mu(\mathbf{x};1,0) = h_{\mu'} \tag{3.36}$$

$$H_\mu(\mathbf{x};1,1) = (x_1 + x_2 + \cdots)^n \tag{3.37}$$

$H_\mu(\mathbf{x};0,0)$ sums $\mathbf{x}^\varphi$ over all fillings $\varphi$ that have no descents and no inversions. This is, all fillings that are weakly increasing in the reading order. This observation proves equation 3.35.

$H_\mu(\mathbf{x};1,0)$ sums $\mathbf{x}^\varphi$ over all fillings $\varphi$ that have no descents. This is, all fillings that are weakly increasing in the reading order on every column, which clearly implies equation 3.36.

$H_\mu(\mathbf{x};1,1)$ is the sum of $\mathbf{x}^\varphi$ over all fillings $\varphi : \mu \to \mathbb{Z}_+$. This is clearly equal to $(x_1 + x_2 + \cdots)^n$. A more algebraic approach (see (Macdonald, 1995)) produces

$$H_\mu(\mathbf{x};0,1) = h_\mu \tag{3.38}$$

Some general properties of Macdonald polynomials that follow from their definition (see (Macdonald, 1995) for properties of the $P_\mu$'s and their corresponding properties of Macdonald polynomials) and will be useful later, are;

$$H_{\mu'}(\mathbf{x}; q, t) = H_\mu(\mathbf{x}; t, q) \tag{3.39}$$

$$H_\mu(\mathbf{x}; q, t) = q^{n(\mu')} t^{n(\mu)} \omega H_\mu(\mathbf{x}; q^{-1}, t^{-1}) \tag{3.40}$$

## 3.2 $q, t$-Kostka polynomials

The *$q, t$-Kostka polynomials* are the coefficients that appear upon expressing the Macdonald polynomial $H_\mu$ in terms of Schur polynomials:

**Definition 3.2.1.** Let $\mu$ be a partition of size $n$. The $q, t$-Kostka coefficients $K_{\lambda\mu}(q, t)$ $(\lambda \vdash n)$ are the ones appearing in the expression:

$$H_\mu(\mathbf{x}; q, t) = \sum_{\lambda \vdash n} K_{\lambda\mu}(q, t) s_\lambda(\mathbf{x})$$

From equation 3.2, it is immediate that the coefficient of $s_\lambda$ in the expansion of any symmetric polynomial $f(\mathbf{x})$ as a linear combination of Schur polynomials, is equal to $\langle f, s_\lambda \rangle$. It follows that

$$K_{\mu\lambda}(q, t) = \langle H_\mu(\mathbf{x}; q, t), s_\lambda(\mathbf{x}) \rangle \tag{3.41}$$

Equation 3.35 gives

$$K_{\lambda\mu}(0, 0) = \delta_{\lambda(n)} \quad (n = |\mu|) \tag{3.42}$$

Equations 2.28 and 3.3 show that $\langle s_\lambda, h_\mu \rangle = K_{\lambda\mu}$ and so the coefficient of $s_\lambda$ in the expansion of $h_\mu$ in terms of Schur polynomials is $K_{\lambda\mu}$. Thus from 3.38 and 3.36:

$$K_{\lambda\mu}(0, 1) = K_{\lambda\mu} \tag{3.43}$$

$$K_{\lambda\mu}(1, 0) = K_{\lambda\mu'} \tag{3.44}$$

A bijective argument using the RSK algorithm produces;

$$K_{\lambda\mu}(1, 1) = \langle (x_1 + x_2 + \cdots)^n, s_\lambda \rangle = f^\lambda \tag{3.45}$$

Indeed, this is a consequence of the equivalent relation

$$(x_1 + x_2 + \cdots)^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda \tag{3.46}$$

which follows from observing that the left hand side is the "weighed" sum of the lexicographic words $\binom{\mathbf{a}}{\mathbf{b}}$ with $\mathbf{a} = (1, \ldots, n)$, and the right hand side is the "weighed" sum of all pairs of a standard Young tableau and a semi standard Young tableau of the same shape $\lambda \vdash n$.

Equations 3.39 and 3.40 produce:

$$K_{\lambda\mu'}(q,t) = K_{\lambda\mu}(t,q) \tag{3.47}$$

$$K_{\lambda\mu}(q,t) = q^{n(\mu')}t^{n(\mu)}K_{\lambda'\mu}(q^{-1}, t^{-1}) \tag{3.48}$$

The polynomials $K_{\lambda\mu}(q,t)$ have positive integer coefficients. This integrality property has been proven using different methods (see (Bergeron, 2008) for a list of references), while the positivity is a result of the Schur-positivity of LLT polynomials, recently proven in (Assaf, 2007) and (Grojnowski, Haiman, 2007). The proof in (Assaf, 2007) offers a combinatorial interpretation of the coefficients of LLT polynomials in terms of Schur polynomials, which combined with equation 3.8, produces a purely combinatorial proof of the fact that $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

## 3.3 Cocharge Statistic

In (Lascoux, Schützenberger, 1978), the authors proved a combinatorial formula for the expression of the Hall-Littlewood symmetric polynomial $H_\mu(\mathbf{x}; 0, t)$ as a linear combinatio of Schur polynomials, namely;

**Theorem 3.3.1.** (Lascoux, Schützenberger, 1978) *The Hall-Littlewood symmetric polynomial $H_\mu(\mathbf{x}; 0, t)$ is given in terms of Schur polynomials as follows;*

$$H_\mu(\mathbf{x}; 0, t) = \sum_{\lambda \text{ partition}} \left( \sum_{\tau \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{cc}(\tau)} \right) s_\lambda(\mathbf{x}) \tag{3.49}$$

*where* $\mathrm{SSYT}(\lambda,\mu)$ *is the set of semi standard Young tableaux that have exactly* $\mu_i$ *entries* $i$ *for* $1 \leq i \leq l(\mu)$ *(notice that* $|\mathrm{SSYT}(\lambda,\mu)| = K_{\lambda\mu}$*), and* $\mathrm{cc}(\tau)$ *is the* cocharge *statistic on semi standard Young tableaux.*

Before explaining what the *cocharge* statistic is, it is important to mention that this theorem provides another proof of equations 3.38 and 3.43.

The reason why the description of the cocharge statistic was not included in chapter 1 is that, because of its complicated definition, it would disturb the simplicity of the other statistics presented in the same chapter. In (Haglund, Haiman, Loehr, 2005)'s own words, this statistic now emerges naturally from the simpler concepts of inversions and descents of a filling.

**Definition 3.3.1.** Given any filling $\varphi : \lambda \to \mathbb{Z}_+$, define the *reading word* of $\varphi$, denoted $w(\varphi)$ as the word $w : \{1, \ldots, |\lambda|\} \to \mathbb{Z}_+$ that results from writing the entries of $\varphi$ in the reading order of the cells of $\lambda$.

**Definition 3.3.2.** For any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, define $\mathrm{DR}(\sigma)$ as follows:

$$\mathrm{DR}(\sigma) = \{i \in \{1, \ldots, n-1\} : \sigma(i) > \sigma(i+1)\} \tag{3.50}$$

This is equivalent to the reading descent set $\mathrm{DR}(\xi)$ (defined in the first chapter for bijective fillings), when $\sigma = w(\xi)$.

**Definition 3.3.3.** Let $w : \{1, \ldots, n\} \to \mathbb{Z}_+$ be any word. For a subset $S \subseteq \{1, \ldots, n\}$ define the *subword* $w_S$ as the one that results of considering only the entries $w(i)$ for $i \in S$. For instance, if $w = 162417$, then $w_{\{1,3,5,6\}} = 1217$.

**Definition 3.3.4.** Let $w : \{1, \ldots, n\} \to \mathbb{Z}_+$ be any word. Define the subset $S = \{k_1, \ldots, k_r\} \subseteq \{1, \ldots, n\}$ inductively as follows:

$$k_1 = \max\{k : w(k) = 1\}$$

$$k_{i+1} = \begin{cases} \max\{k < k_i : w(k) = i+1\} & \text{if } \{k < k_i : w(k) = i+1\} \neq \emptyset \\ \max\{k : w(k) = i+1\} & \text{if } \{k < k_i : w(k) = i+1\} = \emptyset \end{cases}$$

The inductive process ends when $k_{r+1}$ can not be defined. The *cocharge* of the word $w$ is given inductively by

$$\mathrm{cc}(w) = \mathrm{cc}(w_S) + \mathrm{cc}(w_{\{1,\dots,n\}\setminus S})$$

$$\mathrm{cc}(\sigma) = \sum_{i \in \mathrm{DR}(\sigma)} m - i \quad \text{if } \sigma \text{ is a permutation of } m$$

Notice that the cocharge of a word is well defined if it has more $i$'s than (or as many $i$'s as) $(i+1)$'s for all $i \geq 1$, since $w_S$ (as in the definition) is always a permutation and $w_{\{1,\dots,n\}\setminus S}$ satisfies this condition as long as $w$ does.

**Definition 3.3.5.** Let $\varphi : \lambda \to \mathbb{Z}_+$ be a filling of a partition $\lambda$. The *cocharge* of $\varphi$ is simply given by

$$\mathrm{cc}(\varphi) = \mathrm{cc}(w(\varphi))$$

Haglund, Haiman and Loehr's proof of theorem 3.3.1 is achieved by rewriting the right hand side of equation 3.49 as follows:

$$\sum_{\lambda \text{ partition}} \left( \sum_{\tau \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{cc}(\tau)} \right) \left( \sum_{\rho \in \mathrm{SSYT}(\lambda)} \mathbf{x}^{\rho} \right) \tag{3.51}$$

Which reduces the proof to finding a bijection between pairs of semi standard Young tableaux $(\tau, \rho) \in \bigcup_{\lambda \vdash n = |\mu|} \mathrm{SSYT}(\lambda, \mu) \times \mathrm{SSYT}(\lambda)$ and fillings $\varphi : \mu \to \mathbb{Z}_+$ with $\mathrm{inv}(\varphi) = 0$, such that the multiset of entries of $\varphi$ is the same as that of $\rho$, and $\mathrm{maj}(\varphi) = \mathrm{cc}(\tau)$.

It can be proven that for any word $w$ that is a reordering of $1^{\mu_1} 2^{\mu_2} \cdots k^{\mu_k}$ for some partition $\mu = (\mu_1, \dots, \mu_k)$, the cocharge of $w$ is equal to the cocharge of $\varepsilon \leftarrow w$, where $\varepsilon$ is the empty filling. Thus the RSK correspondence gives a natural way of rewriting expression 3.51:

$$\sum_{\lambda \text{ partition}} \left( \sum_{\tau \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{cc}(\tau)} \right) \left( \sum_{\rho \in \mathrm{SSYT}(\lambda)} \mathbf{x}^{\rho} \right) = \sum_{\binom{\mathbf{a}}{\mathbf{b}}} t^{\mathrm{cc}(\mathbf{b})} \mathbf{x}^{\mathbf{a}} \tag{3.52}$$

where $\mathbf{x}^{\mathbf{a}} = x_{a_1} \cdots x_{a_n}$ for $\mathbf{a} = (a_1, \dots, a_n)$. The sum on the right hand side is over all lexicographic words $\binom{\mathbf{a}}{\mathbf{b}}$ with positive integer entries such that $\mathbf{b}$ is a reordering of

$1^{\mu_1}2^{\mu_2}\cdots k^{\mu_k}$ ($k = l(\mu)$). Haglund proved that for any $l(\mu) = k$ multisets $M_1, \ldots, M_k$ with $|M_i| = \mu_i$ ($1 \leq i \leq k$), there is one unique filling $\varphi : \mu \to \mathbb{Z}_+$ under the conditions that $\mathrm{inv}(\varphi) = 0$ and that the entries of $\varphi$ in the $i$th row are the elements of $M_i$ for $i = 1, \ldots, k$. Thus for any such lexicographic word $\binom{\mathbf{a}}{\mathbf{b}}$ there is one unique filling $\varphi$ with $\mathrm{inv}(\varphi) = 0$ and such that the entries of $\varphi$ in the $i$th row are the $a_j$'s for which $b_j = i$. The proof of theorem 3.3.1 follows then from an inductive argument that shows that the relation

$$\mathrm{maj}(\varphi) = \mathrm{cc}(\mathbf{b}) \tag{3.53}$$

holds for said unique filling $\varphi$.

## 3.4 Algebraic Approach

The most remarkable non-combinatorial applications of the theory of Macdonald polynomials appear in the theory of representations of the symmetric group. Refer to (Vargas, 2008) for an almost purely algebraic outline of the most important known results concerning coinvariant spaces and the so called Garsia-Haiman modules. While seeking to prove the integrality of $K_{\lambda\mu}(q,t)$, some expressions involving Macdonald polynomials, such as equation 3.37 motivated Garsia and Haiman (Garsia, Haiman, 1996b) to find an interpretation of $H_\mu$ in terms of the representation theory of some bigraded $\mathfrak{S}_n$-modules. One of the most important results of this work, proven almost ten years after it was first conjectured by Garsia and Haiman (Garsia, Haiman, 1993) is that

$$\mathrm{Frob}_{qt}(\mathcal{H}_\mu) = H_\mu \tag{3.54}$$

Where $\mathrm{Frob}_{qt}$ is the bigraded Frobenius characteristic and $\mathcal{H}_\mu$ is the space defined below in this same section. An interesting consequence of this result is that the dimension of $\mathcal{H}_\mu$ is $n!$. This became known for a while as the $n!$ *conjecture*. Although apparently a simple result, it is still awaiting an elementary proof.

The large amount of research in the subject makes it impossible to condense all the important results in one sketch. For this reason, and in order to keep an elementary approach, all is given here is a basic introduction to what was previously known as the $n!$ conjecture.

For a partition $\mu \vdash n$, define the determinant;

$$\Delta_\mu(\mathbf{x},\mathbf{y}) = \det(x_k^j y_k^i)_{\substack{1\le k\le n \\ (i,j)\in\mu}} \tag{3.55}$$

Notice that every entry of the matrix on the right hand side is indexed by a pair $(k,(i,j))$ and so in order for this definition to make sense it is necessary to use a particular order for the cells of $\mu$. The lexicographic order is the most commonly used, even though what follows does not depend on the chosen ordering. For $\mu = (2,2,1)$, one has

$$\Delta_{(2,2,1)}(\mathbf{x},\mathbf{y}) = \det \begin{pmatrix} 1 & x_1 & x_1^2 & y_1 & x_1y_1 \\ 1 & x_2 & x_2^2 & y_2 & x_2y_2 \\ 1 & x_2 & x_2^2 & y_2 & x_2y_2 \\ 1 & x_2 & x_2^2 & y_2 & x_2y_2 \\ 1 & x_2 & x_2^2 & y_2 & x_2y_2 \end{pmatrix} \tag{3.56}$$

In the cases $\mu = (n)$ and $\mu = (1^n)$, this is equivalent to the Vandermonde determinant. Define the operators $\partial_{x_i}$ and $\partial_{y_j}$ as the partial derivative operators given by

$$\partial_{x_i} g(\mathbf{x},\mathbf{y}) = \frac{\partial g}{\partial x_i}(\mathbf{x},\mathbf{y})$$

$$\partial_{y_j} g(\mathbf{x},\mathbf{y}) = \frac{\partial g}{\partial y_j}(\mathbf{x},\mathbf{y})$$

and set $\partial\mathbf{x} = (\partial_{x_1},\partial_{x_2},\ldots)$, $\partial\mathbf{y} = (\partial_{y_1},\partial_{y_2},\ldots)$. The now proven (Haiman 2001) $n!$ conjecture states the following:

**Theorem 3.4.1** (n! theorem). *The space $\mathcal{H}_\mu$, defined by*

$$\mathcal{H}_\mu = \{f(\partial\mathbf{x},\partial\mathbf{y})\Delta_\mu(\mathbf{x},\mathbf{y}) : f(\mathbf{x},\mathbf{y}) \in \mathbb{C}[\mathbf{x},\mathbf{y}]\} \tag{3.57}$$

*has dimension $n!$.*

Haiman's proof of this theorem is far from elementary, but it shows stronger facts, which among other results, imply that the $q,t$-Kostka coefficients have positive integer coefficients. An explicit basis of $\mathcal{H}_\mu$ is yet to be found.

### Nabla Operator

The linear *nabla operator* $\nabla$ on symmetric polynomials is given in terms of its action on Macdonald polynomials $H_\mu(\mathbf{x}; q, t)$:

$$\nabla(H_\mu) = q^{n(\mu')} t^{n(\mu)} H_\mu \tag{3.58}$$

Since Macdonald polynomials generate the space of symmetric polynomials with coefficients in $\mathbb{C}(q,t)$, equation 3.58 defines $\nabla(f)$ for any symmetric polynomial $f$.

The application of $\nabla$ to the families of symmetric functions studied in the previous chapter, yield some interesting results (see (Loehr, Warrington, 2008) for a list of them), and some important conjectures (see chapter 4) have been established.

The nabla operator was introduced in (Bergeron, Garsia, 1999; Bergeron, Garsia, Haiman, Tesler, 1999) as a way to simplify some expressions that appear in the Frobenius characteristics of the intersection of Garsia-Haiman modules. Thus it plays an important role in the algebraic approach of this theory. One of the most important relations is the fact that $\nabla(e_n)$ is the Frobenius characteristic of the *diagonal coinvariant ring* (Haiman 2003), i.e., the one defined by $D_n = \mathbb{C}[\mathbf{x}, \mathbf{y}]/I$ where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ and $I$ is the ideal of polynomials $f(\mathbf{x}, \mathbf{y})$ with no constant term that are invariant under the diagonal action of the symmetric group given by $\sigma f(\mathbf{x}, \mathbf{y}) = f(\sigma\mathbf{x}, \sigma\mathbf{y})$ for $\sigma \in \mathfrak{S}_n$. This in particular was shown to imply that the dimension of the diagonal coinvariant ring is equal to $(n+1)^{n-1}$, which was previously known as the $(n+1)^{n-1}$ *conjecture*. Haiman's proof of this result uses the same techniques he developed to prove the $n!$ conjecture.

## 3.5 $q,t$-Catalan numbers

The $q,t$-Catalan polynomials (Garsia, Haiman 1996a) appear as the Hilbert series of the antisymmetric part $R_n^\epsilon$ of the diagonal coinvariant ring. They are given in terms of the nabla operator by

$$C_n(q,t) = \langle e_n, \nabla e_n \rangle \tag{3.59}$$

A remarkable combinatorial formula for $C_n(q,t)$ was shown to hold by Haglund and Garsia (Garsia, Haglund, 2002), giving an elementary proof of the positivity and integrality of its coefficients, as well as directly showing that it is a $t$-analog of the Carlitz-Riordan $q$-Catalan numbers.

**Theorem 3.5.1.** ((Garsia, Haglund, 2002), reformulated as in (Haiman 2003)) *Denote by $b(\lambda)$ the number of cells $u$ in a partition $\lambda$ for which*

$$\mathrm{leg}(u) \leq \mathrm{arm}(u) \leq \mathrm{leg}(u) + 1$$

*(see figure 3.1). Let $\delta_n$ ($n = |\lambda|$) be the* staircase partition $(n-1, n-2, \ldots, 1)$. *Then the $q,t$-Catalan polynomial is equal to the sum*

$$C_n(q,t) = \sum_{\lambda \subseteq \delta_n} q^{\binom{n}{2} - |\lambda|} t^{b(\lambda)} \tag{3.60}$$

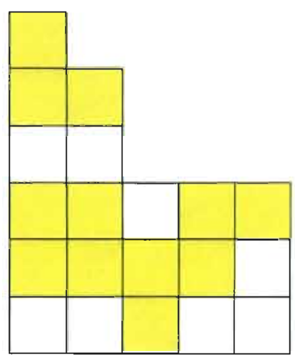


**Figure 3.1** Cells $u$ such that $\mathrm{leg}(u) \leq \mathrm{arm}(u) \leq \mathrm{leg}(u) + 1$.

The symmetry

$$C_n(q,t) = C_n(t,q) \tag{3.61}$$

can be easily shown to be true from the original definition (equation 3.59). However, no simple bijection is known to prove this equality from the combinatorial formula 3.60.

# CHAPTER IV

# FURTHER DEVELOPMENT AND OPEN PROBLEMS

These are some of the most important conjectures and open problems that have been proposed in the theory of Macdonald polynomials. Some of them are purely combinatorial while others require some algebraic concepts for their understanding. Some of the statements include terms that have not been defined in this work, for which the reader is advised to refer to the cited literature.

1. Finding elementary proofs of the $n!$ and $(n+1)^{n-1}$ conjectures.

2. (Bergeron, Garsia, Haiman, Tesler, 1999) For all $\lambda$ and $\mu$, the polynomial $(-1)^{\kappa(\lambda')} \langle \nabla(s_\lambda), s_\mu \rangle$ has positive integer coefficients, with $\kappa(\lambda) = \binom{l(\lambda)}{2} + \sum_{\lambda_i < (i-1)} (i - 1 - \lambda_i)$.

3. (Bergeron, Bergeron, Garsia, Haiman, Tesler, 1999) For all partitions $\mu$ and all cells $(i,j) \in \mu$, the bigraded Frobenius characteristic of the space $\mathcal{H}_{\mu \setminus \{(i,j)\}}$ is given by the symmetric function $H_{\mu/ij}$.

4. (Haglund, Haiman, Loehr, Remmel, Ulyanov, 2005) A combinatorial formula for $\nabla(e_n)$:

$$\nabla(e_n(\mathbf{x})) = \sum_{\lambda \subseteq \delta_n} \sum_{\tau \in \mathrm{SSYT}((\lambda+1^n)\setminus\lambda)} t^{|\delta_n \setminus \lambda|} q^{\mathrm{dinv}(\tau)} \mathbf{x}^\tau$$

where $\lambda + 1^n = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_n + 1)$ and $\mathrm{dinv}(\tau)$ counts the number of *d-inversions* of $\tau$, i.e., the number of pairs of cells $((i,j),(k,l))$ such that $\tau(i,j) > \tau(k,l)$ and either

   (a) $k + l = i + j$ and $i < k$, or,

(b) $k+l=i+j-1$ and $i>k$.

5. Define the "higher" $q,t$-Catalan numbers by the formula

$$C_n^{(m)}(q,t) = \langle e_n, \nabla^m e_n \rangle$$

Then they are given by

$$C_n^{(m)}(q,t) = \sum_{\lambda \subseteq m\delta_n} q^{m\binom{n}{2}-|\lambda|} t^{b^{(m)}(\lambda)}$$

where $m\delta_n = (m(n-1), m(n-2), \ldots, m)$ and $b^{(m)}(\lambda)$ is the number of cells $u \in \lambda$ such that $\mathrm{leg}(u) \le \mathrm{arm}(u) \le \mathrm{leg}(u) + m$.

6. (Haglund, Haiman, Loehr, Remmel, Ulyanov, 2005) A combinatorial formula for $\langle e_1^n, \nabla(e_n)\rangle$ (known to be the Hilbert series of the diagonal coinvariant ring) indexed by *parking functions*:

$$D_n(q,t) = \langle e_1^n, \nabla(e_n)\rangle = \sum_f q^{w(f)} t^{\mathrm{dinv}(f)}$$

The index $f$ in the sum varies over all *parking functions* on $\{1,\ldots,n\}$. A parking function is simply a function $f : \{1,\ldots,n\} \to \{1,\ldots,n\}$ satisfying

$$|f^{-1}(\{1,\ldots,k\})| \ge k \text{ for } k = 1,\ldots,n$$

The *weight* of $f$ is given by $w(f) = \binom{n+1}{2} - \sum_i f_i$. It is easy to show that every parking function on $\{1,\ldots,n\}$ is in unique correspondence with a standard Young tableau $\tau$ of $(\lambda + 1^n)\backslash\lambda$ for some $\lambda \subseteq \delta_n$, such that $f(i)$ is the column occupied by the entry $i$ in $\tau$. This said, the parameter $\mathrm{dinv}(f)$ is simply $\mathrm{dinv}(\tau)$. See (Haiman 2003).

7. A general combinatorial formula for $\nabla s_\lambda$ in terms of nested labelled *Dyck paths* was very recently conjectured in (Loehr, Warrington, 2008). See (Loehr, Warrington, 2008) for a complete list of proven results and current conjectures involving the nabla operator.

8. Although a purely combinatorial proof that $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$ has already been presented in (Assaf, 2007), it is still an open problem to find a simple combinatorial formula for the $q,t$-Kostka polynomials of the form

$$K_{\lambda\mu}(q,t) = \sum_{\tau \in \mathrm{SYT}(\lambda)} q^{\alpha(\tau,\mu)} t^{\beta(\tau,\mu)}$$

for some statistics $\alpha$ and $\beta$. The existence of such formula is suggested from the fact that $K_{\lambda\mu}(1,1) = f^{\lambda} = |\mathrm{SYT}(\lambda)|$.

The resolution of any of these open problems would lead to remarkable discoveries within the theories of Macdonald Polynomials and Garsia-Haiman modules, and numerous applications to representation theory and several other important research topics in Algebra and Combinatorics.

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