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Le premier chapitre documente les régularités du marché du travail, à la fois, dans les fréquences des cycles d'affaires et dans les fréquences des cycles économiques à moyen terme. Il est montré que les variables du marché du travail sont plus persistantes et sont environ deux fois plus volatiles dans les fréquences des cycles à moyen terme que dans celles des cycles d'affaires. Il est également montré qu'une inversion se produit dans les élasticités de la tension du marché du travail et du salaire par rapport à la productivité du travail, respectivement, dans les fréquences des cycles économiques à moyen terme. Ensuite, pour évaluer la capacité du modèle canonique du marché de travail avec frictions de recherche et d'appariement (ou modèle DMP, en référence aux travaux de Diamond 1982, Mortensen 1982 et Pissarides 1985) à reproduire conjointement les régularités observées dans ces deux fréquences, un ensemble de trois stratégies de calibration couramment utilisées dans la littérature (Shimer 2005 ; Hagedorn and Manovskii 2008 ; Pissarides 2009, respectivement SS, HMS et PS) a été appliqué. Pour cet exercice, les données trimestrielles et annuelles de l'économie américaine ont tour à tour été considérées. Les résultats montrent que le modèle DMP fonctionne relativement mieux dans les fréquences des cycles d'affaires que dans celles à moyen terme en termes de volatilités, en réponse à un choc de productivité d'une ampleur plausible. Autrement dit, ce résultat signifie que la capacité du modèle DMP à reproduire la volatilité du marché du travail observée dans les données réelles s'amenuise dans les fréquences des cycles à moyen terme. Nous trouvons que les performances du modèle DMP diffèrent selon les calibrations. Nous montrons que cibler l'élasticité des salaires par rapport à la productivité du travail (comme dans HMS) ou celle de la tension du marché du travail par rapport à la productivité du travail (comme dans PS) améliore les performances du modèle en termes de volatilités, mais pas en termes de co-mouvements à la fois dans les fréquences des cycles d'affaires et dans celles à moyen terme. Enfin, nous constatons que le modèle DMP ne parvient pas à reproduire avec succès l'inversion des élasticités observée dans les fréquences des cycles économiques à moyen terme.

Le deuxième chapitre est un article co-écrit avec mon directeur de recherche, dans lequel nous identifions et évaluons quantitativement un nouveau mécanisme de propagation et d'amplification des chocs de productivité sur les variables du marché du travail dans un modèle macroéconomique dans lequel la croissance économique à long terme est déterminée de manière endogène par l'investissement en R&D. Nous trouvons que la croissance endogène est cruciale pour expliquer quantitativement la dynamique à moyen terme des variables du marché du travail. L'effet principal dans notre modèle passe par l'interaction entre les frictions sur le marché du travail et les activités de R&D et d'innovation. Nous montrons que la réaction du secteur de la R&D aux chocs de productivité amplifie substantiellement les fluctuations sur le marché du travail, principalement en raison des effets sur les anticipations de l'évolution future de la productivité dans le secteur des biens finaux.

Le troisième chapitre s'inscrit dans la suite du chapitre précédent. Dans ce chapitre, nous étudions les implications, sur la dynamique du marché du travail, d'un processus de croissance économique basé sur la R&D. Nous examinons comment l'introduction du secteur de R&D aide le modèle de recherche et d'appariement du marché du travail (modèle DMP) à tenir compte des principaux faits stylisés sur les fluctuations du marché du travail. Nous montrons qu'un processus de R&D basé sur la connaissance, c'est-à-dire qui utilise le travail des scientifiques comme intrant, entraîne un puissant mécanisme d'amplification des chocs de productivité dans le modèle DMP. En particulier, cela permet au modèle de générer des volatilités dans les variables du marché du travail qui sont consistantes avec celles observées dans les données américaines. En revanche, nous trouvons que ce mécanisme d'amplification est atténué lorsque la R&D n'utilise que des biens finaux comme intrant, comme dans le modèle *lab equipment* (voir Acemoglu 2009, ch. 13). Nous montrons que, dans ce cas, les performances du modèle en termes de volatilité des variables du marché du travail sont médiocres. La principale caractéristique de nos résultats se résume par le fait que la main-d'oeuvre ne se déplace que lentement entre les secteurs de la R&D et de la production du bien final, alors que les dépenses en équipements peuvent s'ajuster immédiatement. Nous montrons analytiquement que ce canal d'amplification est différent et complémentaire au canal de *fundamental surplus* habituel étudié par Ljungqvist et Sargent (2017, *American Economic Review*, 107 (9) : pp. 2630-2665).

Mots-clés : Dynamique du marché du travail ; Croissance endogène ; R&D.

ABSTRACT

This thesis is divided in three chapters. It explores macroeconomic dynamics over the medium-term business cycles with a special focus on the labor market variables. In the first chapter, I document key regularities of the labor market dynamics over both the business cycle frequencies (6-32 quarters) and the medium-term frequencies (6-200 quarters). I show that the standard Diamond–Mortensen–Pissarides (DMP) search-and-matching model of the labor market in its most currently used calibrations cannot successfully account for the main stylized facts concerning these business cycle and medium-term dynamics. In the second chapter, which is a joint work with Pavel Ševčík, we identify and quantitatively evaluate a new propagation and amplification mechanism of productivity shocks on the labor market variables in a medium-term cycles framework with endogenous growth. The main effect goes through interaction between labor market frictions and R&D and innovation activities. We show that the R&D sector reaction to productivity shocks substantially amplifies the fluctuations on the labor market, primarily due to the effects on the expectations of the future evolution of productivity in the final goods sector. The mechanism is especially important for explaining the labor market fluctuations at the medium-term frequencies. In the third chapter, we examine how introduction of the R&D sector helps the DMP model to account for the main stylized facts on the labor market fluctuations. We show that, in order to have a powerful amplification effect enabling the model to reproduce the observed volatilities of the labor market quantities, it is crucial that the R&D sector uses labor as production input. In contrast, this amplification mechanism is muted when R&D necessitates only final goods input, as in the lab equipment setup. The key feature behind our result is the fact that labor only moves slowly between the R&D and production sectors, whereas expenditures on lab equipment can adjust immediately.

Mots-clés: Labor market dynamics; endogenous growth; R&D.

INTRODUCTION

Cette thèse étudie de manière conjointe la dynamique macroéconomique dans les fréquences des cycles d'affaires (6–32 trimestres) et dans les fréquences des cycles économiques à moyen terme (6–200 trimestres), en accordant une attention particulière aux variables du marché du travail. Elle se structure autour de trois chapitres.

Dans le premier chapitre intitulé *The Medium-Term Cyclical Behavior of Equilibrium Unemployment and Vacancies*, nous adressons deux questions principales. La première est: Quelles sont les principales régularités du marché du travail dans les fréquences des cycles d'affaires et dans celles à moyen terme? La deuxième est: Une version calibrée du modèle canonique du marché de travail avec frictions de recherche et d'appariement (ou modèle DMP, en référence aux travaux de Diamond 1982, Mortensen 1982, et Pissarides 1985) peut-elle reproduire ces régularités? Pour répondre à ces questions, nous documentons les régularités du marché du travail dans ces deux fréquences, en utilisant tour à tour les données trimestrielles et annuelles de l'économie américaine. Nous montrons que les variables du marché du travail sont plus persistantes et sont environ deux fois plus volatiles dans les fréquences des cycles à moyen terme que dans les fréquences des cycles d'affaires. Nous documentons également une inversion dans les élasticités de la tension du marché du travail et du salaire par rapport à la productivité du travail, respectivement, dans les fréquences des cycles économiques à moyen terme. Ensuite, en utilisant trois stratégies de calibration différentes (Shimer 2005; Hagedorn et Manovskii 2008; Pissarides 2009, respectivement SS, HMS et PS), nous montrons qu'en termes de volatilités, le modèle DMP est relativement plus performant dans les fréquences des cycles d'affaires que dans celles à moyen terme. Lorsque nous comparons les performances du modèle DMP sous les différentes calibrations considérées, nous trouvons que le modèle est capable de générer des volatilités plus proches de celles observées dans les données empiriques, à la fois dans les fréquences des cycles d'affaires que dans celles à moyen terme, sous les calibrations HMS et PS que sous la calibration SS. En termes d'autocorrélations de premier ordre, le modèle DMP sous les trois calibrations (HMS, PS et SS) arrive à reproduire relativement bien ces moments empiriques dans les fréquences des cycles d'affaires et dans celles à moyen terme. Cependant, nous constatons qu'il ne parvient pas à reproduire avec succès les autocorrélogrammes plus complets des variables clés du marché du travail tels que le chômage et les postes vacants. Nous trouvons également que le modèle échoue à reproduire les comouvements entre la productivité du travail et la tension sur le marché du travail, à la fois, dans les fréquences des cycles d'affaires que dans celles à moyen terme. Nous montrons que cibler l'élasticité du salaire par rapport à la productivité du travail (comme c'est le cas avec la calibration HMS) ou cibler l'élasticité de la tension du marché du travail par rapport à la productivité du travail (comme c'est le cas avec la calibration PS) améliorent les performances du modèle en termes de volatilités, mais pas en termes de comouvements. Ce résultat est en partie cohérent avec l'argument selon lequel le modèle de recherche et d'appariement sur le marché du travail peut générer une volatilité élevée dans les variables du marché du travail même en présence d'une forte volatilité des salaires s'il est calibré d'une manière appropriée (voir Hagedorn et Manovskii 2008, p. 1695). Enfin, nous constatons que le modèle DMP ne parvient pas à reproduire l'inversion des élasticités observée sur données américain dans les fréquences des cycles économiques à moyen.

Dans le deuxième chapitre intitulé *Medium-term Business Cycles and Labor Market Search*, nous examinons la question suivante: Comment les activités de R&D, l'innovation et les progrès technologiques qui en découlent affectent la dynamique du marché du travail dans les fréquences des cycles d'affaires et dans celles à moyen terme? Pour répondre à cette question, nous documentons d'abord empiriquement la dynamique de l'investissement en R&D et des principales variables macroéconomiques dans ces deux fréquences. Nous trouvons que la R&D, la productivité du travail, les postes vacants, la tension du marché du travail et les salaires sont procycliques dans les fréquences des cycles d'affaires et dans celles à moyen terme, tandis que le chômage et les séparations d'emploi sont contracycliques. Nous trouvons aussi que la R&D affiche une corrélation contemporaine statistiquement et économiquement significative avec certaines variables clés du marché du travail, notamment le chômage, les postes vacants, la tension du marché du travail et les salaires dans les fréquences du cycle économique et dans celles à moyen terme. Nous documentons aussi une avance (*lead*) de la R&D sur la productivité du travail, les salaires et le chômage dans les fréquences des cycles économiques à moyen terme. Cette avance est absente dans les fréquences des cycles d'affaires. Ces évidences empiriques suggèrent que le fait de se concentrer uniquement sur les fluctuations qui surviennent dans les périodicités comprises entre 6 et 32 trimestres peut éclipser une partie substantielle de la dynamique liant le marché du travail et l'activité économique globale. Par ailleurs, en s'inscrivant dans la lignée des travaux de Comin and Gertler (2006) et Comin (2009), ces évidences confortent l'idée que les sources des fluctuations économiques à court terme sont interconnectées avec les sources de la croissance économique de long terme. Partant de ces faits stylisés, nous construisons un modèle de croissance endogène avec des frictions de recherche et d'appariement sur le marché du travail. Nous utilisons ce modèle pour comprendre le rôle des interactions entre ces frictions, les déterminants de la croissance économique de long terme et les chocs de productivité totale des facteurs (TFP) dans l'explication de la dynamique du marché du travail. Comparé à un modèle où la croissance est exogène, nous trouvons que l'introduction du marché du travail spécialisé en R&D accroît l'élasticité de la tension du marché du travail par rapport à la productivité. Cela permet à la version stochastique du modèle de mieux propager et d'amplifier considérablement les effets des chocs de TFP sur les variables du marché du travail. Par rapport aux mécanismes de transmission classiques étudiés dans la littérature sur la dynamique du marché du travail en présence des frictions de recherche et d'appariement, notre modèle présente un nouveau canal. Une augmentation à la période courante de la TFP dans le secteur du bien final augmente la productivité future des biens intermédiaires et donc la valeur de l'innovation. Cela pousse à son tour les innovateurs à exiger aujourd'hui plus de la main-d'oeuvre spécialisée. En conséquence, à l'équilibre, davantage de travailleurs vont se spécialiser dans les activités de la R&D au lieu de chercher un emploi dans le secteur du bien final, ce qui entraîne une baisse du chômage à la période courante. La découverte plus importante de nouvelles variétés d'intrants intermédiaires qui en résulte dans le secteur de R&D améliore encore la productivité dans le secteur du bien final au cours des périodes suivantes, déclenchant ainsi un puissant effet de rétroaction qui conduit à des réponses persistantes aux fluctuations de la productivité. Grâce à ce canal, une version calibrée de notre modèle est capable de générer une volatilité et une persistance quantitativement adéquates dans les variables du marché du travail mais aussi dans d'autres variables macroéconomiques standard dans les fréquences des cycles d'affaires et dans celles à moyen terme en réponse à des chocs de TFP d'une ampleur réaliste.

Dans le troisième chapitre intitulé *Labor Market Dynamics and R&D-Based Growth*, nous étudions la dynamique du marché du travail dans un cadre similaire au modèle standard des cycles d'affaires réels (modèle RBC), mais qui intègre en outre les frictions de recherche et d'appariement et un processus de croissance basé sur la R&D. Dans une étude influente, Cogley et Nason (1995) ont montré que de nombreux modèles RBC ont de faibles mécanismes de propagation internes et doivent s'appuyer sur des sources externes de dynamique pour reproduire la dynamique de la production globale observée aux états-Unis. Merz (1995) et Andolfatto (1996) soutiennent que le modèle de recherche et d'appariement sur le marché du travail génère un mécanisme de propagation quantitativement important des fluctuations économiques dans le contexte des modèles RBC. Cependant, Shimer (2005) a montré qu'en réponse à des chocs de productivité d'une ampleur plausible, un modèle de recherche et d'appariement sur le marché du travail génère trop peu de volatilité dans les variables du marché du travail par rapport à celle qui est observée sur données réelles. Une vaste littérature ultérieure a proposé plusieurs modifications et stratégies de calibration alternatives pour aider le modèle de base à rendre compte de ces volatilités. Ljungqvist et Sargent (2017, 2021) montrent qu'une grande partie de ces extensions se résument à un seul mécanisme commun qui consiste à réduire le surplus généré par un emploi, lequel fait l'objet du partage entre une firme et un travailleur. Le présent chapitre apporte trois contributions principales à la littérature existante. Premièrement, nous montrons qu'un modèle de recherche RBC avec un processus de croissance basé sur la R&D peut rendre compte de la dynamique observée du marché du travail. Deuxièmement, nous examinons formellement dans quelle mesure la manière dont le secteur de R&D est introduit influence la capacité du modèle de recherche et d'appariement à amplifier les effets des chocs de productivité sur les variables du marché du travail. Nous montrons que, pour disposer d'un puissant mécanisme d'amplification permettant au modèle de reproduire la volatilité observée sur le marché du travail, il est crucial que le modèle comporte un processus de R&D basé sur la connaissance. Autrement dit, le travail doit être un intrant important dans la production d'innovation. En revanche, lorsque la croissance est basée sur un processus qui utilise les biens finaux comme seul intrant de l'activité de R&D, le modèle devient incapable de reproduire avec succès la volatilité du marché du travail. La clé pour comprendre ce résultat est le fait que la main-d'oeuvre circule lentement entre les secteurs de la R&D et de la production. Troisièmement, nous dérivons une expression analytique de l'élasticité de la tension du marché du travail par rapport à la productivité dans le cas où le processus de R&D est basé sur la connaissance. Cette expression analytique montre clairement que le mécanisme exploité dans notre modèle est différent et complémentaire de celui qui a été étudié par Ljungqvist et Sargent (2017, 2021). Nous montrons que les effets de la R&D sur le marché du travail ne passent pas par la réduction du « surplus fondamental » créé par un emploi. Ils introduisent plutôt un terme supplémentaire, à valeur positive, dans l'expression de l'élasticité de la tension du marché du travail par rapport à la productivité.

CHAPTER I

THE MEDIUM-TERM CYCLICAL BEHAVIOR OF EQUILIBRIUM UNEMPLOYMENT AND VACANCIES

ABSTRACT

This paper investigates the labor market dynamics at both business cycle and medium-term frequencies. I first show that labor market variables are more persistent and are approximately twice as volatile over the medium-term cycle than over the business cycle. I also document an inversion in the labor productivity elasticities of the vacancy–unemployment ratio and wage, respectively, over the medium-term cycle. Then, I put the textbook Diamond–Mortensen–Pissarides (DMP) model to the twin test of generating the observed business cycle and medium-term fluctuations in labor market variables in response to a productivity shock of plausible magnitude. I apply three variants of calibrations: Shimer's (2005) calibration strategy, a small surplus calibration approach proposed by Hagedorn and Manovskii (2008), and a calibration approach that uses a fixed matching cost as in Pissarides (2009). I find that the DMP model performs better over the business cycles than over the medium-term cycles in terms of volatilities and fails to reproduce the "elasticities inversion" in all three calibrations over the medium-term cycle.

Keywords: Business cycles; Medium-term; Labor market dynamics.
JEL Codes: E24, E32.

1.1 Introduction

This paper studies labor market dynamics over both business cycle and medium-term frequencies. The canonical search-and-matching model has been criticized for failing to generate enough volatility in labor market variables in response to shocks of a plausible magnitude over business cycles (Costain and Reiter 2003, 2008; Shimer 2005) which is referred to as the so-called "labor market puzzle." The two most popular solutions that have been proposed to solve this puzzle are (i) the small surplus calibration approach (Hagedorn and Manovskii, 2008); and (ii) the introduction of a fixed matching cost (Pissarides, 2009).[1] One way to look at the volatility puzzle is to compute the elasticity of labor market tightness with respect to the aggregate productivity.[2] In data, this elasticity can be computed as $(\sigma_\theta/\sigma_p) \times \text{corr}(\theta, p)$, where σ_θ is the standard deviation of labor market tightness, σ_p is the standard deviation of labor productivity, and $\text{corr}(\theta, p)$ is

[1]Various other solutions have been proposed in the literature to account for labor market volatility over the business cycles, including: upfront costs for firms to secure credit (Wasmer, 2004); sticky wage (Hall, 2005a); one-the-job search (Mortensen and Nagypál, 2007); cost of delay for firms that participate in alternating-offer bargaining (Hall and Milgrom, 2008); staggered Nash wage bargaining (Gertler and Trigari, 2009); frictional financial markets (Petrosky-Nadeau and Wasmer, 2013); technological learning-by-doing (Gervais et al., 2015); hiring costs (Coles and Mortensen, 2015, 2016); diminishing marginal revenue product of labor (Dao and Delacroix, 2017); government mandated unemployment compensation and layoff costs (Ljungqvist and Sargent, 2017); skill accumulation and skill loss (Lalé, 2018). See Mortensen and Pissarides (1999), Yashiv (2007), and Pissarides (2011) for survey, or the textbooks by Pissarides (2000), Shimer (2010), Cahuc, Carcillo, and Zylberberg (2014), and Petrosky-Nadeau and Wasmer (2017), among others, for a detailed exposition.

[2]For example, see Shimer (2005, p. 40), Hagedorn and Manovskii (2008, p. 1695), Pissarides (2009, p. 1364), Hagedorn and Manovskii (2011, p. 606) and, more recently, Ljungqvist and Sargent (2017, p. 2631).

the correlation between labor market tightness (θ) and labor productivity.[3] This paper examines the puzzle empirically while considering the medium-term cycles in the analysis.

The medium-term cycle is defined as the sum of the high- and medium-frequency variations in the data. High-frequency variations are usually referred to as conventional business cycles (i.e., cycles with periods smaller than 32 quarters as in Burns and Mitchell 1946), while the medium frequency corresponds to cycles with periods between 32 and 200 quarters. Accordingly, medium-term variations in the data are computed by applying a band pass (BP) filter that isolates fluctuations with periods between 32 and 200 quarters. A popular alternative to this is the Hodrick-Prescott (HP) filter (Hodrick and Prescott, 1997). Following Baxter and King (1999), I opt for the BP filter because it allows to be precise in frequency domain terms.

There are at least two reasons that justify scholars' interest in examining the labor market over medium-term cycles. First, the empirical and theoretical evidence from Comin and Gertler (2006) and Comin (2009) suggests that (i) the high- and medium-frequency realities are not orthogonal; (ii) the medium-term fluctuations, which generate the persistence of macro variables, are also not exogenous; and (iii) business cycles and medium-term cycles are intimately connected, since they could be driven by the same underlying temporary shock.[4] Second, and more importantly, based on the spectral properties of a set of US macroeconomic variables, Beaudry, Galizia, and Portier (2020) recently argued that a significant proportion of the fluctuations in labor market variables may come from some cyclical force with a periodicity of more than 32 quarters. It is, therefore, interesting to explore how search-and-matching frictions affect labor market dynamics over medium-term cycles in response to productivity shocks of a plausible magnitude. First, I systematically document the labor market dynamics at both business cycle and medium-term frequencies. Second, I examine the ability of the textbook Diamond–Mortensen–Pissarides (DMP) model to generate the observed fluctuations in labor market variables over both business cycles and medium-term cycles. To do this, I use setups proposed by Hagedorn and Manovskii (2008), Pissarides (2009), and Shimer (2005) (hereafter, HMS, PS, and SS) to generate the simulated data. Then, based on US data and the simulated data, I examine the properties of labor market variables over both the conventional business cycle and medium-term frequencies.

My main results in terms of volatilities, persistence, and comovement are summarized as follows. First, as in Shimer (2005), I find that the volatility puzzle is present at both the business cycle and medium-term frequencies in response to shocks of a plausible magnitude. Second, from an "inter-setup comparison" perspective, I find that the DMP model, under the HMS and PS calibrations, performs better than the SS in terms of the replications of empirical volatilities over both business cycle and medium-term frequencies.

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[3] Schultz and Vaile (1923, p. 170) show that the coefficient of correlation can be used as an index of elasticity. Without loss of generality, consider the following linear functional form: $y_t = ax_t + b$. As a result, the index of elasticity reads $(\mathrm{d}y/\mathrm{d}x) \times (x/y) = a$, where the slope, a, is equal to $(\sigma_y/\sigma_x) \times \mathrm{corr}(y, x)$. Also see Moore (1922) and Vaile (1922) for more discussions.

[4] Two notes of caution merit discussion here. First, Blanchard (1997) suggests naming macroeconomic changes that unfold over periods of 15 to 30 years as medium-run phenomena (see Beaudry, 2005, p. 1137). However, in contrast to Comin and Gertler (2006), the medium term in Blanchard's (1997) approach is mainly confined to the medium-frequency variations in the data. Second, Rotemberg (2003) argues that the high- and medium-frequency realities are orthogonal. Unlike Rotemberg (2003), Comin and Gertler (2006) argue that the high- and medium-term fluctuations in macro variables are part of the same reality.

However, from an "intra-setup comparison" perspective, I find that the HMS and PS perform better over the business cycle than over the medium-term cycle. Third, in terms of first-order autocorrelations, all three setups (HMS, PS, and SS) perform relatively well over both the business cycle and medium-term cycle. However, the model fails to reproduce autocorrelograms beyond four lags of certain key labor market variables, including unemployment and job vacancies. Fourth, HMS, PS, and SS fail to match the comovement between the labor productivity and the vacancy–unemployment ratio over both business cycles and medium-term cycles. Targeting the elasticity of wages with respect to labor productivity (as in HMS) or that of labor market tightness with respect to labor productivity (as in PS) improves the performance of the model in terms of volatilities but not in terms of comovements. Fifth, in data, I find that the elasticity of tightness with respect to labor productivity is higher in the business cycles than in the medium-term cycles, whereas the elasticity of wages with respect to labor productivity is lower in the business cycles than in the medium-term cycles. This evidence ("elasticities inversion") is a challenge to reproduce in the search-and-matching model of the labor market because a lower elasticity of wages means that the variations in productivity are less absorbed by those in the price of labor and will thus be more prominently reflected in the variations of quantities, which results in a higher elasticity of labor market tightness. I show that the DMP model fails to reproduce the observed inversion in the labor productivity elasticities of the vacancy–unemployment ratio and wage, respectively, over the medium-term cycle.

This paper is primarily linked with two main literatures. First, it is related to the literature on labor markets with search-and-matching frictions (Merz 1995; Andolfatto 1996; Shimer 2005; Costain and Reiter 2008; Hagedorn and Manovskii 2008; Pissarides 2009, and subsequent studies). These papers address the business cycle fluctuations of the labor market, while my paper primarily focuses on both business cycle and medium-term fluctuations. It is worth noting, however, that Shimer (2005) and Costain and Reiter (2008) apply the HP filter with a high value for the smoothing parameter ($\lambda = 10^5$) to obtain the cyclical components of labor market variables. This is similar to including variations with a somewhat longer periodicity than 32 quarters in the business cycle fluctuations. However, these authors do not systematically compare the labor market dynamics over both the high- and medium-frequency horizon, as is done in Comin and Gertler's (2006) framework.

Second, my work has many points in common with the recent literature, which argues that medium-term fluctuations can be explained by the same factors as business cycle fluctuations.[5] Contrary to these studies, my paper points out the role of search-and-matching frictions in the labor market. Moreover, in contrast to the conventional business cycle literature, I follow Comin and Gertler (2006) and consider the *medium-term cycles* in the analysis. According to this perspective, Comin and Gertler (2006) derived a series of stylized facts on medium-term cycles for a set of usual macro variables that appear in business cycle analysis. Thus, this paper complements the existing literature by providing evidence for the labor market dynamics over the medium-term cycles.

[5]For example, see Blanchard 1997; Evans et al. 1998; Solow 2000; Comin and Gertler 2006; Comin 2009; Comin et al. 2014; Schwark 2014; Anzoategui et al. 2019a; Anzoategui et al. 2019b; Matthes, Lubik, and Verona 2019; Beaudry, Galizia, and Portier 2020; Schüler 2020, among others.

The rest of the paper is organized as follows. Section 1.2 discusses the procedure for extracting the medium-term cycles. Section 1.3 presents data. Section 1.4 documents the empirical regularities associated with medium-term cycles in the US labor market. Section 1.5 outlines the baseline model. Section 1.6 summarizes the results in terms of volatilities and comovements. The implications of my findings for search-and-matching models are also discussed in this section. Finally, Section 1.7 concludes.

1.2 Methodology

In this section, I first review two of the most popular univariate methodologies that are usually applied in the macro literature to decompose a time series into a growth (or trend) component and a cyclical component: (i) the Hodrick-Prescott (HP) filter, which is based on the penalized least squares approach (for example, see Whittaker 1923; Hodrick and Prescott 1980, 1997; Hodrick 2020); (ii) the band pass (BP) filter, which is based on the frequency domain analysis (see Baxter and King 1999, and Christiano and Fitzgerald 2003, among others). Next, I briefly explain why the BP filter is preferred for extracting the medium-term component and discuss how the medium-term cycles are computed in the data.

As mentioned in the introduction, the previous studies on labor market search-and-matching frictions basically considered the labor market dynamics at either business cycle (for example, see Hagedorn and Manovskii 2008) or medium-term frequencies (for example, see Shimer 2005) but did not focus on the systematic differences and patterns that emerge when these two frequencies are jointly examined. This paper attempts to systematically investigate these patterns and document the labor market volatility over both business cycle and medium-term cycles. As a detrending method, Shimer (2005) and Hagedorn and Manovskii (2008) used the HP filter with different parameterizations ($\lambda = 10$ to the power 5 for Robert Shimer, and $\lambda = 1,600$ for Marcus Hagedorn and Iourii Manovskii).[6]

The HP filter is a penalized approach that identifies the cyclical component c_t and the growth component g_t, for $t = 1, 2, ..., T$, as the solution to the minimization of the following loss function (see Hodrick and Prescott 1997, p. 3; Guay and St-Amant 2005, p. 136; Guerrero 2008, p. 189; Phillips and Jin 2021, p. 476, among others, for more details):

$$\{g_t^\star\}_{t=1}^T = \underset{\{g_t\}_{t=1}^T}{\operatorname{argmin}} \left\{ \sum_{t=1}^T (y_t - g_t)^2 + \lambda \sum_{t=1}^T \left[\nabla^2(g_t)\right]^2 \right\} \tag{1.1}$$

where y_t denotes the natural logarithm of a given macroeconomic time series, ∇ denotes the difference operator, and $\nabla^2(g_t)$ denotes the second difference of g_t, with $\nabla = 1 - L$ and L the lag operator defined by $L g_t = g_{t-1}$. The residual $(y_t - g_t)$ is referred to as the business cycle component. According to this formula, the HP growth component is the result of a trade-off between minimizing the variance of the cyclical component and keeping the growth rate of the trend constant. This trade-off is governed by the parameter λ. The parameter λ is a positive constant that penalizes variability or smoothness in the growth component

[6] Also see Shimer (2010, pp. 162–163), and Petrosky-Nadeau, Zhang, and Kuehn (2018, p. 2224), among others, who follow the standard practice in the macro-literature and consider a value of $\lambda = 1,600$.

series. The larger the value of λ, the smoother the solution series is. For a sufficiently large λ, at the optimum all the $\nabla^2(g_t)$ must be arbitrarily near some constant β and, therefore, the g_t arbitrarily near $g_0 + \beta t$. This implies that the limit of the solutions to problem (1.1) as λ approaches infinity is the least squares fit of a linear time trend model. The associated matrix notation of the penalized least squares problem (1.1) is

$$\min_{\mathbf{g}} HP(\lambda) = (\mathbf{y} - \mathbf{g})^{\mathrm{T}}(\mathbf{y} - \mathbf{g}) + \lambda(\mathbf{H}\,\mathbf{g})^{\mathrm{T}}(\mathbf{H}\,\mathbf{g}), \tag{1.2}$$

where $\mathbf{y} = (y_1, \ldots, y_T)$, $\mathbf{g} = (g_1, \ldots, g_T)^{\mathrm{T}}$, and $\mathbf{H}$ denotes a $(T-2) \times T$ matrix given by

$$\mathbf{H} = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \end{pmatrix}. \tag{1.3}$$

The solution to the minimization problem is obtained by successively computing the derivative of $HP(\lambda)$ with respect to $\mathbf{g}$, then equating the derivative evaluated at $\mathbf{g} = \mathbf{g}^{\star}$ to zero, and, finally, solving the subsequent equation

$$\mathbf{g}^{\star} = \left(\mathbf{I}_T + \lambda\,\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1} \mathbf{y}. \tag{1.4}$$

The derivative of $HP(\lambda)$ evaluated at $\mathbf{g} = \mathbf{g}^{\star}$ is a symmetric positive definite matrix. Accordingly, the solution (1.4) is a minimum. The HP filter represents a reasonable approach for estimating trends in the standard business cycle analysis. However, it is important to note that the solution $\mathbf{g}^{\star}$ implies that the $T \times T$ matrix must be inverted. This may cause instability and create a lack of precision in the numerical solution when T is large. Thus, the HP filter is viewed as an approach that does not provide an efficient computation tool. Moreover, the HP procedure depends heavily on the smoothing parameter, lambda (λ), and there are some variations in its preferred value across studies, making their results not directly comparable. Using quarterly data, Shimer (2005) and Hagedorn and Manovskii (2008) apply a value of 100,000 and 1,600 for lambda, respectively. Moreover, Ravn and Uhlig (2002, p. 375) recommend a value of approximately $\lambda = 1,600/4^4$ (or 6.25) for annual data, which is close to the value of 10 given by Baxter and King (1999, p. 590) but significantly different from the value of 100 used by Backus and Kehoe (1992, p. 886) or of 400 applied by Correia, Neves, and Rebelo (1992, p. 560). Hodrick (2020) (see Section 3 of his paper) discusses some of the criticisms of the HP filter that have emerged over the years.

In contrast to Shimer (2005) and Hagedorn and Manovskii (2008), in this paper, I apply the strategy proposed by Comin and Gertler (2006), which consists of using a BP filter because it allows to be more precise in frequency domain terms.[7] An intuitive way of illustrating the BP filter is to consider a time series that may

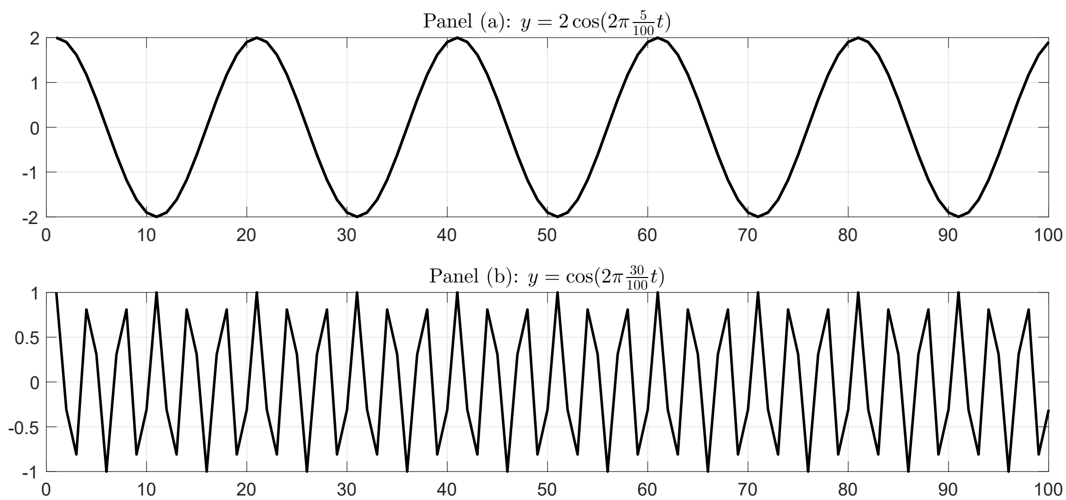
[7]The BP filter has a rich history in modern macroeconomics, and has been analyzed by Englund, Persson, and Svensson (1992), Hassler et al. (1994), Baxter and King (1999), and Christiano and Fitzgerald (2003), among others. For a more detailed description, see, for instance, the work of Murray (2003), and Guay and St-Amant (2005).

be generated from the following function:

$$y_t = A\cos(2\pi\omega t) \tag{1.5}$$

where t denotes time, A denotes a height that measures the amplitude of the cycle, 2π denotes a constant that measures the period of the cycles, and ω denotes an oscillation frequency index that measures the number of occurrences of a repeating event per time.

Figure 1.1: Frequency components with artificial data



Note: Figure 1.1 plots two frequency components that were generated from the artificial data for 100 observations. Panel (a) was produced with $y = \cos(2\pi\frac{6}{100}t)$. Panel (b) was produced with $y = \cos(2\pi\frac{32}{100}t)$.

As illustrated in Figure 1.1, in Panel (a), where $\omega = 6/100$, six cycles can be counted over the time span of 100, i.e. there are six cycles between 0 and 2π. In contrast, in Panel (b), where $\omega = 32/100$, there are 32 cycles over a time span of 100. Baxter and King (1999, p. 576) consider a BP linear filter that eliminates the very slow-moving (or "trend") component and very high-frequency ("seasonal" or "irregular") components while retaining intermediate (or business cycle) components. Referring to the National Bureau of Economic Research (NBER) business cycle chronology (see Burns and Mitchell 1946), the business cycles fluctuations are defined as periodic components whose frequencies lie between one and a half and eight years per cycle. Periodic components with lengths that are longer than eight years are identified with the trend, and those that have periodic components of less than one and a half years are identified using the irregular component. Let λ denote time in quarterly frequency that elapses per cycle, such that:

$$\lambda = \frac{2\pi}{\omega} \tag{1.6}$$

Then, to find the value of ω that corresponds to a high frequency cycle length of one and a half years, I set $\lambda = 6$ quarters. Thus, I obtain $\omega_H = \pi/3$. Similarly, to find the value of ω that corresponds to a low-frequency cycle length of eight years, I set $\lambda = 32$ quarters. Hence, I obtain $\omega_L = \pi/16$. Let $B(\omega)$ be the BP filter. Based on the classical definition of business cycles proposed by Burns and Mitchell (1946), I apply the following restrictions on the filter to extract the business cycle component:

$$\begin{aligned} B(\omega) &= 1, \quad \text{for } \omega \in [\pi/16, \pi/3] \cup [-\pi/3, \pi/16] \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{1.7}$$

Equation (1.7) entails that any periodic components with a frequency within the interval $[\pi/16, \pi/3]$ can pass through the filter unchanged, as the filter multiplies them by one. The interval $[0, \pi/16]$ is associated with the trend component, and $[\pi/3, \pi]$ is associated with the irregular fluctuations. Periodic components with a frequency that lies within $[0, \pi/16]$ and $[\pi/3, \pi]$ are eliminated, as the filter multiplies them with zero.

A useful feature of the BP filter is that it can easily be changed to accommodate data sampled at different frequencies by simply changing the values of the lower and upper bounds. Thus, I opt for the BP filter to implement the procedure proposed by Comin and Gertler (2006). To apply this procedure, I first convert all the data into growth rates by taking the log differences. As in Shimer (2005, p. 27), I also take the log differences of unemployment. Second, I use a BP filter that is constrained to produce stationary outcomes when applied to growing time series and allows to specify the frequencies of the data that one wishes to isolate. Because I apply a band filter that is an optimal approximation to the ideal filter but for finite series, I pad the series by forecasting and backcasting. The padding allows the minimization of biases that are likely to arise at sample endpoints. Third, I apply the BP filter to the growth rate data. Hence, I obtain a measure of the trend growth rates that corresponds to frequencies of 200 quarters and below. Finally, I cumulate the trend growth rates to obtain the measures of the trends in log levels. In turn, the medium-term cycle in the data is computed by applying the BP filter for the second time, which isolates fluctuations with frequencies between 6 and 200 quarters. The difference between the medium-term cycle and the medium-frequency component gives the business cycle component, i.e. the variation at frequencies between 6 and 32 quarters, which is the measure of the cycles in conventional analysis (see Stock and Watson 1999, p. 11).

1.3 Data

This section presents the data that I use to examine the medium-term cycles in the US labor market. The data are in quarterly frequencies from 1951:01 to 2006:01, except when noted otherwise. They include a set of variables that are useful for characterizing the labor market dynamics (unemployment, job vacancies, labor market tightness, job finding, job separation, labor productivity, and wage). The series of unemployment is quarterly rate directly constructed from the averages of the monthly Bureau of Labor Statistics (BLS) series. The job vacancies are Index of Help-Wanted Advertising in Newspapers from the Conference Board divided by the labor force from the BLS based on the Current Population Survey (CPS). Vacancies measures the rate of vacancy postings. The Help-Wanted Advertising Index and labor force are quarterly averages directly constructed from monthly series.[8] The series for job finding and job separation are obtained from the Shimer

[8]Abraham (1987) and, more recently, Barnichon (2010) discuss how the Help-Wanted Index is constructed.

database.[9] Labor market tightness is the ratio of job vacancies to unemployment. As in Shimer (2005), labor productivity is measured in the data as real average output per person in the nonfarm business sector constructed by the BLS. The real output is obtained from the the BLS. Following den Haan and Kaltenbrunner (2009) and the common practice in the macro-labor literature, wage is measured by the compensation for all employed persons divided by the number of employed. Wage is deflated using GDP deflator.[10] The quarterly series of compensation for All Employed Persons, employment, real output, and current output are obtained from the BLS. All series are expressed in logs.

Following Comin and Gertler (2006), I also consider annual data because medium-term fluctuations seem considerably more persistent than the conventional business cycle component. The annual data for unemployment, labor productivity, and wages are directly constructed from the averages of the monthly BLS series. The annual series of job vacancies is an average of the quarterly vacancies series. Labor market tightness in annual frequencies is obtained by computing the ratio of the annual series of job vacancies to the annual series of unemployment. Finally, the annual job finding and job separation rates are obtained from the observed average quarterly job finding and job separation rates provided by Shimer by following the transition probability tree over four quarterly sub-periods. The annual data span the period of 1951–2006 and are expressed in logs.

1.4 Empirical evidence

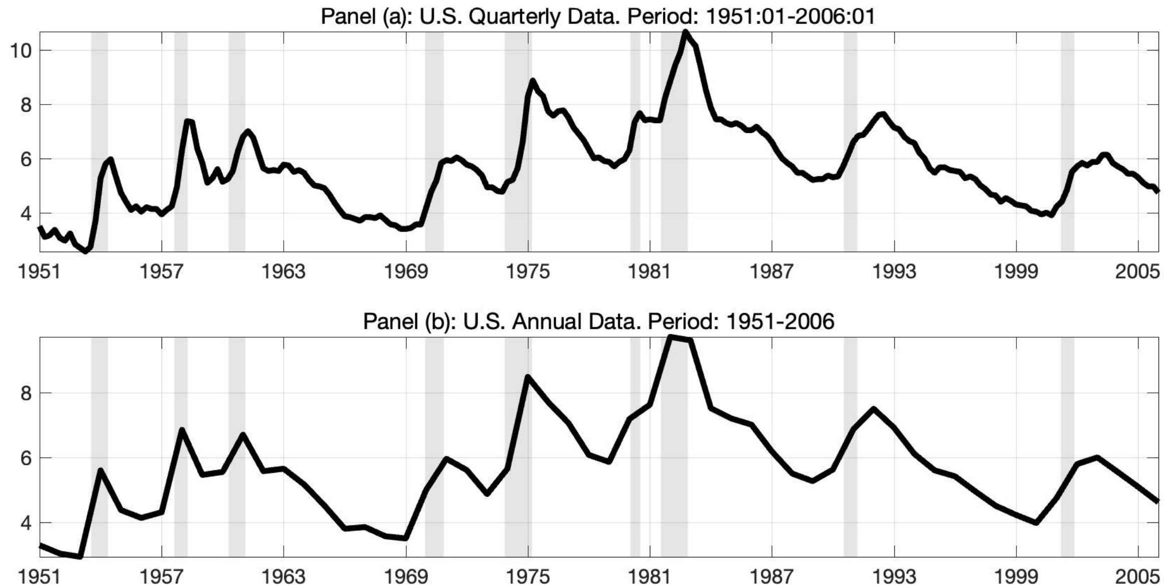
This section presents empirical evidence associated with medium-term cycles in the US labor market. The behavior of the unemployment rate in the US economy is a good illustration of the medium-term cycles. Panel (a) in Figure 1.2 shows the evolution of the US unemployment rate, in quarterly frequencies between 1951 and 2006. The unemployment rate appears to be relatively low in the 50s and 60s, then increases for roughly the next 15 years, and from the 90s, returns to the lower levels. These fluctuations occur with a periodicity far greater than a decade and are attributed to the medium-frequency component of the medium-term cycle. The main conclusion that emerges from Panel (a) in Figure 1.2 for the quarterly series is the same as that for Panel (b) in Figure 1.2 for the annual series of the unemployment rate in the US economy.

In the economics literature, several authors have argued that the unemployment rate is subject to fluctuations in lower frequencies. Blanchard and Summers (1987) argue that shocks have much more persistent effects on unemployment than standard theories can possibly explain. Blanchard (1997) argues that modern economies are characterized by medium-run evolutions that are distinct from either business cycle fluctuations or steady-state growth. Berentsen, Menzio, and Wright (2011) address the low-frequency patterns in unemployment but focus on monetary policy issues by examining how monetary factors account quantitatively for low-frequency labor market behavior. Figure 1.3 plots the medium-frequency component and the medium-term cycle for the unemployment rate. The dashed line denotes the percent deviation from trend for the medium-term

[9]Shimer database url: `sites.google.com/site/robertshimer/research/flows`.

[10]I use the ratio nominal/real GDP to infer a price index (which, in this case, is a version of the GDP deflator) and then apply this price index to deflate the wage series.

Figure 1.2: Unemployment rate in level. US data



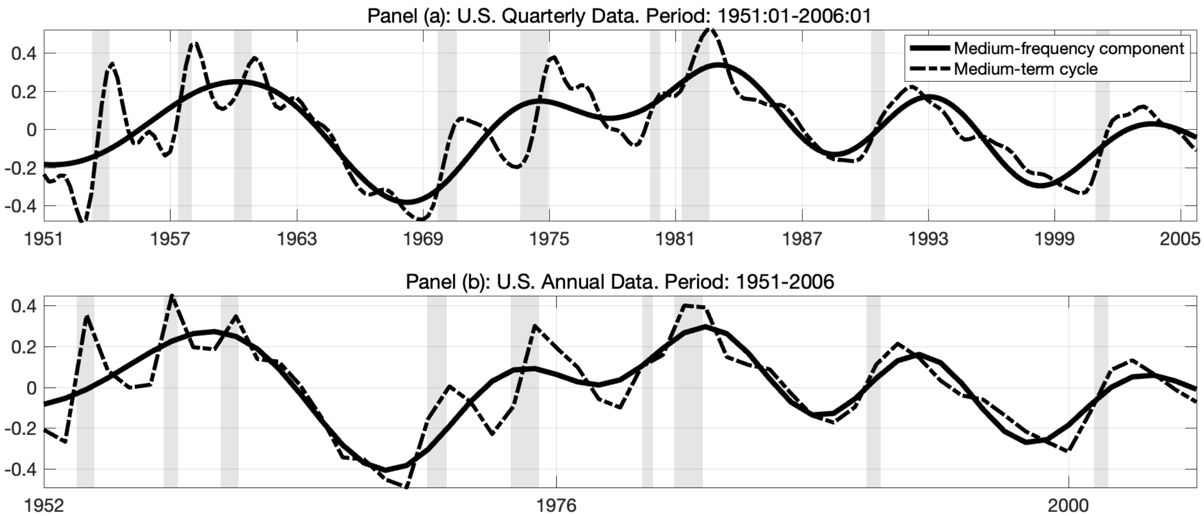
Note: Figure 1.2 plots the US unemployment rate between 1951 and 2006. The data are quarterly average (Panel (a)) and annual average (Panel (b)) calculated from the monthly BLS data. The shaded areas represent recession periods as determined by the National Bureau of Economic Research (NBER). Referring to the medium-term cycle literature (see Comin and Gertler (2006, p. 529) and the subsequent literature), I also consider annual data because medium-term fluctuations seem considerably more persistent than are typically considered in the business cycle literature.

cycle, i.e. the variation in the data at frequencies between 6 and 200 quarters. The solid line represents the medium-frequency component, i.e. the variation in the data at frequencies between 32 and 200 quarters. The difference between the two lines is the business cycle component. Put differently, Figure 1.3 confirms the intuition gained from the visual inspection of the evolution of the unemployment rate in Figure 1.2.

Moreover, Figure 1.4 plots the medium-term cycles for labor market tightness, job vacancies, labor productivity, job separation, and job finding between 1951 and 2006 at both quarterly and annual frequencies. All labor market variables clearly exhibit a pronounced medium-frequency component. This evidence reinforces the intuition gained from the visual inspection regarding the unemployment rate over medium-term cycles, and confirms that the medium-frequency component also accounts for a substantial part of the fluctuations of other important labor market variables.

Finally, Figure 1.5 presents a set of scatter plots of the key relationships in the macro-labor literature. In each panel (i.e., quarterly and annual frequencies), I specifically plot the correlation patterns in both business cycles and medium-term cycles between 1951 and 2006. Panel (a.1) and Panel (b.1) display the correlation between job vacancies and unemployment in percentage, which is the so-called Beveridge curve. The correlation of the percentage deviation of job vacancies and unemployment from the trend is -0.93 over the business cycle frequencies and -0.72 over the medium-term cycles for the quarterly data. For the annual data, the correlation of job vacancies and unemployment is -0.94 over the business cycle frequencies and -0.83 over

Figure 1.3: Medium-term variations for the unemployment rate



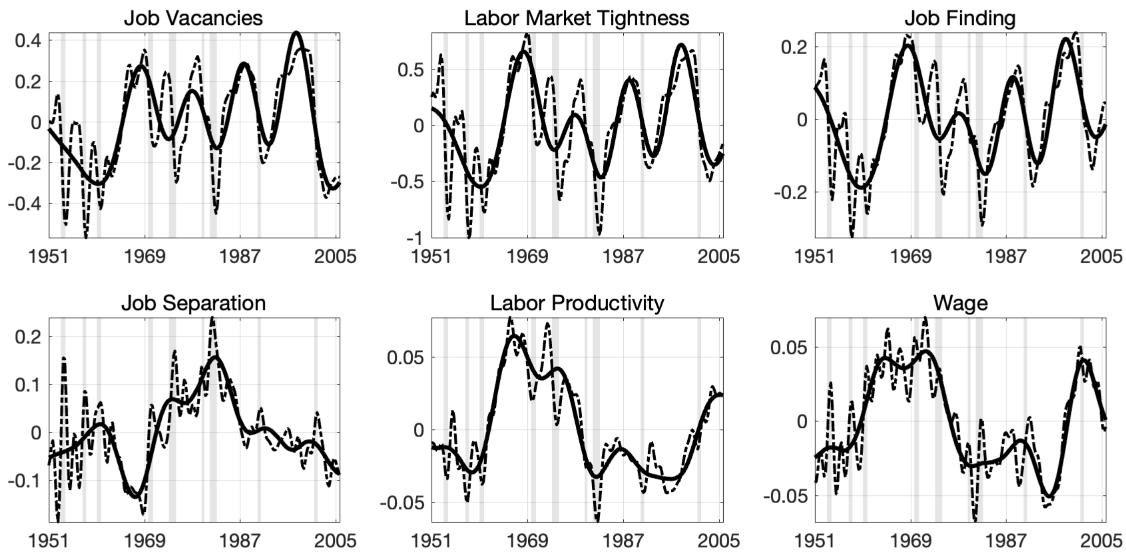
Note: Figure 1.3 plots the medium-term cycle along with the associated medium-frequency component for the unemployment rate between 1951 and 2006. The data are both in quarterly and annual frequencies. The dashed line denotes the percent deviation of the variable from the trend for the medium-term cycle. The solid line denotes the medium-frequency component. The difference between the dashed line and the solid line is the business cycle component, which corresponds to the conventional measure of the cycle in the business cycle literature. The shaded areas represent recession periods as determined by the NBER.

the medium-term cycles. Moreover, Panel (a.2) and Panel (b.2) display the positive correlation between labor market tightness and labor productivity. The correlation of the percentage deviation of labor market tightness and labor productivity from the trend is 0.44 over the business cycle frequencies and 0.28 over the medium-term cycles for the quarterly data. Finally, for the annual data, the correlation of labor market tightness and labor productivity is 0.42 over the business cycle frequencies and 0.33 over the medium-term cycles. Shimer (2005) discusses the labor market volatility and reports that the textbook search-and-matching model exhibits a contemporaneous correlation between labor market tightness and labor productivity of 1 when the data show a weaker contemporaneous tightness–productivity correlation, around 0.23 between 1951–2003. The general conclusion that emerges at both quarterly and annual frequencies is that the same correlation patterns are present in the medium-term cycle as in the business cycle, but they are weaker. All the correlations displayed in Figure 1.5 are statistically significant.

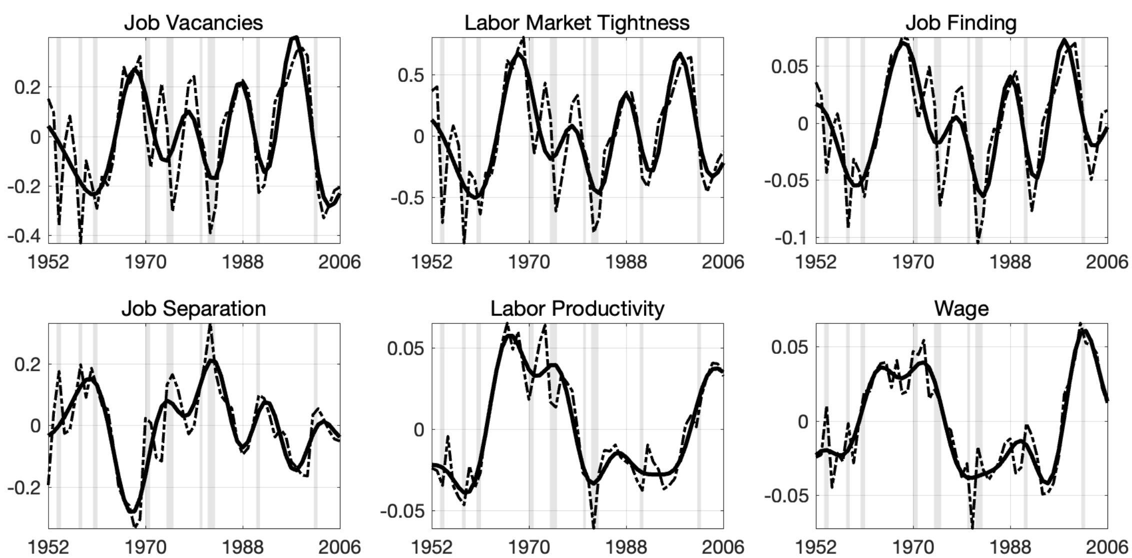
To fully appreciate the empirical evidence obtained from the previous visual inspection, I compute a set of formal statistics that characterize the medium-term fluctuations in labor market variables. Table 1.1 reports the first-order autocorrelations. All the variables exhibit high persistence over the medium-term cycle, with more persistence concentrated in the medium-frequency component. This is particularly visible for the annual data, where the autocorrelation of the medium-term cycle is three times more than that of the business cycle. For instance, the first-order autocorrelation for the annual unemployment rate is 0.021 over the business cycle frequencies and 0.66 over the medium-term cycles. Moreover, Figure 1.6 shows that the autocorrelation over the business cycle frequencies decreases much faster than that of the medium-term cycle. This pattern is consistently present across other labor market variables.

Figure 1.4: Medium-term variations for some key labor market variables

Panel (a): U.S. Quarterly Data. Period: 1951:01-2006:01

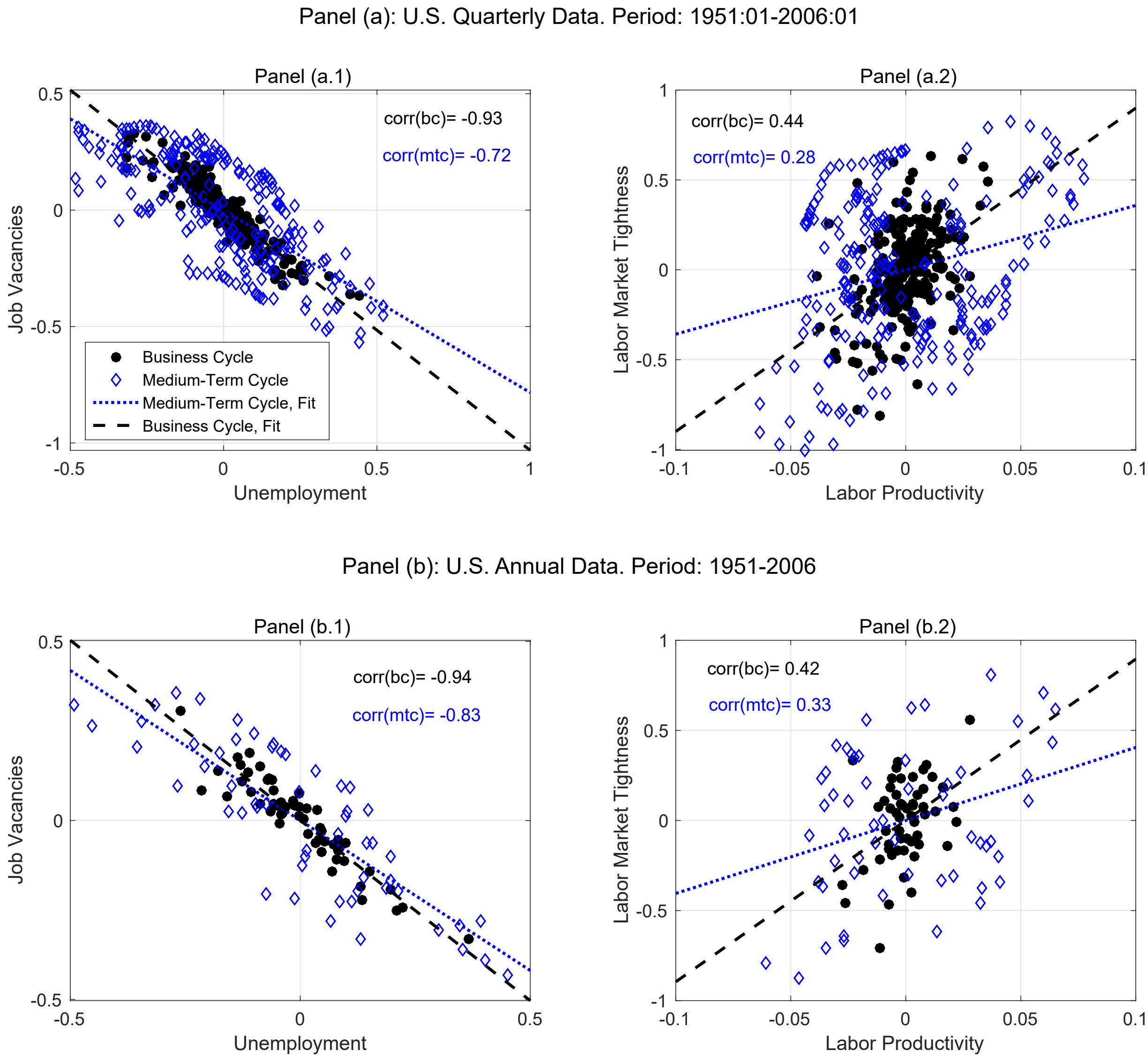


Panel (b): U.S. Annual Data. Period: 1951-2006



Note: Figure 1.4 plots the medium-term cycle along with the associated medium-frequency component for job vacancies, labor market tightness, job finding, job separation, labor productivity, and wage between 1951 and 2006, at both quarterly and annual frequencies. For each panel, the dashed line denotes the percent deviation of the variable from the trend for the medium-term cycle. The solid line denotes the medium-frequency component. The difference between the dashed line and the solid line is the high-frequency component, which corresponds to the conventional measure of the cycle in the business cycle literature. The shaded areas represent recession periods as determined by the NBER.

Figure 1.5: Correlation patterns for some key labor market variables



Note: Figure 1.5 plots the correlation patterns for certain key labor market variables at both business cycle frequencies and medium-term cycles. Every point corresponds to a quarter between 1951:01 and 2006:01 in Panel (a) and to a year between 1951 and 2006 in Panel (b).

Table 1.2 reports the volatilities of the medium-term cycle and its high- and medium-frequency components. The medium-frequency component is systematically more volatile than the business cycle component, resulting in the medium-term cycle displaying higher standard deviations than the conventionally measured business cycle. This evidence supports the significance of the extremely persistent fluctuations presented in Table 1.1.

Panel (b) in Table 1.3 reports the cyclical elasticity of labor market tightness with respect to labor productivity, $\varepsilon(\theta, p)$, and the cyclical elasticity of wage with respect to labor productivity, $\varepsilon(w, p)$, over the medium-

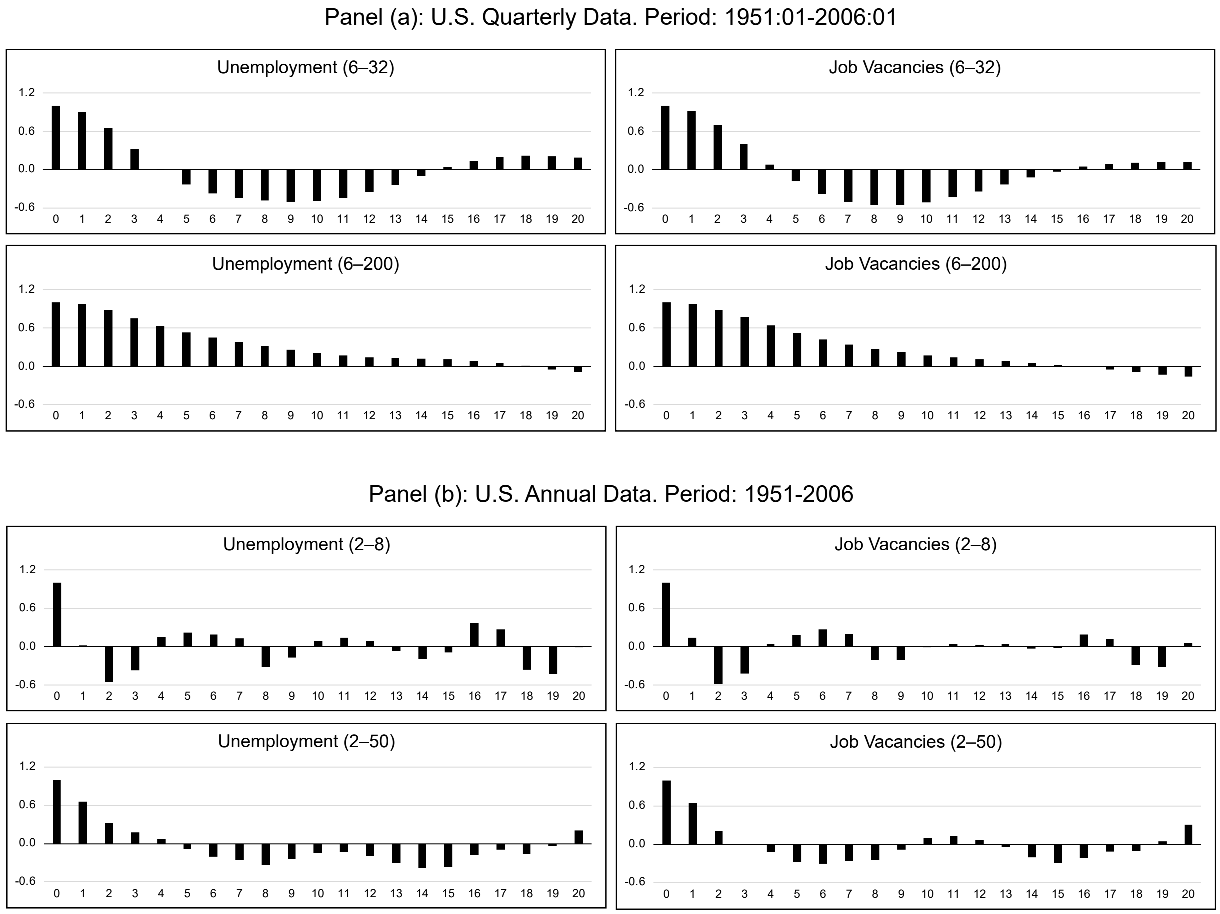
Table 1.1: First-order autocorrelations. US data: 1951–2006

	Medium-term cycle	Business cycle component	Medium-frequency component
Panel (a): US Quarterly Data	6–200	6–32	32–200
Unemployment	0.967	0.903	0.994
Job vacancies	0.970	0.919	0.994
Labor market tightness	0.965	0.913	0.993
Job finding	0.969	0.913	0.994
Job separation	0.944	0.834	0.997
Labor productivity	0.983	0.898	0.998
Wage	0.989	0.928	0.998
Panel (b): US Annual Data	0-50	0–8	8–50
Unemployment	0.658	0.021	0.915
Job vacancies	0.644	0.135	0.896
Labor market tightness	0.630	0.075	0.900
Job finding	0.685	0.159	0.894
Job separation	0.710	0.087	0.922
Labor productivity	0.866	0.077	0.967
Wage	0.887	0.264	0.963

Note: Table 1.1 reports the first-order autocorrelations of the key labor variables over the medium-term cycle as well as the associated business cycle and medium-frequency components in both quarterly and annual frequencies.

term cycle, as well as for the associated business cycle and medium-frequency components. Elasticities are computed from the quarterly and annual data using Table 1.2 and Panel (a) in Table 1.3, respectively. The results show that the cyclical elasticity of labor market tightness with respect to labor productivity at business cycle frequencies is higher than that at medium-term frequencies, i.e. $\varepsilon_{0-8}(\theta, p) > \varepsilon_{0-50}(\theta, p)$. These patterns are observed for both quarterly and annual data. Moreover, the cyclical elasticity of wage with respect to labor productivity at business cycle frequencies is lower than that at medium-term frequencies, i.e. $\varepsilon_{0-8}(w, p) < \varepsilon_{0-50}(w, p)$. These patterns (or *inversions*) are also observed for both the quarterly and the annual data. Traditionally, in reference to the so-called "Shimer puzzle" (Shimer, 2005), one of the challenges of search-and-matching models has been to reproduce the amplitude of cyclical elasticities at business cycle frequencies. Now, based on the evidence presented in Table 1.3, two contrasting facts emerge: The elasticity of tightness with respect to labor productivity is higher in the business cycles than in the medium-term cycles, whereas the elasticity of wages with respect to labor productivity is lower in the business cycles than in the medium-term cycles. Subsequently, the challenge involved with search-and-matching models in a medium-term setup is at least threefold: First, the amplitude of the observed cyclical elasticities must be reproduced at business cycle frequencies; second, the amplitude of these cyclical elasticities must be reproduced at medium-term frequencies; third, the following contrasting patterns, that are associated with the cyclical elasticity of labor market tightness with respect to labor productivity *versus* the cyclical elasticity of wage with respect to

Figure 1.6: Autocorrelograms for unemployment and vacancies



labor productivity, must be reproduced: (i) $\varepsilon_{0-8}(\theta, p) > \varepsilon_{0-50}(\theta, p)$; (ii) $\varepsilon_{0-8}(w, p) < \varepsilon_{0-50}(w, p)$.[11]

Furthermore, to complete the data analysis section, it is worth noting that the results from Table 1.3 suggest wages are less sensitive to labor productivity than labor market tightness, which exhibits higher elasticities. These results provide additional support to the evidence recently discussed by Abraham, Haltiwanger, and Rendell (2019), who argue that wages do not increase faster than the labor market tightness.[12] This empirical evidence is valid at both the business cycle and the medium-term frequencies.

[11]Intuitively, the lower elasticity of wages implies that the variations in productivity are less absorbed by those in the price of labor and will thus be more prominently reflected in the variations of quantities, which results in a higher elasticity of labor market tightness. Although this can be considered intuitive, it is a challenge to reproduce this evidence in the search-and-matching model of the labor market, as I will show in the next section of the paper.

[12]Also see Blanchard 2018, Ball and Mazumder 2019, Helbling, Jaumotte, and Sommer 2006, Auer, Borio, and Filardo 2017, Krueger 2018, and Abraham, Haltiwanger, and Rendell 2020, among others, who examine why wages are more stable than the labor market tightness.

Table 1.2: Standard deviations. US data: 1951–2006

	Medium-term cycle	Business cycle component	Medium-frequency component
Panel (a): US Quarterly Data	6–200	6–32	32–200
Unemployment	21.99	12.16	18.24
Job vacancies	23.77	13.44	20.01
Labor market tightness	42.44	25.20	34.34
Job finding	13.16	7.23	11.01
Job separation	8.121	4.658	6.770
Labor productivity	3.266	1.246	3.071
Wage	2.948	1.027	2.682
Panel (b): US Annual Data	0-50	0–8	8–50
Unemployment	21.35	11.41	17.89
Job vacancies	21.47	12.27	17.62
Labor market tightness	40.93	23.37	33.48
Job finding	4.445	2.362	3.751
Job separation	13.51	6.886	11.56
Labor productivity	3.294	1.103	3.098
Wage	2.952	0.962	2.793

Note: Table 1.2 reports the percentage standard deviations from the baseline trend of the key macro and labor variables over the medium-term cycle, as well as the associated business cycle and medium-frequency components, in both quarterly (Panel (a)) and annual data frequencies (Panel (b)).

1.5 Model

This section basically follows Pissarides (2000, ch. 1). I consider a stochastic discrete time version of the DMP search-and-matching model with aggregate uncertainty. The value functions depend only on the labor productivity. The existence of a unique equilibrium in such a setup is straightforward.[13] I use the model to investigate the labor market dynamics over the medium-term cycles.

Workers and Firms. The economy is populated by a unit measure of infinitely lived workers and a continuum of infinitely lived firms. Workers can be either unemployed (u) or employed (n). There is free entry for firms. Firms attract unemployed workers by posting a vacancy at the flow cost c. Once matched, workers and firms separate exogenously with probability s per period. The output per unit of employed workers is p. Unemployed workers get flow utility z from nonmarket activity. Employed workers are paid a wage w. Firms make accounting profits $p - w$ per employed worker for each period in which they operate. Workers and firms split the surplus from a match according to the Nash bargaining solution. The bargaining power of workers is $\beta \in (0, 1)$.

Search and matching.

[13]For more details, see Pissarides (1985, p. 683), Mortensen and Nagypál (2007, p. 331), and Hagedorn and Manovskii (2008, p.1694), among others.

Table 1.3: Labor productivity contemporaneous correlation and elasticities

	Medium-term cycle	Business cycle component	Medium-frequency component
(a) *Correlation with labor prod.*			
Panel (a.1): Quarterly Data			
$\text{corr}(\theta, p)$	0.276	0.445	0.224
$\text{corr}(w, p)$	0.874	0.696	0.922
Panel (a.2): Annual Data			
$\text{corr}(\theta, p)$	0.326	0.423	0.322
$\text{corr}(w, p)$	0.866	0.702	0.888
(b) *Elasticities*			
Panel (b.1): Quarterly Data			
$\varepsilon(\theta, p)$	3.581	8.993	2.503
$\varepsilon(w, p)$	0.789	0.574	0.805
Panel (b.2): Annual Data			
$\varepsilon(\theta, p)$	4.051	8.962	3.477
$\varepsilon(w, p)$	0.776	0.612	0.801

Note: Table 1.3 reports the contemporaneous correlation with respect to labor productivity for both labor market tightness ($\text{corr}(\theta, p)$) and wage ($\text{corr}(w, p)$) in Panel (a) and the elasticity with respect to labor productivity of both labor market tightness ($\varepsilon(\theta, p)$) and wage ($\varepsilon(w, p)$) in Panel (b). Each of these moments is reported over both business cycle and medium-term cycle frequencies, and is computed from quarterly and annual data, respectively. The elasticity is computed using Table 1.2 and Panel (a) in Table 1.3 as follows: $(\sigma_x/\sigma_p) \times \text{corr}(x, p)$, where x is the standard deviation of the variable of interest (labor market tightness and wage respectively), σ_p is the standard deviation of labor productivity, and $\text{corr}(x, p)$ is the correlation between the variable of interest (x) and the labor productivity (p) (see Hagedorn and Manovskii 2011, p. 606).

Let v be the number of vacancies posted by firms. I define the labor market tightness as the ratio of vacancies to unemployment ($\theta = v/u$). A constant returns to scale matching function governs the number of new matches (starting to produce output in the next period) such that $m(u, v) \leq \min(u, v)$. The probability of an unemployed worker being matched with a vacancy in the next period is $f(\theta) = m(1, \theta)$. The probability of a vacancy being filled in the next period is $q(\theta) = f(\theta)/\theta$. The law of motion for employment is $n' = (1 - s)n + m(u, v)$.

Equilibrium. The firm's value of a job is denoted by J; the firm's value of an unfilled vacancy is denoted by V; the worker's value of having a job is represented by W, and the worker's value of being unemployed is denoted by U. The following Bellman equations describe the equilibrium conditions of the model:

$$J(p) = p - w(p) + \delta(1-s)E_pJ(p') \tag{1.8}$$

$$V(p) = -c(p) + \delta q(\theta(p))E_pJ(p') \tag{1.9}$$

$$W(p) = w(p) + \delta\left((1-s)E_pW(p') + sE_pU(p')\right) \tag{1.10}$$

$$U(p) = z + \delta\left(f(\theta(p))E_pW(p') + (1-f(\theta(p)))E_pU(p')\right) \tag{1.11}$$

$$\beta J(p) = (1-\beta)(W(p) - U(p)) \tag{1.12}$$

where $\delta \in (0,1)$ is the workers' and firms' common discount factor. The notation $X(p)$ entails that a variable X is a function of the aggregate productivity level p, and $E_pX(p')$ denotes the next period's expected value of X, which is dependent on the current state p. Labor productivity p is an exogenous state variable that evolves according to $\ln p' = (1-\rho)\ln\overline{p} + \rho\ln p + \epsilon'$, where $\epsilon' \sim N\left(0, \sigma_\epsilon^2\right)$, and $\overline{p}$ is the steady-state value of labor productivity. Free entry implies $V(p) = 0$ for all p; therefore, $c(p) = \delta q(\theta(p))E_pJ(p')$. Nash bargaining implies that a worker and a firm split the surplus $S(p) = J(p) + W(p) - U(p)$ such that $J(p) = (1-\beta)S(p)$, $W(p) - U(p) = \beta S(p)$, and wages are given by $w(p) = \beta p + (1-\beta)z + c(p)\beta\theta(p)$.

1.6 Model evaluation

In this section, I describe the calibration of the the model's parameters, proceed to numerical simulations, and discuss the ability of the textbook DMP model to generate medium-term cycles. I use three distinct calibration techniques. First, I use Shimer's (2005) calibration approach. I call this framework the *Shimer setup* (SS). Second, I follow Hagedorn and Manovskii (2008) and impose a calibrated value for the nonmarket activity of a worker which is close to market productivity, within only a few percentage points, and a low value for the bargaining power of worker. I call this framework the *Hagedorn–Manovskii setup* (HMS). To complement my analysis, I introduce fixed matching costs ($h(p)$) in the canonical model based on the evidence provided by Pissarides (2009). An important property of this reformulation is that the vacancy posting cost $c(p)$ is replaced by the matching cost function $c(p) + h(p)q(\theta(p))$, which falls in tightness (see Pissarides 2009, p. 1364). I call this variant of the model the *Pissarides setup* (PS). The benchmark parameterization is based on the model time period length of one quarter, which is the same as that in most of the business cycle literature.

Moreover, I use an alternative parameterization (annual calibration) that is based on the model time period length of one year. My motivation for considering two different calibration strategies is that the fluctuations I am modeling are considerably more persistent than are typically considered in the conventional macro-labor literature, and the time aggregation of the quarterly calibrated model is not straightforward. As a result, considering a separate annual calibration and comparing the results of the yearly simulated model to those of the quarterly model provides a consistency check for the performance of the textbook DMP model over the medium-term cycles.

1.6.1 Baseline calibration

There are ten, nine, and eleven parameters to calibrate for the SS, HMS, and PS, respectively. For each setup, the calibration has been appropriately updated to match the quarterly US data over the period of 1951:01–2006:01. The parameters values for the baseline calibration are presented in Table 1.4.

Table 1.4: Baseline calibration. US quarterly data: 1951:01–2006:01

Parameter		Shimer	Hagedorn & Manovskii	Pissarides
External calibration				
Value of nonmarket activity	z	0.400	...	0.820
Matching function elasticity	α	0.720	...	0.500
Workers' bargaining power	β	0.720	...	0.500
Discount rate	δ	0.996	0.996	0.996
Separation rate	s	0.027	0.027	0.027
Cost of vacancy	c	...	0.546	...
Stochastic process				
Persistence of shock process	ρ	0.9612	0.7358	0.9624
Standard deviation of innovation	σ_ϵ	0.0108	0.0030	0.0105
Mean of shock process	$\overline{p}$	1.0000	1.0000	1.0000
Internal calibration				
Cost of vacancy	c	0.219	...	0.169
Matching function parameter	μ	0.456	0.890	0.459
Value of nonmarket activity	z	...	0.903	...
Workers' bargaining power	β	...	0.092	...
Fixed matching cost	h	...	...	0.272
Calibration target (Moment)				
Average job finding rate	f	0.453	0.453	0.453
Average labor market tightness	θ	0.975	0.975	0.975
Data prod. elasticity of tightness	$\varepsilon(\theta, p)$	...	...	8.993
Data prod. elasticity of wages	$\varepsilon(w, p)$	...	0.574	...
Std. dev. of labor productivity	σ_p	0.0125	0.0125	0.0125
Persistence of labor productivity	σ_p	0.8977	0.8977	0.8977
Steady state (Model implied value)				
Average job finding rate	$\overline{f}$	0.453	0.453	0.453
Average labor market tightness	$\overline{\theta}$	0.975	0.975	0.975
Model prod. elasticity of tightness	$\varepsilon(\theta, p)$	1.416	9.515	8.993
Model prod. elasticity of wages	$\varepsilon(w, p)$	0.958	0.574	0.965

Shimer setup (SS). In this setup, the matching process in the labor market is described by a standard Cobb–Douglas function: $m(v, u) = \mu v^{1-\alpha} u^{\alpha}$, where $\mu > 0$ denotes the matching efficiency, and $\alpha \in (0, 1)$ denotes the matching function elasticity. The vacancy posting cost c is assumed to be constant. There are 10 parameters to calibrate. I fix a first set of eight parameters to the values provided by Shimer (2005). I assign the values of these parameters as follows: (value of nonmarket activity) $z = 0.40$; (matching function elasticity) $\alpha = 0.72$; (workers' bargaining power) $\beta = 0.72$ (Hosios (1990) efficiency condition); (discount rate) $\delta = 0.996$; and (separation rate) $s = 0.027$. I calibrate the values of the remaining two parameters to replicate selected empirical moments as in Shimer (2005). Thus, I set the values of parameters μ (matching function parameter) and c (cost of vacancy) to pin down the average job-finding rate (f) and the average labor market tightness (θ), respectively. I choose the persistence of the productivity (ρ) and the standard deviation of innovation (σ_ϵ) to reproduce the persistence and the standard deviation of labor productivity, respectively. The mean of the shock process is normalized to one $(\overline{p} = 1)$.

Hagedorn–Manovskii setup (HMS). In contrast to Shimer (2005), who used a Cobb–Douglas matching function, Hagedorn and Manovskii (2008) opted for the specification $m(v,u) = vu/(v^\mu + u^\mu)^{1/\mu}$, $\mu > 0$, in reference to den Haan, Ramey, and Watson (2000) (hereafter, HRW). The HRW matching function ensures that the probability of finding a job and filling a vacancy lies between 0 and 1. Moreover, it is important to emphasize that Hagedorn and Manovskii (2008) recognize the presence of capital in their setup. However, since the authors assume that the firms in the model can buy and sell capital in a competitive market, the wage bargain is not affected by the presence of capital. The matching cost function is given by $c = c^K p + c^W p^{\varepsilon(w,p)}$, where c^K denotes the capital flow cost of posting a vacancy, and c^W denotes the labor flow cost of posting a vacancy. There are nine parameters to calibrate. I follow Hagedorn and Manovskii (2008) and assign the values for the first set of six parameters as follows: (discount rate) $\delta = 0.996$; (separation rate) $s = 0.027$, implying a steady-state unemployment rate $u = s/(s+f) = 0.057$; (cost of vacancy when $p = 1$) $c = c^K p + c^W p^{\varepsilon(w,p)} = 0.5461$.[14] I choose the persistence of the productivity (ρ) and the standard deviation of innovation (σ_ϵ) to reproduce the persistence and the standard deviation of labor productivity, respectively. The mean of the shock process is normalized to one $(\overline{p} = 1)$. I calibrate the remaining three parameters to replicate selected empirical moments as in Hagedorn and Manovskii (2008). I first set the value of μ (matching function parameter) to pin down the average job-finding rate (f) on the quarterly data. I choose the values for parameters z (value of nonmarket activity) and β (workers' bargaining power) to match the data on the average value for labor market tightness (θ) and the elasticity of wages with respect to productivity $(\varepsilon(w,p))$, respectively. As mentioned in the Introduction, I use the equation $\varepsilon(w,p) = (\sigma_\theta/\sigma_p) \times \text{corr}(\theta, p)$ to compute the elasticity of wage with respect to labor productivity. In this equation, σ_θ is the standard deviation of the labor market tightness, σ_p is the standard deviation of the labor productivity, and $\text{corr}(\theta, p)$ is the correlation between labor market tightness and labor productivity. The advantage of this approach is that I can find a direct counterpart of the value of this model elasticity in the data. Using US data (1951:01–2006:01), I find $\varepsilon(w,p) = 0.574$ over the business cycle. In contrast, Hagedorn and Manovskii (2008)

[14]The capital flow cost of posting a vacancy is given by $c^K = (F_K K/F)((1-u)/(1-u+v))(F/(1-u))$, where F denotes the income, $(F_K K/F)$ denotes the capital income share, u denotes the unemployment rate, and v denotes the job vacancies rate. Since labor productivity is normalized to one $(p = 1)$, it follows that $(1-u)/F = 1 - (F_K K/F)((1-u)/(1-u+v))$. As a result, (capital flow cost of posting a vacancy) $c^K = 0.4597$. Moreover, the labor flow cost of posting a vacancy is given by $c^W = c^L W q$, where c^L denotes the average labor cost of hiring one worker, which is between 3–4.5% of the quarterly wages of a new hire according to the findings of Silva and Toledo (2009a), W denotes the average wage, which is equal to two-thirds of the national income $(W = 2/3F)$, and q denotes the probability for a firm to fill a job vacancy, which is $q = f/\theta = 0.4646$. As a result, (labor flow cost of posting a vacancy) $c^W = 0.0864$.

econometrically estimate this elasticity from the BLS data (1951:01–2004:04) and find that $\varepsilon(w, p) = 0.449$.

Pissarides setup (PS). In this setup, the matching process in labor market is described by a standard Cobb–Douglas function: $m(v, u) = \mu v^{1-\alpha} u^{\alpha}$, where $\mu > 0$ denotes the matching efficiency, and $\alpha \in (0, 1)$ denotes the matching function elasticity. As mentioned in the previous section, in the PS, the constant vacancy posting cost c is replaced by the matching cost function $c + h \times q(\theta)$, which falls in labor market tightness θ, where c denotes the proportional job creation cost, h denotes the fixed matching cost, and $q(\theta)$ denotes the probability of a vacancy being filled in the next period. There are 11 parameters to calibrate. I fix the first set of eight parameters according to the values provided by Pissarides (2009). I calibrate the values of these parameters as follows: (value of nonmarket activity) $z = 0.80$;[15] (matching function elasticity) $\alpha = 0.5$;[16] (workers' bargaining power) $\beta = 0.5$ (see Hosios (1990) efficiency condition); (discount rate) $\delta = 0.996$; and (separation rate) $s = 0.035$. Pissarides uses the Hosios (1990) condition to fix the value of β (workers' bargaining power) and, moreover, refers to the findings of Shimer (2012) to set the value of s (separation rate). I choose the persistence of the productivity (ρ) and the standard deviation of innovation (σ_ϵ) to reproduce the persistence and the standard deviation of labor productivity, respectively. The mean of the shock process is normalized to one ($\overline{p} = 1$). I calibrate the values of the remaining three parameters to replicate selected empirical moments as in Pissarides (2009). I set the values of parameters c (cost of vacancy) and μ (matching function parameter) to pin down the average job-finding rate (f) and the average labor market tightness (θ), respectively, as in Pissarides (2009). I fix h (fixed matching cost) to match the productivity elasticity of tightness in the data, $\varepsilon(\theta, p)$.

1.6.2 Alternative calibration

Alternatively, as explained in Section 1.3, I follow Comin and Gertler (2006) and consider an annual calibration.[17] All the moments I used as calibration targets are now annual averages in the US data (1951–2006). The parameters are calibrated using the same targets as in my baseline calibration but have been adequately adjusted for annual observations.

Shimer setup (SS). I set: (value of nonmarket activity) $z = 0.71$; (matching function elasticity) $\alpha = 0.72$; (workers' bargaining power) $\beta = 0.72$ (Hosios (1990) efficiency condition); (discount rate) $\delta = 0.996$; (separation rate) $s = 0.035$; (persistence of shock process) $\rho = 0.7571$; (standard deviation of innovation) $\sigma_\epsilon = 0.0085$; (mean of shock process) $\overline{p} = 1$. I calibrate the two remaining parameters c (cost of vacancy) and μ (matching function parameter) to pin down the average job-finding rate (f) and the average labor market tightness (θ) in the annual frequencies, respectively.

Hagedorn–Manovskii setup (HMS).

[15] At $h = 0$ (fixed matching cost), the value of z required to match the θ elasticity in the data is about 0.85 (see Pissarides 2009, p. 1365).

[16] Petrongolo and Pissarides (2001) report estimates of α between 0.3 and 0.5.

[17] Another example of using annual calibration in the contexte of macro-labor literature is Krusell and Rudanko (2013). The authors study the impact of a monopoly union on the setting of wage in the labor market with search-and-matching frictions.

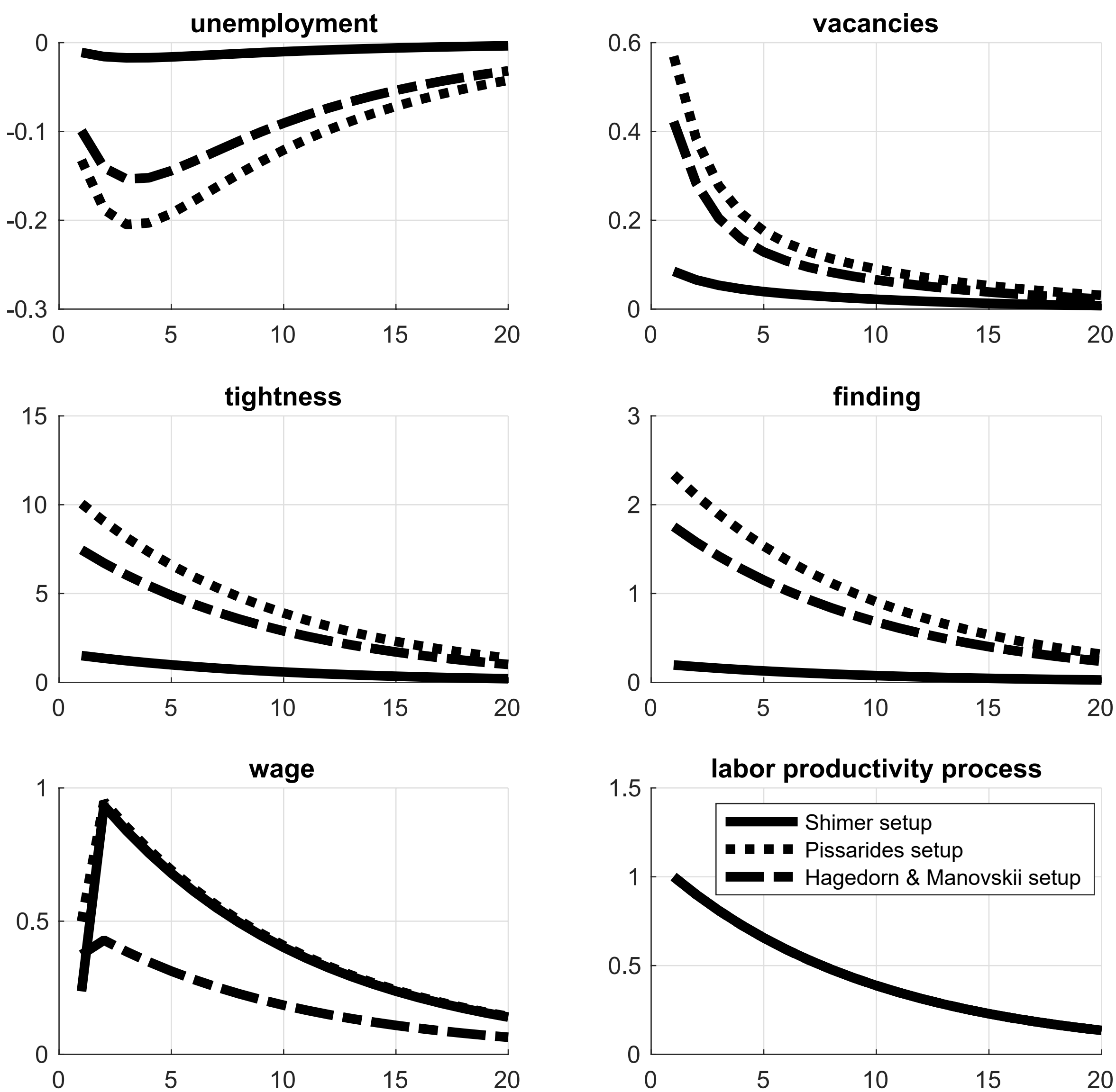
Table 1.5: Alternative calibration. US annual data: 1951–2006

Parameter		Shimer	Hagedorn & Manovskii	Pissarides
External calibration				
Value of nonmarket activity	z	0.400	...	0.820
Matching function elasticity	α	0.720	...	0.500
Workers' bargaining power	β	0.720	...	0.500
Discount rate	δ	0.984	0.984	0.984
Separation rate	s	0.056	0.051	0.035
Cost of vacancy	c	...	0.497	...
Stochastic process				
Persistence of shock process	ρ	0.7571	0.7571	0.7571
Standard deviation of innovation	σ_ϵ	0.0085	0.0085	0.0085
Mean of shock process	$\overline{p}$	1.0000	1.0000	1.0000
Internal calibration				
Cost of vacancy	c	0.218	...	0.259
Matching function parameter	μ	0.866	5.088	0.872
Value of nonmarket activity	z	...	0.950	...
Workers' bargaining power	β	...	0.289	...
Fixed matching cost	h	...	...	0.099
Calibration target (moments)				
Average job finding rate	f	0.858	0.858	0.858
Average labor market tightness	θ	0.967	0.967	0.967
Data prod. elasticity of tightness	$\varepsilon(\theta, p)$	...	...	8.962
Data prod. elasticity of wages	$\varepsilon(w, p)$	...	0.612	...
Std. dev. of labor productivity	σ_p	0.011	0.011	0.011
Persistence of labor productivity	σ_p	0.077	0.077	0.077
Steady state (Model implied value)				
Average job finding rate	f	0.858	0.858	0.858
Average labor market tightness	θ	0.967	0.967	0.967
Model prod. elasticity of tightness	$\varepsilon(\theta, p)$	0.130	1.320	8.962
Model prod. elasticity of wages	$\varepsilon(w, p)$	0.752	0.612	0.608

I set: (discount rate) $\delta = 0.996$; (separation rate) $s = 0.051$; (cost of vacancy when $p = 1$) $c = 0.497$; (persistence of shock process) $\rho = 0.7571$; (standard deviation of innovation) $\sigma_\epsilon = 0.0085$; (mean of shock process) $\overline{p} = 1$. I calibrate the three remaining parameters z (cost of vacancy), β (workers' bargaining power) and μ (matching function parameter) to pin down the average value for labor market tightness (θ), the elasticity of wages with respect to productivity $\varepsilon(w, p)$, and the average job-finding rate (f) in annual frequencies, respectively.

Pissarides setup (PS). I calibrate the first set of five parameters as follows: (value of nonmarket activity) $z = 0.82$; (matching function elasticity) $\alpha = 0.5$; (workers' bargaining power) $\beta = 0.5$; (discount rate

)δ = 0.984; and (separation rate) s = 0.035. Next, I set the values of parameters c (cost of vacancy) and μ (matching function parameter) to pin down the average job-finding rate (f) and the average labor market tightness (θ) as in Pissarides (2009). I fix h (fixed matching cost) to match the productivity elasticity of tightness in the data, $\varepsilon(\theta, p)$. I choose the persistence of the productivity (ρ) and the standard deviation of innovation (σ_ϵ) to reproduce the persistence and the standard deviation of labor productivity, respectively. The mean of the shock process is normalized to one ($\overline{p} = 1$).



Note: Figure 1.7 plots the model impulse responses to the productivity shocks of a 1% deviation from the steady state. The solid line in each panel depicts the response in the Shimer setup (SS). The dotted line depicts the response in the Pissarides setup (PS). The dashed line depicts the response in the Hagedorn–Manovskii (HMS). The y-axis scale is in percentage deviations from the steady state. The x-axis scale is the time horizon. The period length is one quarter.

Table 1.6: Standard deviations: Data *versus* model

	Medium-term cycle				Business cycle component			
	Data	SS	HMS	PS	Data	SS	HMS	PS
Panel (a): Quarterly data								
Unemployment	21.99	1.083	7.856	7.291	12.16	0.458	5.388	5.020
Vacancies	23.77	3.408	15.22	14.27	13.44	1.724	10.98	10.27
Tightness	42.44	4.340	19.56	18.33	25.20	2.022	13.72	12.87
Finding	13.16	1.215	10.00	9.164	7.226	0.566	7.014	6.436
Productivity	3.266	2.672	1.719	1.771	1.246	1.246	1.246	1.246
Wage	2.948	2.589	1.012	1.714	1.027	1.201	0.733	1.244
Model's performance	1.000	0.142	0.515	0.488	1.000	0.120	0.648	0.615
Panel (b): Annual data								
Unemployment	21.35	0.566	11.19	5.032	11.41	0.434	8.739	3.865
Vacancies	21.47	2.113	22.56	11.59	12.27	1.819	20.74	10.44
Tightness	40.93	2.264	23.69	12.55	23.37	1.791	18.96	9.917
Finding	4.445	0.634	15.29	5.646	2.362	0.501	12.52	4.463
Productivity	3.294	1.395	1.395	1.395	1.103	1.103	1.103	1.103
Wage	2.952	1.201	1.189	1.1370	0.962	0.874	0.836	0.955
Model's performance	1.000	0.087	0.798	0.398	1.000	0.127	1.222	0.597

Note: Table 1.6 reports the percentage standard deviations. Quarterly data is presented in Panel (a), and annual data is presented in panel (b). Columns SS, HMS, and PS present the simulated standard deviations for the DMP model with the SS, HMS, and PS calibrations, respectively. The business cycle component is defined as fluctuations with a periodicity between 6 and 32 quarters for the quarterly data and with a periodicity between two and eight years for the annual data. The medium-term cycles denote fluctuations with a periodicity between 6 and 200 quarters for the quarterly data and with a periodicity between 2 and 50 years for the annual data. The medium-term and business cycle variations in both the real and simulated data are computed by applying the routine presented in Section 1.2. The *Model's performance* row compares the ratio of the average for model volatilities to the average for data volatilities.

1.6.3 Results

I solve the model numerically using perturbation methods. I approximate the equilibrium system by log-linearization and simulate 10,000 long-period artificial data.[18] Based on US data and simulated data, I

[18]Petrosky-Nadeau and Zhang (2017) solve the standard DMP search model under the HM calibration using both an accurate global projection algorithm (AGPA) and log-linearization (local perturbation solution), and argue that an AGPA is critical for quantifying the basic moments of the DMP model. Log-linearization understates the mean and volatility of unemployment, but overstates the volatility of labor market tightness as well as the magnitude of the unemployment–vacancy correlation. Moreover, using both projection and perturbation methods, to numerically solve an RBC search model, Lan (2018) finds that, while the projection method is more accurate in terms of Euler residuals, the second moments are, however, not significantly different. Unlike Petrosky-Nadeau and Zhang (2017), Atolia, Gibson, and Marquis (2018) find that the second moments in a search model with curvature in the preferences are slightly larger when the less accurate linear approximation is used. This finding reinforces the claim made by Lan (2018), suggesting that the accuracy of approximation, while important for statistical fit, has little impact on the second moments

Table 1.7: First-order autocorrelations: Data *versus* model

	Medium-term cycle				Business cycle component			
	Data	SS	HMS	PS	Data	SS	HMS	PS
Panel (a): Quarterly data								
Unemployment	0.967	0.976	0.976	0.977	0.903	0.921	0.922	0.922
Vacancies	0.970	0.883	0.829	0.820	0.919	0.881	0.868	0.866
Tightness	0.965	0.930	0.928	0.930	0.913	0.898	0.896	0.897
Finding	0.969	0.930	0.926	0.930	0.913	0.898	0.894	0.897
Productivity	0.983	0.930	0.931	0.931	0.898	0.898	0.898	0.898
Wage	0.989	0.956	0.948	0.964	0.928	0.902	0.901	0.904
Panel (b): Annual data								
Unemployment	0.658	0.460	0.432	0.457	0.021	0.145	0.128	0.142
Vacancies	0.644	0.165	-0.031	0.028	0.135	-0.093	-0.199	-0.173
Tightness	0.630	0.389	0.365	0.390	0.075	0.078	0.0613	0.079
Finding	0.685	0.389	0.315	0.390	0.159	0.078	0.025	0.079
Productivity	0.866	0.389	0.389	0.389	0.077	0.077	0.077	0.077
Wage	0.887	0.588	0.660	0.678	0.264	0.301	0.403	0.430

Note: Table 1.7 reports the first-order autocorrelations for the variables of interest. Quarterly data is presented in Panel (a), and annual data is presented in panel (b). Columns SS, HMS, and PS present simulated first-order autocorrelations for the DMP model with Shimer (SS), Hagedorn–Manovskii (HMS), and Pissarides (PS) calibrations, respectively. The business cycle component is defined as fluctuations with a periodicity between 6 and 32 quarters for the quarterly data and with a periodicity between two and eight years for the annual data. The medium-term cycles denote fluctuations with a periodicity between 6 and 200 quarters for the quarterly data and with a periodicity between 2 and 50 years for the annual data. The medium-term and business cycle variations in both the real and simulated data are computed by applying the routine presented in Section 1.2.

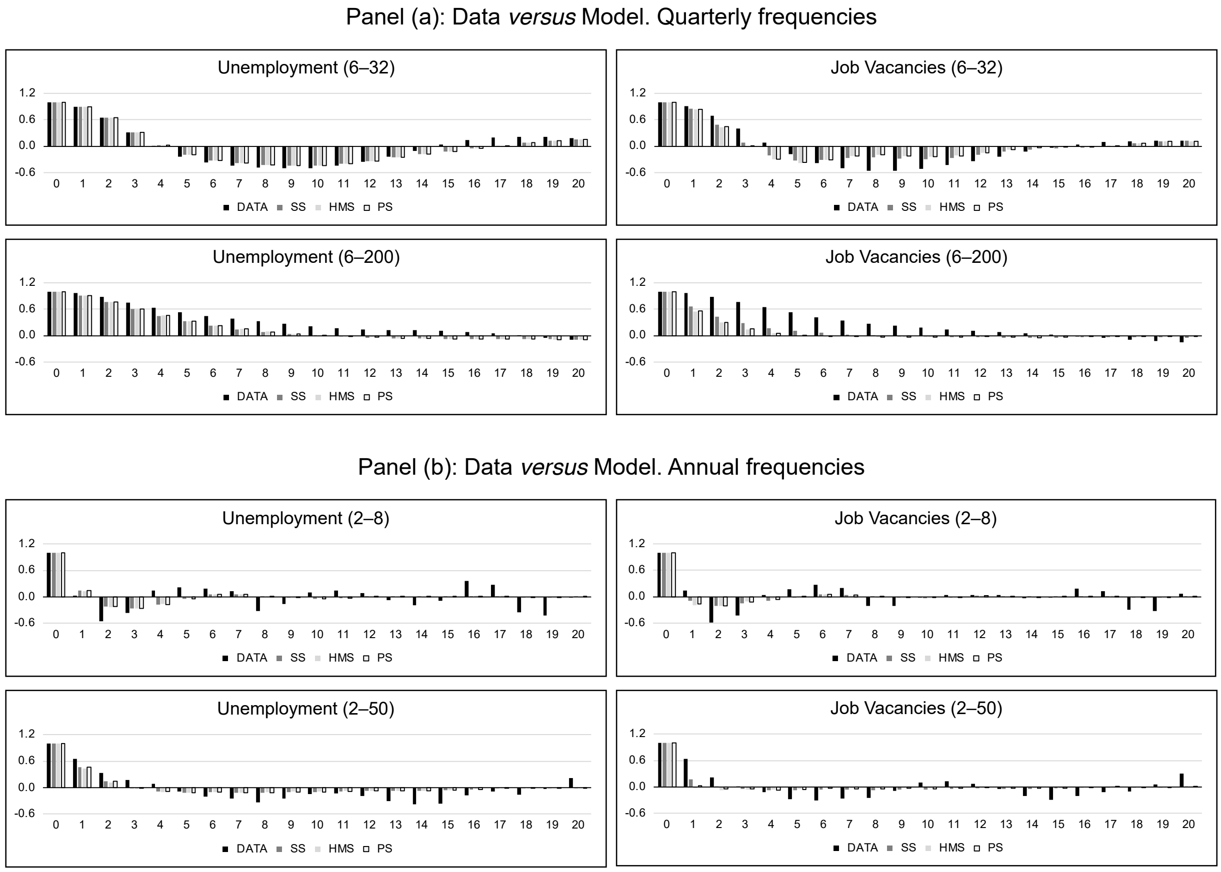
compute the moments of the variables over both business cycle and medium-term frequencies.

Impulse response functions. Figure 1.7 describes how certain key labor market variables respond to productivity shocks of a plausible magnitude. In response to a productivity shock of the same magnitude, PS generates more volatility and more persistence than SS and HMS. These results are consistent with the findings from the macroeconomics-labor literature. Moreover, it is worth noting that the volatility of wage is relatively lower in HMS than in SS and PS, as HMS fixes the workers' bargaining power to match the elasticity of wage with respect to labor productivity.

Standard deviations. Table 1.6 reports the percent standard deviations of the variables over the business cycle and the medium-term cycle. As in Shimer (2005), I find that the volatility puzzle is present at both

of the model. In this paper, I use a local perturbation solution as a first attempt at investigating the empirical evidence associated with the labor market dynamics over the medium-term cycles. An interesting exercise for future work would be to carry out a comparative analysis of the performance of the perturbation method and other methods, including the AGPA over both business cycle and medium-term frequencies.

Figure 1.8: Autocorrelograms for unemployment and vacancies



the business cycle and medium-term frequencies in response to shocks of a plausible magnitude. The DMP model, under the HMS and PS calibrations, performs well compared to the SS over both business cycle and medium-term frequencies. However, the ability of HMS and PS to match the data weakens over the medium-term cycle.

Persistence. In terms of the first-order autocorrelations, Table 1.7 reveals that all three setups (HMS, PS, and SS) perform relatively well over both the business cycle and the medium-term cycle. However, examining more complete autocorrelograms for unemployment and job vacancies in Figure 1.8 reveals that the DMP model is unable to produce sufficient persistence over both the business cycle and the medium-term cycle.

Comovement. Table 1.8 shows that HMS, PS, and SS fail to match the comovement between the labor productivity and the vacancy–unemployment ratio over both the business cycles and the medium-term cycles. Targeting the elasticity of wages with respect to labor productivity (as in HMS) or that of labor market tightness with respect to labor productivity (as in PS) improves the performance of the model in terms of volatilities but not in terms of comovements.

Table 1.8: Contemporaneous correlation and elasticities: Data *versus* model

	Medium-term cycle				Business cycle component			
	Data	SS	HMS	PS	Data	SS	HMS	PS
(a) Contemporaneous correlation								
(a.1) *Quarterly Data*								
corr(θ, p)	0.276	0.749	0.741	0.746	0.445	0.873	0.864	0.871
corr(w, p)	0.874	0.988	0.896	0.932	0.696	0.994	0.976	0.966
(a.2) *Annual Data*								
corr(θ, p)	0.326	0.390	0.361	0.390	0.423	0.080	0.077	0.997
corr(w, p)	0.866	0.964	0.874	0.814	0.702	0.949	0.807	0.702
(b) Elasticities								
(b.1) *Quarterly Data*								
$\varepsilon(\theta, p)$	3.581	1.227	9.657	9.389	8.993	1.416	9.515	8.993
$\varepsilon(w, p)$	0.789	0.961	0.591	0.968	0.574	0.958	0.574	0.965
(b.2) *Annual Data*								
$\varepsilon(\theta, p)$	4.051	0.609	1.002	4.456	8.962	0.130	1.320	8.962
$\varepsilon(w, p)$	0.776	0.746	0.438	0.694	0.612	0.752	0.612	0.608

Note: Table 1.8 reports the contemporaneous correlations with respect to labor productivity for labor market tightness (corr(θ, p)) and wage (corr(w, p)), respectively, in Panel (a) and the elasticities with respect to labor productivity of labor market tightness ($\varepsilon(\theta, p)$) and wage ($\varepsilon(w, p)$), respectively, in Panel (b). Columns SS, HMS, and PS present the simulated moments for the DMP model with the SS, HMS, and PS calibrations, respectively. The business cycle component is defined as fluctuations with a periodicity between 6 and 32 quarters for the quarterly data and with a periodicity between two and eight years for the annual data. The medium-term cycles denote fluctuations with a periodicity between 6 and 200 quarters for the quarterly data and with a periodicity between 2 and 50 years for the annual data. The medium-term and business cycle variations in both the real and simulated data are computed by applying the routine presented in Section 1.2.

Elasticities. Panel (b) in Table 1.8 shows that the DMP model suffers from an anomaly over the medium-term cycle. While the elasticity of labor market tightness with respect to labor productivity is high, HMS and PS reproduce only 35.7% and 33.2% of the unemployment volatility on quarterly data over the medium-term cycle (see Table 1.6). The reason for this is that the correlation between labor market tightness and labor productivity is almost three times higher in the model than in the data. This anomaly is more apparent for quarterly frequencies than for annual frequencies. Further, it is worth noting that the DMP model fails to reproduce the key ratios $\varepsilon_{0-8}(\theta, p)/\varepsilon_{0-50}(\theta, p)$ and $\varepsilon_{0-8}(w, p)/\varepsilon_{0-50}(w, p)$, which are equal to 2.51 and 0.73 in the data, respectively, but between 0.96 and 1.15 and between 0.97 and 1.00, respectively, in the model.

In sum, my findings suggest that the textbook DMP model fails the twin test of generating the observed business cycle and medium-term fluctuations in labor market variables in response to shocks of a plausible magnitude.

1.7 Conclusion

What are key regularities of labor market over the medium-term cycles? Can a calibrated version of the textbook Diamond–Mortensen–Pissarides (DMP) model reproduce these regularities? To address these issues, I first systematically document the differences and patterns across the business cycle and medium-term frequencies. I show that labor market variables are more persistent and are approximately twice as volatile over the medium-term cycle than over the business cycle. I document an inversion in the labor productivity elasticities of the vacancy–unemployment ratio and wage, respectively, over the medium-term cycle.

Then, using a set of three calibration strategies (Shimer 2005; Hagedorn and Manovskii 2008; Pissarides 2009, respectively SS, HMS, and PS) for both quarterly and annual data, I show that the DMP model performs better over the business cycles than over the medium-term cycles in terms of volatilities. Specifically, I find that the volatility puzzle is present at both the business cycle and medium-term frequencies in response to shocks of a plausible magnitude. From an "inter-setup comparison" perspective, I show that the DMP model, under the HMS and PS calibrations, performs better as compared to the SS calibration in terms of the replications of empirical volatilities over both business cycle and medium-term frequencies. From an "intra-setup comparison" perspective, I find that the HMS and PS perform better over the business cycle than over the medium-term cycle. While all three setups (HMS, PS, and SS) perform relatively well over both the business cycle and the medium-term cycle in terms of first-order autocorrelations, I show that they fail to reproduce autocorrelograms of certain key labor market variables, including unemployment and job vacancy for four periods and more, and fail to match the comovement between the labor productivity and the vacancy–unemployment ratio over both business cycles and medium-term cycles. Interestingly, I show that targeting the elasticity of wages with respect to labor productivity (as in HMS) or that of labor market tightness with respect to labor productivity (as in PS) improves the performance of the model in terms of volatilities, but not in terms of comovements over both business cycles and medium-term cycles. This finding is consistent with the critique of the "wage absorption" mechanism, which argues the search-and-matching model can generate a volatile labor market despite a high volatility of wages (see Hagedorn and Manovskii 2008, p. 1695). Finally, I find that, in general, the DMP model fails to reproduce the observed inversion in the labor productivity elasticities of the vacancy–unemployment ratio and wage, respectively, over the medium-term cycle.

Following Comin and Gertler (2006), and Comin (2009), among others, an immediate direction for future research would be to examine how interactions between search-and-matching frictions and long-term growth determinants can enhance the ability of the DMP model to match the data over the medium-term cycles.

CHAPTER II

MEDIUM-TERM BUSINESS CYCLES AND LABOR MARKET SEARCH

ABSTRACT

We study short- and medium-term dynamics of the labor market. We first empirically document fluctuations and co-movements of key macroeconomic and labor market variables. We show that while the business cycle dynamics are important, a substantial part of the labor market adjustments happens also at lower "medium-term" frequencies. To study the role of interactions between labor market frictions and the determinants of growth for explaining these medium-term labor market dynamics we build a macroeconomic model in which long-term economic growth is endogenously determined by R&D investment. Our results suggest that endogenous growth is crucial for quantitatively explaining the medium-term dynamics of labor market variables. In particular, we show that the R&D sector substantially amplifies and propagates the responses of the labor market variables to shocks of plausible magnitude at both business cycle and medium-term frequencies.

Keywords: Labor market dynamics, Medium-term cycles, Endogenous growth, R&D.
JEL Codes: E24, E32, J64, O30.

2.1 Introduction

How do research and development activities, innovation, and the resulting technological progress affect the short- and medium-term dynamics of the labor market? To answer this question, we first empirically document fluctuations and co-movements of R&D investment and the key labor market and macroeconomic variables. We then build a model of endogenous growth with search-and-matching frictions in the labor market that helps us understand the role of interactions between these frictions, the determinants of long-term economic growth, and total factor productivity (TFP) shocks for explaining the labor market dynamics.

In contrast to most of the existing macroeconomics-labor literature, we do not focus only on the business cycle fluctuations, usually defined as those with frequencies between 6 and 32 quarters, but we follow Comin and Gertler (2006) and study also the medium-term fluctuations with frequencies between 32 and 200 quarters.[1] Our results show that while the short-term dynamics are important, a substantial part of the labor market adjustments happens also at these medium-term frequencies. Several of the patterns we find for labor market variables are consistent with those found by Comin and Gertler (2006) for a set of other standard macroeconomic variables. Labor market variables are twice as volatile in the medium-term frequencies than in the business cycle frequencies. R&D, labor productivity, job vacancies, labor market tightness, and wages are pro-cyclical in both business cycle and medium-term frequencies, while unemployment and job separations are counter-cyclical. R&D displays statistically and economically significant contemporaneous correlation with some key labor market variables including unemployment, vacancies, labor market tightness, and wages

[1] In practice the differentiation between business cycle and medium-term fluctuations is obtained by filtering the time series with two band pass filters, each enabling us to isolate different frequencies of the data. See Section 2.2 for details.

in both business cycle and medium-term frequencies. There is a lead of R&D over labor productivity, wages, and unemployment in the medium-term frequencies. This lead is almost absent in the business cycle frequencies. Our empirical evidence sheds some new light on the analysis of labor market fluctuations and suggests that limiting the focus only to fluctuations up to 32 quarters may miss a substantial part of the dynamics linking the labor market and aggregate economic activity. It also provides support to the idea that the sources of economic fluctuations are interconnected with the sources of long-term growth, in line with Comin (2009).

Our model extends Romer's (1990) model of endogenous growth through expanding variety of inputs to allow for interactions between R&D activity and the labor market. Specifically, we assume that final good producers purchase intermediate inputs and hire workers on a labor market characterized by search-and-matching frictions. New varieties of intermediate goods are generated by innovators, who combine final good and specialized labor as inputs to R&D activity. Innovators hire workers on a separate labor market that is not subject to search-and-matching frictions. However, in order to enter this labor market, workers must become specialists in R&D, which is costly. In absence of shocks the model exhibits a deterministic balanced growth path (BGP) along which the final output growth rate, the share of workers specializing in R&D, and the unemployment rate are constant and endogenously determined. Compared to a model in which growth is exogenous, the introduction of the market for R&D-specialized labor raises the elasticity of labor market tightness with respect to productivity in the final good sector. This allows the stochastic version of the model to better propagate and substantially amplify the effects of TFP shocks on labor market quantities - namely unemployment - and other macroeconomic variables.

In addition to standard transmission mechanisms studied in the macro-labor literature, our model features a new channel: A current increase in TFP in the final good sector raises the future productivity of intermediate goods and thus the value of innovation. This in turn pushes the innovators to demand more specialized labor today. As a result, in equilibrium more workers will become R&D specialists instead of searching for a job in the final good sector and the current unemployment falls. The resulting higher discovery of new varieties of intermediate inputs further enhances the productivity in the final good sector in the subsequent periods, triggering a powerful feedback effect that leads to persistent responses to productivity fluctuations. Thanks to this channel, a calibrated version of our model is able to generate quantitatively adequate volatility and persistence in labor market and other standard macro variables in both business cycle and medium-term frequencies in response to TFP shocks of realistic magnitude.

Our paper is related to a growing body of literature that stresses the importance of considering the short-term behavior of the economy in connection to its longer-term dynamics for understanding economic fluctuations (see e.g., Blanchard 1997, Evans et al. 1998, Solow 2000, Comin and Gertler 2006, Comin et al. 2014, Schwark 2014, Anzoategui et al. 2019b, Beaudry, Galizia, and Portier 2020, Schüler 2020). In contrast to these papers, we specifically focus on labor market dynamics and the interaction between labor market frictions and R&D activity.

Labor market fluctuations have been the interest of extensive empirical and theoretical literature (see e.g., Mortensen and Pissarides 1994, Merz 1995, Andolfatto 1996, Shimer 2005, Hagedorn and Manovskii 2008, and the subsequent literature). Papers in this literature somewhat differ in terms of the frequencies at which the labor market fluctuations are measured, but we are not aware of a study systematically documenting the differences and patterns across the business cycle and medium-term frequencies. Our work thus complements the existing evidence by providing such an overview. One of the challenges for models relying on the search-and-matching framework has been to reproduce the observed magnitude of volatility of key labor market variables, such as unemployment and vacancies, in response to productivity shocks of realistic size. The existing papers proposed several solutions which, as shown by Ljungqvist and Sargent (2017), mainly come down to reducing the "fundamental surplus" a job creates.[2] This in turn increases the elasticity of labor market tightness with respect to productivity and thus amplifies fluctuations in unemployment. We explore a different, complementary mechanism. In our model, the elasticity of labor market tightness with respect to productivity is high not because of low fundamental surplus, but because of existence of an alternative labor market to which unemployed workers can turn, albeit at a cost. Crucially, the opportunities on this alternative labor market are procyclical and persistent, leading to amplified and persistent responses of unemployment to productivity shocks.

Finally, our framework shares some features with the news shocks literature that considers perceptions and revisions of beliefs about the future as an important source of business cycle fluctuations (see Beaudry and Portier 2006, 2007, and the subsequent literature). Our work differs from these papers by connecting the beliefs to determinants of long-term economic growth. In our framework, the future innovation possibility frontier increases in responses to persistent positive TFP shocks. This in turn spurs R&D activity that further enhances the expectations of future productivity. Accordingly, a current TFP shock induces also effects similar to news about future technologies.

2.2 Medium-term cycles – empirical evidence

This section presents some empirical evidence about medium-term cycles in the US, which are defined as the sum of the high- and medium-frequency fluctuations. High-frequency fluctuations are usually referred to as conventional business cycles (i.e., cycles with periods smaller than 32 quarters as in Burns and Mitchell 1946).[3] The medium-frequency fluctuations correspond to cycles with periods between 32 and 200 quarters.

In practice the high- and medium-frequency components of the medium-term cycle are obtained by applying two band pass filters with different bandwidths to the data. The first band pass filter allows us to identify the *medium-term cycle* (i.e., fluctuations with periodicity of 6-200 quarters). The second band pass filter

[2]These modifications of the matching model range from different calibration strategies (Hagedorn and Manovskii, 2008), introduction of nominal rigidities (Hall, 2005b), financing frictions (Wasmer, 2004; Petrosky-Nadeau and Wasmer, 2013), fixed costs (Pissarides, 2009), and different bargaining protocols (Hall and Milgrom, 2008) to including firm heterogeneity and decreasing returns to scale (Elsby and Michaels, 2013).

[3]Stock and Watson (1999) are pointing out that the shortest full cycle (peak to peak) from NBER business cycle reference dates is 6 quarters, and about 90% of these cycles are no longer than 32 quarters. Baxter and King (1999) recommend using a band pass filter that admits frequency components between 6 and 32 quarters for isolating business cycle fluctuations.

allows us to isolate the *medium-frequency component* (i.e., fluctuations with periodicity of 32-200 quarters). The difference between the medium-term cycle and its medium-frequency component then corresponds to the *high-frequency component* or the conventional measure of the business cycle (i.e., fluctuations with periodicity of 6-32 quarters). Band pass filters are a particularly appropriate method for identifying stochastic trends in our context because they specify with precision the frequencies in which the trends are isolated. Other popular methods, such as the Hodrick and Prescott's (1997) filter, allow to control the degree of smoothing of the trend, but the mapping into the frequency domain is less straightforward.

As in Comin and Gertler (2006), before applying the band pass filters we transform all raw data series into growth rates by taking log differences. Our implementation of band pass filters follows Baxter and King (1999). We use approximate band pass filters that are constrained to produce stationary outcomes when applied to growing time series and allow to specify frequencies of the data that one wishes to isolate. Because we are using band pass filters that are optimal approximations to the ideal filter but for finite series, we pad the series by forecasting and back-casting. The padding allows to minimize biases that are likely to arise at sample endpoints. By applying the filters to the growth rate data we obtain measures of trend growth rates in the specified frequencies of interest. We accumulate these trend growth rates to obtain measures of trends in log levels. The medium-term cycle is then computed as deviations of the log-level data with respect to the baseline log-level trend in the 6-200 quarters periodicity. The medium-frequency component is computed as deviations of the log-level data with respect to the log-level trend in the 32-200 quarters periodicity.

Comin and Gertler (2006) and Comin (2009) derived a series of stylized facts on medium-term cycles for a set of usual macro variables that appear in the business cycle analysis. We update their empirical evidence and extend it with a set of labor market variables. Our data are in quarterly frequencies from 1951:01 to 2006:01, except when noted otherwise. The data include two sets of variables.[4] The first set includes: gross domestic product (GDP), consumption, investment, total factor productivity (TFP), and research and development (R&D) investment. GDP, consumption, and investment are from the Bureau of Economic Analysis (BEA) National Income and Product Account (NIPA) tables. These series are deflated by the GDP deflator and expressed in per capita terms after dividing by the civilian non-institutionalized population aged 16 and over. Consumption includes non-durables and services. Investment is nonresidential. We use the quarterly TFP series from Fernald (2014).[5] The quarterly series of R&D investment is from BEA and is deflated using the R&D chain-type price index.

The second set of variables characterizes the labor market dynamics. It includes labor productivity, job vacancies, unemployment, labor market tightness, and wages. Following Shimer (2005), labor productivity is measured as the real average output per person in the non-farm business sector. This series is directly available from the Bureau of Labor Statistics (BLS). Job vacancies are measured by the Index of Help-Wanted Advertising in Newspapers from the Conference Board divided by labor force from BLS based on Current Population Survey (CPS). Unemployment is quarterly rate directly constructed from averages of monthly

[4]Appendix 4 describes the data set in more detail.

[5]Fernald's (2014) TFP series is the quarterly version of the annual series developed by Basu, Fernald, and Kimball (2006).

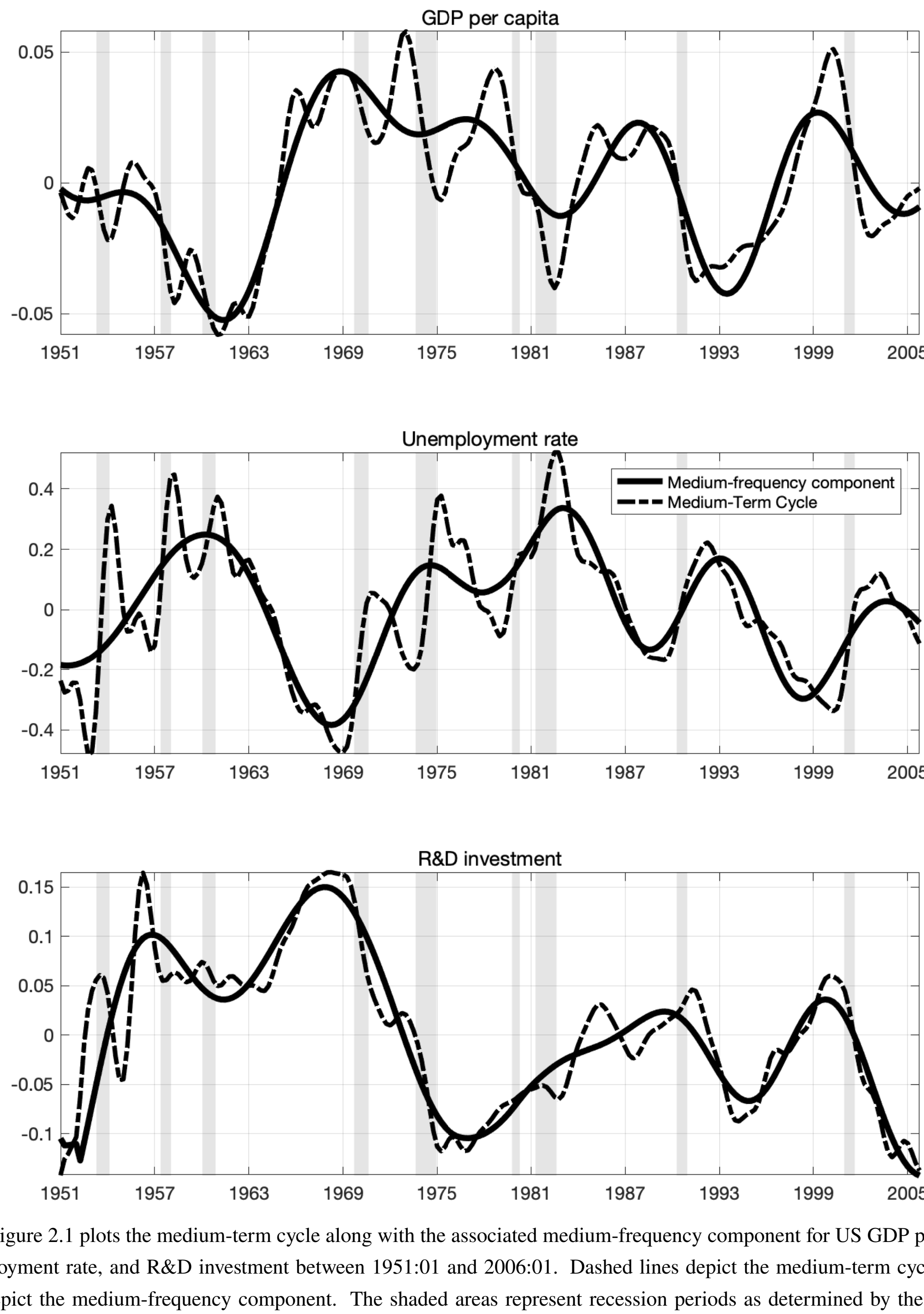
BLS series. Wages are non-farm business compensation divided by number of employed from the BLS. Labor market tightness is the ratio of job vacancies and unemployment.

Following Comin and Gertler (2006), we also consider annual data because medium-term fluctuations seem considerably more persistent than typically considered. Annual data for per capita GDP, consumption, investment, TFP, labor productivity, wages and, unemployment are obtained from St. Louis Fed database. The annual series of R&D is non-federally funded R&D expenditures as reported by the National Science Foundation (NSF). The NSF R&D series is deflated using the implicit GDP deflator. The annual job vacancies series is an average calculated from Robert Shimer's data. Finally, the annual job finding and job separation rates are obtained from the observed average quarterly job finding and job separation rates provided by Shimer by following the transition probability tree over four quarterly sub-periods. Annual data span the period 1953 to 2006. The definition of the medium-term cycle in annual data corresponds to the fluctuations with periodicity of 0-50 years, with high- and medium-frequency components corresponding respectively to the fluctuations with periodicity of 0-8 years and 8-50 years.

Figure 2.1 gives a first look at the medium-term cycle in three important variables in the US data, per capita GDP, the unemployment rate, and R&D investment. For each variable the dashed line depicts the medium-term cycle expressed as percentage deviation from the baseline trend at periodicity between 6 and 200 quarters. The solid line depicts the medium-frequency component (i.e., the percentage deviation from the trend at periodicity between 32 and 200 quarters). The difference between the two lines is the high-frequency component. We observe that, for the three variables, the medium-frequency component accounts for a substantial part of the fluctuations between 6 and 200 quarters. Clearly, studying the medium-term fluctuations is important for understanding the dynamics of goods and labor markets. Moreover, it is crucial for the dynamics of innovation activities, where the medium-frequency component accounts for almost the entirety of the fluctuations. Unemployment displays medium-term fluctuations which move in the opposite direction to the medium-term fluctuations of GDP per capita. Conversely, the medium-term fluctuations of R&D investment are procyclical. In the light of this evidence, it appears that by limiting the horizon of fluctuations to 32 quarters the typical business cycle analysis reveals only a small part of the relationship between the GDP and the labor market and it seems virtually unable to link these fluctuations to the R&D dynamics. Figure 2.2 confirms that the medium-frequency component also accounts for a substantial part of fluctuations of other important macroeconomic variables. This is particularly true for the labor market variables such as labor market tightness, vacancies, labor productivity, and wages. We now present a set of formal statistics characterizing the medium-term business cycles for a number of key macro variables.

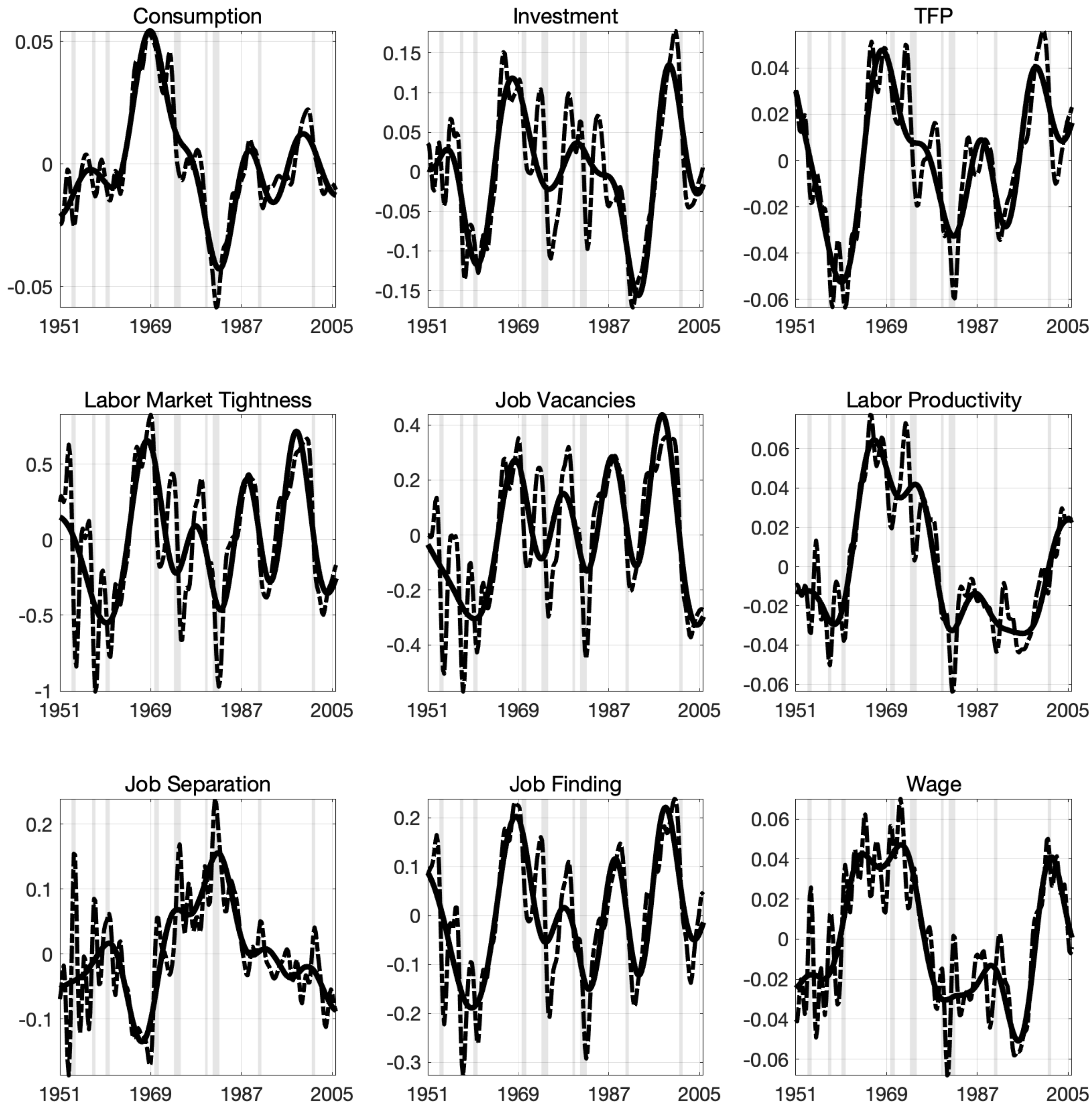
Volatility. Table 2.1 reports the standard deviations of macroeconomic variables in the medium-term cycle and in its high- and medium-frequency components expressed in terms of percentage deviations from the trend. The medium-frequency component is systematically more volatile than the high-frequency component, resulting in the medium-term cycle displaying higher standard deviations than the conventionally measured business cycle. The labor market variables, in particular, are approximately twice as volatile in the medium-term cycle than in its high-frequency component. One comment is in order concerning the trend-cycle

Figure 2.1: The medium-term cycle. US quarterly data 1951:01–2006:01



Note: Figure 2.1 plots the medium-term cycle along with the associated medium-frequency component for US GDP per capita, unemployment rate, and R&D investment between 1951:01 and 2006:01. Dashed lines depict the medium-term cycle. Solid lines depict the medium-frequency component. The shaded areas represent recession periods as determined by the National Bureau of Economic Research (NBER).

Figure 2.2: Medium-term variations for some key macro variables. US quarterly data: 1951:01–2006:01



Note: Figure 2.2 plots the medium-term cycle along with the associated medium-frequency component for consumption, investment, TFP, labor market tightness, job vacancies, labor productivity, job separation, job finding, and wages between 1951:01 and 2006:01. Dashed lines depict the medium-term cycle. Solid lines depict the medium-frequency component. The shaded areas represent recession periods as determined by the National Bureau of Economic Research (NBER).

Table 2.1: Standard deviations. US data: 1951–2006

	Medium-term cycle	High-frequency component	Medium-frequency component
US Quarterly Data	6–200	6–32	32–200
GDP	3.30	1.49	2.97
Consumption	2.25	0.75	2.14
Investment	8.07	4.15	6.94
TFP	2.90	1.26	2.63
R&D	8.87	3.28	8.57
Labor productivity	3.27	1.25	3.07
Wage	2.95	1.03	2.81
Unemployment	21.99	12.16	18.24
Job vacancies	23.77	13.45	20.01
Labor market tightness	42.44	25.20	34.34
Job finding	13.16	7.23	11.01
Job separation	8.12	4.66	6.77
US Annual Data	0–50	0–8	8–50
GDP	3.50	1.38	3.21
Consumption	2.33	0.69	2.22
Investment	8.42	3.96	7.38
TFP	3.00	1.16	2.76
R&D	7.66	2.37	7.16
Labor productivity	3.31	1.12	3.11
Wage	3.12	0.98	2.96
Unemployment	21.50	11.26	18.40
Job vacancies	25.52	14.00	20.96
Labor market tightness	43.37	24.30	35.83
Job finding	3.86	1.96	3.31
Job separation	19.09	8.86	16.91

Note: The standard deviations are expressed in terms of percentage deviations from trend.

decomposition of the labor market variables. The existing labor literature generally employed a HP-filter to obtain the trend and the cyclical component of these variables. However, the papers vary in the chosen value for the smoothing parameter λ of the filter. While some studies (e.g. Hagedorn and Manovskii, 2008) use the standard value in business cycle analysis ($\lambda = 1,600$), others (e.g. Shimer, 2005; Costain and Reiter, 2008) use much higher value ($\lambda = 10^5$). This is akin to including in the cyclical fluctuations those with a longer periodicity than 32 quarters. Nonetheless, we are not aware of a paper systematically investigating the patterns and differences across the fluctuations at the business cycle and medium-term frequencies.

Persistence. Table 2.2 reports the first-order auto-correlations. All variables are very persistent over the medium-term cycle with more persistence concentrated in the medium-frequency component. This is

Table 2.2: First-order auto-correlations. US data: 1951–2006

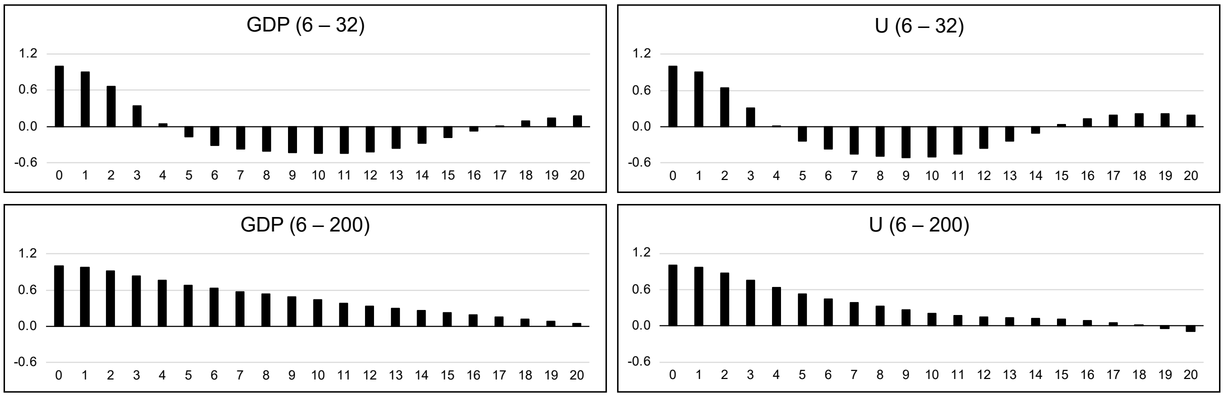
	Medium-term cycle	High-frequency component	Medium-frequency component
US Quarterly Data	6–200	6–32	32–200
GDP	0.978	0.908	0.996
Consumption	0.989	0.922	0.997
Investment	0.976	0.927	0.994
TFP	0.983	0.930	0.996
R&D	0.984	0.913	0.996
Labor productivity	0.983	0.898	0.998
Wage	0.989	0.928	0.998
Unemployment	0.967	0.903	0.994
Job vacancies	0.970	0.919	0.994
Labor market tightness	0.965	0.913	0.993
Job finding	0.969	0.913	0.994
Job separation	0.944	0.834	0.997
US Annual Data	0–50	0–8	8–50
GDP	0.822	0.118	0.944
Consumption	0.889	0.216	0.953
Investment	0.766	0.227	0.919
TFP	0.836	0.105	0.965
R&D	0.849	0.049	0.940
Labor productivity	0.862	0.071	0.966
Wage	0.894	0.268	0.964
Unemployment	0.726	0.121	0.916
Job vacancies	0.691	0.173	0.897
Labor market tightness	0.683	0.145	0.897
Job finding	0.719	0.182	0.902
Job separation	0.752	0.118	0.920

particularly visible in the annual data, where the auto-correlation of the high-frequency component is lower than $1/3$ of that of the medium-term cycle.[6] Looking at more complete auto-correlograms for GDP and unemployment in Figure 2.3 reveals that the auto-correlation of the high-frequency component decays much faster than that of the medium-term cycle. The pattern is consistently present across the majority of macro and labor variables.

Cyclicality. Table 2.3 reports contemporaneous correlations between GDP per capita and other variables. In line with the recent literature (Barlevy 2007, Ouyang 2011, Fabrizio and Tsolmon 2014, Anzoategui et al.

[6]The pattern is consistent with the quarterly data. When we take quarterly series and compute the annual auto-correlation (auto-correlation of order $t-4$) over medium-term cycles, we obtain the numbers of the same magnitude as for the annual series.

Figure 2.3: Auto-correlograms. US quarterly data: 1951:01–2006:01



Note: Figure 2.3 plots the auto-correlograms of GDP per capita (GDP) and unemployment rate (U) in quarterly data for the high-frequency component (6-32) and the medium-term cycle (6-200).

2019b), our results show that R&D is pro-cyclical. We show that this is the case at both business cycle and medium-term frequencies. Consumption, investment, TFP, labor productivity, job vacancies, labor market tightness, and job finding are also pro-cyclical at both business cycle and medium-term frequencies. However, unemployment and job separation are counter-cyclical.

Lead of R&D. Table 2.4 reports the correlation between R&D at year $t-5$ and other variables at year t. Our results confirm the finding of Comin and Gertler (2006) that accounting for medium-frequency component in the analysis modifies the timing of the cross-correlogram for a number of key macro variables including GDP per capita, consumption, and TFP. In particular, there is a lead of R&D in the medium-frequency component, but this lead is almost absent in the high-frequency component. We extend these findings to a set of labor market variables including wages, and unemployment. In line with Comin and Gertler (2006), we focus in the table on the annual data as the pattern is not evident in a single lead common to all variables in the quarterly data. For illustration, Figure 2.4 gives the detail of the cross-correlogram between two key variables (GDP and Unemployment) and R&D in the quarterly data.

Labor market. Figure 2.5 shows scatter plots of the key relationships in the macro-labor literature. In each panel, every point corresponds to a quarter between 1951 and 2006. We plot data for the medium-term cycle and its high-frequency component (business cycle). Panel (a) displays the Beveridge curve, Panel (b) the the Okun's law, Panel (c) the relationship between labor market tightness and labor productivity, and Panel (d) the relationship between labor market tightness and consumption. The general conclusion that emerges is that the same correlation patterns are present in the medium-term cycle as in the business cycle, but they are weaker, except for the Okun's law.[7]

[7]All correlations displayed in Figure 2.5 are statistically significant.

Table 2.3: Contemporaneous correlation with GDP. US data: 1951–2006

	Medium-term cycle	High-frequency component	Medium-frequency component
US Quarterly Data	6–200	6–32	32–200
Consumption	0.898	0.856	0.921
Investment	0.669	0.817	0.641
TFP	0.855	0.882	0.830
R&D	0.484	0.594	0.489
Labor productivity	0.700	0.723	0.698
Wage	0.686	0.663	0.701
Unemployment	-0.878	-0.858	-0.895
Job vacancies	0.625	0.936	0.509
Labor market tightness	0.804	0.914	0.772
Job finding	0.830	0.877	0.814
Job separation	-0.710	-0.678	-0.718
US Annual Data	0–50	0–8	8–50
Consumption	0.931	0.857	0.944
Investment	0.697	0.858	0.665
TFP	0702	0.821	0.677
R&D	0.597	0.374	0.635
Labor productivity	0.754	0.729	0.755
Wage	0.747	0.673	0.757
Unemployment	-0.932	-0.863	-0.953
Job vacancies	0.561	0.812	0.512
Labor market tightness	0.792	0.869	0.788
Job finding	0.774	0.800	0.784
Job separation	-0.880	-0.820	-0.898

2.3 The Model

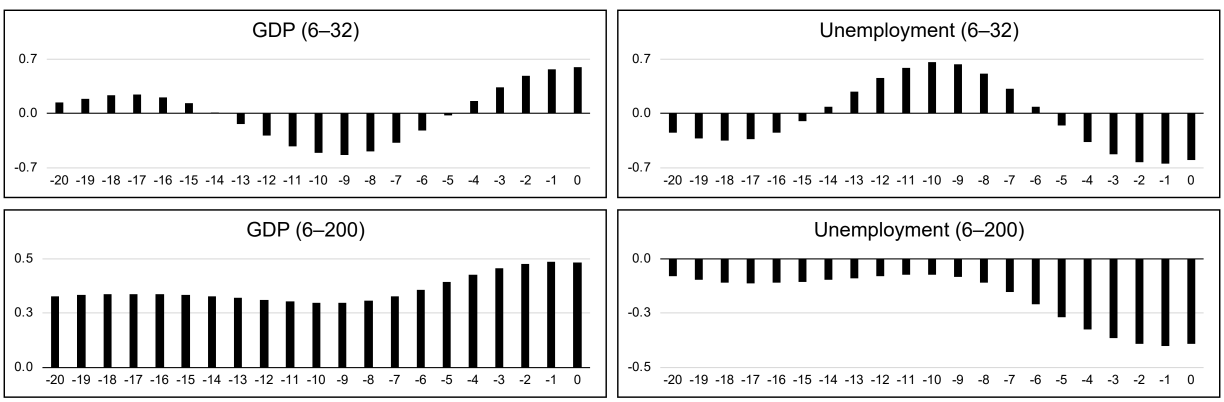
In this Section, we develop a model to understand the documented key empirical facts on labor market dynamics over medium-term cycles. The basic setup is an endogenous growth model in which growth happens through expanding variety of intermediate inputs along the lines of Romer (1990). We explicitly build in a frictional labor market with firms posting vacancies and unemployed searching for jobs as in Mortensen and Pissarides (1994). To generate economic fluctuations we introduce shocks to the total factor productivity in the final good production as in the real business cycle (RBC) literature (see e.g. Kydland and Prescott, 1995). We study the propagation and amplification mechanisms and we particularly focus on the short- and medium-term fluctuations of the labor market variables.

There are four types of agents: final good producers, intermediate goods producers, households, and government. There are three sectors: final good, intermediate goods, and R&D. Time is discrete and goes from

Table 2.4: Cross-correlation with R&D at $t-5$. US data: 1951–2006

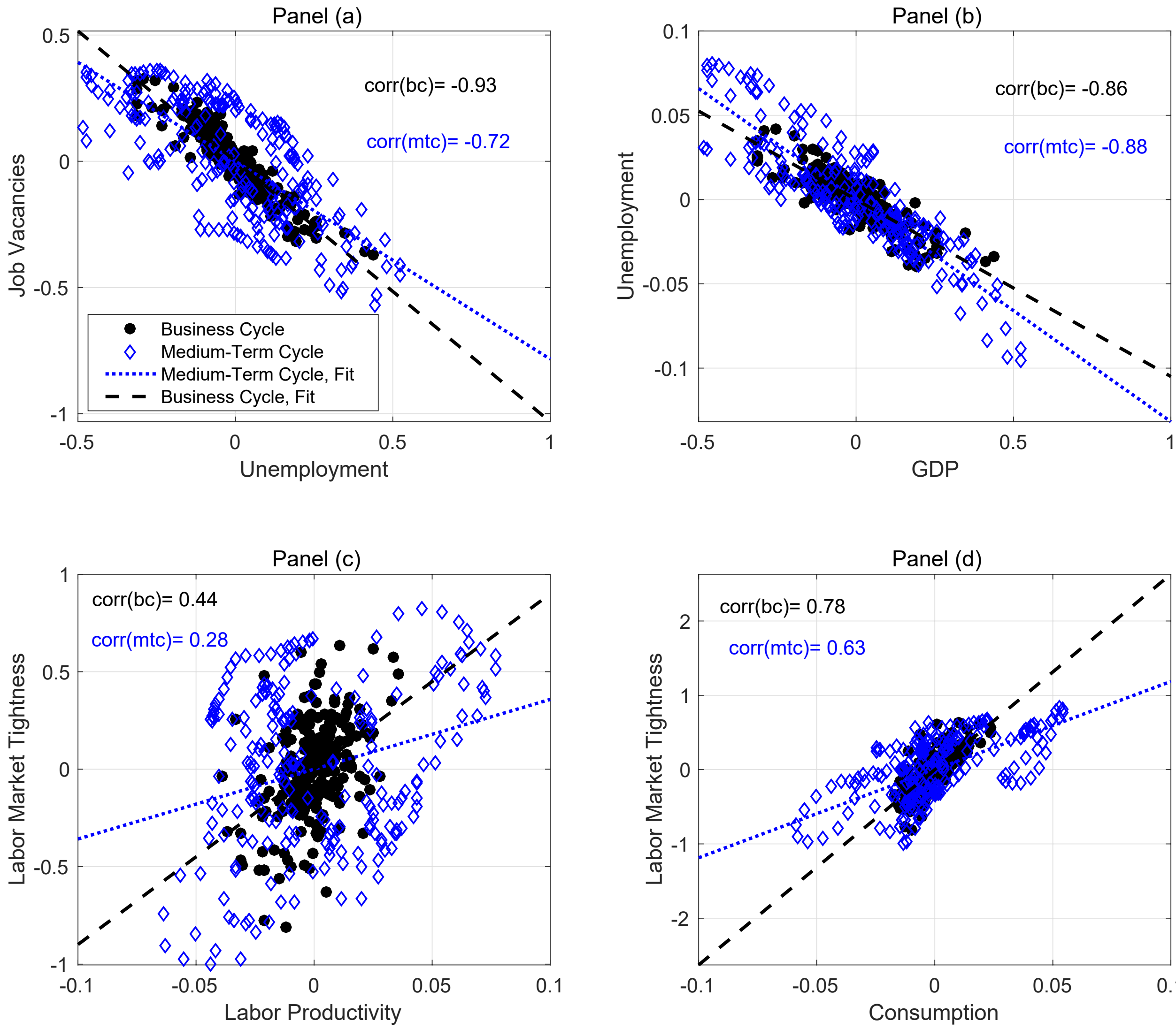
	Medium-term cycle	High-frequency component	Medium-frequency component
US Annual Data	0–50	0–8	8–50
Per capita GDP	0.494	0.110	0.549
Consumption	0.590	0.055	0.657
Investment	0.074	0.202	0.054
TFP	0.518	-0.002	0.590
Labor productivity	0.616	-0.070	0.698
Wage	0.585	-0.185	0.684
Unemployment	-0.308	-0.087	-0.352
Job vacancies	-0.115	0.027	-0.111
Labor market tightness	0.089	0.059	0.121
Job finding	0.169	0.149	0.194
Job separation	-0.253	-0.125	-0.285

Figure 2.4: Cross-correlograms with R&D. US quarterly data: 1951:01–2006:01



Note: Figure 2.4 plots the cross-correlograms of GDP per capita (GDP) and unemployment rate (U) with R&D at different leads in quarterly data for the high-frequency component (6-32) and the medium-term cycle (6-200).

zero to infinity. In anticipation of the recursive formulation of agents' problems we denote with prime the next period variables. The aggregate state of the economy at the beginning of the period is given by vector $\Omega \equiv \left(\zeta, N, K, L, L_R^{(-1)}, A\right)$, where ζ is an aggregate TFP shock, N is the number of varieties of intermediate goods available to be used in production of the final good in the current period, K is the aggregate stock of physical capital, L is the aggregate number of workers hired by the final good producers, $L_R^{(-1)}$ is the aggregate number of workers hired by the R&D sector in the previous period, and A is the aggregate households loans to innovators in the current period. We now formally describe the problems of agents and the market structure in each of the sectors of the economy. The final good is taken as the numeraire, and its price is normalized to one.



Note: Figure 2.5 plots correlation patterns for some key labor market variables at both business cycle frequencies and medium-term cycles. Each point corresponds to a quarter between 1951:01 and 2006:01.

2.3.1 Final good sector

The final good is produced by a continuum of measure one of identical and perfectly competitive firms according to production function

$$y = \zeta z^{1-\psi} \left(k^{\alpha} l^{1-\alpha}\right)^{\psi}. \tag{2.1}$$

ζ is the aggregate TFP shock following an AR(1) process in logs, k and l are the respective inputs of capital and labor services, and z is a constant elasticity of substitution (CES) aggregate of intermediate goods

$$z = \left[\int_0^N x(i)^{1-\psi} \, \mathrm{d}i\right]^{\frac{1}{1-\psi}}, \tag{2.2}$$

where $x(i)$ is the quantity of intermediate good of variety i. The parameters governing the elasticity of final output with respect to different inputs and the elasticity of substitution between the varieties of intermediate goods are such that $0 < \alpha < 1$ and $0 < \psi < 1$.

A representative final good producer contracts the quantities of production inputs as to maximize its profits. Intermediate goods and physical capital are available on spot markets, whereas workers must be hired on the labor market one period in advance by posting job vacancies because of search-and-matching frictions. When making these decisions, the firm takes the rental rate for physical capital R, and the price of each variety i of intermediate good $p(i)$ as given. Capital depreciates at rate $\delta_k \in (0,1)$.

Let $V(l,\Omega)$ be the value function of the final good producer that has hired l workers, v the number of vacancies to be posted for the next period, and κ the parameter that determines per vacancy posting cost. Let also $\beta\Lambda(\Omega,\Omega')$ be the firm's appropriate discount factor.[8] The problem of the final good producer can be written recursively as

$$\begin{aligned} V(l,\Omega) = \max_{x(i),k,l',v} \Big\{ & \zeta z^{1-\psi}\left(k^{\alpha} l^{1-\alpha}\right)^{\psi} - \int_0^N p(i)\,x(i)\,di - Rk \\ & -\omega l - \kappa (N')^{\frac{1}{1-\alpha}} v + \beta E\left[\Lambda(\Omega,\Omega')\,V(l',\Omega')\right]\Big\}, \end{aligned} \tag{2.3}$$

subject to the CES aggregate of intermediate goods (2.2), and the law of motion of the firm's employment

$$l' = (1-\delta_l)\,l + q(\theta)\,v, \tag{2.4}$$

where δ_l is the exogenous job separation rate and $q(\theta)$ is the probability of filling a vacancy expressed as a function of the labor market tightness θ, and subject to the law of motion of the aggregate state and pricing functions

$$\Omega' = G_{\Omega}(\Omega), \tag{2.5}$$

$$p(i) = P(i,\Omega), \tag{2.6}$$

$$R = R(\Omega), \tag{2.7}$$

$$\omega = \omega(\Omega), \tag{2.8}$$

$$\theta = \theta(\Omega). \tag{2.9}$$

In equation (2.3) the firm's total vacancy posting cost, $\kappa (N')^{1/1-\alpha} v$, is assumed to be proportional to the innovation frontier given by the number of the intermediate good varieties that will become available in the next period to an appropriate power. This assumption is akin to that of Pissarides (2000, *ch. 3*), who assumed vacancy posting cost to be proportional to labor productivity. As will be seen later, this specification ensures the existence of a well-defined deterministic BGP.[9] The corresponding decision rules for the final

[8] We assume that households are proprietors of the firms. In equilibrium, $\Lambda(\Omega,\Omega')$ will be equal to the marginal rate of substitution $u'(c')/u'(c)$.

[9] Along a deterministic BGP the vacancy posting cost in our model is effectively proportional to the combined multi-factor

good producer are functions

$$x(i) = f_{x(i)}\left(l, \Omega\right), \tag{2.10}$$

$$k = f_k\left(l, \Omega\right), \tag{2.11}$$

$$l' = f_l\left(l, \Omega\right), \tag{2.12}$$

$$v = f_v\left(l, \Omega\right). \tag{2.13}$$

2.3.2 Intermediate goods sector

A continuum of differentiated varieties of intermediate goods is produced by monopolistically competitive firms. Production function for each variety of intermediate good is identical and linear. Namely, one unit of variety i can be produced at the constant marginal cost $\mu > 0$ units of the final good. Each intermediate good producer maximizes the value of expected discounted monopolistic profits by choosing the price of its output subject to the final good sector demand schedule. Denoting by $W\left(i; \Omega\right)$ the value of discounted profits of producer i, the maximization problem can be written recursively as

$$W\left(i; \Omega\right) = \max_{p(i)} \left\{p\left(i\right) x\left(i\right) - \mu x\left(i\right) + \left(1 - \delta_N\right) \beta E\left[\Lambda\left(\Omega, \Omega'\right) W\left(i; \Omega'\right)\right]\right\}, \tag{2.14}$$

subject to $x\left(i\right) = f_{x(i)}\left(l, \Omega\right)$. In equation (2.14) δ_N is the probability that the variety i of the intermediate good becomes obsolete and thus unusable in the future. The decision rule for the intermediate good producer is a pricing function

$$p(i) = g_p\left(i, \Omega\right). \tag{2.15}$$

2.3.3 R&D sector

New varieties of intermediate goods can be invented by combining labor input (l_R) and final goods (a). We assume that labor for the R&D sector is hired on a separate competitive spot market that is free from search-and-matching frictions. The newly invented varieties become available for use as production inputs in the final good sector in the following period. The aggregate level of knowledge (the number of existing varieties) evolves according to

$$\frac{N'}{N} = \left(1 - \delta_N\right) + \left(1 - \delta_N\right) \chi\left(l_R\right)^{\iota} \left(N^{-1/1-\alpha} a\right)^{1-\iota}, \tag{2.16}$$

where δ_N is the rate of obsolescence of technologies, ι is the elasticity of new intermediate goods with respect to R&D labor input, $(1 - \iota)$ is the elasticity of new intermediate goods with respect to R&D final goods investment, and χ is a congestion externality taken as given by the individual innovators. The productivity of a is inversely proportional to N to an appropriate power. This feature of the R&D production function implies that on a BGP the R&D final goods investment (a) will be proportional to $N^{1/(1-\alpha)}$. As in Comin

productivity of capital and labor.

and Gertler (2006), we assume that while the congestion externality χ is regarded as given by the individual innovators, it depends on aggregate conditions in the economy

$$\chi = \eta \left[L_R{}^{-\iota} \left(N^{1/1-\alpha} A \right)^{\iota - 1} \right]^{1-\phi}, \tag{2.17}$$

where $\eta > 0$ is a technology coefficient and $0 < \phi \leq 1$ is a curvature parameter. Equation (2.17) implies a congestion externality of R&D on economic growth. Successful innovations are harder to make as the aggregate level of R&D activity increases.[10]

Entry into the R&D sector is free; any firm or individual can initiate research. A successful inventor of a new variety receives a fully enforced perpetual patent on production of this variety. The value of owning the patent is the value of expected discounted monopolistic profits of the intermediate good producer given by equation (2.14). Free entry implies that the research sector will attract additional hires and investment as long as the expected future marginal benefit from developing new intermediate goods in R&D sector is higher than the associated current marginal cost. As a result, the free entry equilibrium condition implies

$$\iota \chi N \left(\frac{N^{-1/1-\alpha} a}{l_R} \right)^{1-\iota} (1 - \delta_N) \beta E \left\{ \Lambda (\Omega, \Omega') W (i; \Omega') | \Omega \right\} = \omega_R, \tag{2.18}$$

$$(1 - \iota) \chi N^{1 - \frac{1}{1-\alpha}} \left(\frac{N^{-1/1-\alpha} a}{l_R} \right)^{-\iota} (1 - \delta_N) \beta E \left\{ \Lambda (\Omega, \Omega') W (i; \Omega') | \Omega \right\} = 1. \tag{2.19}$$

The left hand side of equation (2.18) is the expected future benefit of hiring one more unit of R&D labor. This additional unit of R&D labor leads to invention of $\iota \chi N \left(N^{-1/1-\alpha} a / l_R \right)^{1-\iota}$ new varieties of intermediate goods, each with a net present discounted value of profits given by (2.14). The right hand side of equation (2.18) is the current cost of hiring one more unit of R&D labor, which is simply given by the wage rate on the R&D labor market ω_R. The interpretation of equation (2.19) is analogous for marginal benefit and cost of R&D final goods investment. Equations (2.18) and (2.19) show how the model generates pro-cyclical R&D. For instance, during a boom, the value of a new intermediate good, W, increases. Since the profit flow from intermediate goods rises, the benefit from creating new varieties of these goods goes up, labor and investment demand of the research sector will increase in response. The hiring and investment decision rules for the innovators are functions

$$l_R = f_{l_R} (\Omega), \tag{2.20}$$

$$a = f_a (\Omega). \tag{2.21}$$

2.3.4 Households

The representative household consists of a continuum of measure one of members, each of whom can be either employed in the final good production sector, employed in the R&D sector, or unemployed. Because

[10]See Eicher and Turnovsky (2000) for more details on the importance of congestion effects in endogenous growth models.

of search-and-matching frictions on the final good sector labor market, household members have to secure their jobs in that sector one period in advance. The R&D labor is traded on a spot market. However, in order to be able to supply R&D labor, household members need to go through a training and become specialists, which is costly. Once trained, they remain competent in R&D labor as long as they stay employed in the R&D sector. Once they quit that sector, their R&D skill fully depreciates.

As a result, at the beginning of period fraction l of household members have a job in the final good sector, fraction l_R take jobs in the R&D sector, and the remaining fraction $1 - l - l_R$ are unemployed and spend the period searching for a new job. Among the members who are employed by the R&D sector, mass n are newly entering this sector and must go through the training. The distinction between employed and unemployed members of the household is important for the characterization of the labor market equilibrium. However, we assume that the household provides perfect insurance to its members in terms of consumption. Consequently, we can write the utility maximization problem at the household level and abstract from the intra-household transfers among the individual members.

The representative household consumes, saves by either investing into physical capital or lending to innovators (investing in R&D), and supplies labor services to the final good and R&D sectors. The household maximizes its expected inter-temporal utility subject to the budget constraint, the laws of motion of the household's capital stock, employment, and aggregate variables. The per-period utility function $u(c)$ is strictly increasing, strictly concave, and satisfies the usual Inada conditions. We denote c as the consumption, k as the stock of physical capital, l as the labor supply to the final good sector, l_R as the labor supply to the R&D sector, n as the mass of workers newly entering the R&D sector, and a as the one-period household loans to innovators. Let $Q\left(k, l, l_R^{(-1)}, a, \Omega\right)$ be the household's value function. To lighten the notation let $\Omega^\star \equiv \left\{k, l, l_R^{(-1)}, a, \Omega\right\}$ be the vector of household's state variables. The household's problem can be written recursively as

$$Q\left(\Omega^\star\right) = \max_{\left\{\substack{c,k',l,\\ l_R,n,a'}\right\}} \left\{u\left(c\right) + \beta E\left\{Q\left(\Omega^{\star\prime}\right)|\Omega^\star\right\}\right\} \tag{2.22}$$

subject to

$$\begin{aligned} c + k' + a' + \frac{1}{2} c_R n^2 = {} & \omega l + \omega_R l_R + b\left(1 - l - l_R\right) \\ & + \left[R + (1-\delta_k)\right]k + (1+r)a + T + \Pi^i + \Pi^f, \end{aligned} \tag{2.23}$$

$$l' = (1-\delta_l)l + f\left(\theta\right)\left(1 - l - l_R\right), \tag{2.24}$$

$$l_R = (1-\delta_{lR}) l_R^{(-1)} + n, \tag{2.25}$$

$$c, k' \geq 0, c_R = \overline{c}_R N^{\frac{1}{1-\alpha}}, \tag{2.26}$$

$$r = r\left(\Omega\right), \tag{2.27}$$

$$T = \tau\left(\Omega\right), \tag{2.28}$$

and subject to the law of motion of the aggregate state (2.5), and the pricing equations (2.6)–(2.9).

Equation (2.23) is the budget constraint, where the left hand side corresponds to household's expenditures and the right hand side to the revenues. The last term on the expenditure side is the cost of training new workers to become R&D specialists. We assume the cost quadratic in the quantity of workers undertaking the training and $c_R > 0$ is a parameter governing the its magnitude.[11] On the revenue side, ω is the wage in the final good sector, ω_R is the wage in the R&D sector, b is unemployment benefit, R is the capital rental rate, r is the payoff on the loans, and T are lump-sum taxes. We assume the household is the residual claimant of the profits generated by the firms in the intermediate and final good sectors Π^i and Π^f respectively, where Π^i is taken net of innovators' expenses.

Equation (2.24) is the law of motion of employment in the final good sector, where $\delta_l \in (0, 1)$ is the exogenous job destruction rate in the sector and $f(\theta)$ is the probability of finding a new job for the unemployed expressed as a function of labor market tightness. The sunk cost nature of the training cost implies that the fraction of workers specialized in R&D labor evolves dynamically according to the law of motion that is given by equation (2.25), even though the market for R&D labor is spot. $\delta_{lR} \in (0, 1)$ is the exogenous job destruction rate in the R&D sector. The decision rules that solve household's problem are functions

$$c = h_c\left(\Omega^\star\right), \tag{2.29}$$

$$k' = h_k\left(\Omega^\star\right), \tag{2.30}$$

$$l' = h_l\left(\Omega^\star\right), \tag{2.31}$$

$$l_R = h_{l_R}\left(\Omega^\star\right), \tag{2.32}$$

$$n = h_n\left(\Omega^\star\right), \tag{2.33}$$

$$a' = h_a\left(\Omega^\star\right). \tag{2.34}$$

2.3.5 Government

The government finances unemployment benefits by collecting lump-sum taxes to maintain a balanced budget in every period

$$(1 - L - L_R)\, b + \tau\left(\Omega\right) = 0, \tag{2.35}$$

where unemployment benefits, b, are assumed proportional to the current innovation frontier given by the number of intermediate good varieties that are available as inputs in production to an appropriate power

$$b = \bar{b}N^{\frac{1}{1-\alpha}}, \tag{2.36}$$

[11] While, in principle, this formulation would imply a positive retraining cost also if the chosen net flow of workers into the R&D sector was negative, the sunk cost nature of the training cost means that in equilibrium it is always optimal for the household to choose $n \geq 0$.

where $\bar{b} > 0$ is a parameter. This assumption makes unemployment benefits proportional to the job productivity and ensures existence of a well-defined BGP.

2.3.6 Frictional labor market and wage determination in the final good sector

The market for labor in the final good sector is characterized by search-and-matching frictions. Employment relationship consists of a worker and a final good producer who engage in production until the relationship is severed. As in Cahuc and Wasmer (2001) and Elsby and Michaels (2013), final good producers need to post job vacancies one period ahead of production in order to recruit workers. Wages, however, are determined by bargaining between the firms and the workers only after the current-period TFP shock is realized. In such framework, firm's employment is a predetermined variable but the wage rate and firm's demand for renting capital are not predetermined.

Each job can be in one of two states, filled or vacant. On the household side, workers can be employed in the final good or R&D sector or unemployed and searching for a job. The total number of vacancies posted by the firms is v. Let $u \equiv 1 - L - L_R$ be the measure of the pool of unemployed workers searching for a job. Following Mortensen and Pissarides (1994) we model the number of new hires, m, as a function of the number of vacancies and the number of searching workers

$$m = m(v, u), \tag{2.37}$$

where $m(\cdot)$ is the matching function that is homogeneous of degree one, increasing in each of its arguments, concave, and continuously differentiable. Let $\theta \equiv v/u$ be the labor market tightness. Then, the probability for a final good producer to fill a vacancy is

$$q(\theta) \equiv m\left(1, \frac{1}{\theta}\right), \tag{2.38}$$

with $\partial q/\partial\theta < 0$. Similarly, the probability for an unemployed worker to find a job is

$$f(\theta) \equiv m(\theta, 1), \tag{2.39}$$

with $\partial f/\partial\theta > 0$. As is clear from problems (2.3) and (2.22), both final good producers and workers take functions f and q as given and form anticipation on θ in function of the aggregate state according to equation (2.9).

A realized job match yields economic surplus that is shared between the firm and workers. We assume that the wage that pins down the surplus sharing is determined by a Nash bargain. The bargaining takes place after the realization of the current-period aggregate TFP shock. In particular, let $J(\cdot)$ and $H(\cdot)$ be, respectively, the firm's final surplus and worker's final surplus as a function of the bargained wage and the aggregate state, and let $\xi \in (0,1)$ be the worker's bargaining power. Then, the bargained wage is

$$\omega = \arg\max_{\omega} G\left\{H(\omega,\Omega), J(\omega,\Omega); \xi\right\}, \tag{2.40}$$

where $G(\cdot)$ is the joint surplus written as a Nash product.[12] Given the firm's problem (2.3), $J(\cdot)$ is given by the firm's marginal value of the labor input. Similarly, $H(\cdot)$ can be expressed as the marginal value of employment in the household's problem (2.22).[13] A solution to problem (2.40) is the negotiated wage that is a weighted arithmetic average of reservation wages

$$\omega = \xi\overline{\overline{\omega}}(\Omega) + (1-\xi)\underline{\underline{\omega}}(\Omega), \tag{2.41}$$

where $\overline{\overline{\omega}}(\Omega)$ is the highest wage that the final good producer is willing to pay for each additional worker and $\underline{\underline{\omega}}(\Omega)$ is the lowest wage at which an unemployed would be willing to accept a new job in state of nature Ω.

2.3.7 Timing

Figure 2.6 illustrates the timing of the model. The economy starts period t with the vector of predetermined endogenous state variables $\left(N, K, L, L_R^{(-1)}, A\right)$ and a realization of the exogenous aggregate TFP shock ζ. Then final good sector firms and their employees bargain over the current wage ω. Simultaneously, firms rent the available physical capital and purchase the available intermediate goods for inputs in production. Innovators hire labor and use the available loans to finance final good expenditures as inputs in R&D. Next, production y is realized and the R&D activity determines the number of intermediate good varieties available for use as inputs in production in the next period N'. Once production is determined and production factors are paid, firms decide the number of vacancies to post v. The currently unemployed workers search for jobs and the number of new realized hires depends on the current labor market tightness θ. The matching process together with exogenous destruction of a part of the already existing jobs, determines the next period employment level L'. Finally, households decide the desired levels of consumption, investment in physical capital, and investment in R&D. The investment determines stock of physical capital and the quantity of final good for R&D input available for renting in the next period (K',A').

2.3.8 Equilibrium

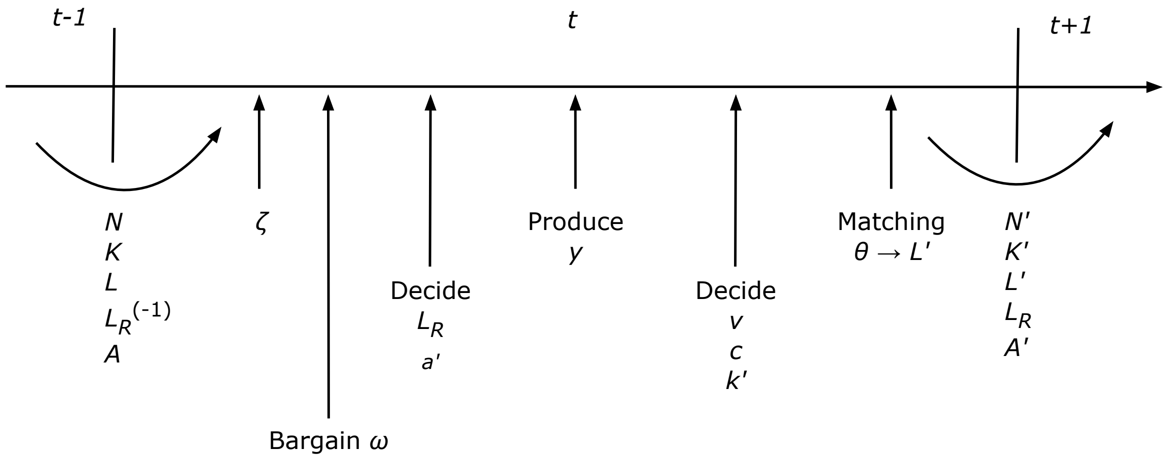
We are now ready to define an equilibrium.

Definition. A recursive equilibrium is a collection of value functions $V(l,\Omega), W(i;\Omega), Q(\Omega^\star)$ and decision rules $f \equiv \left\{f_{x(i)}(l,\Omega), f_k(l,\Omega), f_l(l,\Omega), f_v(l,\Omega), f_{l_R}(\Omega), f_a(\Omega)\right\}$, $g \equiv \left\{g_p(i;\Omega)\right\}$, and $h \equiv \{h_c(\Omega^\star),$ $h_k(\Omega^\star), h_l(\Omega^\star), h_{l_R}(\Omega^\star), h_n(\Omega^\star), h_a(\Omega^\star)\big\}$, government fiscal policy $t \equiv \{b(\Omega),\ \tau(\Omega)\}$, a collection

[12]See Ljungqvist and Sargent (2012a, p. 1194).

[13]In practice, expressions of $J(\cdot)$ and $H(\cdot)$ are obtained by the envelope conditions with respect to labor in the firm's and household's problems. See Gertler and Trigari (2009) for a similar treatment of frictional labor market in a dynamic stochastic general equilibrium model.

Figure 2.6: Model timing



of pricing and aggregator functions $p \equiv \{P(i,\Omega), R(\Omega), r(\Omega), \omega(\Omega), \ \omega_R(\Omega), \theta(\Omega), L_R(\Omega)\}$, and law of motion for the aggregate state $G_\Omega(\Omega)$ such that

- given p, $G_\Omega(\Omega)$, and t, the final good producers and the intermediate goods producers maximize their profits, and the households maximize their utility, that is, the value functions solve problems (2.3), (2.14), and (2.22), and f, g, and h are the associated decision rules,

- given p and $G_\Omega(\Omega)$, the labor and final good input demand in the R&D sector is determined by the free entry equilibrium conditions (2.18) and (2.19),

- given p and $G_\Omega(\Omega)$, the fiscal policy balances government's budged, that is, equation (2.35) holds,

- the wage rate in the final good sector is determined by the bargaining between the firms and workers, that is, $\omega(\Omega)$ solves (2.40),

- aggregate consistency holds:
  (i) the law of motion of aggregate state is consistent with the individual decision rules, the markets for labor in the R&D sector and for capital clear, and the mass of filled vacancies in the production sector

corresponds to the mass of workers with job in that sector

$$G_{\Omega}(\Omega) = \begin{pmatrix} \exp\left[(1-\rho)\ln\bar{\zeta} + \rho\ln\zeta + \epsilon_{\zeta}\right] \\ (1-\delta_N) N \left\{1 + \chi h_{l_R}\left(\tilde{\Omega}^{\star}\right)^{\iota} \left[\frac{h_a(\tilde{\Omega}^{\star})}{N^{1/(1-\alpha)}}\right]^{1-\iota}\right\} \\ h_k\left(\tilde{\Omega}^{\star}\right) \\ h_l\left(\tilde{\Omega}^{\star}\right) \\ h_{l_R}\left(\tilde{\Omega}^{\star}\right) \\ h_a\left(\tilde{\Omega}^{\star}\right) \end{pmatrix} = \begin{pmatrix} \exp\left[\left[(1-\rho)\ln\bar{\zeta} + \rho\ln\zeta + \epsilon_{\zeta}\right]\right. \\ (1-\delta_N) N \left\{1 + \chi f_{l_R}(\Omega)^{\iota} \left[\frac{f_a(\Omega)}{N^{1/(1-\alpha)}}\right]^{1-\iota}\right\} \\ f_k(L,\Omega) \\ f_l(L,\Omega) \\ f_{l_R}(\Omega) \\ f_a(\Omega) \end{pmatrix}, \tag{2.42}$$

where $\tilde{\Omega}^{\star} \equiv \left\{K, L, L_R^{(-1)}, A, \Omega\right\}$,

(ii) the pricing function for intermediate goods is consistent with the intermediate goods producer's decision rule and the market tightness is consistent with the hiring decisions

$$P(i;\Omega) = g_p(i;\Omega), \tag{2.43}$$

$$\theta(\Omega) = \frac{f_v(L,\Omega)}{1 - L - L_R(\Omega)}, \tag{2.44}$$

(iii) the market for final good clears

$$\begin{aligned} \zeta\left[\int_0^N f_{x(i)}(L,\Omega)\,di\right]^{1-\psi}\left(K^{\alpha}L^{1-\alpha}\right)^{\psi} - \int_0^N \mu f_{x(i)}(L,\Omega)\,di = \\ h_c\left(\tilde{\Omega}^{\star}\right) + h_k\left(\tilde{\Omega}^{\star}\right) - (1-\delta_K)K + h_a\left(\tilde{\Omega}^{\star}\right) + \frac{1}{2}c_R h_n\left(\tilde{\Omega}^{\star}\right)^2 \\ + \kappa\left\{(1-\delta_N)N\left[1 + \chi\left(f_{l_R}(\Omega)\right)^{\iota}\left(N^{-1/(1-\alpha)}f_a(\Omega)\right)^{1-\iota}\right]\right\}^{\frac{1}{1-\alpha}} f_v(L,\Omega) \end{aligned} \tag{2.45}$$

2.3.9 Balanced growth path

The non-stochastic version of the model features a well-defined balanced growth path (BGP) along which the mass of available varieties of intermediate inputs N grows at a constant endogenous rate $\bar{g}_N$. The final good output, consumption, stock of capital, goods expenditures in R&D investment, and wages in both final good and R&D sectors grow at constant rate $\bar{g}_y = \bar{g}_N^{\frac{1}{1-\alpha}}$. The share of workers employed in the R&D sector, unemployment rate, labor market tightness, prices of intermediate goods, rental price of capital, and interest rate are constant. We use extensively the BGP in calibration by requiring that, along the BGP, the model matches a set of empirical moments.

Moreover, by dividing any growing variable x by an appropriate factor, the model rewritten in terms of normalized variables $\tilde{x} \equiv x/N^{\frac{1}{1-\alpha}}$ becomes difference-stationary.[14] This facilitates solving the model. When conducting the simulations, we accumulate and add back all effects of changes in N in order to obtain simulated data in levels before calculating any statistics.

2.4 Calibration

We solve the model numerically using perturbation methods.[15] In this section, we describe the calibration of the model's parameters. In the next section, we proceed to numerical simulations and discuss the ability of the model to generate medium-term cycles. We use two distinct calibrations. The baseline calibration considers model time period length of one quarter as in much of the business cycle literature. The second parameterization (annual calibration) is based on model time period length of one year. We opt for using two different calibration strategies because fluctuations we are modeling are considerably more persistent than typically considered in the macroeconomics-labor literature and the time aggregation of the quarterly calibrated model is not straightforward. Comparing the results of the yearly simulated model to those of the quarterly model also provides a consistency check of the mechanism of interaction of R&D investment and labor market frictions over the medium-term cycles.

2.4.1 Baseline calibration

There are in total 20 parameters to calibrate. Our strategy is to calibrate a first set of nine parameters to the values provided by independent studies and the standard practice in quantitative macroeconomics literature. The final good production function parameter α is chosen to replicate the capital income share of $1/3$. We set the marginal cost of intermediate input production $\mu = (1-\psi)^2$. This is a convenient normalization. The per-period utility function is assumed to have constant relative risk aversion form $u(c) = \left(c^{1-\sigma} - 1\right)/(1-\sigma)$, where 2 is chosen as the coefficient of relative risk aversion, which implies greater risk aversion than logarithmic utility.[16] The capital depreciation rate δ_k is set to 2.6% (approximately 10% per annum) in reference to the standard practice in the quantitative macroeconomics literature (see e.g., Chugh, 2016). The gross growth rate of the innovation possibility frontier along the BGP is given by the law of motion for the number of varieties of intermediate goods (2.16). Following Li and Hall (2020), the obsolescence rate of new technologies, δ_N, is set equal to 6.9% (~25% per annum). This value is also in line with the main conclusions from Pakes and Schankerman (1984) and Huang and Diewert (2011). Following Goolsbee (1998), we fix the value for the elasticity of innovation with respect to labor ι to 0.667. The technology coefficient associated with the innovation process is normalized to one ($\eta = 1$). Using BGP restrictions, the average of the TFP shocks process is set to one and the separation rate in the R&D sector is set to 0.027. Finally, we specify

[14] The transformed model is not trend-stationary because any changes in level of N have permanent effects on the trend.

[15] We use log-linearization around a deterministic BGP and Dynare software package to obtain an approximation to the solution of the system of equations characterizing the equilibrium.

[16] A value of 2 is also chosen in Walsh (2005). A plausible range of alternative values for σ varies between 0.5 and 2.5 (see Table 6.1 in DeJong and Dave (2005) for more details). Aghion and Howitt (1994) and Mortensen (2005), among others, showed that a value for σ greater than 1 guarantees the capitalization effect of growth.

the parameter governing the worker's bargaining power $\xi = 0.45$, which is near the midpoint of the range of values in the macroeconomics-labor literature.[17]

Table 2.5: Baseline calibration. US quarterly data: 1951–2006

1. External Calibration				
Parameter	Description	Parameter value	Source	
α	Capital share in the output	0.333	National Income (BLS)	
σ	Relative risk aversion	2.000	Mortensen (2005)	
δ_k	Capital depreciation rate	0.026	~ 10% per annum (BEA)	
δ_N	Technology depreciation rate	0.069	Li and Hall (2020)	
ι	R&D labor elasticity	0.667	Goolsbee (1998)	
η	Technology coefficient	1.000	Normalization	
δ_R	Separation rate in R&D sector	0.027	BGP restriction	
$\overline{\zeta}$	Average TFP process	1.000	BGP restriction	
ξ	Worker's bargaining power	0.450	Midrange value in macro-labor	
2. Internal Calibration				
Parameter	Target (US data, 1951:01–2006:01)	Parameter value	Target data	Target model
β	Real interest rate	0.998	0.0127	0.0127
ψ	investment–GDP ratio	0.692	0.2067	0.2159
ϕ	Average GDP growth	0.531	1.0054	1.0054
$\overline{c}_R$	R&D workers share (NSF, ν_l)	1.2e6	0.0058	0.0058
δ_l	Average job-finding rate	0.027	0.4529	0.4529
γ	Average labor market tightness	1.371	0.9746	0.9746
κ	Vacancy filling cost in % of wage	0.050	0.1400	0.1400
$\overline{b}$	Average unemployment rate	1.057	0.0566	0.0566
Downhill simplex minimization algorithm:				
ρ_ζ	Persistence TFP from Fernald (2014)	0.932	0.9296	0.8873
$\sigma_{\epsilon\zeta}$	Std. dev. TFP from Fernald (2014)	0.008	0.0126	0.0126
N_0	First observation of TFP level in log	3.203	3.1186	3.1186

The values of the remaining eleven parameters are set to allow the model to replicate selected empirical moments along the BGP. All the moments we use as calibration targets are quarterly averages in the US data over the period 1951Q01–2006Q01. We choose the value for the discount factor β to match the annual long-term real interest rate of 5.0% in reference to the average annual return on the S&P500 index.[18] The elasticity of the production function with respect to intermediate inputs $(1-\psi)$ matches the investment–GDP

[17] See, for instance, Shimer (2005), Hagedorn and Manovskii (2008), Hall and Milgrom (2008), Gertler and Trigari (2009), Gervais et al. (2015), Petrosky-Nadeau, Zhang, and Kuehn (2018), and Drautzburg, Fernández-Villaverde, and Guerrón-Quintana (2021).

[18] DeJong and Dave (2005), and Hornstein, Krusell, and Violante (2005), among others, use the same value.

ratio.[19] The curvature parameter of the law of motion for the technology frontier ϕ is set so that the model reproduces the average output growth rate of 0.54% (approximately 2.16% per annum). Parameter $\overline{c}_R$ is set so that the model matches the aggregate share of workers employed in R&D sector. We use the data from the National Science Foundation (NSF) on employment in R&D. For calibrating our model we consider workers in R&D sector as the people employed by R&D-performing companies who were engaged in scientific or engineering work at a level that required knowledge gained either formally or by experience. The separation rate δ_l in the final good sector ensures an average value for the job finding rate of 0.4529 as estimated from the Robert Shimer's database. We follow den Haan, Ramey, and Watson (2000) and choose the matching function of the form $m(u, v) = uv/(u^{\gamma} + u^{\gamma})^{1/\gamma}$. This specification ensures that matching probabilities always lie in $[0, 1]$ while retaining the properties of monotonicity, concavity, and constant returns to scale. We calibrate the value of parameter γ to match the average value of vacancy–unemployment ratio of 0.975. We obtain this last value from den Haan and Kaltenbrunner (2009) who used the Job Openings and Labor Turnover Survey (JOLTS) data and the Help Wanted index to estimate the quarterly number of vacancies. As in Elsby and Michaels (2013), the vacancy posting cost κ is targeted to per worker hiring cost of 14% of quarterly worker compensation. We set the value of non-market activity for the workers $\overline{b}$ to match the average value for the unemployment rate of 5.66%. The targets for the calibration of the persistence (ρ_{ζ}) and volatility ($\sigma_{\epsilon\zeta}$) of the AR(1) process governing the productivity shocks are the first-order auto-correlation and volatility of the high-frequency component of Fernald's (2014) TFP series. Because the (value-added) TFP is endogenous in our model, we chose the values for these parameters that minimize the distance between the moments in the simulated data and their empirical counterpart. Finally, we set the initial mass of available intermediate goods in period zero (N_0) to match the TFP level in the first quarter of 1951 as measured by Fernald (2014).

There are two possibly less standard aspects in our calibration that seem important for the quantitative implications of the model. The first is the calibration of the innovation technology. We used available estimates in recent economic literature, the average GDP growth rate and the average share of workers employed in R&D over our sample period to assign values to the parameters governing R&D investment productivity and the incentives to become R&D-specialized worker. It is reassuring that the calibration implies the model is able to closely match the non-targeted average ratio of R&D investments to GDP along the BGP, as can be seen in Table 2.6. Second, the calibration of the value of non-market activity for workers can have important influence on the ability of the model to amplify the effects of fluctuations on the labor market quantities. While not explicitly targeted, our calibration implies the value for the replacement ratio of 0.9. Reassuringly, this is in the ballpark of values used in the recent literature.[20]

[19]Gomme and Rupert (2007) advocated the use of investment–GDP ratio rather than capital–output ratios. The periodic revisions to the capital stock data are so large, and the conceptual questions about what should be included in the capital stock are so difficult to satisfactorily answer, that estimates of the capital–output ratios are too unreliable to use as a calibration target.

[20]Although our replacement ratio is larger than the value of 0.75 in Pissarides (2009), it is relatively close to 0.85 in Petrosky-Nadeau, Zhang, and Kuehn (2018) or to 0.88 as estimated by Christiano, Eichenbaum, and Trabandt (2016). Our value for the replacement ratio is, however, lower than the value of 0.97 used by Hagedorn and Manovskii (2008).

Table 2.6: Non-targeted moments

	Value	
Moments	Data/Literature	Model
R&D/GDP	0.0245	0.0274
Replacement ratio	0.75 – 0.97	0.899

2.4.2 Alternative calibration

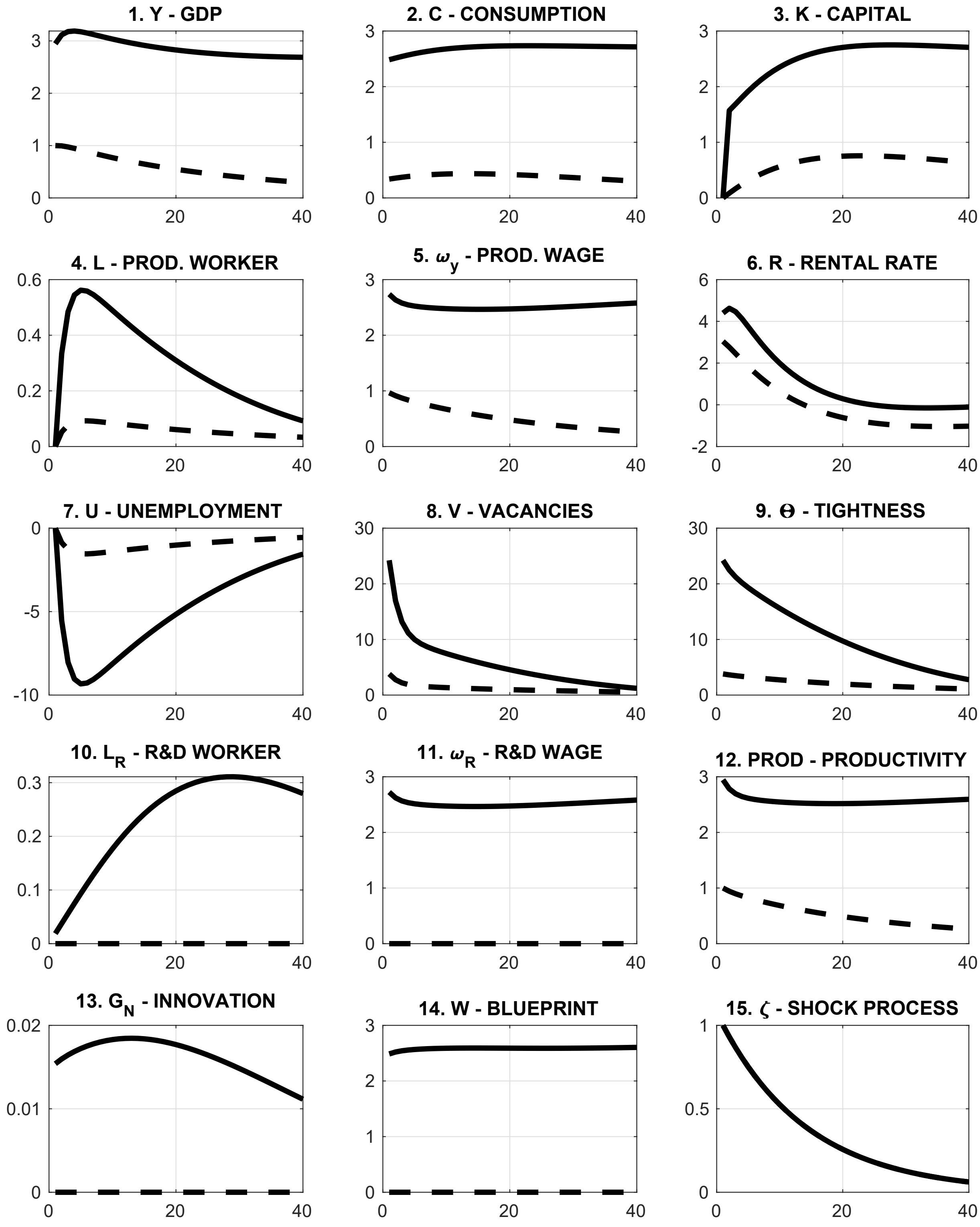
Following Comin and Gertler (2006) we also consider an annual calibration. The moments used as calibration targets are now annual averages in the US data (1953–2006). As summarized in Table 2.7, we use the same targets as in our baseline calibration for most parameters, only adequately adjusted for annual observations. The annual job finding and separation rates are obtained from the average quarterly job finding and separation rates provided in by Robert Shimer by following the transition probability tree over four quarterly sub-periods. This yields $f = 0.8547$ and $\delta_l = 0.052$ at the annual frequency. Differently from the quarterly calibration, the worker's bargaining power parameter ξ is now internally calibrated. We set its value to 0.0424 in order to obtain the same replacement ratio of 0.9 as implied by the quarterly calibration. The annual series for TFP from which are calculated the target moments for calibration of ρ_ζ, $\sigma_{\epsilon\zeta}$, and N_0 is from BLS.

2.5 Results

We follow the RBC literature by examining how well a single shock to TFP, presumed to be the principal driving force of economic fluctuations, can explain the unconditional patterns in the data. The RBC literature focused on the ability of technology shocks to account for short-term fluctuations. We instead explore the ability of our model to account for both short- and medium-term fluctuations.

2.5.1 Impulse response functions

We first analyze impulse response functions to gain some insight into the endogenous propagation mechanism of the model. In order to highlight the importance of endogenous growth for the propagation and amplification of shocks we report two sets of impulse response functions in Figure 2.7. One generated from our benchmark model with endogenous growth (solid lines), and another generated from a model without R&D sector and with constant mass of one of the varieties of intermediate goods (dashed lines). In that version the long-run growth is generated by exogenous constant growth in the TFP of the final good production sector. This model thus corresponds to a standard RBC model with search-and-matching frictions in the labor market (RBC-SM). We re-calibrate the RBC-SM model to match the relevant subset of the same empirical moments as our benchmark model. The magnitude of the shock is of 1% deviation from the BGP and the impulse responses are depicted as percentage deviations from the models' BGPs.



Note: Figure 2.7 plots the model impulse responses to a TFP shock of 1% deviation from the BGP. The solid line in each panel depicts the response in our benchmark model with endogenous growth. The dashed line depicts the response in the RBC-SM model. The y-axis scale is in percentage deviations from a BGP. The x-axis scale is the time horizon. The period length is one quarter.

Table 2.7: Alternative calibration. US annual data: 1953–2006

Parameter	Description	Parameter value	Source	
1. External Calibration				
α	Capital share in the output	0.333	National Income (BLS)	
μ	Cost of int. goods production	$(1-\psi)^2$	Normalization	
σ	Relative risk aversion	2.000	Mortensen (2005)	
δ_k	Capital depreciation rate	0.100	BEA	
δ_N	Technology depreciation rate	0.250	Li and Hall (2020)	
ι	R&D labor elasticity	0.667	Goolsbee (1998)	
η	Technology coefficient	1.000	Normalization	
δ_R	Separation rate in R&D sector	0.052	BGP restriction	
$\overline{\zeta}$	Average TFP process	1.000	BGP restriction	
2. Internal Calibration				
Parameter	Target (US data, 1953–2006)	Parameter value	Target data	Target model
β	Real interest rate	0.994	0.0500	0.0500
ψ	investment–GDP ratio	0.694	0.2068	0.2142
ϕ	Average gross output growth	0.261	1.0216	1.0216
$\overline{c}_R$	R&D workers share (NSF, ν_l)	9.8e4	0.0058	0.0058
δ_l	Average job-finding rate	0.052	0.8547	0.8547
γ	Average labor market tightness	0.890	0.9389	0.9389
ξ	Replacement ratio	0.042	0.9000	0.9000
κ	Vacancy filling cost in % of wage	0.824	0.4530	0.4530
$\overline{b}$	Average unemployment rate	0.350	0.0573	0.0573
Downhill simplex minimization algorithm:				
ρ_ζ	Persistence of TFP (BLS)	0.820	0.1055	-0.0155
$\sigma_{\epsilon\zeta}$	Std. dev. of TFP (BLS)	0.012	0.0116	0.0116
N_0	First observation of TFP level in log	4.036	3.9884	3.9884

A positive TFP shock raises the productivity of intermediate inputs, capital, and labor in the final good production sector. As a result, for given prices, the demand for intermediate inputs and capital would increase. In equilibrium, this leads to a higher quantity of each existing variety of intermediate goods being used as input in production of the final good and to an increase in the price index of intermediate inputs. Notice however that on impact there is not an immediate increase in the number of varieties of intermediate goods since this mass is predetermined by previous period R&D efforts. Similarly, given that on impact the stock of capital is predetermined by previous period investment decisions (panel 3), the rise in demand for capital is entirely compensated by a sharp increase in the equilibrium rental rate of capital (panel 6). Because hires of workers by the final good firms are also predetermined, the quantity of labor input in the final good sector stays also unchanged on impact (panel 4). Nevertheless, the current jobs create a higher surplus to be shared between the firms and workers reflecting into a higher wage rate in the final good sector (panel 5).

Because the stochastic process governing TFP shocks is persistent, a high realization of the shock today creates expectations of higher productivity shocks also in the following periods. This increases the expected future productivity of intermediate inputs, capital, and labor, leading to a rise in the value of intermediate goods producer firms (panel 14), investment and future capital stock (panel 3), and the final good firms' incentives to post vacancies (panel 8). The higher value of intermediate goods firms spurs more innovation (panel 13) by increasing incentives to commit more resources to R&D (panel 10). In particular, the innovators demand more of specialized R&D labor. As a result, more workers become R&D specialists instead of searching for a job in the final good sector, the unemployment falls, and wages in the R&D sector rise (panel 11).[21]

Moreover, a TFP shock produces effects similar to those of news about future technologies. The higher discovery of new varieties of intermediate inputs further enhances the productivity of the final good sector in all subsequent periods (panel 12), triggering a powerful feedback effect that leads to large and persistent responses of macroeconomic and labor variables to productivity fluctuations. This amplification mechanism generates responses of much larger magnitude in both the impact period and over the subsequent periods compared to those in the RBC-SM model in which the mechanism is absent.

Finally, notice that, in the benchmark model, GDP, consumption, capital stock and wages do not go back to their original BGP levels in the long run. Instead, they stay permanently higher on a new BGP level. This is a consequence of a permanent increase in productivity due to the rise of the number of available intermediate input varieties. By contrast, in the RBC-SM model the variables simply revert to their original BGP levels in the long run.

2.5.2 Medium-term cycles

We now quantitatively evaluate the capacity of the model to generate the medium-term fluctuations patterns observed in the US data. We use simulation of the model to generate artificial time series of the same length as the available US data. We generate 1,000 such simulations and for each simulation we compute counterparts to the empirical moments studied in Section 2.2. We then report average values of these statistics and compare them to the empirical evidence.[22] As when studying the impulse response functions, we also report the statistics generated by the simulations of the exogenous growth model (RBC-SM) for comparison purposes.

[21]The value of L_R and U are modified already on impact but the magnitudes are small relative to the following periods rendering it difficult to clearly visualize in Figure 2.7. The adjustment of L_R is gradual due to high sunk costs of R&D training.

[22]We treat the artificial data exactly as we do the real data. In particular, in each simulation, we re-accumulate non-stationary time series by adding trend growth back. We then apply the same transformations and procedures to the artificial data as those described in Section 2.2 to isolate medium-term cycle and its high- and medium-frequency components.

Table 2.8: Standard deviations: Data *versus* model:

	Medium-term cycle			High-frequency component		
	Data	Benchmark	RBC-SM	Data	Benchmark	RBC-SM
Quarterly Data						
GDP	3.30	3.47	2.49	1.49	1.51	1.28
Consumption	2.25	2.65	1.13	0.75	0.90	0.35
Investment	8.07	7.16	6.77	4.15	3.67	3.85
TFP	2.90	2.29	2.12	1.26	1.26	1.26
R&D	8.87	2.93	···	3.28	1.05	···
Productivity	3.27	2.42	2.28	1.25	1.13	1.20
Wage	2.95	2.23	2.20	1.03	1.00	1.15
Weighted wage[a]	···	2.22	···	···	0.99	···
Unemployment	21.99	20.62	3.91	12.16	8.33	1.81
Vacancies	23.77	27.01	5.29	13.45	15.75	3.33
Tightness	42.44	44.90	8.56	25.20	20.48	4.39
Annual Data						
GDP	3.50	3.18	1.30	1.38	1.13	1.07
Consumption	2.33	2.25	0.39	0.69	0.74	0.17
Investment	8.42	6.06	4.07	3.96	2.20	3.52
TFP	3.00	2.16	1.32	1.16	1.16	1.16
R&D	7.66	2.68	···	2.37	0.91	···
Productivity	3.31	2.03	1.29	1.12	1.13	1.10
Wage	3.12	0.74	0.62	0.98	0.32	0.50
Weighted wage	···	0.77	···	···	0.33	···
Unemployment	21.50	24.50	1.95	11.26	9.54	1.12
Vacancies	25.52	30.78	3.15	14.00	21.57	2.73
Tightness	43.37	45.26	3.66	24.30	18.98	2.26

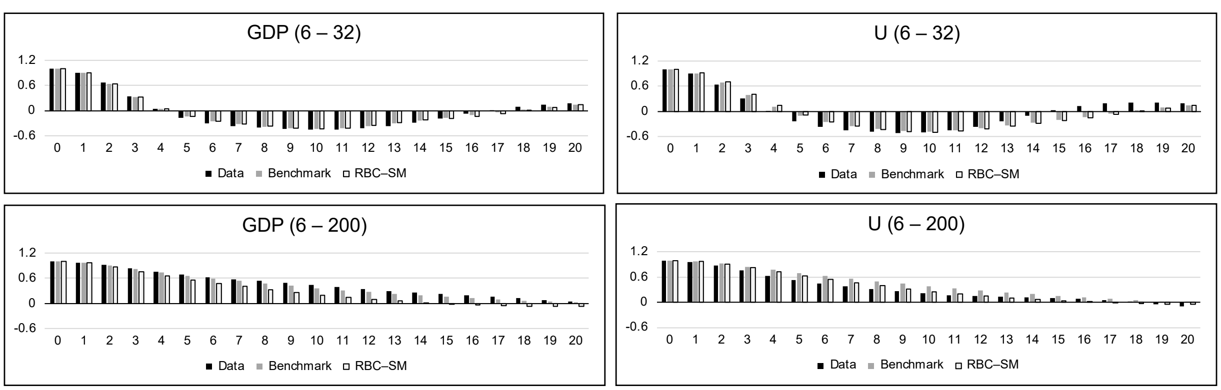
Note: The reported model standard deviations are average statistics over 1,000 simulations of a sample size corresponding to the data. [a]Weighted wage is the average wage across final good and R&D sectors weighted by the employment shares of each sector.

Volatility. Table 2.8 reports the percent standard deviations of the variables over the medium-term cycle, and its high-frequency component (the traditional business cycle). Overall, the benchmark model does a reasonably good job in capturing the breakdown of the variation between the high- and medium-frequency components. Importantly, the benchmark model does particularly well in reproducing the volatility of labor market quantities at both medium-term and business cycle frequencies and this is in stark contrast with the poor performance of the RBC-SM model along this dimension. The main amplification mechanism behind this success is the interaction between the effects of the TFP shock on expected future productivity in the final good sector and the demand for specialized R&D labor. As we have explained with the impulse response functions, a positive TFP shock indirectly raises innovators' demand for specialized labor. This creates new opportunities to which the unemployed workers can turn instead of searching for a job in the final good sector, and thus decreases unemployment. As a result, the elasticity of labor market tightness to productivity in our benchmark model is higher than in the RBC-SM model, which lacks this mechanism. The fact that the rise in demand for specialized R&D labor is pro-cyclical and persistent leads to amplified and persistent responses of labor market quantities to TFP shocks. It is worth to emphasize that this mechanism is different from those proposed in most of macro-labor literature. In particular, the higher elasticity of labor market tightness in our model is not obtained through a reduction of what Ljungqvist and Sargent (2017) call the fundamental surplus of a job.

For other variables, the quarterly calibrated benchmark model generates volatility that is relatively close to that observed in the data over both the medium-term cycle and its high-frequency component with exception of R&D, which is less volatile in the model than in the data. The performance of the benchmark model is slightly better than that of the RBC-SM model, especially over the medium-term cycle. This difference is more apparent for the annually calibrated versions, which also do a poorer job in reproducing the volatility of labor productivity and wages.

Persistence. Table 2.9 shows that both the benchmark and the RBC-SM model are able to generate high first-order auto-correlation on the quarterly basis. However, examining more in detail the auto-correlograms for GDP and unemployment in Figure 2.8 reveals that the auto-correlation of the medium-term cycle decays faster in the RBC-SM model than in the benchmark model. This is also clearly visible in the bottom panel of Table 2.9, which shows that, on annual basis, the RBC-SM model is not able to produce sufficient persistence in the medium-term cycle.

Figure 2.8: Auto-correlograms: Data *versus* model



Note: Figure 2.8 plots the auto-correlograms of GDP per capita (GDP) and unemployment (U) in quarterly data and model simulated data for the high-frequency component (6-32) and the medium-term cycle (6-200). The reported model auto-correlations are average statistics over 1,000 simulations of a sample size corresponding to the data.

Table 2.9: First-order autocorrelations: Data *versus* model

	Medium-term cycle			High-frequency component		
	Data	Benchmark	RBC-SM	Data	Benchmark	RBC-SM
Quarterly Data						
GDP	0.978	0.974	0.965	0.908	0.900	0.897
Consumption	0.989	0.980	0.985	0.922	0.895	0.910
Investment	0.976	0.968	0.959	0.927	0.906	0.897
TFP	0.983	0.959	0.956	0.930	0.887	0.894
R&D	0.984	0.979	···	0.913	0.895	···
Productivity	0.983	0.965	0.962	0.898	0.879	0.892
Wage	0.989	0.979	0.975	0.928	0.912	0.917
Weighted wage[a]	···	0.968	···	···	0.882	···
Unemployment	0.967	0.979	0.975	0.903	0.912	0.917
Vacancies	0.970	0.943	0.937	0.919	0.853	0.860
Tightness	0.965	0.970	0.964	0.913	0.887	0.894
Annual Data						
GDP	0.822	0.809	0.123	0.118	0.142	-0.221
Consumption	0.889	0.812	0.714	0.216	0.012	-0.009
Investment	0.766	0.822	0.047	0.227	0.312	-0.226
TFP	0.836	0.622	0.014	0.105	-0.016	-0.239
R&D	0.849	0.802	···	0.049	0.011	···
Productivity	0.862	0.558	0.060	0.071	-0.140	-0.241
Wage	0.894	0.769	0.519	0.268	0.042	-0.127
Weighted wage	···	0.724	···	···	-0.037	···
Unemployment	0.726	0.769	0.548	0.121	0.042	-0.117
Vacancies	0.691	0.334	-0.036	0.173	-0.214	-0.333
Tightness	0.683	0.735	0.480	0.145	-0.010	-0.162

Note: The reported model auto-correlations are average statistics over 1,000 simulations of a sample size corresponding to the data. [a]Weighted wage is the average wage across final good and R&D sectors weighted by the employment shares of each sector.

Cyclicality. Figure 2.9 shows that the model captures quite well the cyclical co-movements of variables in the data at both the business cycle and medium-term cycle frequencies. However, in the medium-term cycle, labor productivity, vacancies, and wages tend to be more pro-cyclical in the model than in the data. The model also generates too much cross-correlation between past GDP and investment, vacancies, and labor market tightness over the medium-term cycle.

Lead of R&D. The R&D dynamics is important for amplification and propagation of shocks in our model. We now verify whether the co-movement of R&D with the key variables is also in line with the data. As we stressed in Section 2.2, taking in account the medium-term fluctuations in the annual data reveals a lead of R&D over several variables. By contrast, this lead is almost absent in the high-frequency component. Table 2.10 shows that the benchmark model is able to reproduce this pattern, to some extent, for GDP, consumption, TFP, labor productivity, and unemployment.

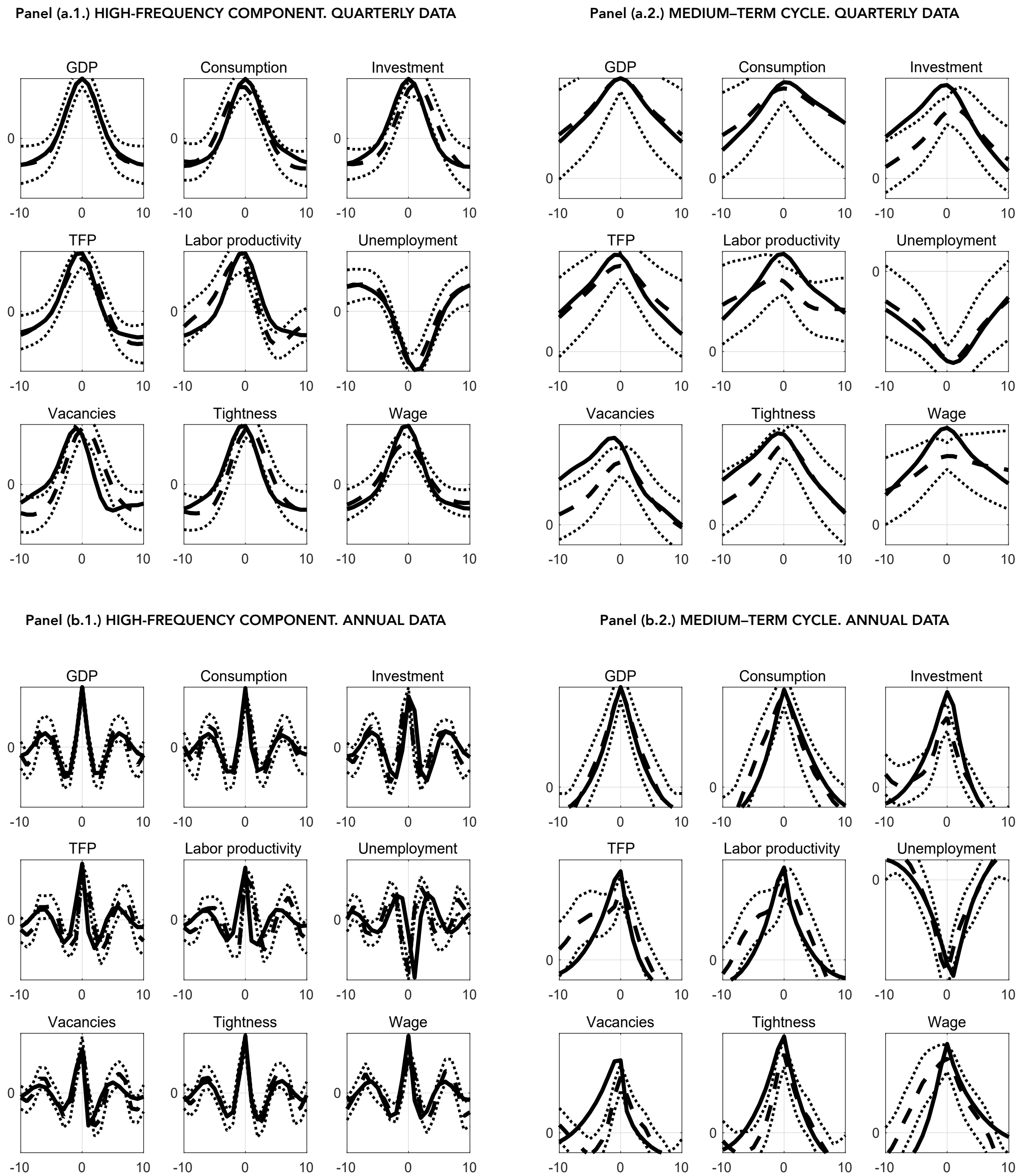
Figure 2.10 complements the evaluation by looking at detailed cross-correlograms of GDP per capita and unemployment rate with R&D at different leads in the quarterly real and simulated data. We can see that, in line with the data, the quarterly-calibrated benchmark model is able to generate a positive cross-correlation for up to 20 quarters' lead of R&D over GDP in the medium-term cycle and a negative cross-correlation for up to 20 quarters' lead of R&D over unemployment in the medium-term cycle. Moreover, these cross-correlations decay to zero by the lead of only 4 quarters in the high-frequency components. However, in the medium-term cycle, the model generates cross-correlations that are stronger than found in the real data, especially at closer leads.

Labor Market. In order to evaluate how well the model is able to explain the key labor market relationships we summarize each of them by the slope of the regression line and the correlation coefficient implied by the scatter plot of the relationship in the data. While the slope is informative about the elasticity of the relationship, the correlation coefficient is informative about its strength. Figure 2.11 shows the regression lines obtained from the real data (black) and the simulated data from the benchmark (red) and the RBC-SM (blue) models for the medium-term cycle (dashed) and its high-frequency component (solid). Table 2.11 then reports the associated slope and correlation coefficients.

In terms of the Beveridge curve (Panel (a) of Figure 2.11) both models seem to perform similarly, counterfactually generating a higher elasticity of vacancies to unemployment in the medium-term cycle than in its high-frequency component. The correlation between vacancies and unemployment in the simulated data is in line with the one observed in the real data in the medium-term cycle but it is much weaker for the high-frequency component.

In terms of the Okun's law (Panel (b) of Figure 2.11) the benchmark model generates elasticity of unemployment to GDP that is close to that observed in the real data in the medium-term cycle. By contrast, this elasticity is too high in the RBC-SM model. In both models there is virtually no differential in the slope of the regression line for the medium-term cycle and for its high-frequency component. Looking at the

Figure 2.9: Cross correlation with GDP per capita: Data *versus* model



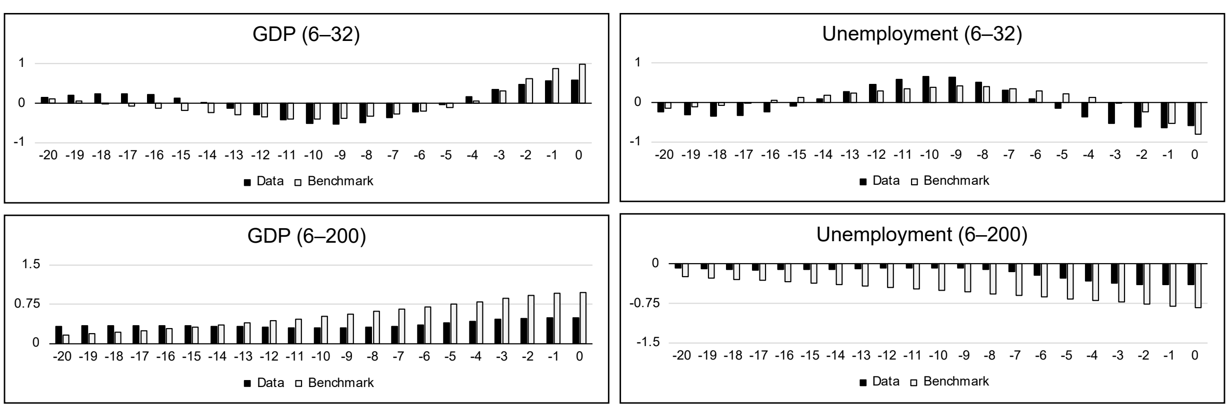
Note: Figure 2.9 plots the cross correlation with GDP per capita of the high-frequency component and the medium-term cycle at different leads and lags. The dashed lines depict the cross correlation in the data with 95% confidence bands (dotted lines). The solid lines depict the cross correlation in the benchmark model.

Table 2.10: Cross-correlation with R&D at $t - 5$: Data *versus* model

	Medium-term cycle		High-frequency component	
	Data	Benchmark	Data	Benchmark
Annual Data				
GDP	0.494	0.182	0.110	0.126
Consumption	0.590	0.095	0.055	0.118
Investment	0.074	0.248	0.202	0.163
TFP	0.518	0.268	-0.002	0.047
Labor productivity	0.616	0.172	-0.070	0.033
Wage	0.585	-0.047	-0.185	0.086
Weighted wage	···	-0.043	···	0.088
Unemployment	-0.308	-0.153	-0.087	-0.178
Vacancies	-0.115	0.282	0.027	-0.018
Labor market tightness	0.089	0.274	0.059	0.072

Note: The reported model cross-correlations are average statistics over 1,000 simulations of a sample size corresponding to the data. [a]Weighted wage is the average wage across final good and R&D sectors weighted by the employment shares of each sector.

Figure 2.10: Cross-correlation of GDP and unemployment with R&D: Data *versus* model



Note: Figure 2.10 plots the cross-correlation of GDP per capita (GDP) and unemployment (U) with R&D in quarterly data and model simulated data for the high-frequency component (6-32) and the medium-term cycle (6-200). The reported model cross-correlations are average statistics over 1,000 simulations of a sample size corresponding to the data.

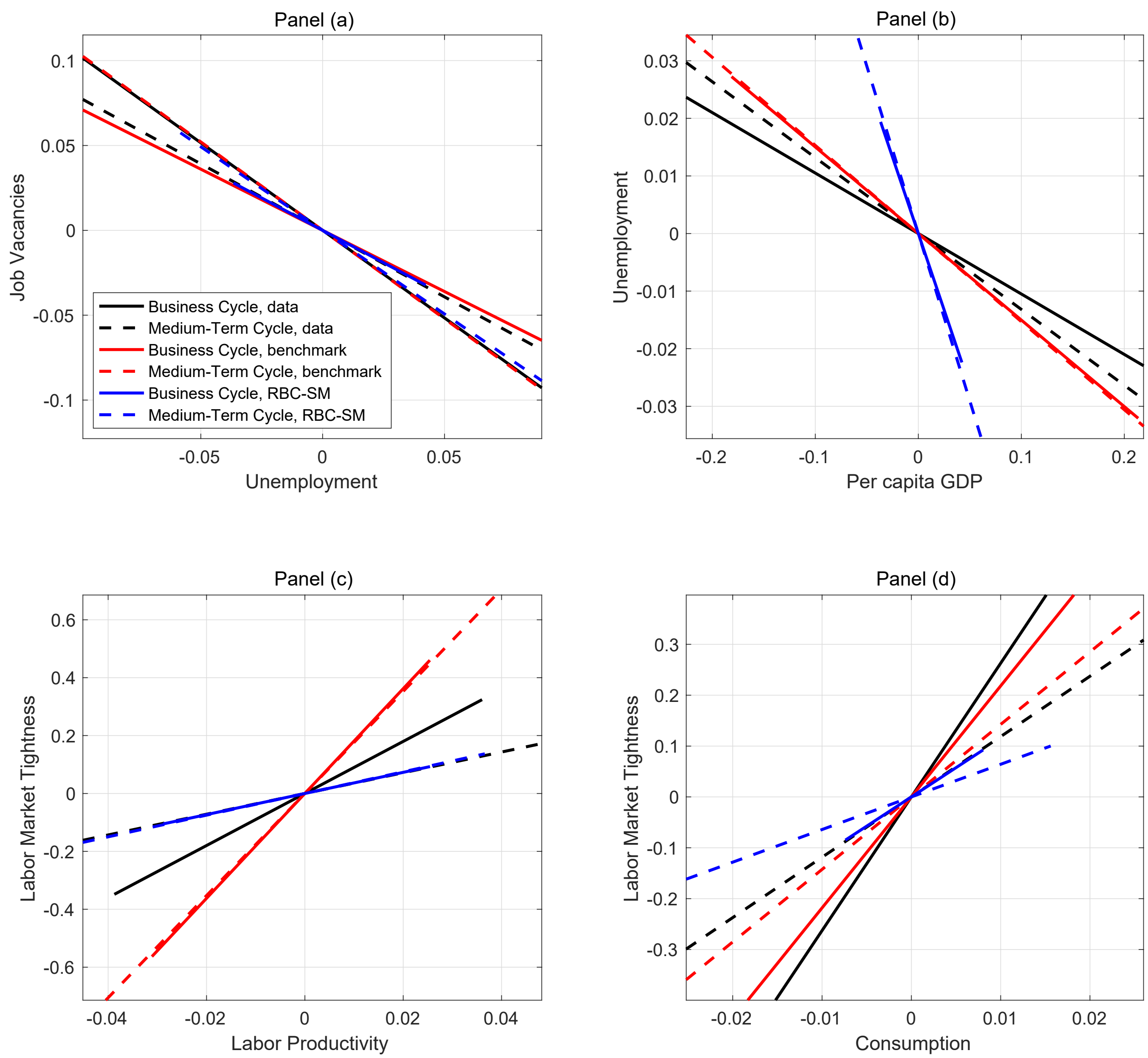
correlation coefficients, we see that both models imply a strong relationship between the two variables, which is consistent with the real data, with a slightly better performance of the benchmark model.

The relationship between labor productivity and labor market tightness (Panel (c) of Figure 2.11) in the benchmark model implies a higher elasticity of the tightness to productivity than in the data. However, the differential in the slope of the regression line between the medium-term cycle and its high-frequency component is quantitatively in line with the real data. By contrast, the RBC-SM model implies too low elasticity of tightness to productivity, that explains its failure to produce enough volatility in the labor market quantities. The RBC-SM model also fails to produce any significant differential in the value of this elasticity between the medium-term cycle and its high-frequency component. The correlation coefficients for this relationship implied by both models are too high compared to the data. This is typical in search-and-matching models of the labor market with flexible wages. In the plain RBC-SM model there is a straightforward one-to-one relationship between labor productivity and labor market tightness. In the benchmark model, however, labor market tightness in the final goods sector is also affected by the number of workers who decide to specialize in R&D labor and the hiring incentives for the innovators. Because specializing in R&D labor involves quadratic adjustment costs, and past discoveries have permanent effects on the value of future innovation, the final goods labor market is affected by these effects more gradually. This maps into a lower correlation between the labor productivity and the tightness in the medium-term cycle.

Finally, for the relationship between labor market tightness and consumption (Panel (d) of Figure 2.11) the benchmark model implies an elasticity between the two variables that is close to that in real data both in the medium-term cycle and its high-frequency component. By contrast, the elasticity implied by the RBC-SM model is too low. Both models imply slightly higher correlation coefficients than those observed in the real data for this relationship.

2.6 Conclusion

A growing body of literature stresses that in order to further improve our understanding of economic fluctuations, we need to consider the short-term behavior of the economy in connection to its longer-term dynamics. This paper provides an attempt to quantitatively evaluate the role of the R&D activity and innovation dynamics for explaining the short- and medium-term fluctuations of the labor market. Our results show that the expectations of the future evolution of productivity growth through innovation can significantly amplify and propagate the effects of current productivity shocks on the labor market variables. Moreover, the medium-term search-and-matching framework developed here complements the recent work emphasizing the role of the fundamental surplus in increasing the volatility of labor market variables over the business cycle (Ljungqvist and Sargent 2017, 2021). In our setup, the higher elasticity of unemployment with respect to labor market tightness is not obtained through a reduction of the fundamental surplus, but through the fact that the rise in demand for specialized R&D labor is pro-cyclical and leads to amplified and persistent responses of labor market quantities to shocks of a plausible magnitude. In the context of the macroeconomics-labor literature, this may reorient the interest towards investigating the role of interactions of growth and search



Note: Figure 2.11 plots regression lines for correlation patterns for key labor market relationships. Solid lines correspond to the high-frequency component. Dashed lines correspond to medium-term cycle. Black lines correspond to the linear fit in the US quarterly data. Red lines correspond to the linear fit in the data simulated with the benchmark model. Blue lines correspond to linear fit in the data simulated with the RBC-SM model.

Table 2.11: Labor market relationships: Data *versus* model

	Medium-term cycle			High-frequency component		
	Data	Benchmark	RBC-SM	Data	Benchmark	RBC-SM
Slope						
Unemp.–vacancies	-0.784	-1.042	-0.985	-1.032	-0.773	-0.763
GDP–unemployment	-0.132	-0.153	-0.581	-0.105	-0.154	-0.530
Productivity–tightness	3.581	17.638	3.745	8.993	18.161	3.650
Consumption–tightness	11.865	14.258	6.416	26.289	21.842	11.500
Correlation						
Unemp.–vacancies	-0.725	-0.759	-0.706	-0.933	-0.421	-0.411
GDP–unemployment	-0.878	-0.894	-0.900	-0.858	-0.851	-0.743
Productivity–tightness	0.276	0.827	0.999	0.445	0.997	1.000
Consumption–tightness	0.629	0.767	0.852	0.778	0.964	0.915

frictions for the labor market policies.

An important and interesting question that we leave for future research is how interactions between R&D activities and search-and-matching frictions affect welfare costs of business cycles (see e.g., Barlevy 2004, and Krebs 2007, among others).

CHAPTER III

LABOR MARKET DYNAMICS AND R&D-BASED GROWTH

ABSTRACT

We study the implications of R&D-based economic growth for labor market dynamics. We examine how the introduction of the R&D sector helps the search-and-matching model of the labor market to account for the main stylized facts on labor market fluctuations. We show that knowledge-driven R&D, which uses scientists' labor as input, entails a powerful amplification mechanism of productivity shocks. In particular, it enables the model to generate volatilities of labor market quantities that are in line with those observed in the US data in response to productivity shocks of plausible magnitude. By contrast, this amplification mechanism is muted when R&D necessitates only final goods input, as in the lab equipment setup. In that case, the model does a poor job in terms of generating labor market volatility. The key feature behind our result is the fact that labor only moves slowly between the R&D and production sectors, whereas expenditures on lab equipment can adjust immediately. We show that this amplification channel is different and complementary to the usual fundamental surplus channel studied by Ljungqvist and Sargent (2017, *American Economic Review*, 107 (9): pp. 2630-2665).

Keywords: Labor market dynamics, Growth, R&D technology.
JEL Codes: E24, E32, J64, O30.

3.1 Introduction

This paper studies the labor market dynamics in a framework that is similar to the standard real business cycle (RBC) model but that also incorporates search-and-matching frictions and an R&D-based growth process. In an influential study, Cogley and Nason (1995) showed that many RBC models have weak internal propagation mechanisms and must rely on external sources of dynamics to replicate the observed US output dynamics. Merz (1995) and Andolfatto (1996) argued that the labor market search-and-matching environment embodies a quantitatively important propagation mechanism for economic fluctuations in the context of a RBC model. However, Shimer (2005) showed that, in response to productivity shocks of plausible magnitude, a textbook search-and-matching model of the labor market generates too little volatility in labor market quantities compared with real data. A vast subsequent literature proposed several modifications and alternative calibrations of the basic model that help account for these volatilities.[1] Ljungqvist and Sargent (2017, 2021) demonstrated that a large bulk of these extensions boil down to a single common mechanism

[1]Hall (2005a,b), Costain and Reiter (2008) among others show that the wage stickiness helps the textbook search-and-matching model to account for the observed labor market dynamics. Costain and Reiter (2008) note that match-specific productivity shocks can improve the search model's performance. Hagedorn and Manovskii (2008) show that an alternative calibration of the textbook search-and-matching model can generate large fluctuations in labor market variables. Atolia, Gibson, and Marquis (2018) show that the Hagedorn-Manovskii calibration strategy may get the RBC search model closer to the data, but additional amplification mechanisms are needed. A large number of studies have proposed alternative mechanisms to account for labor market volatilities over business cycles. These mechanisms include: endogenous separations and costly capital mobility (den Haan, Ramey, and Watson, 2000); upfront cost for firms to secure credit (Wasmer, 2004); one-the-job search (Mortensen and Nagypál, 2007); cost of delay for firms that participate in alternating-offer bargaining (Hall and Milgrom, 2008); staggered Nash wage bargaining (Gertler and Trigari, 2009); fixed matching cost (Pissarides, 2009) heterogeneous-worker mechanism (Robin 2011; Murtin and Robin 2018); frictional

that consists of making small the surplus generated by a job and left to be shared between a firm and a worker. In this paper, we propose that, under certain conditions, R&D-driven endogenous growth entails a different and complementary amplification mechanism that is relevant for explaining labor market dynamics in the short and medium term.

Our paper makes three main contributions to the literature. First, we show that an RBC model with search-and-matching frictions in the labor market and with an R&D-based growth process can account for the observed labor market dynamics. Second, we formally examine how the way in which the R&D sector is introduced influences the ability of the search-and-matching model to amplify the effects of productivity shocks on labor market quantities. We show that, to have a powerful amplification mechanism that enables the model to reproduce the observed labor market volatility, the model should feature a knowledge-driven R&D process. That is, labor must be an important input in the innovation production. By contrast, when R&D-based growth results from the use of final goods as the only input to R&D activity (lab equipment setup), the model does a poor job in terms of generating labor market volatility. The key to understanding this result is the fact that labor reallocates only slowly between the R&D and production sectors. After a change in the relative productivity of the two sectors, workers do not quit their existing jobs easily to move immediately to the other sector. As a result, most adjustments go through changes in unemployment and labor market tightness, making these quantities more responsive to productivity shocks. Third, we derive an analytical expression for the elasticity of labor market tightness to productivity in the setup with knowledge-driven R&D. This analytical expression makes clear that the mechanism exploited in our paper is different and complementary to the one that has been shown by Ljungqvist and Sargent (2017, 2021) to encompass the vast majority of search-and-matching model extensions studied in the macro-labor literature. In particular, the effects of R&D on the labor market do not pass through the reduction of the "fundamental surplus" created by a job. Rather, they introduce an additional term in the expression of the labor market tightness elasticity with respect to productivity.

This paper builds on three literatures. We follow the literature on R&D-based growth process to model the innovation possibility frontier (R&D technology). In the general formulation of the model, we specify an innovation possibility frontier that uses both research labor (i.e., a knowledge-driven R&D specification as in Romer 1990), and final good (i.e., a lab equipment specification as in Acemoglu 2009, ch. 13) as inputs. We explicitly build in a frictional labor market with firms posting vacancies and unemployed searching for jobs as in Mortensen and Pissarides (1994), Merz (1995), Andolfatto (1996), and subsequent vast macro-labor literature. Finally, we follow the RBC literature to model both household decisions and the economy's production sector. To generate economic fluctuations, we introduce shocks to the total factor productivity (TFP) process in the final good production sector, as in Kydland and Prescott (1995). We use the den Haan and Kaltenbrunner (2009) database to compute empirical moments and characterize the US labor market

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financial markets (Petrosky-Nadeau and Wasmer, 2013); technological learning-by-doing (Gervais et al., 2015); hiring costs (Coles and Mortensen 2015, 2016); diminishing marginal revenue product of labor (Dao and Delacroix, 2017); government mandated unemployment compensation and layoff costs (Ljungqvist and Sargent, 2017); skill accumulation and skill loss (Lalé, 2018). See Mortensen and Pissarides (1999), Yashiv (2007), and Pissarides (2011) for survey, or textbooks by Pissarides (2000), Shimer (2010), Cahuc, Carcillo, and Zylberberg (2014), and Petrosky-Nadeau and Wasmer (2017) among others for a detailed exposition.

dynamics. We carefully calibrate the model to target some key moments in the US economy, including the investment/GDP ratio (Gomme and Rupert 2007, p. 494), an appropriate value of worker compensation (Hall and Milgrom 2008, p. 1654), and the R&D share of labor force (National Science Foundation, NSF).

The rest of the paper is organized as follows. Section 3.2 outlines our framework, a simple dynamic stochastic general equilibrium model with search-and-matching frictions and endogenous growth process. Section 3.4 discusses the main insights into the internal propagation mechanism of the model. Section 3.3 presents quantitative results from the calibrated version of the model. Section 3.5 concludes.

3.2 Model

Our framework can be viewed as a combination of a model of endogenous technical change with R&D-based growth process and an RBC model with search-and-matching labor market frictions. The economy consists of four types of agents (households, final good producers, intermediate goods producers, and government) and three sectors (final goods, intermediate goods, and R&D). Time is discrete and goes from zero to infinity. In what follows, we formally describe the problems of agents and the market structure in each of the sectors of the economy. The final good is taken as the numeraire, and its price is normalized to one.

3.2.1 Labor market

The market for labor in the final good sector is characterized by search-and-matching frictions. There is a matching technology that determines the number of job matches, m_t, as a function of vacant jobs and unemployed workers:

$$m_t = m(v_t, u_t), \tag{3.1}$$

where v_t denotes the number of vacancies open at time t, and u_t denotes the number of searching workers. Function m is assumed to be increasing in both arguments, jointly concave, continuously differentiable, and linearly homogeneous. The labor market tightness, θ_t, is given by v_t/u_t. Vacancies are filled at rate $m\left(1, 1/\theta_t\right) = m_t/v_t$ and unemployed workers switch to employment at rate $m\left(\theta_t, 1\right) = m_t/u_t$. A realized job match yields an economic surplus that is shared between firms and workers. We follow the literature and assume that the worker and the firm in a matched pair engage in Nash bargaining over the joint surplus, the outcome of which is the negociated wage. Let H_t be the worker surplus, J_t be the firm surplus, and $\xi \in (0, 1)$ the worker's relative bargaining power. Then, the bargained wage is:

$$\omega_t^\star = \arg\max_{\omega_t} G\left\{H_t(\omega_t), J_t(\omega_t); \xi\right\}, \tag{3.2}$$

where $G(\cdot)$ is the joint surplus written as a Nash product (see e.g., Ljungqvist and Sargent 2012b, p. 1194), H_t is the marginal value of employment in the household optimization problem, and J_t is the marginal value of filled job in the firm's optimization problem.

3.2.2 R&D sector

Entry into the R&D sector is free; any firm or individual can initiate research. New varieties of intermediate goods can be invented using labor and the final consumption good (lab equipment) as inputs for developing new products. To finance lab equipment purchases, innovators use loans from households. We assume that labor for the R&D sector is hired on a separate competitive spot market that is free from search-and-matching frictions. The newly invented varieties become available for use as production inputs in the final good sector in the following period. The number of existing varieties (or innovation possibilities frontier) evolves according to:

$$N_{t+1} = (1-\delta_N) N_t + (1-\delta_N) \chi_t N_t \left(l_{R,t}\right)^{\iota} \left(N_t^{-1/1-\alpha} a_t\right)^{1-\iota}, \tag{3.3}$$

where δ_N is the rate of obsolescence of technologies, N_t is the number of varieties in period t, $l_{R,t}$ is the number of workers employed in the R&D sector, a_t is the expenditures on lab equipment, α is the elasticity of final output Cobb-Douglas production function with respect to physical capital, ι is the elasticity of innovation goods with respect to labor input in the R&D sector, $(1-\iota)$ is the elasticity of innovation goods with respect to lab equipment expenditures, and χ_t is a congestion externality taken as given by the individual innovators. In equation (3.3), the term N_t simply indicates that researchers need to stand on the shoulder of past giants, that is, the future inventions are developed on the basis of today's technology frontier. The productivity of a_t is inversely proportional to N_t to an appropriate power. This feature of the R&D production function implies that on a balanced growth path (BGP), the expenditures a_t will be proportional to $N_t^{1/(1-\alpha)}$. The congestion externality χ_t depends on aggregate conditions in the economy:

$$\chi_t = \eta \left[{l_{R,t}}^{-\iota} \left(N_t^{-1/1-\alpha} a_t\right)^{\iota-1} \right]^{1-\phi}, \tag{3.4}$$

where $\eta > 0$ is a technology coefficient, and $0 < \phi \leq 1$ is a curvature parameter. Equation (3.4) implies that coming up with successful innovations increases in difficulty as the aggregate level of R&D activity increases.[2]

Our formulation of the law of motion of the technology frontier contains the knowledge-driven R&D and the lab equipment models as special cases with $\iota = 1$ and $\iota = 0$, respectively. In the next sections of the paper, we embed the search-and-matching frictions and R&D sector into a standard RBC model, in which the number of unemployed workers, vacancies, and investment decisions are determined by interaction of optimizing households and firms. We then compare the implications of the knowledge-driven R&D model and the lab equipment formulation for the labor market dynamics.

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[2]See Segerstrom (1998), Jones (1999), Eicher and Turnovsky (2000), Chu and Cozzi (2019) among others for more details on the importance of congestion effects in endogenous growth models.

3.2.3 Households

The representative household consists of a continuum of measure one of members. For each period t, household members are engaged in only one of the following three activities: working in the final good production sector, l_t; working in the R&D sector, $l_{R,t}$; searching for jobs (unemployment), $1 - l_t - l_{R,t}$. The distinction between employed and unemployed members of the household is important for the characterization of the labor market equilibrium. However, we assume that the household provides perfect insurance to its members in terms of consumption. Consequently, we can write the utility maximization problem at the household level and abstract from the intra-household transfers among the individual members. The representative household consumes, saves by either investing into physical capital or lending to innovators, and supplies labor services to the final good and R&D sectors. Let c_t be the consumption, k_t the stock of physical capital, l_t the labor supply to the final good sector, $l_{R,t}$ the labor supply to the R&D sector, and a_t the one-period household loans to innovators. The representative household derives utility from consumption and maximizes the expected discounted flow of utility:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}, \tag{3.5}$$

where $\beta \in (0, 1)$ is the time discount factor, and E_0 is an expectation operator conditional on the information available at period 0. The per-period utility function, $u(c_t)$, is assumed to be strictly increasing, strictly concave, and continuously differentiable. Given initial conditions $k_0, l_0, a_0 > 0$, the representative household chooses contingent plans $\{c_t, k_{t+1}, l_{t+1}, l_{R,t}, n_t, a_t\}_0^\infty$ subject to the budget constraints and the laws of motion of the household's capital stock and employment for all possible contingencies and dates:

$$\begin{aligned} c_t + k_{t+1} + \frac{1}{2} c_{R,t} n_t^2 + a_{t+1} = \omega_t l_t + \omega_{R,t} l_{R,t} + b_t \left(1 - l_t - l_{R,t}\right) \\ + [R_t + (1 - \delta_k)] k_t + (1 + r_t) a_t + \Pi_t^i + \Pi_t^f + T_t, \end{aligned} \tag{3.6}$$

$$l_{t+1} = (1 - \delta_l) l_t + m\left(\theta_t, 1\right)\left(1 - l_t - l_{R,t}\right), \tag{3.7}$$

$$l_{R,t} = (1 - \delta_{lR}) l_{R,t-1} + n_t, \tag{3.8}$$

$$c_t, k_{t+1} \geq 0,\ c_{R,t} = \bar{c}_R N_t^{\frac{1}{1-\alpha}},\ b_t = \bar{b} N_t^{\frac{1}{1-\alpha}}, \tag{3.9}$$

where ω_t is the wage in the final good sector, $\omega_{R,t}$ is the wage in the R&D sector, b_t is the unemployment benefit, R_t is the capital rental rate, r_t is the payoff on the loans, and T_t is lump-sum taxes. Capital depreciates at rate $\delta_k \in (0, 1)$. The household is the residual claimant of the profits generated by firms in the intermediate and final good sectors, (Π_t^i, Π_t^f). Equation (3.7) is the law of motion of employment in the final good sector, where $\delta_l \in (0, 1)$ is the exogeneous job destruction rate in the sector, and $m\left(\theta_t, 1\right)$ is the probability of finding a new job for the unemployed expressed as a function of labor market tightness. Finally, we assume that l_R is a distinct form of labor than l. To supply R&D labor, household members need to go through a training, which is costly. The cost of training (c_R) that the household has to pay is quadratic in the mass of agents undertaking the training, which we denote by n. Once trained, the household members are competent for R&D labor as long as they stay employed in the R&D sector. Once they quit that sector, their R&D skill fully depreciates. If they wish to reintegrate the R&D sector later, they have to go through the training again. Thus, for the representative household, even though the market for l_R is spot, the fraction of agents able to supply R&D labor evolves dynamically according to the law of motion that is given by equation (3.8), where $\delta_{lR} \in (0, 1)$ is the exogenous job destruction rate in the R&D sector.

3.2.4 Intermediate goods sector

A continuum of differentiated varieties of intermediate goods is produced by monopolistically competitive firms. Production function for each variety of intermediate good is identical and linear. Namely, one unit of variety i can be produced at the constant marginal cost $\mu > 0$ units of the final good. The optimization problem for the intermediate good firm i consists to maximize the following value of expected discounted profits by choosing $x_t(i)$ and $p(i)$:

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_{t-1,t} (1 - \delta_N)^t \{p_t(i) x_t(i) - \mu x_t(i)\}, \tag{3.10}$$

subject to the final good sector demand schedule. We assume that households are proprietors of the firms, where $\beta^t \Lambda_{t-1,t}$ is the firm's stochastic discount factor, such as $\Lambda_{t-1,t} = 0$ if $t = 0$, and $\Lambda_{t-1,t} = u'(c_t)/u'(c_{t-1})$ if $t > 0$.

3.2.5 Final good sector

The final good is produced by a continuum of measure one of identical and perfectly competitive firms according to production function:

$$y_t = \zeta_t z_t^{1-\psi} \left(k_t^{\alpha} l_t^{1-\alpha}\right)^{\psi}, \tag{3.11}$$

where ζ_t is an aggregate TFP shock, k_t and l_t are the respective inputs of capital and labor services, and z_t is a constant elasticity of substitution (CES) aggregate of intermediate goods:

$$z_t = \left[\int_0^{N_t} x_t(i)^{1-\psi} \,\mathrm{d}i\right]^{\frac{1}{1-\psi}}, \tag{3.12}$$

where $x_t(i)$ is the quantity of intermediate good of variety i. The parameters governing the elasticities of final output with respect to different inputs and the elasticity of substitution between the varieties of intermediate goods are such that $0 < \alpha < 1$ and $0 < \psi < 1$. A representative final good firm contracts the quantities of production inputs to maximize the present value of cash flows. Intermediate goods and physical capital are available on spot markets, whereas workers must be hired on the labor market one period in advance by posting job vacancies because of search-and-matching frictions. When making these decisions, the firm takes the rental rate for physical capital R_t, and the price of each variety i of intermediate good $p_t(i)$ as given. By contrast, the wage rate ω_t is determined by bargaining with the workers who are matched to the vacancies posted in the previous period. Let $\beta^t \Lambda_{t-1,t}$ be the firm's appropriate stochastic discount factor. The representative firm in the final good sector maximizes the expected discounted flow of profit:

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_{t-1,t} \left\{ \zeta_t z_t^{1-\psi} \left(k_t^{\alpha} l_t^{1-\alpha} \right)^{\psi} - \int_0^{N_t} p_t (i) x_t (i) \, di - R_t k_t - \omega_t l_t - \kappa \left(N_{t+1} \right)^{\frac{1}{1-\alpha}} v_t \right\}, \tag{3.13}$$

subject to the CES aggregator of intermediate goods (3.12), and the law of motion of the firm's employment:

$$l_{t+1} = (1 - \delta_l) \, l_t + m \left(1, \frac{1}{\theta_t} \right) v_t, \tag{3.14}$$

where κ is the parameter that determines per vacancy posting cost, δ_l is exogenous job separation rate, and $m(1, 1/\theta_t)$ is the probability of filling a vacancy expressed as a function of the labor market tightness θ_t. In equation (3.13) the firm's total vacancy posting cost, $\kappa \left(N_{t+1} \right)^{1/1-\alpha} v_t$, is assumed to be proportional to the innovation frontier given by the number of the intermediate good varieties that become available in the next period to an appropriate power. This assumption is akin to that of Pissarides (2000, *ch. 3*), who assumed vacancy posting cost to be proportional to labor productivity. This specification ensures the existence of a well-defined deterministic BGP equilibrium.

3.2.6 Government

The government finances unemployment benefits by collecting lump-sum taxes to maintain a balanced budget in every period and every contingency:

$$\left(1 - L_t - L_{R,t} \right) b_t = T_t, \tag{3.15}$$

where unemployment benefits, b_t, are assumed proportional to the current innovation frontier given by the number of intermediate good varieties that are available as inputs in production to an appropriate power:

$$b_t = \overline{b} N_t^{\frac{1}{1-\alpha}}, \tag{3.16}$$

where $\overline{b} > 0$ is a parameter. This assumption makes unemployment benefits proportional to the job productivity and ensures existence of a well-defined BGP equilibrium.

3.2.7 Equilibrium

An equilibrium is a set of state-contingent plans for quantities $\{c_t, k_t, l_t, l_{R,t}, n_t, a_t, x_t(i), u_t, v_t, \Pi_t^i, \Pi_t^f, T_t\}_{t=0}^{\infty}$, job matching rates $\{m(\theta_t, 1), m(1, 1/\theta_t)\}_{t=0}^{\infty}$, and prices $\{\omega_t, p_t(i), R_t, r_t, \omega_{R,t}\}_{t=0}^{\infty}$, such that:

- Given job matching rates and prices, plans $\{c_t, k_{t+1}, l_{t+1}, l_{R,t}, n_t, a_{t+1}\}_{t=0}^{\infty}$ solve the household's optimization problem.
- Given job matching rates and prices $\{\omega_t, R_t, r_t, \omega_{R,t}\}_{t=0}^{\infty}$, plans $\{x_t(i), p_t(i)\}_{t=0}^{\infty}$ solve the intermediate

good firm's optimization problem.

- Given job matching rates and prices, plans $\{k_t, l_{t+1}, v_t, x_t(i)\}_{t=0}^{\infty}$ solve the final good firm's optimization problem.
- Given job matching rates and prices, the government budget constraint is satisfied for all contingencies and all dates.
- Given job matching rates and prices, $\{l_{R,t}, a_t\}_{t=0}^{\infty}$ satisfy free entry equilibrium conditions into the R&D sector.
- Job matching rates are given by the matching function $m(v_t, u_t) = m(\theta_t, 1)u_t = m(1, 1/\theta_t)v_t$, where $u_t = 1 - l_t - l_{R,t}$ and $\theta_t = v_t/u_t$ for all contingencies and all dates.
- $\{w_t\}_{t=0}^{\infty}$ is determined by the Nash bargaining between firms and workers in the final sector. $m(v_t, u_t) = m(\theta_t, 1)u_t = m(1, 1/\theta_t)v_t$.
- Markets for final goods, capital, and loans clear.

3.3 Quantitative evaluation

In this section, we first describe the calibration of model parameters. Then we proceed to numerical exercises and discuss the relative importance of lab equipment and knowledge-driven R&D specifications for labor market dynamics.

3.3.1 Calibration

The model is characterized by five sets of parameters: (i) those characterizing a technological innovation process, (ii) those describing household preferences, (iii) those specifying the production technology, (iv) those describing the labor market dynamics, and (v) those characterizing the stochastic shocks in TFP. Parameter values are chosen to be consistent with the standard practice in quantitative macroeconomics literature. All the moments we use as calibration targets are quarterly averages in the US data over the period 1951Q01–2006Q01. Our data come from den Haan and Kaltenbrunner (2009). We carefully calibrate knowledge-driven R&D and lab equipment versions of the model to match the same sets of empirical moments.[3] Table 3.1 summarizes the baseline values for the key parameters of the model.

Technological innovation process. The gross growth rate of the innovation possibility frontier, at a deterministic steady state, is given by $g_N = 1 - \delta_N + \chi\,(l_R)^{\iota}\,(a)^{1-\iota}$, where l_R denotes the number of workers in the R&D sector, and parameter ι denotes the elasticity of g_N with respect to l_R. We consider two versions of the model: knowledge-driven R&D process ($\iota = 1$) and the pure lab equipment model ($\iota = 0$). Following

[3]In order to be consistent with matching the same set of empirical moments, the parameter values need to be adjusted when we move from $\iota = 1$ to $\iota = 0$.

Table 3.1: Baseline calibration

(i) Innovation process

	ι	c_R	δ_N	δ_{lR}	η	ϕ
Knowledge-driven R&D	1	6.8e5	0.069	0.036	1	0.544
Lab equipment technology	0	...	0.069	...	1	0.108

(ii) Household preferences

	β	σ
Knowledge-driven R&D	0.998	2
Lab equipment technology	0.998	2

(iii) Production technology

	α	δ_k	ψ
Knowledge-driven R&D	1/3	0.026	0.698
Lab equipment technology	1/3	0.026	0.692

(iv) Labor market dynamics

	$\overline{b}$	γ_m	γ	δ_l	κ	ξ
Knowledge-driven R&D	1.036	0.772	0.5	0.027	0.096	0.3
Lab equipment technology	1.003	1.371	0.5	0.027	0.090	0.3

(v) TFP process

	ρ_ζ	$\sigma_{\epsilon\zeta}$
Knowledge-driven R&D	0.87	0.0072
Lab equipment technology	0.87	0.0073

Li and Hall (2020), the obsolescence rate of new technologies, δ_N, is set equal to 6.9% (~25% per annum). An estimated δ_N of 6.9 aligns with the main conclusions from Pakes and Schankerman (1984, p. 80) and Huang and Diewert (2011, p. 407). In the knowledge-driven R&D model, parameter $\overline{c}_R$ is set so that the model matches the aggregate share of workers employed in the R&D sector. We use data from the NSF on employment in R&D. For calibrating our model, we consider workers in the R&D sector as people employed by R&D-performing companies who were engaged in scientific or engineering work at a level that required knowledge, gained either formally or by experience. The curvature parameter of the law of motion for the technology frontier ϕ is set so that the model reproduces the average output growth rate of 0.60% (approximately 2% per annum). The technology coefficient associated with the innovation process is normalized to one ($\eta = 1$). Using BGP restrictions, the separation rate in the R&D sector, δ_{lR}, is set to 0.027. Finally, we set the initial mass of available intermediate goods in period zero to match the TFP level in the first quarter of 1951, as measured by Fernald (2014).

Preferences. The per-period utility function is assumed to have constant relative risk aversion form $u(c) = \left(c^{1-\sigma} - 1\right)/(1-\sigma)$, where 2 is chosen as the coefficient of relative risk aversion, which implies greater risk aversion than logarithmic utility. A value of 2 is also chosen in Walsh (2005, p. 838). The plausible range of alternative values for σ varies between 0.5 and 2.5. Aghion and Howitt (1994, p. 487) and Mortensen (2005, p. 249), among others, showed that a value for σ greater than 1 guarantees the capitalization effect of growth. We fix the value for the discount factor β to match the annual real interest rate of 5.0% in reference to the average annual return on the S&P500 index.

Production technology. The final good production function parameter α is chosen to replicate the capital income share of 1/3. The capital depreciation rate δ_k is set to 2.6% (approximately 10% per annum) in reference to the standard practice in the quantitative macroeconomics literature (see e.g., Chugh 2016, p. 124). The elasticity of the production function with respect to intermediate inputs matches the investment–GDP ratio ($\psi = 0.698$). Gomme and Rupert (2007, p. 494) advocated the use of investment–GDP ratio rather than capital–output ratios. The periodic revisions to the capital stock data are so large, and the conceptual questions about what should be included in the capital stock are so difficult to satisfactorily answer that estimates of the capital–output ratios are too unreliable to use as a calibration target.

Labor market dynamics. We follow Petrongolo and Pissarides (2001) and choose a standard Cobb-Douglas specification for the matching function: $\overline{m} = \gamma_m \overline{u}^{\gamma}\, \overline{v}^{1-\gamma}$, where $\gamma_m > 0$ denotes the matching function constant, and $\gamma \in (0,1)$ denotes the matching function elasticity. We calibrate the value of the matching function parameter, $\gamma_m > 0$, to match the data on the average value of vacancy–unemployment ratio of 0.975. We obtain this last value from den Haan and Kaltenbrunner (2009), who used the Job Openings and Labor Turnover Survey data and the Help Wanted index to estimate the quarterly number of vacancies. We set the matching function elasticity (γ) to 0.5. The separation rate δ_l in the final good sector ensures an average value for the job finding rate of 0.5953, as estimated by Shimer (2005). The worker's bargaining weight, ξ, targets the value of volatility in the unfiltered unemployment rate in data. As in Hall and Milgrom (2008), Silva and Toledo (2009b), and Elsby and Michaels (2013), among others, the cost of vacancy κ is targeted to generate a κ/q ratio equal to 14% of quarterly worker compensation. We set the value of non-market activity for the workers $\overline{b}$ to match an average value for the unemployment rate of 5.66%, which implies a replacement ratio of 0.88.

Stochastic shocks in TFP. The model includes one exogenous aggregate shock. The TFP process is assumed to follow a mean zero AR(1) in logs with innovation ε_{ζ}, where $\xi_{\zeta} \sim N(0, \sigma_{\epsilon\zeta})$:

$$\ln \zeta_t = (1 - \rho_{\zeta})\overline{\zeta} + \rho_{\zeta} \ln \zeta_{t-1} + \varepsilon_{\zeta,t}. \tag{3.17}$$

We calibrate the autocorrelation coefficient ρ_{ζ} and the standard deviation $\sigma_{\epsilon\zeta}$ to match volatility and the first-order autocorrelation of the TFP shocks process estimated from the quarterly series provided by Fernald (2014). Notice that in our model, TFP is endogenous and equal to $TFP_t = \zeta_t z_t^{1-\psi}$. To calibrate the parameters of the TFP shock process, we use model simulations to estimate the persistence and volatility of TFP innovations from the model-generated data. We then adjust the values of ρ_{ζ} and $\sigma_{\epsilon\zeta}$ to minimize the distance between these simulated moments and their empirical counterpart. Using BGP restrictions, the average TFP process is set equal to 1. We find that $\sigma_{\epsilon\zeta} = 0.0072$ and $\rho_{\zeta} = 0.87$, which are consistent with the standard practice in the quantitative macroeconomics literature (see e.g., den Haan, Ramey, and Watson, 2000).

3.3.2 Results

To assess the role played by the specification of R&D growth process for labor market fluctuations, we follow the RBC literature. We examine how well a single shock (i.e., a TFP shock), presumed to be the principal driving force, can explain the unconditional patterns in the data. To compute the numerical solution, we solve the model by loglinearizing around the deterministic BGP.[4] We generate artificial time series of the same length as the actual data time series, and we repeat the simulation 1,000 times. In each simulation, we reaccumulate nonstationary time series by adding trend growth back. Specifically, to get the business cycle component from artificial series, we respectively proceed as follows: (i) we convert all the artificial data into growth rates by taking log differences; (ii) we pad the series by forecasting and backcasting; (iii) we apply a

[4]Petrosky-Nadeau and Zhang (2017) solve the standard DMP search model under the HM calibration using both an accurate global projection algorithm (AGPA) and loglinearization (local perturbation solution), and argue that an AGPA is critical for quantifying the basic moments of the DMP model. Loglinearization understates the mean and volatility of unemployment, but overstates the volatility of labor market tightness as well as the magnitude of the unemployment–vacancy correlation. Moreover, using both projection and perturbation methods, to numerically solve an RBC search model, Lan (2018) finds that while the projection method is more accurate in terms of Euler residuals, the second moments are however not significantly different. Unlike Petrosky-Nadeau and Zhang (2017), Atolia, Gibson, and Marquis (2018) find second moments in a search model with curvature in preferences are slightly larger when the less-accurate linear approximation is used. This finding bolster the claim made by Lan (2018) suggesting that the accuracy of approximation, while important for statistical fit, has little impact on the second moments of the model. In this paper, we use a local perturbation solution to basically explore implications of R&D-based growth process for labor market dynamics.

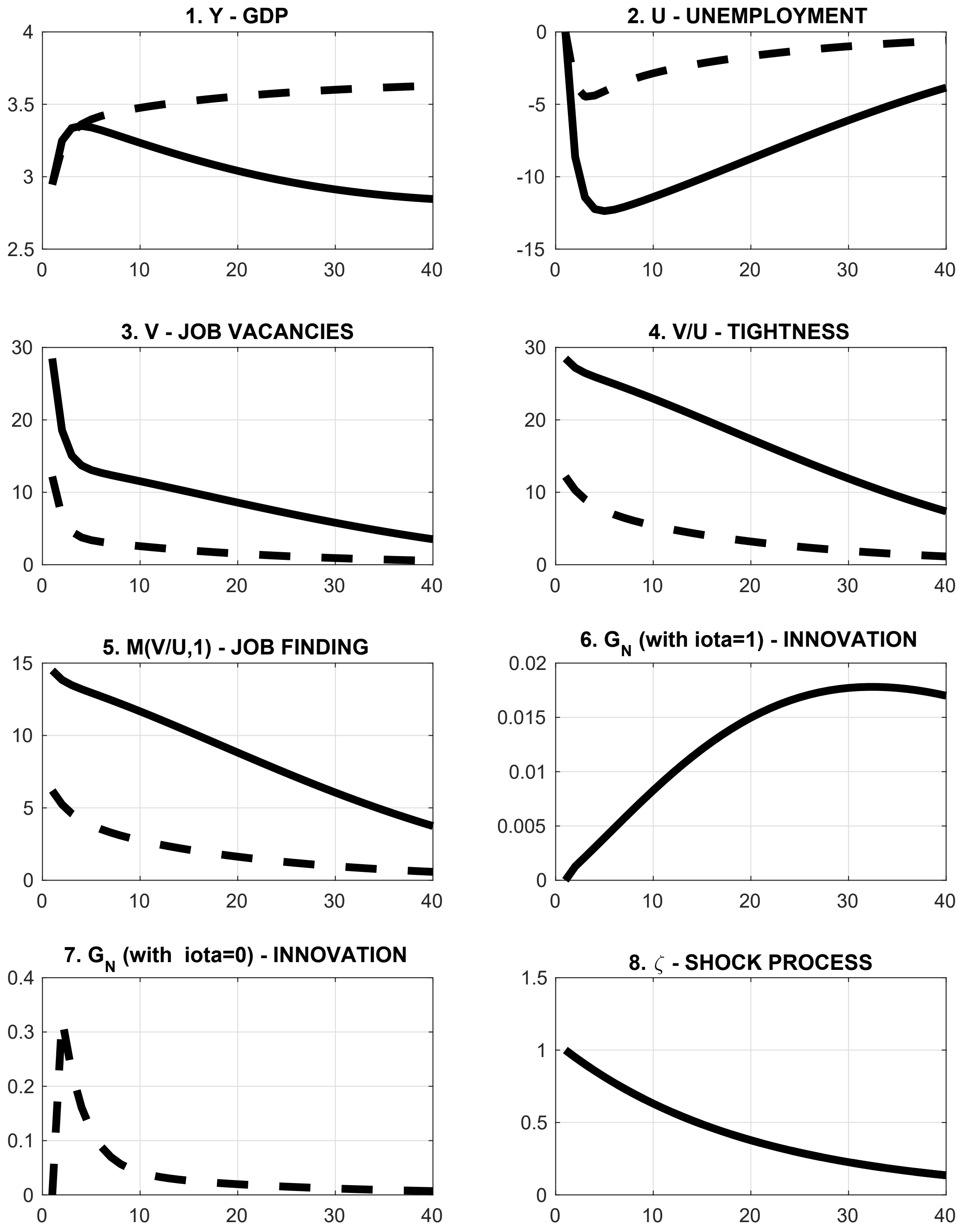
band pass filter to the growth rate data to obtain a measure of trend growth; (iv) we cumulate the trend growth rates to obtain measures of the trends in log levels; (v) finally, we use the associated cyclical components to compute moments.

Impulse response functions. We first analyze impulse response functions to gain some intuition about the endogenous propagation mechanism of the model. Figure 3.1 describes how the key labor market variables respond to TFP shocks of a plausible magnitude. To highlight the importance of the specification of the R&D growth process for the amplification of shock, we report two sets of IRFs: one generated from the knowledge-driven R&D model and another generated from the lab equipment model. The magnitude of the shock is equal to one standard deviation. The solid line in each panel denotes the response of the respective variable in the model with the knowledge-driven R&D process. By contrast, the dashed line corresponds to the model with the lab equipment technology. The y-axis displays deviations from BGP, whereas the x-axis displays the time horizon. The period length is one quarter.

On impact, a positive TFP shock raises production in the final good production sector (panel 1). Because the stochastic process governing TFP shocks is persistent, a high realization of the shock today also creates expectations of higher productivity shocks in the following periods (equation (3.10)). This increases the final good firms' incentives to post vacancies (panel 3) and raises labor market tightness (panel 4). In the following periods, hiring goes up and, consequently, unemployment falls (panel 2).

Table 3.2: Business cycle properties: Standard deviations

	US data		Model outcomes			
	(1)	(2)	(3)	(4)	(5)	(6)
			$\iota = 1$	$\iota = 1$	$\iota = 0$	$\iota = 0$
Variable	σ_i	σ_i/σ_y	σ_i	σ_i/σ_y	σ_i	σ_i/σ_y
GDP	0.013	1.000	0.015	1.000	0.014	1.000
TFP	0.012	0.923	0.012	0.807	0.012	0.861
Unemployment	0.122	9.354	0.094	6.298	0.042	3.014
Job vacancies	0.134	10.34	0.147	9.917	0.066	4.741
Labor market tightness	0.252	19.38	0.210	14.14	0.095	6.768
Job finding	0.072	5.562	0.107	7.197	0.048	3.443
Overall model's performance in terms of labor market variability	1.000	. . .	0.841	. . .	0.402	. . .



Note: Figure 3.1 plots the model impulse responses for a unit variance shock to TFP. The solid line in each panel denotes the response of the respective variable in the model with knowledge-driven R&D process. By contrast, the dashed line corresponds to the model with lab equipment technology. The period length is one quarter.

The expected higher productivity in the future leads to a rise in the value of intermediate good producer firms, which spurs further innovation. In the lab equipment model, this is done by rapidly increasing the spending on lab equipment, leading to a spike in the growth of innovation possibility frontier in period 2 (panel 7). By contrast, in the knowledge-driven R&D model, the adjustment is smoother because of slow movement of labor between the production and R&D sectors, leading to innovation possibility frontier that increases until period 40 (panel 6). Notice that, early on, the response of innovation possibility frontier in the lab equipment model is stronger than that of the knowledge-driven R&D model. However, the almost monotonous rise of the innovation IRF in the knowledge-driven R&D model amplifies the effects of the expected rise in future productivity. This is the main difference between the two parameterizations with $\iota = 0$ and $\iota = 1$. This difference explains the relative importance of the knowledge-driven R&D specification for generating labor market volatility in response to a plausible TFP shock. In conclusion, results from IRFs show that, in contrast to the lab equipment technology, the knowledge-driven R&D specification amplifies the economy's response to a TFP shock and helps in generating more volatility and persistence in labor market variables.

We stress that the key feature behind this result is the fact that labor only moves slowly between the R&D and production sectors because of two reasons. First, the quadratic training costs make rapidly reallocating a large mass of workers from unemployment to the R&D sector costly. This effect is important from the quantitative point of view. It allows the model to have empirically relevant fraction of workers employed in the R&D sector in the BGP. Second, the model implies that in equilibrium, workers must be indifferent between searching for a job in the production sector or entering immediately the R&D labor force. Given that wages rise in both sectors in response to a positive TFP shock, a worker who has a secured job in the production sector will not quit her job to switch to an R&D job. Therefore, no voluntary quits would occur, preventing the reallocation of a large mass of workers from production to R&D, even if the training costs were null. With c_R set to zero, we would obtain qualitatively the same IRFs as those displayed in Figure 3.1.

The slow movement of labor between the production and R&D sectors is at the origin of the amplified and persistent response of the labor market to TFP shocks in the knowledge-driven R&D model because it forces most of the adjustment to pass through variations in unemployment and, thus, labor market tightness. By contrast, in the lab equipment model, an increase in R&D productivity stemming from a positive and persistent TFP shock triggers an immediate and large spike in expenditures on lab equipment. This flow is, however, quickly resorted back to its BGP level. Because expenditures on lab equipment are much easier to adjust, they absorb a large part of response to the variation in productivity, leading to labor market responses that are of lesser magnitude than in the knowledge-driven R&D model.

Volatilities. To fully appreciate the ability of the model to match empirical evidence, we compute a set of formal statistics that characterize the business cycle fluctuations for a number of key labor market variables. Column (1) of Table 3.2 reports standard deviations based on US data. The data come from den Haan and Kaltenbrunner (2009). Standards deviations are based on quarterly filtered data. Column (2) expresses these relative to the standard deviation of output. Columns (3) and (4) report the results from the knowledge-driven R&D model. Finally, Columns (5) and (6) report the results from the lab equipment model. The standard

Table 3.3: Business cycle properties: Comovements

		GDP	TFP	u	v	v/u	$m(v/u, 1)$
Panel (a): Quarterly autocorrelation							
US data		0.935	0.901	0.903	0.919	0.913	0.913
Knowledge-driven R&D, $\iota = 1$		0.910	0.900	0.914	0.875	0.904	0.904
Lab equipment, $\iota = 0$		0.907	0.900	0.914	0.875	0.904	0.904
Panel (b): Correlation matrix							
US data	GDP	1.000	0.695	-0.788	0.882	0.851	0.839
	TFP		1.000	-0.529	0.6777	0.617	0.548
	u			1.000	-0.933	-0.982	-0.957
	v				1.000	0.984	0.949
	v/u					1.000	0.969
	f						1.000
Knowledge-driven R&D, $\iota = 1$	GDP	1.000	0.959	-0.867	0.853	0.987	0.987
	TFP		1.000	-0.701	0.951	0.982	0.982
	u			1.000	-0.490	-0.787	-0.787
	v				1.000	0.923	0.923
	v/u					1.000	0.999
	f						1.000
Lab equipment, $\iota = 0$	GDP	1.000	0.981	-0.836	0.884	0.994	0.994
	TFP		1.000	-0.725	0.947	0.989	0.989
	u			1.000	-0.491	-0.788	-0.788
	v				1.000	0.923	0.923
	v/u					1.000	0.999
	f						1.000

deviations in Columns (3)–(6) are computed from artificial data generated by the model economy after filtering. Our finding is that the knowledge-driven R&D model implies more labor market volatility than the lab equipment model. For example, the knowledge-driven R&D model can account for approximately 77% and 83% of volatilities in unemployment and job vacancies, respectively, whereas the lab equipment model can only reproduce approximately 34% and 37% of volatilities in the same variables. The last row in Table 3.1 presents the overall model's performance (OMP) in terms of labor market variability. The OMP is computed as follows. First, for each of Columns (1), (3), and (5), we compute the arithmetic mean for the statistics associated with unemployment, job vacancies, labor market tightness, and job finding, respectively. Then, we report the ratio of this average relative to its value in the real data. The results show that the knowledge-driven R&D model reproduces approximately 84% of labor market volatility found in the data, whereas the lab equipment model only generates approximately 40% of labor market volatility.

Persistence. Panel (a) in Table 3.3 reports the statistics on the persistence of our variables of interest. The results suggest that both the knowledge-driven R&D model and the lab equipment model match empirical quarterly autocorrelations quite well. In reference to the main issue of our investigation, these results show that in terms of persistence, both the knowledge-driven R&D model and the lab equipment model do a good job.

Contemporaneous cross-correlations. Finally, Panel (b) in Table 3.3 reports the contemporaneous cross-correlations between our variables of interest. Qualitatively, both the knowledge-driven R&D model and the lab equipment model perform quite well. In particular, both models replicate the negative correlation between GDP and unemployment (Okun's law) and the negative correlation between unemployment and vacancies (Beveridge curve). Table 3.3 also reveals a well-known shortcoming of models that use only one shock to generate the fluctuations: the cross-correlations of the variables of interest with the TFP generated by the model are greater than those in the data.

Table 3.4: R&D share of GDP (in %)

US data	Model with knowledge-driven R&D process	Model with lab equipment specification
2.447	2.740	2.744

R&D share of GDP. Before concluding, we briefly discuss an important result about the implications of models for the size of the R&D sector in the economy. Table 3.4 shows that both versions of our model generate an R&D spending share of GDP of about 2.7% compared to a value of 2.5 on US data.[5] Notably,

[5]Barlevy (2007, p. 1149) and Mand (2019, p. 48) also reports a similar number for the US R&D share of GDP.

the R&D spending share of GDP generated by our model was not a direct target of the calibration exercise.

3.4 Discussions

To gain insight into how the model generates labor market volatilities when the gross growth rate of the innovation possibility frontier is governed by a knowledge-driven R&D process, we examine the elasticity of labor market tightness with respect to productivity in the BGP. Ljungqvist and Sargent (2017, 2021) argued that a single intermediate channel, namely, the fundamental surplus, can explain how economic forces in the search-and-matching model of the labor market affect the responsiveness of unemployment to changes in productivity. In this section, we show that including a knowledge-driven R&D process into the search-and-matching framework yields a different, complementary mechanism to the fundamental surplus channel.

As discussed in the Introduction, one of the challenges for search-and-matching models has been to generate enough volatility of key labor market variables in response to shocks of a plausible magnitude to be consistent with the real-world data. Existing papers have reconfigured the basic search-and-matching model in various ways and proposed several solutions to this puzzle. Ljungqvist and Sargent (2017) showed that most of these solutions come down to reducing the fundamental surplus a job creates. This in turn increases the elasticity of labor market tightness with respect to productivity. In our setup with knowledge-driven R&D, an analytical expression for this elasticity can be derived. This expression shows that the elasticity is high not because of a low fundamental surplus, but because of existence of an alternative R&D labor market to which unemployed workers can turn, albeit at a cost, and expectations of future evolution of the productivity.

To derive the analytical expression for the elasticity of labor market tightness with respect to labor productivity in the model's BGP, we follow the protocol proposed by Ljungqvist and Sargent (2017). This entails five steps: The first step is to calculate the equilibrium value of labor market tightness in the final good sector. The second step is to determine an explicit equation for the relationship between unemployment and vacancies (the Beveridge curve). The third step consists of jointly determining explicit expressions for the derivatives of unemployment and the mass of workers entering the R&D sector with respect to labor market tightness. The fourth step is to calculate the elasticity of unemployment with respect to labor market tightness. Finally, in the fifth step, we can use these expressions to calculate the elasticity of labor market tightness with respect to productivity.

Because explicit analytical expressions for all needed terms cannot be obtained in the general version of the model, we make two assumptions that ensure tractability before we proceed. In the paragraphs that follow, a bar over variable's name denotes its detrended value in the deterministic steady state. The deterministic steady state in detrended variables corresponds to a BGP of the original model.[6]

Assumption 1. (*Matching technology.*) Let $\overline{u}$ denote the number of unemployed workers searching for a job

[6]In our model, along a BGP, the output, consumption, investment and wages grow at a constant rate, whereas unemployment, the share of workers employed in R&D, and the interest rate are constant. Detrending involves dividing the growing variables by $N^{1/(1-\alpha)}$.

in a deterministic steady state. Let $\overline{v}$ denote the number of vacant job in a deterministic steady state. Let $\gamma_m > 0$ denote a scale parameter reflecting the efficiency of the matching process. Let $\overline{m}$ denote the number of successful job matches (new hires), in a deterministic steady state. Given the scale parameter γ_m, the matching function that maps $\overline{u}$ and $\overline{v}$ into new hires $\overline{m}$, $m(\overline{u}, \overline{v}; \gamma_m) : \mathbb{R}_+^2 \to \mathbb{R}_+$, has a Cobb-Douglas form:

$$\overline{m} = \gamma_m \overline{u}^{\gamma}\, \overline{v}^{1-\gamma}. \tag{3.18}$$

Assumption 2. (*Knowledge-driven R&D process.*) The law of motion for the innovation possibility frontier depends only on labor input ($\iota = 1$).

We now can go through the five steps mentioned above to arrive at the analytical expression for the elasticity of labor market tightness with respect to productivity. We summarize each step into a proposition. The proofs are presented in the Appendix.

Proposition 1. *Given the firm's value of a filled job* (3.13)*, the Nash bargaining solution* (3.2)*, and the zero profit condition, in a deterministic steady state, the equilibrium value of labor market tightness in the final good sector, $\overline{\theta}$, is*

$$\left[(1-\alpha)\left(\frac{\overline{y}}{\overline{l}}\right) - \overline{b}\right] = \frac{1}{(1-\xi)\,\overline{q}}\left[\overline{r} + \delta_l \overline{g}_N^{\frac{1}{1-\alpha}} + \overline{g}_N^{\frac{1}{1-\alpha}} \xi \overline{\theta}\,\overline{q} - \left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right)\right]\overline{\kappa}. \tag{3.19}$$

Proof. See Appendix 6.1. □

In a standard search-and-matching model, $\overline{g}_N^{\frac{1}{1-\alpha}} = 1$.

Proposition 2. *Let $\overline{n}$ denote the mass of agents newly entering in the R&D labor market in a deterministic steady state. In a deterministic steady state, $\overline{\theta}$ determines unemployment in the final good sector by the condition that the measure of workers laid off in a period, $\delta_l(1 - \overline{u} - \overline{l}_R)$, must equal the measure of unemployed workers gaining employment in that sector, $\overline{\theta}\,\overline{q}(\overline{\theta})\overline{u}$, which implies that*

$$\overline{u} = \frac{\delta_l}{\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})} - \left(\frac{\delta_l}{\delta_{lR}}\right)\frac{\overline{n}(\overline{\theta})}{\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})}. \tag{3.20}$$

Proof. See Appendix 6.2. □

In a standard search-and-matching model, $\overline{n} = 0$.

Proposition 3. *Under Assumptions 1 and 2, explicit expressions for the derivative of $\overline{u}$ with respect to $\overline{\theta}$ and the derivative of $\overline{n}$ with respect to $\overline{\theta}$ are jointly determined by the system of equations (3.21–3.22)*

$$\frac{d\overline{u}}{d\overline{\theta}} = -(1-\gamma)\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l + \overline{\theta}\overline{q}(\overline{\theta})} - \left[\frac{\overline{u}}{\delta_{lR} - \overline{n}(\overline{\theta})}\right]\frac{d\overline{n}(\overline{\theta})}{d\overline{\theta}} \tag{3.21}$$

$$\frac{d\,\overline{n}(\overline{\theta})}{d\overline{\theta}} = \Theta_1 + \Theta_2 \frac{d\overline{u}}{d\overline{\theta}} \tag{3.22}$$

where

$$\gamma = -\frac{\overline{\theta}}{\overline{q}(\overline{\theta})}\overline{q}'(\overline{\theta})$$

$$\overline{q}'(\overline{\theta}) \equiv \frac{d\,\overline{q}(\overline{\theta})}{d\overline{\theta}}$$

$$\Theta_1 = \Phi_1\Phi_2\Psi_{\#,2} + \Phi_1 A_{\omega,6} - \Phi_1\overline{H}(1-\gamma)\,\overline{q}(\overline{\theta}) - \Phi_1\overline{f}A_{h,5} > 0$$

$$\Theta_2 = \Phi_1\Phi_2\Psi_{\#,3} + \Phi_1 A_{\omega,5} + \Phi_1\overline{f}A_{h,4} > 0.$$

Proof. See Appendix 6.3. □

In a standard search-and-matching model $\Theta_1 = \Theta_2 = 0$. As a result,

$$\frac{\mathrm{d}\,\overline{n}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = 0\cdot \tag{3.23}$$

Therefore, the system of equations (3.21–3.22) reduces to a single equation

$$\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} = -\,(1-\gamma)\,\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})}\cdot \tag{3.24}$$

By analogy, equation (3.24) is similar to that presented at the beginning of Section II in Ljungqvist and Sargent (2017).

Proposition 4. *Under Assumptions 1 and 2, in a deterministic steady state, the elasticity of unemployment with respect to market tightness is*

$$\begin{aligned}\varepsilon(\overline{u},\overline{\theta}) = &-\,(1-\gamma)\,(1-\overline{u})\\ &+\left\{\frac{(1-\gamma)\,\delta_l}{\delta_{lR}\left[\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})\right]}\right\}\overline{n}(\overline{\theta})\\ &-\left\{\frac{\Theta_1\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right] - \Theta_2\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})}{\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right]\left[\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}\right]}\right\}\overline{\theta}\cdot\end{aligned} \tag{3.25}$$

where $\Theta_1 > 0$ and $\Theta_2 > 0$ are deep structural parameters.

Proof. See Appendix 6.4. □

In a standard search-and-matching model

$$\overline{n}(\overline{\theta}) = \Theta_1 = \Theta_2 = 0. \tag{3.26}$$

As a result, equation (3.25) reduces to[7]

$$\varepsilon(\overline{u}, \overline{\theta}) = -(1-\gamma)(1-\overline{u})\,. \tag{3.27}$$

Proposition 5. *In a deterministic steady state, the elasticity of labor market tightness with respect to labor productivity is*

$$\varepsilon(\overline{\theta}, \overline{y}_{\#}) = \underbrace{\Upsilon_1 \frac{\overline{y}_{\#}}{\overline{y}_{\#} - \overline{b}}}_{\textit{FS block}} - \underbrace{\Upsilon_2 \varepsilon\left(\overline{r}, \overline{y}_{\#}\right)}_{\textit{NLE block}} + \underbrace{\Upsilon_3 \varepsilon\left(\overline{g}_N, \overline{y}_{\#}\right) + \Upsilon_4 \left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right)}_{\textit{RD block}} \tag{3.28}$$

where

$$\Upsilon_1 = \Xi_{\#} \left[\frac{\overline{q}(\overline{\theta})}{\gamma\left(\overline{r} + \delta_l\right) + \xi\, \overline{\theta}\, \overline{q}(\overline{\theta})} \right]$$

$$\Upsilon_2 = \left[\frac{\overline{r}}{\gamma\left(\overline{r} + \delta_l\right) + \xi\, \overline{\theta}\, \overline{q}(\overline{\theta})} \right]$$

$$\Upsilon_3 = \frac{(1-\delta_l) + \xi \overline{\theta}\overline{q}(\overline{\theta})}{\gamma\left(\overline{r} + \delta_l\right) + \xi\, \overline{\theta}\, \overline{q}(\overline{\theta})} \overline{g}_N^{\frac{1}{1-\alpha}}$$

$$\Upsilon_4 = \left[\frac{\overline{y}_{\#}}{\gamma\left(\overline{r} + \delta_l\right) + \xi\, \overline{\theta}\, \overline{q}(\overline{\theta})} \right] \left[\frac{\gamma}{\overline{\theta}\, \overline{q}(\overline{\theta})} (1-\delta_l) + \xi \right]$$

$$\varepsilon\left(\overline{r}, \overline{y}_{\#}\right) = 1 + \frac{\delta_k}{\overline{r}}$$

$$\varepsilon\left(\overline{g}_N, \overline{y}_{\#}\right) = \frac{\delta_k + \overline{r}}{(1+\overline{r})\,\sigma}.$$

Proof. We compute the elasticity of labor market tightness with respect to labor productivity by using the equilibrium value of the labor market tightness in the final good sector determined by equation (3.19) in *Proposition 1* and collecting results from *Propositions 2–4*. See Appendix 6.5. □

Equation (3.28) includes three main components. The first component is for the fundamental surplus block (FS block), as in Ljungqvist and Sargent (2017). The second block collects the nonlinear effects of having capital input in the production function of final goods sector (NLE block). Finally, the third component includes the R&D implications (RD block) for labor market dynamics. The RD block increases the size of the elasticity of labor market tightness with respect to productivity. This analytical evidence is consistent with our quantitative results from the calibrated version of the model. By contrast, the NLE block reduces

[7]See equation (14) in Ljungqvist and Sargent (2017, p. 2636).

the ability of the search-and-matching model for generating labor market volatility. This result is consistent with empirical evidence from Atolia, Gibson, and Marquis (2018).

In a standard search-and-matching model which does not include R&D activities, we obtain $\overline{g}_N^{\frac{1}{1-\alpha}} = 1$, and $\varepsilon\left(\overline{g}_N, \overline{y}_\#\right) = 0$. Moreover, when the model does not include a nonlinear production function with capital input, $\varepsilon\left(\overline{r}, \overline{y}_\#\right) = 0$. Thus, in absence of R&D activities and capital input, equation (3.28) reduces to

$$\varepsilon(\overline{\theta}, \overline{y}_\#) = \Upsilon_1 \frac{\overline{y}_\#}{\overline{y}_\# - \overline{b}}. \tag{3.29}$$

Equation (3.29) is similar to equation (15) in Ljungqvist and Sargent (2017).

3.5 Conclusion

In this paper, we provided a first-pass analysis of the implications of knowledge-driven R&D for labor market dynamics. We can think of many interesting extensions left for future research. For example, one can ask how the eventual skill bias of technological progress affects this amplification mechanism, or how this mechanism interacts with the introduction of labor-replacing technologies (robotization and automation) along the lines of Autor, Katz, and Kearney (2006), Acemoglu and Restrepo (2017), or Jaimovich and Siu (2020).

CONCLUSION

Cette thèse a examiné la dynamique macroéconomique à moyen terme, avec une attention particulière sur les variables du marché du travail. Dans le premier chapitre, nous avons documenté les principales régularités du marché du travail dans les fréquences des cycles d'affaires (6–32 trimestres) et dans les fréquences des cycles économiques à moyen terme (6–200 trimestres). Nous avons montré, en utilisant tour à tour les données annuelles et les données trimestrielles de l'économie américaine, que le modèle canonique du marché du travail avec frictions de recherche et d'appariement (modèle DMP) est incapable de reproduire avec succès ces régularités. Nous avons également montré que ce résultat est invariant à la stratégie de calibration utilisée. Dans le deuxième chapitre, nous avons proposé un nouveau mécanisme de propagation et d'amplification des chocs de productivité qui aide le modèle DMP à reproduire la dynamique du marché du travail dans les fréquences des cycles d'affaires et dans celles à moyen terme. Nous avons montré que l'effet principal de ce mécanisme repose sur l'interaction entre les frictions sur le marché du travail dans le secteur de production et les activités de R&D et d'innovation. Enfin dans le troisième chapitre, nous avons examiné dans quelle mesure le processus de croissance économique basé sur la R&D influence la capacité du modèle DMP à reproduire avec succès les principaux faits stylisés sur les fluctuations du marché du travail. Nous avons montré qu'un processus de R&D axé sur la connaissance, c'est-à-dire qui utilise le travail des scientifiques comme intrant, entraîne un puissant mécanisme d'amplification des chocs de productivité dans le modèle. En revanche, nous avons trouvé que ce mécanisme d'amplification est atténué lorsque la R&D n'utilise que le bien final (disons les equipements) comme intrant. Ces résultats sont générés principalement par le fait que la main-d'oeuvre ne se déplace que lentement entre les secteurs de la R&D et de la production du bien final, alors que les dépenses en équipements peuvent s'ajuster immédiatement.

CHAPTER IV

DATA APPENDIX

This Appendix contains additional details on data used in the chapter 2.

4.1 GDP, consumption, and investment

The series for GDP, consumption and investment are from the Bureau of Economic Analysis (BEA) National Income and Product Account (NIPA) historical tables, see Tables 1.1.4; 1.1.5; 1.1.6 (Url to query: `apps.bea.gov/iTable/iTable.cfm?reqid=19&step=2#panel-2`).

- Consumption includes nondurables goods and services.
- Investment is nonresidential.
- These series are based on the BEA chained-dollar estimates (index 2012) and expressed in per capita terms after dividing by the civilian non-institutionalized population aged 16 and over (Fred St Louis Fed, source code: LNU00000000; `fred.stlouisfed.org/series/CNP16OV`). The BEA chained-dollar estimates are obtained by multiplying the chain-type quantity index for an aggregate variable by its value in current dollars in the reference year and dividing by 100: `Real Value=((Nominal Value)/(Price Indexes))*100`.

The series for GDP, consumption and investment expressed in nominal values are from BEA NIPA Table 1.1.5. The series for GDP, consumption and investment expressed in real values are from BEA NIPA Table 1.1.6. The price indices series is from BEA NIPA Table 1.1.4.

4.2 Total factor productivity

The measure of the total factor productivity (TFP), also known as multifactor productivity, is published by the Bureau of Labor Statistics (BLS). The BLS measure of TFP is available only at the annual frequency (see Fred St Louis Fed, Source code: MPU4910012; `fred.stlouisfed.org/series/MFPNFBS`). The TFP series in quarterly frequencies is from Fernald (2014).[1] The Fernald's TFP series is the quarterly version of the annual series developed by Basu, Fernald, and Kimball (2006).

4.3 Research and Development

Our measure of R&D investment is from the BEA. This series refers to the gross private domestic investment (Fixed investment: Nonresidential, Intellectual property products, Research and Development, Billions of dollars, Annual, Seasonally adjusted annual rate). This data can be directly downloaded from the Fred St Louis Fed website (code: Y006RC1Q027SBEA, `fred.stlouisfed.org/series/Y006RC1Q027SBEA`). The series of R&D investment is deflated using the R&D chain-type price index (code: Y694RG3Q086SBEA, `fred.stlouisfed.org/series/Y694RG3Q086SBEA`).[2] The BEA R&D series is available at both quarterly and annual frequencies.

Notice that the National Science Foundation (NSF) also reports a time series on R&D, available here: `ncses.nsf.gov/pubs/nsb20203/recent-trends-in-u-s-r-d-performance`. While the BEA series measures the R&D investment, the NSF series measures the cost of R&D or R&D expenditures and is only available at annual frequency. The BEA measures of R&D investment are designed to capture the value of R&D, not the current costs of R&D as in the NSF series. Table 4.1 shows that key moments associated to NSF R&D and BEA R&D, in annual frequencies, are similar.

4.4 The labor market variables

Time series for the labor market variables are from den Haan and Kaltenbrunner (2009) database and Robert Shimer website (`sites.google.com/site/robertshimer/research/flows`), except when noted otherwise. The labor market variables include labor productivity, job vacancies, unemployment, labor market tightness, and wage. As in Shimer (2005), labor productivity is measured in the data as real average output per person in the nonfarm business sector constructed by the BLS. Real output is from the the BLS (code: PRS85006163; `fred.stlouisfed.org/series/PRS85006163`). The job vacancies are Index of Help-Wanted Advertising in Newspapers from the Conference Board divided by labor force from BLS based on Current Population Survey (CPS). Vacancies measures the rate of vacancy postings. Help-Wanted Advertising Index and labor force are quarterly averages directly constructed from monthly series. Abraham (1987) describes how the Help-Wanted Index is constructed. The series on unemployment is quarterly rate directly constructed from averages of monthly BLS series. Wage is nonfarm business compensation (divided by the number of employed) from the BLS. Labor market tightness is the ratio of job vacancies and unemployment.

Moreover, in the calibration exercise, we use statistics on the number of R&D scientists and engineers as reported by the NSF to measure the number of workers in the R&D sector (`nsf.gov/statistics/induswork-`

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[1] See the following websites which present the quarterly series on TFP for the US business sector constructed by John G. Fernald: (1) `frbsf.org/economic-research/indicators-data/total-factor-productivity-tfp`; (2) `johnfernald.net/TFP`.

[2] `Real Value=((Nominal Value)/(Price Index))*100`.

Table 4.1: BEA R&D vs NSF R&D. US annual data: 1953–2006

	Bureau of Economic Analysis BEA	National Science Foundation NSF
Mean for data in level		
R&D	1.053e+11	1.064e+11
R&D per capita	571.2	579.4
Standard deviation for data in log		
R&D	0.871	0.826
R&D per capita	0.639	0.595
Standard deviation for data in log-difference		
R&D	0.042	0.041
R&D per capita	0.043	0.042
Volatility in filtered data		
High-frequency	0.024	0.024
Medium-frequency	0.072	0.075
Medium-term cycle	0.077	0.079
Persistence in filtered data		
High-frequency	0.049	0.070
Medium-frequency	0.849	0.861
Medium-term cycle	0.940	0.945

Note: Series of R&D are in real terms.

`force`). The number of R&D scientists and engineers denotes the total number of people domestically employed by R&D-performing companies who were engaged in scientific or engineering work at a level that required knowledge, gained either formally or by experience, of engineering or of the physical, biological, mathematical, statistical, or computer sciences equivalent to at least that acquired through completion of a 4-year college program with a major in one of those fields. The company R&D in this case is the industrial R&D performed within company facilities funded from all sources except the federal government.

CHAPTER V

BALANCED GROWTH PATH

This Appendix contains additional details on the model used in the chapter 2.

The production function for the final good reads

$$y = A_{\#} N \zeta^{\frac{1}{\psi}} k^{\alpha} l^{1-\alpha}, \tag{5.1}$$

where $A_{\#} \equiv \left\{(1-\psi)^{1/\psi} \left[\mu/(1-\psi)\right]^{-1/\psi}\right\}^{1-\psi} > 0$, and ζ is a stationary shock process. Next by taking equation (5.1) in log, and then in first difference we get

$$(\ln y' - \ln y) = (\ln N' - \ln N) + \alpha(\ln k' - \ln k) + (1-\alpha)(\ln l' - \ln l). \tag{5.2}$$

Along a deterministic BGP, $\zeta = 1$ labor will grow at rate $g_l = 1$. Capital will grow at the same rate as output because the real interest rate is constant. Using these facts, we find

$$(1-\alpha)(\ln y' - \ln y) = (\ln N' - \ln N). \tag{5.3}$$

Hence, along the BGP, output divided by the scaling factor $N^{1/(1-\alpha)}$ does not grow, i.e. it is stationary

$$y' N'^{-\frac{1}{1-\alpha}} = y N^{-\frac{1}{1-\alpha}}. \tag{5.4}$$

Thus we can induce stationarity in the production function by dividing both sides of the equation (5.1) by the scaling factor $N^{1/(1-\alpha)}$:

$$\begin{aligned} \widetilde{y} &= A_{\#} \zeta^{\frac{1}{\psi}} N^{-\frac{\alpha}{1-\alpha}} \left[\frac{k}{N^{\frac{1}{1-\alpha}}}\right]^{\alpha} N^{\frac{\alpha}{1-\alpha}} l^{1-\alpha} \\ &= A_{\#} \zeta^{\frac{1}{\psi}} \widetilde{k}^{\alpha} l^{1-\alpha}, \end{aligned} \tag{5.5}$$

where $A_{\#} \equiv \left\{(1-\psi)^{1/\psi} \left[\mu/(1-\psi)\right]^{-1/\psi}\right\}^{1-\psi} > 0$. Equation (5.4) implies:

$$g_N = g_y^{1-\alpha}, \tag{5.6}$$

where g_y is the gross growth rate of output.

CHAPTER VI

MATHEMATICAL PROOFS

The appendix is built on the set of stationarized equilibrium conditions in a deterministic steady state. A bar over variable's name denotes its detrended value in the deterministic steady state. The deterministic steady state in detrended variables corresponds to a BGP of the original model.

6.1 Proof of Proposition 4.1

Proof. Let $\overline{J}$ be the firm's value of a filled job in a deterministic steady state

$$\overline{J} = \beta \, \overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}} \left[1 - \beta \left(1 - \delta_l\right) \overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}} \right]^{-1} \left[\left(1-\alpha\right) \left(\frac{\overline{y}}{\overline{l}} \right) - \overline{\omega} \right]. \tag{6.1}$$

Given the first order condition of the firm's problem with respect to job vacancies,

$$\overline{J} = \frac{\overline{\kappa}}{\overline{q}} \, \overline{g}_N^{\frac{1}{1-\alpha}} \tag{6.2}$$

substituting equation (6.2) into equation (6.1) yields

$$\overline{\omega} = \left(1-\alpha\right) \left(\frac{\overline{y}}{\overline{l}} \right) - \left[\frac{1}{\beta} \overline{g}_N^{\frac{\sigma}{1-\alpha}} - \left(1-\delta_l\right) \overline{g}_N^{\frac{1}{1-\alpha}} \right] \frac{\overline{\kappa}}{\overline{q}}. \tag{6.3}$$

Given values of the discount factor β and the rental rate for physical capital $\overline{R}$ in a deterministic steady state

$$\beta = \overline{g}_N{}^{\frac{\sigma}{1-\alpha}} \left[\overline{R} + \left(1-\delta_k\right) \right]^{-1} \tag{6.4}$$

$$\overline{R} = \left(1+\overline{r}\right) - \left(1-\delta_k\right) \tag{6.5}$$

equation (6.3) can be expressed as

$$\overline{\omega} = \left(1-\alpha\right) \left(\frac{\overline{y}}{\overline{l}} \right) - \left[\left(1+\overline{r}\right) - \left(1-\delta_l\right) \overline{g}_N^{\frac{1}{1-\alpha}} \right] \frac{\overline{\kappa}}{\overline{q}}. \tag{6.6}$$

In a standard search-and-matching model, $\overline{g}_N$ is implicitly set to one.[1] Equation (6.6) can also be written as follows to highlight the terms that are added in a medium-term search RBC model compared with the standard search-and-matching model

$$\overline{\omega} = (1-\alpha)\left(\frac{\overline{y}}{\overline{l}}\right) - \left[\overline{r} + \delta_l \overline{g}_N^{\frac{1}{1-\alpha}} - \left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right)\right]\frac{\overline{\kappa}}{\overline{q}}. \tag{6.7}$$

Moreover, using Nash bargaining procedure, we arrive at the following equation that maximizes the match surplus split between a matched firm and worker

$$\overline{\omega} = \overline{b} + \xi\left[(1-\alpha)\left(\frac{\overline{y}}{\overline{l}}\right) - \overline{b} + \overline{g}_N^{\frac{1}{1-\alpha}}\overline{\kappa}\,\overline{\theta}\right]. \tag{6.8}$$

Therefore, equations (6.7) and (6.8) for the wage jointly determine the equilibrium value of the labor market tightness, $\overline{\theta}$

$$\left[(1-\alpha)\left(\frac{\overline{y}}{\overline{l}}\right) - \overline{b}\right] = \frac{1}{(1-\xi)\,\overline{q}}\left[\overline{r} + \delta_l \overline{g}_N^{\frac{1}{1-\alpha}} + \overline{g}_N^{\frac{1}{1-\alpha}}\xi\overline{\theta}\,\overline{q} - \left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right)\right]\overline{\kappa}\,. \tag{6.9}$$

□

6.2 Proof of Proposition 4.2

Proof. Considering the following equilibrium conditions in a deterministic steady state to compute the explicit analytical expression for the derivative of the unemployment with respect to the labor market tightness, $\mathrm{d}\overline{u}/\mathrm{d}\overline{\theta}$. Let $\overline{f}$ be the probability of finding a new job for the unemployed in a deterministic steady state

$$\overline{f} = \frac{\overline{m}}{\overline{u}} \tag{6.10}$$

Let $\overline{q}$ be the probability of filling a job vacancy for the firm in a deterministic steady state

$$\overline{q} = \frac{\overline{m}}{\overline{v}} \tag{6.11}$$

Let

$$0 = -\delta_l \overline{l} + \overline{f}\,\overline{u} \tag{6.12}$$

be the law of motion for employment in a deterministic steady state. Let $\overline{u}$ be the unemployment in a deterministic steady state

$$\overline{u} = 1 - \overline{l} - \overline{l}_R. \tag{6.13}$$

[1]See e.g. Hagedorn and Manovskii (2008, p. 1695), Ljungqvist and Sargent (2017, p. 2636), Ljungqvist and Sargent (2021, p. 16).

Next, combining (6.10) and (6.11) yields

$$\overline{f} = \overline{\theta}\overline{q}(\overline{\theta}) \tag{6.14}$$

and combining (6.12), (6.13), and (6.14) yields

$$\delta_l(1 - \overline{u} - \overline{l}_R) = \overline{\theta}\overline{q}(\overline{\theta})\overline{u}. \tag{6.15}$$

After few simple mathematical manipulations, we obtain

$$\overline{u} = \frac{\delta_l}{\delta_l + \overline{\theta}\overline{q}}\left(1 - \overline{l}_R\right). \tag{6.16}$$

Given in a deterministic steady state $\delta_{lR}\overline{l}_R = \overline{n}$ (i.e., the mass of workers who leave R&D sector is equal to the mass of workers who newly enter to the R&D sector), equation (6.16) can be also expressed as follows

$$\begin{aligned}\overline{u} &= \frac{\delta_l}{\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})}\left[1 - \frac{\overline{n}(\overline{\theta})}{\delta_{lR}}\right] \\ &= \frac{\delta_l}{\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})} - \left(\frac{\delta_l}{\delta_{lR}}\right)\frac{\overline{n}(\overline{\theta})}{\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})}.\end{aligned} \tag{6.17}$$

where $\overline{\theta}$ determines the unemployment in the final good sector. □

6.3 Proof of Proposition 4.3

Proof. To derive explicit expressions of the derivative of $\overline{u}$ with respect to $\overline{\theta}$ and the derivative of $\overline{n}$ with respect to $\overline{\theta}$, we proceed in three main steps as follows

1. We determine $\mathrm{d}\overline{u}/\mathrm{d}\overline{\theta}$ as a function of $\mathrm{d}\overline{n}/\mathrm{d}\overline{\theta}$, see equation (6.19)

2. We determine $\mathrm{d}\overline{n}/\mathrm{d}\overline{\theta}$ as a function of $\mathrm{d}\overline{u}/\mathrm{d}\overline{\theta}$, see equation (6.124)

3. We solve the system of structural equations (6.19–6.124) to obtain equation (6.131), which gives the explicit expression of the derivative of $\overline{u}$ with respect to $\overline{\theta}$ and equation (6.132), which gives the explicit expression of the derivative of $\overline{n}$ with respect to $\overline{\theta}$.

Fist, we compute the derivative of $\overline{u}$ with respect to $\overline{\theta}$ as a function of $\mathrm{d}\overline{n}/\mathrm{d}\overline{\theta}$

$$
\begin{aligned}
\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} &= -\frac{\delta_l\left[\overline{q}(\overline{\theta})+\overline{\theta}\overline{q}'(\overline{\theta})\right]}{\left[\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})\right]^2} - \left(\frac{\delta_l}{\delta_{lR}}\right)\frac{\left[\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})\right]\overline{n}'(\overline{\theta}) - \overline{n}(\overline{\theta})\left[\overline{q}(\overline{\theta})+\overline{\theta}\overline{q}'(\overline{\theta})\right]}{\left[\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})\right]^2} \\
&= -\frac{\delta_l\left[\overline{q}(\overline{\theta})+\overline{\theta}\overline{q}'(\overline{\theta})\right]}{\left[\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})\right]^2}\left[1-\frac{\overline{n}(\overline{\theta})}{\delta_{lR}}\right] - \left(\frac{\delta_l}{\delta_{lR}}\right)\frac{\overline{n}'(\overline{\theta})}{\left[\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})\right]} \\
&= -\frac{\left[\overline{q}(\overline{\theta})+\overline{\theta}\overline{q}'(\overline{\theta})\right]}{\left[\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})\right]}\overline{u} - \left(\frac{\delta_l}{\delta_{lR}}\right)\frac{\overline{n}'(\overline{\theta})}{\left[\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})\right]} \\
&= -\left[1+\frac{\overline{\theta}\overline{q}'(\overline{\theta})}{\overline{q}(\overline{\theta})}\right]\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})} - \left(\frac{\delta_l}{\delta_{lR}}\right)\frac{\overline{n}'(\overline{\theta})}{\left[\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})\right]}\cdot
\end{aligned}
\tag{6.18}
$$

Alternatively, equation (6.18) can be rewritten as follows

$$
\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} = -\,(1-\gamma)\,\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})} - \left[\frac{\overline{u}}{\delta_{lR}-\overline{n}(\overline{\theta})}\right]\frac{\mathrm{d}\,\overline{n}(\overline{\theta})}{\mathrm{d}\overline{\theta}} \tag{6.19}
$$

where the matching function (see Assumption 1) implies that

$$
\overline{q}(\overline{\theta}) \equiv \frac{\overline{m}}{\overline{v}} = \gamma_m \overline{\theta}^{-\gamma} \tag{6.20}
$$

$$
\overline{q}'(\overline{\theta}) \equiv \frac{\mathrm{d}\,\overline{q}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = -\gamma_m\,\gamma\,\overline{\theta}^{-\gamma-1} \Rightarrow \gamma = -\frac{\overline{\theta}}{\overline{q}(\overline{\theta})}\overline{q}'(\overline{\theta})\cdot \tag{6.21}
$$

Notice that in a standard search-and-matching model

$$
\left[\frac{\overline{u}}{\delta_{lR}-\overline{n}(\overline{\theta})}\right]\frac{\mathrm{d}\,\overline{n}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = 0 \Rightarrow \frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} = -\,(1-\gamma)\,\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\,\overline{q}(\overline{\theta})}\cdot \tag{6.22}
$$

Second, we now compute the derivative of $\overline{n}$ with respect to $\overline{\theta}$ as a function of $\mathrm{d}\overline{u}/\mathrm{d}\overline{\theta}$ using equation (6.23), (6.24), and (6.25)

$$
\overline{H}^R = \overline{c}_R\,\overline{n} \tag{6.23}
$$

$$
\overline{H}^R = \left[1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\,(1-\delta_{lR})\right]^{-1}\left(\overline{\omega}_R-\overline{b}-\overline{f}\,\overline{H}\right) \tag{6.24}
$$

$$\overline{H} = \beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left[1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_l-\overline{f}\right)\right]^{-1}\left(\overline{\omega}-\overline{b}\right) \tag{6.25}$$

we obtain the following expression of the mass of workers who newly enter to the R&D sector in a deterministic steady state

$$\overline{n} = \frac{1}{\overline{c}_R}\left\{\left[1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_{lR}\right)\right]^{-1}\left(\overline{\omega}_R-\overline{b}-\overline{f}\,\overline{H}\right)\right\} \tag{6.26}$$

Next

$$\begin{aligned}\frac{\mathrm{d}\,\overline{n}(\overline{\theta})}{\mathrm{d}\overline{\theta}} =& \frac{1}{\overline{c}_R}\beta\left(1-\delta_{lR}\right)\left\{\left[1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_{lR}\right)\right]^{-2}\left(\overline{\omega}_R-\overline{b}-\overline{f}\,\overline{H}\right)\right\}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}}\\ &+\frac{1}{\overline{c}_R}\left[1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_{lR}\right)\right]^{-1}\left[\frac{\mathrm{d}\,\overline{\omega}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}}-\overline{H}\frac{\mathrm{d}\,\overline{f}(\overline{\theta})}{\mathrm{d}\overline{\theta}}-\overline{f}\frac{\mathrm{d}\,\overline{H}(\overline{\theta})}{\mathrm{d}\overline{\theta}}\right]\\ =& \frac{1}{\overline{c}_R}\left[1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_{lR}\right)\right]^{-1}\times\cdots\\ &\cdots\times\left\{\frac{\mathrm{d}\,\overline{\omega}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}}-\overline{H}\frac{\mathrm{d}\,\overline{f}(\overline{\theta})}{\mathrm{d}\overline{\theta}}-\overline{f}\frac{\mathrm{d}\,\overline{H}(\overline{\theta})}{\mathrm{d}\overline{\theta}}-\left[\beta\left(1-\delta_{lR}\right)\overline{c}_R\,\overline{n}\right]\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}}\right\}\end{aligned} \tag{6.27}$$

6.3.1 Derivative of $\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}$ with respect to θ

Given the law of motion of the technology frontier

$$\overline{g}_N = 1-\delta_N+\overline{\chi}\,\overline{l}_R^{\,\iota}\,\overline{a}^{1-\iota} \tag{6.28}$$

it follows that

$$\left(\overline{g}_N\right)^{\frac{-(\sigma-1)}{1-\alpha}} = \left(1-\delta_N+\overline{\chi}\,\overline{l}_R^{\,\iota}\,\overline{a}^{1-\iota}\right)^{\frac{-(\sigma-1)}{1-\alpha}} \tag{6.29}$$

$$\begin{aligned}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} =& -\left(\frac{\sigma-1}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{-(\sigma-1)}{1-\alpha}-1}\times\cdots\\ &\cdots\times\left[\overline{l}_R(\overline{\theta})^{\iota}\,\overline{a}^{1-\iota}\frac{\mathrm{d}\,\overline{\chi}}{\mathrm{d}\overline{\theta}}+\overline{\chi}\,\overline{a}^{1-\iota}\frac{\mathrm{d}\,\overline{l}_R(\overline{\theta})^{\iota}}{\mathrm{d}\overline{\theta}}+\overline{\chi}\,\overline{l}_R(\overline{\theta})^{\iota}\frac{\mathrm{d}\,\overline{a}^{1-\iota}}{\mathrm{d}\overline{\theta}}\right]\end{aligned} \tag{6.30}$$

Notice that $\sigma = 1$ (i.e., a logarithmic utility) implies that $\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}/\mathrm{d}\overline{\theta} = 0$. In our benchmark calibration, we chose the value of 2 for the coefficient of relative risk aversion, σ, which implies greater risk aversion than logarithmic utility. A value of 2 is also chosen in Walsh (2005, p. 838). A plausible range of alternative values for σ varies between 0.5 and 2.5 (see Table 6.1 in DeJong and Dave (2005) for more details). Aghion and Howitt (1994, p. 487), and Mortensen (2005, p. 249), among others, show a value for σ greater than one guarantees the capitalization effect of growth.

Let $\overline{\chi}$ be the congestion externality in a deterministic steady state. Let $\overline{u}$ be the unemploymen in a deterministic steady state. Let $0 = -\delta_l \overline{l} + \overline{q}\,\overline{v}$ be the law of motion for employment in a deterministic steady state. Let $\overline{\theta}$ be the labor market tightness in a deterministic steady state. Let $\overline{a}$ be the expenditures on lab equipment in a deterministic steady state. Let $\overline{W}$ be the value of a new intermediate good in a deterministic steady state.

$$\overline{\chi} = \eta \left[\overline{l}_R^{\,-\iota} \left(\overline{a}\right)^{-(1-\iota)}\right]^{1-\phi} \tag{6.31}$$

$$\overline{u} = 1 - \overline{l} - \overline{l}_R \tag{6.32}$$

$$0 = -\delta_l \overline{l} + \overline{q}\,\overline{v} \tag{6.33}$$

$$\overline{\theta} = \frac{\overline{v}}{\overline{u}} \tag{6.34}$$

$$\overline{a} = (1-\iota)\left(1-\delta_N\right)\beta \overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}} \left[1-\left(1-\delta_N\right) g_N^{-1}\right] \overline{W} \tag{6.35}$$

$$\overline{W} = C_{\#} \left[1-\left(1-\delta_N\right)\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\right]^{-1} \zeta^{\frac{1}{\psi}} \overline{k}^{\alpha} \overline{l}^{1-\alpha} \tag{6.36}$$

thus, combining equation (6.28) and equations (6.13), (6.28), (6.31), (6.33), (6.34), (6.35), and (6.36) yields

$$\begin{aligned}
\overline{g}_N &= 1 - \delta_N + \overline{\chi}\,\overline{l}_R^{\,\iota}\,\overline{a}^{1-\iota} \\
&= 1 - \delta_N + \eta \left[\overline{l}_R^{\,-\iota} \left(\overline{a}\right)^{-(1-\iota)}\right]^{1-\phi} \overline{l}_R^{\,\iota}\,\overline{a}^{1-\iota} \\
&= 1 - \delta_N + \eta\,\overline{l}_R^{\,\iota\phi}\,\overline{a}^{(1-\iota)\phi} \\
&= 1 - \delta_N + \eta \left[1 - \frac{1}{\delta_l}\,\overline{u}(\overline{\theta})\,\overline{\theta}\,\overline{q}(\overline{\theta}) - \overline{u}(\overline{\theta})\right]^{\iota\phi} \overline{a}^{(1-\iota)\phi}
\end{aligned} \tag{6.37}$$

where from equations (6.32), (6.33), and (6.34), we get

$$\overline{l}_R = 1 - \frac{1}{\delta_l}\,\overline{u}(\overline{\theta})\,\overline{\theta}\,\overline{q}(\overline{\theta}) - \overline{u}(\overline{\theta}). \tag{6.38}$$

Finally, after few simple mathematical manipulations, we obtain

$$\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}} = \left\{1 - \delta_N + \eta \left[1 - \frac{1}{\delta_l}\,\overline{u}(\overline{\theta})\,\overline{\theta}\,\overline{q}(\overline{\theta}) - \overline{u}(\overline{\theta})\right]^{\iota\phi} \overline{a}^{(1-\iota)\phi}\right\}^{\frac{-(\sigma-1)}{1-\alpha}} \tag{6.39}$$

where given equations (6.35) and (6.36), $\overline{a}$ also depends on θ. Equation (6.39) cannot be solved analytically for cases where $\iota < 1$. We therefore proceed under Assumption 2 ($\iota = 1$), and equation (6.39) reads

$$\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}} = \left\{1 - \delta_N + \eta \left[1 - \frac{1}{\delta_l}\,\overline{u}(\overline{\theta})\,\overline{\theta}\,\overline{q}(\overline{\theta}) - \overline{u}(\overline{\theta})\right]^{\phi}\right\}^{\frac{-(\sigma-1)}{1-\alpha}} \tag{6.40}$$

$$\begin{aligned}
\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} &= -\eta\phi\left(\frac{\sigma-1}{1-\alpha}\right)\overline{g}_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\left[1 - \frac{1}{\delta_l}\,\overline{u}(\overline{\theta})\,\overline{\theta}\,\overline{q}(\overline{\theta}) - \overline{u}(\overline{\theta})\right]^{-(1-\phi)} \times \cdots \\
&\qquad \cdots \times \left[-\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} - \frac{1}{\delta_l}\,\overline{u}\,\overline{q}(\overline{\theta}) - \frac{1}{\delta_l}\,\overline{u}\,\overline{\theta}\,\overline{q}'(\overline{\theta}) - \frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\right] \\
&= \eta\phi\left(\frac{\sigma-1}{1-\alpha}\right)\overline{g}_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\left[1 - \frac{1}{\delta_l}\,\overline{u}(\overline{\theta})\,\overline{\theta}\,\overline{q}(\overline{\theta}) - \overline{u}(\overline{\theta})\right]^{-(1-\phi)} \times \cdots \\
&\qquad \cdots \times \left[\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + \frac{1}{\delta_l}\,\overline{u}\,\overline{q}(\overline{\theta}) + \frac{1}{\delta_l}\,\overline{u}\,\overline{\theta}\,\overline{q}'(\overline{\theta}) + \frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\right] \\
&= \eta\phi\left(\frac{\sigma-1}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{-(\sigma-\alpha)}{1-\alpha}}\left(\bar{l}_R\right)^{-(1-\phi)} \times \cdots \\
&\qquad \cdots \times \left[\left(1 + \frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + \frac{1}{\delta_l}\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right]
\end{aligned} \tag{6.41}$$

and, finally,

$$\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} = \Psi_{\#,2} + \Psi_{\#,3}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} \tag{6.42}$$

where

$$\Psi_{\#,1} = \eta\phi\left(\frac{\sigma-1}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{-(\sigma-\alpha)}{1-\alpha}}\left(\bar{l}_R\right)^{-(1-\phi)} > 0 \tag{6.43}$$

$$\Psi_{\#,2} = \Psi_{\#,1}\left[\frac{1}{\delta_l}\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right] > 0 \tag{6.44}$$

$$\Psi_{\#,3} = \Psi_{\#,1}\left(1 + \frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right) > 0 \tag{6.45}$$

and from (6.38)

$$\frac{\mathrm{d}\,\bar{l}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} = -\frac{\Psi_{\#,2}}{\Psi_{\#,1}\,\bar{l}_R} - \frac{\Psi_{\#,3}}{\Psi_{\#,1}\,\bar{l}_R}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}. \tag{6.46}$$

6.3.2 Derivative of $\overline{\omega}_R(\overline{\theta})$ with respect to θ

Let $\overline{\omega}_R$ be the wage rate on the R&D labor market in a deterministic steady state

$$\overline{\omega}_R = \iota\overline{\chi}\left(\frac{\overline{a}}{\overline{l}_R}\right)^{1-\iota}(1-\delta_N)\,\beta\overline{\Lambda}\,\overline{g}_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\,\overline{W}. \tag{6.47}$$

Using equations (6.31) under Assumption 1

$$\overline{\chi} = \eta\left(\overline{l}_R^{\,-\iota}\right)^{1-\phi}$$

we get

$$\begin{aligned}\overline{\omega}_R =&\overline{\chi}\,(1-\delta_N)\,\beta\overline{\Lambda}\,\overline{g}_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\,\overline{W}\\ =&\eta\left(\overline{l}_R\right)^{-(1-\phi)}(1-\delta_N)\,\beta\left(\overline{g}_N\right)^{\frac{-(\sigma-\alpha)}{1-\alpha}}\,\overline{W}.\end{aligned} \tag{6.48}$$

As a result,

$$\frac{\mathrm{d}\,\overline{\omega}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} =\overline{\omega}_R\left\{\left(\frac{1}{\overline{l}_R}\right)\frac{\mathrm{d}\,\overline{l}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} - \left[\left(\frac{(\sigma-\alpha)}{1-\alpha}\right)\frac{1}{\overline{g}_N}\right]\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} + \left(\frac{1}{\overline{W}}\right)\frac{\mathrm{d}\,\overline{W}(\overline{\theta})}{\mathrm{d}\overline{\theta}}\right\}. \tag{6.49}$$

Derivative of $\overline{l}_R(\overline{\theta})$ with respect to θ

See equation (6.46).

Derivative of $\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}$ with respect to θ

In reference to equations (6.28), (6.30), and (6.46)

$$\begin{aligned}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} &= \eta\phi\left(\frac{\sigma-\alpha}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{-(\sigma-\alpha)}{1-\alpha}-1}\left(\overline{l}_R\right)^{-(1-\phi)}\times\cdots\\ &\cdots\times\left[\left(1+\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + \frac{1}{\delta_l}\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right]\\ &=\Psi_{\#,5}+\Psi_{\#,6}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\end{aligned} \tag{6.50}$$

where

$$\Psi_{\#,4} = \eta\phi\left(\frac{\sigma-\alpha}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{-(\sigma-\alpha)}{1-\alpha}-1}\left(\overline{l}_R\right)^{-(1-\phi)} > 0 \tag{6.51}$$

$$\Psi_{\#,5} = \Psi_{\#,4}\left[\frac{1}{\delta_l}\ (1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right] > 0 \tag{6.52}$$

$$\Psi_{\#,6} = \Psi_{\#,4}\left(1+\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right) > 0. \tag{6.53}$$

Derivative of $\overline{W}(\overline{\theta})$ with respect to θ

Using equations (6.36), (6.54), (6.55), (6.56), (6.57)

$$\overline{W} = C_{\#}\left[1-(1-\delta_N)\,\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\right]^{-1}\zeta^{\frac{1}{\psi}}\overline{k}^{\alpha}\overline{l}^{1-\alpha}$$

$$\alpha\frac{\overline{y}}{\overline{k}} - (\overline{r}+\delta_k) = 0 \tag{6.54}$$

$$\overline{g}_N^{\frac{\sigma}{1-\alpha}} - \beta\left[\overline{R}+(1-\delta_k)\right] = 0 \tag{6.55}$$

$$1+\overline{r} = \overline{R}+(1-\delta_k) \tag{6.56}$$

$$\overline{y} = A_{\#}\overline{\zeta}^{\frac{1}{\psi}}\overline{k}^{\alpha}\overline{l}^{1-\alpha} \tag{6.57}$$

we get

$$\begin{aligned}
\overline{W} =& C_{\#}\left[1-(1-\delta_N)\,\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\right]^{-1}\zeta^{\frac{1}{\psi}}\overline{k}^{\alpha}\overline{l}^{1-\alpha}\\
=& C_{\#}\left[1-(1-\delta_N)\,\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\right]^{-1}\zeta^{\frac{1}{\psi}}\left\{\left[\frac{1}{\alpha\,A_{\#}}\overline{\zeta}^{\frac{-1}{\psi}}\left(\frac{1}{\beta}\overline{g}_N^{\frac{\sigma}{1-\alpha}}+\delta_k-1\right)\right]^{\frac{-1}{1-\alpha}}\overline{l}\right\}^{\alpha}\overline{l}^{1-\alpha}\\
=& C_{\#}\left[1-(1-\delta_N)\,\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\right]^{-1}\left\{\left[\frac{1}{\alpha\,A_{\#}}\overline{\zeta}^{\frac{-(1-\alpha)}{\alpha\psi}}\overline{\zeta}^{\frac{-1}{\psi}}\left(\frac{1}{\beta}\overline{g}_N^{\frac{\sigma}{1-\alpha}}+\delta_k-1\right)\right]^{\frac{-1}{1-\alpha}}\right\}^{\alpha}\overline{l}\\
=& C_{\#}\left(\frac{1}{\alpha\,A_{\#}}\right)^{\frac{-\alpha}{1-\alpha}}\overline{\zeta}^{\left(\frac{1}{\psi}\right)\left(\frac{1}{1-\alpha}\right)}\left[1-(1-\delta_N)\,\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}\right]^{-1}\left(\frac{1}{\beta}\overline{g}_N^{\frac{\sigma}{1-\alpha}}+\delta_k-1\right)^{\frac{-\alpha}{1-\alpha}}\left[\frac{1}{\delta_l}\overline{u}\overline{\theta}\overline{q}(\overline{\theta})\right]
\end{aligned} \tag{6.58}$$

where we use equations (6.54), (6.55), (6.56), (6.57) to express $\overline{k}$ in terms of $\overline{\theta}$

$$\overline{k}(\overline{\theta}) = \left\{\frac{1}{\alpha\,A_{\#}}\overline{\zeta}^{\frac{-1}{\psi}}\left[\frac{1}{\beta}\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}}+\delta_k-1\right]\right\}^{\frac{-1}{1-\alpha}}\overline{l}(\overline{\theta}) \tag{6.59}$$

and we use equations (6.33) and (6.34) to express $\overline{l}$ as follows

$$\overline{l}(\overline{\theta}) = \frac{1}{\delta_l}\overline{u}\overline{\theta}\overline{q}(\overline{\theta}). \tag{6.60}$$

As a result

$$\begin{aligned}\frac{\mathrm{d}\,\overline{W}(\overline{\theta})}{\mathrm{d}\overline{\theta}} =&\overline{W}(\overline{\theta})\left\{\left[\frac{\beta\,(1-\delta_N)}{1-(1-\delta_N)\,\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}}\right]\frac{\mathrm{d}\,\overline{g}_N\,(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}}{\mathrm{d}\overline{\theta}}\right\}\\ &-\overline{W}(\overline{\theta})\left\{\frac{1}{\beta}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\frac{1}{\beta}\overline{g}_N^{\frac{\sigma}{1-\alpha}}+\delta_k-1}\right)\frac{\mathrm{d}\,\overline{g}_N\,(\overline{\theta})^{\frac{\sigma}{1-\alpha}}}{\mathrm{d}\overline{\theta}}-\left[\left(\frac{1}{\overline{u}}\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}+\frac{1}{\overline{\theta}}-\frac{\gamma}{\overline{\theta}}\right]\right\}\\ =&\,\overline{W}(\overline{\theta})\left\{\left[\frac{\beta\,(1-\delta_N)}{1-(1-\delta_N)\,\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}}\right]\frac{\mathrm{d}\,\overline{g}_N\,(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}}{\mathrm{d}\overline{\theta}}\right\}\\ &+\overline{W}(\overline{\theta})\left\{\frac{1}{\beta}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{1-\delta_k-\frac{1}{\beta}\overline{g}_N^{\frac{\sigma}{1-\alpha}}}\right)\frac{\mathrm{d}\,\overline{g}_N\,(\overline{\theta})^{\frac{\sigma}{1-\alpha}}}{\mathrm{d}\overline{\theta}}+\left[\left(\frac{1}{\overline{u}}\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}+\frac{1}{\overline{\theta}}-\frac{\gamma}{\overline{\theta}}\right]\right\}.\end{aligned} \tag{6.61}$$

Derivative of $\overline{g}_N\,(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}$ with respect to θ

We refer to equation (6.50) to compute $\mathrm{d}\,\overline{g}_N\,(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}/\mathrm{d}\overline{\theta}$ in this step.

Derivative of $\overline{g}_N\,(\overline{\theta})^{\frac{\sigma}{1-\alpha}}$ with respect to θ

In reference to equations (6.28), (6.30), and (6.46)

$$\begin{aligned}\frac{\mathrm{d}\,\overline{g}_N\,(\overline{\theta})^{\frac{\sigma}{1-\alpha}}}{\mathrm{d}\overline{\theta}} &= -\eta\phi\left(\frac{\sigma}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{\sigma}{1-\alpha}-1}\left(\overline{l}_R\right)^{-(1-\phi)}\times\cdots\\ &\cdots\times\left[\left(1+\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}+\frac{1}{\delta_l}\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right]\\ &=-\,\Psi_{\#,8}-\Psi_{\#,9}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\end{aligned} \tag{6.62}$$

where

$$\Psi_{\#,7}=\eta\phi\left(\frac{\sigma}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{\sigma}{1-\alpha}-1}\left(\overline{l}_R\right)^{-(1-\phi)}>0 \tag{6.63}$$

$$\Psi_{\#,8}=\Psi_{\#,7}\left[\frac{1}{\delta_l}\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right]>0 \tag{6.64}$$

$$\Psi_{\#,9}=\Psi_{\#,7}\left(1+\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right)>0. \tag{6.65}$$

Explicit expression for d $\overline{W}(\overline{\theta})/d\overline{\theta}$

Equation (6.61) can be rewritten as follows

$$\frac{\mathrm{d}\,\overline{W}(\overline{\theta})}{\mathrm{d}\overline{\theta}}=A_{g,1}\frac{\mathrm{d}\,\overline{g}_N\,(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}}{\mathrm{d}\overline{\theta}}+A_{g,2}\frac{\mathrm{d}\,\overline{g}_N\,(\overline{\theta})^{\frac{\sigma}{1-\alpha}}}{\mathrm{d}\overline{\theta}}+A_{g,3}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}+A_{g,4} \tag{6.66}$$

where

$$A_{g,1} = \overline{W}(\overline{\theta}) \left[\frac{\beta\,(1-\delta_N)}{1-(1-\delta_N)\,\beta g_N^{\frac{-(\sigma-\alpha)}{1-\alpha}}} \right] > 0 \tag{6.67}$$

$$A_{g,2} = \overline{W}(\overline{\theta}) \left(\frac{1}{\beta}\right) \left(\frac{\alpha}{1-\alpha}\right) \left(\frac{1}{1-\delta_k - \frac{1}{\beta}\overline{g}_N^{\frac{\sigma}{1-\alpha}}} \right) > 0 \tag{6.68}$$

$$A_{g,3} = \overline{W}(\overline{\theta}) \left(\frac{1}{\overline{u}}\right) > 0 \tag{6.69}$$

$$A_{g,4} = \overline{W}(\overline{\theta}) \left(\frac{1}{\overline{\theta}}\right) (1-\gamma) > 0. \tag{6.70}$$

Now, using equation (6.62) it follows that

$$\frac{\mathrm{d}\,\overline{W}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = A_{g,1} \frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} - A_{g,5}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} - A_{g,6} \tag{6.71}$$

where

$$A_{g,5} = A_{g,2}\Psi_{\#,9} - A_{g,3} > 0 \tag{6.72}$$

$$A_{g,6} = A_{g,2}\Psi_{\#,8} - A_{g,4} > 0 \tag{6.73}$$

Finally, using equation (6.50), we get

$$\frac{\mathrm{d}\,\overline{W}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = A_{g,7}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + A_{g,8} \tag{6.74}$$

where

$$A_{g,7} = A_{g,1}\Psi_{\#,6} - A_{g,5} > 0 \tag{6.75}$$

$$A_{g,8} = A_{g,1}\Psi_{\#,5} - A_{g,6} > 0 \tag{6.76}$$

Explicit expression for d $\overline{\omega}_R(\overline{\theta})/d\overline{\theta}$

Using equation (6.74), equation (6.49) can be rewritten as follows

$$\frac{\mathrm{d}\,\overline{\omega}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} = \overline{\omega}_R \left\{ \left(\frac{1}{\overline{l}_R}\right) \frac{\mathrm{d}\,\overline{l}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} + \left[\left(\frac{-(\sigma-\alpha)}{1-\alpha}\right) \frac{1}{\overline{g}_N} \right] \frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-\alpha)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} \right\} + A_{\omega,1}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + A_{\omega,2} \tag{6.77}$$

where

$$A_{\omega,1} = \frac{\overline{\omega}_R}{\overline{W}} A_{g,7} > 0 \tag{6.78}$$

$$A_{\omega,2} = \frac{\overline{\omega}_R}{\overline{W}} A_{g,8} > 0 \tag{6.79}$$

Now using equation (6.50) it follows

$$\frac{\mathrm{d}\,\overline{\omega}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} = \overline{\omega}_R \left(\frac{1}{\overline{l}_R}\right) \frac{\mathrm{d}\,\overline{l}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} + A_{\omega,3} \frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + A_{\omega,4} \tag{6.80}$$

where

$$A_{\omega,3} = A_{\omega,1} - \overline{\omega}_R \left[\left(\frac{(\sigma - \alpha)}{1 - \alpha} \right) \frac{1}{\overline{g}_N} \right] \Psi_{\#,6} > 0 \tag{6.81}$$

$$A_{\omega,4} = A_{\omega,2} - \overline{\omega}_R \left[\left(\frac{(\sigma - \alpha)}{1 - \alpha} \right) \frac{1}{\overline{g}_N} \right] \Psi_{\#,5} > 0 \tag{6.82}$$

Finally, using equation (6.46), we get

$$\frac{\mathrm{d}\,\overline{\omega}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} = - A_{\omega,5} \frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} - A_{\omega,6} \tag{6.83}$$

where

$$A_{\omega,5} = \overline{\omega}_R \left(\frac{1}{\overline{l}_R}\right) \frac{\Psi_{\#,3}}{\Psi_{\#,1}\,\overline{l}_R} - A_{\omega,3} > 0 \tag{6.84}$$

$$A_{\omega,6} = \overline{\omega}_R \left(\frac{1}{\overline{l}_R}\right) \frac{\Psi_{\#,2}}{\Psi_{\#,1}\,\overline{l}_R} - A_{\omega,4} > 0. \tag{6.85}$$

6.3.3 Derivative of $\overline{f}(\overline{\theta})$ with respect to θ

To analytically solve equation (6.27), we need also to determine the explicit expression of $\mathrm{d}\,\overline{f}(\overline{\theta})/\mathrm{d}\overline{\theta}$. Equations (6.10), (6.11), and (6.34) yield

$$\overline{f} = \overline{\theta}\,\overline{q}(\overline{\theta}) \tag{6.86}$$

Then

$$\begin{aligned} \frac{\mathrm{d}\,\overline{f}(\overline{\theta})}{\mathrm{d}\overline{\theta}} &= \overline{q}(\overline{\theta}) + \overline{\theta}\overline{q}'(\overline{\theta}) \\ &= \overline{q}(\overline{\theta}) + \overline{\theta} \frac{\mathrm{d}\,\overline{q}(\overline{\theta})}{\mathrm{d}\overline{\theta}}. \end{aligned} \tag{6.87}$$

Finally

$$\frac{\mathrm{d}\,\overline{f}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = (1-\gamma)\,\overline{q}(\overline{\theta}) \tag{6.88}$$

where from (6.21)

$$\gamma = -\frac{\overline{\theta}}{\overline{q}(\overline{\theta})}\frac{\mathrm{d}\,\overline{q}(\overline{\theta})}{\mathrm{d}\overline{\theta}}. \tag{6.89}$$

6.3.4 Derivative of $\overline{H}(\overline{\theta})$ with respect to θ

Using equations (6.8), (6.25), (6.57), and (6.59)

$$\begin{aligned}
\frac{\mathrm{d}\,\overline{H}(\overline{\theta})}{\mathrm{d}\overline{\theta}} =& \overline{H}(\overline{\theta})\left(\frac{1}{\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}}\right)\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} \\
&- \overline{H}(\overline{\theta})\left[\frac{1}{1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_l-\overline{f}\right)}\right]\left[\beta\left(1-\delta_l-\overline{f}\right)\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} + \beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\frac{\mathrm{d}\,\overline{f}(\overline{\theta})}{\mathrm{d}\overline{\theta}}\right] \\
&+ \overline{H}(\overline{\theta})\left(\frac{1}{\overline{\omega}-\overline{b}}\right)\frac{\mathrm{d}\,\overline{\omega}(\overline{\theta})}{\mathrm{d}\overline{\theta}}
\end{aligned} \tag{6.90}$$

where

$$\overline{H}(\overline{\theta}) = \beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left[1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_l-\overline{f}\right)\right]^{-1}\left(\overline{\omega}-\overline{b}\right) \tag{6.91}$$

$$\begin{aligned}
\overline{\omega} =& (1-\xi)\overline{b} + \xi\left[(1-\alpha)\left(\frac{\overline{y}}{\overline{l}}\right) + \overline{g}_N^{\frac{1}{1-\alpha}}\overline{\kappa}\,\overline{\theta}\right] \\
=& (1-\xi)\overline{b} + \xi\left\{(1-\alpha)\,A_{\#}\overline{\zeta}^{\frac{1-\alpha}{\psi}}\left(\frac{1}{\alpha}\right)^{\alpha}\left[\frac{1}{\beta}\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}} + \delta_k - 1\right]^{\frac{-\alpha}{1-\alpha}} + \overline{g}_N^{\frac{1}{1-\alpha}}\overline{\kappa}\overline{\theta}\right\}
\end{aligned} \tag{6.92}$$

Derivative of $\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}$ with respect to θ

In reference to equations (6.28), (6.30), and (6.46), and given equation (6.41)

$$\begin{aligned}
\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} =& \eta\phi\left(\frac{\sigma-1}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{-(\sigma-\alpha)}{1-\alpha}}\left(\overline{l}_R\right)^{-(1-\phi)} \times \cdots \\
&\cdots \times \left[\left(1+\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + \frac{1}{\delta_l}\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right] \\
=& \Psi_{\#,2} + \Psi_{\#,3}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}
\end{aligned} \tag{6.93}$$

where

$$\Psi_{\#,1} = \eta\phi\left(\frac{1-\sigma}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{-(\sigma-\alpha)}{1-\alpha}}\left(\overline{l}_R\right)^{-(1-\phi)} > 0 \tag{6.94}$$

$$\Psi_{\#,2} = \Psi_{\#,1}\left[\frac{1}{\delta_l}\;(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right] > 0 \tag{6.95}$$

$$\Psi_{\#,3} = \Psi_{\#,1}\left(1+\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right) > 0. \tag{6.96}$$

Derivative of $\overline{f}(\overline{\theta})$ with respect to θ

See equation (6.88).

Derivative of $\overline{\omega}$ with respect to θ

Using equation (6.92), it follows that

$$\begin{aligned}\frac{\mathrm{d}\,\overline{\omega}(\overline{\theta})}{\mathrm{d}\overline{\theta}} =& -\alpha\xi\left(\frac{1}{\beta}\right)\left[A_{\#}\overline{\zeta}^{\frac{1-\alpha}{\psi}}\left(\frac{1}{\alpha}\right)^{\alpha}\right]\left[\frac{1}{\beta}\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}}+\delta_k-1\right]^{\frac{-\alpha}{1-\alpha}-1}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}}}{\mathrm{d}\overline{\theta}}\\ &+\left(\overline{\kappa}\overline{\theta}\right)\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{\theta}}+\overline{g}_N^{\frac{1}{1-\alpha}}\overline{\kappa}\end{aligned} \tag{6.97}$$

1. **Derivative of $\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}}$ with respect to θ**

   In reference to equations (6.62)

$$\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}}}{\mathrm{d}\overline{\theta}} = \left(\frac{\sigma}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{\sigma}{1-\alpha}-1}\overline{\chi}\,\frac{\mathrm{d}\,\overline{l}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}}. \tag{6.98}$$

$$\begin{aligned}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}}}{\mathrm{d}\overline{\theta}} &= -\eta\phi\left(\frac{\sigma}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{\sigma}{1-\alpha}-1}\left(\overline{l}_R\right)^{-(1-\phi)}\times\cdots\\ &\cdots\times\left[\left(1+\frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}+\frac{1}{\delta_l}\;(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right]\\ &= -\Psi_{\#,8}-\Psi_{\#,9}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\end{aligned} \tag{6.99}$$

   where

$$\Psi_{\#,7} = \eta\phi\left(\frac{\sigma}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{\sigma}{1-\alpha}-1}\left(\overline{l}_R\right)^{-(1-\phi)} > 0 \tag{6.100}$$

$$\Psi_{\#,8} = \Psi_{\#,7}\left[\frac{1}{\delta_l}\ (1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right] > 0 \tag{6.101}$$

$$\Psi_{\#,9} = \Psi_{\#,7}\left(1 + \frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right) > 0. \tag{6.102}$$

2. **Derivative of $\overline{g}_N(\overline{\theta})^{\frac{1}{1-\alpha}}$ with respect to θ**

Using equations (6.28), (6.30), and (6.46)

$$\begin{aligned}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{\theta}} &= -\eta\phi\left(\frac{1}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{1}{1-\alpha}-1}\left(\overline{l}_R\right)^{-(1-\phi)} \times \cdots \\ &\quad \cdots \times \left[\left(1 + \frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right)\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + \frac{1}{\delta_l}\ (1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right] \\ &= -\,\Psi_{\#,11} - \Psi_{\#,12}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\end{aligned} \tag{6.103}$$

where

$$\Psi_{\#,10} = \eta\phi\left(\frac{1}{1-\alpha}\right)\left(\overline{g}_N\right)^{\frac{1}{1-\alpha}-1}\left(\overline{l}_R\right)^{-(1-\phi)} > 0 \tag{6.104}$$

$$\Psi_{\#,11} = \Psi_{\#,10}\left[\frac{1}{\delta_l}\ (1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})\right] > 0 \tag{6.105}$$

$$\Psi_{\#,12} = \Psi_{\#,10}\left(1 + \frac{1}{\delta_l}\,\overline{\theta}\,\overline{q}(\overline{\theta})\right) > 0. \tag{6.106}$$

3. **Explicit expression for $\mathbf{d}\,\overline{\omega}(\overline{\theta})/\mathbf{d}\overline{\theta}$**

Using equations (6.99) and (6.103), equation (6.97) can be rewritten as follows

$$\begin{aligned}\frac{\mathrm{d}\,\overline{\omega}(\overline{\theta})}{\mathrm{d}\overline{\theta}} &= -\,\alpha\,\xi\left(\frac{1}{\beta}\right)\left[A_{\#}\overline{\zeta}^{\frac{1-\alpha}{\psi}}\left(\frac{1}{\alpha}\right)^{\alpha}\right]\left[\frac{1}{\beta}\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}} + \delta_k - 1\right]^{\frac{-1}{1-\alpha}}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}}}{\mathrm{d}\overline{\theta}} \\ &\quad + \left(\overline{\kappa}\overline{\theta}\right)\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{\theta}} + \overline{g}_N^{\frac{1}{1-\alpha}}\overline{\kappa} \\ &= -\,A_{v,1}\left(-\Psi_{\#,8} - \Psi_{\#,9}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\right) - \left(\overline{\kappa}\overline{\theta}\right)\left(\Psi_{\#,11} + \Psi_{\#,12}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\right) \\ &\quad - \left(\overline{g}_N^{\frac{1}{1-\alpha}}\overline{\theta}\right)\gamma\left(\frac{1}{\overline{\theta}}\right)\overline{q}(\overline{\theta}) + \overline{g}_N^{\frac{1}{1-\alpha}}\overline{\kappa}\end{aligned} \tag{6.107}$$

where

$$A_{v,1} = \alpha\,\xi\left(\frac{1}{\beta}\right)\left[A_{\#}\overline{\zeta}^{\frac{1-\alpha}{\psi}}\left(\frac{1}{\alpha}\right)^{\alpha}\right]\left[\frac{1}{\beta}\overline{g}_N(\overline{\theta})^{\frac{\sigma}{1-\alpha}} + \delta_k - 1\right]^{\frac{-1}{1-\alpha}} > 0. \tag{6.108}$$

Finally we obtain

$$\frac{\mathrm{d}\,\overline{\omega}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = A_{v,2}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + A_{v,3} \tag{6.109}$$

where

$$A_{v,2} = A_{v,1}\Psi_{\#,9} - \left(\overline{\kappa}\overline{\theta}\right)\Psi_{\#,12} > 0 \tag{6.110}$$

$$A_{v,3} = A_{v,1}\Psi_{\#,8} + \overline{g}_N^{\frac{1}{1-\alpha}}\overline{\kappa} - \left(\overline{\kappa}\overline{\theta}\right)\Psi_{\#,11} - \left(\overline{g}_N^{\frac{1}{1-\alpha}}\overline{\theta}\right)\gamma\left(\frac{1}{\overline{\theta}}\right)\overline{q}(\overline{\theta}) > 0. \tag{6.111}$$

Explicit expression for $\mathbf{d}\,\overline{H}(\overline{\theta})/\mathbf{d}\overline{\theta}$

Equation (6.90) can be rewritten as follows

$$\begin{aligned}
\frac{\mathrm{d}\,\overline{H}(\overline{\theta})}{\mathrm{d}\overline{\theta}} =& \overline{H}(\overline{\theta})\left(\frac{1}{\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}}\right)\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} \\
&- \overline{H}(\overline{\theta})\left[\frac{1}{1-\beta\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_l-\overline{f}\right)}\right]\left[\beta\left(1-\delta_l-\overline{f}\right)\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} + \beta\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\frac{\mathrm{d}\,\overline{f}(\overline{\theta})}{\mathrm{d}\overline{\theta}}\right] \\
&+ \overline{H}(\overline{\theta})\left(\frac{1}{\overline{\omega}-\overline{b}}\right)\frac{\mathrm{d}\,\overline{\omega}(\overline{\theta})}{\mathrm{d}\overline{\theta}} \\
=& A_{h,1}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} - A_{h,2}\frac{\mathrm{d}\,\overline{f}(\overline{\theta})}{\mathrm{d}\overline{\theta}} + A_{h,3}\frac{\mathrm{d}\,\overline{\omega}(\overline{\theta})}{\mathrm{d}\overline{\theta}}
\end{aligned} \tag{6.112}$$

where

$$A_{h,1} = \overline{H}(\overline{\theta})\left(\frac{1}{\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}}\right) - \overline{H}(\overline{\theta})\left[\frac{1}{1-\beta\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_l-\overline{f}\right)}\right]\beta\left(1-\delta_l-\overline{f}\right) > 0 \tag{6.113}$$

$$A_{h,2} = \overline{H}(\overline{\theta})\left[\frac{1}{1-\beta\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\left(1-\delta_l-\overline{f}\right)}\right]\beta\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}} > 0 \tag{6.114}$$

$$A_{h,3} = \overline{H}(\overline{\theta})\left(\frac{1}{\overline{\omega}-\overline{b}}\right) > 0. \tag{6.115}$$

Given equation (6.88)

$$\frac{\mathrm{d}\,\overline{H}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = A_{h,1}\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}} + A_{h,3}\frac{\mathrm{d}\,\overline{\omega}(\overline{\theta})}{\mathrm{d}\overline{\theta}} - A_{h,2}(1-\gamma)\,\overline{q}(\overline{\theta}) \tag{6.116}$$

Finally, using equations (6.93) and (6.109), we get

$$\frac{\mathrm{d}\,\overline{H}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = -A_{h,4}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + A_{h,5} \tag{6.117}$$

where

$$A_{h,4} = A_{h,1}\Psi_{\#,3} - A_{h,3}A_{v,2} > 0 \tag{6.118}$$

$$A_{h,5} = A_{h,3}A_{v,3} - A_{h,2}(1-\gamma)\,\overline{q}(\overline{\theta}) - A_{h,1}\Psi_{\#,2} > 0. \tag{6.119}$$

6.3.5 Explicit expression for $\mathrm{d}\overline{u}/\mathrm{d}\overline{\theta}$ and $\mathrm{d}\overline{n}/\mathrm{d}\overline{\theta}$

Now, we are ready to compute the explicit expression for equations (6.19) and (6.27). To do so, we express equation (6.27) as a function $\mathrm{d}\overline{u}/\mathrm{d}\overline{\theta}$.

$$\frac{\mathrm{d}\,\overline{n}(\overline{\theta})}{\mathrm{d}\overline{\theta}} =\Phi_1\left[\frac{\mathrm{d}\,\overline{\omega}_R(\overline{\theta})}{\mathrm{d}\overline{\theta}} - \overline{H}\frac{\mathrm{d}\,\overline{f}(\overline{\theta})}{\mathrm{d}\overline{\theta}} - \overline{f}\frac{\mathrm{d}\,\overline{H}(\overline{\theta})}{\mathrm{d}\overline{\theta}} - \Phi_2\frac{\mathrm{d}\,\overline{g}_N(\overline{\theta})^{\frac{-(\sigma-1)}{1-\alpha}}}{\mathrm{d}\overline{\theta}}\right] \tag{6.120}$$

where

$$\Phi_1 = \frac{1}{\overline{c}_R}\left[1-\beta\,\overline{g}_N^{\frac{-(\sigma-1)}{1-\alpha}}\,(1-\delta_{lR})\right]^{-1} \tag{6.121}$$

$$\Phi_2 = \left[\beta\,(1-\delta_{lR})\,\overline{c}_R\,\overline{n}\right] \tag{6.122}$$

Using equations (6.42) (6.83), (6.88), and (6.117), it follows that

$$\begin{aligned}\frac{\mathrm{d}\,\overline{n}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = \Phi_1\Phi_2\left(\Psi_{\#,2} + \Psi_{\#,3}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}}\right) + \Phi_1\left(A_{\omega,5}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + A_{\omega,6}\right) - \Phi_1\overline{H}(1-\gamma)\,\overline{q}(\overline{\theta})\\ -\Phi_1\overline{f}\left(-A_{h,4}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} + A_{h,5}\right).\end{aligned} \tag{6.123}$$

After rearranging, we get

$$\frac{\mathrm{d}\,\overline{n}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = \Theta_1 + \Theta_2\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} \tag{6.124}$$

where

$$\Theta_1 = \Phi_1\Phi_2\Psi_{\#,2} + \Phi_1 A_{\omega,6} - \Phi_1\overline{H}(1-\gamma)\,\overline{q}(\overline{\theta}) - \Phi_1\overline{f}A_{h,5} > 0 \tag{6.125}$$

$$\Theta_2 = \Phi_1\Phi_2\Psi_{\#,3} + \Phi_1 A_{\omega,5} + \Phi_1\overline{f}A_{h,4} > 0. \tag{6.126}$$

Now solve for explicit expressions of $\mathrm{d}\overline{u}/\mathrm{d}\overline{\theta}$ and $\mathrm{d}\overline{n}/\mathrm{d}\overline{\theta}$. Using equations (6.19) and (6.124), we get

$$\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} = -\left[1+\Theta_2\frac{\overline{u}}{\delta_{lR}-\overline{n}(\overline{\theta})}\right]^{-1}\left\{(1-\gamma)\,\frac{\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}+\left[\frac{\Theta_1}{\delta_{lR}-\overline{n}(\overline{\theta})}\right]\right\}\overline{u} \tag{6.127}$$

$$\Rightarrow \frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} = -\left[\frac{\delta_{lR}-\overline{n}(\overline{\theta})}{\delta_{lR}-\overline{n}(\overline{\theta})+\Theta_2\overline{u}}\right]\left\{(1-\gamma)\,\frac{\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}+\left[\frac{\Theta_1}{\delta_{lR}-\overline{n}(\overline{\theta})}\right]\right\}\overline{u} \tag{6.128}$$

Equation (6.128) can also be written as follows to highlight the terms that are added in a medium-term search RBC model compared with the standard search-and-matching model

$$\begin{aligned}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} = &-(1-\gamma)\,\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}\\ &+(1-\gamma)\left\{1-\left[\frac{\delta_{lR}-\overline{n}(\overline{\theta})}{\delta_{lR}-\overline{n}(\overline{\theta})+\Theta_2\overline{u}}\right]\right\}\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}-\left[\frac{\Theta_1}{\delta_{lR}-\overline{n}(\overline{\theta})+\Theta_2\overline{u}}\right]\overline{u}\end{aligned} \tag{6.129}$$

$$\begin{aligned}\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} = &-(1-\gamma)\,\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}\\ &+(1-\gamma)\,\Theta_2\left[\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_{lR}-\overline{n}(\overline{\theta})+\Theta_2\overline{u}}\right]\frac{\overline{u}}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}-\left[\frac{\Theta_1}{\delta_{lR}-\overline{n}(\overline{\theta})+\Theta_2\overline{u}}\right]\overline{u}\end{aligned} \tag{6.130}$$

$$\frac{\mathrm{d}\overline{u}}{\mathrm{d}\overline{\theta}} = -(1-\gamma)\,\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}-\left\{\frac{\Theta_1\left[\delta_l+\overline{\theta}\overline{q}(\overline{\theta})\right]-\Theta_2\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})}{\left[\delta_l+\overline{\theta}\overline{q}(\overline{\theta})\right]\left[\delta_{lR}-\overline{n}(\overline{\theta})+\Theta_2\overline{u}\right]}\right\}\overline{u} \tag{6.131}$$

Combining (6.124) and (6.131) yields

$$\begin{aligned}\frac{\mathrm{d}\,\overline{n}(\overline{\theta})}{\mathrm{d}\overline{\theta}} = \Theta_1&-\Theta_2\,(1-\gamma)\,\frac{\overline{u}\,\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}\\ &-\Theta_2\left\{\frac{\Theta_1\left[\delta_l+\overline{\theta}\overline{q}(\overline{\theta})\right]-\Theta_2\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})}{\left[\delta_l+\overline{\theta}\overline{q}(\overline{\theta})\right]\left[\delta_{lR}-\overline{n}(\overline{\theta})+\Theta_2\overline{u}\right]}\right\}\overline{u}.\end{aligned} \tag{6.132}$$

□

6.4 Proof of Proposition 4.4

Proof. The elasticity of unemployment with respect to market tightness is

$$\varepsilon(\overline{u},\overline{\theta}) = -\left[1+\Theta_2\frac{\overline{u}}{\delta_{lR}-\overline{n}(\overline{\theta})}\right]^{-1}\left\{(1-\gamma)\,\frac{\overline{q}(\overline{\theta})}{\delta_l+\overline{\theta}\overline{q}(\overline{\theta})}+\left[\frac{\Theta_1}{\delta_{lR}-\overline{n}(\overline{\theta})}\right]\right\}\overline{\theta} \tag{6.133}$$

where $\varepsilon(\overline{u},\overline{\theta}) = (\mathrm{d}\overline{u}/\mathrm{d}\overline{\theta})(\overline{\theta}/\overline{u})$. Using equation (6.17), it follows that

$$\frac{\delta_l}{\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})} = \overline{u}\left[1 - \frac{\overline{n}(\overline{\theta})}{\delta_{lR}}\right]^{-1}. \tag{6.134}$$

Then combining (6.133) and (6.134) yields

$$\begin{aligned}
\varepsilon(\overline{u},\overline{\theta}) &= -\left[1 + \Theta_2 \frac{\overline{u}}{\delta_{lR} - \overline{n}(\overline{\theta})}\right]^{-1}\left\{(1-\gamma)\left[1 - \frac{\delta_l}{\delta_l + \overline{\theta}\overline{q}(\overline{\theta})}\right] + \left[\frac{\Theta_1}{\delta_{lR} - \overline{n}(\overline{\theta})}\right]\overline{\theta}\right\} \\
&= -\left[\frac{\delta_{lR} - \overline{n}(\overline{\theta})}{\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}}\right]\left\{(1-\gamma)\left[1 - \frac{\delta_l}{\delta_l + \overline{\theta}\overline{q}(\overline{\theta})}\right] + \left[\frac{\Theta_1}{\delta_{lR} - \overline{n}(\overline{\theta})}\right]\overline{\theta}\right\} \\
&= -\left[\frac{\delta_{lR} - \overline{n}(\overline{\theta})}{\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}}\right]\left\{(1-\gamma)\left[1 - \frac{\delta_{lR}\,\overline{u}}{\delta_{lR} - \overline{n}(\overline{\theta})}\right] + \left[\frac{\Theta_1}{\delta_{lR} - \overline{n}(\overline{\theta})}\right]\overline{\theta}\right\} \\
&= -\left[\frac{1}{\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}}\right]\left\{(1-\gamma)\left[\delta_{lR} - \overline{n}(\overline{\theta}) - \delta_{lR}\,\overline{u}\right] + \Theta_1\overline{\theta}\right\}.
\end{aligned} \tag{6.135}$$

After rearranging we get

$$\varepsilon(\overline{u},\overline{\theta}) = -\left[\frac{1}{\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}}\right]\left\{(1-\gamma)\left[\delta_{lR}(1-\overline{u}) - \overline{n}(\overline{\theta})\right] + \Theta_1\overline{\theta}\right\}. \tag{6.136}$$

Using equation (6.131), we get

$$\varepsilon(\overline{u},\overline{\theta}) = -(1-\gamma)\frac{\overline{\theta}\,\overline{q}(\overline{\theta})}{\delta_l + \overline{\theta}\overline{q}(\overline{\theta})} - \left\{\frac{\Theta_1\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right] - \Theta_2(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})}{\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right]\left[\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}\right]}\right\}\overline{\theta} \tag{6.137}$$

$$\varepsilon(\overline{u},\overline{\theta}) = -(1-\gamma)\left[1 - \frac{\delta_l}{\delta_l + \overline{\theta}\overline{q}(\overline{\theta})}\right] - \left\{\frac{\Theta_1\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right] - \Theta_2(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})}{\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right]\left[\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}\right]}\right\}\overline{\theta} \tag{6.138}$$

Combining (6.17) and (6.138) yields

$$\begin{aligned}
\varepsilon(\overline{u},\overline{\theta}) = &-(1-\gamma)(1-\overline{u}) \\
&+ (1-\gamma)\left(\frac{\delta_l}{\delta_{lR}}\right)\left[\frac{\overline{n}(\overline{\theta})}{\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})}\right] \\
&- \left\{\frac{\Theta_1\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right] - \Theta_2(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})}{\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right]\left[\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}\right]}\right\}\overline{\theta}
\end{aligned} \tag{6.139}$$

Finally, after rearranging we get

$$\begin{aligned}\varepsilon(\overline{u},\overline{\theta}) = &- (1-\gamma)\,(1-\overline{u})\\ &+ \left\{ \frac{(1-\gamma)\,\delta_l}{\delta_{lR}\left[\delta_l + \overline{\theta}\,\overline{q}(\overline{\theta})\right]} \right\} \overline{n}(\overline{\theta})\\ &- \left\{ \frac{\Theta_1\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right] - \Theta_2\,(1-\gamma)\,\overline{u}\,\overline{q}(\overline{\theta})}{\left[\delta_l + \overline{\theta}\overline{q}(\overline{\theta})\right]\left[\delta_{lR} - \overline{n}(\overline{\theta}) + \Theta_2\overline{u}\right]} \right\} \overline{\theta}\end{aligned} \tag{6.140}$$

□

In a standard search-and-matching model[2]

$$\overline{n}(\overline{\theta}) = \Theta_1 = \Theta_2 = 0. \tag{6.141}$$

6.5 Proof of Proposition 4.5

Proof. To generate big responses of unemployment to productivity changes, search-and-matching models require forces that can make market tightness $\overline{\theta}$ highly elastic with respect to productivity. To compute the explicit expression of the elasticity of labor market tightness with respect to labor productivity, see equation (6.160), we first use the equilibrium value of the labor market tightness in the final good sector, $\overline{\theta}$, determined by equation (6.9) in *Proposition 4.1*. Equation (6.9) can be written as

$$\left(\frac{1-\xi}{\overline{\kappa}}\right)\left(\overline{y}_{\#} - \overline{b}\right) = \frac{1}{\overline{q}}\left(\overline{r} + \delta_l \overline{g}_N^{\frac{1}{1-\alpha}}\right) + \xi \overline{g}_N^{\frac{1}{1-\alpha}}\overline{\theta} - \frac{1}{\overline{q}}\left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right) \tag{6.142}$$

In turn, equation (6.142) can be rewritten as follows

$$\left(\frac{1-\xi}{\overline{\kappa}}\right)\left(\overline{y}_{\#} - \overline{b}\right) = \frac{1}{\overline{q}}\left(\overline{r} + \delta_l\right) + \xi\overline{\theta} - \left[\frac{1}{\overline{q}}\left(1 - \delta_l\right) - \xi\overline{\theta}\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right) \tag{6.143}$$

where

$$\overline{y}_{\#} = (1-\alpha)\left(\frac{\overline{y}}{\overline{l}}\right)$$

$$\overline{g}_N = 1 - \delta_N + \overline{\chi}\,\overline{l}_R^{\,\iota}\,\overline{a}^{1-\iota}.$$

In a standard search-and-matching model

$$\overline{g}_N^{\frac{1}{1-\alpha}} = 1 \Rightarrow \left(\frac{1-\xi}{\overline{\kappa}}\right)\left(\overline{y}_{\#} - \overline{b}\right) = \frac{1}{\overline{q}}\left(\overline{r} + \delta_l\right) + \xi\overline{\theta}$$

[2]See e.g. equation (14) in Ljungqvist and Sargent (2017, p. 2636).

Differentiation

$$\left(\frac{1-\xi}{\kappa}\right)\frac{\mathrm{d}\overline{y}_{\#}}{\mathrm{d}\overline{\theta}} = -\left(\overline{r}+\delta_l\right)\frac{\overline{q}'(\overline{\theta})}{\overline{q}^2} + \left(\frac{1}{\overline{q}}\right)\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{\theta}} + \xi - \left[\frac{\overline{q}'(\overline{\theta})}{\overline{q}^2}\left(1-\delta_l\right) - \xi\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right) - \left[\frac{1}{\overline{q}}\left(1-\delta_l\right) + \xi\overline{\theta}\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{\theta}} \tag{6.144}$$

$$\left(\frac{1-\xi}{\kappa}\right)\frac{\mathrm{d}\overline{y}_{\#}}{\mathrm{d}\overline{\theta}} = -\left(\overline{r}+\delta_l\right)\frac{\overline{q}'(\overline{\theta})}{\overline{q}^2} + \left(\frac{1}{\overline{q}}\right)\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{\theta}} + \xi + \left[\frac{\overline{q}'(\overline{\theta})}{\overline{q}^2}\left(1-\delta_l\right) - \xi\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right) - \left[\frac{1}{\overline{q}}\left(1-\delta_l\right) + \xi\overline{\theta}\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{\theta}} \tag{6.145}$$

Using equation (6.21), it follows that

$$\left(\frac{1-\xi}{\kappa}\right)\frac{\mathrm{d}\overline{y}_{\#}}{\mathrm{d}\overline{\theta}} = \frac{\gamma\left(\overline{r}+\delta_l\right)}{\overline{\theta}\,\overline{q}(\overline{\theta})} + \xi + \left(\frac{1}{\overline{q}}\right)\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{\theta}} + \left[\frac{\gamma\left(1-\delta_l\right)}{\overline{\theta}\,\overline{q}(\overline{\theta})} - \xi\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right) - \left[\frac{1}{\overline{q}}\left(1-\delta_l\right) + \xi\overline{\theta}\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{\theta}} \cdot \tag{6.146}$$

$$\left(\frac{1-\xi}{\kappa}\right)\frac{\mathrm{d}\overline{y}_{\#}}{\mathrm{d}\overline{\theta}} = \frac{\gamma\left(\overline{r}+\delta_l\right)}{\overline{\theta}\,\overline{q}(\overline{\theta})} + \xi + \left[\frac{1}{\overline{q}(\overline{\theta})}\right]\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{\theta}} + \left[\frac{\gamma\left(1-\delta_l\right)}{\overline{\theta}\,\overline{q}(\overline{\theta})} + \xi\right]\left(1 - \overline{g}_N^{\frac{1}{1-\alpha}}\right) - \left[\frac{1}{\overline{q}(\overline{\theta})}\left(1-\delta_l\right) + \xi\overline{\theta}\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{\theta}} \cdot \tag{6.147}$$

$$\left(\frac{1-\xi}{\kappa}\right) = \left[\frac{\gamma\left(\overline{r}+\delta_l\right)}{\overline{\theta}\,\overline{q}(\overline{\theta})} + \xi\right]\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\overline{y}_{\#}} + \left[\frac{1}{\overline{q}(\overline{\theta})}\right]\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{y}_{\#}} + \left[\frac{\gamma\left(1-\delta_l\right)}{\overline{\theta}\,\overline{q}(\overline{\theta})} - \xi\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}} - 1\right) - \left[\frac{1}{\overline{q}(\overline{\theta})}\left(1-\delta_l\right) + \xi\overline{\theta}\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{y}_{\#}} \cdot \tag{6.148}$$

$$
\begin{aligned}
\left(\frac{1-\xi}{\overline{\kappa}}\right) = &\left[\frac{\gamma\,(\overline{r}+\delta_l)}{\overline{\theta}\,\overline{q}(\overline{\theta})}+\xi\right]\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\overline{y}_{\#}}+\left[\frac{1}{\overline{q}(\overline{\theta})}\right]\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{y}_{\#}} \\
&+\left[\frac{\gamma\,(1-\delta_l)}{\overline{\theta}\,\overline{q}(\overline{\theta})}+\xi\right]\left(1-\overline{g}_N^{\frac{1}{1-\alpha}}\right)-\left[\frac{1}{\overline{q}(\overline{\theta})}\,(1-\delta_l)+\xi\overline{\theta}\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{y}_{\#}}\cdot
\end{aligned}
\tag{6.149}
$$

Using equation (6.142)

$$
\begin{aligned}
\frac{1-\xi}{\overline{\kappa}} =&\frac{1}{\overline{q}(\overline{\theta})}\left\{\left[(1+\overline{r})-(1-\delta_l)\overline{g}_N^{\frac{1}{1-\alpha}}\right]+\xi\overline{g}_N^{\frac{1}{1-\alpha}}\overline{\theta}\,\overline{q}(\overline{\theta})\right\}\frac{1}{\overline{y}_{\#}-\overline{b}} \\
=&\ \Xi_{\#}\frac{1}{\overline{y}_{\#}-\overline{b}}
\end{aligned}
\tag{6.150}
$$

where

$$
\Xi_{\#} = \frac{1}{\overline{q}(\overline{\theta})}\left\{\left[(1+\overline{r})-(1-\delta_l)\overline{g}_N^{\frac{1}{1-\alpha}}\right]+\xi\overline{g}_N^{\frac{1}{1-\alpha}}\overline{\theta}\,\overline{q}(\overline{\theta})\right\}.
\tag{6.151}
$$

Using equation (6.149), it follows that

$$
\begin{aligned}
\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\overline{y}_{\#}} =&\Xi_{\#}\left[\frac{\overline{\theta}\,\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{1}{\overline{y}_{\#}-\overline{b}} \\
&-\left[\frac{\overline{\theta}\,\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[\frac{1}{\overline{q}(\overline{\theta})}\right]\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{y}_{\#}} \\
&+\left[\frac{\overline{\theta}\,\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[\frac{\gamma}{\overline{\theta}\,\overline{q}(\overline{\theta})}\,(1-\delta_l)+\xi\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}}-1\right) \\
&+\left[\frac{\overline{\theta}\,\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[\frac{1}{\overline{q}(\overline{\theta})}\,(1-\delta_l)+\xi\overline{\theta}\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{y}_{\#}}\cdot
\end{aligned}
\tag{6.152}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\overline{y}_{\#}} =&\Xi_{\#}\left[\frac{\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{\overline{\theta}}{\overline{y}_{\#}-\overline{b}} \\
&-\left[\frac{\overline{\theta}}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{y}_{\#}} \\
&+\left[\frac{1}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[\gamma\,(1-\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}}-1\right) \\
&+\left[\frac{\overline{\theta}}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[(1-\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{y}_{\#}}\cdot
\end{aligned}
\tag{6.153}
$$

Using equation (6.54)

$$\begin{aligned}\overline{r} &= \alpha \frac{\overline{y}}{\overline{k}} - \delta_k \\ &= (1-\alpha)\left(\frac{\overline{y}}{\overline{l}}\right)\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\overline{l}}{\overline{k}}\right) \\ &= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\overline{l}}{\overline{k}}\right)\overline{y}_{\#}\end{aligned} \tag{6.154}$$

Then,

$$\begin{aligned}\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{y}_{\#}} &= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\overline{l}}{\overline{k}}\right) \\ &= \frac{1}{(1-\alpha)\left(\frac{\overline{y}}{\overline{l}}\right)}\alpha\left(\frac{\overline{y}}{\overline{k}}\right) \\ &= \frac{\overline{r}+\delta_k}{\overline{y}_{\#}}\end{aligned} \tag{6.155}$$

Now, using equations (6.55) and (6.56)

$$\begin{aligned}\overline{g}_N^{\frac{1}{1-\alpha}} &= \beta^{\frac{1}{\sigma}}\left[\overline{R}+(1-\delta_k)\right]^{\frac{1}{\sigma}} \\ &= \beta^{\frac{1}{\sigma}}\left(1+\overline{r}\right)^{\frac{1}{\sigma}}\end{aligned} \tag{6.156}$$

Then,

$$\begin{aligned}\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{y}_{\#}} &= \frac{1}{\sigma}\left(\frac{1}{1+\overline{r}}\right)\overline{g}_N^{\frac{1}{1-\alpha}}\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{y}_{\#}} \\ &= \frac{1}{\sigma}\left[\frac{\delta_k+\overline{r}}{(1+\overline{r})\overline{y}_{\#}}\right]\overline{g}_N^{\frac{1}{1-\alpha}}\end{aligned} \tag{6.157}$$

Using equation (6.153), it follows that

$$
\begin{aligned}
\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\overline{y}_\#} =& \Xi_\# \left[\frac{\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{\overline{\theta}}{\overline{y}_\# - \overline{b}} - \left[\frac{\overline{\theta}}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left(\frac{\overline{r}+\delta_k}{\overline{y}_\#}\right)\\
&+\left[\frac{1}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[\gamma\,(1-\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}}-1\right)\\
&+\left[\frac{\overline{\theta}}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[(1-\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})\right]\left\{\frac{1}{\sigma}\left[\frac{\delta_k+\overline{r}}{(1+\overline{r})\overline{y}_\#}\right]\overline{g}_N^{\frac{1}{1-\alpha}}\right\}\\
&=\Xi_\#\left[\frac{\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{\overline{\theta}}{\overline{y}_\#-\overline{b}}\\
&-\left[\frac{\overline{\theta}}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left(\frac{\overline{r}+\delta_k}{\overline{y}_\#}\right)+\left[\frac{\gamma\,(1-\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}}-1\right)\\
&+\left[\frac{\overline{\theta}}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left(\frac{\overline{r}+\delta_k}{\overline{y}_\#}\right)\frac{1}{\sigma}\left[\frac{(1-\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}{1+\overline{r}}\right]\overline{g}_N^{\frac{1}{1-\alpha}}\\
&=\Xi_\#\left[\frac{\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{\overline{\theta}}{\overline{y}_\#-\overline{b}}+\left[\frac{\gamma\,(1-\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}}-1\right)\\
&+\left[\frac{\overline{\theta}}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left(\frac{\overline{r}+\delta_k}{\overline{y}_\#}\right)\left[\frac{1}{\sigma}\left(\frac{(1-\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}{1+\overline{r}}\right)\overline{g}_N^{\frac{1}{1-\alpha}}-1\right].
\end{aligned}
\tag{6.158}
$$

We are now ready to compute the elasticity of labor market tightness with respect to labor productivity as

$$
\begin{aligned}
\varepsilon(\overline{\theta},\overline{y}_\#) =& \Xi_\#\left[\frac{\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{\overline{y}_\#}{\overline{y}_\#-\overline{b}}-\left[\frac{\overline{y}_\#}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{\mathrm{d}\overline{r}}{\mathrm{d}\overline{y}_\#}\\
&+\left[\frac{\overline{y}_\#}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[\frac{\gamma}{\overline{\theta}\,\overline{q}(\overline{\theta})}\,(1-\delta_l)+\xi\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}}-1\right)\\
&+\left[\frac{\overline{y}_\#}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[(1-\delta_l)+\xi\overline{\theta}\overline{q}(\overline{\theta})\right]\frac{\mathrm{d}\overline{g}_N^{\frac{1}{1-\alpha}}}{\mathrm{d}\overline{y}_\#}\\
&=\Xi_\#\left[\frac{\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\frac{\overline{y}_\#}{\overline{y}_\#-\overline{b}}-\left[\frac{\overline{r}}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\varepsilon\left(\overline{r},\overline{y}_\#\right)\\
&+\left[\frac{\overline{y}_\#}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\right]\left[\frac{\gamma}{\overline{\theta}\,\overline{q}(\overline{\theta})}\,(1-\delta_l)+\xi\right]\left(\overline{g}_N^{\frac{1}{1-\alpha}}-1\right)\\
&+\left[\frac{(1-\delta_l)+\xi\overline{\theta}\overline{q}(\overline{\theta})}{\gamma\,(\overline{r}+\delta_l)+\xi\,\overline{\theta}\,\overline{q}(\overline{\theta})}\overline{g}_N^{\frac{1}{1-\alpha}}\right]\varepsilon\left(\overline{g}_N,\overline{y}_\#\right).
\end{aligned}
\tag{6.159}
$$

$$
\varepsilon(\overline{\theta},\overline{y}_\#) = \Upsilon_1\frac{\overline{y}_\#}{\overline{y}_\#-\overline{b}} - \Upsilon_2\varepsilon\left(\overline{r},\overline{y}_\#\right)+\Upsilon_3\varepsilon\left(\overline{g}_N,\overline{y}_\#\right)+\Upsilon_4\left(\overline{g}_N^{\frac{1}{1-\alpha}}-1\right)
\tag{6.160}
$$

where

$$\Upsilon_1 = \Xi_\# \left[\frac{\overline{q}(\overline{\theta})}{\gamma\,(\overline{r} + \delta_l) + \xi\,\overline{\theta}\,\overline{q}(\overline{\theta})} \right] \tag{6.161}$$

$$\Upsilon_2 = \left[\frac{\overline{r}}{\gamma\,(\overline{r} + \delta_l) + \xi\,\overline{\theta}\,\overline{q}(\overline{\theta})} \right] \tag{6.162}$$

$$\Upsilon_3 = \frac{(1 - \delta_l) + \xi\overline{\theta}\overline{q}(\overline{\theta})}{\gamma\,(\overline{r} + \delta_l) + \xi\,\overline{\theta}\,\overline{q}(\overline{\theta})} \overline{g}_N^{\frac{1}{1-\alpha}} \tag{6.163}$$

$$\Upsilon_4 = \left[\frac{\overline{y}_\#}{\gamma\,(\overline{r} + \delta_l) + \xi\,\overline{\theta}\,\overline{q}(\overline{\theta})} \right] \left[\frac{\gamma}{\overline{\theta}\,\overline{q}(\overline{\theta})}\,(1 - \delta_l) + \xi \right]. \tag{6.164}$$

Moreover, given equation (6.155) and (6.157), respectively, we have

$$\varepsilon\,(\overline{r}, \overline{y}_\#) = 1 + \frac{\delta_k}{\overline{r}} \tag{6.165}$$

$$\varepsilon\left(\overline{g}_N, \overline{y}_\#\right) = \frac{\delta_k + \overline{r}}{(1 + \overline{r})\,\sigma}. \tag{6.166}$$

□

In a standard search-and-matching model with capital stock $\overline{g}_N^{\frac{1}{1-\alpha}} = 1$ and $\varepsilon\left(\overline{g}_N, \overline{y}_\#\right) = 0$. Moreover, in a standard search-and-matching model without capital stock, see equation (15) in Ljungqvist and Sargent (2017), $\varepsilon\,(\overline{r}, \overline{y}_\#) = 0$, $\overline{g}_N^{\frac{1}{1-\alpha}} = 1$, and $\varepsilon\left(\overline{g}_N, \overline{y}_\#\right) = 0$.

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