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RÉSUMÉ

Mes travaux se situent au cœur de l'interaction récente entre la combinatoire algébrique et la théorie de la représentation. Un des problèmes principaux consiste à décrire de manière explicite la décomposition en composantes irréductibles d'espaces de polynômes harmoniques diagonaux. La problématique est motivée par des questions issues de plusieurs domaines, allant de la théorie des fonctions symétriques, à la théorie des nœuds, en passant par la géométrie algébrique et la physique théorique. On y retrouve notamment une décomposition partielle en bicaractère des espaces diagonaux harmonique de même que des espaces diagonaux harmoniques multivariés. Un nouvel objet combinatoire est aussi présenté, ainsi que des ramifications en vertu de la combinatoire classique.

Dans cette thèse par article, on donne une description explicite des bicaractères irréductibles indexée par des équerres, dans la décomposition en composantes irréductibles des espaces de polynômes harmoniques diagonaux (en deux jeux de variables, et en un nombre quelconque de jeux de variables). Les outils utilisés ont permis de répondre partiellement au problème 3.11 de (Haglund, 2008), qui consistent en une bijection inversant certaines statistiques sur les chemins de Schröder. Pour atteindre cet objectif, un nouvel objet combinatoire a été introduit. Dans le dernier article, on examine les liens inhérents entre celui-ci et la combinatoire des espaces diagonaux harmonique ainsi que la combinatoire classique en général.

INTRODUCTION

Les espaces de polynômes harmoniques diagonaux ont des applications en théorie des nœuds, en physique théorique, en géométrie algébrique en plus d'être caractérisés comme des ensembles de solutions de certaines équations différentielles. Ce sont des modules munis à la fois d'une action du groupe symétrique et d'une action du groupe général linéaire.

Ce travail porte sur certains aspects de la décomposition de ces espaces en composantes irréductibles pour ces deux actions de groupes, plus spécialement dans le cas de deux jeux de variables. Certains opérateurs, dont l'opérateur nabla et les opérateurs delta, présentés par Bergeron, Garsia, Haiman et Tesler ont les polynômes de Macdonald comme vecteurs propres. Il a été démontré par Haiman que l'opérateur nabla appliqué à des fonctions symétriques bien choisies décrit le caractère, au sens de la théorie de la représentation, des espaces diagonaux harmoniques. Dans ce travail, je donne des formules combinatoires pour certains bicaractères irréductibles de ces modules.

Les espaces de polynômes harmoniques diagonaux ont une interprétation combinatoire en termes de fonctions de stationnement. Les chemins de Schröder sont un sous-ensemble des fonctions de stationnement. Dans « Toward a Schurification of Parking Function Formulas via Bijections with Young Tableaux », qui correspond au chapitre 2, une réponse partielle au problème 3.11 de (Haglund, 2008) est présenté. Étant donné la symétrie entre les variables q et t dans la série génératrice des chemins de Schröder, l'auteur cherche une bijection explicite entre chemins de Schröder qui inverse les statistiques aire et bounce. En outre, on met de l'avant un algorithme qui donne lieu à un regroupement de certaines fonctions de stationnement en les associant à un bicaractère irréductible. Conséquemment, on obtient des formules combinatoires explicites pour ces bicaractères.

Le chapitre 3 traite de « Explicit Formulas for Characters of the Module of Multivariate Diagonal Harmonics », où j'ai introduit un nouvel objet combinatoire qui permet de trouver facilement les polynômes représentant certains bicaractères irréductibles des espaces harmoniques diagonaux multivariés. Les bicaractères sont associés à une paire constituée d'un chemin nord-est dans une grille en forme d'escalier et un tableau de Young.

On termine avec le chapitre 4 où je discute d'un troisième article qui explore les liens entre l'objet présenté au chapitre 3 et des objets classiques de combinatoires. Certaines statistiques de la combinatoire classique sont raffinées. En outre, on présente une bijection entre les permutations qui évitent les motifs 132 et 312 et les permutations qui évitent les motifs 213 et 231. Cette bijection préserve l'indexe major, le nombre de descentes ainsi que le Q-tableau au sens de l'algorithme Robinson-Schensted change le $P-$tableau en sont tableau d'évacuation. On termine avec un lien surprenant qui permet d'écrire la formule du chapitre 3 à l'aide paires de chemins nord-est dans une grille en forme d'escalier.

CHAPITRE I

DÉFINITIONS ET NOTATIONS

1.1 Mots, permutations, compositions, partage et tableaux

Un *alphabet*, A, est un ensemble. Chaque élément de cet ensemble se nomme *lettre*. Un *mot*, m, est une suite de lettre, si aucune confusion n'est possible, on omet généralement les virgules. Un mot est de *longueur* n, s'il contient n lettres, noté $|m| = n$. Pour une lettre a dans A et un mot m, le nombre d'occurrences de a dans m est noté $|m|_a$. L'ensemble des mots de longueur n est noté A^n et l'ensemble de tous les mots de longueur fini composé des lettres de l'alphabet A est noté A^*. Un *sous-mot* est une sous-suite du mot tandis qu'un *facteur* est une sous-suite consécutive du mot. Un *préfixe* est un facteur qui contient la première lettre du mot et un *suffixe* est un facteur qui contient la dernière lettre du mot. On dénote mA^* l'ensemble des mots de toutes longueurs qui ont m comme préfixe, on dénote $A^n u$ est l'ensemble des mots de longueur $n + |m|$ qui ont m comme suffixe et on dénote A^*mA^* l'ensemble des mots de toute longueur qui ont m comme facteur. Il est possible de considérer l'ensemble des mots constitué de lettres et de mots. Par exemple, EN, NE, DE, ED, ND, DN, EE, NN et DD sont tous les mots de $\{N, E, D\}^2$, mais l'ensemble des mots constitué du mot NE et de la lettre D, dénoté $\{NE, D\}^2$ contient DNE, NED, $NENE$ et DD.

L'*image miroir* d'un mot m , dénoté m^r, est le mot obtenu en lisant les lettres à partir de la fin. Par exemple, $(EENNNEN)^r = NENNNEE$. Lorsque l'alphabet contient deux lettres, l'involution $\overline{m}$ échange les lettres. Par exemple, pour l'alphabet $\{N, E\}$ on trouve $\overline{EENNNEN} = NNEEENE$. De plus, on défini le *mélange* de deux mots m et w comme l'ensemble des mots de longueur $|m| + |w|$

qui contient les sous-mots distincts m et w. Par exemple, le mélange de 34 et 21 est l'ensemble $\{3421, 3241, 3214, 2341, 2314, 2134\}$.

Une permutation est une autobijection de l'ensemble $\{1, 2, \ldots, n\}$. L'ensemble de ces permutations est dénoté $\mathbb{S}_n$. Soit σ un élément de $\mathbb{S}_k$ on dit que π *évite le motif* σ si tous les sous-mots de π de longueur k n'ont pas le même ordre relatif que σ. Par exemple, $\pi = 34125$ évite les motifs 132 and 321, mais contient le motif 213 puisque $\pi(1)\pi(3)\pi(5) = 315$ a le même ordre relatif que 213. On dénote $A_n(\sigma_1, \sigma_2, \ldots, \sigma_k)$ l'ensemble des permutations de longueur n qui évite simultanément les motifs σ_1, σ_2, …, σ_k. Selon l'exemple précédent $\pi \in A_5(132, 321)$, mais $\pi \notin A_5(213, 321)$, car il n'évite pas le motif 213. On remarque que les permutations de n qui évitent 132 et 312 sont des mélanges de $n-k, n-k+1, \ldots, n$ et $n-k-1, n-k-2, \ldots, 1$.

Une *descente* d'une permutation π est une position i dans π tel que $\pi(i) > \pi(i+1)$. L'ensemble des descentes de la permutation π est dénoté $\mathrm{Des}(\pi)$ et la cardinalité de cet ensemble est dénoté $\mathrm{des}(\pi)$. L'*index majeur*, dénoté maj est la somme des descentes. Par exemple, pour $\pi = 34125$ on trouve $\mathrm{Des}(\pi) = \{2\}$, $\mathrm{des}(\pi) = 1$ et $\mathrm{maj}(\pi) = 2$, car $\pi(2) = 4 > 1 = \pi(3)$.

Le *complément* de la permutation π est dénotée π^c et est défini par $\pi^c(i) = \pi(n+1-i)$. L'image miroir d'une permutation est définie de la même façon de l'image miroir d'un mot. On dénote w_0 la permutation $n, n-1, \ldots, 2, 1$ et il est établi que $w_0\pi = \pi^c$ et $\pi w_0 = \pi^r$. Il a été démontré dans (Simon et Schmidt, 1985) que si π évite les motifs σ_1, σ_2, …, σ_k alors la permutation π^{-1} évite les motifs σ_1^{-1}, σ_2^{-1}, …, σ_k^{-1}, la permutation π^c évite les motifs σ_1^c, σ_2^c, …, σ_k^c et la permutation π^r évite les motifs σ_1^r, σ_2^r, …, σ_k^r.

Une *composition* de n, disons $c = (c_1, c_2, \ldots, c_k)$, est une suite d'entiers positifs non nuls, dont la somme est n. Les parenthèses et les virgules sont généralement omises. Une *part* est un élément de la suite, le nombre d'éléments dans la suite est la *longueur*, noté ℓ et la taille, dénoté $|c|$, est la somme des parts. Soit deux compositions c et d, on dit que c est un *raffinement* de d, dénoté $c \preccurlyeq d$, si d peut-être obtenue en additionnant des parts adjacentes de c. Les compositions munies de cette relation forment un treillis. Il existe une bijection entre les sous-ensembles de $\{1, 2, \ldots, n-1\}$, disons S, et les compositions de n, noté $C(S)$. En outre, on dénote ides(π) la composition associée à l'ensemble de descente de l'inverse de π. En d'autres mots, $\text{ides}(\pi) = C(\text{Des}(\pi^{-1}))$.

Un *partage* de n, est une composition dont les parts sont décroissantes. Il peut être représenté par un diagramme de Ferrers. Un partage $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$ est dit en forme d'*équerre* si $\lambda_i = 1$ pour tout $2 \leq i \leq k$. Le remplissage des n cases d'un diagramme de Ferrers par n entrées, généralement des nombres, se nomme *tableau*. Un tableau est dit *semi-standard* si les entrées des lignes sont croissantes et les entrées des colonnes sont strictement croissantes. On dit que c'est un *tableau de Young standard* si les entrées sont en bijection avec l'ensemble $\{1, 2, \ldots, n\}$ et si les entrées des lignes ainsi que des colonnes sont strictement croissantes (voir figure 1.1 pour un exemple). Le *conjugué* d'un partage (respectivement d'un tableau) est la réflexion de celui-ci par la droite $x = y$ (voir figure 1.1 pour un exemple). La *forme* d'un tableau, τ, est le partage associé à ce tableau, dénoté $\lambda(\tau)$. L'ensemble des tableaux semi-standard de forme λ est noté $SSYT(\lambda)$ et l'ensemble des tableaux standard de forme λ est noté $SYT(\lambda)$. Une *descente* d'un tableau est une entrée i telle que l'entrée $i+1$ est sur une ligne au-dessus. Comme pour les permutations on dénote Des l'ensemble des descentes, des le nombre de descentes et maj dénote l'index majeur d'un tableau, c'est-à-dire la somme de ces descentes. Par exemple, dans la figure 1.1 les descentes sont $\{2, 3, 7, 9\}$, donc l'indexe majeur est de 21.

10				
4				
3	8	9		
1	2	5	6	7

7			
6			
5	9		
2	8		
1	3	4	10

Figure 1.1 À gauche, un tableau de Young standard, P, de forme 5311. À droite P' le conjugué de P de forme 42211

Le *tableau d'évacuation* d'un tableau P de forme λ, dénoté $\text{ev}(P)$, est le tableau obtenu en numérotant dans l'ordre inverse les cases évacuées. Pour évacuer une case, on choisit la case c qui a la plus petite entrée et on la remplace par un point. Si l'entrée au-dessus du point est plus petite que l'entrée à droite du point on glisse le point vers le haut et l'on remplace la case vacante par l'entrée qui était en haut du point. Si l'entrée au-dessus du point est plus grande que l'entrée à droite du point, on glisse le point vers la droite et l'on remplace la case vacante par l'entrée qui était à droite du point. On recommence le processus jusqu'à ce qu'il n'y ait plus d'entrée au-dessus et à la droite du point. La case évacuée est la dernière case qui a contenu le point. Pour un exemple, voir la figure 1.2. Lorsque P est en forme d'équerre on trouve le tableau d'évacuation simplement en changeant l'entrée i pour $n+2-i$, lorsque $i > 1$ et en réordonnant les colonnes et les lignes pour obtenir un tableau de Young standard. Pour plus d'information, une personne peut se référer à (Bjorner et Brenti, 2005) ou à (Sagan, 2001).

La première évacuation :

$$\begin{array}{|c|c|} \hline 4 & \\ \hline 2 & 5 \\ \hline 1 & 3 \\ \hline \end{array} \to \begin{array}{|c|c|} \hline 4 & \\ \hline 2 & 5 \\ \hline \bullet & 3 \\ \hline \end{array} \to \begin{array}{|c|c|} \hline 4 & \\ \hline \bullet & 5 \\ \hline 2 & 3 \\ \hline \end{array} \to \begin{array}{|c|c|} \hline \bullet & \\ \hline 4 & 5 \\ \hline 2 & 3 \\ \hline \end{array} \to \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 2 & 3 \\ \hline \end{array}$$

Les évacuations successives sont :

$$\begin{array}{|c|c|} \hline 4 & \\ \hline 3 & 5 \\ \hline \end{array}, \qquad \begin{array}{|c|c|} \hline 4 & 5 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline 5 \\ \hline \end{array}$$

Le tableau d'évacuation est :

$$\begin{array}{|c|c|} \hline 5 & \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$$

Figure 1.2 Tableau d'évacuation

L'algorithme Robinson-Schensted est une bijection entre les permutations et les paires de tableaux de Young standard de mêmes formes. Le P-tableau d'une permutation π est dénoté $P(\pi)$ et le Q-tableau d'une permutation π est dénoté $Q(\pi)$. Le Q-tableau enregistre dans laquelle les cases sont créées. Les tableaux sont construits de la façon suivante. On insère $\pi(1)$ dans la première ligne première colonne du P-tableau et 1 dans la première ligne et première colonne du Q-tableau. Ensuite, à l'étape i on insère $\pi(i)$ dans la première ligne du P-tableau. Si $\pi(i)$ est la plus grande entrée de la première ligne on le met au bout de la ligne, créant ainsi une nouvelle case et on insère i dans la case correspondante du Q-tableau. Autrement, on insère $\pi(i)$ dans la case contenant la plus petite entrée qui est plus grande que $\pi(i)$, disons E_1, et on insère E_1, de la même façon, sur la deuxième ligne. Ainsi, lorsque E_j est inséré au bout de la ligne $j+1$, créant une nouvelle case, on inscrit i dans la case correspondante du Q-tableau (voir figure 1.3).

Dans la prochaine section, certains liens entre les fonctions symétriques et les tableaux seront établis.

Première insertion :

$\mathrm{P}=\boxed{5}\ \ \mathrm{Q}=\boxed{1}$.

Deuxième insertion :

$\mathrm{P}=\boxed{5} \leftarrow 3$, $\quad \mathrm{P}=\boxed{3}\ \overset{\leftarrow 5}{}$, $\quad \mathrm{P}=\begin{array}{|c|}\hline 5\\ \hline 3\\ \hline\end{array} \quad \mathrm{Q}=\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}$.

Troisième insertion :

$\mathrm{P}=\begin{array}{|c|}\hline 5\\ \hline 3\\ \hline\end{array} \leftarrow 4$, $\quad \mathrm{P}=\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 3 & 4\\ \hline\end{array} \quad \mathrm{Q}=\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3\\ \hline\end{array}$.

Quatrième insertion :

$\mathrm{P}=\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 3 & 4\\ \hline\end{array} \leftarrow 1$, $\quad \mathrm{P}=\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{\leftarrow 3} \\ \hline 1 & 4\\ \hline\end{array}$, $\quad \mathrm{P}=\begin{array}{|c|c|}\multicolumn{1}{c}{} & \multicolumn{1}{c}{\leftarrow 5} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 4\\ \hline\end{array}$, $\quad \mathrm{P}=\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 4\\ \hline\end{array} \quad \mathrm{Q}=\begin{array}{|c|c|}\cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3\\ \hline\end{array}$.

Cinquième insertion :

$\mathrm{P}=\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 4\\ \hline\end{array} \leftarrow 2$, $\quad \mathrm{P}=\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{\leftarrow 4} \\ \hline 1 & 2\\ \hline\end{array}$, $\quad \mathrm{P}=\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 3 & 4 \\ \hline 1 & 2\\ \hline\end{array} \quad \mathrm{Q}=\begin{array}{|c|c|}\cline{1-1} 4 & \multicolumn{1}{c}{} \\ \hline 2 & 5 \\ \hline 1 & 3\\ \hline\end{array}$.

Figure 1.3 Les tableaux $P(53412)$ et $Q(53412)$.

1.2 Fonctions symétriques, fonctions quasisymétriques

Les notations pour les fonctions symétriques sont les mêmes que celles retrouvées dans le livre de Macdonald (Macdonald, 1995). L'*anneau des polynômes symétriques* est l'ensemble des polynômes invariant sous l'action de permutation des variables. Si f est un polynôme symétrique à n variables, alors pour tout $\sigma \in \mathbb{S}_n$ on trouve $f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$. L'*anneau des fonctions symétriques* peut être vu comme l'ensemble des fonctions symétriques à plusieurs variables. C'est un anneau gradué qui a les *fonctions symétriques élémentaires*, dénoté e_λ pour base. Ils sont définit par $e_n(X) = \sum_{i_1<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$ et $e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k}$. Les *Fonctions de Schur* sont une autre base des fonctions symétriques. Ils sont définis par la somme $s_\lambda = \sum_{\tau\in \mathrm{SST}(\lambda)} x_\tau$, où $x_\tau := \prod_{c\in\tau} x_c$.

Voici un exemple :

$$\begin{aligned} s_{21}(x,y,z,\ldots) &= \ytableaushort{2,11} + \ytableaushort{3,11} + \ytableaushort{2,12} + \ytableaushort{3,13} + \ytableaushort{3,22} + \ytableaushort{3,23} + \ytableaushort{3,12} + \ytableaushort{2,13} + \cdots \\ &= x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + 2xyz + \cdots \end{aligned}$$

On remarque que la condition de croissance stricte sur les colonnes implique que le nombre de variables doit être supérieur ou égal aux nombre de parts du partage, autrement il est impossible de remplir le tableau de Young semi-standard.

Lorsqu'on a deux variables q et t, on trouve $s_\lambda(q,t) = 0$, si $\ell(\lambda) \geq 3$. Pour un partage de la forme (a,b), avec a et b dans $\mathbb{N}$ les fonctions de Schur sont des polynômes de la forme $s_{a,b}(q,t) = (qt)^b(q^{a-b} + q^{a-b-1}t + \cdots + t^{a-b})$.

Si l'on considère les fonctions symétriques avec coefficients dans le corps de fraction $\mathbb{Q}(q,t)$, les fonctions symétriques élémentaires et les fonctions de Schur sont également des bases. En outre, les *polynômes de Macdonald*, dénoté $\tilde{H}_\mu$, forme une base de cet anneau. Ils sont les fonctions propres de plusieurs opérateurs, notamment l'opérateur ∇ et les opérateurs Δ'_{e_k}, introduit dans (Bergeron et Garsia, 1999) et (Bergeron *et al.*, 1999). On les définit par leurs valeurs propres, ainsi on trouve :

$$\nabla(\tilde{H}_\mu) = \prod_{(i,j)\in\mu} q^i t^j \tilde{H}_\mu \quad \text{et} \quad \Delta'_{e_m}(\tilde{H}_\mu) = e_m\left[\sum_{(i,j)\in\mu} q^i t^j - 1\right]\tilde{H}_\mu.$$

Les crochets sont pour la notation de pléthysme. Une personne curieuse pourrait se référer à (Bergeron, 2009) pour plus d'information.

Le *produit scalaire de Hall* est définit comme suit sur les fonctions de Schur à variables dans $X = \{x_1, x_2, \ldots\}$, pour f et g dans $\mathbb{Q}(q,t)$

$$\langle f(q,t)s_\lambda(X), g(q,t)s_\mu(X)\rangle = \begin{cases} f(q,t)g(q,t) & \text{si } \lambda = \mu \\ 0 & \text{sinon} \end{cases}$$

Celui-ci sera principalement utilisé pour trouver les coefficients de fonctions de Schur, à variables en X, dans une fonction symétrique donnée. Lorsque les coefficients dans $\mathbb{Q}(q,t)$ sont eux même symétriques, c'est-à-dire que $f(q,t) = f(t,q)$, il est possible de décomposer ceux-ci en fonctions de Schur. On utilise la notation de restriction $|_{\text{1Part}}$ (respectivement, $|_{\text{équerres}}$) pour donner la restriction de la décomposition en fonctions de Schur dans les variables q et t, au fonctions de Schur en les variables q et t qui sont indexées par des partages d'une seule part (respectivement, en forme d'équerre). Par exemple :

$$\left\langle \sum_{\mu,\rho} c_{\rho,\mu} s_\mu(q,t) s_\rho(X), s_\lambda(X) \right\rangle |_{\text{1Part}} = \sum_{\mu \text{ a une seule part}} c_{\lambda,\mu} s_\mu(q,t), \;\; c_{\lambda,\mu} \in \mathbb{Q}$$

$$\left\langle \sum_{\mu,\rho} c_{\rho,\mu} s_\mu(q,t) s_\rho(X), s_\lambda(X) \right\rangle |_{\text{équerre}} = \sum_{\mu \text{ est en forme d'équerre}} c_{\lambda,\mu} s_\mu(q,t), \;\; c_{\lambda,\mu} \in \mathbb{Q}.$$

On utilisera également la *règle de Pieri dual*. On considère l'égalité $\langle e_k(X)g(X), h(X)\rangle = \langle g(X), e_k^\perp h(X)\rangle$. La règle de Pieri dual est une représentation combinatoire de $e_k^\perp s_\lambda$. C'est la somme des s_μ, sur tous les partages μ obtenue en enlevant du partage λ, k cases, dans k lignes distinctes. La figure 1.4 est un exemple.

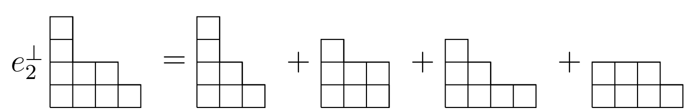


Figure 1.4 $e_2^\perp s_{4311} = s_{3211} + s_{331} + s_{421} + s_{43}$

Ceci sera utilisé sur les variables q et t plutôt que X.

L'ensemble des fonctions quasi symétriques forment un anneau similaire à l'anneau des fonctions symétriques. Plutôt que de considérer les invariants par l'action du groupe symétrique sur les variables, on considère les invariants pour l'échange de deux variables consécutives lorsque l'une des deux variables n'est pas présente. Celles-ci sont indexées par des compositions plutôt que des partages. La base monomiale, notée M_c, est définie comme suit, pour $c = c_1, c_2, \ldots, c_k$:

$$M_c = \sum_{x_1 < x_2 < \cdots < i_k} x_{i_1}^{c_1} x_{i_2}^{c_2} \cdots x_{i_k}^{c_k}.$$

Une autre base est la base quasisymétrique fondamentale, définie comme suit :

$$F_c = \sum_{c \preccurlyeq d} M_d.$$

Pour plus d'information sur les fonctions quasisymétriques, se référer à (Kurt *et al.*, 2013).

1.3 Espaces diagonaux harmoniques et espace diagonaux harmonique multivariées

En théorie de la représentation des groupes, un groupe agit sur un module. Celui-ci est ensuite décomposé en sous-espaces respectant l'action de groupe. S'il est impossible d'obtenir un sous-espace non trivial, on dit qu'il est irréductible. Les caractères sont des invariants qui encodent cette information, donc les espaces irréductibles sont associés aux caractères irréductibles. Les bicaractères correspondent à la considération de deux actions de groupes. Les espaces diagonaux harmoniques sont munis d'une action du groupe symétrique, noté $\mathbb{S}_n$, à droite. Dans (Haiman, 2002) il est démontré que les caractères des espaces diagonaux harmoniques sont donnés par $\nabla(e_n)$. Pour plus d'information sur les espaces diagonaux harmoniques, consultez (Haglund, 2008).

Les travaux de François Bergeron (Bergeron, 2020) montrent que les espaces diagonaux harmoniques multivariés possèdent aussi une action du groupe général linéaire de degré k, noté GL_k, à gauche. Soit $X = (x_{i,j})_{i,j}$, pour (τ, σ) dans $GL_k \times \mathbb{S}_n$, l'action de $GL_k \times \mathbb{S}_n$ sur l'anneau de polynôme $\mathbb{Q}[X]$ est définie comme suit :

$$(\tau, \sigma).F(X) = F(\tau \cdot X \cdot \sigma)$$

Avec cette action l'auteur définit $\mathcal{E}_{n,n}^{\langle k \rangle}$ comme le plus petit sous-module de $\mathbb{Q}[X]$ qui contient le déterminant de Vandermonde, est fermé pour les opérateurs de polarisation $\sum_{j=0}^{n} x_{r,j}\partial_{x_{s,j}}$ et est fermé sous toutes les dérivées partielles $\partial_{x_{s,j}}$.

Avec cette généralisation, on peut voir les espaces diagonaux harmoniques comme des $GL_2 \times \mathbb{S}_n$-modules notés $\mathcal{E}_{n,n}^{\langle 2 \rangle}$. Les bicaractères de ces représentations sont des produits de fonctions symétriques. Ainsi, la décomposition en bicaractères irréductibles est équivalente à la décomposition en produits de fonctions de Schur. Trouver une décomposition en caractères irréductibles pour l'action du groupe symétrique des espaces diagonaux harmoniques est toujours un problème irrésolu.

1.4 Chemins

Avant de définir les différents types de chemins utilisés dans cette thèse, la notation pour les q-analogues doit être établie.

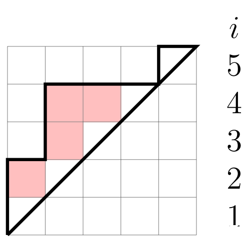
$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, [n]!_q := \prod_{i=1}^{n} [i]_q, \qquad \text{et} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[n-k]!_q [k]!_q}.$$

On définit un *Pas* comme le segment entre deux points entiers positifs du plan cartésien qui ne touche à aucun autre point entier. On utilise des pas nord de (a, b) vers $(a, b+1)$, des pas est de (a, b) vers $(a+1, b)$ et des pas diagonales de (a, b) vers $(a+1, b+1)$. Une *grille* $n \times k$ est l'ensemble des coordonnées entière (a, b) avec $0 \leq a \leq k$ et $0 \leq b \leq n$. Un *chemin* dans une grille $n \times k$ est une succession de pas qui débute au point $(0, 0)$ et termine au point (n, k). On utilise parfois la notation en mot pour représenter un chemin. Dans ce cas, les pas nord sont représentés par N, les pas est par E et les pas diagonaux par D. Par abus, on note la représentation en mot d'un chemin γ par γ. Une *case* est une coordonnée entière (a, b) de la grille telle que $(a-1, b)$ et $(a, b+1)$ sont également des coordonnées entières dans la grille. On dit, donc, qu'une case (a, b) est sous un chemin si les coordonnées (a, b) sont strictement sous le chemin. De plus, on dit qu'une case est au-dessus d'une diagonale si la coordonnée (a, b) est au-dessus ou sur la diagonale. Les *lignes* d'une grille ou d'un chemin sont numéroté du bas vers le haut en commençant avec 1. Une *colonne* du chemin γ, est un facteur $N^j E^k$ tel que $k > 0$ et $j > 0$. Par exemple, le chemin de la figure 1.5 a 3 colonnes. Dans une grille $n \times n$, on nomme *diagonale principale* la diagonale qui débute au point

$(0, 0)$ et qui termine au point (n, n).

Un *chemin de Dyck* de taille n est l'ensemble des chemins constitués de pas nord et de pas est dans une grille $n \times n$, qui se maintient au-dessus de la diagonale principale. L'*aire* d'un chemin de Dyck γ noté aire(γ), est l'ensemble des cases entre le chemin et la diagonale principale (le chemin à gauche dans la figure 1.5 en est un exemple). On dit qu'un chemin γ effectue un retour à la diagonale s'il existe un préfixe, γ_1 de γ tel que γ_1 est un chemin de Dyck de taille $0 < k < n$. On nomme *la suite touch de* γ, dénoté Touch(γ), la suite $(\gamma_1, \ldots, \gamma_k)$ de facteurs de γ tels que $\gamma = \gamma_1 \cdots \gamma_k$, tous les γ_i sont des chemins de Dyck et tous les γ_i ne contiennent aucun retour à la diagonale. La notion est généralement utilisée pour le *vecteur touch* qui est la suite $(|\gamma_1|_N, \ldots, |\gamma_k|_N)$. Par exemple dans la figure 1.5, Touch$(NNENNNEEENE) = (NNENNEEE, NE)$ et touch$(NNENNEEENE) = (4, 1)$.

Le *chemin bounce* de γ est le chemin de Dyck qui reste sous le chemin γ et qui change de direction si et seulement s'il touche le chemin ou la diagonale principale. La statistique *bounce* est la somme de toutes les positions des retours à la diagonale. Dans ce cas-ci, les lignes sont numérotées du haut vers le bas. On remarque que par la définition de retour à la diagonale, la position ne peut jamais être n. Le côté droit de la figure 1.5 montre un exemple. Les *pics* d'un chemin de Dyck, γ, sont les points où le chemin bounce change de direction en touchant au chemin γ.



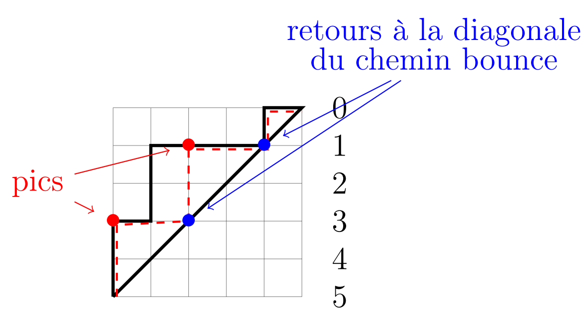


Figure 1.5 À gauche, l'aire du chemin de Dyck $NNENNEEENE$ a une valeur de 4. À droite, la statistique bounce du chemin $NNENNEEENE$ est 4.

Un *chemin de Schröder* de taille n est un chemin constitué de pas nord, est et diagonale dans une grille $n \times n$ et qui se maintient au-dessus de la diagonale principale. L'ensemble des chemins de Schröder de taille n contenant d pas diagonaux est dénoté $\mathrm{Sch}_{n,d}$. Si on élimine les pas diagonaux d'un chemin de Schröder, α, on obtient un chemin de Dyck, dénoté $\Gamma(\alpha)$. Par exemple, pour le chemin de Schröder, α de la figure 1.6, le chemin $\Gamma(\alpha)$ est celui de la figure 1.5. L'ensemble des chemins de Schröder avec d pas diagonaux qui ont un suffixe de la forme NE^j sera noté $\widetilde{\mathrm{Sch}}_{n,d}$.

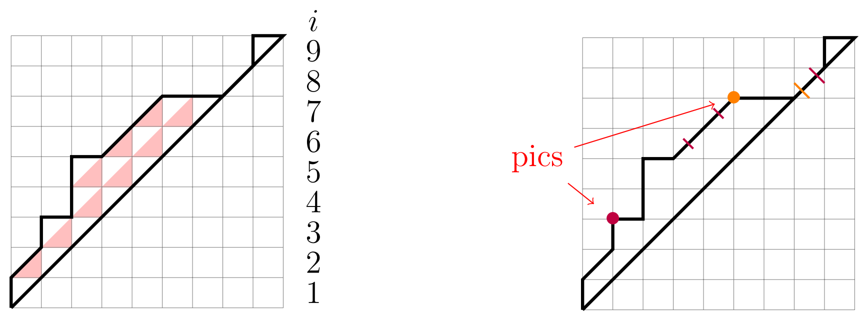


Figure 1.6 À gauche, un chemin de Schröder $NDNENNEDDEEDNE$ d'aire 9. À droite, le chemin, γ, donne $\mathrm{numph}(\gamma) = 4$ et $\mathrm{bounce}(\gamma) = 8$.

L'*aire* d'un chemin de Schröder est le nombre de cases (on remarque que la définition d'une case forme un triangle) entre le chemin et la diagonale principale. La figure 1.6 offre un exemple. Les *pics* d'un chemin de Schröder, α sont les pics du chemin de Dyck $\Gamma(\alpha)$ auquel on a remis les pas diagonaux. Si un pic touche un pas diagonal, il doit être situé en haut du pas diagonal. Pour chacun des pas diagonaux on dénombre le nombre de pics situés sous cette diagonale. La somme de tous ces nombre constitue la statistique numph. La statistique *bounce* d'un chemin de Schröder, α est la somme du bounce de $\Gamma(\alpha)$ et de la statistique *numph* (voir la figure 1.6 pour un exemple). La suite touch peut également être définie pour les chemins de Schröder en changeant «chemin de Dyck» pour «chemin de Schröder» dans la définition de retour à la diagonale.

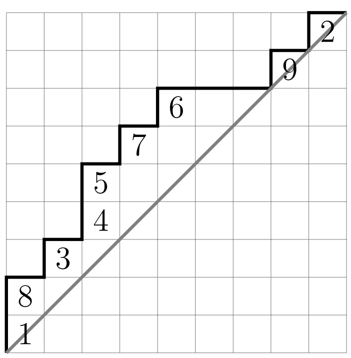
Dans (Haglund, 2004) les égalités suivantes sont démontrées :

$$\mathrm{Sch}_{n,d}(q,t) := \sum_{\gamma \in \mathrm{Sch}_{n,d}} q^{\mathrm{bounce}(\gamma)} t^{\mathrm{area}(\gamma)} = \langle \nabla(e_n), e_{n-d} h_d \rangle,$$

$$\widetilde{\mathrm{Sch}}_{n,d}(q,t) := \sum_{\gamma \in \widetilde{\mathrm{Sch}}_{n,d}} q^{\mathrm{bounce}(\gamma)} t^{\mathrm{area}(\gamma)} = \langle \nabla(e_n), s_{d+1,1^{n-d-1}} \rangle.$$

C'est formules sont symétriques en les variables q et t, ainsi elles peuvent s'exprimer comme une somme $\sum_{\lambda,\mu} c_{\lambda,\mu} s_\lambda(q,t) s_\mu(X)$, où les coefficients $c_{\lambda,\mu}$ sont dans $\mathbb{N}$. La notation $\widetilde{\mathrm{Sch}}_{n,d}(q,t)|_{1\mathrm{Part}}$ (respectivement, $\widetilde{\mathrm{Sch}}_{n,d}(q,t)|_{\text{équerres}}$) signifie qu'on se restreint aux partages λ dans la décomposition en fonctions de Schur qui ont une seule part (respectivement, qui ont la forme d'une équerre).

Une *fonction de stationnement* de taille n, est un couple constitué d'un chemin de Dyck de taille n et d'une permutation de n, w, telle que w_i est inscrit sur la ligne i, immédiatement à droite du chemin. En outre, tout facteur de w dans une même colonne ne contient aucune descente. Pour des exemples, voir la figure 1.7. On dénote l'ensemble des fonctions de stationnement de taille n par $\mathcal{P}_{n,n}$.



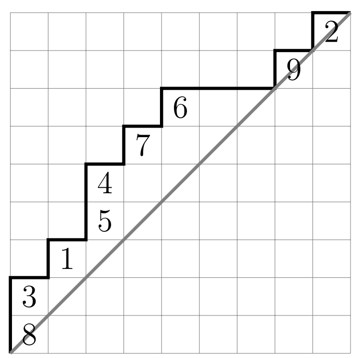


Figure 1.7 À gauche, la fonction de stationnement (γ, w) où $w = 183457692$. Les facteurs de chaque colonne (18, 3, 45, 7, 6, 9, 2) ne contiennent aucune descente. À droite, ce n'est pas une fonction de stationnement, puisque le facteur de la première colonne est 83, donc il contient une descente (on peut dire la même chose de 54).

Le *mot de lecture* d'une fonction de stationnement, est obtenue en lisant les lettres de w en suivant les diagonales parallèles à la diagonale principale, du haut vers le bas, en commençant avec la diagonale la plus éloignée de la diagonale principale. Par exemple le mot de lecture de la figure de gauche 1.7 est 675438291. L'aire d'une fonction de stationnement est l'aire de son chemin de Dyck. Pour calculer la statistique d'*inversion diagonale*, dénoté dinv, on considère pour chaque lettre de w, disons w_i, une diagonale, parallèle à la diagonale principale, qui part du centre du pas nord qui lui est associé et la diagonale un pas nord plus bas. Pour chaque lettre, w_j, situé en haut de w_i tel que $w_i < w_j$ sur la même diagonale ou $w_i > w_j$ et w_j est une diagonale en dessous de la diagonale de w_i contribue 1 à la statistique d'inversion diagonale. Pour un exemple, voir la figure 1.8. Les chemins

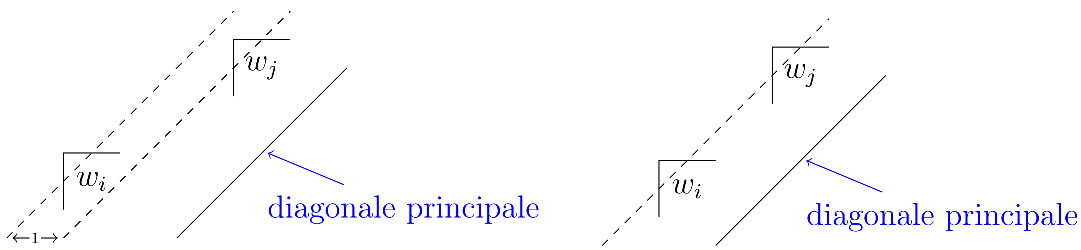


Figure 1.8 À gauche, le couple (i, j) contribue 1 si $i < j$, $w_i > w_j$ et la diagonale de w_i est au-dessus de la diagonale de w_j. À droite, le couple (i, j) contribue 1 si $i < j$, $w_i < w_j$ et les deux lettres sont sur la même diagonale.

de Schröder avec d pas diagonaux peuvent être représentés par des fonctions de stationnements tels que le mot de lecture est un mélange de $n-d+1, \cdots, n$ et de $n-d, \cdots, 1$. Il suffit de remplacer les facteurs NE étiquetés par une lettre de $\{n-d+1, \cdots, n\}$ par des pas diagonaux. La fonction de stationnement de la figure de gauche 1.7 représente le chemin de Schröder de la figure 1.6. De façon équivalente on peut représenter les chemins de Schröder par des fonctions de stationnement dont la permutation évite 132 et 312. Pour plus de détails, il est possible de se référer à (Haglund, 2008).

Enfin, les fonctions de stationnement sont liées aux modules diagonaux harmoniques via le théorème shuffle conjecturé dans (Haglund *et al.*, 2005) et prouvé dans (Carlsson et Mellit, 2018) :

$$\nabla(e_n) = \sum_{(\gamma,w)\in\mathcal{P}_{n,n}} t^{\mathrm{dinv}(\gamma,w)} q^{\mathrm{aire}(\gamma)} F_{\mathrm{ides}(w)}(X).$$

CHAPITRE II

BICARACTÈRES DES ESPACES DIAGONAUX HARMONIQUE

2.1 Résumé de l'article

Les coefficients de la caractéristique de Frobenius bigraduée des espaces diagonaux harmoniques $\mathcal{F}_{DH}$ exprime la multiplicité des caractères irréductibles de $\mathbb{S}_n$ pour l'action de permutations de variables. Plusieurs travaux de recherches ont pour objectif d'exprimer ces coefficients en terme d'objet combinatoire. Les travaux de François Bergeron (Bergeron, 2020) permettent de considérer $\mathcal{F}_{DH}(X; q, t)$ comme le caractère d'un $(GL_2 \times \mathbb{S}_n)$-module. Il en découle une décomposition en bicaractère qui prend la forme d'une somme positive de produit de fonctions de Schur. En outre, Hogancamp, (Hogancamp, 2017), démontre que ces caractères correspondent à la série de Hilbert-Poincaré de l'homologie de Khovanov-Rozansky des $(n, mn + 1)$ nœuds sur le tore.

L'article présenté dans ce chapitre, soumis à «The Electronic Journal of Combinatorics», répond en partie à cette question par le biais de formules combinatoires explicites d'une décomposition partielle en bicaractères irréductibles des espaces diagonaux harmoniques. En effet, le théorème principal présente une équation simple pour les cas suivants :

Théorème 1 : Si $\mu \in \{(d, 1^{n-d}) \mid 1 \leq d \leq n\}$ et $\nu \vdash n$, alors :

$$\langle \nabla(e_n), s_\mu \rangle|_{\text{équerres}} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{\mathrm{maj}(\tau)}(q, t) + \sum_{i=2}^{\mathrm{des}(\tau)} s_{\mathrm{maj}(\tau)-i,1}(q, t), \tag{2.1}$$

$$\langle \nabla^m(e_n), s_\nu \rangle|_{\mathrm{1Part}} = \sum_{\tau \in \mathrm{SYT}(\nu)} s_{m\binom{n}{2}-\mathrm{maj}(\tau')}(q,t), \tag{2.2}$$

et :

$$\langle \nabla^m(e_n), e_n \rangle|_{\text{équerres}} = s_{m\binom{n}{2}}(q,t) + \sum_{i=2}^{n-1} s_{m\binom{n}{2}-i,1}(q,t). \tag{2.3}$$

Un résultat classique établie qu'il y a une correspondance entre les composantes irréductibles du groupe symétrique et les tableaux standard. Ainsi, le théorème donne lieu à une meilleure compréhension de la structure de chacune des composantes puisqu'il est possible de restreindre l'énoncé à un tableau de Young standard fixé pour obtenir la multiplicité d'une composante irréductible donnée.

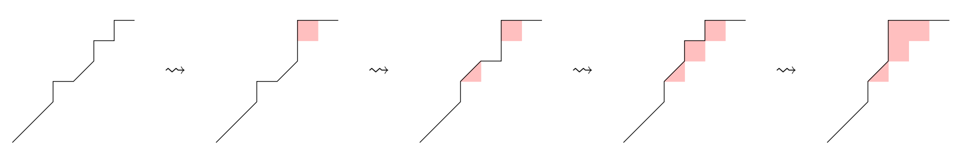
La première somme de L'équation, (2.1) s'établie au moyen d'un algorithme déterministe qui permet d'associer un ensemble de chemins de Schröder à chacune des fonctions de Schur. On remarque d'abord que les fonctions de Schur sur k variables sont indexées par des partages de longueur au plus k. Les fonctions de Schur à deux variables ont donc au plus deux parts. De fait, on a $s_{a,b}(q,t) = q^{a-b}t^{a-b}(q^b + q^{b-1}t + \cdots + qt^{b-1} + t^b)$. Autrement dit, si le monôme q^d a pour coefficient c, non nul, dans la décomposition d'une fonction symétrique $f(q,t)$ si et seulement c est le coefficient de $s_d(q,t)$ dans la décomposition en fonctions de Schur. Pour les formules sur les chemins de Schröder à d diagonales, ceci est équivalent à dire que les fonctions de Schur présente dans la décomposition de $\mathrm{Sch}_{n,d}(q,t)|_{\mathrm{1Part}}$ apparaissent avec la même multiplicité dans la décomposition de $\mathrm{Sch}_{n,d}(q,0)$. Ainsi, l'ensemble des chemins contribuant à $\mathrm{Sch}_{n,d}(q,0)$ sont les chemins dont la valeur de l'aire est 0. Ceux-ci peuvent être représenté par des mots composé des facteurs NE et de la lettre D, $\{NE, D\}^n$, tel que vu à la section 1.4 du chapitre 1. Cependant, plusieurs chemins sont nécessaires pour exprimer $\mathrm{Sch}_{n,d}(q,t)|_{\mathrm{1Part}}$; ce sont ces chemins que l'algorithme rassemble.

L'algorithme φ transforme un chemin de Schröder, γ dans $\{NE, D\}^*$, en une suite de chemins de Schröder $(\gamma_0, \gamma_1, \ldots, \gamma_{\mathrm{bounce}(\gamma)})$. Pour $k = \mathrm{bounce}(\gamma)$, cette suite est construite de la façon suivante :

> *Pour* $\varphi(\gamma)$ *on pose* $\gamma_0 = \gamma$, *Temp*$(\gamma) = (\gamma_0)$, $n = |\gamma|_E + |\gamma|_N + |\gamma|_D$.
>
> *Pour* v *de* 1 *à* k *;*
>
> *Soit* $\gamma_{v-1} = w_1 w_2 \cdots w_n$.
>
> *Soit* i *tel que* $i \leq n-1$, $w_i = E$ *et* $w_{i+1} \neq E$
>
> *et pour tout* j *tel que* $w_j = E$ *alors* $w_{j+1} = E$ *ou* $j \leq i$.
>
> *On pose* $\gamma_v = w_1 \cdots w_{i-1} w_{i+1} w_i w_{i+2} \cdots w_n$.
>
> *Ajouter* γ_v *à la suite de Temp*(γ).
>
> *repéter ;*
>
> *retourner Temp*(γ).

La suite de chemin commence avec un chemin dont la valeur pour l'aire est 0. À chaque étapes subséquente on obtient des chemins dont la valeur pour l'aire diminue de un et la valeur de la statistique bounce augmente de 1. Tous les chemins de la suite sont associés à un terme de la forme $q^{\text{area}(\gamma)} t^{\text{bounce}(\gamma)}$ ce qui permet d'associer à chaque suite une fonction de Schur à deux variables. De plus, on remarque qu'un tel w_i est unique et existe pour autant que la statistique bounce à la valeur 0. En effet, les chemins de $\text{Sch}_{n,d}$ ont la forme $\gamma N E^j$ et il s'agit du dernier pas est de γ.

Pour illuster avec $DDNEDNENE$ comme chemin de départ, l'algorithme produit la suite de chemins de la Figure 2.1.



$$
\begin{array}{ccccccccccc}
 & & DDNEDNENE & & DDNEDNNEE & & DDNDENNEE & & DDNDNENEE & & DDNDNNEEE \\
s_4(q,t) & = & t^4 & + & qt^3 & + & q^2t^2 & + & q^3t & + & q^4
\end{array}
$$

Figure 2.1 Suite de chemins obtenue à partir du chemin $\gamma_0 = DDNEDNENE$. La somme $\sum_{\gamma \in \varphi(\gamma_0)} q^{\text{area}(\gamma)} t^{\text{bounce}(\gamma)} = s_4(q,t)$.

Cet algorithme sous-tend une autobijection partielle entre les chemins de Schröder qui inverse les valeurs des statistiques pour l'aire et pour bounce. Celle-ci est donnée par la Proposition 1 répondant en partie au problème ouvert 3.11 de Haglund cité dans (Haglund, 2008) reformulé de diverses façons dans la communauté depuis les dix dernières années. En effet, l'algorithme induit une autobijection de l'ensemble de chemins $\cup_{\gamma\in\{NE,D\}^n}\varphi(\gamma)$ inclus dans les chemins de Schröder telle que $\text{area}(\gamma_i) = \text{bounce}(\gamma_{\text{bounce}(\gamma_i)+\text{area}(\gamma_i)-i})$ et $\text{bounce}(\gamma_i) = \text{area}(\gamma_{\text{bounce}(\gamma_i)+\text{area}(\gamma_i)-i})$. On remarque que cet ensemble est décrit explicitement par :

$$\{uvEwE^j \mid u \in \{NE, D\}, v, w \in \{N, D\}, |v|_N \geq 1, |u|_E+j+1 = |u|_N+|v|_N+|w|_N\}.$$

Grâce à cet algorithme, la Proposition 2 donne lieu, entre autres, à l'énoncé :

$$\langle \nabla e_n, s_{d+1,1^{n-d-1}}\rangle|_{1\text{Part}} = \sum_{\substack{\gamma\in\{NE,D\}^{n-1}NE \\ |\gamma|_D=d}} s_{\text{bounce}(\gamma)}(q,t).$$

En comparaison, l'équation de Stanley-Lusztig (Stanley, 1979) :

$$\langle \nabla e_n|_{q=0}, s_{(d+1,1^{n-d-1})}\rangle = \sum_{\tau\in\text{SYT}(d+1,1^{n-d-1})} t^{\text{maj}(\tau)} = \sum_{\tau\in\text{SYT}(d+1,1^{n-d-1})} s_{\text{maj}(\tau)}(0,t).$$

Celle-ci suggère qu'il doit y avoir une bijection, $\mathcal{M}_{n,d}$, entre l'ensemble des chemins de Schröder de taille n avec $d-1$ diagonale dont la valeur de l'aire est zéro, dénoté $\widetilde{\text{Sch}}_{n,d-1,(0)}$ et l'ensemble des tableaux de Young standard de la forme $(d, 1^{n-d})$. Pour un tel τ dans $\text{SYT}(d, 1^{n-d})$ on a $\text{maj}(\tau) = \text{bounce}(\mathcal{M}_{n,d}(\tau))$. Cette bijection est donnée par la Proposition 7. Elle est définie via une famille d'application $\{\mathcal{M}_{n,d}\}$, où $\mathcal{M}_{n,d} : \text{SYT}(d, 1^{n-d}) \to \widetilde{\text{Sch}}_{n,d-1,(0)}$, et $\mathcal{M}_{n,d}(\tau) = \gamma_1\gamma_2\cdots\gamma_n$, avec $\gamma_n = NE$, $\gamma_{n-i} = NE$ si $i \in \text{Des}(\tau)$ et $\gamma_{n-i} = D$ sinon. (Voir à gauche de la Figure 2.2 pour un exemple.)

Les inverses de ces bijections sont les $\mathcal{R}_{n,d} : \widetilde{\text{Sch}}_{n,d-1,(0)} \to \text{SYT}(d, 1^{n-d})$ où $\mathcal{R}_{n,d}(\gamma) = \tau$. Le tableau τ est ici caractèrisé par son ensemble de descentes, $\text{Des}(\tau) = \{n - i \mid 1 \leq i \leq n-1, \gamma_i = NE \in \text{Touch}(\gamma)\}$. (Voir à droite de la

Figure 2.2 pour un exemple).

Pour les tableaux en forme d'équerre cette caractérisation est unique. De plus, si $\text{area}(\gamma) = 0$ et $\text{Touch}(\gamma) = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, alors pour tout i, γ_i est dans $\{NE, D\}$.

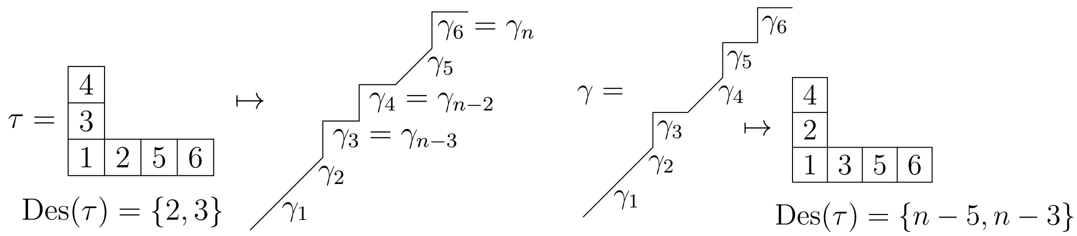


Figure 2.2 À gauche, un exemple pour $\mathcal{M}_{6,4}$. À droite, un exemple pour $\mathcal{R}_{6,4}$.

En outre, pour τ dans $\text{SYT}(d+1, n-d-1)$, la composition $\varphi \circ \mathcal{M}_{n,d}$ identifie tous les chemins de Schröder associé à la fonction de Schur $s_{\text{maj}(\tau)}(q,t)$.

Quant à elle, la deuxième somme de l'Équation (2.1) est obtenue de façon similaire par le biais de la Proposition 9 en utilisant les applications $\mathcal{Q}_{n,d}$, $\mathcal{S}_{n,d}$ et $\Pi_{n,d-1}$ qui lient les chemins de Schröder dont l'aire est égale à 1 à des tableaux de Young standard en forme d'équerre. Enfin, l'Équation (2.2) et l'Équation (2.3) sont le résultat d'une caractérisation des fonctions de stationnement dont la statistique d'inversions diagonales a pour valeur 0 ou 1. L'article se termine sur une interprétation en termes de cristaux.

2.2 Relations entre cet article et les autres articles présentés

Dans le chapitre suivant, le principe d'inclusion-exclusion est appliqué au théorème principal de ce chapitre afin d'obtenir un relèvement aux espaces diagonaux harmoniques multivariés. Ceci établi une partie de l'équation présentée pour exprimer certains bicaractères des espaces diagonaux harmoniques multivariée.

En outre, l'article présenté au chapitre 4 donne une deuxième interprétation de la proposition 7. En effet, les tableaux de Young standard obtenus par la bijection $\mathcal{M}_{n,d}$ sont les Q-tableaux, au sens de l'algorithme Robinson-Schensted, du mot de lecture de la représentation par des fonctions de stationnement des chemins de Schröder correspondant. D'autres liens seront précisés dans les chapitres correspondants.

2.3 Toward a Schurification of Parking Function Formulas via bijections with Young Tableaux

2.3.1 Abstract

This paper contains a partial answer to the open problem 3.11 stated by James Haglund in *"The q,t-Catalan numbers and the space of diagonal harmonics"(volume 41 of University Lecture Series, American Mathematical Society (2008))*. That is to find an explicit automorphism on Schröder paths that exchanges the statistics area and bounce. This paper started as an attempt to write the sum over m-Schröder paths with a fixed number of diagonal steps into Schur functions in the variables q and t. Some results have been generalized to parking functions, and some bijections were made with standard Young tableaux giving way to partial combinatorial formulas in the basis $s_\mu(q,t)s_\lambda(X)$ for $\nabla(e_n)$ (respectively, $\nabla^m(e_n)$), when μ and λ are hooks (respectively, μ is of length one). We also give an explicit algorithm that gives all the Schröder paths related to a Schur function $s_\mu(q,t)$ when μ is of length one. In a sense, it is a partial decomposition of Schröder paths into crystals.

2.3.2 Introduction

In this paper, Proposition 1 gives a partial answer to the open problem 3.11 of (Haglund, 2008). This problem asks for an explicit bijection $\tilde{\varphi}$ on Schröder paths, such that for a given path γ we have $\text{bounce}(\gamma) = \text{area}(\tilde{\varphi}(\gamma))$ and $\text{area}(\gamma) = \text{bounce}(\tilde{\varphi}(\gamma))$, where bounce and area are statistics on paths recalled in Section 2.3.3. But the aim of this paper is to decompose parking functions and Schröder path formulas obtained by statistics on paths, in terms of the basis $s_\mu(q,t)s_\lambda(X)$. It is then used in (Wallace, 2019a) (Chapter 3) to give explicit combinatorial formulas for the modules of multivariate diagonal harmonics. In other words, the combinatorics of parking functions are used to elevate the understanding of the structure of the modules of diagonal harmonics. This combinatorial representation was first known as the Shuffle Conjecture. It was introduced in (Haglund *et al.*,

2005) and proven by Carlson and Mellit in (Carlsson et Mellit, 2018), (Mellit, 2016). It was shown beforehand in (Garsia et Haiman, 1996) and (Haiman, 2002) that the Frobenius transformation of its graded characters may be expressed as $\nabla^m(e_n)$, where ∇ is the Macdonald eigenoperator introduced in (Bergeron et Garsia, 1999), and e_n is the n-th elementary symmetric function, both recalled in Section 2.3.3, along with classical combinatorial tools. More precisely, we will give a partial decomposition of parking functions and Schröder path formulas in terms of the basis $s_\mu(q,t)s_\lambda(X)$, we then find a bijection between the paths used in the decomposition and standard Young tableaux proving the following :

Theorem 1 : If $\mu \in \{(d, 1^{n-d}) \mid 1 \leq d \leq n\}$ and $\nu \vdash n$, then :

$$\langle \nabla(e_n), s_\mu \rangle|_{\text{hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{\mathrm{maj}(\tau)}(q,t) + \sum_{i=2}^{\mathrm{des}(\tau)} s_{\mathrm{maj}(\tau)-i,1}(q,t), \tag{2.4}$$

$$\langle \nabla^m(e_n), s_\nu \rangle|_{\text{1Part}} = \sum_{\tau \in \mathrm{SYT}(\nu)} s_{m\binom{n}{2}-\mathrm{maj}(\tau')}(q,t), \tag{2.5}$$

and :

$$\langle \nabla^m(e_n), e_n \rangle|_{\text{hooks}} = s_{m\binom{n}{2}}(q,t) + \sum_{i=2}^{n-1} s_{m\binom{n}{2}-i,1}(q,t), \tag{2.6}$$

where τ' is the conjugate of the tableau τ.

This will be done by characterizing particular parking functions, in Section 2.3.6, which leads to Equation (2.5). In Section 2.3.7, we restrict the characterization on Schröder paths. In Section 2.3.8, we give bijections between subsets of Schröder paths and Standard Young tableaux, and use them to prove Equation (2.4) and Equation (2.6). Moreover, in Section 2.3.5, we exhibit an explicit algorithm that gives all the Schröder paths associated to a Schur function in the variables q and t when μ is of length one. We will briefly explain, in Section 2.3.10, what it means in terms of the theory of crystals. We end with a list of open problems in Section 2.3.11.

2.3.3 Combinatorial Tools

The notions discussed in this section are classical and are recalled to set notations. An alphabet, A, is a set. The elements of that set are called *letters*. A *word* is a finite sequence of elements of A, we usually omit the parentheses and the commas. The empty word is denoted ε. The number of letters in a word w is called the *length*, denoted $|w|$, the number of occurrences of the letter a in w is denoted $|w|_a$. The set of words of length n in the alphabet A is denoted A^n, we denote A^* the set $\cup_n A^n$. A *factor* of w is a consecutive subsequence of w. Additionally, if we are interested in word ending with a fixed factor u, we will denote the set A^*u, and u is called a suffix. If we want those words to be of length, $n+|u|$ we will denote the set $A^n u$. Likewise, a factor at the beginning of a word is called a prefix, and the set of words with prefix u is denoted uA^*. For a word $w = w_1 w_2 \cdots w_k$, w^n is the concatenation on m copies of w, and $w^{-1} = w_k \cdots w_2 w_1$. For two words u and w the set $u \sqcup\!\sqcup w$ is the set containing all words such that u and w are subsequences. We call these words *shuffles*. A *permutation* of n can be represented as words of $\{1, \ldots, n\}^n$ with all distinct letters. The *descent set of a permutation* $w = w_1 \cdots w_n$, denoted $\mathrm{Des}(w)$, is the set of i's such that $w_i > w_{i+1}$. The cardinality of the set will be denoted $\mathrm{des}(w)$. The *Major index of a permutation*, denoted $\mathrm{maj}(w)$, is by definition $\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i$. To avoid confusion we will write the *inverse of a permutation* w, $\mathrm{inv}(w)$.

A *partition* of n is a decreasing sequence of positive integers it can be represented by a Ferrers diagram (see Figure **??**). Each number in the sequence is called a *part*, and, if it has k parts, it is of *length* k denoted $\ell(\lambda) = k$. If $\lambda = \lambda_1, \cdots, \lambda_k$ and $n = \sum_i \lambda_i$, we say λ is of *size* n, denoted $|\lambda| = n$. Although the notation $|\cdot|$ is used for words and partitions, it will be clear by context which one is used. For λ a *Ferrers diagram* of shape $\lambda = \lambda_1, \lambda_2, \cdots, \lambda_k$ is a left justified pile of boxes having λ_i boxes in the i-th row. We will use the French notation ; hence, the second row lies on top of the first row (see Figure **??**). We can see them as a subset of $\mathbb{N} \times \mathbb{N}$ if we put the bottom left corner of the diagram to the origin. In this setting, we can associate the bottom left corner of a box to the coordinate it lies on. We say a partition is *hook-shaped* if it has the shape $(a, 1, \cdots, 1) = (a, 1^k)$, where $a, k \in \mathbb{N}$. The *conjugate* of a partition λ, (or a diagram) is denoted λ', and is its reflection through the line $x = y$ (see Figures **??**). A *tableau* is a filling of a

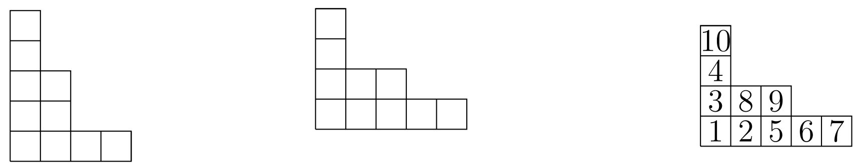


Figure 2.3 To the left an example of $\lambda = 42211$. In the center an example of the conjugate $\lambda' = 5311$. To the right an example of $\tau \in \mathrm{SYT}(5311)$, with descent set $\{2, 3, 7, 9\}$ and Major index $2 + 3 + 7 + 9 = 21$.

diagram by positive integers, the number in each box is called an *entry*. The *size* of a tableau relates to the size of the diagram it fills. It is said to be a *semi-standard Young tableau* if all entries are weakly increasing in rows and strictly increasing in columns. A *standard Young tableau* is a tableaux of size n, such that all numbers from 1 to n appear exactly once and all entries are strictly increasing in rows and columns. If a tableau is a filling of the diagram associated to the partition λ, it is said to be of *shape* λ. The set of standard Young tableaux of shape λ is denoted $\mathrm{SYT}(\lambda)$. The *descent set of a tableau* τ, denoted $\mathrm{Des}(\tau)$ is the set of entries i such that $i + 1$ lies in a higher row. The cardinality of the descent set of τ is denoted $\mathrm{des}(\tau)$, and the sum of the elements in the descent set is the *Major index* denoted $\mathrm{maj}(\tau)$ (see Figure 4.10). Again, it will be clear by context if the descent set and the Major index are used on words or tableaux. Since each box of τ is associated to its own entry, we will write $c \in \tau$ when we refer to the entry c in the tableau τ. We will use the notation x_τ for the monomial $\prod_{c \in \tau} x_c$.

For a possibly infinite set of variables, $X = \{x_1, \ldots, x_n\}$, the *elementary symmetric functions* $e_n(X)$ are the sum of all square-free monomials of degree n in the set of variables X. The symmetric function e_λ is simply $e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$. The elementary symmetric functions form a basis of the symmetric functions. Another basis is the set of Schur functions. For λ a partition the *Schur function* $s_\lambda(X) = \sum x_\tau$, where the sum is over all semi-standard Young tableau of shape λ. The Schur basis in the X variables is self-dual for the Hall scalar product, denoted $\langle -, - \rangle$. We will use this notation when we want to display the coefficient of a particular Schur function. Note that the Schur functions in the variables q and t are coefficients and can go in and out of the scalar product. We will sometimes call Schur functions index by partitions that are hook-shaped, hook-shaped Schur functions or, simply, hook Schur functions. It will also be useful to remember that $e_n = s_{1^n}$. Furthermore, the *complete homogeneous symmetric functions* are a basis such that $h_n(X) = s_{(n)}(X)$ and $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$. We simply write e_λ for $e_n(X)$, s_λ for $s_\lambda(X)$ and h_λ for $h_\lambda(X)$ (not for $e_\lambda(q,t)$, $h_\lambda(q,t)$ or $s_\lambda(q,t)$). A curious reader could look at (Macdonald, 1995).

The modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$ form another base of the ring of symmetric functions. In (Bergeron et Garsia, 1999) Bergeron and Garsia introduce the operator ∇ defined with the modified Macdonald polynomials as eigenfunctions, with eigenvalues $\prod_{(i,j)\in\mu} q^i t^j$. The Shuffle Theorem, proven by Carlson and Mellit (see (Carlsson et Mellit, 2018) and (Mellit, 2016)), gives a combinatorial formula for $\nabla^m(e_n)$. This formula uses path combinatorics.

2.3.4 Path Combinatorics

Before we can state the Shuffle Theorem, we need more classical definitions, relating to path combinatorics. More details on these classical notions can be found in (Haglund, 2008).

The following q-analogues will be very useful :

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, [n]!_q := \prod_{i=1}^{n} [i]_q, \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[n-k]!_q [k]!_q}.$$

Let $\mathcal{C}_k^n$ be the set of paths composed of north and east steps, in an $(n-k) \times k$ grid starting at the bottom left corner. The *area* of a path is the number of boxes under the path. A classical result relates the Gaussian Polynomials to path combinatorics. Indeed, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\gamma \in \mathcal{C}_k^n} q^{\mathrm{area}(\gamma)}$. A path $\gamma \in \mathcal{C}_k^n$ can be identified with a word, w_γ in $\{N, E\}^n$ such that $|w_\gamma|_E = k$. To facilitate reading we will frequently refer to γ when we talk about w_γ (see Figure 4.16 for an example).

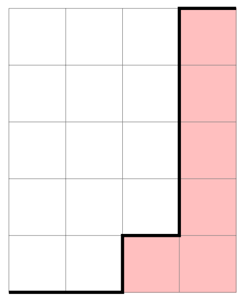


Figure 2.4 Path of area 6 in a 5×4 grid, with word representation $EENENNNNE$

In an $n \times m$ grid, *the main diagonal* is the diagonal starting at the bottom left corner and finishing at the top right corner. A *Dyck path* of size (n, m) is a path composed of north and east steps, starting at the bottom left corner in an $n \times m$ grid such that the path always stays over or on the main diagonal. The set of such paths is denoted $\mathcal{D}_{n,m}$. A classical result makes it possible to represent Dyck paths by words in $\{N, E\}^*$ such that for all γ_i, prefix of γ, we have $|\gamma_i|_N \geq \frac{m}{n}|\gamma_i|_E$. The lines of the grid are numbered from bottom to top. A line i is said to *contain an east step* if the factor starting with the i-th north step and ending the letter before the $i+1$-th north step contains an east step. A *column of a path*, γ, is a factor N^jE^k, for intergers j and k such that $\gamma = uN^jE^kw$, where $u, w \in N\{N, E\}^*E \cup \{\varepsilon\}$. For example, in Figure 2.5, the path has 3 columns.

The *area of a Dyck path* will be the number of boxes under the path and over the main diagonal (see left of Figure 2.5 for an example). The area of the line i, denoted a_i, is the number of boxes in the line i that are between the path and the main diagonal. Obviously, the area of a path is the sum of the a_i's. The path γ is said to have a *return to the main diagonal* if there is γ_i, a non-trivial prefix of γ, such that γ_i is a Dyck path and the end point of γ_i lies on the main diagonal of γ. The *touch sequence of a path* γ, denoted $\mathrm{Touch}(\gamma)$, is defined as a sequence $(\gamma_1, \ldots, \gamma_k)$ of factors of γ such that $\gamma = \gamma_1 \cdots \gamma_k$, all γ_i are Dyck paths, all γ_i contain no return to the main diagonal, and $\gamma_1 \cdots \gamma_i$ is a returns to the main diagonal for γ. The sequence $(\frac{n}{n+m}|\gamma_1|, \ldots, \frac{n}{n+m}|\gamma_k|)$ (or $(\frac{1}{2}|\gamma_1|, \ldots, \frac{1}{2}|\gamma_k|)$ when $n = m$) defines the touch vector $\mathrm{touch}(\gamma)$. The touch vector describes all the *touch points*. For example, in Figure 2.5, $\mathrm{Touch}(NNENNNEEENE) = (NNENNEEE, NE)$ and touch($NNENNEEENE$) $= (4,1)$. The *bounce path* of a path $\gamma \in \mathcal{D}_{n,n}$ will be the Dyck path that remains under the path γ and changes direction if and only if it touches the path γ or the main diagonal. The *bounce vector* is the vector containing the positions of the return to the main diagonal, starting from the top, of the bounce path. For the bounce vector, the lines are numbered from the top starting at 0. Finally, the *bounce statistic* is the sum of the integer in the bounce vector minus n. It is, usually, simply called bounce (see Figure 2.5 for an example). Note that the bounce statistic is not defined for Dyck paths in an $n \times m$ grid with $m \neq n$. In theses cases we use the diagonal inversion statistic which will be discussed at the end of this section.

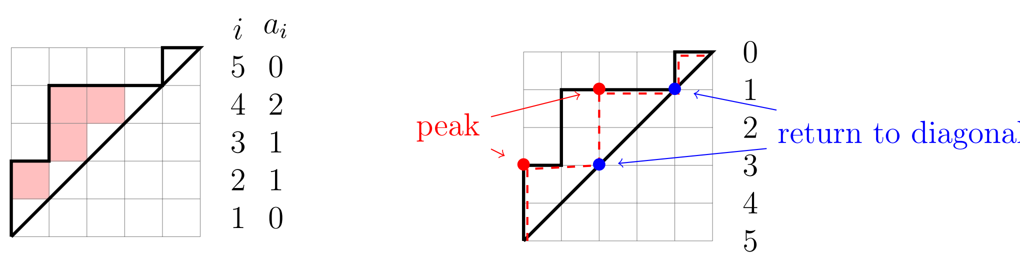


Figure 2.5 To the left, a Dyck path of area 4, and word representation $NNENNEEENE$. To the right, the dotted red path is the bounce path and its vector is $(0,1,3,5)$ and the bounce statistic is $1+3=4$.

A *Schröder path* of size (n, rn) is a path composed of north, east and diagonal steps in an $n \times rn$ grid such that the path always stay over the main diagonal starting at the bottom left corner. In respect to cartesian coordinates, a diagonal step corresponds to adding $(1,1)$. The set of paths containing d diagonal steps is denoted $\mathrm{Sch}^{(r)}_{n,d}$. These paths can also be represented by words in the alphabet $\{N, E, D\}^*$ such that for every prefix γ_i of γ we have $|\gamma_i|_N \geq r|\gamma|_E$. Clearly $\mathcal{D}_{n,rn} = \mathrm{Sch}^{(r)}_{n,0}$. Moreover, the path obtained by deleting all diagonal steps in a Schröder path is a Dyck path. For a Schröder path, π this new path will be denoted $\Gamma(\pi)$. For example, the path π in Figure 2.6 is such that $\Gamma(\pi)$ is the path seen in Figure 2.5. We will also frequently use another subset of Schröder paths :

$$\widetilde{\mathrm{Sch}}_{n,d} = \{\gamma \in \mathrm{Sch}_{n,d} \mid \gamma = wNE^j, w \in \{D, N, E\}^*\}.$$

The *area statistic of a Schröder path* is fairly the same as the other definitions of the area statistic. Instead of counting the squares, we count the number of *lower triangles* under the path and over the main diagonal, where a lower triangle is the lower half of a square cut in two starting by the botom left corner and ending at the top right corner (see left of Figure 2.6 for an example).

In (Haglund, 2008), Haglund defines a bounce statistic for Schröder paths in an $n \times n$ grid. We first define the set of *peaks* of the path $\Gamma(\gamma)$ defined previously. These are the set of lattice points at the beginning of an east step such that the bounce path of $\Gamma(\gamma)$ switches from a north step to an east step. By extension the peaks of γ are the lattice points found by reinserting the diagonal steps in $\Gamma(\gamma)$. If a peak touches a diagonal step it must be placed higher. The number of peaks of the path γ, with multiplicity, that lie below each diagonal step is the statistic *numph*, denoted $\mathrm{numph}(\gamma)$ (this is equivalent to the number of diagonal step, with multiplicity above each peak). The *bounce statistic* will be extended to a Schröder path, γ, by the formula (see Figure 2.6 for an example) :

$$\mathrm{bounce}(\gamma) = \mathrm{bounce}(\Gamma(\gamma)) + \mathrm{numph}(\gamma).$$

Finally, touch points can be defined for Schröder path, simply change Dyck path for Schröder paths in the definition.

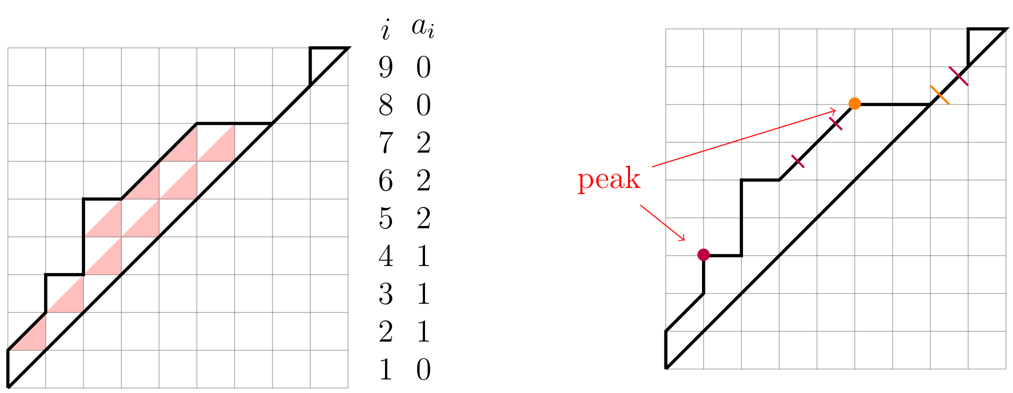


Figure 2.6 To the left a Schröder path of area 9, and word representation $NDNENNEDDEEDNE$. To the right the path, γ, yields $\mathrm{numph}(\gamma) = 4$ and $\mathrm{bounce}(\gamma) = 4 + 4 = 8$.

The generating function of the Schröder paths is defined by :

$$\mathrm{Sch}_{n,d}(q,t) = \sum_{\gamma \in \mathrm{Sch}_{n,d}} q^{\mathrm{bounce}(\gamma)} t^{\mathrm{area}(\gamma)},$$

and the generating function of the Schröder paths ending with NE is defined by :

$$\widetilde{\mathrm{Sch}}_{n,d}(q,t) = \sum_{\gamma \in \widetilde{\mathrm{Sch}}_{n,d}} q^{\mathrm{bounce}(\gamma)} t^{\mathrm{area}(\gamma)}.$$

Since the subset $\widetilde{\mathrm{Sch}}$ is chosen to work with the bounce statistic which is not defined for $n \times nm$ grids when $m \neq 1$, we will not define $\mathrm{Sch}^{(m)}_{n,d}$. We will define $\widetilde{\mathrm{Sch}}^{(m)}_{n,d}(q,t)$ as follows :

$$\widetilde{\mathrm{Sch}}^{(m)}_{n,d}(q,t) = \sum_{k=d}^{n} (-1)^{k-d} \mathrm{Sch}^{(m)}_{n,k}(q,t).$$

The reason is due to the fact that $s_{d+1,1^{n-d-1}} = \sum_{k=d}^{n} (-1)^{k-d} e_{n-d} h_d$, which is used in Equation (2.7) due to Haglund and Equation (2.8) due to Mellit. An (n, mn)-*parking function* is a pair consisting of and a (n, mn)-Dyck path and a permutation of n, w, for which we write w_i on the line i of the Dyck path. Moreover, all factors of w in a given column of the path must contain no descents (see Figure 2.7 for examples). The set of all (n, mn)-parking function is denoted $\mathcal{P}_{n,mn}$.

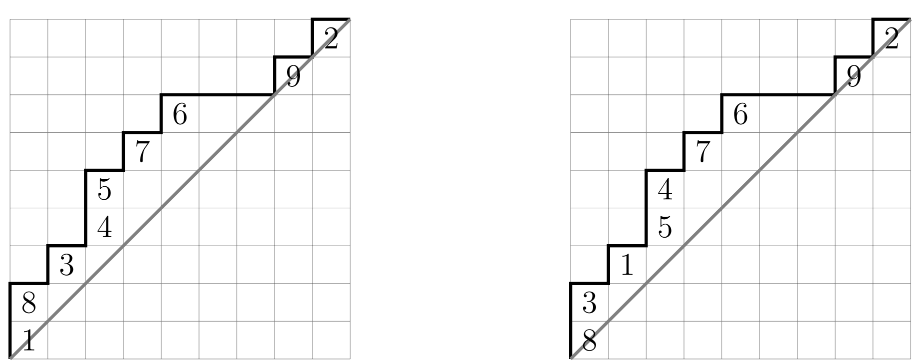


Figure 2.7 To the left a parking function with $w = 183457692$. The factors in each column are 18, 3, 45, 7, 6, 9, 2, and contain no descents. To the right the path is NOT a parking function, since $w = 831457692$ and the factor in the first column is 83 and 54 has a descent.

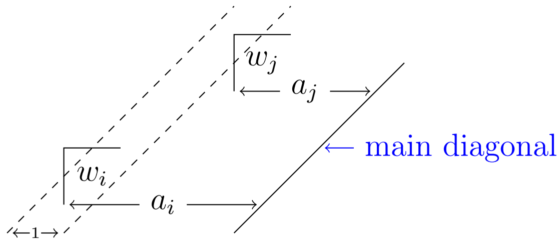
The *reading word* is obtained by reading the letters of w (which are written immediately to the right of each north step) in regard to the diagonals parallel to the main diagonal starting from top right corner to the bottom left corner and starting with the diagonal that is the farthest from the main diagonal. For example, the reading word to the left of Figure 2.7 is 675438291. The reading word of the parking function (γ, w) is denoted $\mathrm{read}(\gamma, w)$.

The *area of a parking function* is the area of its Dyck path. The *diagonal inversion statistic*, of a parking function in $\mathcal{P}_{n,mn}$, (sometimes called dinv for short) is given by the formula $\sum_{i<j} d_i(j)$, where :

$$d_i(j) = \begin{cases} \chi(w_i < w_j)\max(0, r - |a_i - a_j|) & \text{if } i < j \\ +\chi(w_i > w_j)\max(0, m - |a_j - a_i + 1|) & \\ 0 & \text{if } i \geq j. \end{cases}$$

The diagonal inversion statistic of the parking function (γ, w) is denoted $\mathrm{dinv}(\gamma, w)$. Note that all the definitions make sense if w is not a permutation (some authors use words but these can be regrouped with permutations as representatives). Equivalently, for a $(\gamma, w) \in \mathcal{P}_{n,mn}$ we can consider the diagonal inversion of $(\tilde{\gamma}, \tilde{w})$, where $\tilde{\gamma}$ is the (mn, mn)-Dyck path obtained by repeating all north steps m times and for $w = w_1 \cdots w_n$ we have $\tilde{w} = w_1^m \cdots w_n^m$ (here $\tilde{w}$ is not a permutation). In this case we can consider the sum $d_i(j) = \sum_{t=1}^{m} d_i^t(j)$, where $d_i^t(j)$ is calculated with $\tilde{\gamma}$, for the t-th copy of w_i.

A visual representation of the diagonal inversion statistic for $(\gamma, w) \in \mathcal{P}_{n,n}$ is obtained by considering one diagonal parallel to the main diagonal on each north step. For the north step on line j if the diagonal crosses the north step on line i, with $i < j$ and $w_i < w_j$, then the pair (i, j) contributes one to the diagonal inversion statistic. If the diagonal immediately over the diagonal crossing the north step on line j crosses the line i, with $i < j$ and $w_i > w_j$, then the pair (i, j) contributes one to the diagonal inversion statistic (see Figure **??**). The Schröder paths in an $n \times mn$ grid with d diagonal steps can be represented by parking functions (γ, w) such that $\mathrm{read}(\gamma, w) \in \{n-d+1, \cdots, n\} \sqcup \{n-d, \cdots, 1\}$. Indeed, by definition of parking functions, if w_i is in $\{n - d + 1, \cdots, n\}$, then the north



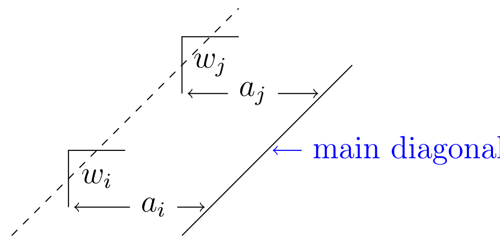


Figure 2.8 To the left, the pair (i, j) contributes 1 if $i < j$ and $w_i > w_j$. To the right, the pair (i, j) contributes 1 if $i < j$ and $w_i < w_j$.

step at line i is followed by an east step. Therefore, for all w_i in $\{n-d+1, \cdots, n\}$ one can change the factor NE on line i for a D and unlabel the path. This procedure gives us a Schröder path with d diagonal steps. Conversely, all D steps of a Schröder path can be changed for NE factors and tagged in the reading order by the letters in $\{n-d+1, \cdots, n\}$ and all the north steps can be tagged in the reading order by letters in $\{n-d, \cdots, 1\}$. This bijection will be mostly used for proofs. Hence, we will often refer to Schröder paths by their parking function description.

In (Thomas et Williams, 2018), Thomas and Williams proved that the zeta map, denoted ζ, is a bijection on rational parking functions, that preserves statistics. In the $n \times n$ case it is such that $\operatorname{dinv}(\gamma, w) = \operatorname{area}(\zeta(\gamma, w))$ and $\operatorname{area}(\gamma, w) = \operatorname{bounce}(\zeta(\gamma, w))$. This will be used implicitly in the following way : if one can decompose of Schröder paths formulas for d diagonal steps into Schur functions in the variables q and t, in terms of area and bounce, the decomposition of the formulas in terms of the statistics of diagonal inversions and area will be the same. For more on m-Schröder paths see (Haglund, 2008) and (Song, 2005).

In this paper we will give explicit decompositions in Schur functions in the variables q and t for $\langle \nabla^m(e_n), e_n \rangle|_{\text{hook}}$, $\langle \nabla^m e_n, s_\mu \rangle|_{\text{1Part}}$ and $\langle \nabla e_n, s_{d+1,1^{n-d-1}} \rangle|_{\text{hook}}$ by using Corollary 2.4 in (Haglund, 2004) :

Theorem : Let n, d be positive integers such that $n \geq d$. Then :

$$\widetilde{\operatorname{Sch}}_{n,d}(q,t) = \langle \nabla e_n, s_{d+1,1^{n-d-1}} \rangle, \tag{2.7}$$

and :

$$\mathrm{Sch}_{n,d}(q,t) = \langle \nabla e_n, e_{n-d} h_d \rangle.$$

The following equalities will also be used and can be inferred from Mellit's proof found in (Mellit, 2016) of the compositional shuffle conjecture of (Bergeron *et al.*, 2016). Let n, d, m be positive integer such that $n \geq d$. Then :

$$\widetilde{\mathrm{Sch}}^{(m)}_{n,d}(q,t) = \langle \nabla^m e_n, s_{d+1,1^{n-d-1}} \rangle. \tag{2.8}$$

and :

$$\nabla^m(e_n) = \sum_{(\gamma,w)\in\mathcal{P}_{n,nm}} t^{\mathrm{dinv}(\gamma,w)} q^{\mathrm{area}(\gamma)} F_{\mathrm{co}(\mathrm{Des}(\mathrm{inv}(\mathrm{read}(\gamma,w))))}(X),$$

where F_c is the fundamental quasisymmetric function index by the composition c and for S a subset of $\{1,\ldots,n-1\}$, $\mathrm{co}(S)$ is the composition associated to S. We can infer the last result from (Stanley, 1979) and (Haiman, 2002) :

$$\nabla(e_n)|_{q=0} = \sum_{\tau\in\mathrm{SYT}(n)} t^{\mathrm{maj}(\tau)} s_{\lambda(\tau)}. \tag{2.9}$$

2.3.5 Algorithm on Schröder Paths Related to Schur Functions Index by One Part Partitions

It was proven in (Haiman, 2002) that $\nabla(e_n)$ is the character of the $GL_2 \times \mathbb{S}_n$-module of diagonal harmonics. Hence, the polynomials $\mathrm{Sch}_{n,d}(q,t)$ and $\widetilde{\mathrm{Sch}}_{n,d}(q,t)$ are symmetric in q,t and can be written as a sum of Schur functions evaluated in q,t. The restriction of a symmetric function to the sum of Schur functions indexed by only one part (respectively, hook-shaped Schur functions) will be denoted by $|_{\mathrm{1Part}}$ (respectively, $|_{\mathrm{hooks}}$). For example, if $f = \sum_{\lambda\in C} c_\lambda s_\lambda$, then the restriction to one part is $f|_{\mathrm{1Part}} = \sum_{\lambda\in C,\ell(\lambda)=1} c_\lambda s_\lambda$ (respectively, the restriction to hooks is $f|_{\mathrm{hooks}} = \sum_{\lambda\in C,\lambda=(a,1^b)} c_\lambda s_\lambda$). In this section, we will give a simple formula for the Schur functions indexed by one part partitions contained in the development of $\mathrm{Sch}_{n,d}(q,t)$. This will be done by proving an algorithm that allows us to describe all the paths of $\widetilde{\mathrm{Sch}}_{n,d}$ relating to the restrictions to Schur functions indexed by one part in the Schur function decomposition of $\widetilde{\mathrm{Sch}}_{n,d}(q,t)$.

Let us first notice that Schur functions on a set of k ordered variables are indexed by a partition with length smaller or equal to k. This follows from the combinatorial definition of Schur functions, since the filling of the first column of a semi-standard Young tableau must be strictly increasing and the first column has the same number of boxes to fill than the number of parts of the partition. Hence, Schur functions in two variables have at most two parts. Furthermore, a Schur function in two variables is such that $s_{a,b}(q,t) = q^{a-b}t^{a-b}(q^b+q^{b-1}t+\cdots+qt^{b-1}+t^b)$. Ergo, for c an integer, the monomial q^c has a non-zero coefficient in the decomposition of a symmetric function $f(q,t)$ if and only if the decomposition in Schur functions contains the term $s_c(q,t)$. This is equivalent to stating that the indices of the Schur functions in the variables q and t appearing in $\mathrm{Sch}_{n,d}(q,t)|_{\mathrm{1Part}}$ are the same as the the powers of q appearing in $\mathrm{Sch}_{n,d}(q,0)$. The only paths that contribute to monomials with non-zero coefficients in $\mathrm{Sch}_{n,d}(q,0)$ are the paths γ such that $\mathrm{area}(\gamma) = 0$. Hence, these are the set of paths $\{NE, D\}^n$.

The algorithm φ takes a path, γ in $\{NE, D\}^*$ as input, and returns a sequence of paths $(\gamma_0, \gamma_1, \ldots, \gamma_{\mathrm{bounce}(\gamma)})$, obtained as follows :

First set $\mathrm{Temp}(\gamma) = (\gamma_0),\ k = |\gamma|_E + |\gamma|_N + |\gamma|_D$.
For $v = 1$ to $\mathrm{bounce}(\gamma)$;
  Let $\gamma_{v-1} = w_1 w_2 \cdots w_k$.
  Let $i \leq n-1$ be such that $w_i = E$, $w_{i+1} \neq E$
    and $w_j = E$ implies $w_{j+1} = E$ or $j \leq i$.
  Set $\gamma_v = w_1 \cdots w_{i-1} w_{i+1} w_i w_{i+2} \cdots w_k$.
  Append γ_v to the sequence $\mathrm{Temp}(\gamma)$.
repeat ;
return $\mathrm{Temp}(\gamma)$.

Schröder paths end with and east step, hence the condition "$w_j = E$ implies

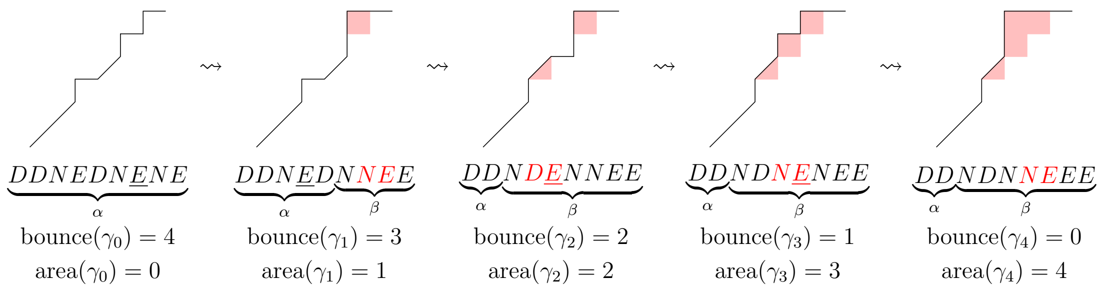


Figure 2.9 The sequence $\varphi(DDNEDNENE)$. Underlined is w_i and in red are the two letters that are swaped.

$w_{j+1} = E$ or $j \leq i$" states that w_i is the last east step that is followed by a north or diagonal step. The only case in which this dose not happen is when all the east steps are at the end. When that happens the bounce statistic is 0. Therefore, the i of the algorithm exists if, for each step of the algorithm, the new path reduces the bounce statistic by 1. This will be shown in Lemma 3.

We first need to prove this algorithm provides us with a sequence of Schröder paths.

Lemma 1 : For all $\gamma \in \{NE, D\}^*$, the elements of the sequence $\varphi(\gamma)$ are Schröder paths. Moreover, if $\gamma \in \mathrm{Sch}_{n,d}$, then each $\gamma_i \in \varphi(\gamma)$ is an element of $\mathrm{Sch}_{n,d}$.

Proof. Recall that for a path γ to be a Schröder path we must have $|\omega|_N \geq |\omega|_E$ for all prefix ω of γ. Because γ_0 is a Schröder path, it is sufficient to show that if γ_i is a Schröder path, then the path γ_{i+1}, obtained by parsing one time through the algorithm, is also a Schröder path. Let $\gamma_i = w_1 w_2 \cdots w_k$, then $\gamma_{i+1} = w_1 \cdots w_{i-1} w_{i+1} w_i w_{i+2} \cdots w_k$ and the only prefixes that are different are $w_1 \cdots w_{i-1} w_{i+1}$ compared to $w_1 \cdots w_{i-1} w_i$ and $w_1 \cdots w_{i-1} w_{i+1} w_i$ compared to $w_1 \cdots w_{i-1} w_i w_{i+1}$. The last pair has the same letters ordered differently, so $|w_1 \cdots w_{i-1} w_{i+1} w_i|_N \geq |w_1 \cdots w_{i-1} w_{i+1} w_i|_E$ if and only if $|w_1 \cdots w_{i-1} w_i w_{i+1}|_N \geq$

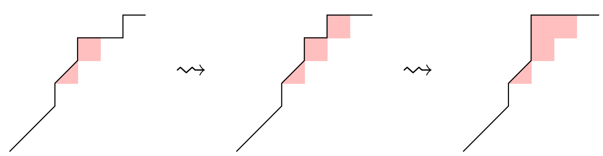
$|w_1 \cdots w_{i-1}w_iw_{i+1}|_E$. Now for the first pair of prefixes we have :

$$\begin{aligned} |w_1 \cdots w_{i-1}w_{i+1}|_N &\geq |w_1 \cdots w_{i-1}w_i|_N, \text{ since } w_i = E \text{ and } w_{i+1} \in \{N, D\}, \\ &\geq |w_1 \cdots w_{i-1}w_i|_E, \text{ since } \gamma_i \text{ is a Schröder path,} \\ &> |w_1 \cdots w_{i-1}w_{i+1}|_E, \text{ since } w_i = E \text{ and } w_{i+1} \neq E. \end{aligned}$$

Therefore, γ_{i+1} is indeed a Schröder path.

Finally, for γ a Schröder path we can move east steps to the left at least a number of times equal to bounce. Indeed, the peaks are associated to an east step and the numph statistic gives the number of diagonal steps over that east step (to the right of that E in the word representation). Because the bounce path associated to $\Gamma(\gamma)$ changes direction at the peak when it hits an east step, except for the last entry, the vector associated to bounce($\Gamma(\gamma)$) gives the number of north steps over that east step (to the right of that E in word representation). □

We give an example of $\varphi(\gamma)$ for γ a Schröder path not in $\{NE, D\}$. In Figure 2.10



bounce(γ_0) = 1 bounce(γ_1) = 1 bounce(γ_2) = 0
area(γ_0) = 2 area(γ_1) = 3 area(γ_2) = 4

Figure 2.10 The sequence $\varphi(DDNDNEENE)$.

the bounce statistic does not decrease evenly throughout the iterations of the algorithm. In Figure 2.9 each iteration increases the area statistic by exactly one and decreases the bounce statistic by exactly one. We will show in Lemma 3 that this is not a coincidence if γ_0 is a Schröder path of area 0. But first, we need to show a result on the prefixes relating to the paths in $\varphi(\gamma)$.

Lemma 2 : Let γ be in $\{NE, D\}^*$ and $\varphi(\gamma) = (\gamma_0, \ldots, \gamma_{\text{bounce}(\gamma)})$. If γ_i as a prefix α in $\{NE, D\}$, then α is a prefix of γ. Moreover, if α is the longest prefix of γ_i such that α is in $\{NE, D\}$, then for β such that $\alpha\beta = \gamma_i$ we have $\beta = \omega E^{|\beta|_E - 1}$, where ω is a word in the alphabet $\{N, E, D\}^*$. Finally, if $\gamma_i = w_1 \cdots w_{j-2} w_j w_{j-1} w_{j+1} \cdots w_k$ and $\gamma_{i-1} = w_1 \cdots w_{j-2} w_{j-1} w_j w_{j+1} \cdots w_k$, then w_{j-1} is a letter in ω.

Proof. By induction on i. If $i = 0$, then $\alpha = \gamma_0 = \gamma$ and $\beta = \varepsilon$. For $i > 0$ let α (respectively,α') be the longest prefix of γ_i (respectively,γ_{i-1}) such that α is in $\{NE, D\}^*$ (respectively,$\alpha' \in \{NE, D\}^*$) and β (respectively,β') be such that $\alpha\beta = \gamma_i$ (respectively,$\alpha'\beta' = \gamma_{i-1}$). By induction we know that α' is a prefix of γ, and $\beta' = \omega' E^{|\beta'|_E - 1}$. By definition of each iteration of the algorithm, there is j such that $\gamma_i = w_1 \cdots w_{j-2} w_j w_{j-1} w_{j+1} \cdots w_k$ and $\gamma_{i-1} = w_1 \cdots w_{j-2} w_{j-1} w_j w_{j+1} \cdots w_k$.

If $|\alpha'| = l \leq j - 2$, then $\beta' = w_{l+1} \cdots w_{j-2} w_{j-1} w_j w_{j+1} \cdots w_k$ and $\alpha = \alpha'$. By definition of the algorithm $w_j \in \{N, D\}$. Due to $\beta' = \omega' E^{|\beta'|_E - 1}$, we must have that w_j and w_{j-1} are both letters of ω'. Ergo, the suffix $E^{|\beta'|_E - 1}$ of β' is unchanged in β. Hence, $\beta = \omega E^{|\beta'|_E - 1} = \omega E^{|\beta|_E - 1}$ and $|\omega|_E = |\omega'|_E$.

If $|\alpha'| = l > j - 2$, then by definition of the algorithm $w_{j-1} = E$, because $\alpha' \in \{NE, D\}^*$, $w_{j-2} = N$. Consequently,$\gamma_i = w_1 \cdots w_{j-3} N w_j E w_{j+1} \cdots w_k$ and $\alpha = w_1 \cdots w_{j-3}$ is in $\{NE, D\}^*$ and is a prefix of α'. Thus, it is a prefix of γ. This means $\beta = N w_j E w_{j+1} \cdots w_k$. Each iteration of the algorithm swaps the rightmost east step that is not followed by an east step to the right. In consequence, $w_j \in \{N, D\}$ and we must have ω such that $\beta = \omega E^{|\beta|_E - 1}$, with w_{j-1} is a letter of ω ; otherwise the letters w_j and w_{j-1} would not have been swapped. $\square$

Lemma 3 : Let γ be in $\{NE, D\}^*$ and $\varphi(\gamma) = (\gamma_0, \ldots, \gamma_{\mathrm{bounce}(\gamma)})$. Then, for all i, such that $0 \leq i < \mathrm{bounce}(\gamma)$, the following equalities hold :

$$\begin{aligned} \mathrm{area}(\gamma_i) + 1 &= \mathrm{area}(\gamma_{i+1}), \\ \mathrm{bounce}(\gamma_i) &= \mathrm{bounce}(\gamma_{i+1}) + 1, \\ \mathrm{area}(\gamma_i) &= \mathrm{bounce}(\gamma_{\mathrm{bounce}(\gamma)-i}), \text{ and,} \\ \mathrm{bounce}(\gamma_i) &= \mathrm{area}(\gamma_{\mathrm{bounce}(\gamma)-i}). \end{aligned}$$

Proof. Let us first notice that the algorithm changes EN for NE or ED for DE. In both cases, this adds exactly one lower triangle under the path. Therefore,

$\text{area}(\gamma_i) + 1 = \text{area}(\gamma_{i+1})$.

For the second condition, let $\gamma_i = w_1 \cdots w_{j-1}w_jw_{j+1}w_{j+2} \cdots w_k$. By definition of the algorithm φ, we know that $\gamma_{i+1} = w_1 \cdots w_{j-1}w_{j+1}w_jw_{j+2} \cdots w_k$, $w_j = E$ and $w_{j+1} \in \{N, D\}$. By Lemma 2, we know $\gamma_{i+1} = \alpha\beta$ with $\alpha \in \{NE, D\}^*$, so there is a return to the main diagonal of the bounce path between α and β. Hence, the following east step is associated to a peak. By Lemma 2, $\beta = \omega E^{|\beta|_E - 1}$, w_j is a letter of ω and ω contains exactly one east step. Consequently, there is a peak at w_j in γ_{i+1}. Let $\gamma_i = \alpha'\beta'$, if w_j is a letter in ω', the same reasoning leads to a peak at w_j in γ_i. If w_j is a letter in α', there is a peak at w_j in γ_i, since $\alpha' \in \{NE, D\}^*$. Thus, $NE = w_{j-1}w_j \in \text{Touch}(\alpha)$.

If $w_{j+1} = D$, then $\text{bounce}(\Gamma(\gamma_i)) = \text{bounce}(\Gamma(\gamma_{i+1}))$ because Γ discards the diagonal steps. Recall that the peaks of a Schröder path, γ, are also obtained from $\Gamma(\gamma)$, in consequence, the peaks in γ_i and γ_{i+1} are associated to the same east steps. Recall that numph is the number of diagonal steps, with multiplicity, positioned after a peak (higher if you consider the path itself rather than the word representation). The diagonal step w_{j+1} is after the peak at w_j in γ_i and before the peak at w_j in γ_{i+1}. All other peaks and diagonal steps remain unchanged. Hence, $\text{numph}(\gamma_i) = \text{numph}(\gamma_{i+1}) + 1$.

If $w_{j+1} = N$, then the peak at w_j moves one position to the right in the word representation (one line higher if you consider the path itself). By definition, at this point, the bounce path returns to the main diagonal and goes north to the next east step, which, by Lemma 2, are all after ω. Therefore, the peak at w_j does not move to a line already containing a peak unless ω ends with w_jD^l. In this last case, the peak at w_j contributed 1 to $\text{bounce}(\gamma_i)$. In any case, the peak on the first line of $\Gamma(\gamma_{i+1})$, contributes 0 to bounce. Consequently, in both cases, $\text{bounce}(\Gamma(\gamma_i)) = \text{bounce}(\Gamma(\gamma_{i+1}))+1$. Additionally, all the east steps keep the same number of diagonal steps positioned after them. Hence, all the peaks keep the same number of diagonal steps positioned after them and $\text{numph}(\gamma_i) = \text{numph}(\gamma_{i+1})$.

Considering the path γ in $\{NE, D\}^*$ has an area equal to zero, the third and fourth conditions follows from the first two conditions. $\square$

We now present a map that will be useful for the discussion on crystals in Section 2.3.10. For γ in $\{NE, D\}^*$ we have $\varphi(\gamma) = (\gamma_0, \ldots, \gamma_{\text{bounce}(\gamma)})$. With this notation we define the map :

$$\tilde{\varphi} : \{\gamma \in \text{Sch}_{\,n,d-1} \mid \text{area}(\gamma) = 0\} \to \{\gamma \in \text{Sch}_{\,n,d-1} \mid \text{area}(\gamma) = 1\}$$
$$\gamma = \gamma_0 \mapsto \gamma_1$$

Hence $\tilde{\varphi}$ gives the first iteration of the algorithm φ. This next lemma will be used in the proof of Theorem 1.

Lemma 4 : Let d be an integer such that $1 \leq d \leq n-1$, then the image of the map $\tilde{\varphi}$ is given by the set $\{uD^jNNEE, vNDEDD^jNE \mid u \in \{NE, D\}^{n-d-2}, v \in \{NE, D\}^{n-d-1}\}$.

Proof. Follows from the definition of φ. □

In order to give the decomposition in Schur functions evaluated in the variables q and t, for $\langle \nabla(e_n), s_{d+1,1^{n-d-1}} \rangle|_{1\text{Part}}$, we will show that for a path γ in $\{NE, D\}^*$, the sum $\sum_{\pi \in \varphi(\gamma)} q^{\text{bounce}(\pi)} t^{\text{area}(\pi)}$ is a Schur function in the variables q and t. For the general result to hold, we need for each pair of different paths to have an empty intersection of theire sets generated by φ.

Lemma 5 : Let γ, π be in $\{NE, D\}^*$ such that $\gamma \neq \pi$ then $\varphi(\gamma) \cap \varphi(\pi) = \emptyset$.

Proof. Let us first notice that the algorithm changes EN for NE or ED for DE. Therefore, the relative order of the north and diagonal steps does not change for all paths in $\varphi(\gamma)$ and $\varphi(\pi)$. Hence, γ_0 and π_0 have the same relative order in regard to the north and diagonal steps which uniquely determine paths of $\{NE, D\}^*$. Consequently, $\varphi(\gamma) \cap \varphi(\pi) = \emptyset$. □

We can now present a bijection that exchanges the statistics area and bounce. This partially solves open problem 3.11 of (Haglund, 2008).

Proposition 1 : Let n be a positive integer and $\varphi(\{NE, D\}^n)$ be $\cup_{\gamma \in \{NE,D\}^n} \varphi(\gamma)$.

There is a bijection, Ω_n, of $\varphi(\{NE, D\}^n)$ onto itself. For all $n \geq 1$ we have, $\text{area}(\gamma_i) = \text{bounce}(\Omega_n(\gamma_i))$ and $\text{bounce}(\gamma_i) = \text{area}(\Omega_n(\gamma_i))$.

Proof. By Lemma 5, for $\gamma \in \varphi(\{NE, D\}^n)$ there is a unique γ_0 and a unique i such that $\gamma \in \varphi(\gamma_0)$ and $\gamma = \gamma_i$. Thus, we can define $\Omega_n(\gamma) = \gamma_{\text{bounce}(\gamma_i)+\text{area}(\gamma_i)-i}$. The result is a consequence of Lemma 3. □

The following Lemma gives us a full of set representatives of Schur functions indexed by one part.

Lemma 6 : Let $A_d = \{\gamma \in \{NE, D\}^n \mid |\gamma|_D = d\}$, there is a bijection $\theta : A_d \to \mathcal{C}_d^n$ such that $\theta(NE) = N$ and $\theta(D) = E$. Moreover, for $\gamma \in A_d$ we have $\text{bounce}(\gamma) = \text{area}(\theta(\gamma)) + \binom{n-d}{2}$.

Proof. In A_d, the factor NE, can be changed for a letter. A path γ in A_d is a word of length n with d occurrences of one letter and $n - d$ occurrences of the other letter. A path in $\mathcal{C}_d^n$ can be represented by a word with d occurrences of the letter E and $n - d$ occurrences of the letter N. Hence, θ merely relabels the letters and is a bijection.

Furthermore, in A_d all east steps are associated to a peak ; therefore, the i-th diagonal steps contribute the number of factors NE before the i-th diagonal steps to numph. But that number is the number of boxes under the i-th north step in $\theta(\gamma)$. Thus, $\text{numph}(\gamma) = \text{area}(\theta(\gamma))$. Finally, $\text{bounce}(\Gamma(\gamma)) = \binom{n-d}{2}$ for all paths γ in A_d because all the $n - d$ east steps return to the main diagonal. □

The next proposition will be generalized for parking functions by Proposition 5 and generalized for the restriction to Schur functions indexed by a hook-shaped partition, evaluated in the variables q and t by Theorem 1. Although the generalizations will not account for all the paths related to each Schur functions.

Proposition 2 : For γ in $\{NE, D\}^*$, we have :

$$\sum_{\gamma_i \in \varphi(\gamma)} q^{\text{bounce}(\gamma_i)} t^{\text{area}(\gamma_i)} = s_{\text{bounce}(\gamma)}(q, t), \tag{2.10}$$

$$\langle \nabla e_n, e_{n-d}h_d\rangle|_{1\text{Part}} = \sum_{\substack{\gamma\in\{NE,D\}^n\\ |\gamma|_D=d}} s_{\text{bounce}(\gamma)}(q,t) = \sum_{\gamma\in\mathcal{C}_d^n} s_{\text{area}(\gamma)+\binom{n-d}{2}}(q,t), \text{ and,} \tag{2.11}$$

$$\langle \nabla e_n, s_{d+1,1^{n-d-1}}\rangle|_{1\text{Part}} = \sum_{\substack{\gamma\in\{NE,D\}^{n-1}NE\\ |\gamma|_D=d}} s_{\text{bounce}(\gamma)}(q,t) = \sum_{\gamma\in\mathcal{C}_d^{n-1}} s_{\text{area}(\gamma)+\binom{n-d}{2}}(q,t). \tag{2.12}$$

Proof. Equation (2.10) follows from Lemma 3. For the first equality of Equation (2.11), we notice that $s_a(q,t) = q^a + q^{a-1}t + \cdots + qt^{a-1} + t^a$, and, thus, by Haglund's Theorem, a Schur function indexed by a one part partition in $\langle \nabla e_n, e_{n-d}h_d\rangle$ can be associated to a path γ in $\text{Sch}_{n,d}$ such that $\text{area}(\gamma) = 0$. But these are in $\{NE, D\}^n$ and have d diagonal steps. For this reason, by Equation (2.10), the equality holds. The second equality of Equation (2.11) follows from Lemma 6. Finally, for Equation (2.12) we only need to notice that paths of $\mathcal{C}_d^n$ ending with a north step are in bijection with paths of $\mathcal{C}_d^{n-1}$ and have the same area. The result is a consequence of Haglund's Theorem, Lemma 6 and Equation (2.10). □

We end this section with a result needed for the generalization of this last proposition (Theorem 1).

Corollary 1 : Let d be an integer and γ be a path in $\widetilde{\text{Sch}}_{n,d}$. Then, the path γ is such that $\gamma = \gamma' NDED^jNE$ or $\gamma = \gamma' NED^jNNEE$, $j \geq 0$, with $\gamma' \in \{NE, D\}^*$ if and only if $\text{area}(\gamma) = 1$ and γ contributes to a Schur function indexed by a partition of length 1 in $\langle \nabla(e_n), s_{d+1,1^{n-d-1}}\rangle$.

Proof. If $\gamma = \gamma' NDED^jNE$ or $\gamma = \gamma' NED^jNNEE$), with $\gamma' \in \{NE, D\}^*$, then, by Lemma 4, γ is in the image of $\tilde{\varphi}$ and the result follows from Proposition 2. If $\text{area}(\gamma) = 1$ and γ contributes to a Schur function indexed by a partition of length 1, by Lemma 4, γ is in the image of $\tilde{\varphi}$ and the result follows from Proposition 2. □

2.3.6 From Parking Functions Formulas to Schur Functions

The aim of this section is to give a combinatorial formula for $\nabla^m(e_n)$ restricted to Schur functions indexed by one part partitions in the variables q and t. We will denote this restriction $\nabla^m(e_n)|_{\text{1Part}}$. In this section we will be using the diagonal inversion statistic, since bounce is not defined for parking functions. This is the main obstacle to knowing all the paths related to each Schur functions, in the variables q and t in the formula will prove for $m \geq 2$. We mentionned earlier on that the Schur functions indexed by one part are the only Schur functions with term q^0t^a and q^at^0. Hence we need for the diagonal inversion statistic or the area statistic to be 0. To obtain a formula for $\nabla^m(e_n)|_{\text{1Part}}$, we will give the necessary and sufficient conditions for a parking function to have a diagonal inversion statistic of 0. We will also determine the necessary and sufficient conditions for a parking function to have a diagonal inversion statistic of 1 if m is greater or equal to 2.

The next three results are technical and will mostly be used to discard some reoccurring cases.

Claim 1 : Let (γ, w) be in $\mathcal{P}_{n,nm}$ such that γ has a factor of the form $\gamma' = NE^pN$, with $1 \leq p \leq m$, and with its north steps associated to w_i and w_{i+1}. Then, if $w_i > w_{i+1}$, $d_i(i+1) = p - 1$ and if $w_i < w_{i+1}$, $d_i(i+1) = p$.

Proof. By definition :

$$d_i(i+1) = \chi(w_i < w_{i+1}) \max(0, m - |a_i - a_{i+1}|) + \chi(w_i > w_{i+1}) \max(0, m - |a_{i+1} - a_i + 1|).$$

Hence, by hypothesis $a_i = a_{i+1} + p - m$. Therefore :

$$d_i(i+1) = \chi(w_i < w_{i+1})(p) + \chi(w_i > w_{i+1})(p-1).$$

□

Lemma 7 : Let (γ, w) be in $\mathcal{P}_{n,mn}$ such that γ has a factor of the form $\gamma' = NE^pN$, with $p > m$. Then, $\text{dinv}(\gamma, w) \geq m - 1$.

Proof. Let w_i and w_{i+1} be the letters of w associated to the factor γ'. Since the path is continuous and over the main diagonal, there exists $j_1, \ldots, j_m$ such that the k-th copy (from the top) of w_{i+1} is to the north on the same diagonal than the north step associated to the letter w_{j_k} in w. Note that the j_k's are not necessarily distinct, ergo, $d_{j_k}(i+1) = d_s^t(i+1)$ for $s = j_k$ and j_k is the t-th copy (from the top) of w_s. Which means that for $1 \leq k \leq m-1$ when $w_{j_k} > w_{i+1}$ we get $d_{j_k}(i+1) \geq 1$ (see left of Figure 2.11) and when $w_{j_k} < w_{i+1}$ the copy $k+1$ of w_{i+1} is to the north and one diagonal lower than w_{j_k}. Therefore, $d_{j_k}(i+1) \geq 1$ (see right of Figure 2.11). Hence :

$$\operatorname{dinv}(\gamma, w) = \sum_{s=1}^{n-1} \sum_{t=1}^{m} \sum_{r>l}^{n} d_s^t(r) \geq \sum_{k=1}^{m-1} d_{j_k}(i+1) \geq m-1.$$

□

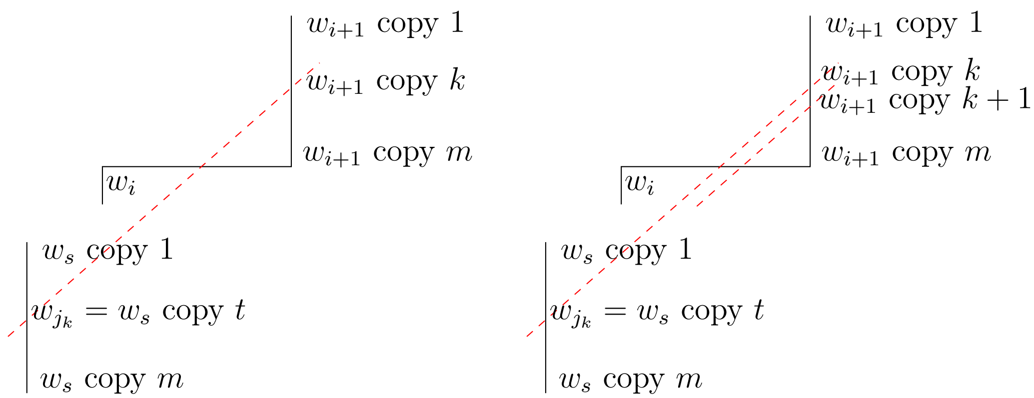


Figure 2.11 To the left the case $w_{j_k} > w_{i+k}$. To the right case $w_{j_k} < w_{i+k}$.

Lemma 8 : Let (γ, w) be in $\mathcal{P}_{n,mn}$. If there is a factor of γ, that is of the form $\gamma' = NE^pN$ associated to the lines $i-1$ and i such that $p \geq 2$, then there is k such that $d_k(i) = 1$, if $m = 1$ and $d_k(i) \geq 1$, if $m > 1$.

Proof. We will work with $\tilde{\gamma}$ and $\tilde{w}$ (remember $\tilde{\gamma}$ is the (mn, mn)-Dyck path obtained by repeating all north steps m times and $\tilde{w} = w_1^m \cdots w_n^m$). Therefore we can use a Dyck path in an $mn \times mn$ grid. Let us suppose there is no such k. Dyck paths have the property of always having more north steps than east steps for all prefixes. Hence, there is a line j in γ such that $j < i$ and the north step on line j_s is on the same diagonal than the north step on line i_1. We can assume j_s is the upper bound of such lines.

By hypothesis, $d_j(i) = 0$ for (γ, w), in consequence, $d_{j_s}(i_1) = 0$ and we must have $\tilde{w}_{j,s} > \tilde{w}_{i,1}$, the contrary would lead to $d_{j_s}(i_1) = 1$. Since $p \geq 2$, there is l such that $j \leq l < i$ and the line l_r is one diagonal over the diagonal passing through the north step at line i_1 in $(\tilde{\gamma}, \tilde{w})$. We can assume l is the smallest line satisfying these properties. Note that if $j = l$, then $r = s - 1$ and if $j \neq l$, then $s = 1$ (see Figure 2.12). Again, $\tilde{w}_{l,r} < \tilde{w}_{i,1}$ or else we would have $d_{l_r}(i_1) = 1$. This means $l \neq j$ and $l \neq j + 1$, since $w_j > w_i > w_l$. So there must be at least one east step between the lines j_1 and $j + 1_m$. If there is just one, then the line $j + 1_m$ is on the same diagonal than the lines i_1 and j_1 contradicting that j is the upper bound. If there are two or more east steps between the lines j_1 and $j + 1_m$, then the path goes under the diagonal passing through the north steps at the line j_1 and at the line i_1. But it must cross it again before the line l_r because the path is continuous and the line l_r is over the diagonal. Which contradicts again that j is the upper bound. $\square$

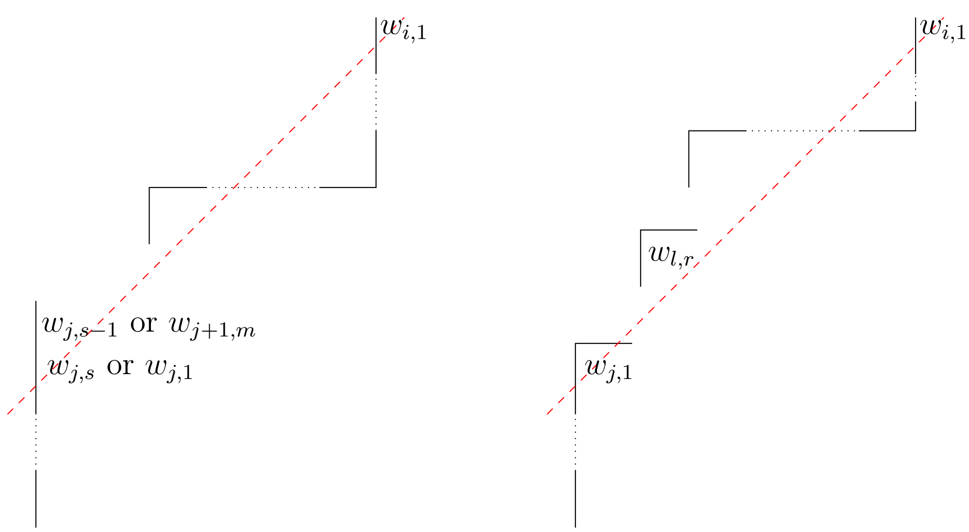


Figure 2.12 To the left the case $j = l$. To the right the case $j \neq l$.

We can now state a first condition.

Lemma 9 : Let (γ, w) be in $\mathcal{P}_{n,n}$. If $\operatorname{dinv}(\gamma, w) = 0$, then $\operatorname{read}(\gamma, w) = w^{-1}$. Additionally, all factors of γ, of the form $\gamma' = NE^pN$ are such that $p \in \{0, 1\}$.

Proof. If $p \leq 1$, then by Claim 1, for all factors of γ of the form NE^pN we must have $p = 1$ when $w_i > w_{i+1}$ and $p = 0$ when $w_i < w_{i+1}$, since $\operatorname{dinv}(\gamma, w) = 0$. If $\gamma' = NE^pN$ is a factor of γ associated to lines i and $i+1$ such that and $p > 1$, then, by Lemma 8, there is k such that $d_k(i+1) = 1$. Therefore, $p \not> 1$ because $\operatorname{dinv}(\gamma, w) = 0$. Finally, $\operatorname{read}(\gamma, w) = w^{-1}$, is a direct consequence of $p \leq 1$. $\square$

The same result is also true for general parking functions when $m \geq 2$. The following gives somewhat of a generalization.

Lemma 10 : Let a and m be integers such that $2 \leq a \leq m$ and (γ, w) be in $\mathcal{P}_{n,nm}$. If $\operatorname{dinv}(\gamma, w) = a - 2$, then $\operatorname{read}(\gamma, w) = w^{-1}$. Additionally, all factors of γ, of the form $\gamma' = NE^pN$ are such that $p \leq m$.

Proof. By hypothesis $\operatorname{dinv}(\gamma, w) = a-2$. If $p > m$, then, by Lemma 7, $a-2 \geq m-1$, which contradicts $a \leq m$. Hence, all factors of γ of the form $\gamma' = NE^pN$ are such that $p \leq m$. Thereafter, all factors of the form $\gamma_{i,j} = NE^{p_i}NE^{p_{i+1}} \cdots NE^{p_{j-1}}N$, with $i < j$, satisfy $|\gamma_{i,j}|_E = \sum_{k=i}^{j-1} p_k \leq (j-i)m = m|\gamma_{i,j}|_N$. Consequently, for all $i < j$ we read w_j before w_i in $\operatorname{read}(\gamma, w)$ and $\operatorname{read}(\gamma, w) = w^{-1}$ as stated. $\square$

Obviously, $\operatorname{read}(\gamma, w) \neq w^{-1}$ in general. But sometimes $\operatorname{read}(\gamma, w) = w^{-1}$ even without the condition on diagonal inversions. The last part of the proof gave us a weaker yet more general statement.

Claim 2 : Let m be an integer and (γ, w) be in $\mathcal{P}_{n,nm}$. If all factors of γ, of the form $\gamma' = NE^pN$ are such that $p \leq m$, then $\operatorname{read}(\gamma, w) = w^{-1}$.

Sadly Lemma 10 does not apply for $m = 2$, when the diagonal inversion statistic has value 1. Therefore, we have to prove it separately.

Lemma 11 : Let (γ, w) be in $\mathcal{P}_{n,2n}$. If $\operatorname{dinv}(\gamma, w) = 1$, then $\operatorname{read}(\gamma, w) = w^{-1}$. Additionally, all factors of γ, of the form $\gamma' = NE^pN$ are such that $p \leq 2$.

Proof. Suppose there is a factor of γ of the form NE^pN such that $p > 2$, associated to lines $j-1$ and j. We can assume j to be the smallest line satisfying that property. Let us consider $\tilde{\gamma}$ the path found by doubling each north step in γ and $\tilde{w}$ the word $w_{1,1}w_{1,2}w_{2,1}w_{2,2}\cdots w_{n,1}w_{n,2}$. Considering the path is continuous and $p > 2$, the path goes over the diagonal passing through $w_{j,1}$. The path ends under that diagonal, ergo there must be i such that $w_{j,2}$ is on the same diagonal as $w_{i,1}$ or $w_{i,2}$ and $w_{i+1,1}$ is strictly over the diagonal passing through $w_{j,1}$. The three cases possible are illustrated by Figure 2.13.

For the first case, if $w_i < w_j$, then the pairs (i_1, j_1) and (i_2, j_2) both contribute to dinv. If $w_i > w_j$, then the pairs $(i+1_2, j_1)$ and (i_1, j_2) both contribute to dinv. Thus, the diagonal inversion statistic cannot be equal to 1.

For the second case, if $w_{i+1} < w_j$, then the pairs $(i+1_2, j_1)$ and (i_1, j_2) both contribute to dinv, since $w_i < w_{i+1}$. If $w_{i+1} > w_j$, then the pairs $(i+1_1, j_1)$ and $(i+1_2, j_2)$ both contribute to dinv. Hence, the diagonal inversion statistic cannot be equal to 1.

For the last case, if $w_i < w_j$, then the pairs (i_1, j_1) and (i_2, j_2) both contribute to dinv. If $w_i > w_j$, then the pairs $(i+1_2, j_2)$ and (i_1, j_2) both contribute to dinv because $w_i < w_{i+1}$. Ergo, the diagonal inversion statistic cannot be equal to 1. Therefore, $p \leq 2$ for all factors of γ of the form NE^pN and, by Claim 2, we get $\text{read}(\gamma, w) = w^{-1}$. $\square$

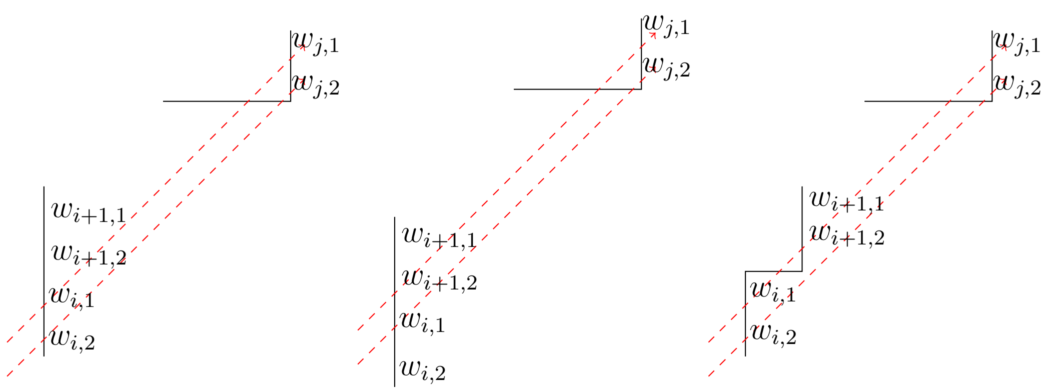


Figure 2.13 To the left case 1. In the middle case 2. To the right case 3

By definition, it is fairly easy to see that for (γ, w) the descent set of w is related to the number of columns of γ. We state the following claim in order to avoid repetition.

Claim 3 : Let (γ, w) be in $\mathcal{P}_{n,nm}$, $1 \leq m$. Then, the number of descents of w plus 1 is smaller or equal to the number of distinct columns. Additionally, the descents are at the top of a column.

Proof. If w_i and w_{i+1} are in the same columns, then by definition of parking functions we must have $w_i < w_{i+1}$. Therefore, descents must be at the top of a column. The last column cannot have descents, since the top of that column is w_n and we know the last letter of a permutation cannot be a descent. □

The previous result relates the number of distinct columns to the descent set of w. But the following relates the number of distinct columns to the descent set of $\text{read}(\gamma, w)^{-1}$.

Lemma 12 : Let m and n be integers, (γ, w) be in $\mathcal{P}_{n,nm}$ and $T(\gamma)$ be the number of distinct columns. Let σ be the permutation such that $\sigma.(\text{read}(\gamma, w)^{-1}) = w$. Then :

$$\text{dinv}(\gamma, w) \geq \begin{cases} T(\gamma) - \text{des}(\text{read}(\gamma, w)^{-1}) - 1 & \text{if } \sigma(n) = n, \\ T(\gamma) - \text{des}(\text{read}(\gamma, w)^{-1}) - 2 & \text{if } \sigma(n) \neq n. \end{cases}$$

Proof. We will show that the letter at the top of a column contribute at least 1 to dinv unless they are in the descent set of $\text{read}(\gamma, w)^{-1}$, in the last column, or the last letter of $\text{read}(\gamma, w)^{-1}$. Notice that if $\sigma(n) = n$, then last letter of $\text{read}(\gamma, w)^{-1}$ is in the last column, so we only need to subtract it once. Let $\text{read}(\gamma, w)^{-1} = v_1 v_2 \cdots v_n$ and let v_i be at the top of a column. If v_i is not in the last column and $i \neq n$, then by definition of the reading word and because the path is continuous, we have , in $(\tilde{\gamma}, \tilde{w})$, these three cases : the last copy from the top of v_{i+1} is to the north and on the same diagonal than a copy of v_i, let us say the k-th copy from the top (see left part of Figure 2.14), the letter v_{i+1} is to south and on one of the diagonals crossing one of the $m - 1$ first copies of v_i (see the middle part of Figure 2.14), let us say the p-th copy, or v_{i+1} is to the south one diagonal higher than the first copy of v_i (see right of Figure 2.14). Let σ be the permutation that send $\text{read}(\gamma, w)^{-1}$ to w. When i is not in the descent set of $\text{read}(\gamma, w)^{-1}$, our first case yields $d_{\sigma(i)}(\sigma(i+1)) = k$, by definition of the diagonal inversion statistics. The same reasoning shows $d_{\sigma(i+1)}(\sigma(i)) = p+1$ for the second case and $d_{\sigma(i+1)}(\sigma(i)) = 1$ for the last case. $\square$

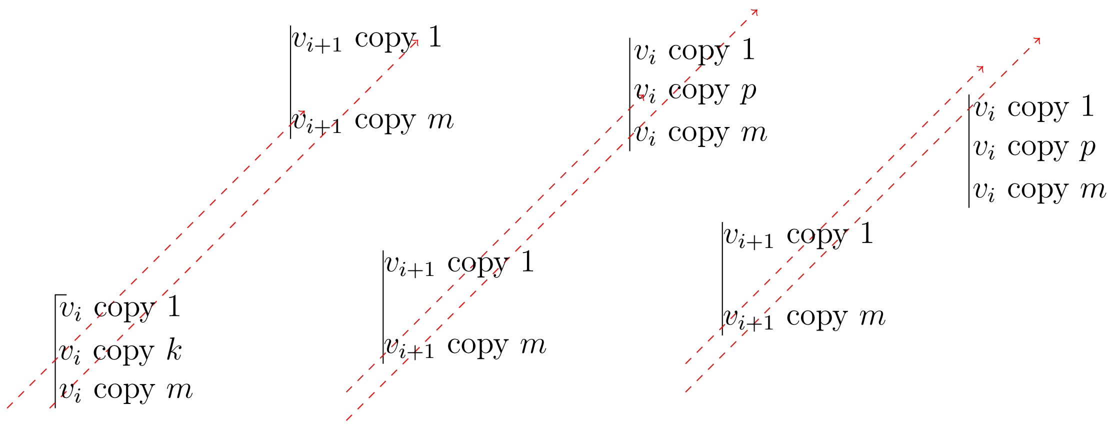


Figure 2.14 Three cases when v_i is not in the last column and $i \neq n$.

We can now state necessary and sufficient conditions for the diagonal inversion statistic to be equal to zero.

Proposition 3 : Let (γ, w) be in $\mathcal{P}_{n,nm}$, $1 \leq m$. Then, $\text{dinv}(\gamma, w) = 0$ if and only if the following conditions hold :

- The path γ can be written as $\gamma' E^j$ where all factors of γ' of length 2 have at most one east step.
- If $\{i_1, \ldots, i_k, n\}$ is the set of all lines containing an east step, then $\{i_1, \ldots, i_k\}$ is the descent set.
- $\text{read}(\gamma, w) = w^{-1}$.

Proof. If $\text{dinv}(\gamma, w) = 0$, then by Claim 1 and Lemma 7, we have the first condition. If $\gamma' = NE^pN$ is a factor of γ associated to w_i and w_{i+1}, then by now proven first condition and Claim 1, $w_i > w_{i+1}$. Hence, the position i is a descent. But, by Claim 3, we know that the number of descents plus 1 is greater or equal to the number of columns and that w_n contains an east step, ergo the second condition. The last condition follows from Lemma 9 and Lemma 10. Therefore, $\mathrm{dinv}(\gamma, w) = 0$ does imply the stated conditions.

Conversely, by Claim 3, the descents are at the top of each column, since n cannot be a descent. Therefore, by the first condition and by Claim 1, we know that, for all lines i, with an east step, we are at the top of a column and $d_i(i+1) = 0$. For all lines i with an east step, and, all lines j such that $j > i+1$, we know that $a_i + (j-i)m \geq a_j \geq a_i + (j-i)(m-1)$, since there is at most one east step between each north step. This leads to :

$$(j-i)m + 1 \geq |a_j - a_i + 1| \geq (j-i)(m-1) + 1 \geq 2(m-1) + 1 \geq m,$$

and :

$$(j-i)m \geq |a_j - a_i| \geq (j-i)(m-1) \geq 2(m-1) \geq m,$$

Therefore, $d_i(j) = 0$ for all i and j. Hence, $\mathrm{dinv}(\gamma, w) = 0$. □

Note that the second statement of the previous proposition implies that the number of descents in w is equal to the number of distinct columns plus 1. Looking at the specialization $q = 0$ is equivalent to looking only at the parking functions (γ, w) such that $\mathrm{dinv}(\gamma, w) = 0$, and, thus, we need the area of theses parking functions.

Proposition 4 : Let (γ, w) be in $\mathcal{P}_{n,nm}$, $1 \leq m$ such that $\mathrm{dinv}(\gamma, w) = 0$. Then :

$$\mathrm{area}(\gamma, w) = m\binom{n}{2} - \mathrm{des}(w)n + \mathrm{maj}(w).$$

Proof. Let $\{i_1, \ldots, i_k, n\}$ be the lines with east steps. We know that these are tops of columns and by the previous proposition we know that $\{i_1, \ldots, i_k\}$ is the descent set. By the previous proposition, we also know that :

$$\begin{aligned}
\mathrm{area}(\gamma, w) &= m\binom{n}{2} - \sum_{j=1}^{k}(n - i_j) \\
&= m\binom{n}{2} - n\sum_{j=1}^{k} 1 + \sum_{j=1}^{k} i_j \\
&= m\binom{n}{2} - \mathrm{des}(w)n + \mathrm{maj}(w)
\end{aligned}$$

□

This last proposition allows us to give a proper formula for $\nabla^m(e_n)|_{q=0}$. Which is just an extension of the Stanley-Lusztig formula.

Proposition 5 : For integers n, m such that $1 \leq n, m$

$$\nabla^m(e_n)|_{1\mathrm{Part}} = \sum_{\tau \in \mathrm{SYT}(n)} s_{\mathrm{maj}(\tau)+(m-1)\binom{n}{2}}(q,t) s_{\lambda(\tau)}(X), \text{and,} \tag{2.13}$$

$$\nabla^m e_n|_{q=0} = t^{(m-1)\binom{n}{2}} \sum_{w \in \mathbb{S}_n} t^{\binom{n}{2} - \mathrm{des}(w)n + \mathrm{maj}(w)} F_{\mathrm{co}(\mathrm{Des}(\mathrm{inv}(w^{-1})))}(X) \tag{2.14}$$

$$= t^{(m-1)\binom{n}{2}} \sum_{\tau \in \mathrm{SYT}(n)} t^{\mathrm{maj}(\tau)} s_{\lambda(\tau)}(X). \tag{2.15}$$

Proof. The first line of Equation (2.14) follows from Proposition 4 and Proposition 3. Consequently, by the Equation inferred from (Stanley, 1979) and (Haiman, 2002) (see Equation (2.9)), we have :

$$\sum_{w \in \mathbb{S}_n} t^{\binom{n}{2} - \mathrm{des}(w)n + \mathrm{maj}(w)} F_{\mathrm{co}(\mathrm{Des}(\mathrm{inv}(w^{-1})))}(X) = \sum_{\tau \in \mathrm{SYT}(n)} t^{\mathrm{maj}(\tau)} s_{\lambda(\tau)}(X).$$

Therefore, the second line of Equation (2.14) holds. For Equation (2.13), we only need to notice that $\nabla^m(e_n)$ is symmetric in q,t and $s_\lambda(q,t) = 0$ if $\ell(\lambda) > 2$ and $s_{a,b}(q,t) = (qt)^b(q^{a-b} + q^{a-b-1}t + \cdots + qt^{a-b-1} + t^{a-b}$. Hence, $s_{a,b}(0,t) = 0$ if $b \neq 0$ and $s_a(0,t) = t^a$. Ergo, we have the stated result by Equation (2.14). □

From this last proposition and Proposition 2 we can obtain the following q-analogues that are used in (Wallace, 2019b).

Corollary 2 : Let n, m, d be integers, then :

$$\langle \nabla^m(e_n), s_{d+1,1^{n-d-1}} \rangle|_{t=0} = q^{(m-1)\binom{n}{2}+\binom{n-d}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q = q^{m\binom{n}{2}-\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_{q^{-1}}$$

Proof. We recall that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\gamma \in \mathcal{C}_k^n} q^{\mathrm{area}(\gamma)}$. In consequence, by Proposition 2 and Proposition 5, we have the first equality. The second equality follows from $\binom{n}{2} - \binom{d+1}{2} = d(n-d-1) + \binom{n-d}{2}$ and $q^{d(n-d-1)} \begin{bmatrix} n-1 \\ d \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n-1 \\ d \end{bmatrix}_q$. □

This next lemma emulates Proposition 3 for diagonal inversion statistics values of one.

Lemma 13 : Let (γ, w) be in $\mathcal{P}_{n,nm}$, $2 \leq m$ and $T(\gamma)$ be the number of distinct columns. Then, $\mathrm{dinv}(\gamma, w) = 1$ if and only if one of the following conditions applies :

• All factors of γ of the form NE^pN are such that $p \leq 1$ and $T(\gamma) = \mathrm{des}(w) + 2$.

• Exactly one factor of γ of the form NE^pN is such that $p = 2$ all other such factors satisfy $p \leq 1$ and $T(\gamma) = \mathrm{des}(w) + 1$.

Proof. We will start by proving the statement for $m \geq 3$. If $\mathrm{dinv}(\gamma, w) = 1$, by Lemma 10, we have $\mathrm{read}(\gamma, w) = w^{-1}$. Thus,by Lemma 12 and Claim 3, $\mathrm{des}(w) + 1 \leq T(\gamma) \leq \mathrm{des}(w) + 2$.

By Claim 3, if $T(\gamma) = \mathrm{des}(w) + 1$, then $w_i > w_{i+1}$ for all i at the top of a column. Hence, by Lemma 10, $p \leq m$. In consequence, by Claim 1, there is exactly one factor of γ of the form NE^pN is such that $p = 2$, all other such factors satisfy $p \leq 1$. Again, by Claim 3, if $T(\gamma) = \mathrm{des}(w) + 2$, there is exactly one line i containing an east step such that $w_i < w_{i+1}$. Ergo, by Claim 1, all factors of γ of the form NE^pN are such that $p \leq 1$.

If all factors of γ of the form NE^pN are such that $p \leq 1$ and $T(\gamma) = \mathrm{des}(w) + 2$, then, by Claim 3, we know there is exactly one i at the top of a column such that $w_i < w_{i+1}$. Moreover, $p \leq 1$ for all factors of γ of the form NE^pN, so, by Claim 1, $\mathrm{dinv}(\gamma, w) = 1$, if $d_i(j) = 0$ for all $j \geq i+2$. But $p \leq 1$ means $m \geq a_{i+1} - a_i \geq m-1$. Hence, for $j > i$, $|a_j - a_i| \geq (m-1)(j-i) \geq m$ and $|a_j - a_i + 1| \geq (m-1)(j-i) + 1 \geq m$, since $j - i \geq 2$ and $m \geq 3$. Consequently, $d_i(j) = 0$ for all $j \geq i+2$ and $\mathrm{dinv}(\gamma, w) = 1$.

If exactly one factor of γ of the form NE^pN is such that $p = 2$ all other such factors satisfy $p \leq 1$ and $T(\gamma) = \mathrm{des}(w) + 1$. By Claim 3, for all i at the top of a column $w_i > w_{i+1}$. Additionally, exactly one factor of γ of the form NE^pN is such that $p = 2$, all other such factors satisfy $p \leq 1$. In consequence, by Claim 1, $\mathrm{dinv}(\gamma, w) = 1$ if $d_i(j) = 0$ for all $j \geq i+2$. But $p \leq 1$ means $m \geq a_{i+1} - a_i \geq m-1$ and $p = 2$ means $m \geq a_{i+1} - a_i \geq m-2$. Ergo, for $j > i$, $|a_j - a_i + 1| \geq (m-1)(j-i) + 1 \geq m$ because $j - i \geq 2$ and $m \geq 3$. Thus, $d_i(j) = 0$ for all $j \geq i+2$ and $\mathrm{dinv}(\gamma, w) = 1$.

For $m = 2$ the proof is the same, we only need to change references of Lemma 10 to Lemma 11. □

Note that for $m = 1$ the previous condition do not hold (see Figure 2.15, Figure 2.16 and Figure 2.17) but, in Section 2.3.8, we manage to obtain a Proposition 5 type formula for the restriction to hook Schur functions in the variables q and t, by using Schröder paths and the bounce statistic.

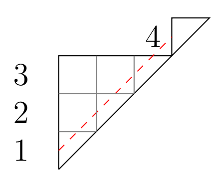


Figure 2.15 The diagonal inversion statistic is 1 yet $p > 2$.

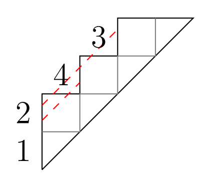


Figure 2.16 The diagonal inversion statistic is 2, yet $T(\gamma) = \mathrm{des}(1243) + 2 = 3$ and $p \leq 1$.

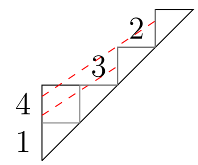


Figure 2.17 The diagonal inversion statistic is 2 yet exactly one factor of the form NE^pN is such that $p = 2$ all others are such that $p \leq 1$ and $T(\gamma) = \mathrm{des}(1432)+1 = 3$.

2.3.7 Restriction to m-Schröder Paths

This section is dedicated to the restriction to Schröder paths. With this restriction we can give the necessary and sufficient conditions for a path to have a diagonal inversion statistic value of one. Proposition 3 can be restated in terms of Schröder paths.

Corollary 3 : Let (γ, w) be in $\mathrm{Sch}_{n,d}^{(m)}$, $1 \leq m$. Then, $\mathrm{dinv}(\gamma, w) = 0$ if and only if, one of the following conditions apply :

- The path γ can be written as $\gamma' E^j$ where all factors of γ' of length 2 have at most one east step.
- If $\{i_1, \ldots, i_k, n\}$ is the set of all horizontal lines containing an east step, then $\{w_{i_1}, \ldots, w_{i_k}\} \subseteq \{n-d+1, \ldots, n\}$.
- $\mathrm{read}(\gamma, w) = w^{-1}$.

Proof. Condition one and three are consequences of Proposition 3. By definition of Schröder paths $\mathrm{read}(\gamma, w) \in \{n-d+1, \ldots, n\} \sqcup \{n-d, \ldots, 1\}$. Therefore, $\mathrm{read}(\gamma, w)^{-1} = w \in \{n, \ldots, n-d+1\} \sqcup \{1, \ldots, n-d\}$ and the descents of w are the positions of $n-d+1, \ldots, n$ in w. Hence, the result follows from Proposition 3. $\square$

The restriction to m-Schröder paths allow us to write the formula of Proposition 5 in terms of paths in a rectangular grid as we did for $m = 1$ in Proposition 2.

Corollary 4 : Let n, d and m be positive integer such that $n \geq d$ and $m > 1$.

Then :

$$\begin{aligned}\mathrm{Sch}^m_{n,d}(q,t)|_{1\mathrm{Part}} =&\langle \nabla^m e_n, e_{n-d}h_d\rangle|_{1\mathrm{Part}}\\ =&\sum_{\gamma\in\mathcal{C}^{n-1}_{d-1}} s_{(m\binom{n}{2}-\binom{d}{2}-\mathrm{area}(\gamma))}(q,t)+\sum_{\gamma\in\mathcal{C}^{n-1}_{d}} s_{(m\binom{n}{2}-\binom{d+1}{2}-\mathrm{area}(\gamma))}(q,t)\end{aligned}$$

Additionally :

$$\mathrm{Sch}^m_{n,d}(q,t)|_{1\mathrm{Part}} = \langle \nabla^m e_n, s_{d+1,1^{n-d-1}}\rangle|_{1\mathrm{Part}} = \sum_{\gamma\in\mathcal{C}^{n-1}_{d}} s_{(m\binom{n}{2}-\binom{d+1}{2}-\mathrm{area}(\gamma))}(q,t). \tag{2.16}$$

Proof. Due to $\sum_{\gamma\in\mathcal{C}^{n-1}_d}\mathrm{area}(\gamma)=\sum_{\gamma\in\mathcal{C}^{n-1}_d}(n-1-d)d-\mathrm{area}(\gamma)$ and $\binom{n}{2}-\binom{d+1}{2}=\binom{n-d}{2}+(n-1-d)d$, the result follows from Proposition 2 and Proposition 5. □

The main proof of this section is a very technical case-by-case proof. It will be used to prove the main theorem via Corollary 6.

Proposition 6 : Let (γ,w) be in $\mathrm{Sch}_{n,d}$ and let $T(\gamma)$ be the number of distinct columns. Then, $\mathrm{dinv}(\gamma,w)=1$ if and only if one of the following conditions applies :

• All factors of γ of the form NE^pN are such that $p\le 1$ and $T(\gamma)=\mathrm{des}(w)+2$.
• Exactly one factor of γ of the form $\gamma'=NE^pN$ is such that $p=2$, and all other such factors satisfy $p\le 1$ and $T(\gamma)=\mathrm{des}(w)+1$.
• Exactly one factor of γ of the form $\gamma'=NE^pN$ is such that $p>2$ and γ' is associated to lines $n-1$, n, $w_n\in\{n-d+1,\dots,n\}$ and all other such factors satisfy $p\le 1$ and $T(\gamma)=\mathrm{des}(w)+2$ if $w_{n-1}\in\{n-d,\dots,1\}$ or $T(\gamma)=\mathrm{des}(w)+1$ if $w_{n-1}\in\{n-d+1,\dots,n\}$.

Proof. If $\mathrm{dinv}(\gamma,w)=1$ and all factors of γ of the form $\gamma'=NE^pN$ are such that $p\le 1$, then, by Claim 1, there is exactly one line i at the top of a column such that $w_i<w_{i+1}$. Additionally, by Claim 3, we know that $T(\gamma)\ge\mathrm{des}(w)+1$. By Claim 2 and Lemma 12, we have $T(\gamma)\le\mathrm{des}(w)+2$. Since w_n is at the top of its own column, i and n are not descents and the top of all the other columns are descents. Thus, $T(\gamma)=\mathrm{des}(w)+2$.

For the remaining cases, if $\mathrm{dinv}(\gamma, w) = 1$, then, by Lemma 8, we know there is at most one factor of γ say $\gamma' = NE^pN$ associated to the lines i and $i+1$ such that $p > 1$ and there is k such that $d_k(i+1) = 1$.

If $w_k > w_{i+1}$, then the north step at line k is on the diagonal above the north step at the line $i{+}1$. Moreover, the path is continuous and the north step at line i is over the diagonal passing through the north steps at the line k and $i{+}1$, and, thus, there exist $l < k$ such that the north step at the line $i+1$ and the north step at the line l are on the same diagonal. Assuming l is the biggest such l. We know, $w_l > w_{i+1}$ because $d_l(i+1) = 0$. This means w_k is read before w_{i+1} and w_{i+1} is read before w_l. By definition of Schröder paths $\mathrm{read}(\gamma, w) \in \{n{-}d{+}1, \ldots, n\} \sqcup \{n{-}d, \ldots, 1\}$. Hence, $w_l \in \{n-d+1, \ldots, n\}$. There is at most one east step between the line l and the line $l+1$, so w_{l+1} is read before w_l. Ergo, $w_l > w_{l+1}$. Therefore, w_{l+1} is not in the same column as w_l. Consequently, w_{l+1} is on the same diagonal as w_l, since there is at most one factor of the form NE^rN with $r > 1$. This contradicts that l is the highest line such that $l < k$, l and $i+1$ are on the same diagonal. (See left of Figure **??**.)

If $w_k < w_{i+1}$, $p \geq 2$ and there is an east step between the lines k and $k+1$, then $d_k(i+1) = 1$ implies the lines k and $i+1$ are crossed by the same diagonal. By Lemma 8, there is exactly one east step between the lines k and $k+1$, and, thus, they are on the same diagonal and $k \neq i$. Considering $\mathrm{dinv}(\gamma, w) = 1$ we need $d_k(k+1) = 0$ and $d_{k+1}(i+1) = 0$. Therefore, $w_k > w_{k+1}$ and $w_{k+1} > w_{i+1}$ which is absurd. (See the middle of Figure **??**.)

For the case $w_k < w_{i+1}$, $p \geq 2$, $k = i-1$ and there is no east step between the lines k and $k{+}1$. Notice that if $k = i{-}1$, there are $i{+}1{-}k = 2$ north steps. Since $d_k(i{+}1) = 1$, we know k and $i{+}1$ are on the same diagonal. Hence, there is 2 east step between the north step at the line k and the north step at the line $i+1$. In addition, letters on the same diagonal are separated by the same number of east steps than north steps. Thus, $p = 2$. Additionally, by Claim 3, $T(\gamma) \geq \mathrm{des}(w) + 1$. Moreover, w_i is read before w_{i+1}, since they are separated by more than one east step and $d_i(i+1) = 0$. Thus,$w_i > w_{i+1}$. Furthermore, all descent in w contribute to a different column. Hence, if $T(\gamma) > \mathrm{des}(w) + 1$ we must have a change of column at a line l, $l \neq i$, such that l is not a descent. Ergo $w_l < w_{l+1}$ and because there is at most one east step between w_l and w_{l+1}, by Lemma 8, we must have $d_l(l+1) = 1$ which is absurd. Therefore, $T(\gamma) = \text{des}(w) + 1$.

If $w_k < w_{i+1}$, $p \geq 2$, $k \neq i-1$ and there is no east step between the lines k and $k+1$, then $k \neq i$ and $w_k < w_{k+1}$. Moreover, $d_k(i+1) = 1$ implies the lines k and $i+1$ are crossed by the same diagonal. Consequently, the north step at line $k+1$ is on the diagonal over the north step at the line $i+1$. Hence, $w_{k+1} < w_{i+1}$, since $d_{k+1}(i+1) = 0$. Thus, w_{k+1} is read before w_{i+1} and w_{i+1} is read before w_k in $\text{read}(\gamma, w)$. We know $\text{read}(\gamma, w) \in \{n-d+1, \ldots, n\} \sqcup \{n-d, \ldots, 1\}$, ergo, $w_{i+1} \in \{n-d+1, \ldots, n\}$ and $w_{k+1}, w_k \in \{n-d, \ldots, 1\}$. If $i+1 \neq n$ and w_{i+2} is in the same column as w_{i+1}, then w_{i+2} is read before w_{i+1} and $w_{i+1} < w_{i+2}$. But this is impossible because w_{i+1} is in the set $\{n-d+1, \ldots, n\}$. Therefore, w_{i+2} and w_{i+1} are on the same diagonal and $w_{i+1} > w_{i+2}$, since $d_{i+1}(i+2) = 0$. Due to $\text{dinv}(\gamma, w) = 1$ and $d_k(i+1) = 1$, we have $d_k(i+2) = 0$. The north step at line $i+2$ is on the same diagonal as the north step at the line k, ergo $w_k > w_{i+2}$. For this reason, w_{i+2} is read before w_k. But, $w_k \in \{n-d, \ldots, 1\}$ means $w_{i+2} > w_k$ which is impossible. So, $i+1 = n$. (See the right of Figure 2.18.)

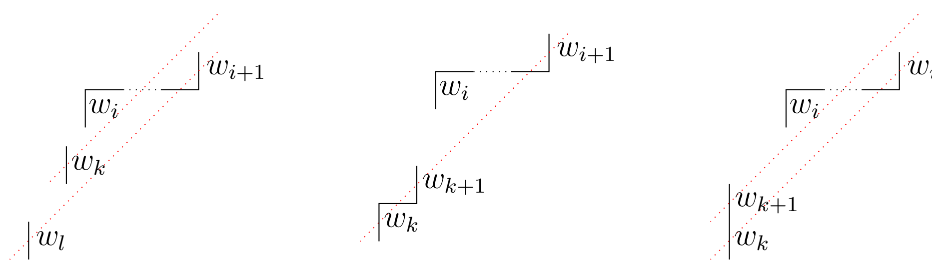


Figure 2.18 To the left case when $w_k > w_{i+1}$. In the center the case when $w_k < w_{i+1}$ and $p \geq 2$. To the right the case $w_k < w_{i+1}$, $p \geq 2$ and $k \neq i-1$

If $w_k < w_{i+1}$, $i+1 = n$, $p > 2$ and there is no east step between the lines k and $k+1$, then, by Claim 3, $T(\gamma) \geq \text{des}(w) + 1$. As in the previous case $w_{i+1} \in \{n-d+1, \ldots, n\}$. Additionally, w_i is read before w_{i+1}, considering they are separated by more than one east step. Hence, $w_i < w_{i+1}$. Furthermore, all descent in w contribute to a different column and the letters w_{n-1}, w_n are in different columns (recall $i+1 = n$ and there are p east steps between w_{n-1} and w_n). This means $T(\gamma) \geq \text{des}(w)+2$ if $w_{n-1} \in \{n-d, \ldots, 1\}$ and $T(\gamma) \geq \text{des}(w)+1$ if $w_{n-1} \in \{n-d+1, \ldots, n\}$. In both cases if the inequality is strict, we must have a change of column at a line l such that $l \neq n-1$ and l is not a descent, ergo, $w_l < w_{l+1}$. Since there is at most one east step between w_l and w_{l+1}, by Lemma 8, we must have $d_l(l+1) = 1$ which is absurd. Therefore, $T(\gamma) = \text{des}(w) + 2$ if $w_{n-1} \in \{n-d, \ldots, 1\}$ and $T(\gamma) = \text{des}(w) + 1$ if $w_{n-1} \in \{n-d+1, \ldots, n\}$.

Conversely, if all factors of γ of the form NE^pN are such that $p \leq 1$ and $T(\gamma) = \text{des}(w)+2$, then there is exactly one line i at the top of a column such that $i \neq n$ and $w_i < w_{i+1}$. By Claim 1, $d_i(i+1) = 1$ and $d_l(l+1) = 0$ for all $l \neq i$ because $p \leq 1$. For the same reason, all lines j and k such that $k - j \geq 2$, have a number of north steps greater or equal to the number on east steps between them and $d_j(k) = 0$, unless the north step on lines j and k are on the same diagonal. But when the north step at the line k and the north step at the line j are on the same diagonal, $k > j$ and $p \leq 1$ we know the line j is associated to a factor of γ of the form NEN and is at the top of a column. By Claim 2, we have $\text{read}(\gamma, w) = w^{-1}$, and, thus, $w_j \in \{n-d+1, \ldots, n\}$ and w_k is read before w_j. Consequently, $w_j > w_k$ and $\text{dinv}(\gamma, w) = 1$.

If exactly one factor of γ of the form $\gamma' = NE^pN$ is such that $p = 2$, and all other such factors satisfy $p \leq 1$ and $T(\gamma) = \text{des}(w) + 1$, then by Claim 3, all lines j at the top of a column are such that j is in the descent set of w. Hence, if w_i and w_{i+1} are associated to the factor $\gamma' = NE^2N$ of γ, then w_i is on the diagonal above w_{i+1} and $w_i > w_{i+1}$, so $d_i(i+1) = 1$. For the same reasons as in the previous case for all j and k such that $j \neq i$ and $k \neq i+1$, then $d_j(k) = 0$. Therefore, $\text{dinv}(\gamma, w) = 1$.

Let us now consider the case when exactly one factor of γ if of the form $\gamma' = NE^pN$ with $p > 2$ and γ' is associated to lines $n-1$, n and all other such factors satisfy $p \leq 1$ and $T(\gamma) = \text{des}(w)+2$. If we take out the last north step and the last east step and call the new path $\tilde{\gamma}$, then, by Proposition 3, $\text{dinv}(\tilde{\gamma}, w_1 \cdots w_{n-1}) = 0$. Hence, for all $1 \leq j < k \leq n-1$ we have $d_j(k) = 0$. Because $d_j(k)$ is a local property, it is also true for (γ, w). By continuity of the path, since $p > 2$, there is a line k such that w_k and w_{k+1} are in the same column and the north step a line n is on the same diagonal than the north step at the line k. Thus, we read w_{k+1} before w_n and w_n before w_k in $\text{read}(\gamma, w)$. Moreover, $w_{k+1} > w_k$, and, therefore, $w_k \in \{n-d, \ldots, 1\}$ and $w_n > w_{k+1} > w_k$, is a consequence of $w_n \in \{n-d+1, \ldots, n\}$. So,

$d_k(n) = 1$ and $d_{k+1}(n) = 0$. All other factors of γ of the form $\gamma'' = NE^{p'}N$ satisfy $p' \leq 1$, if there is j distinct from k such that w_j is on the same diagonal than w_n, then $w_j, w_{j+1}, \ldots, w_k$ are all on the same diagonal and $w_j, w_{j+1}, \ldots, w_{k-1}$ are at the top of their column. Furthermore, the condition $T(\gamma) = \mathrm{des}(w) + 2$ if $w_{i+1} \in \{n-d, \ldots, 1\}$ or $T(\gamma) = \mathrm{des}(w) + 1$ if $w_{i+1} \in \{n-d+1, \ldots, n\}$ forces all letters of w at the top of a column except for w_{n-1} and w_n to be in $\{n-d+1, \ldots, n\}$. Consequently, $w_j > w_{j+1} > \cdots > w_{k-1} > w_n > w_k$ and $d_l(n) = 0$, for all $j \leq l \leq k-1$. Therefore, $\mathrm{dinv}(\gamma, w) = 1$. $\square$

The next corollary can also be deduced from the more general Lemma 13. We only state it here, so one can notice that Proposition 6 is hiding a general statement for $\mathrm{Sch}^{(m)}_{n,d}$.

Corollary 5 : Let m be an integer such that $m \geq 2$, (γ, w) be in $\mathrm{Sch}^{m}_{n,d}$ and let $T(\gamma)$ be the number of distinct columns. Then, $\mathrm{dinv}(\gamma, w) = 1$ if and only if one of the following conditions applies :

• All factors of γ of the form NE^pN are such that $p \leq 1$ and $T(\gamma) = \mathrm{des}(w) + 2$.

• Exactly one factor of γ of the form $\gamma' = NE^pN$ is such that $p = 2$, and all other such factors satisfy $p \leq 1$ and $T(\gamma) = \mathrm{des}(w) + 1$, then $\mathrm{dinv}(\gamma, w) = 1$.

Proof. The proof of the previous proposition can be extended to $\tilde{\gamma}$, since $w_i = w_j$ only if they are in the same column. $\square$

The following is the restriction to unlabelled Dyck paths.

Corollary 6 : Let m be an integer such that $m \geq 1$, (γ, w) be in $\mathrm{Sch}^{m}_{n,0}$ and let $T(\gamma)$ be the number of distinct columns. Then, $\mathrm{dinv}(\gamma, w) = 1$ if and only if one of the following conditions applies :

• All factors of γ of the form NE^pN are such that $p \leq 1$ and $T(\gamma) = \mathrm{des}(w) + 2$.

• Exactly one factor of γ of the form $\gamma' = NE^pN$ is such that $p = 2$, and all other such factors satisfy $p \leq 1$ and $T(\gamma) = \mathrm{des}(w) + 1$, then $\mathrm{dinv}(\gamma, w) = 1$.

Proof. For $m = 1$, the proof is a direct consequence of $\mathrm{read}(\gamma, w) = n \cdots 1$ and Proposition 6. For $m > 1$ the proof follows from the last corollary.

Recall, from the proof of Corollary 3, that when $m \geq 3$ a path (γ, w) of $\mathrm{Sch}_{n,d}(m)$ is such that $\mathrm{des}(w) = d$. Hence the last two corollaries can be stated with nicer formulas. $\square$

2.3.8 Bijections With Tableaux

From Equation (2.14) of Proposition 5, one could wonder which tableau is associated to which path. In this section, we will first show a bijection between standard Young tableaux of shape $(d, 1^{n-d})$ and the subset of Schröder paths $\{\gamma \in \widetilde{\mathrm{Sch}}_{n,d-1} \mid \mathrm{area}(\gamma) = 0\}$. Afterwards, we exhibit a bijection between the set of paths $\{\gamma \in \widetilde{\mathrm{Sch}}_{n,d-1} \mid \mathrm{area}(\gamma) = 1\}$ and pairs containing a standard Young tableaux of shape $(d, 1^{n-d})$ and a number i, $0 \leq i \leq n-d$. This last bijection will allow us to write the sum over these paths with the area and bounce statistics in terms of hook-shaped Schur functions in the variables q and t. In other words, we will obtain an explicit combinatorial formula for the expansion in Schur functions of $\langle \nabla(e_n), s_\mu \rangle|_{\mathrm{hooks}}$. Before we start, we shall also notice that these bijections could easily be extended to paths ending with a diagonal step, by using the bijection between paths with d diagonal steps that end with the factor NE and paths with $d+1$ diagonal steps that end with a D step.

Recall in Section 2.3.4 we defined the touch points of a path. Notice that for a path γ if $\mathrm{area}(\gamma) = 0$ and $\mathrm{Touch}(\gamma) = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, then for all i, γ_i is in $\{NE, D\}$. Let's define the sets $\widetilde{\mathrm{Sch}}_{n,d,(i)}$ by :

$$\widetilde{\mathrm{Sch}}_{n,d,(i)} = \{\gamma \in \widetilde{\mathrm{Sch}}_{n,d} \mid \mathrm{area}(\gamma) = i\}.$$

Let $\{\mathcal{M}_{n,d}\}$ be a family of maps :

$$\begin{aligned} \mathcal{M}_{n,d} : \mathrm{SYT}(d, 1^{n-d}) &\to \widetilde{\mathrm{Sch}}_{n,d-1,(0)} \\ \tau &\mapsto \gamma_1\gamma_2\cdots\gamma_n, \end{aligned}$$

with $\gamma_n = NE$, $\gamma_{n-i} = NE$ if $i \in \mathrm{Des}(\tau)$ and $\gamma_{n-i} = D$ otherwise (see Figure 2.19 for an example). Let $\{\mathcal{R}_{n,d}\}$ be a family of maps :

$$\begin{aligned} \mathcal{R}_{n,d} : \widetilde{\mathrm{Sch}}_{\,n,d-1,(0)} &\to \mathrm{SYT}(d, 1^{n-d}) \\ \gamma &\mapsto \tau, \end{aligned}$$

with $\mathrm{Des}(\mathcal{R}_{n,d}(\gamma)) = \{n-i \mid 1 \leq i \leq n-1, \gamma_i = NE \in \mathrm{Touch}(\gamma)\}$ (see Figure 2.20 for an example).

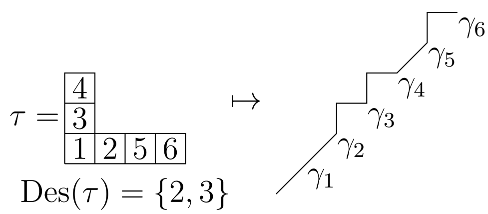


Figure 2.19 An example of the application of map $\mathcal{M}_{6,4}$.

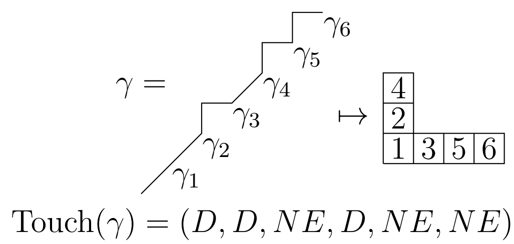


Figure 2.20 An example of the application of map $\mathcal{R}_{6,4}$.

Lemma 14 : The families of maps $\{\mathcal{M}_{n,d}\}$ and $\{\mathcal{R}_{n,d}\}$ are well defined.

Proof. We have already seen that γ in $\widetilde{\mathrm{Sch}}_{n,d-1,(0)}$ is represented by a word in $\{NE, D\}^{n-1}NE$ such that $|\gamma|_D = d-1$. Moreover, $\mathcal{M}_{n,d}(\tau)$ is in $\{NE, D\}^{n-1}NE$ by construction. For $\tau \in \mathrm{SYT}(d, 1^{n-d})$ the descent set of the tableau τ is a subset of $n-d$ elements in $\{1, \ldots, n-1\}$, since for $j \neq 1$ in the first column $j-1$ is lower or to the right of j by definition of hook-shaped standard tableaux. Hence, there are $n-d+1$, i's such that $\gamma_{n-i} = NE$ and $|\mathcal{M}_{n,d}(\tau)|_D = d-1$.

For the map $\mathcal{R}_{n,d}$, notice that hooked-shaped tableaux are uniquely defined by their descent set. Furthermore, there are $n-d+1$ north steps in γ and $\gamma_n = NE \in \mathrm{Touch}(\gamma)$. Thus, for $1 \leq i \leq n-1$ there are $n-d$ factors γ_i such that $\gamma_i = NE$ and $\mathrm{des}(\mathcal{R}_{n,d}(\tau)) = n-d$. Ergo, $\mathcal{R}_{n,d}(\gamma) \in \mathrm{SYT}(d, 1^{n-d})$. $\square$

Proposition 7 : Let n, d be integers such that $0 \leq d \leq n$. The map $\mathcal{M}_{n,d}$ is a bijection from the set of standard tableaux of shape $(d, 1^{n-d})$ to the subset of Schröder paths $\widetilde{\mathrm{Sch}}_{n,d-1,(0)}$, with inverse $\mathcal{R}_{n,d}$. Moreover, the map $\mathcal{M}_{n,d}$ is such that $\mathrm{maj}(\tau) = \mathrm{bounce}(\mathcal{M}_{n,d}(\tau))$ and the map $\mathcal{R}_{n,d}$ is such that $\mathrm{maj}(\mathcal{R}_{n,d}(\gamma)) = \mathrm{bounce}(\gamma)$.

Proof. For the first statement, we only need to show that $\mathcal{M}_{n,d}$ and $\mathcal{R}_{n,d}$ are inverse maps. Let τ be in $\mathrm{SYT}(d, 1^{n-d})$ if i is in $\mathrm{Des}(\tau)$ (respectively, i is not in $\mathrm{Des}(\tau)$), then the map $\mathcal{M}_{n,d}$ sends i to $\gamma_{n-i} = NE$ in $\mathrm{Touch}(\mathcal{M}_{n,d}(\tau))$ (respectively, to $\gamma_{n-i} = D$) and $\mathcal{R}_{n,d}$ sends $\gamma_{n-i} = NE$ in $\mathrm{Touch}(\mathcal{M}_{n,d}(\tau))$ to i in $\mathrm{Des}(\mathcal{R}_{n,d}(\mathcal{M}_{n,d}(\tau)))$ (respectively,to i not in $\mathrm{Des}(\mathcal{R}_{n,d}(\mathcal{M}_{n,d}(\tau))))$. Hence, $\mathcal{R}_{n,d}(\mathcal{M}_{n,d}(\tau)) = \tau$, since hooked-shaped tableaux are uniquely determined by their descent set. For γ in $\widetilde{\mathrm{Sch}}_{n,d-1,(0)}$, the proof of $\mathcal{M}_{n,d}(\mathcal{R}_{n,d}(\gamma)) = \gamma$ is similar. Additionally, in $\widetilde{\mathrm{Sch}}_{n,d-1,(0)}$ all east steps are associated to a peak. For $\gamma = \gamma_1\gamma_2\cdots\gamma_n$, if $\gamma_{n-i} = NE$ and there are k diagonal steps after the factor γ_{n-i}, then the peak associated to that factor contributes k to numph and there is a return to the main diagonal after the factor γ_{n-i} that contributes $i-k$ to $\mathrm{bounce}(\Gamma(\gamma))$. Moreover, the peak $\gamma_n = NE$ contributes nothing to $\mathrm{bounce}(\gamma)$ because it is the end of the path. Hence, $\mathrm{maj}(\tau) = \mathrm{bounce}(\mathcal{M}_{n,d}(\tau))$ and $\mathrm{maj}(\mathcal{R}_{n,d}(\gamma)) = \mathrm{bounce}(\gamma)$, since both maps associate the factor $\gamma_{n-i} = NE$ in the touch sequence to i in the descent set. $\square$

This last proposition gives a combinatorial formula for $\langle \nabla e_n, s_{d,1^{n-d}}\rangle$ in terms of the Major index as in Proposition 5 although we now know to which paths the top weights are associated (see Section 2.3.10 for more on this).

Corollary 7 : Let n, d be positive integer such that $n \geq d$. Then :

$$\langle \nabla e_n, s_{d,1^{n-d}}\rangle|_{1\mathrm{Part}} = \sum_{\tau\in\mathrm{SYT}(d,1^{n-d})} s_{\mathrm{maj}(\tau)}(q,t).$$

Proof. By Proposition 2 and Proposition 7. $\square$

In order to get the same type of formula for the restriction on hook-shaped partitions, we will partition the set $\widetilde{\mathrm{Sch}}_{n,d-1,(1)}$. We need maps to do so. Let τ be in

$\mathrm{SYT}(d, 1^{n-d})$, then we define the path $\gamma_\tau = \gamma_1\gamma_2\cdots\gamma_{n-1}$ where :

$$\gamma_{n-i} = \begin{cases} NE & \text{if } i = 1 \text{ or } i \in \mathrm{Des}(\tau)\backslash\{\max(\mathrm{Des}(\tau))\} \\ D & \text{if } i \notin \mathrm{Des}(\tau) \cup \{1\} \\ NNEE & \text{if } i = \max(\mathrm{Des}(\tau)) \text{ and } 1 \in \mathrm{Des}(\tau) \\ NDE & \text{if } i = \max(\mathrm{Des}(\tau)) \text{ and } 1 \notin \mathrm{Des}(\tau). \end{cases}$$

Let $V_{n,d}$ be a collection of sets defined by :

$$V_{n,d} = \{\gamma \in \widetilde{\mathrm{Sch}}_{n,d-1} \mid \gamma = D^j NNEEu \text{ or } \gamma = D^j NDEu, j \geq 0, u \in \{NE, D\}^* NE\}.$$

Our first family of maps $\mathcal{S}_{n,d}$ is defined as follows, for $n - d \geq 1$:

$$\begin{aligned} \mathcal{S}_{n,d} : \mathrm{SYT}(d, 1^{n-d}) &\to V_{n,d} \\ \tau &\mapsto \gamma_\tau. \end{aligned}$$

The second family of maps is defined as follows, for $n - d \geq 1$:

$$\begin{aligned} \mathcal{T}_{n,d} : V_{n,d} &\to \mathrm{SYT}(d, 1^{n-d}) \\ \gamma &\mapsto \tau_\gamma, \end{aligned}$$

where for $U = \{1\}$ if $\gamma = D^j NDEv$, $U = \emptyset$ if $\gamma = D^j NNEEu$, $u \in \{NE, D\}^*$, $v \in \{NE, D\}^* NE$ we have :

$$\mathrm{Des}(\tau_\gamma) = \{n - i \mid \gamma_i = Nw \in \mathrm{Touch}(\gamma), w \in \{NEE, E, DE\}^*\}\backslash\{U\}.$$

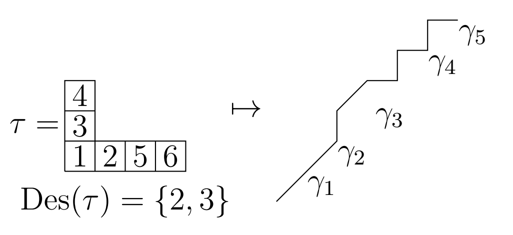


Figure 2.21 Example of the map $\mathcal{S}_{6,4}$

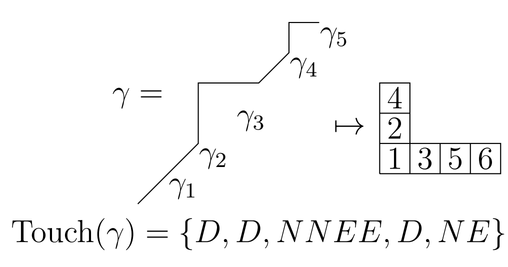


Figure 2.22 Example of the map $\mathcal{T}_{6,4}$

Lemma 15 : Let n, d be positive integers such that $n-d \geq 1$, the maps $\mathcal{S}_{n,d}$ and $\mathcal{T}_{n,d}$ are well defined.

Proof. The factors γ_i are all Schröder paths ans the concatenation of Schröder paths is a Schröder path. Moreover, the factor γ_{n-1} of γ_τ contains a north step and all the other north steps are related to an element in the descent set. Hence, we have $n-d+1$ north steps and γ_τ is an element of $\widetilde{\mathrm{Sch}}_{n,d-1}$. Additionally, the construction of γ_τ is based on four mutually exclusive conditions that define γ_{n-i}. If $i = \max(\mathrm{Des}(\tau))$, then $n-i \leq n-k$ for all $k \in \mathrm{Des}(\tau)$. Thus, by construction the path starts with D^jNNEE or D^jNDE. Since there is only one maximum of the descent set the map $\mathcal{S}_{n,d}$ is well defined.

Notice that for all paths in $V_{n,d}$ there is exactly one factor in $\{NNEE, NDE\}$ and all other are in $\{NE, D\}$. Hence, $\mathrm{Touch}(\gamma)$ has $n-1$ factors because it is equivalent to counting the numbers of north and diagonal steps minus one. Therefore, the descent set is included in $\{1, \ldots, n-1\}$. Moreover, there are $n-d+1$ north step and one is in the factor γ_{n-1}. Ergo, $1 \in \{n-i : \gamma_i = Nw \in \mathrm{Touch}(\gamma), w \in \{NEE, E, DE\}^*\}$.

If $\gamma = D^jNNEEu$, $u \in \{NE, D\}^*$, then there are $n-d+1$ north steps for $n-d$ factors containing a north step. Consequently, $\mathrm{des}(\tau_\gamma) = n-d$ and it uniquely determines a hooked-shaped tableau in $\mathrm{SYT}(d, 1^{n-d})$.

If $\gamma = D^jNDEv, v \in \{NE, D\}^*NE$, then there are $n-d+1$ north steps for $n-d+1$ factors containing a north step. But U takes out one from the descent set. Hence, the descent set has $n-d$ elements and it uniquely determines a hooked-shaped tableau in $\mathrm{SYT}(d, 1^{n-d})$. Consequently, $\mathcal{T}_{n,d}$ is well defined. $\square$

The interesting thing about these maps is that they preserve statistics as shown in the next lemma.

Lemma 16 : Let n, d be positive integers such that $n-d \geq 1$. The maps $\mathcal{S}_{n,d}$ and $\mathcal{T}_{n,d}$ are bijective maps and $\mathcal{T}_{n,d} = \mathcal{S}_{n,d}^{-1}$. Moreover, the maps $\mathcal{T}_{n,d}$ and $\mathcal{S}_{n,d}$ preserve statistics in the following way :

$$\mathrm{bounce}(\gamma) = \mathrm{maj}(\mathcal{T}_{n,d}(\gamma)) - \mathrm{des}(\mathcal{T}_{n,d}(\gamma)),$$

$$\mathrm{bounce}(\mathcal{S}_{n,d}(\tau)) = \mathrm{maj}(\tau) - \mathrm{des}(\tau).$$

Proof. Let τ be a Standard Young tableau in of shape $(d, 1^{n-d})$, $\mathcal{S}_{n,d}(\gamma) = \gamma_\tau$, $\gamma_\tau = \gamma_1\gamma_2\cdots\gamma_{n-1}$. For $i \geq 2$, if the factor $\gamma_{n-i} = NE$, then the map $\mathcal{T}_{n,d}$ send that factor to $i \in \mathrm{Des}(\mathcal{T}_{n,d}(\gamma_\tau))$. But the map $\mathcal{S}_{n,d}$ gives us $\gamma_{n-i} = NE$ when $i \in \mathrm{Des}(\tau)$. If $\gamma_{n-i} = NNEE$, then the map $\mathcal{T}_{n,d}$ sends that factor to $1, i \in \mathrm{Des}(\mathcal{T}_{n,d}(\gamma_\tau))$ and the map $\mathcal{S}_{n,d}$ gives $\gamma_{n-i} = NNEE$ when $1, i \in \mathrm{Des}(\mathcal{T}_{n,d}(\gamma_\tau))$. Finally, if $\gamma_{n-i} = NDE$, then the map $\mathcal{T}_{n,d}$ sends that factor to $i \in \mathrm{Des}(\mathcal{T}_{n,d}(\gamma_\tau))$ and the image of map $\mathcal{S}_{n,d}$ is $\gamma_{n-i} = NDE$ when $i \in \mathrm{Des}(\mathcal{T}_{n,d}(\gamma_\tau))$. Thus,$\mathcal{T}_{n,d}(\mathcal{S}_{n,d}(\tau)) = \tau$. The proof of $\mathcal{S}_{n,d}(\mathcal{T}_{n,d}(\gamma)) = \gamma$ is similar.

The proof that $\mathrm{bounce}(\mathcal{S}_{n,d}(\tau)) = \mathrm{maj}(\tau) - \mathrm{des}(\tau)$ is almost the same as the proof in Proposition 7. Since there are only γ_{n-1} factors, we need to subtract one to each contribution to $\mathrm{bounce}(\Gamma(\gamma))$. Moreover, if $\gamma_{n-i} = NNEE$ and there are k diagonal steps after the factor γ_{n-i}, then the peak associated to that factor contributes k to numph and there is a return to the main diagonal after the factor γ_{n-i} that contributes $i-k-1$ to $\mathrm{bounce}(\Gamma(\gamma))$. Finally, if $\gamma_{n-i} = NDE$ and there are k diagonal steps after the factor γ_{n-i}, then the peak associated to that factor contributes k to numph and there is a return to the main diagonal after the factor γ_{n-i} that contributes $i-k-1$ to $\mathrm{bounce}(\Gamma(\gamma))$. But $1 \notin \mathrm{Des}(\mathcal{T}_{n,d}(\gamma)$ which mean we do not add $1-1=0$ to $\mathrm{Des}(\mathcal{T}_{n,d}(\gamma)$. Hence, $\mathrm{maj}(\tau) - \mathrm{des}(\tau) = \mathrm{bounce}(\mathcal{S}_{n,d}(\tau))$ and $\mathrm{maj}(\mathcal{T}_{n,d}(\gamma)) - \mathrm{des}(\mathcal{T}_{n,d}(\gamma)) = \mathrm{bounce}(\gamma)$, because both maps associate the factor $\gamma_{n-i} = NE$ to i in the descent set. Consequently, $\mathrm{bounce}(\mathcal{S}_{n,d}(\tau)) = \mathrm{maj}(\tau) - \mathrm{des}(\tau)$. The map $\mathcal{S}_{n,d}$ is a bijection of inverse $\mathcal{T}_{n,d}$, and, thus, we also have $\mathrm{bounce}(\gamma) = \mathrm{maj}(\mathcal{T}_{n,d}(\gamma)) - \mathrm{des}(\mathcal{T}_{n,d}(\gamma))$. □

To extend the maps $\mathcal{S}_{n,d}$ and $\mathcal{T}_{n,d}$ we need to partition the paths of $\widetilde{\mathrm{Sch}}_{n,d,(1)}$. Notice that $\widetilde{\mathrm{Sch}}_{n,n-2,(1)} = \{D^{n-2}NNEE, D^iNDED^jNE \mid i+j = n-3\}$ and $\widetilde{\mathrm{Sch}}_{n,n,(1)} = \widetilde{\mathrm{Sch}}_{n,n-1,(1)} = \emptyset$.

For $d = n-2$, we define $\Pi_{n,d-1}$ to be the identity map. For $n-d+1 \geq 3$ and

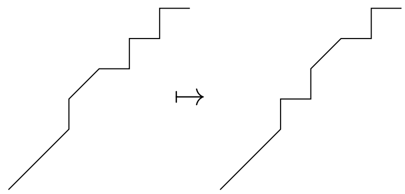


Figure 2.23 Example of the map $\Pi_{6,3}$

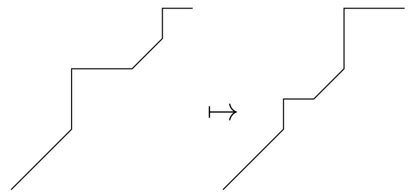


Figure 2.24 Example of the map $\Pi_{6,3}$

$j \leq 0$ let :

$$\begin{array}{lcl}
\Pi_{n,d-1} : \widetilde{\text{Sch}}_{\,n,d-1,(1)} & \to & \widetilde{\text{Sch}}_{\,n,d-1,(1)} \\
uNNEED^jNEv & \mapsto & uNED^jNNEEv \\
uNDED^jNEv & \mapsto & uNED^jNDEv \\
uNNEED^jNE & \mapsto & uNED^jNNEE \\
D^iNEuD^jNNEE & \mapsto & D^iNNEEuD^jNE \\
D^iNEuNDED^jNE & \mapsto & D^iNDEuNED^jNE
\end{array}$$

For u in $\{NE, D\}^*$ and v in $\{NE, D\}^*NE$.

Proposition 8 : For all integers n and d such that $n - d \geq 1$. For each τ in $\text{SYT}(d, 1^{n-d})$ we can associate a set $\{\Pi^k_{n,d-1}(\gamma_\tau)\}_{1\leq k\leq n-d}$. These sets are a partition of the set $\widetilde{\text{Sch}}_{n,d-1,(1)}$. Additionally, $\Pi_{n,d-1}$ is cyclic of order $n - d$.

Proof. We shall first notice that the map is well defined, since there is exactly one factor N^2E^2 or NDE in path of area 1. Secondly, $\gamma_\tau = D^iN^2E^2D^jNEuNE$ or $\gamma_\tau = D^iNDED^jNEuNE$ and the action of $\Pi_{n,d-1}$ is to exchange the factor N^2E^2 (respectively, NDE) with the factor next NE factor. Because there are $n - d + 1$ north steps in γ_τ, if N^2E^2 is a factor of γ_τ (respectively, NDE) there are $n - d - 1$ (respectively, $n - d$) factors NE above the factor N^2E^2 (respectively, NDE) and $\Pi^{n-d-1}_{n,d-1}(\gamma_\tau) = D^iNED^jNEuN^2E^2$ (respectively, $\Pi^{n-d-1}_{n,d-1}(\gamma_\tau) = D^iNED^jNEu'NDED^kNE$, where $u = u'NED^k$). Therefore, $\Pi^{n-d}_{n,d-1}(\gamma_\tau) = \gamma_\tau$.

Let γ be a path in $\widetilde{\text{Sch}}_{n,d-1,(1)}$. Then, there is a unique factor N^2E^2 or NDE which corresponds to the line of area 1. Hence, $\gamma = D^jNEu'N^2E^2u''NE$ (respectively, $\gamma = D^jNEu'NDEu''NE$, $\gamma = D^jNEu'N^2E^2$), with u' and u'' in $\{NE, D\}$. Let k

be the number of north steps before the factor N^2E^2 (respectively,NDE, N^2E^2 for $k = n-d-1$). Consequently, $\Pi^k_{n,d-1}(\gamma_0) = \gamma$, for $\gamma_0 = D^jN^2E^2u'NEu''NE$ (respectively,$\gamma_0 = D^jNDEu'NEu''NE$, $\gamma_0 = D^jN^2E^2u'NE$). Thus, $\mathcal{T}_{n,d}(\gamma_0) = \tau_{\gamma_0}$. So, γ is in the set $\{\Pi^k_{n,d-1}(\gamma_{\tau_{\gamma_0}})\}$.

Finally, let γ be in $\{\Pi^k_{n,d-1}(\gamma_\tau)\} \cap \{\Pi^k_{n,d-1}(\gamma_\rho)\}$, then there is k such that $\gamma = \Pi^k_{n,d-1}(\gamma_\tau)$ and l such that $\gamma = \Pi^l_{n,d-1}(\gamma_\rho)$. Hence, $\Pi^{n-d-k+l}_{n,d-1}(\gamma_\rho) = \gamma_\tau$. By previous statements we know $k = l$; it is the number of north steps before the factor N^2E^2 of NDE in γ. Ergo,$\gamma_\rho = \gamma_\tau$. Thus,by Lemma 16, $\rho = \mathcal{T}_{n,d}(\gamma_\rho) = \mathcal{T}_{n,d}(\gamma_\tau) = \tau$. Therefore $\{\Pi^k_{n,d-1}(\gamma_\tau)\} \cap \{\Pi^k_{n,d-1}(\gamma_\rho)\} = \emptyset$ if $\rho \neq \tau$. □

One might see similarities between the next lemma and Lemma 3, since the bounce statistic increases by exactly one with each iteration of $\Pi_{n,d-1}$. But it is worth mentioning that in this case the area remains one. Thus, this is not an extension of the algorithm seen in Section 2.3.5 for paths with a bounce statistic value of zero.

Lemma 17 : Let $U = \{uNED^jN^2E^2, uNDEDD^jNE \mid u \in \{NE, D\}^*\}$. The map $\Pi_{n,d-1}$ is such that $\text{bounce}(\Pi_{n,d-1}(\gamma)) = \text{bounce}(\gamma) + 1$ for all $\gamma \in \widetilde{\text{Sch}}_{n,d,(1)} \backslash U$.

Proof. If γ as a factor N^2E^2, then the map $\Pi_{n,d-1}$ swaps the factor N^2E^2 with the next factor NE and all factors return to the main diagonal in $\Pi_{n,d-1}(\gamma)$. Hence, the number of peak under each diagonal step is left unchanged and $\text{numph}(\gamma) = \text{numph}(\Pi_{n,d-1}(\gamma))$. Moreover, $\text{area}(\gamma) = 1$, and, therefore, there is some k such that $\Gamma(\gamma) = (NE)^{n-d-k}N^2E^2(NE)^{k-1}$. Thus, we obtain $\text{bounce}(\Gamma(\gamma)) = \binom{n-d+2}{2} - k$ and $\text{bounce}(\Gamma(\Pi_{n,d-1}(\gamma))) = \binom{n-d+2}{2} - k + 1$.

If γ as a factor NDE, then $\text{bounce}(\Pi_{n,d-1}(\Gamma(\gamma))) = \text{bounce}(\Gamma(\gamma))$. Since the map $\Pi_{n,d-1}$ swaps the factor NDE with the next factor NE and all factors return to the main diagonal, in $\Pi_{n,d-1}(\gamma)$ there is one more peak below the diagonal step coming from the factor NDE than in γ. Moreover, the number of peaks below all the other diagonal steps remains unchanged. Ergo, $\text{numph}(\gamma) + 1 = \text{numph}(\Pi_{n,d-1}(\gamma))$. □

With this partition we can put forward a bijection between tableaux and Schröder paths.

Proposition 9 : Let $\mathcal{Q}_{n,d}$ be a map between the set of pairs in the product set $\mathrm{SYT}(d,1^{n-d}) \times \{0,1,\cdots,n-d-1\}$ and the set $\widetilde{\mathrm{Sch}}_{n,d-1,(1)}$, defined by the composition $\mathcal{Q}_{n,d}(\tau,i) = \Pi^i_{n,d-1}(\mathcal{S}_{n,d}(\tau))$. Then, the map $\mathcal{Q}_{n,d}$ is a bijection. Moreover, $\mathrm{bounce}(\mathcal{Q}_{n,d}(\tau,i)) = \mathrm{maj}(\tau) - \mathrm{des}(\tau) + i$ for all i in $\{0,1,\cdots,n-d-2\}$.

Proof. By Lemma 16 and Proposition 8, this map is well defined. Furthermore, for γ in $\widetilde{\mathrm{Sch}}_{n,d-1,(1)}$ there is a unique τ in $\mathrm{SYT}(d,1^{n-d})$ an a unique integer i such that $0 \leq i \leq n-d-1$ and $\Pi^i_{n,d-1}(\gamma_\tau) = \gamma$. Moreover, $\mathcal{S}_{n,d}$ is a bijection of inverse $\mathcal{T}_{n,d}$. Hence, $\mathcal{T}_{n,d}(\gamma_\tau) = \tau$ and the pre-image of γ is unique. Ergo $\mathcal{Q}_{n,d}$ is a bijection.

By the previous lemma $\mathrm{bounce}(\Pi^i_{n,d-1}(\gamma_\tau)) - i = \mathrm{bounce}(\gamma_\tau)$ if $\Pi^i_{n,d-1}(\gamma_\tau)$ is not an element of $U = \{uNED^jNNEE, uNDED^jNE \mid u \in \{NE,D\}^*\}$. The proof of Proposition 8 shows that $\Pi^i_{n,d-1}(\gamma_\tau)$ is an element of U if and only if $i = n-d-1$. Hence, we only need to show $\mathrm{bounce}(\gamma_\tau) = \mathrm{maj}(\tau) - \mathrm{des}(\tau)$. But this is true by Lemma 16. □

We now restate and prove our main theorem.

Theorem : If $\mu \in \{(d,1^{n-d}) \mid 1 \leq d \leq n\}$ and $\nu \vdash n$, then :

$$\langle \nabla(e_n), s_\mu \rangle|_{\text{hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{\mathrm{maj}(\tau)}(q,t) + \sum_{i=2}^{\mathrm{des}(\tau)} s_{\mathrm{maj}(\tau)-i,1}(q,t), \tag{2.17}$$

$$\langle \nabla^m(e_n), s_\nu \rangle|_{1\text{Part}} = \sum_{\tau \in \mathrm{SYT}(\nu)} s_{m\binom{n}{2} - \mathrm{maj}(\tau')}(q,t), \tag{2.18}$$

and :

$$\langle \nabla^m(e_n), e_n \rangle|_{\text{hooks}} = s_{m\binom{n}{2}}(q,t) + \sum_{i=2}^{n-1} s_{m\binom{n}{2}-i,1}(q,t). \tag{2.19}$$

Note that $\mathrm{maj}(1^n) = \binom{n}{2}$, $\binom{n}{2} - \mathrm{maj}(\tau') = \mathrm{maj}(\tau)$ and $\mathrm{des}(\tau) = n - 1 - \mathrm{des}(\tau')$.

Proof. Equation (2.5) is true by Proposition 5. For $m = 1$ Equation (2.6) is a direct consequence of Equation (2.4). Notice that if γ is in $\mathrm{Sch}^{(1)}_{n,d}$, then $\gamma E^{(m-1)n}$ is in $\mathrm{Sch}^{(m)}_{n,d}$. Furthermore, the difference between the area of γ and the area of

$\gamma E^{(m-1)n}$ is exactly $(m-1)\binom{n}{2}$. Ergo, for $m>1$ Equation (2.6) follows from Corollary 6.

The first sum on the right side of Equation (2.4) follows from Proposition 5. For the second sum on the right side of Equation (2.4), notice that, by Proposition 9, there is a bijection between paths of area value equal to one with d diagonal steps ending with NE or $NNEE$ and the product set of standard Young tableaux of shape $(d,1^{n-d})$ and the set $\{0,\ldots,n-d-1\}$. Moreover,one can see that the cyclic action of the map $\Pi_{n,d-1}$, proven in Proposition 8, puts $\Pi_{n,d-1}^{n-d-1}(\mathcal{S}_{n,d}(\tau))$ in $\{uNED^jNNEE, vNDED^jNE \mid u\in\{NE,D\}^{n-d-2}, v\in\{NE,D\}^{n-d-1}\}$, since it is of order $n-d$. So, by Corollary 1 and Proposition 5, the set $\{\Pi_{n,d-1}^{k}(\mathcal{S}_{n,d}(\tau))\}_{0\leq k\leq n-d-1}$ contains the only paths that contribute to Schur functions indexed by partition of length 1. Consequently, we need only to consider the set $\{0,\ldots,n-d-2\}$ because the second sum relates to partition of length 2. Additionally, $\mathrm{des}(\tau)=n-d$, for $\tau\in\mathrm{SYT}(d,1^{n-d})$, hence $2\leq i\leq \mathrm{des}(\tau)$ if and only if $n-d-2\geq \mathrm{des}(\tau)-i\geq 0$. Therefore, we sum from 2 to $\mathrm{des}(\tau)$. Except for the paths already contributing to Schur function having only one part, the restriction to a value of one for the area corresponds to hook-shaped Schur functions. Indeed, $s_{a,b}(q,t)=(qt)^b(q^{a-b}+q^{a-b-1}t+\cdots+qt^{a-b-1}+t^{a-b})$, and, thus, the monomial q^ct can only be found when $b\in\{0,1\}$. Ergo, by Proposition 9, we have the stated result. $\square$

We conjecture that the following equation is true for all μ when $m=1$. Although we know, by Lemma 13, that this is not true for $m>1$.

Conjecture 1 : For all $\mu\vdash n$:

$$\langle\nabla(e_n),s_\mu\rangle|_{\mathrm{hooks}}=\sum_{\tau\in\mathrm{SYT}(\mu)}s_{\mathrm{maj}(\tau)}(q,t)+\sum_{i=2}^{\mathrm{des}(\tau)}s_{\mathrm{maj}(\tau)-i,1}(q,t),$$

2.3.9 Inclusion Exclusion

In this section we will see that half the paths in $\widetilde{\mathrm{Sch}}_{n,d,(1)}$ are related to $\widetilde{\mathrm{Sch}}_{n,d-1,(1)}$ and the other half is related to $\widetilde{\mathrm{Sch}}_{n,d+1,(1)}$. Thereafter, this will be used to find a positive formula for an alternating sum. The outcome is needed to prove results on multivariate diagonal harmonics in (Wallace, 2019a) (Chapter 3). Recall that we have $\widetilde{\mathrm{Sch}}_{n,d,(1)} = \{\gamma \in \widetilde{\mathrm{Sch}}_{n,d} \mid \mathrm{area}(\gamma) = 1\}$ and $\widetilde{\mathrm{Sch}}_{n,n,(1)} = \widetilde{\mathrm{Sch}}_{n,n-1,(1)} = \emptyset$. Now, let :

$$\overline{\mathrm{Sch}}_{n,d} = \{\gamma \in \widetilde{\mathrm{Sch}}_{n,d,(1)} \mid \gamma = D^jNNEE\gamma'NENE \text{ or } \gamma = \gamma'NED^jNNEE\gamma'', j \geq 0\},$$

and :

$$\underline{\mathrm{Sch}}_{n,d} = \{\gamma \in \widetilde{\mathrm{Sch}}_{n,d,(1)} \mid \gamma = D^jNNEE\gamma'DNE \text{ or } \gamma = \gamma'NDED^jNE\gamma'', j \geq 0\},$$

where γ' is a possibly empty word in $\{NE, D\}^*$ and γ'' is a word in $\{NE, D\}^*NE$.

Lemma 18 : For $1 \leq d \leq n-4$ we have the following equality $\widetilde{\mathrm{Sch}}_{n,d,(1)} = \overline{\mathrm{Sch}}_{n,d} \cup \underline{\mathrm{Sch}}_{n,d}$. Additionally, $\widetilde{\mathrm{Sch}}_{n,0,(1)} = \overline{\mathrm{Sch}}_{n,0}$, $\widetilde{\mathrm{Sch}}_{n,n-2,(1)} = \underline{\mathrm{Sch}}_{n,n-2} \cup \{D^{n-2}NNEE\}$ and $\widetilde{\mathrm{Sch}}_{n,n-3,(1)} = \overline{\mathrm{Sch}}_{n,n-3} \cup \underline{\mathrm{Sch}}_{n,n-3} \cup \{D^{n-3}NNEENE\}$. Furthermore, for all d, $\overline{\mathrm{Sch}}_{n,d} \cap \underline{\mathrm{Sch}}_{n,d} = \emptyset$.

Proof. A simple check shows that the four cases of $\overline{\mathrm{Sch}}_{n,d}$ and $\underline{\mathrm{Sch}}_{n,d}$ are mutually exclusive. Hence, $\overline{\mathrm{Sch}}_{n,d} \cap \underline{\mathrm{Sch}}_{n,d} = \emptyset$. The cases $d = 0$, $d = n-2$ and $d = n-3$ are related to the maximal number of north steps and diagonal steps. For d general, $1 \leq d \leq n-4$, let γ be in $\widetilde{\mathrm{Sch}}_{n,d,(1)}$. When a Schröder path has an area value of 1, there is a factor $\pi = NNEE$ or $\pi = NDE$ such that $\gamma = u\pi v$, $u \in \{NE, D\}^*$ and $v \in \{NE, D\}^*NE \cup \{\varepsilon\}$. By definition of $\widetilde{\mathrm{Sch}}_{n,d,(1)}$, if $v = \varepsilon$, then $\pi = NNEE$ and γ is in $\overline{\mathrm{Sch}}_{n,d}$. Moreover, if $\pi = NDE$, then there is a factor NE in v and γ is in $\underline{\mathrm{Sch}}_{n,d}$. If $\pi = NNEE$ and $u = D^j$, then v has at least two factors NE, since $d \leq n-4$, and v can end with $NENE$, in which case γ is in $\overline{\mathrm{Sch}}_{n,d}$ or v end with DNE and γ is in $\underline{\mathrm{Sch}}_{n,d}$. If $\pi = NNEE$ and $u \neq D^j$, then there is a factor NE in u and γ is in $\overline{\mathrm{Sch}}_{n,d}$. $\square$

Let d be an integer such that $0 \leq d \leq n-1$. For each d let :

$$\begin{array}{lcl} \rho_{n,d} : \overline{\mathrm{Sch}}_{n,d} & \to & \underline{\mathrm{Sch}}_{n,d+1} \\ \gamma' NED^j NNEE \gamma'' & \mapsto & \gamma' NDED^j NE \gamma'' \\ D^j NNEE \gamma' NENE & \mapsto & D^j NNEE \gamma' DNE \end{array}$$

Lemma 19 : Let d be an integer such that $0 \leq d \leq n-1$. Then, for all d, $\rho_{n,d}$ is a bijection such that $\mathrm{bounce}(\rho(\gamma)) = \mathrm{bounce}(\gamma) - 1$.

Proof. Notice that $\gamma' NED^j N^2E^2\gamma'' \neq D^k N^2E^2\gamma' NENE$ for all k and j, since one has a NE factor before its N^2E^2 factor and not the other. The path as an area value of one, ergo there is only one factor N^2E^2 in γ. Moreover, the map $\rho_{n,d}$ increases the number of diagonal steps by one. Therefore, for all d the map $\rho_{n,d}$ is well defined. For the same reasons the map $\rho_{n,d}^{-1}$ defined by $\rho_{n,d}^{-1}(\gamma' NDED^j NE\gamma'') = \gamma' NED^j N^2E^2\gamma''$ and $\rho_{n,d}^{-1}(D^j N^2E^2\gamma' DNE) = D^j N^2E^2\gamma' NENE$ is well defined. Thus, it is the inverse of $\rho_{n,d}$ and we have a bijection.

Recall that the numph statistic is related to the number of peaks that are in a lower row than the diagonals. When the area value is one, all factors NE and $NNEE$ contain a peak ; they always return to the diagonal. If we compare the path $\gamma = \gamma' NED^j N^2E^2\gamma''$ and the path $\rho_{n,d}(\gamma)$, we see that only one diagonal step was added in $\rho_{n,d}(\gamma)$ and the factor NDE in which it was added has its peak above the diagonal step. Hence, $\mathrm{numph}(\rho_{n,d}(\gamma))$ is equal to $\mathrm{numph}(\gamma)$ plus the number of peaks in γ'. Considering that $\Gamma(\gamma' NED^j N^2E^2\gamma'') = (NE)^{|\gamma'|+1}N^2E^2(NE)^{|\gamma''|}$, the value of $\mathrm{bounce}(\Gamma(\gamma))$ is $\binom{n-d}{2} - |\gamma''|_N - 1$. Moreover, $\Gamma(\gamma' NDED^j NE\gamma'') = (NE)^{n-d-1}$; therefore, $\mathrm{bounce}(\Gamma(\rho_{n,d}(\gamma))) = \binom{n-d-1}{2}$. Because the number of peaks in γ' is equal to $|\gamma'|_E$ which is equal to $|\gamma'|_N$ and $|\gamma'|_N + |\gamma''|_N = n-d-3$, we obtain $\mathrm{bounce}(\gamma) - \mathrm{bounce}(\rho_{n,d}(\gamma)) = 1$.

Now comparing the paths $\gamma = D^j N^2E^2\gamma' NENE$ and $\rho_{n,d}(\gamma)$ we see that the statistic $\mathrm{numph}(\rho_{n,d}(\gamma))$ is equal to $\mathrm{numph}(\gamma)$ plus the number of peaks in $D^j N^2E^2\gamma'$. This is equivalent to counting the number of non-consecutive east steps in the path $D^j N^2E^2\gamma'$, that is $n-d-3$, since there are $n-d$ east steps in γ. We know $\Gamma(D^j N^2E^2\gamma' NENE) = N^2E^2(NE)^{n-d-2}$. The value of $\mathrm{bounce}(\Gamma(\gamma))$ is

$\binom{n-d}{2} - (n-d-1)$. Moreover, in we consider the restriction to north steps and east steps is $\Gamma(D^j N^2 E^2 \gamma' DNE) = N^2E^2(NE)^{n-d-3}$, then $\text{bounce}(\Gamma(\rho_{n,d}(\gamma))) = \binom{n-d-1}{2} - (n-d-2)$. Hence, $\text{bounce}(\gamma) - \text{bounce}(\rho_{n,d}(\gamma)) = 1$. $\square$

We now know some Schröder paths of area 1 are associated to Schur functions in the variables q and t indexed by partitions of length one. This corollary is merely to show that the bijection in the previous lemma sends paths associated to Schur functions indexed by a length one partition to another path associated to Schur functions indexed by a length one partition.

Corollary 8 : Let γ be a path of $\overline{\text{Sch}}_{n,d}$. Then, γ contributes to a Schur function indexed by a partition of length 1 in $\langle \nabla(e_n), s_{d+1,1^{n-d-1}} \rangle$ if and only if $\rho_{n,d}(\gamma)$ contributes to a Schur function indexed by a partition of length 1 in $\langle \nabla(e_n), s_{d+2,1^{n-d-2}} \rangle$. Moreover, if γ satisfies this property, then $\gamma = \gamma' NED^j N^2 E^2$ and $\rho_{n,d}(\gamma) = \gamma' NDED^j NE$, with $\gamma' \in \{NE, D\}^*$.

Proof. By Corollary 1, the definition of $\rho_{n,d}$ and the definition of $\overline{\text{Sch}}_{n,d}$. $\square$

Lemma 20 : Let d and n be integers such that $0 \leq d \leq n-2$. Then the map, $\mathcal{T}_{n,d+2} \circ \rho_{n,d} \circ \mathcal{S}_{n,d+1}$ is a bijection between the sets of tableaux :

$$\{\tau \in \text{SYT}(d+1, 1^{n-d-1}) \mid \{1,2\} \subseteq \text{Des}(\tau)\} \simeq \{\tau \in \text{SYT}(d+2, 1^{n-d-2}) \mid 1 \in \text{Des}(\tau), 2 \notin \text{Des}(\tau)\},$$

such that $\text{maj}(\tau) - \text{des}(\tau) = \text{maj}(\mathcal{T}_{n,d+2} \circ \rho_{n,d} \circ \mathcal{S}_{n,d+1}(\tau)) - \text{des}(\mathcal{T}_{n,d+2} \circ \rho_{n,d} \circ \mathcal{S}_{n,d+1}(\tau)) + 1$. Additionally, $\mathcal{Q}^{-1}_{n,d+2} \circ \rho_{n,d} \circ \mathcal{Q}_{n,d+1}$ is a bijection between the set :

$$\{(\tau, i) \in \text{SYT}(d+1, 1^{n-d-1}) \times \{1, \ldots, n-d-3\} \mid 1 \in \text{Des}(\tau)\},$$

and the set :

$$\{(\tau, i) \in \text{SYT}(d+2, 1^{n-d-2}) \times \{0, \ldots, n-d-4\} \mid 1 \notin \text{Des}(\tau)\},$$

such that $\text{maj}(\tau) - i = \text{maj}(\mathcal{Q}^{-1}_{n,d+2} \circ \rho_{n,d} \circ \mathcal{Q}_{n,d+1}(\tau, i)) - (i-1)$.

Proof. By Corollary 1 we know paths of shape $\gamma' NDED^j NE$ or $\gamma' NED^j N^2 E^2$

are associated to a Schur function of length one. Notice that by definition $\rho_{n,d}$ sends paths of theses shapes to other paths of theses shapes. Hence, we have $\mathcal{Q}_{n,d}^{-1} \circ \rho_{n,d} \circ \mathcal{Q}_{n,d}(\tau, n-d-2) = (\tau', n-d-3)$, where $\mathrm{Des}(\tau') = \mathrm{Des}(\tau)\backslash\{1\}$. At this point one only needs to notice that $\mathcal{Q}_{n,d+2}^{-1} \circ \rho_{n,d} \circ \mathcal{Q}_{n,d+1}(\tau, i) = (\tau', i-1)$, $\mathrm{Des}(\tau') = \mathrm{Des}(\tau)\backslash\{1\}$ and $\mathrm{Des}(\mathcal{T}_{n,d+2} \circ \rho_{n,d} \circ \mathcal{S}_{n,d+1}(\tau)) = \mathrm{Des}(\tau)\backslash\{2\}$. The rest of the proof follows Proposition 9 from the definition of the maps $\rho_{n,d}$, $\mathcal{Q}_{n,d}$, $\mathcal{S}_{n,d}$ and $\mathcal{T}_{n,d}$. □

Proposition 10 : Let k be an integer such that $0 \leq k \leq n-3$ and $\psi : \Lambda_{\mathbb{Q}} \to \mathbb{Q}[q,t]$ be a linear map defined by $\psi(s_\lambda) = q^{|\lambda|}t^{\ell(\lambda)-1}$. If :

$$h_k(q) := \psi\left(\sum_{d=0}^{k}(-1)^{k-d}\langle\nabla(e_n), s_{d+1,1^{n-d-1}}\rangle|_{\text{pure hook}}\right) q^{-k+d}t^{-1},$$

then :

$$\begin{aligned} h_k(q) &= \sum_{\substack{\tau\in\mathrm{SYT}(k+1,1^{n-k-1})\\ \{1,2\}\subseteq\mathrm{Des}(\tau)}} q^{\mathrm{maj}(\tau)-\mathrm{des}(\tau)} + \sum_{\substack{\tau\in\mathrm{SYT}(k+1,1^{n-k-1})\\ 1\in\mathrm{Des}(\tau)}} \sum_{i=2}^{n-k-2} q^{\mathrm{maj}(\tau)-i}, \\ &= \sum_{\substack{\tau\in\mathrm{SYT}(k+2,1^{n-k-2})\\ 1\in\mathrm{Des}(\tau),2\notin\mathrm{Des}(\tau)}} q^{\mathrm{maj}(\tau)-\mathrm{des}(\tau)} + \sum_{\substack{\tau\in\mathrm{SYT}(k+2,1^{n-k-2})\\ 1\notin\mathrm{Des}(\tau)}} \sum_{i=2}^{n-k-2} q^{\mathrm{maj}(\tau)-i}, \end{aligned}$$

where the restriction $|_{\text{pure hook}}$ is the restriction to Schur function indexed by partitions $(a,1)$, with $a \geq 1$.

Proof. Let $\tilde{h}_k(q) = \psi\left(\sum_{d=0}^{k}(-1)^{k-d}\langle\nabla(e_n), s_{d+1,1^{n-d-1}}\rangle|_{\text{hook}}\right) q^{-k+d}t^{-1}$. Due to

Haglund's theorem, Lemma 19 and Lemma 18, for an integer k we have :

$$
\begin{aligned}
\tilde{h}_k(q) &= \sum_{d=0}^{k}(-1)^{k-d} \sum_{\gamma\in\overline{\mathrm{Sch}}_{n,d}} q^{\mathrm{bounce}(\gamma)-k+d} + \sum_{\gamma\in\underline{\mathrm{Sch}}_{n,d}} q^{\mathrm{bounce}(\gamma)-k+d},\\
&= \sum_{d=0}^{k}(-1)^{k-d} \sum_{\gamma\in\overline{\mathrm{Sch}}_{n,d}} q^{\mathrm{bounce}(\rho_{n,d}(\gamma))+1-k+d} + \sum_{d=1}^{k}(-1)^{k-d} \sum_{\gamma\in\underline{\mathrm{Sch}}_{n,d}} q^{\mathrm{bounce}(\gamma)-k+d},\\
&= \sum_{d=0}^{k}(-1)^{k-d} \sum_{\gamma\in\underline{\mathrm{Sch}}_{n,d+1}} q^{\mathrm{bounce}(\gamma)+1-k+d} - \sum_{d=1}^{k}(-1)^{k-d+1} \sum_{\gamma\in\underline{\mathrm{Sch}}_{n,d}} q^{\mathrm{bounce}(\gamma)-k+d},\\
&= \sum_{d=0}^{k}(-1)^{k-d} \sum_{\gamma\in\underline{\mathrm{Sch}}_{n,d+1}} q^{\mathrm{bounce}(\gamma)+1-k+d} - \sum_{d=0}^{k-1}(-1)^{k-d} \sum_{\gamma\in\underline{\mathrm{Sch}}_{n,d+1}} q^{\mathrm{bounce}(\gamma)-k+d+1},\\
&= \sum_{\gamma\in\underline{\mathrm{Sch}}_{n,k+1}} q^{\mathrm{bounce}(\gamma)+1},\\
&= \sum_{\gamma\in\overline{\mathrm{Sch}}_{n,k}} q^{\mathrm{bounce}(\gamma)}.
\end{aligned}
$$

The polynomial $h_k(q)$ only takes into account pure hooks, so we only need to consider the paths $\gamma' NED^j NNEE\gamma''$ and $D^j NNEE\gamma' NENE$, with γ', $\gamma'' \in \{NE, D\}$, as shown in Corollary 8. Thus, by Lemma 16, the map $\mathcal{T}_{n,k+1}$, for $\gamma = D^j NNEE\gamma' NENE$, gives us $\mathcal{T}_{n,k+1}(\gamma)$ is a tableau containing $\{1,2\}$ in its descent set and such that $\mathrm{bounce}(\gamma) = \mathrm{maj}(\mathcal{T}_{n,k+1}(\gamma)) - \mathrm{des}(\mathcal{T}_{n,k+1}(\gamma))$. Additionally, for $\gamma = \gamma' NED^j NNEE\gamma''$, $\mathcal{Q}^{-1}_{n,k+1}(\gamma)$ is a tableau containing $\{1\}$ in its descent set. By Proposition 9, γ is such that $\mathrm{bounce}(\gamma) = \mathrm{maj}(\mathcal{Q}^{-1}_{n,k+1}(\gamma)) - i$, $2 \leq i \leq \mathrm{des}(\mathcal{Q}^{-1}_{n,k+1}(\gamma))$. Moreover, if $\{1\}$ is in the descent set of τ, then the map $\mathcal{Q}_{n,k+1}$ send $(\tau, 0)$ to $D^j NNEE\gamma' NENE$ or $D^j NNEE\gamma' DNE$ the first one was already considered and the last one is in $\underline{\mathrm{Sch}}_{n,d}$. Finally, if $\{1\}$ is in the descent set of τ, then the map $\mathcal{Q}_{n,k+1}$ send (τ, i) to $\overline{\mathrm{Sch}}_{n,d}$ for all $1 \leq i \leq n-d-3$. Hence, we sum over $2 \leq i \leq \mathrm{des}(\mathcal{Q}^{-1}_{n,k+1}(\gamma)) - 1$. The second sum is a consequence of Lemma 20. □

Considering what is known about multivariate diagonal harmonics, it should be possible to extend the results of this section to the case for $\widetilde{\mathrm{Sch}}_{n,d,(i)}$. This generalization would lead to more results on multivariate diagonal harmonics.

2.3.10 Partial Crystal Decomposition

This section is mainly to explain the underlying idea throughout this paper. We can see this as finding the crystal decomposition of the Schröder paths and the parking functions (since their weighted sum relate to modules). In this setting the weights are the statistics area and bounce. We basically found some of the top weights and for some of them gave a map, $\tilde{\varphi}$ that gives the remainder of the crystal (the raising operator). In that setting we can say that for $m = 1$, we can describe all the crystals in the case where the Schur functions are indexed by partitions of length one. When $m > 1$, we can characterize only the top weights. For hooked-shaped Schur functions, we can only depict the top weight, when $m = 1$.

More precisely, the maps $\mathcal{R}_{n,d}$ and $\mathcal{T}_{n,d}$ determine in which crystal the paths lie. The maps $\tilde{\varphi}$, defined by the map φ in Section 2.3.5, give the decomposition according to the top weight. This also is well defined, since for all $\gamma \in \{NE, D\}^{n-1}NE$ we have $\mathcal{T}_{n,d} \circ \Pi \circ \tilde{\varphi}(\gamma) = \mathcal{R}_{n,d}(\gamma)$ (see Figure 2.25 for an example). Notice that map $\mathcal{M}_{n,d}$ (respectively, $\mathcal{Q}_{n,d}$) tells us in which crystal component are the Schröder paths of area value 0 (respectively, area value 1).

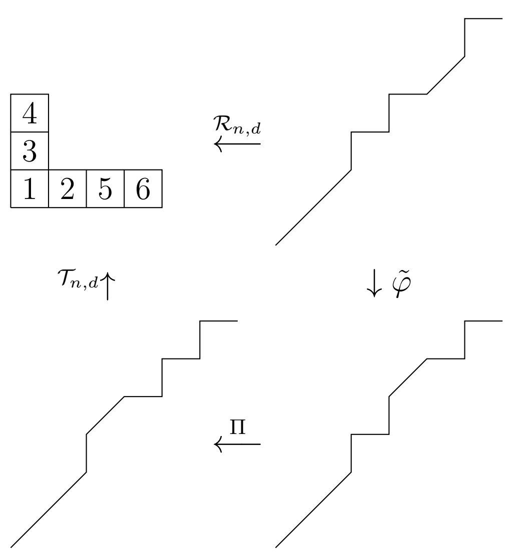


Figure 2.25 An example of $\mathcal{T}_{n,d} \circ \Pi \circ \tilde{\varphi}(\gamma) = \mathcal{R}_{n,d}(\gamma)$.

Using the zeta map, so far we know the top weights for all crystals containing a parking function having a diagonal inversion statistic value of 0, and for all hook-shaped partitions we know all top weights for all crystals containing a parking function having a diagonal inversion statistic value of 1. For crystals containing a parking function having a diagonal inversion statistic value of 0 that are not associated to a hook-shaped partition, we do not know the exact paths. Although, we do know in which subset of parking functions the lowest weight lie. Figure 2.26 gives an overview of what is known so far.

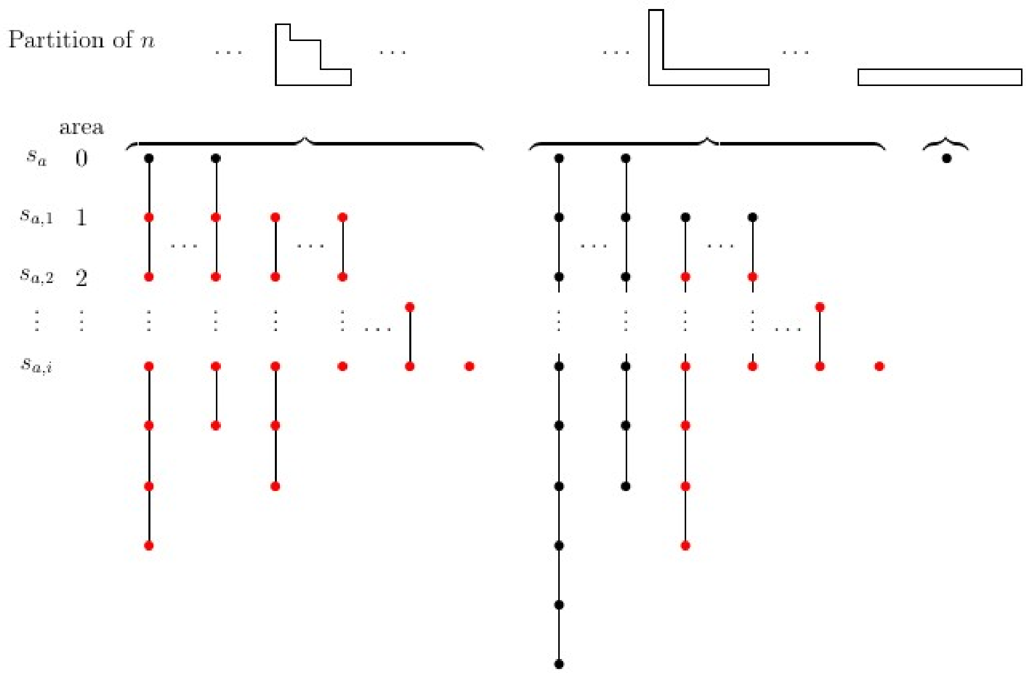


Figure 2.26 The nodes represent paths. Each chain is associated to a Schur function in the variables q and t. The height of the first node determines which Schur function. The partitions determine the Schur function in the variables X. Each chain can be associated to a Standard Young tableau corresponding to the shape of the partition. More than one chain can be associated with the same tableau. When nodes are in black, we know which paths they relate to, in red we do not.

2.3.11 Conclusion and Further Questions

Proving Conjecture 1 would be a great start. Moreover, can one describe nicely the algorithm described in Section 2.3.5, in terms of diagonal inversions and extend it to m-Schröder paths ? Here we are looking for more than just applying the zeta map. This would allow us to know exactly what paths contribute to each Schur function, even when $m \neq 1$. It might be easier to start by the following problem :

Problem 1 : Using the bounce statistic, generalize the algorithm in Section 2.3.5 to all Schröder paths.

This would give the Schröder paths associated to all the Schur functions and not only the one with one part. It would also answer completely Haglund's open problem 3.11 of (Haglund, 2008). One could also generalize the algorithm for labelled Dyck paths, relating to the Delta conjecture, and get a partial decomposition in Schur functions in the variables q and t indexed by partitions of length one. Using Corollary 5 it should be possible to decompose $\nabla^m(e_n)$ into the basis $s_\mu(q,t)s_\lambda(X)$. The following problems could lead to finding a partial decomposition $\nabla^m(e_n)$ into Schur functions in the X, $s_\lambda(X)$, when λ is not a hook. Which is a known hard problem.

Problem 2 : Using Lemma 13 decompose $\nabla^m(e_n)$ into the basis $s_\mu(q,t)F_c(X)$, for μ a hook and c a composition.

Even if the decomposition is in fundamental quasisymmetric functions, it would help get a partial decomposition $\nabla^m(e_n)$ into Schur functions, since it should be easier to regroup the fundamental quasisymmetric functions into Schur functions because there will be fewer coefficients. Extending, the maps in this paper from m-Schröder paths to tableaux with the multiplicity of the descent set, would help decompose completely the Schröder paths into crystals. Of course the extended map must somewhat preserve the area and diagonal inversion statistic or area and bounce statistic through the Major index and the number of descents. Another related problem is :

Problem 3 : Find a bijection between parking functions (γ, w), in an $n \times n$, grid, having diagonal inversion statistic value of 1 and Standard Young tableaux, τ, with a multiplicity related to the Major index of the tableau $(\mathrm{maj}(\tau, i) - i - 1)$ such that $\mathrm{maj}(\tau, i) - i = \mathrm{area}(\mathcal{B}(\tau, i))$.

Using the zeta map, we already have a bijection for such $n \times n$ parking functions (γ, w) such that $\text{read}(\gamma, w) \in \{n-d+1, \ldots, n\} \sqcup \{n-d, \ldots, 1\}$. The idea here is to "extend" that bijection, with multiplicity. We need the multiplicity because there are more than just the parking functions (γ, w) such that $\text{read}(\gamma, w) \in \{n-d+1, \ldots, n\} \sqcup \{n-d, \ldots, 1\}$, that contribute to $\nabla(e_n)$, seen as a sum of parking functions. The said paths are merely representatives. If the solution to Problem 3 is indeed an extension of $\zeta \circ \mathcal{Q}_{n,d}$ this would solve Conjecture 1.

As mentioned in Section 2.3.9 the insight coming from multivariate diagonal harmonics foresees a solution to the problem :

Problem 4 : Find a general map $\rho_{n,d}^{(i)}$ that partitions $\widetilde{\text{Sch}}_{n,d,(i)}$.

This generalization would lead to more results on combinatorial formulas for multivariate diagonal harmonics, like the one in (Wallace, 2019b) (Chapter 3). Actually, any explicit decomposition in terms of Schur functions of parking functions could be lifted with the tools discussed in (Wallace, 2019a) (the long version of (Wallace, 2019b), Chapter 3) and give an explicit combinatorial formula for a partial Schur decomposition of the multivariate diagonal harmonics. Finally, Proposition 5 suggests a bijection between permutations and tableaux with a multiplicity such that $\binom{n}{2} - \text{des}(w)n + \text{maj}(w) = \text{maj}(\tau)$. Research on this last problem could lead to a decomposition of $\nabla^m(e_n)$ altogether. Since it further our knowledge of how fundamental quasi-symmetric functions index by permutations relate to Schur functions.

Problem 5 : Find a combinatorial proof of Proposition 5.

2.3.12 Acknowledgments

Thank you to my advisor, François Bergeron, for introducing me to this beautiful subject and other helpful comments.

CHAPITRE III

BICARACTÈRES DES ESPACES DIAGONAUX HARMONIQUES MULTIVARIÉS

3.1 Résumé de l'article

L'article présenté dans ce chapitre, soumis à «Algebraic Combinatorics» et dont la version courte est publiée dans «Séminaire Lotharingien de Combinatoire : Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)», discute de $GL_k \times \mathbb{S}_n$-modules, dénoté $\mathcal{E}_{n,m}^{\langle k \rangle}$. Lorsque $k = 2$ et $m = n$ ce sont les espaces diagonaux harmoniques du chapitre précédent. Ceux-ci ont été généralisés au cas $k = 3$ dans (Bergeron et Préville-Ratelle, 2012), pour k quelconque et $m = n$ dans (Bergeron, 2013), puis pour k et m générale dans (Bergeron, 2020). Les espaces diagonaux harmoniques multivariés peuvent s'exprimer par des $GL_\infty \times \mathbb{S}_n$-modules dénotés $\mathcal{E}_{n,m}$. Par abus de notation, les bicaractères seront, également, dénotés $\mathcal{E}_{n,m}$ (plus de détails dans la Section 4.3.6).

Dans cette publication on établit des formules combinatoires pour certains bicaractères irréductibles de ces espaces. Pour ce faire, on introduit un nouvel objet combinatoire $T_{n,s}$ qui permet de transformer une somme alternée de polynômes respectant certaines conditions en une somme positive.

On définit $T_{n,s}$ par l'ensemble des chemins dans une grille en forme escalier de taille $n-2$ constitués de pas nord et de pas est qui se terminent à l'un des points dont les coordonnées figurent dans l'ensemble $\{(x, y) \mid x + y = n-2, x \geq 0 \text{ and } y \geq s\}$ et qui débutent à $(0, s)$. Ces chemins peuvent être représentés par des mots de longueurs $n-2-s$ dans l'alphabet $\{N, E\}$, où N représente un pas nord et E représente un pas est. Lorsque $s > n-1$ on pose $s = n-2$, auquel cas $T_{n,s}$

contient seulement le mot vide. Pour un chemin γ dans $T_{n,s}$, l'*aire*, noté $\mathrm{aire}(\gamma)$, est le nombre de carrés qui se situent au sud et à l'est du chemin. La *hauteur*, noté $\mathrm{ht}(\gamma)$, est la coordonnée en y du point où se termine le chemin. Pour un exemple, voir Figures 3.1, Figure 3.2 et Figure 3.3.

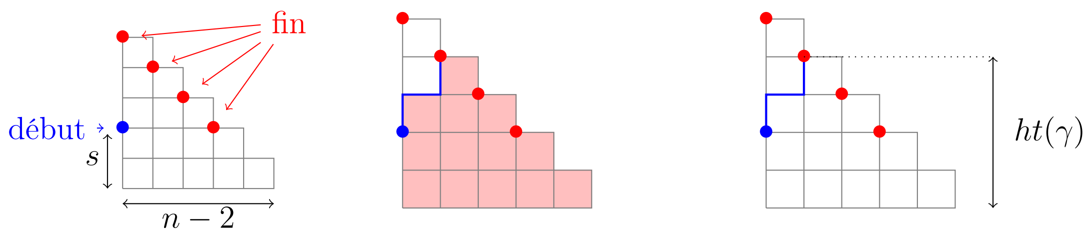


Figure 3.1 $T_{7,2}$ Figure 3.2 $\mathrm{aire}(NEN) = 13$ Figure 3.3 $\mathrm{ht}(NEN) = 4$

La fonction génératrice de cet objet est noté $T_{n,s}(q,t) = \sum_{\gamma \in T_{n,s}} q^{\mathrm{aire}(\gamma)} t^{\mathrm{ht}(\gamma)}$ et est simplifiée par l'expression algébrique suivante (Proposition 11).

$$T_{n,s}(q,t) = \sum_{j=0}^{n-s-2} q^{\binom{s+j+1}{2} + s(n-s-2-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q t^{j+s} = T_{n-s,0}(q,z) t^s q^{(n-s-2)s + \binom{s+1}{2}}, \quad .$$

En particulier, si $s = 0$, alors :

$$T_{n,0}(q,t) = \sum_{j=0}^{n-2} q^{\binom{j+1}{2}} \begin{bmatrix} n-2 \\ j \end{bmatrix}_q t^j = \sum_{k=0}^{n-2} q^{\binom{n-k-1}{2}} \begin{bmatrix} n-2 \\ n-2-k \end{bmatrix}_q t^{n-2-k} = \prod_{i=1}^{n-2} (1 + q^i t)$$

On remarque qu'avec notre convention $T_{n,s}(q,t) = q^{\binom{n-1}{2}} t^{n-2}$, lorsque $s \geq n-2$.

En outre, on découvre, dans la Proposition 12, des conditions pour lesquelles le principe d'inclusion exclusion donne une somme positive. Étant donné une famille d'applications, $\{g_j\}$, telles que $g_j : \mathbb{N} \to \mathbb{Z}$ et $g_j(k) - g_j(k-1) = k + j$ pour tout $k \geq 1$. On a :

$$\sum_{k=0}^{n-j-1} (-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)} = \begin{bmatrix} n-2 \\ j-1 \end{bmatrix}_q q^{g_j(0)},$$

De plus, si $g_j(0) - \binom{j}{2} = g_i(0) - \binom{i}{2}$ pour tout i et j, alors pour j quelconque on trouve :

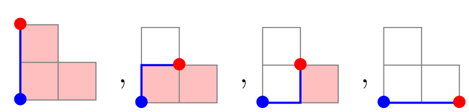
$$\sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}(-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)}t^{j-1} = T_{n,0}(q,t)q^{g_j(0)-\binom{j}{2}},$$

et si $g_j(0)-\binom{j+1}{2} = g_i(0)-\binom{i+1}{2}$ pour tout i et j, alors pour j quelconque on obtient :

$$\sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}(-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)}t^{j-1} = T_{n,0}(q,qt)q^{g_j(0)-\binom{j+1}{2}+1}.$$

Le théorème principal de cet article est une expression algébrique exprimant explicitement les polynômes qui représente certains bicaractères irréductibles des espaces harmoniques diagonaux multivariés. Par exemple, les caractères irréductibles de GL_∞ associées au caractère irréductible $e_4(X) = s_{1^4}(X)$ pour l'action de $\mathbb{S}_4$ dans le caractère $\mathcal{E}_{4,4}$, dont les fonction de Schur, dans les variables $Q = \{q_1, q_2, \ldots\}$ sont indicé par des équerres sont représentés dans la Figure 3.4. Il est également à noter qu'il s'agit d'une conjecture lorsque les partages μ ne sont pas de la forme proposée dans le théorème.

Élements de $T_{4,0}$:



aire(γ) :	3	2	1	0
ht(γ) :	2	1	1	0
équerre(γ) :	$6, 1^0$	$4, 1^1$	$3, 1^1$	$1, 1^2$
$s_{\text{équerre}(\gamma)}$:	s_6	s_{41}	s_{31}	s_{111}

$$\langle \mathcal{E}_{4,4}, e_4\rangle|_{\text{équerres}} = s_6 + s_{41} + s_{31} + s_{111}$$

Figure 3.4 Exemple pour $\langle \mathcal{E}_{4,4}, e_4\rangle|_{\text{équerres}}$ pour $r = 1$ et $n = 4$

Théorème 2 : Si $r = 1$ et $\mu \in \{(n), (n-1,1), (n-2,1,1)\}$ alors :

$$\langle \mathcal{E}_{rn,n}, s_\mu\rangle|_{\text{équerres}} = \sum_{\tau\in \mathrm{SYT}(\mu)} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\text{équerre}(\gamma)}, \tag{3.1}$$

où la première somme est sur tous les tableaux de Young standard, τ, de forme μ, la deuxième somme est sur les chemins γ dans $T_{n,\mathrm{des}(\tau')}$, enfin on définit équerre(γ) par le partage de la forme $\left((r-1)\binom{n}{2}+\mathrm{aire}(\gamma)+\mathrm{ht}(\gamma)-\mathrm{maj}(\tau')+1,1^{n-2-\mathrm{ht}(\gamma)}\right)$.

En outre, lorsque $\mu=1^n$, l'Équation (3.2) est valide pour tout entier r pour lequel l'égalité $\langle\mathcal{E}_{rn,n},s_{k+1,1^{n-k-1}}\rangle=e_k^{\perp}\langle\mathcal{E}_{rn,n},s_{1^n}\rangle$ est vérifiée pour tout k.

De plus, si la restriction à deux variables de l'expression $e_k^{\perp}\langle\mathcal{E}_{n,n},s_\mu\rangle$ est égale au produit scalaire $\langle\Delta'_{e_{n-k-1}}e_n,s_\mu\rangle$ pour tout k, alors l'Équation (3.2) est vrai pour tout μ.

Afin d'obtenir ce résultat, un outil plus général à été développé dans le Lemme 21 et le Lemme 22. Étant donné une fonction symétrique inconnue $G(Q)$, pour $Q=\{q_1,q_2,\ldots\}$, le coefficient de $s_\mu(Q)$ dans $e_i(Q)^{\perp}G(Q)$ représente la multiplicité du caractère $e_i(Q)s_\mu(Q)$ dans le caractère encodé par $G(Q)$, étant donné l'égalité $\langle G,e_is_\mu\rangle=\langle e_i^{\perp}G,s_\mu\rangle$. Si, pour tout i et pour tout μ, l'on connaît la multiplicité de tous les caractères de la forme e_is_μ dans G, il est possible de retrouver G.

Du point de vue combinatoire $e_i^{\perp}s_\mu$ donne la somme des fonctions de Schur s_λ, pour λ obtenu de μ en enlevant i cases dans des lignes différentes. Ainsi si l'on se restreint à l'ensemble des partages de la forme (a,b) où b est fixé (a et k sont quelconques), que l'on note $\mathcal{V}_b$, il est possible de trouver $G(Q)|_{\mathcal{V}_{b+1}}$. C'est-à-dire les composantes irréductibles de $G(Q)$ qui sont indicés par un partage de la forme $(a,b+1,1^k)$.

D'autre part, on trouve dans cet article une méthode pour appliquer $e_k^{\perp}$ directement sur les chemins. En d'autres termes, pour γ dans $T_{n,0}$ qui contribue $s_{\text{équerre}(\gamma)}$ à la décomposition de $\langle\mathcal{E}_{n,n},e_n\rangle|_{\text{équerres}}$ on peut associer $\underline{e_{k+}}^{\perp}(\gamma)$ et $\underline{e_{k-}}^{\perp}(\gamma)$ tels que $e_k^{\perp}s_{\text{équerre}(\gamma)}=s_{\text{équerre}(\underline{e_{k+}}^{\perp}(\gamma))}+s_{\text{équerre}(\underline{e_{k-}}^{\perp}(\gamma))}$.

On rappelle que équerre$(\gamma)=(\mathrm{aire}(\gamma)+\mathrm{ht}(\gamma)-\mathrm{maj}(\tau')+1,1^{n-2-\mathrm{ht}(\gamma)})$, pour $r=1$. L'unique tableau de forme (n) n'a pas de descente, donc $\mathrm{maj}((1^n)')=0$. Ainsi, on considère T_n^k l'ensemble des chemins de $T_{n,0}$ dont $n-2-\mathrm{ht}(\gamma)$ est plus grand à k et pour le chemin γ dans $T_{n,0}$, la suite $\{p_1,p_2,\ldots,p_k\}$ est tel que p_i est le nombre de pas nord avant le i-ème pas est. Ainsi, on défini $\underline{e_{k+}}:T_n^k\to\cup_{\tau\in\mathrm{SYT}(k+1,1^{n-k-1})}\times T_{n,k}$

par le tableau uniquement déterminé par $\mathrm{Des}(\tau') = \{n - i - p_i \mid 1 \leq i \leq k\}$ et le chemin de $T_{n,k}$ dont on a enlevé les k premiers pas est. Similairement, on défini $\underline{e_{k-}} : T_n^{k-1} \backslash \{E^{n-2}\} \to \cup_{\tau \in \mathrm{SYT}(k+1, 1^{n-k-1})} \times T_{n,k}$ par le chemin de $T_{n,k}$ dont on a enlevé les $k-1$ premiers pas est et le premier pas nord et le tableau uniquement déterminé par $\mathrm{Des}(\tau') = \{n - i - p_i \mid 1 \leq i \leq k-1\} \cup \{\max(1, h-k+2)\}$, où h est le nombre de pas est avant le premier pas nord.

Cette application se comprend bien visuellement. Pour $\gamma = NENEENEE$ dans $T_{10,0}$ la Figure 3.5 est un exemple de $\underline{e_{2+}}^{\perp}(\gamma)$ et la Figure 3.6 un exemple de $\underline{e_{2-}}^{\perp}(\gamma)$. La Figure 3.7 permet de visualiser l'application $\underline{e_{2-}}$ dans le cas particulier où $h - k + 2 > 1$.

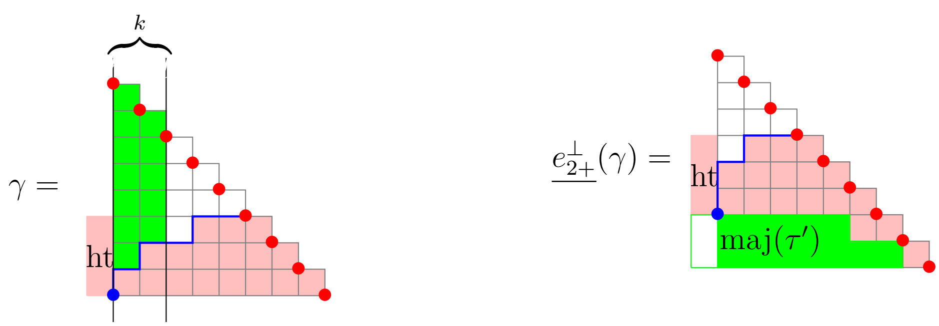


Figure 3.5 L'application $\underline{e_{2+}^{\perp}}$ envoie le chemin $\gamma = NENEENEE$ sur le chemin $NNENEE \in T_{10,\mathrm{des}(\tau')}$, avec $\mathrm{Des}(\tau') = \{6, 8\}$.

.

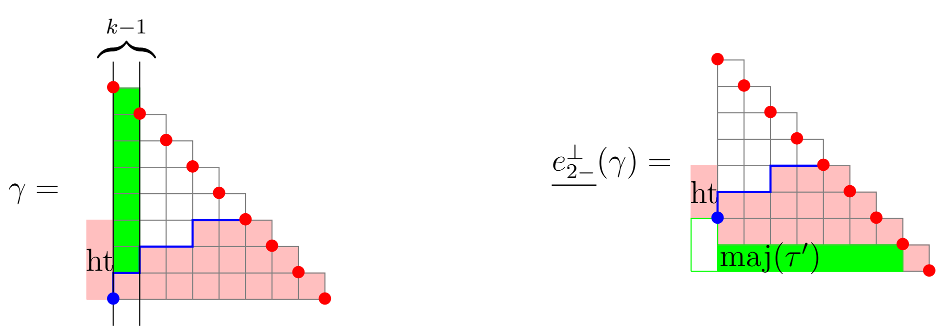


Figure 3.6 L'application $\underline{e_{2-}^{\perp}}$ envoie le chemin $\gamma = NENEENEE$ sur le chemin $NEENEE \in T_{10,\mathrm{des}(\tau')}$ et le tableau $\mathrm{Des}(\tau') = \{1, 8\}$.

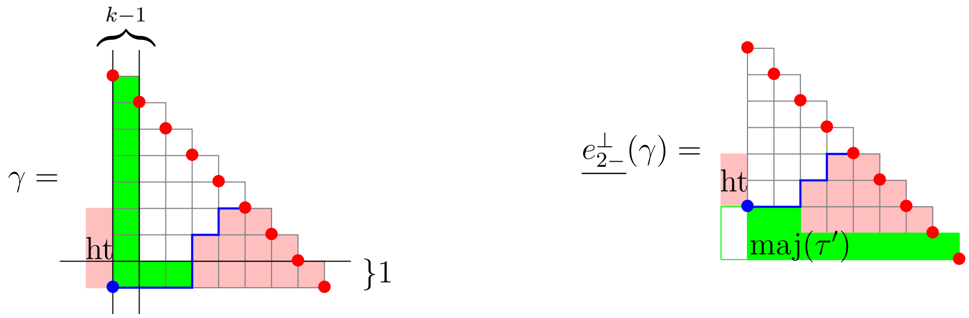


Figure 3.7 L'application $\underline{e_{2-}^{\perp}}$ envoie le chemin $\gamma = EEENNENE$ sur le chemin $EENENE \in T_{10,\mathrm{des}(\tau')}$ et le tableau $\mathrm{Des}(\tau') = \{3, 9\}$.

Ainsi la Proposition 15 établit l'expression algébrique suivante :

$$e_k^{\perp}\left(\langle \mathcal{E}_{n,n}, e_n \rangle|_{\text{équerres}}\right) = \sum_{\gamma \in T_n} s_{\text{équerre}(\underline{e_{k+}^{\perp}}(\gamma))} + s_{\text{équerre}(\underline{e_{k-}^{\perp}}(\gamma))}.$$

De plus, la différence $\sum_{\tau \in \mathrm{SYT}(k,1^{n-k})} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\text{équerre}(\gamma)} - e_{k-1}^{\perp}\left(\langle \mathcal{E}_{n,n}, s_{1^n} \rangle|_{\text{équerres}}\right)$ est Schur positive et est donné explicitement par le Lemme 30. Si l'on considère la conjecture associée au théorème principal, cette proposition suggère une borne inférieure pour les coefficients de $\langle \mathcal{E}_{n,n}, s_\mu \rangle$ lorsque μ est une équerre. Enfin, l'article se termine avec une formule qui commence la deuxième colonne.

3.2 Relations entre cet article et les autres articles présentés

Dans cette publication j'énonce une formule utilisant le principe d'inclusion exclusion, pour relever les bicaractères des espaces diagonaux harmoniques en bicaractère des espaces diagonaux harmonique multivarié. Cette formule donne des bicaractères qui ne pourraient pas être obtenus par un simple plongement. Ensuite, j'utilise cette formule et les résultats obtenus dans «Toward a Schurification of Parking Function Formulas via bijections with Young Tableaux» pour présenter une décomposition partielle en bicaractère des espaces diagonaux multivarié.

On termine avec le chapitre 4 où je discuterais d'un troisième article qui explore les liens entre l'objet introduit au chapitre 3 et des objets classiques de combinatoires (compositions, partages, permutations qui évitent certains motifs). Ce nouvel objet raffine certaines statistiques de la combinatoire classique ou encore permet d'obtenir plus facilement ces statistiques. On termine avec un lien surprenant qui permet d'écrire la formule du chapitre 3 à l'aide de deux chemins nord-est dans une grille en forme d'escalier.

3.3 Explicit combinatorial formulas for some irreducible characters of the $GL_k \times \mathbb{S}_n$-module of multivariate diagonal harmonics

3.3.1 Abstract

We give an explicit combinatorial formula for some irreducible components of $GL_k \times \mathbb{S}_n$-modules of multivariate diagonal harmonics. To this end we introduce a new path combinatorial object $T_{n,s}$ allowing us to give the formula directly in terms of Schur functions. This paper also contains formulas written in terms of Schur functions in the q and t variables for special cases of $\nabla(e_n)$, $\nabla^r(e_n)$ and $\Delta'_{e_k}(e_n)$. We also give an interpretation in terms of paths to the adjoint dual Pieri rule applied on these $GL_k \times \mathbb{S}_n$-characters.

3.3.2 Introduction

The aim of his paper is to describe some features of the characters of "rectangular" $GL_k \times \mathbb{S}_n$-modules, $\mathcal{E}^{\langle k \rangle}_{m,n}$, introduced by F. Bergeron in (Bergeron, 2020). When $k = 2$ and $m = n$, these modules are the modules of diagonal harmonics whose characters have been studied for many years. As shown in (Garsia et Haiman, 1996) and (Haiman, 2002), the Frobenius transformation of its graded characters may be expressed as $\nabla(e_n)$, where ∇ is the Macdonald eigenoperator introduced in (Bergeron et Garsia, 1999) and recalled in Section 3.3.4 and e_n is the n-th elementary symmetric function. A combinatorial interpretation that became known as the Shuffle Conjecture was introduced in (Haglund *et al.*, 2005) and proved recently by Carlson and Mellit (Carlsson et Mellit, 2018), (Mellit, 2016). These characters also intervene in the torus knot link homology and algebraic geometry ; see, for instance (Hogancamp, 2017), (Khovanov, 2007),(Gorsky *et al.*, 2016) and (Oblomkov et Rozansky, 2018). The case $k = 3$ was studied in (Bergeron et Préville-Ratelle, 2012). In recent work (Bergeron, 2020) F. Bergeron made a breakthrough in the multivariate case (k arbitrary). He found interesting relations between various irreducible characters of the modules. This allowed the study of the character of $\mathcal{E}^{\langle k \rangle}_{n,n}$ developed in the elementary symmetric functions for $n \leq 4$ in (Bergeron *et al.*, 2018) (in our notation this is the specialization $\sum c_{\lambda,\mu} s_\lambda(q,1,\ldots,1,0,0,\ldots)s_\mu(X)$, where there are $k-1$, 1's). These relations are also exploited here to obtain our main result. To state our result we briefly fix the required notation. We encode the characters of the irreducible $GL_k \times \mathbb{S}_n$ -modules as products of Schur functions, $s_\lambda(Q)s_\mu(X)$ (see Section 3.3.4 for details). For easier reading we will also write $s_\lambda \otimes s_\mu$ for $s_\lambda(Q)s_\mu(X)$. As shown in (Bergeron, 2013) for the case $m = n$, there is a stability property that makes it possible to avoid mentioning k. Therefore, the character of $\mathcal{E}_{m,n}^{\langle k\rangle}$ can be expressed in the form $\mathcal{E}_{m,n} = \sum_\mu (\sum_\lambda c_{\lambda,\mu} s_\lambda) \otimes s_\mu$. Our aim is to describe some features of $\mathcal{E}_{m,n}$, or equivalently $\langle \mathcal{E}_{m,n}, s_\mu\rangle = \sum_\lambda c_{\lambda,\mu} s_\lambda$. The main result of this paper is a combinatorial description of the multiplicity of $s_\lambda \otimes s_\mu$ in $\mathcal{E}_{n,n}$ when λ is hook-shaped. The result is constructive, in that the hooks are determined combinatorially by a standard Young tableau and certain paths in a staircase shaped grid. More precisely, we give the following combinatorial description (the basic combinatorial notations used in Theorem 2 are recalled in Section 3.3.3).

Theorem 2 : If $r = 1$ and $\mu \in \{(n),(n-1,1),(n-2,1,1)\}$ then :

$$\langle \mathcal{E}_{rn,n}, s_\mu\rangle|_{\text{hooks}} = \sum_{\tau\in\mathrm{SYT}(\mu)} \sum_{\gamma\in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)}, \tag{3.2}$$

where the first sum is over all standard Young tableaux, τ, of shape μ, the second sum is over paths γ in $T_{n,\mathrm{des}(\tau')}$ and $\mathrm{hook}(\gamma) = ((r-1)\binom{n}{2} + \mathrm{area}(\gamma) + \mathrm{ht}(\gamma) - \mathrm{maj}(\tau') + 1, 1^{n-2-\mathrm{ht}(\gamma)})$.

Furthermore, when $\mu = 1^n$, Equation (3.2) holds for all positive integers r that satisfy the equality $\langle \mathcal{E}_{rn,n}, s_{k+1,1^{n-k-1}}\rangle = e_k^\perp \langle \mathcal{E}_{rn,n}, s_{1^n}\rangle$ for all k.

Moreover, if $e_k^\perp\langle \mathcal{E}_{n,n}, s_\mu\rangle =_{\langle 2\rangle} \langle \Delta'_{e_{n-k-1}} e_n, s_\mu\rangle$ is true for all k then Equation (3.2) holds for all μ.

In addition, Proposition 15 gives a lower bound for the coefficients, which gives reason to believe that Equation (3.2) works for all μ when $r = 1$. Moreover, in (Bergeron, 2020) it is conjectured that $e_k^\perp\langle \mathcal{E}_{n,n}, s_\mu\rangle =_{\langle 2\rangle} \langle \Delta'_{e_{n-k-1}} e_n, s_\mu\rangle$ is true for all k. The notation $=_{\langle 2\rangle}$ denotes the restriction $Q = \{q,t\}$. If this conjecture is true then Equation (3.2) holds for all μ when $r = 1$. The combinatorial object $T_{n,s}$ ($T_{n,\mathrm{des}(\tau')}$ in Equation (3.2)) represents a set of paths, these paths and their statistics (area and ht) will be defined in Section 3.3.6. This object was introduced by the author in order to eliminate alternating sums and obtain a Schur positive expression. When $s = 0$, they afford a generating function correlated to the q-Pochhammer symbol $(-qz; z)_n$. The notation $|_{\text{hooks}}$, symmetric functions s_λ, e_n, the operators ∇, Δ' and $e_k^\perp$ will be recollected in Section 3.3.4. Section 3.3.7 will be dedicated to Equation (3.2), and a proposition restricting Theorem 2 to the $GL_2 \times \mathbb{S}_n$-characters mentioned above, which gives formulas in terms of the Major index for $\langle \nabla^r(e_n), s_\mu \rangle|_{\text{hooks}}$, $\langle \Delta'_{e_k}(e_n), e_n \rangle|_{\text{hooks}}$ and $\langle \Delta'_{e_k}(e_n), s_\mu \rangle|_{\text{some hooks}}$. In Section 3.4 we show how the adjoint dual Pieri rule can be applied directly on our paths. Finally, Section 3.4.1 gives a formula similar to Equation (3.2) for shapes $\{(a, 2, 1^k) \mid k \in \mathbb{N}, a \in \mathbb{N}_{\geq 2}\}$.

3.3.3 Combinatorial Tools

In this section the notions are classical, they are recalled here only to set notations. A *partition* of n is a decreasing sequence of positive integers often represented as a Ferrers diagram (see Figure 4.10). Each number in the sequence is called a *part* and if it has k parts it is of length k denoted $\ell(\lambda) = k$. We say λ is of *size* n if $\lambda = \lambda_1, \dots, \lambda_k$ and $n = \sum_i \lambda_i$. For λ a *Ferrers diagram* of shape $\lambda = \lambda_1, \lambda_2, \dots, \lambda_k$ is a left justified pile of boxes having λ_i boxes in the i-th row. We will use the French notation so the second row lies on top of the first row (see Figure 4.10). We can see them as a subset of $\mathbb{N} \times \mathbb{N}$ if we put the bottom left corner of the diagram to the origin. In this setting, we can associate the bottom left corner of a box to the coordinate it lies on. We say a partition is *hook-shaped* if it has the shape $(a, 1, \dots, 1) = (a, 1^k)$, where $a, k \in \mathbb{N}$. If to each box of a diagram we associate an entry, it is called a *tableau.* A tableau is of shape λ if it is a filling by integers of a diagram of shape λ. For λ a partition of n, a tableau of shape λ with distinct entries from 1 to n strictly increasing in rows and columns is a *standard Young tableau.* It is a *semi-standard Young tableau* if the row entries are weakly increasing and the column entries are strictly increasing. The set of all standard Young tableaux of shape μ is denoted $\mathrm{SYT}(\mu)$ and the set of all semi-standard Young tableaux of shape μ is denoted $\mathrm{SST}(\mu)$. The *descent* set of a tableau is the set of entries i such that the entry $i+1$ lies in a row strictly above i. For a tableau τ, the descent set is denoted $\mathrm{Des}(\tau)$ and the cardinality is denoted $\mathrm{des}(\tau)$. The sum of the elements of $\mathrm{Des}(\tau)$ is called the *Major index* and is denoted $\mathrm{maj}(\tau)$ (see Figure 3.10). The *conjugate* of a partition λ, (respectively a diagram λ, a tableau τ) is denoted λ' (respectively a diagram λ', a tableau τ'), and is its reflection through the line $x = y$ (see Figure 3.9). We end this section

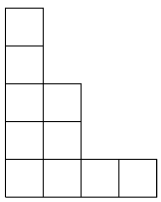


Figure 3.8 $\lambda = 42211$

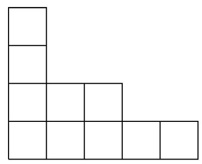


Figure 3.9 $\lambda' = 5311$

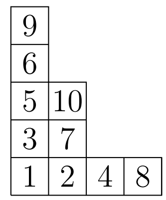


Figure 3.10 $\tau \in \mathrm{SYT}(42211)$; $\mathrm{des}(\tau) = 4$, $\mathrm{maj}(\tau) = 19$ $\mathrm{Des}(\tau) = \{2, 4, 5, 8\}$,

by recalling the definition of the Gaussian polynomials, $\begin{bmatrix} n \\ k \end{bmatrix}_q$. Let :

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}\ ,\ \ [n]!_q = \prod_{i=1}^{n} [i]_q \text{ and } \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[n-k]!_q [k]!_q}.$$

So when $q = 1$ this gives the usual binomial coefficient. It is well known that the Gaussian polynomials are related to the northeast paths of a $k \times (n-k)$ grid : if $\mathcal{C}_k^n$ denotes the set of such paths, and the *area* of a path γ, denoted $\mathrm{area}(\gamma)$, is defined as the number of boxes beneath the path, then $\sum_{\gamma \in \mathcal{C}_k^n} q^{area(\gamma)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

3.3.4 The Space, the Characters, Symmetric Functions and Macdonald Operators

The symmetric function notations are the one used in Macdonald's book (Macdonald, 1995). The *ring of symmetric polynomials* is a set of polynomials which are invariant by permutation of the variables $Y = \{y_1, y_2, \ldots, y_n\}$. The ring of symmetric polynomials is embedded in $\mathbb{Q}[Y]$. In other words, if f is a symmetric polynomial for all $\sigma \in \mathbb{S}_n$, then we have :

$$f(y_1, y_2, \ldots, y_n) = f(y_{\sigma^{-1}(1)}, y_{\sigma^{-1}(2)}, \ldots, y_{\sigma^{-1}(n)}).$$

The *ring of symmetric functions*, denoted Λ, can be thought of as the ring of symmetric polynomials in an infinite set of variables. It is a graded ring and has the *elementary symmetric functions* as a basis. These are defined by $e_n(X) = \sum_{i_1<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$ and $e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k}$. Schur functions also form a basis for the ring of symmetric functions. We now recall the combinatorial definition of Schur functions. For a tableau τ we define $x_\tau := \prod_{c\in\tau} x_c$. The Schur functions are then defined by $s_\lambda = \sum_{\tau\in\mathrm{SST}(\lambda)} x_\tau$. For example, if $x_1 = x$, $x_2 = y$ and $x_3 = z$, we have :

$$\begin{aligned} s_{21}(x,y,z,\ldots) &= \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array} + \cdots \\ &= x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + 2xyz + \cdots \end{aligned}$$

Note that the columns are strictly increasing therefore we need a number of variables greater or equal to the length of the partition.

The ring of symmetric functions in the variables $Q = \{q_1, q_2, \ldots\}$ will be denoted Λ_Q, the ring of symmetric functions in the variables $x_1, x_2, \ldots$ will be denoted Λ. The product of a symmetric function f in Λ_Q and a symmetric function g in Λ will be noted $f \otimes g$. The set of such functions will be noted $\Lambda_Q \otimes \Lambda$. It is easy to see that it is a bigraded ring. If an element of Λ (respectively Λ_Q, $\Lambda_Q \otimes \Lambda$) can be written in the basis of Schur function (respectively the basis $\{s_\lambda\}$, the basis $\{s_\lambda \otimes s_\mu\}$) with coefficients in $\mathbb{N}$ or $\mathbb{N}[q,t]$ (respectively $\mathbb{N}$, $\mathbb{N}$) is said to be Schur positive. The ω linear operator is defined by $\omega(s_\mu(X)) = s_{\mu'(X)}$ which extends to $\Lambda_Q \otimes \Lambda$ by $\omega(s_\lambda \otimes s_\mu) = s_\lambda \otimes s_{\mu'}$.

If $\sum_\lambda c_\lambda s_\lambda$ is a linear combination of Schur functions, and $\mathcal{V}$ is a set of partitions, then we define the notation $|_\mathcal{V}$ by $\sum_\lambda c_\lambda s_\lambda|_\mathcal{V} = \sum_{\lambda\in\mathcal{V}} c_\lambda s_\lambda$. On a symmetric function in the Schur basis, we set the restriction $|_{1\mathrm{Part}}$ (respectively $|_{\mathrm{hooks}}$) to be the partial sum over the Schur functions indexed by partitions having only one part (respectively that are hook-shaped). For example, if $\mathcal{V} = \{111, 32, 6\}$ and

$f = 5s_{111} + 5s_{31} + s_{41} + 2q^6 s_6$ then :

$$f|_{\mathcal{V}} = 5s_{111} + 2q^6 s_6.$$

Before introducing our new combinatorial objects, we provide more details about the modules $\mathcal{E}_{n,n}^{\langle k \rangle}$ and why they are interesting. Let $X = (x_{i,j})_{i,j}$ where $1 \leq i \leq k$, $1 \leq j \leq n$ and let $R_n^{\langle k \rangle} = \mathbb{Q}[X]$ denote the polynomial ring in the variables X. For (τ, σ) in $GL_k \times \mathbb{S}_n$ the group $GL_k \times \mathbb{S}_n$ acts on $R_n^{\langle k \rangle}$ as follows :

$$(\tau, \sigma).F(X) = F(\tau \cdot X \cdot \sigma)$$

With this action we can define $\mathcal{E}_{n,n}^{\langle k \rangle}$ as the smallest submodule of $R_n^{\langle k \rangle}$ that contains the Vandermonde determinant, is closed under all higher polarization operators $\sum_{j=0}^{n} x_{r,j} \partial_{x_{s,j}}$ and is closed under all partial derivatives $\partial_{x_{s,j}}$. As usual we may decompose these modules into a direct sum of irreducible modules. A module can be encoded by its character. Recall that both the irreducible characters for GL_k and the Frobenius transforms of irreducible characters of $\mathbb{S}_n$ are Schur functions. For example, when $n = 4$ we have :

$$\begin{aligned}
\mathcal{E}_{4,4} =& 1 \otimes s_4 + (s_1 + s_2 + s_3) \otimes s_{31} + (s_2 + s_{21} + s_4) \otimes s_{22} \\
&+ (s_{11} + s_{21} + s_{31} + s_3 + s_4 + s_5) \otimes s_{211} + (s_{111} + s_{31} + s_{41} + s_6) \otimes s_{1111}
\end{aligned}$$

If we start with the ring $\mathbb{Q}(q,t)[X]$, we also get a ring with Schur functions as a basis. Additionally, the combinatorial Macdonald polynomials, denoted $\tilde{H}_\mu$, are a basis for Λ with coefficients in $\mathbb{Q}(q,t)$. They appear as eigenvectors of special operators (see (Bergeron, 2009) for more on this), ∇ and the Δ'_{e_m}, introduced in (Bergeron et Garsia, 1999), (Bergeron *et al.*, 1999). These Macdonald operators are defined as follows :

$$\nabla(\tilde{H}_\mu) = \prod_{(i,j)\in\mu} q^i t^j \tilde{H}_\mu \quad \text{and} \quad \Delta'_{e_m}(\tilde{H}_\mu) = e_m \left[\sum_{(i,j)\in\mu} q^i t^j - 1 \right] \tilde{H}_\mu.$$

The brackets are for plethysm. The notion of plethysm is not needed in this paper, but the curious reader could learn more on this in (Bergeron, 2009). The bivariate diagonal harmonic space was proven to have $\nabla(e_n(X))$ as a character in (Haiman, 2002). Ergo $\mathcal{E}_{n,n}$ affords the following specialization :

$$\mathcal{E}_{n,n}^{\langle 2\rangle} = \nabla(e_n(X)).$$

One might notice from our previous example that if we take out the term $s_{111} \otimes s_{1111}$, or equivalently, set $q_1 = q$, $q_2 = t$, $q_j = 0$ for $j \geq 3$ we have $\nabla(e_4)$. That extra term isn't a problem since $s_{111}(q, t, 0, 0, \ldots) = 0$. As noted beforehand, in two variables Schur functions vanish if they have more then two parts. By definition we have $\Delta'_{e_{n-1}}(e_n) = \nabla(e_n)$, which gives the character decomposition of the $GL_2 \times \mathbb{S}_n$ case stated in the introduction. It was proven that the coefficients are symmetric polynomials in the q and t variables ; thus one could write the coefficients as $\sum_{\lambda,\mu} c_{\lambda,\mu} s_\lambda(q, t, 0, \ldots) s_\mu(X)$. More generally, the GL_k characters can be obtained by setting $q_{k+1} = q_{k+2} = \cdots = 0$. A stability property was proven in (Bergeron, 2013), we can therefore set k to infinity and use a more general notation $\mathcal{E}_{n,n}$. Moreover, F.Bergeron conjectures in (Bergeron, 2020) that the restriction to two variables of $e_k^\perp \langle \mathcal{E}_{n,n}, s_\mu\rangle$ is equal to $\langle \Delta'_{e_{n-k-1}} e_n, s_\mu\rangle$ for all k. If this conjecture is true, the Schur positive development of $\mathcal{E}_{n,n}$ (exists since it is a character) gives us the Schur positive development of $\Delta'_{e_k} e_n$. We will also discuss characters of the form $\mathcal{E}_{rn,n}$ which are related to $\nabla^r(e_n)$ when we restrict to GL_2. They are constructed by adding a set of inert variables considered to be of degree 0. For more details see (Bergeron, 2020). We will now extend the Hall scalar product to fit with our notation in the following way, for f, g in Λ_Q :

$$\langle f \otimes s_\lambda, g \otimes s_\mu\rangle = \begin{cases} fg & \text{if } \lambda = \mu \\ 0 & \text{if not} \end{cases}$$

We will sometimes write s_μ instead of $1 \otimes s_\mu$. Looking at our previous example, we can easily see that :

$$\langle \mathcal{E}_4, s_{1111}\rangle = s_{111} + s_{31} + s_{41} + s_6$$

Finally, we also need to recall that the dual Pieri rule describes the multiplication of a Schur function by e_k. The adjoint of the dual Pieri rule for the Hall scalar product, denoted $e_k^\perp$, is defined on the Schur basis and extended linearly. More precisely, $e_k^\perp s_\lambda$ is the sum $\sum_\mu s_\mu$ over all partitions μ obtained by deleting k boxes each lying in a different row (see Figure 3.11).

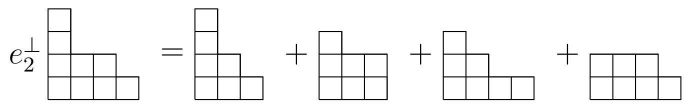


Figure 3.11 $e_2^\perp s_{4311} = s_{3211} + s_{331} + s_{421} + s_{43}$

3.3.5 Lifting to multivariate formulas

The following results give us a way to lift to an alternating sum. We will show later how to obtain positive sums from these.

Claim 4 : Let $G \in \Lambda_Q \otimes \Lambda$, $G = \sum_\mu \mathcal{B}_\mu \otimes s_\mu$, where $\mathcal{B}_\mu = \sum_\lambda b_{\lambda,\mu} s_\lambda(Q)$, $b_{\lambda,\mu} \in \mathbb{N}$. If λ is a partition of shape $(a, b, 1^k)$, with k and a arbitrary, and $b_{\lambda,\mu} \neq 0$ then the coefficient of $s_{(a,b-1)}(Q)$ in $e_{k+1}^\perp(\mathcal{B}_\mu)$ and the coefficient of $s_{(a-1,b-1)}(Q)$ in $e_{k+2}^\perp(\mathcal{B}_\mu)$ are not zero.

Proof. First notice that using the Pieri rule we have $e_{k+1}^\perp s_\lambda = s_{(a-1,b-1,1)} + s_{(a-1,b)} + s_{(a,b-1)}$ and $e_{k+2}^\perp s_\lambda = s_{(a-1,b-1)}$. Since the Pieri rule is linear and $c_{\lambda,\mu} \neq 0$, we must have non-zero coefficients as claimed. □

For the following lemma, we first consider the application $\psi : \Lambda \to \mathbb{Q}[q, t]$ which is defined on the Schur basis by $\psi(s_\lambda) = q^{\lambda_1} t^{\ell(\lambda)-1}$ and extended linearly. The following lemma allows us to lift $\mathcal{E}_{m,n}^{\langle 2 \rangle}$ to $\mathcal{E}_{m,n}$. Note that we cannot obtain all the coefficients of $\mathcal{E}_{m,n}$ this way, but we know which are left out. If we consider the Schur decomposition of $\langle \mathcal{E}_{m,n}, e_n \rangle$, the following lemma shows how to obtain all the coefficients of the Schur functions indexed by partitions of shape $(a, b+1, 1^j)$, with a and j arbitrary and b fixed.

Lemma 21 : Let $G \in \Lambda_Q$ be a symmetric function. If $f_i(q) = \psi((e_i^\perp G)|_{1\text{Part}})$, then :

$$\psi(G|_{\text{hooks}})(q,t) = \sum_{j\geq 0}\sum_{k=0}^{j}(-1)^k f_{j-k}(q)q^{-k}t^j$$

Proof. If λ is such that s_λ has a non-zero coefficient in $G|_{\text{hooks}}$ there exists a k such that $\lambda = (a, 1^k)$. Therefore :

$$e_i^\perp s_\lambda = s_{a,1^{k-i}} + s_{a-1,1^{k-i+1}}.$$

When $\mu = (b, 1^l)$ is not a partition we set $s_\mu = 0$. If one of these 3 conditions apply $b < 0$, $l < 0$ or $b = 0$ and $l \geq 1$ then μ is not a partition. The restriction to one part (or equivalently 1 variables) keeps only the term of $e_i^\perp s_\lambda$ for which $k - i + 1 = 0$ and $a > 1$ or $k - i = 0$. Moreover, if q^a as a non-zero coefficient in f_i then $s_{(a)}$ as a non-zero coefficient in $(e_i^\perp G)|_{1\text{Part}}$, by definition of $f_i(q)$. So by the previous claim q^a is associated to a λ that contributed q^{a+1} in f_{i-1} or q^{a-1} in f_{i+1}. Using the inclusion exclusion principle, the sum $\sum_{k=0}^{j}(-1)^k f_{j-k}(q)q^{-k}$ gives the monomials in f_j that are associated to a term s_λ in $G|_{\text{hooks}}$ which contributes q^{a-1} in f_{j+1}. Consequently λ must be of length $j+1$ in $G|_{\text{hooks}}$ and $\sum_{k=0}^{j}(-1)^k f_{j-k}(q)q^{-k}$ is the coefficient of t^j in $\psi(G|_{\text{hooks}})(q,t)$ has claimed. $\square$

This last lemma can be generalized as follows.

Lemma 22 : Let b be a constant in $\mathbb{N}^*$ and $G \in \Lambda_Q$ be a symmetric function in the variables $Q = \{q_1, q_2, \ldots\}$. Let $\mathcal{V}_b$ be the set of partitions of shape $(a, b, 1^k)$ with k and a arbitrary. If $f_0(q) = \psi(G|_{\mathcal{V}_b}^{\langle 2\rangle})t^{-1}$ and

$$f_i(q) = \psi((e_i^\perp G|_{\mathcal{V}_b})^{\langle 2\rangle}|_{\mathcal{V}_b})t^{-1},$$

then :

$$\psi(G|_{\mathcal{V}_{b+1}})(q,t) = \sum_{j\geq 1}\sum_{k=0}^{j}(-1)^k f_{j-k}(q)q^{-k}t^j$$

Proof. The difference is mainly that if λ is such that s_λ has a non-zero coefficient in $G|_{\mathcal{V}_{b+1}}$ there exists a k such that $\lambda = a, b+1, 1^k$. Therefore :

$$e_i^{\perp} s_\lambda = s_{a,b+1,1^{k-i}} + s_{a,b,1^{k-i+1}} + s_{a-1,b+1,1^{k-i+1}} + s_{a-1,b,1^{k-i+2}}$$

Since we have noticed before that the restriction to 2 variables is equivalent to the restrictions to the sum over Schur functions indexed by partitions of length at most 2. This means that we only keep the terms of $e_i^{\perp} s_\lambda$ such that $k+1 = i$ and $a \geq b$ or $k+2 = i$ and $a > b$. So :

$$(e_i^{\perp} s_\lambda)^{\langle 2 \rangle}|_{\mathcal{V}_b} = \begin{cases} s_{a,b,1^{k-i+1}} & \text{if } k+1 = i \\ s_{a-1,b,1^{k-i+2}} & \text{if } k+2 = i \\ 0 & \text{otherwise.} \end{cases}$$

The remainder of the proof is similar to the previous lemma. □

Note that $\psi(G|_{\text{hooks}})(q,t)$ is the sum of the restriction to $\mathcal{V}_1$ and the restriction to one part partitions ($|_{\mathcal{V}_0}$). This is the reason why, in Section 3.3.6, we will use the convention that $T_{n,s} = \{\epsilon\}$ if $s \geq n-2$. The path ϵ relates to the restriction to one part.

We should also notice that when $b \geq 2$, no formula written in the Schur functions in the variables q and t is known for $\langle \nabla(e_n), s_{k,1^{n-k}} \rangle|_{\mathcal{V}_b}$ at this moment. Given this formula, the lemma gives a way to find the formulas for $\langle \mathcal{E}_{n,n}, s_{1^n} \rangle|_{\mathcal{V}_{b+1}}$.

The restriction of $\mathcal{E}_{rn,n}$ to a set of two variables is predicted to be $\nabla^r(e_n)$ in (Bergeron, 2020). It is also conjectured that the equality $\langle \mathcal{E}_{rn,n}, s_{j+1,1^{n-j}} \rangle = e_j^{\perp} \langle \mathcal{E}_{rn,n}, s_{1^n} \rangle$ holds for all r. Using the combinatorics of m-Schröder paths, the following q-analogue is found in (Wallace, 2020b) (Chapter 2) :

$$f_j^{(r)}(q) := \psi(\langle \nabla^r(e_n), s_{j+1,1^{n-j-1}} \rangle|_{1\text{Part}}) = q^{r\binom{n}{2} - \binom{j+1}{2}} \begin{bmatrix} n-1 \\ j \end{bmatrix}_{q^{-1}}. \tag{3.3}$$

That result and the last lemma will be used to find the conjectured hook components for $\langle \mathcal{E}_{rn,n}, s_{1^n} \rangle$. But first we need to introduce a combinatorial object that will help us transform the alternating sum into a positive sum.

3.3.6 New Combinatorial Object

Our formula for $\langle \mathcal{E}_{rn,n}, s_\mu \rangle|_{\text{hooks}}$ will be formulated in terms of a new object that we denote by $T_{n,s}$. We will often write T_n for $T_{n,0}$. These objects help to transform an alternating sum into a positive sum and has a q,z-analogue, $T_n(q,z)$, realizing the q-Pochhammer symbol (or q-rising factorial) $(-qz;q)_{n-2} = \prod_{i=1}^{n-2}(1+zq^i)$. Notice that the substitution $T_n(q,-zq^{-1})$ brings us back to the usual way of seeing the q-Pochhammer symbol $(z;q)_{n-2} = \prod_{i=0}^{n-3}(1-zq^i)$. Let $T_{n,s}$ denote the set of northeast paths in an $n-2$ staircase shaped grid lying in $\mathbb{N}^2$, starting at $(0,s)$ and ending at a point in the set $\{(x,y) \mid x+y = n-2, x \geq 0 \text{ and } y \geq s\}$. For an example see Figure 3.12. The relevant paths can be represented as words of length $n-s-2$ in the alphabet $\{N, E\}$. For reasons stated earlier when $s > n-1$ we set $s = n-2$. In that case $T_{n,s} = \{\epsilon\}$, where ϵ is the empty word. Note that $\text{area}(\epsilon) = \binom{n-1}{2}$ and $\text{ht}(\epsilon) = n-2$. Notice that for $n < 2$, $T_{n,s} = \emptyset$. The *area* of a path, denoted area, is the number of boxes southeast of the path (see Figure 4.16). The *height* of a path is the y coordinate of its end point (see Figure 3.14). Many

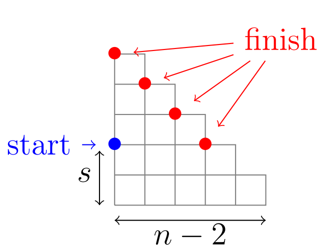


Figure 3.12 $T_{7,2}$

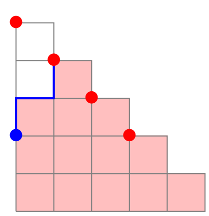


Figure 3.13 $\text{area}(NEN) = 13$

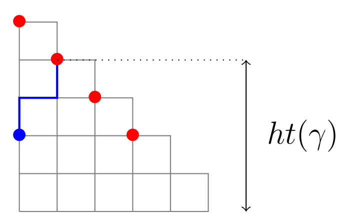


Figure 3.14 $\text{ht}(NEN) = 4$

known objects are in bijection (compositions, pattern avoiding permutations, ...) with $T_{n,s}$, a curious reader could see (Wallace, 2020a) (Chapter 4). Note that the image of $T_{n,s}$ through these bijections preserves many statistics on the associated objects and in some cases they can be used to refine the sets. This new object has the following generating function.

Proposition 11 : Let $T_{n,s}(q,z) = \sum_{\gamma \in T_{n,s}} q^{\text{area}(\gamma)} z^{\text{ht}(\gamma)}$. Then for $r = n-s-2$, we have :

$$T_{n,s}(q,z) = \sum_{j=0}^{r} q^{\binom{s+j+1}{2}+s(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q z^{j+s} = T_{r+2,0}(q,z) z^s q^{rs+\binom{s+1}{2}} \tag{3.4}$$

In particular if $s = 0$ we have :

$$T_{n,0}(q,z) = \sum_{j=0}^{n-2} q^{\binom{j+1}{2}} \begin{bmatrix} n-2 \\ j \end{bmatrix}_q z^j = \sum_{k=0}^{n-2} q^{\binom{n-k-1}{2}} \begin{bmatrix} n-2 \\ n-2-k \end{bmatrix}_q z^{n-2-k} = (-qz;q)_{n-2} \tag{3.5}$$

Note that by our choice of convention when $s \geq n-2$ we have $T_{n,s}(q,z) = q^{\binom{n-1}{2}} z^{n-2}$.

Proof. Starting with $T_{n,0}(q,z)$, we only need to prove the first equality since the second equality is the change of variables $k = n-2-j$ and the last equality is the well known q-binomial theorem. The paths ending at height j are the paths that fit in a $j \times (n-j-2)$ grid. It is known that the q-analogue of these paths with its respective area statistic are the Gaussian q-binomial, $\begin{bmatrix} n-2 \\ j \end{bmatrix}_q$. This leaves us with a staircase of height j which contains $\binom{j+1}{2}$ boxes. In consequence the coefficient of z^j is $q^{\binom{j+1}{2}} \begin{bmatrix} n-2 \\ j \end{bmatrix}_q$. For $T_{n,s}$ we need only to notice that there is a natural bijection with the paths of $T_{n-s,0}$. The only difference is the statistics. The height statistic is exactly s greater in $T_{n,s}$ and the area statistic is exactly $(n-2-s)s + \binom{s+1}{2}$ in $T_{n,s}$. Hence, Equation (3.4). $\square$

We will see that the reason $T_{n,s}(q,z)$ is useful to transform the alternating sums, induced in Lemma 21, into a positive sum is related to the exponents of q in Equation (3.4). The exponents have the following property.

Claim 5 : Let c be an integer and $g_j : \mathbb{N} \to \mathbb{Z}$ be maps such that $g_j(k) - g_j(k-1) = k + j + c$ for all $k \geq 1$. Then $g_j(k) = k(j+c) + \binom{k+1}{2} + g_j(0)$ for all $j \geq 0$ and all $k \geq 1$.

Proof. By definition of the maps, we know that $g_j(1) = j + 1 + c + g_j(0)$. Since $g_j(k) - g_j(k-1) = k + j + c$ by induction we have $g_j(k) = k + j + c + (k-1)(j + c) + \binom{k}{2} + g_j(0)$, which completes the proof. $\square$

The following result shows how this object is used to eliminate an alternating sum and make the relevant formula positive.

Proposition 12 : Let $g_j : \mathbb{N} \to \mathbb{Z}$ be such that $g_j(k) - g_j(k-1) = k + j$ for all $k \geq 1$. Then :

$$\sum_{k=0}^{n-j-1} (-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)} = \begin{bmatrix} n-2 \\ j-1 \end{bmatrix}_q q^{g_j(0)} \tag{3.6}$$

and :

$$\sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{-k+g_0(k)} = 0. \tag{3.7}$$

Moreover, if $g_j(0) - \binom{j}{2} = g_i(0) - \binom{i}{2}$ for all i, j, then for any j we have :

$$\sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} (-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)} z^{j-1} = T_{n,0}(q,z) q^{g_j(0)-\binom{j}{2}}, \tag{3.8}$$

and if $g_j(0) - \binom{j+1}{2} = g_i(0) - \binom{i+1}{2}$ for all i, j, then for any j we have :

$$\sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} (-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)} z^{j-1} = T_{n,0}(q,qz) q^{g_j(0)-\binom{j+1}{2}+1}. \tag{3.9}$$

Before we give the proof, we will give a combinatorial intuition based on the case $g_j(k) = \binom{j+k+1}{2}$. Notice that by the previous proposition we only need to prove the second part. For some fixed k and j we can represent the term $\begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)}$ by the set of paths in $\mathcal{C}_{j+k}^{n-1}$ to which we add $g_j(k)$ boxes and subtract k boxes (see Figure 3.15). There is a bijection between paths ending with a north step in $\mathcal{C}_{j+k+1}^{n-1}$ (blue in Figure 3.16) and paths ending with an east step in $\mathcal{C}_{j+k}^{n-1}$ (red in Figure 3.16). We only need to change the last step. This is an involution and they both account for the same area value (see Figure 3.16 as an example). Since the terms have coefficients, $(-1)^k$ they cancel out pairwise in the sum. So the only steps left are the ones when $k = 0$ and the path in $\mathcal{C}_j^{n-1}$ ends with a north step. Eliminating the last north step does not affect the area because there are no east steps afterwards. Therefore, we can consider the paths in $\mathcal{C}_{j-1}^{n-2}$ with the same statistic, which is what we needed. Note that for $j = 0$ there is no path in $\mathcal{C}_0^{n-1}$ that ends with a north step.

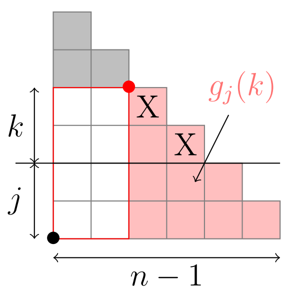


Figure 3.15 Representation of the term $\begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)}$ in the Equation (3.6)

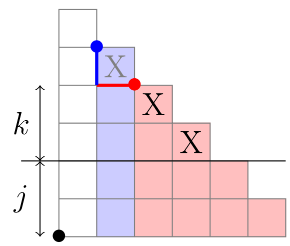


Figure 3.16 Comparing path ending with a north step in $\mathcal{C}_{j+k+1}^{n-1}$ and with an east step in $\mathcal{C}_{j+k}^{n-1}$

Proof. Since $\binom{j+k+1}{2} - \binom{j+1}{2} = \binom{k+1}{2} + kj$, by the previous claim we can rewrite the left hand of Equation (3.6) :

$$q^{g_j(0)-\binom{j+1}{2}} \sum_{k=0}^{n-j-1} (-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+\binom{j+k+1}{2}} \tag{3.10}$$

Let us consider each path of $T_{n+1,0}$ that end over the line of equation $y = j$ and weigh them differently :

$$\hat{T}^{(j)}_{n+1,0}(q,z) := \sum_{\substack{\gamma \in T_{n+1,0} \\ \mathrm{ht}(\gamma) \geq j}} (-qz)^{j-\mathrm{ht}(\gamma)} q^{\mathrm{area}(\gamma)} z^{\mathrm{ht}(\gamma)}$$

Using the same proof as in Proposition 11 we get :

$$\hat{T}^{(j)}_{n+1,0}(q,z) = \sum_{k=0}^{n-j-1} (-1)^k q^{\binom{j+k+1}{2}-k} \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q z^j.$$

Notice that Equation (3.10) is equal to $q^{g_j(0)-\binom{j+1}{2}}\hat{T}^{(j)}_{n+1,0}(q,z)z^{-j}$. Because each paths cross the line $y=j$ we can consider all the paths that touch the line $y=j$ for the first time at $(n-1-j-l,j)$. These end to the northeast of $(n-1-j-l,j)$ to any end point among the set $\{(n-1-j-l+k, j+l-k) \mid 0 \leq k \leq l\}$. But the set of paths starting at $(n-1-j-l,j)$ and ending at $\{(n-1-j-l+k, j+l-k) \mid 0 \leq k \leq l\}$ are the paths of $T_{l+2,0}$ translated by $(n-1-j-l,j)$. Furthermore the paths of height $l-k$ in $T_{l+2,0}$ translated by $(n-1-j-l,j)$ end at $(n-1-j-l+k, j+l-k)$ ergo the related monomials are skewed by $(-qz)^{-l+k}$, in $\hat{T}^{(j)}_{n+1,0}(q,z)$. Therefore, if we consider the part of the path over the line $y=j$ we see that it is weighted by $\hat{T}^{(0)}_{l+2,0}(q,z)$. Take note that $\hat{T}^{(0)}_{l+2,0}(q,z)$ is a polynomial in q and not in q and z since $l-k$ is the height of the path. Furthermore, the paths of $T_{n+1,0}$ crossing at

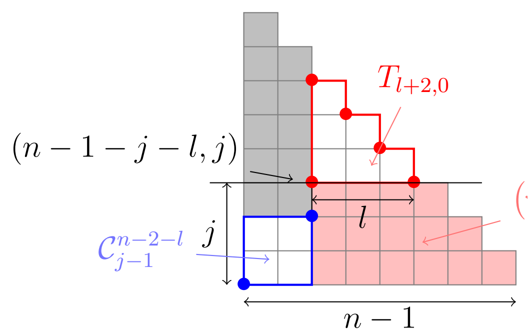


Figure 3.17 Concatenation of a path of $T_{l+2,0}$ and a path of $\mathcal{C}^{n-2-l}_{j-1}$

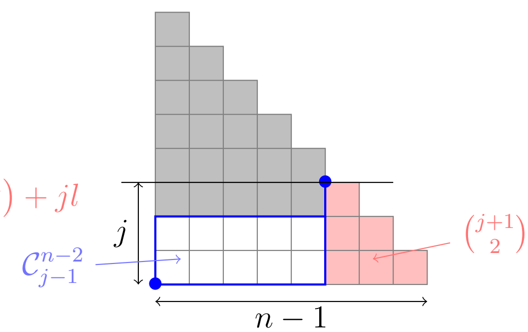


Figure 3.18 The paths that are not cancelled out for j (when $l=0$)

$(n-1-l-j,j)$ are the concatenation of a path of $T_{l+2,0}$ and a path of $\mathcal{C}^{n-2-l}_{j-1}$, since the paths contain the north step that starts at $(n-1-l-j,j-1)$ (see Figure 3.17). Hence, Equation (3.10) is equivalent to :

$$q^{g_j(0)-\binom{j+1}{2}}\sum_{l=0}^{n-j-1}\begin{bmatrix} n-2-l \\ j-1 \end{bmatrix}_q \hat{T}^{(0)}_{l+2,0}(q,z)q^{jl+\binom{j+1}{2}}. \tag{3.11}$$

Now by Proposition 11 we have :

$$T_{l+2,0}(q,-zq^{-1}) = \sum_{k=0}^{l} q^{\binom{l-k+1}{2}}\begin{bmatrix} l \\ l-k \end{bmatrix}_q z^{l-k}(-q)^{-l+k}, \tag{3.12}$$

and by definition of our skewed sum we have :

$$\hat{T}^{(0)}_{l+2,0}(q,z) = \sum_{k=0}^{l}(-1)^{l-k}q^{\binom{l-k+1}{2}-l+k}\begin{bmatrix} l \\ l-k \end{bmatrix}_q z^0, \tag{3.13}$$

therefore comparing Equation (3.12) and Equation (3.13) we get $\hat{T}^{(0)}_{l+2,0}(q,1) = T_{l+2,0}(q,-q^{-1})$. By Proposition 11 $T_{l+2,0}(q,-zq^{-1}) = \prod_{i=0}^{l-1}(1-zq^i)$ thus we have $T_{l+2,0}(q,-q^{-1}) = 0$ if $l > 0$ and 1 if $l = 0$. The equality $\hat{T}^{(0)}_{l+2,0}(q,1) = \hat{T}^{(0)}_{l+2,0}(q,z)$ means it is equivalent to state that $\hat{T}^{(0)}_{l+2,0}(q,z) = 0$ unless $l = 0$. This proves Equation (3.7). Replacing this in Equation (3.11) gives us Equation (3.6) (see Figure 3.18). Equation (3.8) (respectively Equation (3.9)) follows from $g_j(0) - \binom{j}{2} = g_i(0) - \binom{i}{2}$ (respectively $g_j(0) - \binom{j+1}{2} = g_i(0) - \binom{i+1}{2}$) for all i, j and Proposition 11. □

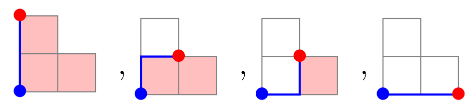
Notice that Equation (3.7) means that we can change the range of Proposition 12 for $0 \leq j \leq n-1$ and obtain the same result. We will see in the next section how to use this to get the formula for the hook components of $\langle \mathcal{E}_{rn,n}, e_n\rangle$ and of $\langle \mathcal{E}_{n,n}, s_\mu\rangle$.

3.3.7 Schur Positive Explicit combinatorial Formula

We can now prove Theorem 2 piece by piece. Figure 3.19 gives an example for $r = 1$ and $n = 4$. For those who are used to seeing $\langle \mathcal{E}^{\langle 2\rangle}_{n,n}, s_\mu\rangle = \langle \nabla(e_n), s_\mu\rangle$ in terms of Dyck paths and Schröder paths, note that each path in $T_{n,k}$ is associated to a subset of Schröder paths. The next proposition is related to the restriction of Equation (3.2) to $\mathcal{E}^{\langle 2\rangle}_{n,n}$. Haglund proved in (Haglund, 2004) that $\langle \nabla(e_n), e_k h_{n-k}\rangle$ could be given in terms of Schröder paths with a given statistic. In (Wallace, 2020b) (Chapter 2) the author gives a bijection between a subset of Schröder paths with d diagonal steps and the set $\mathrm{SYT}(d+1, 1^{n-d-1}) \times \{1, 2, \ldots, n-d-1\}$. The subset is such that the paths end with a north step and the area statistic of that path is equal to 1.

Proposition 13 : If $r = 1$ and $\mu \in \{(k, 1^{n-k}) \mid 1 \leq k \leq n\}$ or if $r > 1$ and $\mu = 1^n$

The elements of $T_{4,0}$:



$area(\gamma)$:	3	2	1	0
$ht(\gamma)$:	2	1	1	0
$hook(\gamma)$:	$6, 1^0$	$4, 1^1$	$3, 1^1$	$1, 1^2$
$s_{hook(\gamma)}$:	s_6	s_{41}	s_{31}	s_{111}

$$\langle \mathcal{E}_{4,4}, e_4 \rangle|_{hooks} = s_6 + s_{41} + s_{31} + s_{111}$$

Figure 3.19 Example of $\langle \mathcal{E}_{4,4}, e_4 \rangle|_{hooks}$

then :

$$\langle \nabla^r(e_n), s_\mu \rangle|_{\text{hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{r\binom{n}{2} - \mathrm{maj}(\tau')}(q,t) + \sum_{i=2}^{\mathrm{des}(\tau)} s_{r\binom{n}{2} - \mathrm{maj}(\tau') - i, 1}(q,t). \quad (3.14)$$

Additionally, for all k we have :

$$\langle \Delta'_{e_k}(e_n), e_n \rangle|_{\text{hooks}} = \sum_{\tau \in \mathrm{SYT}((n-k, 1^k))} s_{\mathrm{maj}(\tau)}(q,t) + \sum_{i=2}^{k} s_{\mathrm{maj}(\tau) - i, 1}(q,t). \quad (3.15)$$

Furthermore, for all $\mu \in \{(n), (n-1,1), (n-2,1,1), (1^n)\}$, $0 \leq k < n-1$, we have :

$$e^{\perp}_{n-k-1} \left(\langle \mathcal{E}_{n,n}, s_\mu \rangle|_{\text{hooks}} \right)^{\langle 2 \rangle} = \sum_{\tau \in \mathrm{SYT}(\mu)} \sum s_{(k-1+\mathrm{area}(\gamma) - \mathrm{maj}(\tau'), 1)}(q,t) + \sum s_{(k+\mathrm{area}(\gamma) - \mathrm{maj}(\tau'))}(q,t). \quad (3.16)$$

Likewise, if the conjecture $e_k^{\perp} \langle \mathcal{E}_{n,n}, s_\mu \rangle =_{\langle 2 \rangle} \langle \Delta'_{e_{n-k-1}} e_n, s_\mu \rangle$ is true for all k, then Equation (3.14) and Equation (3.16) hold for all μ when $r = 1$, and :

$$\langle \Delta'_{e_k}(e_n), s_\mu \rangle|_{\text{some hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} \sum s_{(k-1+\mathrm{area}(\gamma) - \mathrm{maj}(\tau'), 1)}(q,t) + \sum s_{(k+\mathrm{area}(\gamma) - \mathrm{maj}(\tau'))}(q,t), \quad (3.17)$$

where the second sum of Equations (3.16) and Equation (3.17) is over paths in $T_{n,\mathrm{des}(\tau')}$ of height $k-2$ and $k-1$ and the third sum is over paths in $T_{n,\mathrm{des}(\tau')}$ of height $k-1$ and k. Finally, if $t = 0$ or $q = 0$ Equations (3.16) and Equation (3.17) hold for all μ and Equation (3.14) holds for μ general if $r = 1$ and for $r > 1$ if μ

is hook-shaped.

The notation $|_{some\ hooks}$ simply means that some hooks are missing from the sum (i.e. the coefficients of Equation (3.17) constitute a lower bound for the coefficients of $\langle \Delta'_{e_k}(e_n), s_\mu\rangle|_{\text{hooks}}$). Note that Equation (3.14) and Equation (3.15) hold for all q and t when $\mu = 1^n$ and r arbitrary or $r = 1$ and μ is hook-shaped. Additionally, if $k = n-1$ Equations (3.16) and Equation (3.17) hold if the second sum of is over paths in $T_{n,\text{des}(\tau')}$ of height $k-2$ and the third sum is over paths in $T_{n,\text{des}(\tau')}$ of height $k-1$. We will delay the proof of Proposition 13 and Theorem 2 until after Lemma 25. Before we start let us notice that if $r \neq 1$ and $\mu \neq 1^n$ computer experimentation suggests that Equation (3.14) can be extended even though it is incomplete (all the terms of the formula seem to appear but some positive terms of $\langle \mathcal{E}_{rn,n}, s_\mu\rangle|_{\text{hooks}}$ are missing). Let us start by proving the statement for the alternants of $\mathcal{E}_{rn,n}$.

Proposition 14 : For all $r \in \mathbb{N}^*$ such that $\langle \mathcal{E}_{rn,n}, s_{(j+1),1^{n-j}}\rangle = e_j^\perp \langle \mathcal{E}_{rn,n}, s_{1^n}\rangle$ is true for all j we have :

$$\langle \mathcal{E}_{rn,n}, e_n\rangle|_{\text{hooks}} = \sum_{\gamma \in T_n} s_{\text{hook}(\gamma)} \tag{3.18}$$

where hook(γ) is the partition $\left((r-1)\binom{n}{2} + \text{area}(\gamma) + ht(\gamma) + 1, 1^{n-2-ht(\gamma)}\right)$.

Proof. Let $A(q,t) := \psi(\mathcal{E}_{rn,n}, e_n\rangle|_{\text{hooks}})(q,t)$. Recall that F. Bergeron conjectured the equality $\langle \mathcal{E}_{rn,n}, s_{(j+1),1^{n-j-1}}\rangle = e_j^\perp \langle \mathcal{E}_{rn,n}, s_{1^n}\rangle$. Then by Equation (3.3) and Lemma 21 we have :

$$A(q,t) = \sum_{j\geq 0}\sum_{k=0}^{j} (-1)^k \left(q^{r\binom{n}{2} - \binom{j-k+1}{2}} \begin{bmatrix} n-1 \\ j-k \end{bmatrix}_{q^{-1}} \right) q^{-k} t^j$$

Furthermore, $s_{(j+1),1^{n-j-1}}$ makes no sense for $j \geq n$ therefore j parses through all value between 0 and $n-1$ and we can change j for $n-j-1$ and obtain the same result. Then by noticing that $q^{-(j+k)(n-j-k-1)} \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_{q^{-1}}$, that

$\binom{n}{2} - \binom{n-j-k}{2} - (j+k)(n-j-k-1) = \binom{j+k+1}{2}$ and simplifying we get :

$$A(q,t) = q^{(r-1)\binom{n}{2}} \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} (-1)^k \left(q^{\binom{j+k+1}{2}} \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q \right) q^{-k} t^{n-j-1}$$

Considering a twist in variables, we obtain :

$$z^{n-2} A(q,z^{-1}) = q^{(r-1)\binom{n}{2}} \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} (-1)^k \left(q^{\binom{j+k+1}{2}} \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q \right) q^{-k} z^{j-1}$$

Now $\binom{j+k+1}{2} - \binom{j+k}{2} = j+k$ by Proposition 12 (Equation (3.9)) if we set $g_j(k) = \binom{j+k+1}{2}$ we get :

$$z^{n-2} A(q,z^{-1}) = q^{(r-1)\binom{n}{2}+1} T_n(q,qz)$$

Or equivalently :

$$A(q,t) = t^{n-2} q^{(r-1)\binom{n}{2}+1} T_n(q,qt^{-1})$$

In the generating function, $T_{n,0}(q,t)$ the power of the variable t corresponds to the height of the associated path in $T_{n,0}$ and the power of the variable q to the area of the associated path in $T_{n,0}$ we have the interpretation of $t^{n-2} q^{(r-1)\binom{n}{2}+1} T_{n,0}(q,qt^{-1})$ as adding the area, the height and $(r-1)\binom{n}{2}+1$ to the first part and subtracting the height to $n-2$ gives us the remainder of the hook. $\square$

There is only one standard tableau of shape 1^n, its conjugate is (n). Since we have $\mathrm{des}((n)) = 0$ and $\mathrm{maj}((n)) = 0$ the formula of the following lemma coincides with the formula of the previous proposition in the case $r = 1$ and $\mu = 1^n$. For the next lemma, we need to define rev_q. As in (Haglund *et al.*, 2018) $rev_q : \mathbb{Z}[[X]][q] \to \mathbb{Z}[[X]][q]$ is defined by $rev_q(f(x)(q)) = f(x)(q^{-1}) q^{deg_q(f(x)(q))}$.

Lemma 23 : If $e_k^{\perp} \langle \mathcal{E}_{n,n}, s_\mu \rangle =_{\langle 2 \rangle} \langle \Delta'_{e_{n-k-1}} e_n, s_\mu \rangle$ is true for all k then :

$$\langle \mathcal{E}_{n,n}, s_\mu \rangle|_{\text{hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)}, \tag{3.19}$$

where $\mathrm{hook}(\gamma)$ is the partition $(\mathrm{area}(\gamma) + ht(\gamma) - \mathrm{maj}(\tau') + 1, 1^{n-2-ht(\gamma)})$.

Proof. Recall that ω was defined in Section 3.3.4. By applying $\omega \circ rev_q$ to corollary 6.13 of (Haglund *et al.*, 2018) we obtain :

$$\Delta'_{e_{n-k-1}}(e_n)|_{t=0} = \sum_{\tau \in \mathrm{SYT}(n)} q^{k\,\mathrm{des}(\tau')+\binom{n-k}{2}-\mathrm{maj}(\tau')} \begin{bmatrix} \mathrm{des}(\tau) \\ k \end{bmatrix}_q s_{\lambda(\tau)} \tag{3.20}$$

Therefore :

$$f_k^{(\mu)}(q) := \psi(\langle \Delta'_{e_{n-k-1}}(e_n)|_{t=0}, s_\mu \rangle) = \sum_{\tau \in \mathrm{SYT}(\mu)} q^{k\,\mathrm{des}(\tau')+\binom{n-k}{2}-\mathrm{maj}(\tau')} \begin{bmatrix} \mathrm{des}(\tau) \\ k \end{bmatrix}_q$$

and :

$$f_k^{((n))}(q) = q^{k(n-1)+\binom{n-k}{2}-\binom{n}{2}} \begin{bmatrix} 0 \\ k \end{bmatrix}_q$$

Then by Lemma 21 we have :

$$\psi(\langle \mathcal{E}_{n,n}, s_\mu \rangle|_{\mathrm{hooks}}) = \textstyle\sum_{j \geq 0} \sum_{k=0}^{j} (-1)^k \sum_{\tau \in \mathrm{SYT}(\mu)} q^{(j-k)\,\mathrm{des}(\tau')+\binom{n-j+k}{2}-\mathrm{maj}(\tau')} \begin{bmatrix} \mathrm{des}(\tau) \\ j-k \end{bmatrix}_q q^{-k} t^j$$

Let $A_\tau^{(\mu)}(q,t)$ be the part of $\psi(\langle \mathcal{E}_{n,n}, s_\mu \rangle|_{\mathrm{hooks}})$ when the sum is over τ. In other words, we have $\sum_{\tau \in \mathrm{SYT}(\mu)} A_\tau^{(\mu)}(q,t) = \psi(\langle \mathcal{E}_{n,n}, s_\mu \rangle|_{\mathrm{hooks}})$. Much like for the previous proposition j parses from 0 to $\mathrm{des}(\tau)$. So we can change j for $\mathrm{des}(\tau) - j$ and obtain the same result, in consequence simplification gives us :

$$A_\tau^{(\mu)}(q,t) = \textstyle\sum_{j=0}^{\mathrm{des}(\tau)} \sum_{k=0}^{\mathrm{des}(\tau)-j} (-1)^k q^{(\mathrm{des}(\tau)-j-k)\,\mathrm{des}(\tau')+\binom{n-\mathrm{des}(\tau)+j+k}{2}-\mathrm{maj}(\tau')} \begin{bmatrix} \mathrm{des}(\tau) \\ j+k \end{bmatrix}_q q^{-k} t^{\mathrm{des}(\tau)-j}$$

and :

$$A_\tau^{((n))}(q,t) = \sum_{j=0}^{0} \sum_{k=0}^{-j} (-1)^k q^{(-j-k)(n-1)+\binom{n+j+k}{2}-\binom{n}{2}} \begin{bmatrix} 0 \\ j+k \end{bmatrix}_q q^{-k} t^{-j} = 1$$

By recalling that $n - 1 - \mathrm{des}(\tau) = \mathrm{des}(\tau')$ we notice the equality $\binom{n-\mathrm{des}(\tau)+j+k}{2} = \binom{j+k+1}{2} + \binom{\mathrm{des}(\tau')+1}{2} + (j+k)\,\mathrm{des}(\tau')$. Therefore, by setting $g_j(k) = \mathrm{des}(\tau)\,\mathrm{des}(\tau') + \binom{j+k+1}{2} + \binom{\mathrm{des}(\tau')+1}{2} - \mathrm{maj}(\tau')$ we have $g_j(k) - g_j(k-1) = k + j$. Using Equation

(3.6) of Proposition 12 we obtain :

$$A_\tau^{(\mu)}(q,t) = \sum_{j=0}^{\mathrm{des}(\tau)-1} q^{\mathrm{des}(\tau)\,\mathrm{des}(\tau')+\binom{j+1}{2}+\binom{\mathrm{des}(\tau')+1}{2}-\mathrm{maj}(\tau')} \begin{bmatrix}\mathrm{des}(\tau)-1\\ j\end{bmatrix}_q t^{n-1-\mathrm{des}(\tau')-j}$$

Has in the last proposition we change variables and obtain :

$$z^{n-2-\mathrm{des}(\tau')}A_\tau^{(\mu)}(q,z^{-1}) = \sum_{j=0}^{\mathrm{des}(\tau)-1} q^{\mathrm{des}(\tau)\,\mathrm{des}(\tau')+\binom{j+k+1}{2}+\binom{\mathrm{des}(\tau')+1}{2}-\mathrm{maj}(\tau')} \begin{bmatrix}\mathrm{des}(\tau)-1\\ j\end{bmatrix}_q z^{j-1}$$

Finally, by Proposition 11 we get :

$$z^{n-2-\mathrm{des}(\tau')}A_\tau^{(\mu)}(q,z^{-1}) = q^{1+\mathrm{des}(\tau)\,\mathrm{des}(\tau')+\binom{\mathrm{des}(\tau')+1}{2}-\mathrm{maj}(\tau')}T_{\mathrm{des}(\tau)+1,0}(q,qz)$$

Hence :

$$A_\tau^{(\mu)}(q,t) = t^{n-2}q^{1-\mathrm{maj}(\tau')}q^{(\mathrm{des}(\tau)-1)\,\mathrm{des}(\tau')+\binom{\mathrm{des}(\tau')+1}{2}}(qt^{-1})^{\mathrm{des}(\tau')}T_{\mathrm{des}(\tau)+1,0}(q,qt^{-1})$$

Ergo by Equation (3.4) of Proposition 11 we have :

$$A_\tau^{(\mu)}(q,t) = t^{n-2}q^{1-\mathrm{maj}(\tau')}T_{n,\mathrm{des}(\tau')}(q,qt^{-1})$$

and :

$$A_\tau^{((n))}(q,t) = 1 = t^{n-2}q^{1-\binom{n}{2}}q^{\binom{n-1}{2}+n-2}t^{-n+2} = t^{n-2}q^{1-\binom{n}{2}}T_{n,n-1}(q,qt^{-1})$$

$\square$

Notice that Equation (3.19) is a lift of $\Delta'_{e_{n-k-1}}(e_n)|_{t=0}$ therefore Equation (3.16) at $t=0$ is a reinterpretation using our object of Equation (3.20) of Haglund, Rhoades and Shimonozo. Furthermore, $\Delta'_{e_{n-1}}(e_n) = \nabla(e_n)$ thus Equation (3.14) at $t=0$ for general μ and $r=1$ is just a reinterpretation of Equation (3.20) We will now prove that for all $\mu=(d,1^{n-d})$ the restriction to two variables of Equation (3.2) is true independently of F.Bergeron's $e_k^\perp\langle\mathcal{E}_{n,n},s_\mu\rangle =_{\langle 2\rangle} \langle\Delta'_{e_{n-k-1}}e_n,s_\mu\rangle$ for all k conjecture. In other words, the formula gives correctly the coefficients of $s_\mu\otimes s_\lambda$ in $\langle\nabla(e_n),s_\mu\rangle$ when μ and λ are hook-shaped. To this end we recall the following result from (Wallace, 2020b) (Chapter 2) :

Lemma 24 : If $\mu \in \{(d, 1^{n-d}) \mid 1 \leq d \leq n\}$ then :

$$\langle \nabla(e_n), s_\mu \rangle|_{\text{hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{\mathrm{maj}(\tau)}(q,t) + \sum_{i=2}^{\mathrm{des}(\tau)} s_{\mathrm{maj}(\tau)-i,1}(q,t),$$

$$\langle \nabla^r(e_n), s_\mu \rangle|_{1\text{Part}} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{r\binom{n}{2}-\mathrm{maj}(\tau')}(q,t),$$

and :

$$\langle \nabla^r(e_n), e_n \rangle|_{\text{hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{r\binom{n}{2}-\mathrm{maj}(\tau')}(q,t) + \sum_{i=2}^{\mathrm{des}(\tau)} s_{r\binom{n}{2}-\mathrm{maj}(\tau')-i,1}(q,t).$$

Note that $\binom{n}{2} - \mathrm{maj}(\tau') = \mathrm{maj}(\tau)$ and $\mathrm{des}(\tau) = n - 1 - \mathrm{des}(\tau')$.

Lemma 25 : If $\mu \in \{(d, 1^{n-d}) \mid 1 \leq d \leq n\}$ then :

$$\left(\sum_{\tau \in \mathrm{SYT}(\mu)} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)}\right)^{\langle 2 \rangle} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{r\binom{n}{2}-\mathrm{maj}(\tau')}(q,t) + \sum_{i=2}^{\mathrm{des}(\tau)} s_{r\binom{n}{2}-\mathrm{maj}(\tau')-i,1}(q,t),$$

where $\mathrm{hook}(\gamma) = \left((r-1)\binom{n}{2} + \mathrm{area}(\gamma) + \mathrm{ht}(\gamma) - \mathrm{maj}(\tau') + 1, 1^{n-2-\mathrm{ht}(\gamma)}\right)$.

Proof. Notice that the restriction to two variables is equivalent to the restriction to hooks of length 2 or less as seen in Section 3.3.4. We therefore only need to consider the paths of height $n-3$ and $n-2$. The area of a path of height $n-2$ is $\binom{n-1}{2}$ and the hook associated to it has only one part. Furthermore, $(r-1)\binom{n}{2} + \binom{n-1}{2} + n - 2 + 1 = r\binom{n}{2}$. This accounts for the first sum on the right-hand side. The area of a path of height $n-3$ in $T_{n,\mathrm{des}(\tau')}$ starting with exactly p north steps is $\binom{n-1}{2} - j$, where $j = n - 2 - p - \mathrm{des}(\tau')$. The number of north steps at the beginning of the path is bounded by $0 \leq p \leq n - 3 - \mathrm{des}(\tau')$. This is equivalent to $\mathrm{des}(\tau) \geq n - 1 - p - \mathrm{des}(\tau') \geq 2$. Consequently $(r-1)\binom{n}{2} + \binom{n-1}{2} - j + n - 3 + 1 - \mathrm{maj}(\tau') = r\binom{n}{2} - \mathrm{maj}(\tau') - i$ for $i = n - 1 - p - \mathrm{des}(\tau')$ which accounts for the second sum of the right-hand side. $\square$

We can now prove Proposition 13 and Theorem 2.

Proof of Theorem 2. For $r = 1$ we start with the cases where $\mu \in \{(n), (n-1,1), (n-2,1,1)\}$. The descent of a standard tableau of shape $(n-k, 1^k)$ has k elements; thus we have $\mathrm{des}(\tau') = n-1$ if $\tau \in \mathrm{SYT}((n))$, $\mathrm{des}(\tau') = n-2$ if $\tau \in \mathrm{SYT}((n-1,1))$, $\mathrm{des}(\tau') = n-3$ if $\tau \in \mathrm{SYT}((n-2,1,1))$. This implies that the height of the paths related to these tableaux must be greater or equal to $n-3$. In consequence $\ell(\mathrm{hook}(\gamma)) = \ell(\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) - \mathrm{maj}(\tau') + 1, 1^{n-2-\mathrm{ht}(\gamma)}) \leq 2$. Therefore, no term disappears in the restriction to two variables and by Lemma 25 we have $\sum_{\tau \in \mathrm{SYT}(\mu)} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)} = \langle \nabla(e_n), s_\mu \rangle|_{\mathrm{hooks}}$ in these cases. Moreover, it is shown in (Bergeron, 2020) that $\langle \mathcal{E}_{n,n}, s_\mu \rangle = \langle \nabla(e_n), s_\mu \rangle$ when $\mu \in \{(n), (n-1,1), (n-2,1,1), (n-2,2)\}$, which is what we needed. The remainder of the proof is a direct consequence of Proposition 14 and Lemma 23. $\square$

Proof of Proposition 13. It was proven in (Haglund, 2004) that $\langle \nabla(e_n), e_k h_{n-k} \rangle = \langle \Delta_{e_k} e_n, e_n \rangle$, therefore $\langle \nabla(e_n), s_{(k+1,1^{n-k-1})} \rangle = \langle \Delta'_{e_{n-k-1}} e_n, e_n \rangle$. Hence, Equation (3.14) implies the Equation (3.15). Lemma 24 proves that Equation (3.14) is true for $r = 1$ and $\mu \in \{(k, 1^{n-k}) \mid 1 \leq k \leq n\}$ or $r > 1$ and $\mu = 1^n$. It also proves that Equation (3.14) holds for μ, hook-shaped, when $r \geq 1$ and $t = 0$ or $q = 0$. Furthermore, in Lemma 23 the formula is constructed by lifting the Schur functions having only one part. Ergo Equation (3.14), Equation (3.16), and Equation (3.17) hold for all μ when $r = 1$ and $t = 0$ or $q = 0$ since $\Delta'_{e_{n-1}}(e_n) = \nabla(e_n)$. This also implies that Equation (3.17) holds for all μ if F.Bergeron's conjecture is true. Equation (3.16) is the restriction of Theorem 2 to the paths associated to hooks of length one and two. Equation (3.17) is a rewriting of Equation (3.16). This is just to state that the paths ending at height k and $k-1$ in $T_{n,s}$ correspond to the Schur functions with one part in $\Delta'_{e_k}(e_n)$ and the paths ending at height $k-2$ correspond to Schur functions that are indexed by hooks of length 2. $\square$

One may be interested to notice that the hook lengths in Theorem 2 can be computed using only the area statistic and the Major index.

3.4 Adjoint Dual Pieri Rule

In this section, it is shown that $\langle \mathcal{E}_{n,n}, s_{(k+1),1^{n-k-1}} \rangle = e_k^{\perp} \langle \mathcal{E}_{n,n}, s_{1^n} \rangle$ can be done directly in terms of paths (Proposition 15). For k such that $0 \leq k \leq n-2$ and $\mu = (k+1, 1^{n-k-1})$, we consider the following sets of paths :

$$\bigcup_{\tau \in \mathrm{SYT}(\mu)} \{\gamma = N^j \tilde{\gamma} \in T_{n,\mathrm{des}(\tau')} \mid j \geq n-k-\min(\mathrm{Des}(\tau'))\} = T^+_{n,k}$$

$$\bigcup_{\tau \in \mathrm{SYT}(\mu)} \{\gamma = N^j E \tilde{\gamma}, \gamma = N^j \in T_{n,\mathrm{des}(\tau')} \mid j < n-k-\min(\mathrm{Des}(\tau'))\} = T^-_{n,k}$$

$$V_{n,k} = \bigcup_{\substack{\tau \in \mathrm{SYT}(\mu) \\ 1 \in \mathrm{Des}(\tau')}} \left\{\gamma = N^j E \tilde{\gamma}, \gamma = N^j \in T_{n,\mathrm{des}(\tau')} \mid \max\{0, n-k-\min(\mathrm{Des}(\tau') \backslash \{1\}) \leq j\right\}$$

$$\cup \bigcup_{\substack{\tau \in \mathrm{SYT}(\mu) \\ 1 \notin \mathrm{Des}(\tau'), \{n-k+1, \ldots, n-1\} \subset \mathrm{Des}(\tau')}} \left\{\gamma = E^r \tilde{\gamma} \in T_{n,\mathrm{des}(\tau')} \mid r+1 \geq \min(\mathrm{Des}(\tau'))\right\}$$

and

$$T^k_n = \{\gamma \in T_{n,0} \mid n-2-\mathrm{ht}(\gamma) \geq k\}.$$

Notice that for all $\tau \in \mathrm{SYT}(k+1, 1^{n-k-1})$, $\mathrm{des}(\tau') = k$. Hence, all paths of $T_{n,\mathrm{des}(\tau')}$ have at most $n-k-2$ north steps. Therefore, if $1 \in \mathrm{Des}(\tau')$ then $T^+_{n,k} \cap T_{n,\mathrm{des}(\tau')} = \emptyset$ and $T^-_{n,k} \cap T_{n,\mathrm{des}(\tau')} = T^-_{n,k}$. Additionally, one can easily check that for $\mu = (k+1, 1^{n-k-1})$ the sets $T^+_{n,k}$, and $T^-_{n,k}$ are a partition of the set $\bigcup_{\tau \in \mathrm{SYT}(\mu)} T_{n,\mathrm{des}(\tau')}$ and $V_{n,k} \subset T^-_{n,k}$.

For k between 1 and $n-2$, we will now define two families $\{\underline{e^{\perp}_{k-}}\}$ and $\{\underline{e^{\perp}_{k+}}\}$ of maps :

$$\underline{e^{\perp}_{k-}} : T^{k-1}_n \backslash \{E^{n-2}\} \to V_{n,k} \subset T^-_{n,k} \quad \text{and} \quad \underline{e^{\perp}_{k+}} : T^k_n \to T^+_{n,k}$$

For $\gamma \in T_{n,0}$, let us consider the prefix of γ ending with the k-th east step. The prefix exists by definition of T^k_n, since $n-2-\mathrm{ht}(\gamma)$ gives the number of east steps. Let $p_1, \ldots, p_k$ denote the integers such that p_i is the number of north steps before the i-th east step. To this we associate τ', the hook-shaped standard tableau such that $\mathrm{Des}(\tau') = \{n - i - p_i \mid 1 \leq i \leq k\}$. In consequence $\underline{e_{k+}^{\perp}}(\gamma)$ is the path in $T_{n,\mathrm{des}(\tau')}$ given by discarding all the k first east steps of γ. (See Figure 3.20) The

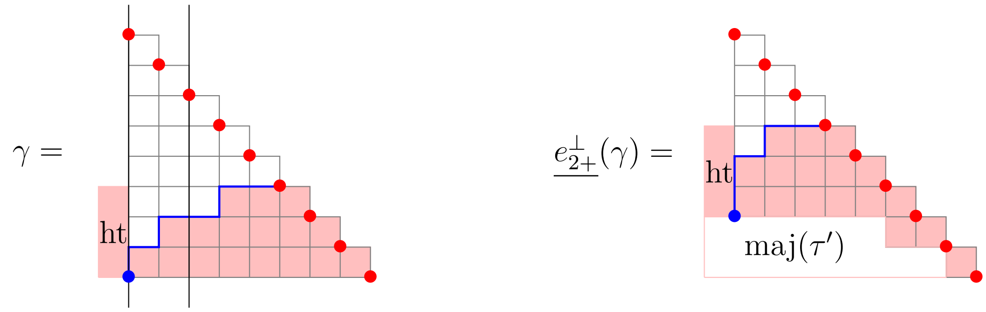


Figure 3.20 The map $\underline{e_{2+}^{\perp}}$ sends the path $\gamma = NENEENEE \in T_{10,0}$ to the path $NNENEE \in T_{10,\mathrm{des}(\tau')}$, with $\mathrm{Des}(\tau') = \{6, 8\}$

map $\underline{e_{k-}^{\perp}}(\gamma)$ is defined in a similar way. For $\gamma \in T_{n,0}$, let us consider the prefix of γ ending with the $k-1$-th east step. We denote by $p_1, \ldots, p_{k-1}$ the integers such that p_i is the number of north steps before the i-th east step. Let h be the number of east steps before the first north step. We choose τ' to be the hook-shaped standard tableau that has the following descent set $\mathrm{Des}(\tau') = \{n - i - p_i \mid 1 \leq i \leq k-1\} \cup \{\max(1, h - k + 2)\}$. Consequently, $\underline{e_{k-}^{\perp}}(\gamma)$ is the path in $T_{n,\mathrm{des}(\tau')}$, given by discarding the $k-1$ first east steps of and the first north step of γ. (See Figure 3.21). Note that we take out the path E^{n-2} because it is associated to the Schur function s_{1^n} and $e_k^{\perp}(s_{1^n})$ has only one term. We will recall that for $r = 1$

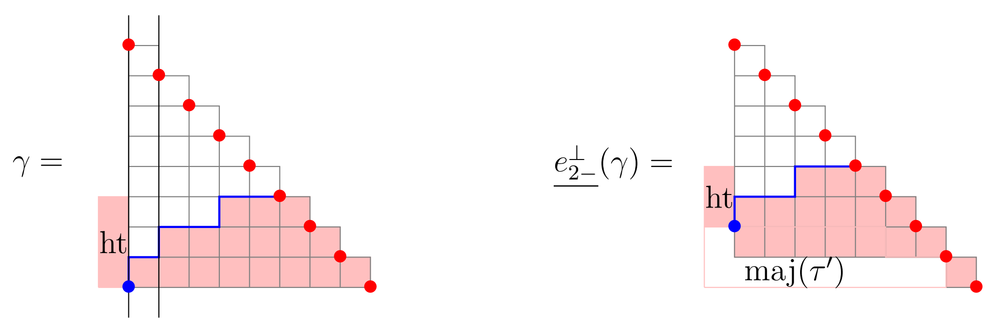


Figure 3.21 The map $\underline{e_{2-}^{\perp}}$ sends the path $\gamma = NENEENEE \in T_{10,0}$ to the path $NEENEE \in T_{10,\mathrm{des}(\tau')}$, with $\mathrm{Des}(\tau') = \{1, 8\}$

the hooks in Theorem 2 are given by :

$$\mathrm{hook}(\gamma) = \left(\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) - \mathrm{maj}(\tau') + 1, 1^{n-2-\mathrm{ht}(\gamma)}\right)$$

Lemma 26 : For all $k \leq n-2$ the map $\underline{e_{k+}^{\perp}}$ is a well-defined map.

Proof. Let us first notice that hook-shaped standard tableaux are uniquely determined by their decent set. Indeed the first column is strictly increasing and all other entries are in the first row. Therefore, an entry, i is in the descent set if and only if $i+1$ is not in the first row. For fixed k, $\gamma \in T_n^k$ has at least k east steps, by definition of T_n^k, since $n-2-\mathrm{ht}(\gamma)$ is the number of east steps. Additionally, we construct k elements in the descent set. This corresponds to the descent set of a unique tableau of shape $(n-k,k)$. The p_i's are weakly increasing and are subtracted from strictly decreasing numbers hence the k numbers of the descent set that we constructed are all distinct and smaller or equal to $n-1$. Moreover, $\min(\mathrm{Des}(\tau')) = n-k-p_k \geq 2$, because $p_k \leq n-2-k$. Finally, the constructed path is of length $n-2-k$ with as at least p_k north steps, making it an element of $T_{n,k}^{+}$. $\square$

Lemma 27 : For all k the map $\underline{e_{k+}^{\perp}}$ is a bijection such that :

$$\mathrm{hook}(\underline{e_{k+}^{\perp}}(\gamma)) = \left(\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) + 1, 1^{n-2-\mathrm{ht}(\gamma)-k}\right).$$

Proof. Let $\gamma, \beta \in T_n$ such that $\underline{e_{k+}^{\perp}}(\gamma) = \underline{e_{k+}^{\perp}}(\beta)$. Let $p_1, \ldots, p_k$ (respectively $r_1 \ldots r_k$) be integers such that p_i (respectively r_i) gives the number of north steps before the i-th east step in γ (respectively β). Because $\underline{e_{k+}^{\perp}}(\gamma) = \underline{e_{k+}^{\perp}}(\beta)$ they are associated to the same descent set, ergo the same tableau (this is only true when the tableau is hook-shaped). We must have $r_i = p_i$ for all i. This means the paths γ and β are identical up to the k-th east step. But the rest of the paths are also identical as a result of $\underline{e_{k+}^{\perp}}(\gamma) = \underline{e_{k+}^{\perp}}(\beta)$. So $\gamma = \beta$ and $\underline{e_{k+}^{\perp}}$ is an injection. Given a path γ in $T_{n,k}^{+}$ with associated tableau τ we can construct $p_1, \ldots, p_k$ by ordering the set $\{n - \pi \mid \pi \in \mathrm{des}(\tau')\}$ and subtracting the i-th number by i. Since $\gamma \in T_{n,k}^{+}$ we know $\gamma = N^{p_k}\gamma'$. Hence, we can construct the path $\beta = N^{p_1}EN^{p_2-p_1}E \cdots EN^{p_k-p_{k-1}}E\gamma'$. It is easy to see that $\underline{e_{k+}^{\perp}}(\beta) = \gamma$ is a consequence of $\underline{e_{k+}^{\perp}}$ taking out the k first east steps. Because we start at height k in $T_{n,\mathrm{des}(\tau')}$, by only erasing east steps we increase the height of $\underline{e_{k+}^{\perp}}(\gamma)$ by exactly k. For this reason we have $\mathrm{ht}(\gamma) = \mathrm{ht}(\underline{e_{k+}^{\perp}}(\gamma)) - k$. Notice that $\mathrm{maj}(\tau') - k$ corresponds to the number of boxes over the part of the path in the first k columns. Therefore,

$\operatorname{area}(\gamma) = \operatorname{area}(\underline{e^{\perp}_{k+}}(\gamma)) - \operatorname{maj}(\tau') + k$ and we have the claimed equality. $\square$

We obtain a similar result for $\underline{e^{\perp}_{k-}}$.

Lemma 28 : For all $k \leq n-2$ the map $\underline{e^{\perp}_{k-}}$ is a well-defined map.

Proof. Has before hook-shaped standard tableaux are uniquely determined by their decent set. For fixed k, $\gamma \in T_n^{k-1} \backslash \{E^n - 2\}$ has at least $k-1$ east steps, by definition. Furthermore, the k numbers of the constructed descent set are smaller or equal than $n-1$. We need to show that they are all distinct and the k elements in the descent set will correspond to the descent set of a unique tableau of shape $(k+1, 1^{n-k-1})'$. Before we do so, let γ be in $T_n^{k-1} \backslash \{E^{n-2}\}$. If $h \leq k-1$ then $\underline{e^{\perp}_{k-}}(\gamma)$ is associated to a tableau such that $1 \in \operatorname{Des}(\tau')$. Since γ as at least $k-1$ east steps by definition of T_n^{k-1}, we know that $p_{k-1} \leq n-k-1$. Hence, $\min(\operatorname{Des}(\tau') \backslash \{1\}) \geq 2$. The p_i's are weakly increasing and are subtracted from strictly decreasing numbers, in consequence the elements created for are descent set are all distinct. Moreover, $n-k+1-p_{k-1} = \min(\operatorname{Des}(\tau') \backslash \{1\})$ and $h \leq k-1$ which yields $p_k - 1 \geq 0$. So we have $p_k \geq n-k+1-\min(\operatorname{Des}(\tau') \backslash \{1\})$. Thus $\underline{e^{\perp}_{k-}}(\gamma)$ is in $V_{n,k}$. If $h > k-1$ then $p_{k-1} = 0$. The height of the path is bounded by the relation $h \leq n-2$ for this reason $n-(k-1) > h-k-2$. Ergo $\underline{e^{\perp}_{k-}}(\gamma)$ can be associated with a tableau such that $\min(\operatorname{Des}(\tau')) = h-k+2 > 1$. The $p_i = 0$ for all i such that $1 \leq i \leq k-1$, thus we have constructed k distinct elements for the descent set. Furthermore, the path begins with $h-k+1$ east steps by definition of the map. Consequently, $\underline{e^{\perp}_{k-}}(\gamma)$ is in $V_{n,k}$. The erased steps do not depend on the maximum value so one might notice that in the case $1 = h-k+2$ we obtain the same tableau and the same path whether we "choose" 1 or $h-k+2$. Therefore, the map is well defined. $\square$

Lemma 29 : For all k the map $\underline{e^{\perp}_{k-}}$ is a bijection such that :

$$\operatorname{hook}(\underline{e^{\perp}_{k-}}(\gamma)) = \left(\operatorname{area}(\gamma) + \operatorname{ht}(\gamma), 1^{n-1-\operatorname{ht}(\gamma)-k}\right).$$

Proof. We start by showing the map is injective, let $\gamma, \beta \in T_n \backslash \{E^{n-2}\}$ such that $\underline{e^{\perp}_{k-}}(\gamma) = \underline{e^{\perp}_{k}}(\beta)$. Let $p_1, \ldots, p_{k-1}$ (respectively $r_1 \ldots r_{k-1}$) be integers such that

p_i (respectively r_i) gives the number of north steps before the i-th east step in γ (respectively β). Let h_γ (respectively h_β) be the number of east steps before the first north step in γ (respectively β). It was proven in the previous lemma that $h = \min(\mathrm{Des}(\tau')) + k - 2$ or $h =\leq k - 1$. Therefore, if $1 = \min(\mathrm{Des}(\tau'))$ we have $h_\gamma \leq k-1$, $h_\beta \leq k-1$ and the proof is very similar to Lemma 27. If $\min(\mathrm{Des}(\tau')) > 1$, we have $h_\gamma - k + 2 = \min(\mathrm{Des}(\tau'))$ and $h_\beta - k + 2 = \min(\mathrm{Des}(\tau'))$. Ergo $h_\gamma = h_\beta$ and all the $p_i = 0$, $1 \leq i \leq k-1$. Consequently γ and β have the same number of east steps before the first north step and the paths after the first north step are the same since $\underline{e^{\perp}_{k-}}(\gamma) = \underline{e^{\perp}_{k}}(\beta)$. Hence, $\gamma = \beta$

We now show that the map is surjective. Let γ be a path in $V_{n,k}$. The set $V_{n,k}$ is a union of sets of paths therefore it can be associated to the tableau τ corresponding to the set it came from in the union. If $1 \in \mathrm{Des}(\tau')$ let $\{1 < d_2 < \cdots < d_k\}$ be the descent set of τ' and let $\gamma = N^j E \tilde{\gamma}$ (respectively $\gamma = N^{n-2-k}$). Remember that by definition of $V_{n,k}$ the length of the path is $n-2-k$. Then the path :

$$N^{n-1-d_k} E N^{d_k - d_{k-1} - 1} E N^{d_{k-1} - d_{k-2} - 1} E \cdots N^{d_3 - d_2 - 1} E N^{j-n+k+d_2} E \tilde{\gamma}$$

(respectively $N^{n-1-d_k} E N^{d_k - d_{k-1} - 1} E N^{d_{k-1} - d_{k-2} - 1} E \cdots N^{d_3 - d_2 - 1} E N^{d_2 - 2}$) is of length $n-2$ since we only added $k-1$ east steps and 1 north steps. Hence, γ is in T_n because $j \geq n-k-d_2$ (respectively $n-k-2 \geq n-k-d_2$) by definition of $V_{n,k}$. Moreover, there are k east steps before $\tilde{\gamma}$ and $j+1$ north steps (respectively $k-1$ east steps and $n-k-1$ north steps) therefore :

$$\underline{e^{\perp}_{k-}}(N^{n-1-d_k} E N^{d_k - d_{k-1} - 1} E N^{d_{k-1} - d_{k-2} - 1} E \cdots N^{d_3 - d_2 - 1} E N^{j-n+k+d_2} E \tilde{\gamma}) = N^j E \tilde{\gamma} = \gamma \in T_{n,\mathrm{des}(\tau')}.$$

$$(\text{respectively,} \underline{e^{\perp}_{k-}}(N^{n-1-d_k} E N^{d_k - d_{k-1} - 1} E N^{d_{k-1} - d_{k-2} - 1} E \cdots N^{d_3 - d_2 - 1} E N^{d_2 - 2}) = N^{n-k-2} = \gamma)$$

If $1 \notin \mathrm{Des}(\tau')$ then $\{d_1 < n-k+1 < \cdots < n-1\}$ is the descent set of τ' and let $\gamma = E^r \tilde{\gamma}$. This means the path :

$$E^{d_1 + k - 2} N E^{r+1-d_1} \tilde{\gamma}$$

is in T_n because $r + 1 \geq d_1$ by definition of $V_{n,k}$. Moreover, there are $r+k-1$ east steps before $\tilde{\gamma}$ and 1 north steps therefore :

$$\underline{e_{k-}^{\perp}}(E^{d_1+k-2}NE^{r+1-d_1}\tilde{\gamma}) = \gamma \in T_{n,\mathrm{des}(\tau')}.$$

Thus $\underline{e_{k-}^{\perp}}$ is a bijection.

By erasing the $k-1$ first east steps and the first north step, we increase the height of $\underline{e_{k-}^{\perp}}(\gamma)$ by exactly $k-1$, since we start at height k in $T_{n,\mathrm{des}(\tau')}$. Hence, $\mathrm{ht}(\gamma) = \mathrm{ht}(\underline{e_{k-}^{\perp}}(\gamma)) - k + 1$. Notice that $\mathrm{maj}(\tau') - k + 1 - \max(1, h - k + 2)$ corresponds to the number of boxes over the part of the path in the first $k-1$ columns and $\max(1, h-k+2)$ correspond to the boxes filled by deleting the first north step. Therefore, $\mathrm{area}(\gamma) = \mathrm{area}(\underline{e_{k-}^{\perp}}(\gamma)) - \mathrm{maj}(\tau') + k - 1$ and we have the claimed equality. □

Note that one could add conditions to $T^{+}_{n,n-1}$, $T^{-}_{n,n-1}$, $V_{n,n-1}$ and T_n^{n-1} to add the case $\mu = (n)$ to the map. The author thinks that it is a lot of commotion just to state that $e_{n-1}^{\perp}(s^{1^n}) = 1$. For the next proposition, we extend our maps in the following way $\underline{e_{k+}^{\perp}}(\gamma) = \emptyset$ if $\gamma \in T_n \backslash T_n^k$, $\underline{e_{k-}^{\perp}}(\gamma) = \emptyset$ if $\gamma \in T_n \backslash T_n^{k-1}$ and $s_{\mathrm{hook}(\emptyset)} = 0$. Observe that $\emptyset$ is not the empty word.

Proposition 15 : For all $k \leq n-2$, we have :

$$e_k^{\perp}\left(\langle \mathcal{E}_{n,n}, e_n\rangle|_{\mathrm{hooks}}\right) = \sum_{\gamma \in T_n} s_{\mathrm{hook}(\underline{e_{k+}^{\perp}}(\gamma))} + s_{\mathrm{hook}(\underline{e_{k-}^{\perp}}(\gamma))}.$$

In addition, $\sum_{\tau \in \mathrm{SYT}(k+1,1^{n-k-1})} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)} - e_k^{\perp}\left(\langle \mathcal{E}_{n,n}, s_{1^n}\rangle|_{\mathrm{hooks}}\right)$ has a Schur positive expansion.

Proof. By Lemma 27 and Lemma 29 $e_k^{\perp} s_{\mathrm{hook}(\gamma)} = s_{\mathrm{hook}(\underline{e_{k+}^{\perp}}(\gamma))} + s_{\mathrm{hook}(\underline{e_{k-}^{\perp}}(\gamma))}$. Furthermore, we have the disjoint union $T^{-}_{n,k} \cup T^{+}_{n,k} = \bigcup_{\tau \in \mathrm{SYT}((k+1,1^{n-k-1}))} T_{n,k}$ so :

$$\sum_{\tau \in \mathrm{SYT}((k+1,1^{n-k-1}))} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)} = \sum_{\gamma \in T^{+}_{n,k}} s_{\mathrm{hook}(\gamma)} + \sum_{\gamma \in T^{-}_{n,k}} s_{\mathrm{hook}(\gamma)}$$

The map $\underline{e_{k+}^{\perp}}$ (respectively $\underline{e_{k-}^{\perp}}$) is an injection from T_n into $T^{+}_{n,k}$ (respectively $T^{-}_{n,k}$) ergo the result holds. □

The last proposition gives reason to believe the main theorem holds for all μ, a hook, since the missing terms should be obtained by the restriction to shapes having two columns. If Theorem 2 is true for all μ, hook-shaped, then the difference would be given by the equation found in the next lemma.

First we will define $W_{n,k} = T^{-}_{n,k} \backslash V_{n,k}$:

$$W_{n,k} = \bigcup_{\substack{\tau\in \mathrm{SYT}(k+1,1^{n-k-1})\\ 1\notin \mathrm{Des}(\tau')}} \left\{\gamma = E^r\tilde{\gamma} \in T_{n,\mathrm{des}(\tau')} \mid 1 < r+1 < \min(\mathrm{Des}(\tau')) < n-k\right\} \tag{3.21}$$

$$\cup \bigcup_{\substack{\tau\in \mathrm{SYT}(k+1,1^{n-k-1})\\ 1\notin \mathrm{Des}(\tau'),\{n-k+1,\ldots,n-1\}\not\subset \mathrm{Des}(\tau')}} \left\{\gamma = E^r\tilde{\gamma} \in T_{n,\mathrm{des}(\tau')} \mid r+1 \geq \min(\mathrm{Des}(\tau'))\right\} \tag{3.22}$$

$$\cup \bigcup_{\substack{\tau\in \mathrm{SYT}(k+1,1^{n-k-1})\\ 1\in \mathrm{Des}(\tau')}} \left\{\gamma = N^jE\tilde{\gamma} \in T_{n,\mathrm{des}(\tau')} \mid 0 \leq j < n-k-\min(\mathrm{Des}(\tau')\backslash\{1\})\right\} \tag{3.23}$$

$$\cup \bigcup_{\substack{\tau\in \mathrm{SYT}(k+1,1^{n-k-1})\\ 1\notin \mathrm{Des}(\tau')}} \left\{\gamma = N^jE\tilde{\gamma} \in T_{n,\mathrm{des}(\tau')} \mid 0 < j < n-k-\min(\mathrm{Des}(\tau'))\}\right\} \tag{3.24}$$

One can easily check that these sets are complementary. Note that Proposition 15 is also a proof of Theorem 2 for the case $(n-1,n)$ since $W_{n,n-2} = \emptyset$.

Lemma 30 : Let $A = \sum_{\tau\in \mathrm{SYT}(k+1,1^{n-k-1})} \sum_{\gamma\in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)} - e_k^{\perp}\left(\langle \mathcal{E}_{n,n}, e_n\rangle|_{\mathrm{hooks}}\right)$ then for $k \geq 2$ we have :

$$A = \sum_{\substack{\tau\in \mathrm{SYT}(k+1,1^{n-k-1})\\ 1\in \mathrm{Des}(\tau)\\ \min(\mathrm{Des}(\tau'))<n-k}} \sum_{r=1}^{\min(\mathrm{Des}(\tau'))-2} \sum_{\gamma\in T_{n-r,k+1}} s_{\mathrm{Shape}\,1(\gamma)} + \sum_{j=1}^{n-k-1-\min(\mathrm{Des}(\tau'))} \sum_{\gamma\in T_{n-1,j+k}} s_{\mathrm{Shape}\,2(\gamma)} \tag{3.25}$$

$$+\sum_{\substack{\tau\in \mathrm{SYT}(k+1,1^{n-k-1})\\ 1\in \mathrm{Des}(\tau)\\ \{n-k+1,\ldots,n-1\}\not\subset \mathrm{Des}(\tau')}} \sum_{r=\min(\mathrm{Des}(\tau'))-1}^{n-k-2} \sum_{\gamma\in T_{n-r,k+1}} s_{\mathrm{Shape}\,1(\gamma)} \tag{3.26}$$

$$+\sum_{\substack{\tau\in \mathrm{SYT}(k+1,1^{n-k-1})\\ 1\notin \mathrm{Des}(\tau)}} \sum_{j=0}^{n-k-1-\min(\mathrm{Des}(\tau')\backslash\{1\})} \sum_{\gamma\in T_{n-1,k+j}} s_{\mathrm{Shape}\,2(\gamma)} \tag{3.27}$$

Where Shape 1(γ) is the partition $\big(\text{area}(\gamma)+\text{ht}(\gamma)+1-\text{maj}(\tau')+kr, 1^{n-2-\text{ht}(\gamma)}\big)$, and Shape 2$(\gamma)$ is the partition $\big(\text{area}(\gamma)+\text{ht}(\gamma)+1-\text{maj}(\tau')+(j+k), 1^{n-2-\text{ht}(\gamma)}\big)$. In particular, for $k=1$ we have :

$$A=\sum_{m=2}^{n-2}\sum_{r=1}^{m-2}\sum_{\gamma\in T_{n-r,2}} s_{\text{Shape }1'(\gamma)}+\sum_{j=1}^{n-2-m}\sum_{\gamma\in T_{n-1,j+1}} s_{\text{Shape }2'(\gamma)} \tag{3.28}$$

Where Shape $1'(\gamma)$ is the partition $\big(\text{area}(\gamma)+\text{ht}(\gamma)+1-m+r, 1^{n-2-\text{ht}(\gamma)}\big)$, and Shape $2'(\gamma)$ is the partition $\big(\text{area}(\gamma)+\text{ht}(\gamma)+2-m+j, 1^{n-2-\text{ht}(\gamma)}\big)$.

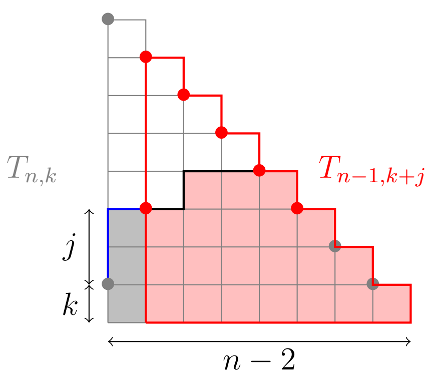


Figure 3.22 Comparing the path $NNEENEE = N^2E\tilde{\gamma} \in T_{10,1}$ and the path $\tilde{\gamma} \in T_{9,3}$. Same height $\text{ht}(N^2E\tilde{\gamma}) = \text{ht}(\tilde{\gamma})$, though $\text{area}(N^2E\tilde{\gamma}) = \text{area}(\tilde{\gamma})+3$.

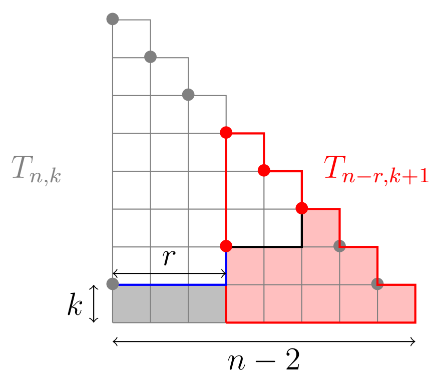


Figure 3.23 Comparing the path $EEENEEN = E^3N\tilde{\gamma} \in T_{10,1}$ and the path $\tilde{\gamma} \in T_{7,2}$. Same height, $\text{ht}(E^3N\tilde{\gamma}) = \text{ht}(\tilde{\gamma})$ though $\text{area}(E^3N\tilde{\gamma}) = \text{area}(\tilde{\gamma})+3$.

The last equality is a consequence of $W_{n,k}=T^{-}_{n,k}\backslash V_{n,k}$. Up to a slight change in the area statistic, the paths $\gamma\in T_{n,k}$ such that $\gamma=N^jE\tilde{\gamma}$ are the same as the paths $\tilde{\gamma}\in T_{n-1,j+k}$ (see Figure 3.22). To obtain the same hook-shaped partition we only need to add $j+k$ to the area. Hence, the sum of Line (3.27) corresponds to the set in Line (3.23) and the second sum of Line (3.25) to the set in Line (3.24).

Proof. We know that $T^k_n = T^+_{n,k}\cup T^-_{n,k}$, so by Proposition 15, Lemma 27 and

Lemma 29 we have :

$$
\begin{aligned}
A &= \sum_{\gamma\in T^{+}_{n,k}} s_{\mathrm{hook}(\gamma)} + \sum_{\gamma\in T^{-}_{n,k}} s_{\mathrm{hook}(\gamma)} - \sum_{\gamma\in T^{+}_{n,k}} s_{\mathrm{hook}(\gamma)} - \sum_{\gamma\in V_{n,k}} s_{\mathrm{hook}(\gamma)}, \\
&= \sum_{\gamma\in T^{-}_{n,k}} s_{\mathrm{hook}(\gamma)} - \sum_{\gamma\in V_{n,k}} s_{\mathrm{hook}(\gamma)}, \\
&= \sum_{\gamma\in W_{n,k}} s_{\mathrm{hook}(\gamma)}.
\end{aligned}
$$

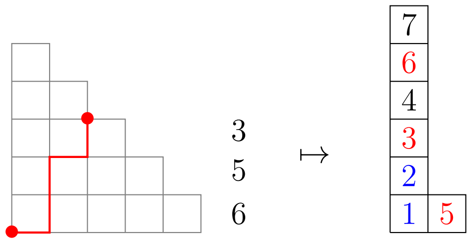
Similarly, the paths $\gamma \in T_{n,k}$ such that $\gamma = E^r N\tilde{\gamma}$ are the same as the paths $\tilde{\gamma} \in T_{n-r,k+1}$ (see Figure 3.23). To obtain the same hook-shaped partition we only need to add rk to the area. Consequently the first sum of Line (3.25) corresponds to the set in Line (3.21) and the sum of Line (3.26) to the set in line (3.22). Note that for Line (3.25) and Line (3.26) the case $r = n-k-2$ corresponds to the $T_{k+2,k+1}$ which correspond to the paths E^{n-2-k} in $T_{n,k}$ that as area $k(n-k-2)$ greater than area(ϵ), where ϵ is the only path of $T_{k+2,k+1}$. This works with the convention $T_{k+2,k+1} = T_{k+2,k}$. For the restriction to $k = 1$ one only needs to notice that for all tableaux τ in $\mathrm{SYT}(2, 1^{n-2})$ the descent set contains only one element. Therefore, two of the sums are empty and the result follows. □

3.4.1 Bijections and Starting the Second Column

If we dismiss the first part, we can see hook-shaped partitions, as partitions with one column. We don't have a formula for the restriction to partitions that have two columns, but we can start the second column. Before we prove our formula for the restriction to shapes $\{(a, 2, 1^k) \mid k \in \mathbb{N}, a \in \mathbb{N}_{\geq 2}\}$, we need preliminary result.

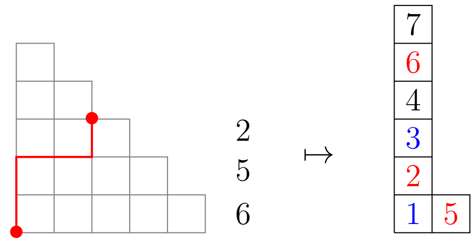
Let $T^{E}_{n,0,h}$ be the subset of paths of $T_{n,0}$ that start with an east step and have height h. Let $\mathrm{SYT}(k+1, 1^{n-k-1})_S$ be the set of tableaux of shape $(k+1, 1^{n-k-1})$ for which the descent set contains the set S. For a path γ let n_i be the number of east steps before the i-th north step. Define $\Phi_k : T^{E}_{n,0,n-k-3} \to \mathrm{SYT}(k+1, 1^{n-k-1})_{\{1,2\}}$ by $\Phi_k(\gamma)$ is the unique hook-shape tableau having $\{(n-i-n_i+1) | 1 \leq i \leq$

$\mathrm{ht}(\gamma)\} \cup \{1,2\}$ as a descent set (see Figure 3.24).



$\mathrm{Des}(\Phi_1(ENNEN)) = \{1,2,3,5,6\}$

Figure 3.24 An example of the map Φ_1 when $n = 7$ and $k = 1$. With $ENNEN \in T^E_{7,0,3}$.



$\mathrm{Des}(\Omega^1_1(NNEEN)) = \{1,2,3,5,6\}$

Figure 3.25 An example of the map Ω^1_1 when $n = 7$, $k = 1$ and $j = 1$. With $NNEEN \in T^{(1)}_{7,0,3}$.

Lemma 31 : The map Φ_k is a well-defined bijective map. Additionally, for $\gamma \in T^E_{n,0}$ we have :

$$\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) + 1 = \mathrm{maj}(\Phi_k(\gamma)) - \mathrm{des}(\Phi_k(\gamma))$$

In particular the image of Φ_k is the set of tableaux of shape $(k+1, 1^{n-k-1})$ such that the descent set contains the set $\{1,2\}$.

Proof. For all hook-shaped tableaux, the descent set is given by subtracting one to each entry that does not lie in the first row. Because the remainder of the entries are in the first row the descent set uniquely determines a hook-shaped tableau. Moreover, a path, γ of $T^E_{n,0,n-k-3}$ starts with one east step and has exactly $k+1$ east steps. Ergo for all i such that $1 \leq i \leq n-k-3$ we have $1 \leq n_i \leq k+1$. This yields $n - i + 1 > n - i - n_i + 1 \geq 3$. For all path γ, the set $\{n_1, \ldots, n_{n-k-3}\}$ is an increasing sequence of positive integers therefore the $n-k-3$ elements of the descent set created, are all distinct values of $\{1, \ldots, n-1\}$. Hence, Φ_k associates γ to a unique tableau of shape $(k+1, 1^{n-k-1})$ consequently the maps, Φ_k is well defined. Let $\gamma, \pi \in T^E_{n,0}$ be such that $\Phi_k(\gamma) = \Phi_k(\pi)$. We have previously seen that the height of the path determines the shape of the tableau thus γ and π are of the same height. Let :

$$\{1 < 2 < d_1 < d_2 < \cdots < d_{n-k-3}\} = \mathrm{Des}(\Phi_k(\gamma)) = \mathrm{Des}(\Phi_k(\pi)). \tag{3.29}$$

By construction we must have $d_{n-k-i-2} = n-i-n_i+1$ and therefore we know the number of east step before each north step which uniquely determines a path. Ergo $\gamma = \pi$. Now we show that the map is surjective. Let $\tau \in \mathrm{SYT}(k+1, 1^{n-k-1})_{\{1,2\}}$ with $\mathrm{Des}(\tau) = \{1 < 2 < d_1 < d_2 < \cdots < d_{n-k-3}\}$. Then the path :

$$\gamma = E^{n-d_{n-k-3}} N E^{d_{n-k-3}-d_{n-k-4}-1} N \cdots E^{d_2-d_1-1} N E^{d_1-3},$$

is in $T^E_{n,0,n-k-3}$ since $d_{n-k-3} \leq n-1$ and the path has $n-k-3$ north steps and $n-2$ steps. Moreover, $\Phi_k(\gamma) = \tau$. Finally, the area of a path γ is equal to $\sum_{i=1}^{n-k-3} n-i-1-n_i$, $\mathrm{maj}(\Phi_k(\gamma)) = 3 + \sum_{i=1}^{n-k-3} n-i-n_i+1$ and $\mathrm{Des}(\Phi_k(\gamma)) = n-k-1$ which yields :

$$\begin{aligned}
\mathrm{ht}(\gamma) + 1 + \mathrm{area}(\gamma) &= n-k-2 + \sum_{i=1}^{n-k-3} (n-i-n_i-1) \\
&= -n+k+4 + \sum_{i=1}^{n-k-3} (n-i-n_i+1) \\
&= -(\mathrm{Des}(\Phi_k(\gamma)) - 2) + 1 + (\mathrm{maj}(\Phi_k(\gamma)) - 3) \\
&= \mathrm{maj}(\Phi_k(\gamma)) - \mathrm{Des}(\Phi_k(\gamma))
\end{aligned}$$

$\square$

Let $T^{(j)}_{n,0,h}$ be the set of paths of $T_{n,0}$ that start with a north step, end with exactly j north steps and has height h. For $0 \leq k \leq n-3$, we also define $\Omega^j_k : T^{(j)}_{n,0,n-k-3} \to \mathrm{SYT}(k+1, 1^{n-k-1})_{\{1,\ldots,j+2,n-1\}}$ by $\Omega^j_k(\gamma)$ is the unique hook-shape tableau having $\{(n-i-n_i) | 1 \leq i \leq \mathrm{ht}(\gamma)\} \cup \{1, j+2\}$ as a descent set (see Figure 3.25). As before, in a path γ the n_i's are the number of east steps before the i-th north step. Note that $0 \leq j \leq n-k-3$, since the path cannot have more north steps than its height.

Lemma 32 : The maps Ω^j_k are well defined bijective maps. Additionally, for $\gamma \in T^{(j)}_{n,0,n-k-3}$ we have :

$$\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) + 1 = \mathrm{maj}(\Omega^j_k(\gamma)) - (j+2)$$

Furthermore the image of Ω_k^j is the set of all tableaux, τ, of shape $(k+1, 1^{n-k-1})$, such that $\{1, 2, \ldots, j+2, n-1\} \subseteq \mathrm{Des}(\tau)$.

Proof. As before, the descent set uniquely determines a hook-shaped tableau. Furthermore, a path of height $n-k-3$ has exactly $k+1$ east steps. Hence, for all i, $0 \leq n_i \leq k+1$. Which yields $n-i \geq n-i-n_i \geq n-k-i-1 \geq 2$ as a result of $i \leq n-k-3$. In particular the paths start with a north step ergo $n_1 = 0$ and $n-1 \in \mathrm{Des}(\Omega_k^j(\gamma))$. Additionally, for all i such that $\mathrm{ht}(\gamma) - j + 1 \leq i \leq \mathrm{ht}(\gamma)$ the value of n_i is $k+1$, since the paths and with exactly j north steps. In consequence we have $j+1 \geq n-i-n_i = n-k-1-i \geq 2$. For all path γ, the set $\{n_1, \ldots, n_{\mathrm{ht}(\gamma)}\}$ is an increasing sequence of positive integers thus $n-i-n_i > n-(i+1)-n_{i+1}$ and the $n-k-3$ elements of the descent set created are all distinct. We then have j distinct values between 2 and $j+1$ for this reason $\{2, \ldots, j+1, n-1\}$ is a subset of $\mathrm{Des}(\Omega_k^j(\gamma))$. Finally $n_{h(\gamma)-j} > k+1$, by definition of $T^{(j)}_{n,0,n-k-3}$. Therefore, $j+2$ is not one of the $n-k-3$ elements created and the map Ω_k^j associates γ to a unique tableau of shape $(k+1, 1^{n-k-1})$. Consequently it is well defined. To show the maps are bijective we can use the same proof as in Lemma 31 if we swap the descent set on Line (3.29) for $\{1 < d_1 < \cdots < d_j < j+2 < d_{j+1} < \cdots < d_{\mathrm{ht}(\gamma)-1} < n-1\}$ and the path on Line (3.4.1) for $NE^{n-d_{n-k-4}-2}NE^{d_{n-k-4}-d_{n-k-5}-1}N \cdots E^{d_{j+2}-d_{j+1}-1}NE^{d_{j+1}-j-2}N^j$. The area of a path γ is equal to $\sum_{i=1}^{\mathrm{ht}(\gamma)} n-i-n_i-1$, $\mathrm{maj}(\Omega_k^j(\gamma)) = 3+j+\sum_{i=1}^{\mathrm{ht}(\gamma)} n-i-n_i$. Ergo :

$$
\begin{aligned}
\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) + 1 &= \mathrm{ht}(\gamma) + 1 + \sum_{i=1}^{\mathrm{ht}(\gamma)} (n-i-1-n_i) \\
&= 1 + \sum_{i=1}^{\mathrm{ht}(\gamma)} (n-i-n_i) \\
&= \mathrm{maj}(\Omega_k^j(\gamma)) - (j+2)
\end{aligned}
$$

$\square$

Notice that $T^E_{n,0} \cup \bigcup_{j,k} T^{(j)}_{n,0,n-k-3} = T_{n,0}$. By lifting the formula for hook-shaped Schur functions in two variables, we get a first formula for the alternant restricted to the shape $R = \{(a, 2, 1^k) \mid k \in \mathbb{N}, a \in \mathbb{N}_{\geq 2}\}$.

Proposition 16 : For $\mathcal{R} = \{(a, 2, 1^k) \mid k \in \mathbb{N}, a \in \mathbb{N}_{\geq 2}\}$, if $\langle \mathcal{E}_{n,n}, s_{(k+1),1^{n-k-1}} \rangle = e_k^{\perp} \langle \mathcal{E}_{n,n}, s_{1^n} \rangle$, then :

$$\langle \mathcal{E}_{n,n}, e_n \rangle|_{\mathcal{R}} = \sum_{k=1}^{n-4} \sum_{i=2}^{n-k-2} \sum_{\substack{\tau \in \mathrm{SYT}(k+1, 1^{n-k-1}) \\ 1 \in \mathrm{Des}(\tau) \\ \{1,\ldots,i,n-1\} \not\subseteq \mathrm{Des}(\tau)}} s_{\mathrm{Shape}(\gamma)} \tag{3.30}$$

$$= \sum_{k=1}^{n-4} \sum_{i=2}^{n-k-2} \sum_{\substack{\tau \in \mathrm{SYT}(k+1, 1^{n-k-1}) \\ 1 \in \mathrm{Des}(\tau) \\ \{1,\ldots,i\} \not\subseteq \mathrm{Des}(\tau)}} s_{\mathrm{Shape}(\gamma)} + \sum_{\substack{\tau \in \mathrm{SYT}(k+1, 1^{n-k-1}) \\ n-1 \notin \mathrm{Des}(\tau) \\ \{1,\ldots,i\} \subseteq \mathrm{Des}(\tau)}} s_{\mathrm{Shape}(\gamma)} \tag{3.31}$$

where Shape(γ) gives the partition $\left(\mathrm{maj}(\tau) - i, 2, 1^{k-1}\right)$.

Proof. Let $h_k(q) = \psi\left(\sum_{d=0}^{k} (-1)^{k-d} (e_d^{\perp} \langle \mathcal{E}_{n,n}, e_n \rangle|_{\mathcal{V}_1})^{\langle 2 \rangle}\right) q^{k-d} t^{-1}$, following the notations in Lemma 22. Note that $|_{\mathcal{V}_1}$ is the same as $|_{\mathrm{hooks}}$. According to (Wallace, 2020b) (Chapter 2), we have :

$$h_k = \sum_{\substack{\tau \in \mathrm{SYT}(k+1, 1^{n-k-1}) \\ \{1,2\} \subseteq \mathrm{Des}(\tau)}} q^{\mathrm{maj}(\tau) - \mathrm{des}(\tau)} + \sum_{\substack{\tau \in \mathrm{SYT}(k+1, 1^{n-k-1}) \\ 1 \in \mathrm{Des}(\tau)}} \sum_{i=2}^{n-k-2} q^{\mathrm{maj}(\tau) - i}$$

Let $g_k(q) = \psi\left((\sum_{d=0}^{k} (-1)^{k-d} e_d^{\perp} \langle \mathcal{E}_{n,n}, e_n \rangle|_{\mathcal{V}_1})^{\langle 2 \rangle}|_{\mathcal{V}_1}\right) q^{k-d} t^{-1}$. Since there is only one tableau of shape (n) and it has an empty descent set, we already know from Equation (3.16) that :

$$g_k(q) = \sum_{d=0}^{k} (-1)^{k-d} \sum_{\substack{\gamma \in T_{n,0} \\ \mathrm{ht}(\gamma) = n-d-2}} q^{\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) + 1 + k - d} + \sum_{\substack{\gamma \in T_{n,0} \\ \mathrm{ht}(\gamma) = n-d-3}} q^{\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) + 1 + k - d}$$

$$= \sum_{\substack{\gamma \in T_{n,0} \\ \mathrm{ht}(\gamma) = n-k-3}} q^{\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) + 1}$$

Recall the remark under Equation (3.16) states that if $d = 0$ we do not account for the paths of height $n-2$. Notice k is only defined for $0 \leq k \leq n-3$. More over :

$$(e_k^{\perp}(\langle \mathcal{E}_{n,n}, e_n\rangle|_{\mathcal{V}_2})^{\langle 2\rangle})|_{\mathcal{V}_1} = (e_k^{\perp}\langle \mathcal{E}_{n,n}, e_n\rangle)^{\langle 2\rangle}|_{\mathcal{V}_1} - (e_k^{\perp}(\langle \mathcal{E}_{n,n}, e_n\rangle|_{\mathcal{V}_1})^{\langle 2\rangle})|_{\mathcal{V}_1}$$

Then by Lemma 22 we have :

$$\Psi(\langle \mathcal{E}_{n,n}, e_n\rangle|_{\mathcal{V}2})(q,t) = \sum_{k=0}^{n-3}(h_k(q) - g_k(q))t^k.$$

Since for the sets $T^{(j)}_{n,0,n-k-3}$, $T^{E}_{n,0,n-k-3}$ partitions $T_{n,0,n-k-3}$, we can apply the maps Φ_k and Ω_k^{i-2} on the paths of g_k. Ergo by Lemma 31 and Lemma 32 we can cancel out all the negative terms and obtain the result has stated. $\square$

Before we give a simpler interpretation in terms of $T_{n,k}$, we need a new map. Let β_d be a family of maps, $\beta_d : \mathrm{SYT}(d+1, 1^{n-d-1})_{\{1\}} \to T_{n,0,n-d-2}$ defined, for $\mathrm{Des}(\tau) = \{1 < r_2 \cdots < r_{n-d-1}\}$, by :

$$\beta_d(\tau) = E^{n-1-r_{n-d-1}} N E^{r_{n-d-1}-r_{n-d-2}-1} N \cdots N E^{r_{n-d-i+1}-r_{n-d-i}-1} N \cdots E^{r3-r_2-1} N E^{r_2-2}.$$

(See Figure 3.26 for an example.)

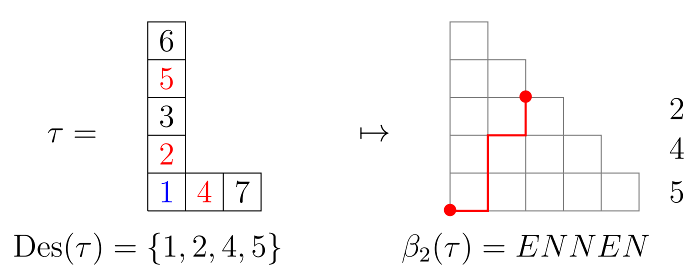


Figure 3.26 An example of the map β_2 when $n=7$ and $d=2$. The image $\beta_2(\tau)$ is in $T_{7,0,3}$.

Lemma 33 : The maps β_d are well-defined bijections. Moreover, for all τ we have $\mathrm{maj}(\tau) = \mathrm{area}(\beta_d(\tau)) + \mathrm{ht}(\beta_d(\tau)) + 1$.

Proof. For all τ we have the image by β_d is a path with $n-d-2$ north steps by construction. So it as height $n-d-2$. The number of steps of the path is :

$$(n-d-2)+(n-1-r_{n-d-1})+(r_2-2)+\sum_{i=1}^{n-d-3} r_{n-d-i}-r_{n-d-i-1}-1=n-2$$

So the image is a path of $T_{n,0,n-d-2}$. Given $\beta_d(\tau)=\beta_d(\pi)$ one could reverse engineer and find $\mathrm{Des}(\tau)=\mathrm{Des}(\pi)$. Conversely, the staircase shape of the grid $T_{n,0}$ assures us that the row area is a set of distinct numbers in $\{1,\ldots,n-2\}$. Hence, to each path of height $n-d-2$ there is a tableau with a descent set corresponding to adding one to each element of the set of row area and appending the element 1. Hence, β_d is indeed a bijection. Let n_i be the number of east steps before the i-th north step of $\beta(\tau)_d$. Then $n_i=n-r_{n-d-i}-i$ by definition of the map β_d. The area of a path is equal to $\sum_{i=1}^{\mathrm{ht}(\beta_d(\tau))} n-i-1-n_i$ therefore $\mathrm{area}(\beta_d(\tau))=\sum_{i=1}^{\mathrm{ht}(\beta_d(\tau))}(r_{n-d-i}-1)=(\mathrm{maj}(\tau)-1)-\mathrm{ht}(\beta_d(\tau))$. □

We can now improve the aspect of our formula.

Proposition 17 : For $R=\{(a,2,1^k)\mid k\in\mathbb{N}, a\in\mathbb{N}_{\geq 2}\}$, if $\langle\mathcal{E}_{n,n},s_{(k+1),1^{n-k-1}}\rangle = e_k^{\perp}\langle\mathcal{E}_{n,n},s_{1^n}\rangle$, then :

$$\langle\mathcal{E}_{n,n},e_n\rangle|_{\mathcal{R}}=\sum_{i=2}^{n-3}\sum s_{\mathrm{Shape}\,T(\gamma)},$$

where the second sum is over paths $\gamma\in T_{n,0}$, such that $\gamma\neq N\tilde{\gamma}N^{i-1}$ and $i\leq \mathrm{ht}(\gamma)\leq n-3$. Additionally, $\mathrm{Shape}\,T(\gamma)=\big(\mathrm{area}(\gamma)+\mathrm{ht}(\gamma)+1-i,2,1^{n-3-\mathrm{ht}(\gamma)}\big)$.

Proof. For a fixed k and i the sum in Equation (3.30) is over all tableaux in $\mathrm{SYT}(k{+}1,1^{n-k-1})_{\{1\}}$ for which the descent set doesn't contain the subset $\{1,2,\ldots,i,n-1\}$. Using the bijection in Lemma 33, we obtain a sum over all paths in $T_{n,0,n-k-2}$ that cannot be written as $N\tilde{\gamma}N^{i-1}$. For $\gamma\in T_{n,0,n-k-2}$ $n-3-\mathrm{ht}(\gamma)=k-1$ and

$\mathrm{maj}(\tau) - i = \mathrm{area}(\beta_k(\tau)) + \mathrm{ht}(\beta_k(\tau)) + 1 - i$ we have :

$$\begin{aligned}
\langle \mathcal{E}_{n,n}, e_n \rangle|_{\mathcal{R}} &= \sum_{k=1}^{n-4} \sum_{i=2}^{n-k-2} \sum_{\substack{\gamma \in T_{n,0,n-k-2} \\ \gamma \neq N\tilde{\gamma}N^{i-1}}} s_{\mathrm{Shape}\, T(\gamma)} \\
&= \sum_{i=2}^{n-3} \sum_{k=1}^{n-i-2} \sum_{\substack{\gamma \in T_{n,0,n-k-2} \\ \gamma \neq N\tilde{\gamma}N^{i-1}}} s_{\mathrm{Shape}\, T(\gamma)} \\
&= \sum_{i=2}^{n-3} \sum_{\substack{\gamma \in T_{n,0} \\ \gamma \neq N\tilde{\gamma}N^{i-1} \\ i \leq \mathrm{ht}(\gamma) \leq n-3}} s_{\mathrm{Shape}\, T(\gamma)}
\end{aligned}$$

□

We have not found a way to show that $e_1^{\perp}(\langle \mathcal{E}_{n,n}, e_n \rangle|_{\mathcal{R}})|_{\text{hooks}}$ is equivalent to Equation (3.28). Such an equivalence would prove the case $\mu = 2, 1^{n-2}$.

3.4.2 Conclusion and Further Questions

It would be interesting to show that the formulas hold for all μ when $r = 1$ and all r when $\mu = 1^n$. As we mentioned previously, this could be obtained, for μ a hook, by having a formula for the restriction to Schur functions indexed by partitions with two columns. If one could write the q, t statistics of the m-Schröder paths in terms of Schur functions we could have a more general formula restricting only to shape $(a, b, 1^j)$ with a and b arbitrary.

Acknowledgments

I would like to thank François Bergeron for sharing a draft of (Bergeron, 2020) and for related useful discussions. Thank you to Franco Saliola for proofreading and useful comments. This is the extended version of the FPSAC 2019 contribution

(Wallace, 2019b).

CHAPITRE IV

CHEMINS, TABLEAUX ET PERMUTATION ÉVITANT UN MOTIF

4.1 Résumé de l'article

L'article présenté dans ce chapitre, soumis à «Enumerative Combinatorics and Applications», porte sur une bijection, Γ, entre les permutations qui évitent les motifs 213 et 231 et les permutations qui évitent les motifs 132 et 312. On y retrouve aussi des bijections entre des objets combinatoires connues et les chemins dans une grille escalier $T_{n,0}$ préservant et raffinant les statistiques sur ces objets. Cette bijection permet de faire certains liens entre les chemins de $T_{n,0}$, les fonctions de stationnement et les tableaux de Young standard.

Il a déjà été établi par Schensted, (Schensted, 1961), et Schützenberger, (Schützenberger, 1963), que pour une permutation π telle que $P(\pi) = P$ et $Q(\pi) = Q$ sont les tableaux obtenus via l'algorithme Robinson-Schensted, noté $\pi \xrightarrow{R-S} (P, Q)$ alors pour l'inverse on a $\pi^{-1} \xrightarrow{R-S} (Q, P)$, pour l'image miroir de la permutation on trouve $\pi^r \xrightarrow{R-S} (P', \mathrm{ev}(Q)')$ et pour le complément de la permutation on a $\pi^c \xrightarrow{R-S} (\mathrm{ev}(P)', Q')$. Ainsi, pour P et Q fixé, les compositions de l'inverse, l'image miroir et le complément produisent la majorité des permutations pour lesquels il est possible de former une paire parmi P, P', $\mathrm{ev}(P)$, $\mathrm{ev}(P)'$, Q, Q', $\mathrm{ev}(Q)$ et $\mathrm{ev}(Q)'$, avec l'algorithme Robinson-Schensted. Afin de composer toutes les paires, il manque une bijection, Γ, telle que $\Gamma(\pi) \xrightarrow{R-S} (\mathrm{ev}(P), Q)$ (voir figure 4.1). Cet article établi la bijection est proposée entre les permutations qui évitent les motifs 213 et 231 et les permutations qui évitent les motifs 132 et 312. Avant d'énoncer les théorèmes principaux, deux autres bijections sont nécessaires.

On considère l'application $\Pi_{A(213,231)}$ de l'ensemble des chemins dans une grille

$$\begin{tikzcd}[column sep=small]
{} \arrow[r, "(\cdot)^r"] & \begin{array}{l} A(132,231), \\ (Q,\mathrm{ev}(P)) \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(213,312), \\ (\mathrm{ev}(Q)'\,\mathrm{ev}(P)') \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(213,312), \\ (\mathrm{ev}(Q),P) \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(132,231), \\ (Q',P') \end{array} \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & {} \arrow[l, "(\cdot)^r"'] \\
{} \arrow[r, "(\cdot)^c"] & \begin{array}{l} A(132,312), \\ (\mathrm{ev}(P),Q) \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, dashed, red, "\Gamma(\cdot)"'] & \begin{array}{l} A(213,231), \\ (\mathrm{ev}(P)',\mathrm{ev}(Q)') \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] \arrow[d, leftrightarrow, dashed, red, "\Gamma(\cdot)"'] & \begin{array}{l} A(213,231), \\ (P,\mathrm{ev}(Q)) \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, dashed, red, "\Gamma(\cdot)"'] & \begin{array}{l} A(132,312), \\ (P',Q') \end{array} \arrow[d, leftrightarrow, dashed, red, "\Gamma(\cdot)"'] & {} \arrow[l, "(\cdot)^c"'] \\
{} \arrow[r, "(\cdot)^c"] & \begin{array}{l} A(213,231), \\ \pi \xrightarrow{R-S} (P,Q) \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(132,312), \\ (P',\mathrm{ev}(Q)') \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(132,312), \\ (\mathrm{ev}(P),\mathrm{ev}(Q)) \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(213,231), \\ (\mathrm{ev}(P)',Q') \end{array} \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & {} \arrow[l, "(\cdot)^c"'] \\
{} \arrow[r, "(\cdot)^r"] & \begin{array}{l} A(213,312), \\ (Q,P) \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] & \begin{array}{l} A(132,231), \\ (\mathrm{ev}(Q)',P') \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] & \begin{array}{l} A(132,231), \\ (\mathrm{ev}(Q),\mathrm{ev}(P)) \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] & \begin{array}{l} A(213,312), \\ (Q',\mathrm{ev}(P')) \end{array} & {} \arrow[l, "(\cdot)^r"']
\end{tikzcd}$$

Figure 4.1 L'application manquante, Γ

en forme d'escalier de taille $n-2$, noté $T_{n,0}$, vers l'ensemble des permutations de longueur $n-1$ qui évite les motifs 213 et 231, noté $A_{n-1}(213,231)$. Soit un chemin γ dans $T_{n,0}$ de hauteur $n-k-2$, on inscrit les nombres de 1 à $n-1$ sur le chemin de la façon suivante :

— On appose les étiquettes de 1 à k, en ordre croissant, de droite à gauche, sur tous les sommets de $N\gamma$ qui sont immédiatement précédés d'un pas est ($\multimap$).

— On appose les étiquettes de $k+1$ à $n-1$, en ordre croissant, du bas vers le haut, sur les sommets de $N\gamma$ qui sont immédiatement précédés par un pas nord ($\Uparrow$).

— La permutation $\Pi_{A(213,231)}(\gamma)$ est obtenue en lisant les étiquettes à partir de la fin du chemin et sera noté $\gamma_{A(213,231)}$ (voir la figure 4.2 pour un exemple).

On note γ^* le chemin γ auquel on appose les étiquettes de 1 à $n-1$ en commençant par la fin du chemin. Il est évident que la lecture des étiquettes de $N\gamma*$ précédés d'un pas est de droite à gauche et ensuite les étiquettes précédées d'un pas nord de bas en haut donne l'inverse de la permutation (voir figure 4.3). Puisque l'inverse d'une permutation qui évite les motifs 213 et 231 est une permutation qui évite 213 et 312, il est, ainsi, possible de définir $\Pi_{A(213,312)}$.

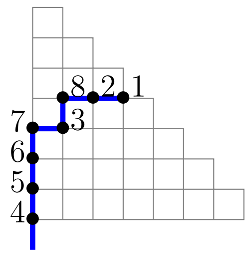


Figure 4.2 Le chemin $NNNENEE_{\pi(213,231)}$, $\Pi_{213,231}(NNNENEE) = 12837654$

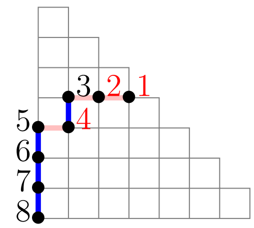


Figure 4.3 Le chemin $NNNENEE^*$ $\pi^{-1}_{NNNENEE(213,231)} = 12487653$

Cette bijection permet d'utiliser les chemins pour obtenir plusieurs statistiques sur les permutations. En particulier, l'aire du chemin correspond à l'index Major de la permutation $\Pi_{A(213,231)}(\gamma)$, la hauteur de γ donne le nombre de descentes de la permutation $\Pi_{A(213,231)}(\gamma)$ ainsi que le nombre de descentes de son inverse. Par ailleurs, l'ensemble des descentes de la permutation et de son inverse sont donnés explicitement en fonction du chemin. Avec $\Pi_{A(213,312)}$ deux chemins ayant la même hauteur donnent lieu à deux permutations dans la même classe de Knuth. Enfin, la signature de la permutation associée à γ est $(-1)^{\mathrm{aire}(\gamma^r)}$. On obtient les fonctions génératrices suivantes :

$$\sum_{\pi\in A_{n-1}(213,231)} q^{\mathrm{maj}(\pi)}t^{\mathrm{des}(\pi)} = \prod_{i=1}^{n-2}(1+tq^i),$$

$$\sum_{\pi\in A_{n-1}(213,231)} \mathrm{sign}(\pi)q^{\mathrm{maj}(\pi)}t^{\mathrm{des}(\pi)} = \prod_{i=1}^{n-2}(1+q^it(-1)^{n+i-1}),$$

$$\sum_{\pi\in A_{n-1}(213,312)} q^{\mathrm{maj}(\pi)}t^{\mathrm{des}(\pi)} = \sum_{k=1}^{n-2}\binom{n-2}{k}q^{\binom{k+1}{2}+k(n-1-k)}t^k,$$

$$\sum_{\sigma\in K(\pi)} q^{\mathrm{maj}(\pi)} = q^{\binom{\mathrm{des}(\pi)}{2}}\begin{bmatrix} n-2 \\ \mathrm{des}(\pi)\end{bmatrix}_q, \quad \text{pour } \pi\in A(213,312).$$

Les bijections $\Pi_{A(213,231)}$ et $\Pi_{A(213,312)}$ permettent également d'obtenir le P-tableau et le Q-tableau de l'algorithme Robinson-Schensted directement à partir du chemin, ainsi que leurs tableaux d'évacuation. En particulier, on trouve l'égalité

$\mathrm{maj}(Q(\Pi_{A(213,231)}(\gamma))) = \mathrm{aire}(\gamma)$.

Similairement, on considère $\Pi_{A(132,312)}$ de $T_{n,0}$ vers $A_{n-1}(132,312)$. Soit un chemin γ dans $T_{n,0}$ de hauteur $n-k-2$, on inscrit les nombres de 1 à $n-1$ sur le chemin de la façon suivante :

— On appose les étiquettes de 1 à k, en ordre croissant, du bas vers le haut, tous les sommets de γE qui sont immédiatement suivis d'un pas nord (↥).

— On appose les étiquettes de $k+1$ à $n-1$, en ordre croissant, de droite à gauche, les sommets de γE qui sont immédiatement suivis par un pas est (•–).

— La permutation $\Pi_{A(132,312)}(\gamma)$ est obtenue en lisant les étiquettes à partir de la fin du chemin et sera noté $\gamma_{A(132,312)}$ (voir la figure 4.4 pour un exemple).

Encore une fois γ^* permet de définir $\Pi_{A(132,231)}$. Il en découle des résultats simi-

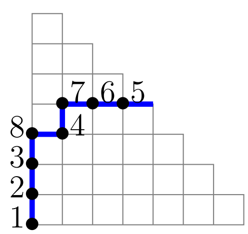


Figure 4.4 Le chemin $NNNENEE_{\pi(132,312)}$, $\Pi_{132,312}(NNNENEE) = 56748321$

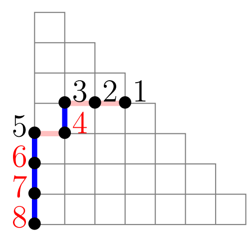


Figure 4.5 Le chemin $NNNENEE^*$ $\pi^{-1}_{NNNENEE(132,312)} = 87641235$

laires aux précédentes bijections $\Pi_{A(213,231)}$ et $\Pi_{A(213,312)}$. Les différences sont que la signature de la permutation qui évite 132 et 312 associés à γ est donnée par $(-1)^{\mathrm{aire}(\gamma)}$. Ainsi que de légères disparités dans les séries génératrices suivantes :

$$\sum_{\pi\in A_{n-1}(132,312)} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = \prod_{i=1}^{n-2}(1+(-q)^i t)$$

$$\sum_{\pi\in A_{n-1}(132,231)} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = \sum_{k=1}^{n-2}\binom{n-2}{k} q^{\binom{k+1}{2}} t^k$$

Enfin, l'action de w_0 sur les éléments de $A(213,231)$ et $A(132,312)$ peut être décrite par des rotations et des réflexions sur le chemin. On trouve alors des formules pour l'index Major et le nombre de descentes de $w_0\pi$, πw_0 et $w_0\pi w_0$ en fonction de π. Par exemple, les figure 4.6, 4.7, 4.8 et 4.9 représente l'action de w_0 pour $\pi = 615432$.

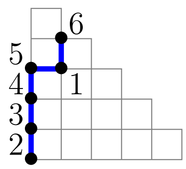


Figure 4.6 $\gamma_{\pi(213,231)}$

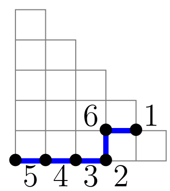


Figure 4.7 $\Pi^{-1}_{A(213,231)}(w_0\pi)$

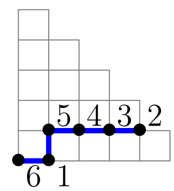


Figure 4.8 $\Pi^{-1}_{A(132,312)}(\pi w_0)$

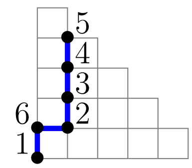


Figure 4.9 $\Pi^{-1}_{A(132,312)}(w_0\pi w_0)$

Il est maintenant possible d'énoncer les théorèmes principaux.

Théorème 3 : L'application $\Gamma = \Pi_{A(132,312)} \circ \Pi^{-1}_{A(213,231)}$ est une bijection de l'ensemble de permutations $A(213, 231)$ vers l'ensemble de permutations $A(132, 312)$. De plus, si π est une permutation qui évite les motifs 213 et 231 et $\pi \stackrel{R-S}{\longrightarrow} (P, Q)$ alors $\Gamma(\pi) \stackrel{R-S}{\longrightarrow} (\mathrm{ev}(P), Q)$. En outre, les statistiques suivantes sont préservées $\mathrm{maj}(\pi) = \mathrm{maj}(\Gamma(\pi))$ et $\mathrm{des}(\pi) = \mathrm{des}(\Gamma(\pi))$.

Pour le second théorème, on considère l'application $\mathcal{M}_{n,d}$, définie au chapitre 2, et la bijection Φ des chemins de Schröder dont la valeur de l'aire est 0 et les chemins de $T_{n,0}$, où les facteurs NE sont envoyés sur E et les facteurs D sur N.

Théorème 4 : Soit α un chemin de Schröder dont la valeur pour l'aire est 0 ayant d pas diagonaux. Si $\mathcal{M}_{n-1,d}(\alpha) = Q$, alors le Q-tableau de $\Pi_{A(132,312)} \circ \Phi(\alpha)$ est Q et le Q-tableau de $\Pi_{A(213,231)} \circ \Phi(\alpha)$ est Q. De plus, $\mathrm{maj}(Q) = \mathrm{bounce}(\alpha)$.

Enfin, on exploite le théorème 3 pour établir la valeur de dinv lorsqu'on applique Γ à la permutation associé à la fonction de stationnement. Subséquemment, on utilise les deux théorèmes ainsi qu'une bijection entre les chemins, γ, de T_n0 et les compositions, $C(\gamma)$, qui préservent l'ordre de raffinement pour établir, dans la proposition 23, les égalités suivantes :

$$\sum_{\pi \in A_{n-1}(132,312)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma \in T_n} t^{\mathrm{aire}(\gamma)} F_{1^{\mathrm{ht}(\gamma)},n-1-\mathrm{ht}(\gamma)}(X),$$

$$\sum_{\pi\in A_{n-1}(132,231)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} t^{\mathrm{aire}(\gamma)} F_{C(\gamma)}(X),$$

$$\sum_{\pi\in A_{n-1}(132,231)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} \prod_{i=1}^{\mathrm{ht}(\gamma)} (1+t^{\mathrm{aireligne}_i(\gamma)}) M_{C(\gamma)}(X),$$

$$\sum_{\pi\in A_{n-1}(213,231)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} t^{\mathrm{aire}(\gamma^r)} F_{n-1-\mathrm{ht}(\gamma),1^{\mathrm{ht}(\gamma)}}(X),$$

$$\sum_{\pi\in A_{n-1}(213,312)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} t^{\mathrm{aire}(\gamma^r)} F_{C(\gamma)}(X),$$

$$\sum_{\pi\in A_{n-1}(213,312)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} \prod_{i=1}^{\mathrm{ht}(\gamma)} (1+t^{\mathrm{aireligne}_i(\gamma)}) M_{C(\gamma^r)}(X).$$

4.2 Interactions entre cet article et les articles présentés précédemment

Dans cet article on explore des bijections entre l'objet combinatoire introduit au chapitre 3 et des objets classiques de combinatoires, comme les compositions et les permutations qui évitent certains motifs. Ces bijections préservent et raffine certaines statistiques de la combinatoire classique donnant lieu à diverses fonctions génératrices.

On rappelle qu'un chemin de Schröder, avec d diagonales, peut être vu comme une fonction de stationnement dont le mot de lecture est un mélange des permutations $n-d+1, n-d+2, \cdots, n$ et $d, d-1, \cdots, 1$. Ces mélanges sont des permutations qui évitent les motifs 132 et 312. La formule de Haglund sur les chemins de Schröder est une décomposition en fonction de Schur. Tandis que l'équation classique sur les fonctions de stationnement est une décomposition en fonction quasi symétrique fondamentale. Puisque les fonctions quasi symétriques fondamentales $F_{ides(w)}$, pour w une permutation qui évite 132 et 312, se retrouve uniquement dans la décomposition des fonctions de Schur indicé par une équerre. Donc chacune de ces permutations est associée à une unique fonction de Schur en forme d'équerre. Il est possible de montrer que si $F_{1^d,n-d}$ a un coefficient non nul dans la décomposition d'une fonction de Schur alors cette fonction de Schur est $s_{n-d,1^d}$. En particulier ce coefficient est 1. Dans le chapitre 2, on démontre une bijection constructive qui permet d'associer les chemins de Schröder dont la valeur de l'aire est 0 à un tableau de Young standard dont la forme est une équerre. Dans cet article on établie que cette bijection donne le Q-tableau du mot de lecture de la fonction de stationnement associé au chemin de Schröder. On déduit également, de façon implicite, que chacune des permutations qui évitent 213 et 231 sont associés à une unique fonction de Schur en forme d'équerre et que leurs Q-tableaux est préservé.

De plus la formule du théorème principal du chapitre 3 peut maintenant s'écrire comme une somme sur une paire de chemins dans $T_{n,s}$, car il a été démontré dans cet article que les tableaux de Young standard en forme d'équerre de taille $n-1$ sont en bijection avec les chemins de $T_{n,0}$, puisque chaque chemin est associé à un unique Q-tableau. L'aire du chemin correspond à l'index Major du tableau et la hauteur concorde avec le nombre de descentes. Par conséquent, pour μ une équerre de taille n, l'équation du théorème principal du chapitre 3 devient :

$$\langle \mathcal{E}_{n,n}, s_\mu \rangle|_{\text{équerres}} = \sum_{\tau \in \mathrm{SYT}(\mu)} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\text{équerre}(\gamma)} = \sum_{\beta \in T_{n+1,0}} \sum_{\gamma \in T_{n,n-1-\mathrm{ht}(\beta)}} s_{\text{équerre}(\gamma)},$$

où $\text{équerre}(\gamma) = \big(\mathrm{aire}(\gamma) + \mathrm{ht}(\gamma) - \binom{n}{2} + \mathrm{aire}(\beta) + 1, 1^{n-2-\mathrm{ht}(\gamma)}\big)$.

4.3 Paths, Tableaux and Pattern Avoiding Permutations

4.3.1 Abstract

We construct a bijection, Γ, between 213, 231 avoiding permutations and 132, 312 avoiding permutations, such that when the Robinson-Schensted algorithm, $R-S$, yields $\pi \overset{R-S}{\longrightarrow} (P,Q)$ then $\Gamma(\pi) \overset{R-S}{\longrightarrow} (\mathrm{ev}(P),Q)$. Parking function formulas are related to the representation of diagonal harmonics spaces and have Schröder paths as a special case. We show the bijection Γ associates Q-tableaux of some pattern avoiding permutations to some special cases of Schröder paths, such that the major index of the Q-tableau is equal to the diagonal inversion statistic of the path. We also put forth a simple equation in terms of paths in a stair shape grid of some parking functions type formula summed over pattern avoiding permutations. As a corollary many two variable generating functions, involving the descents and Major index of pattern avoiding permutations arose from the discussion.

4.3.2 Introduction

Pattern avoiding permutations have been studied in several papers, namely (Schensted, 1961), (Schützenberger, 1963) and (Simon et Schmidt, 1985). Recently many links between some pattern avoiding permutations and paths or other combinatorial objects were also explored such as (Bloom et Elizalde, 2013), and (Stump, 2009).

A permutation, π of $\mathbb{S}_n$ avoids the patterns of the set of permutations $R = \{\sigma_1, \ldots, \sigma_k\}$, denoted $A(R)$, if for all i, there is no order isomorphism between σ_i and any subword of π. For a tableau P, we denote P' its conjugate and $\mathrm{ev}(P)$ its evacuation tableau (for more details see Section 4.3.3). In this paper, we give an explicit bijection between $A(213, 231)$ and $A(132, 312)$ that preserves the descent and Major index statistics. Moreover, we denote $R-S$ the map implied by the Robinson-Schensted algorithm. For a permutation, π, it is already known from Schensted and Schützenberger that if $\pi \overset{R-S}{\longrightarrow} (P,Q)$ then reverse (respectively, the complement and the inverse) of a permutation yields $\pi^r \overset{R-S}{\longrightarrow} (P', \mathrm{ev}(Q)')$ (respectively, $\pi^c \overset{R-S}{\longrightarrow} (\mathrm{ev}(P)', Q')$ and $\pi^{-1} \overset{R-S}{\longrightarrow} (Q, P)$). With these three maps, most of the pairs among P, P', $\mathrm{ev}(P)$, $\mathrm{ev}(P)'$, Q, Q', $\mathrm{ev}(Q)$ and $\mathrm{ev}(Q)'$ that can arise from the Robinson-Schensted algorithm are accounted for (see Figure 4.14). The main result of Section 4.3.5 is to describe "the missing map" that allows all possibilities to be recovered. Preciously, we show that :

Theorem 3 : The map $\Gamma = \Pi_{A(132,312)} \circ \Pi^{-1}_{A(213,231)}$ is a bijection from $A(213, 231)$ to $A(132, 312)$. Furthermore, for $\pi \in A(213, 231)$ if $\pi \overset{R-S}{\longrightarrow} (P, Q)$ then $\Gamma(\pi) \overset{R-S}{\longrightarrow} (\mathrm{ev}(P), Q)$.
Moreover, $\mathrm{maj}(\pi) = \mathrm{maj}(\Gamma(\pi))$ and $\mathrm{des}(\pi) = \mathrm{des}(\Gamma(\pi))$.

Many two variable generating functions are provided along the way. In Section 4.3.6 Proposition 23 uses these generating functions and the consequences of the previous theorem to prove special cases of the decomposition of parking function formulas in fundamental quasisymmetric functions and monomial quasisymmetric functions in terms of paths in a stair case grid.

Finally, the main result of Section 4.3.6 relates to diagonal harmonics. We show that some Schröder paths are in bijection with Q-tableaux associated to $A(213, 231)$ and $A(132, 312)$. This bijection sends the value of the bounce statistic of a Schröder path to the Major index of a Q tableau. The same can be said for the diagonal inversion statistic.

4.3.3 Tableaux, Pattern Avoiding Permutations and the Robinson-Schensted Algorithm

The object of section is mainly to set notation and put forth well-known results.

A permutation, π, is an element of the symmetric group $\mathbb{S}_n$. We will be using the word notation, that is $\pi(1)\pi(2)\cdots\pi(n)$. Let σ be an element of $\mathbb{S}_3$, we say that π *avoids the pattern* σ if for every $i < j < k$, the subword $\pi(i)\pi(j)\pi(k)$ is not order isomorphic to σ. For σ and τ in $\mathbb{S}_3$, we denote by $A(\sigma, \tau)$ the set of permutations that avoid σ and τ. For example, $\pi = 34125$ avoids 132 and 321 but contains the pattern 213 since $\pi(1)\pi(3)\pi(5) = 315$ is order isomorphic to 213.

An *ascent* of the permutation π, is a position i of π such that $\pi(i) > \pi(i-1)$. The number of ascents in π is denoted $\mathrm{asc}(\pi)$. Similarly, a *descent* of π is a position i in π, such that $\pi(i) > \pi(i+1)$. The descent set of π is denoted $\mathrm{Des}(\pi)$ and its cardinality is denoted $\mathrm{des}(\pi)$. For example, 34125 has descent set $\{2\}$ and has 3 ascents (in position 2, 4, 5). The element $w_0 = n, n-1, \ldots, 2, 1$ is called the longest element of $\mathbb{S}_n$. As for paths we can reverse a permutation $\pi \in \mathbb{S}_n$, denoted π^r, where $\pi^r(i) = \pi(n+1-i)$. It is established that $\pi w_0 = \pi^r$. The *complement* of a permutation π, denoted π^c is the permutation such that $\pi^c(i) = n+1-\pi(i)$. It is also common knowledge that $w_0\pi = \pi^c$. It was shown in (Simon et Schmidt, 1985) that if $\pi \in A(\tau, \sigma)$, then $\pi^{-1} \in A(\tau^{-1}, \sigma^{-1})$, $\pi^r \in A(\tau^r, \sigma^r)$ and $\pi^c \in A(\sigma^c, \tau^c)$. A *shuffle* of two words π and σ is the set containing all the word that have π and σ as a subword. For example, the shuffle of 34 and 21 is $\{3421, 3241, 3214, 2341, 2314, 1234\}$.

A composition of n is a sequence of positive integers $c = (c_1, c_2, \ldots, c_k)$ that sum up to n. When no confusion arises the parentheses and the commas are omitted. An element of the sequence is a part and the length of a composition, denoted ℓ, is the number of parts. For compositions c and d we say that c is a *refinement* of d or d is a *coarsening* of c, denoted $c \preccurlyeq d$, if d can be obtained by adding adjacent parts of c. For example, 5123 is a coarsening of 13111121 and 13111121 is a refinement of 5123. The order $\preccurlyeq$ is a lattice. Given a composition of n $c = c_1, c_2, \ldots, c_k$ there is a bijection with subsets of $\{1, 2, \ldots, n-1\}$ denoted $\mathrm{Set}(c) := \{c_1, c_1 + c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1}\}$. It is well known that coarsenings of c correspond to subsets of $\mathrm{Set}(c)$.

A partition is a composition such that the parts are decreasing. Given a partition $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$ of n, a *standard Young tableau* is a bijection from the set $\{1, 2, \ldots, n\}$ to the cells of the Ferrers diagram associated to λ, such that the entries in the rows and columns are strictly increasing (see Figure 4.10 for an example in French notation). The set of standard Young tableaux of shape λ is denoted $\mathrm{SYT}(\lambda)$. The conjugate of a standard Young tableau, P, is denoted P' and is the tableau obtained by reflection through the line $x = y$ (see Figure 4.11

for an example). A *hook-shaped* tableau has only one line and one column. The *descent set* of a tableau is the set of entries i such that $i+1$ lies on a row above i, it will also be denoted Des. We also use the following notations $\mathrm{des}(\tau) = |\,\mathrm{Des}(\tau)|$ and $\mathrm{maj}(\tau) = \sum_{i \in \mathrm{Des}(\tau)} i$. For example, the descent set of the tableau in Figure 4.10 is $\{2, 3, 7, 9\}$ and its Major index is 21.

10				
4				
3	8	9		
1	2	5	6	7

Figure 4.10 A standard Young tableau, P, of shape 42211.

7			
6			
5	9		
2	8		
1	3	4	10

Figure 4.11 P' the conjugate of tableau P in Figure 4.10.

The Robinson-Schensted algorithm is bijection between permutations and pairs of standard Young tableaux (P, Q) with the same shape. We will sometimes denote respectively $P(\pi)$ and $Q(\pi)$, the P-tableau and the Q-tableau associated to permutation π. Given a permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$ we insert $\pi(1)$ in the first row and first column of the tableau P, 1 in the first row and first column of Q. Then at the i-th step we insert entry $\pi(i)$ in P and the numbers i in the corresponding cell in Q. To insert an entry E_1 in a row, we look if it is bigger than all the entries of that row, is so we insert E_1 at the end of the row. If not, we find the cell that as an entry smaller than E_1 immediately to the left and an entry larger than E_1 immediately to the right and replace E_1 with that entry, say E_2 and insert E_2 one row higher (see Figure 4.12 for an example).

Schensted showed, in (Schensted, 1961), that if $P(\pi) = P$ then $P(\pi^r) = P'$. In (Schützenberger, 1963), Schützenberger demonstrated that if $\pi \overset{R-S}{\longrightarrow} (P, Q)$ then $\pi^{-1} \overset{R-S}{\longrightarrow} (Q, P)$. He also proved that $Q(\pi^r) = \mathrm{ev}(Q(\pi))'$. The *evacuation tableau* of a tableau P of shape λ, denoted $\mathrm{ev}(P)$ is the tableau obtained by numbering in reverse order the evacuated cells. To evacuate a cell of a tableau, first choses the cell c that has the smallest entry and replace it with a dot. If the entry over the dot is smaller than the entry to the right of the dot, then slide it down. If not slide to the left the entry to the right of the dot. Stop sliding when there are no more entries on the right or over the dot. This generates a new tableau of shape μ.

First insertion :

$P=\ytableaushort{5}$ $Q=\ytableaushort{1}$.

Second insertion :

$P=\ytableaushort{5} \leftarrow 3$, $P=\ytableaushort{3} \overset{\leftarrow 5}{}$, $P=\ytableaushort{5,3}$ $Q=\ytableaushort{2,1}$.

Third insertion :

$P=\ytableaushort{5,3} \leftarrow 4$, $P=\ytableaushort{5,34}$ $Q=\ytableaushort{2,13}$.

Forth insertion :

$P=\ytableaushort{5,34} \leftarrow 1$, $P=\ytableaushort{5,14} \overset{\leftarrow 3}{}$, $P=\ytableaushort{3,14} \overset{\leftarrow 5}{}$, $P=\ytableaushort{5,3,14}$ $Q=\ytableaushort{4,2,13}$

.

Fifth insertion :

$P=\ytableaushort{5,3,14} \leftarrow 2$, $P=\ytableaushort{5,3,12} \overset{\leftarrow 4}{}$, $P=\ytableaushort{5,34,12}$ $Q=\ytableaushort{4,25,13}$.

Figure 4.12 The (P, Q) tableau resulting from the Robinson-Schensted algorithm applied to permutation 53412.

The evacuated cell is the cell λ/μ (see Figure 4.13 for an example). It so happens that when P is hook-shaped then it is equivalent to change the entry i of P to $n+2-i$ for $i > 1$ and reorder the rows and columns so the result is a standard Young tableau.

For more on this see (Bjorner et Brenti, 2005) and (Sagan, 2001).

Let $\pi \in A(213, 231)$ and (P, Q) be the tableaux obtained from the Robinson-Schensted correspondence. Figure 4.14 shows how applications $(\cdot)^r$, $(\cdot)^c$ and $(\cdot)^{-1}$, give nearly all the possibilities, using the conjugate and the evacuation tableau for pairs with P and Q. The missing map that we will call Γ will be studied in Section 4.3.5 (see Figure 4.14).

This is the map of Theorem 3. It will be used at the end of Section 4.3.6 to find the diagonal inversion statistic of a some parking functions, with a reading word

The first evacuation yields :

$$\begin{array}{cc} 4 & \\ 2 & 5 \\ 1 & 3 \end{array} \rightarrow \begin{array}{cc} 4 & \\ 2 & 5 \\ \bullet & 3 \end{array} \rightarrow \begin{array}{cc} 4 & \\ \bullet & 5 \\ 2 & 3 \end{array} \rightarrow \begin{array}{cc} \bullet & \\ 4 & 5 \\ 2 & 3 \end{array} \rightarrow \begin{array}{cc} 4 & 5 \\ 2 & 3 \end{array}$$

Successive evacuations are :

$$\begin{array}{cc} 4 & \\ 3 & 5 \end{array}, \qquad \begin{array}{cc} 4 & 5 \end{array} \quad \text{and} \quad \begin{array}{c} 5 \end{array}$$

Then the evacuation tableau is :

$$\begin{array}{cc} 5 & \\ 3 & 4 \\ 1 & 2 \end{array}$$

Figure 4.13 Evacuation tableau

in $A(213, 231)$ by using the diagonal inversion statistic of some Schröder paths (these have a reading word in $A(132, 312)$). The next section will set the bases on parking function and Schröder paths.

4.3.4 New Path Combinatorial Object, Schröder Paths and Parking Functions

The aim of this section is to present the path objects and the statistics used in this paper. Introduced in relation to multivariate diagonal harmonics in (Wallace, 2019b) (Chapter 3), T_n is a set of north-east paths in a $n-2$ staircase shaped grid lying in $\mathbb{N}^2$, starting at $(0,0)$ and ending at one coordinate in the set : $\{(x,y) \mid x+y = n-2, x \geq 0 \text{ and } y \geq 0\}$. For an example see Figure 4.15. Notice that for $n < 2$, we set $T_n = \emptyset$.

The *height* of γ, denoted $\mathrm{ht}(\gamma)$, is the y coordinate of its end point (see Figure 4.16). The *vertices of a path* are the integer coordinates the path goes through. In Figure 4.16, the vertices are $(0,0)$, $(0,1)$, $(0,2)$, $(0,3)$, $(1,3)$ and $(1,4)$. We say the i-th east step is associated to the i-th row and the i-th north step is associated to the i-th column. For a path γ in $T_{n,0}$, the *area* of γ, denoted $\mathrm{area}(\gamma)$, is the number of squares southeast of the path (see Figure 4.16). The *row area* of line i of the path γ is denoted $\mathrm{rowarea}_i(\gamma)$. It is the number of squares east of the i-th north step. The *length* of a path is the number of steps.

Let γ be a path of $T_{n,0}$, we denote $\overline{\gamma}$ the path obtained by exchanging north

$$
\begin{tikzcd}
{} \arrow[r, "(\cdot)^r"] & \begin{array}{l} A(132,231), \\ (Q, \mathrm{ev}(P)) \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(213,312), \\ (\mathrm{ev}(Q)' \,\mathrm{ev}(P)') \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(213,312), \\ (\mathrm{ev}(Q), P) \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(132,231), \\ (Q', P') \end{array} \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & {} \arrow[l, "(\cdot)^r"'] \\
{} \arrow[r, "(\cdot)^c"] & \begin{array}{l} A(132,312), \\ (\mathrm{ev}(P), Q) \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, dashed, red, "\Gamma(\cdot)"'] & \begin{array}{l} A(213,231), \\ (\mathrm{ev}(P)', \mathrm{ev}(Q)') \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] \arrow[d, leftrightarrow, dashed, red, "\Gamma(\cdot)"'] & \begin{array}{l} A(213,231), \\ (P, \mathrm{ev}(Q)) \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, dashed, red, "\Gamma(\cdot)"'] & \begin{array}{l} A(132,312), \\ (P', Q') \end{array} \arrow[d, leftrightarrow, dashed, red, "\Gamma(\cdot)"'] & {} \arrow[l, "(\cdot)^c"'] \\
{} \arrow[r, "(\cdot)^c"] & \begin{array}{l} A(213,231), \\ \pi \xrightarrow{R-S} (P, Q) \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(132,312), \\ (P', \mathrm{ev}(Q)') \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(132,312), \\ (\mathrm{ev}(P), \mathrm{ev}(Q)) \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & \begin{array}{l} A(213,231), \\ (\mathrm{ev}(P)', Q') \end{array} \arrow[d, leftrightarrow, "(\cdot)^{-1}"'] & {} \arrow[l, "(\cdot)^c"'] \\
{} \arrow[r, "(\cdot)^r"] & \begin{array}{l} A(213,312), \\ (Q, P) \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] & \begin{array}{l} A(132,231), \\ (\mathrm{ev}(Q)', P') \end{array} \arrow[r, leftrightarrow, "(\cdot)^r"] & \begin{array}{l} A(132,231), \\ (\mathrm{ev}(Q), \mathrm{ev}(P)) \end{array} \arrow[r, leftrightarrow, "(\cdot)^c"] & \begin{array}{l} A(213,312), \\ (Q', \mathrm{ev}(P')) \end{array} & {} \arrow[l, "(\cdot)^r"']
\end{tikzcd}
$$

Figure 4.14 The missing map Γ

steps and east steps. It corresponds to reflecting the path through the line $x = y$ (see Figure 4.17 for an example). Similarly, we denote γ^r the path obtained by reversing the order of the steps. One can consider the reverse as a rotation of the path (see Figure 4.18 for an example).

We will need the following propositions of (Wallace, 2019b) (Chapter 3) :

Proposition : Let $T_n(q,t)$ be the generating function $\sum_{\gamma \in T_n} q^{\mathrm{area}(\gamma)} t^{\mathrm{ht}(\gamma)}$, then $T_n(q,t) = \prod_{i=1}^{n-2}(1 + tq^i)$.

A *Schröder path* of size n is a path composed of north, east and diagonal steps in an $n \times n$ grid starting at the bottom left corner, ending at the top right corner and such that the path always stay over the line $x = y$. The set of Schröder paths, we will use, ends by a north and an east step, and will be denoted $\widetilde{\mathrm{Sch}}_{n,d}$, for d diagonal steps. The *area* statistic of a Schröder path count the number of lower triangles under the path and over the line $x = y$, named the *main diagonal*. Where

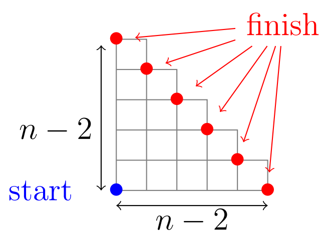


Figure 4.15 $T_{7,0}$

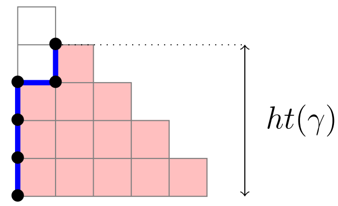


Figure 4.16 area$(NNNEN) = 13$, ht$(NNNEN) = 4$

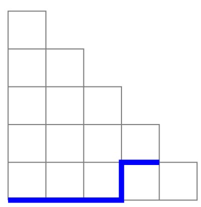


Figure 4.17 $\overline{NNNEN} = EEENE$

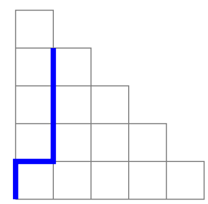


Figure 4.18 $(NNNEN)^r = NENNN$

a lower triangle is the lower half of a square cut in two starting by the bottom left corner and ending at the top right corner (see Figure 4.19 for an example). Rows are numbered from bottom to top and, we name *row area of* i the area east of the north step at row i. For the bounce statistic of a Schröder path with d diagonal steps in an $n \times n$ grid, α, we consider the path α' with no diagonal steps. Then we construct the path bounce path β under α' by going north until we reach an east step then going to the main diagonal then going north until an east step and so on. The vertices where the bounce path goes from north to east are peaks. All return to the diagonal of the bounce path that lies on the top of the row $i > 0$ contribute $n - i - d$ to bounce. If we consider the peaks in α' we can put the peaks in α by putting back the diagonal steps, if a peak touches a diagonal step it is placed over the diagonal step. Then the number of peaks under a diagonal step, with multiplicity, also contributes to bounce. The bounce statistic of 4.19 is 8. For more on this see (Haglund, 2008).

Dyck paths are Schröder paths with no diagonal steps. A *parking function* is a pair (γ, π) constituted of a Dyck path and a permutation, written from bottom to top along the north steps, such that the letters of π are increasing in each column (consecutive north steps). The set of parking functions of size n are denoted $\mathcal{P}_n$.

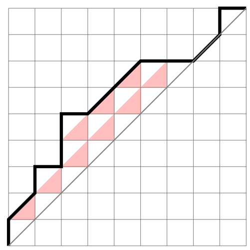


Figure 4.19 Schröder path of area 9

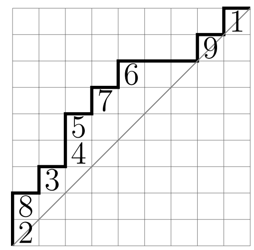


Figure 4.20 Parking function

The reading word of a parking function is the word obtained by reading the letters of π on γ following the diagonals, that are parallel to the main diagonal, from the top of the furthest diagonal to the bottom of the main diagonal. In the special case where the area of the path is 0 the parking function (γ, π) has reading word π^r. Schröder paths, with d diagonal steps, can be seen as parking functions such that the reading word is a shuffle of the words $n-d+1, \ldots, n$ and $n-d, \ldots, 1$ (Figure 4.20 is the parking function associated to the Schröder path of Figure 4.19). Notice that the reading word associated to a Schröder path is a 132 and 312 avoiding permutation. The area of a parking function is the area of its Dyck path and the *diagonal inversion statistic*, denoted dinv is the number of pairs (i, j) such that the north step on row i and the north step on row j are on the same diagonal, $i < j$ and $w_i < w_j$ and the pairs (i, j) such that the north step on row i is one diagonal over the diagonal of the north step on row j, $i < j$ and $w_i > w_j$. For example, the diagonal inversion statistic for Figure 4.20 is 7 since we have the pairs $(1, 8)$, $(3, 4)$, $(5, 6)$, $(5, 7)$, $(2, 9)$, $(3, 9)$ and $(4, 9)$.

Parking functions relate to the Macdonald eigenoperator ∇ introduced by Garsia and Bergeron in (Bergeron et Garsia, 1999). It has acts on symmetric function ("polynomials" in infinitely many variables x_i that remain stable by permutation). The ring of symmetric functions are indexed by partitions and Schur function, denoted s_λ, forme a basis, where $s_\lambda = \sum_{\tau \in \mathrm{SYT}(\lambda)} x^\tau$ and $x^\tau = \prod_{c \in \tau} x_c$. They also encode irreducible characters of representations of $\mathbb{S}_n$ and GL_k. It was shown by Haiman in (Haiman, 2002) that the Frobenius transformation of the graded characters of the modules of diagonal harmonics discussed in Section 4.3.6 are given by $\nabla(s_{1^n})$ and Haglund showed in (Haglund, 2004) that the coefficient of the Schur function $s_{d+1,1^{n-d-1}}$ of $\nabla(s_{1^n})$ is :

$$\sum_{\alpha\in\widetilde{\mathrm{Sch}}_{n,d}} q^{\mathrm{area}(\alpha)}t^{\mathrm{bounce}(\alpha)}. \tag{4.1}$$

It is possible to see diagonal harmonic modules as $GL_2 \times \mathbb{S}_n$-modules. Equation (4.1) relates to the characters of the GL_2 action when we restrict to some irreducible characters for the $\mathbb{S}_n$ action. The standard Young tableaux we will discuss in Section 4.3.6 are of shape $(d+1, 1^{n-d-1})$ and are related to the $\mathbb{S}_n$ action for a given irreducible of the GL_2 action.

A more general statement was proven by Carlsson and Mellit in (Carlsson et Mellit, 2018) :

$$\nabla(s_{1^n}) = \sum_{(\gamma,w)\in\mathcal{P}_n} q^{\mathrm{area}(\gamma,w)}t^{\mathrm{dinv}(\gamma,w)}F_{ides(\mathrm{read}(w))},$$

where F_c is a fundamental quasisymetric function indexed by composition c, $F_c = \sum_{c\preccurlyeq d} M_d$ and the monomial quasisymetric function $M_d = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}^{d_1}x_{i_2}^{d_2}\cdots x_{i_k}^{d_k}$. Fundamental quasisymmetric functions and monomial symmetric functions are a basis for quasisymmetric functions. These can be seen as a refinement of symmetric functions. Finally, $ides(w)$ is a notation for the composition associated to the descent set of the inverse permutation, $C(\mathrm{Des}(w^{-1}))$.

4.3.5 The Missing Map

In this section we display bijections $\Pi_{A(213,231)}$ and $\Pi_{A(132,312)}$ of $T_{n,0}$ to sets of pattern avoiding permutations. Afterwards we show that the map $\Gamma = \Pi_{A(132,312)} \circ \Pi^{-1}_{A(213,231)}$ is the missing map. Moreover we will prove that for any hook-shaped tableaux Q we can find unique $\pi \in A(132,312)$ and $\sigma \in A(213,231)$ such that $\pi \xrightarrow{R-S} (P,Q)$ and $\sigma \xrightarrow{R-S} (R,Q)$.

Before we need some extra tools. Let $\gamma = E^{\gamma_1}NE^{\gamma_2}N\cdots E^{\gamma_{k-1}}NE^{\gamma_k}$ be a path of T_n, with $\gamma_i \geq 0$. We can associate a composition of $n-1$ to γ defined by

$C(\gamma) = (|NE^{\gamma_k}|, |NE^{\gamma_{k-1}}|, \dots, |NE^{\gamma_1}|)$.

Lemma 34 : The map $C : T_n \to \{\text{compositions of } n-1\}$ described above is a bijection. Moreover, $\mathrm{Set}(C(\gamma)) = \{\mathrm{rowarea}_{h(\gamma)}(\gamma), \dots, \mathrm{rowarea}_2(\gamma), \mathrm{rowarea}_1(\gamma)\}$ and $\ell(C(\gamma)) = \mathrm{ht}(\gamma) + 1$.

Proof. It is obvious by construction that $C(\gamma)$ is a composition of $n-1$. Since a path of T_n is uniquely determined by the number of east steps between each north steps C is a bijection. Finally, by definition the set $\mathrm{Set}(C(\gamma))$ is equal to $\{|NE^{\gamma_i} NE^{\gamma_{i+1}} \cdots NE^{\gamma_k}| : 2 \le i \le k\}$ and the row area of line i is equal to the number of step above line i since all east steps above line i are above a square on line i and the number of squares to the east of the end point on line i is the same as the number of north steps above the line i by symmetry of a staircase grid. $\square$

Notice we denote the path obtained by the composition c by γ_c. It is clear by definition that the reverse of a composition, c^r is the path γ_c read in reverse order. In other words $\gamma_{c^r} = \gamma_c^r$ and $C(\gamma^r) = C(\gamma)^r$.

Lemma 35 : Let γ and γ' be paths of T_n then $\gamma = \alpha N \beta$ and $\gamma' = \alpha E \beta$ if and only if $C(\gamma)$ is covered by $C(\gamma')$ in the refinement lattice or equivalently $\mathrm{Set}(C(\gamma')) \subset \mathrm{Set}(C(\gamma))$.

Proof. Let $\alpha = E^{\alpha_1} N \cdots E^{\alpha_{k-1}} N E^{\alpha_k}$ and $\beta = E^{\beta_{k+1}} N \cdots E^{\beta_{l-1}} N E^{\beta_l}$. By definition on the map C the first $l-k-1$ parts and the last $k-1$ parts are the same in $C(\gamma)$ and $C(\gamma')$. The part $l-k$ and $l-k+1$ are $|NE^{\beta_{k+1}}|$ and $|NE^{\alpha_k}|$ in $C(\gamma)$ and the part $l-k$ in $C(\gamma')$ is $|NE^{\beta_{k+1}} E E^{\alpha_k}| = |NE^{\beta_{k+1}}| + |NE^{\alpha_k}|$. Hence $C(\gamma)$ is covered by $C(\gamma')$ in the refinement lattice. If the compositions c and d are such that c cover d in the refinement lattice then $c = c_1, \dots, c_{i-1}, c_i', c_i'', c_{i+1}, \dots, c_k$ and $d = c_1, \dots, c_{i-1}, c_i' + c_i'', c_{i+1}, \dots, c_k$. Hence we know that $\gamma_c = E^{c_k - 1} N E^{c_{k-1}-1} N \cdots E^{c_i'-1} N E^{c_i''-1} \cdots E^{c_2-1} N E^{c_1-1}$ and that $\gamma_d = E^{c_k-1} N E^{c_{k-1}-1} N \cdots E^{c_i'-1} E E^{c_i''-1} \cdots E^{c_2-1} N E^{c_1-1}$, since $E^{c_i'-1} E E^{c_i''-1} = E^{c_i'+c_i''-1}$. $\square$

We can now consider the map $\Pi_{A(213,231)}$ between the set of paths $T_{n,0}$ and permutations of $n-1$ avoiding 231 and 213.

— Starting with 1, label increasingly from right to left all the vertices of the path $N\gamma$ that are immediately preceded by an east step (–•). Let us say there are k such vertices.
— Starting with $k+1$, number increasingly from bottom to top all the vertices of the path $N\gamma$ that are immediately preceded by a north step (•|).
— The permutation $\Pi_{A(213,231)}(\gamma)$ is obtained by reading the numbers, starting at the end of the path and while travelling along the path, and will be denoted $\pi_{\gamma(231,213)}$. We will write $\gamma_{\pi(213,231)}$ for the path gamma labelled as such. (See Figure 4.21 for an example.)

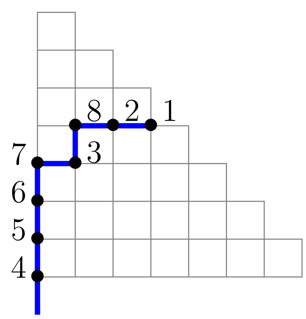


Figure 4.21 The path $NNNENEE_{\pi(213,231)}$, $\Pi_{213,231}(NNNENEE) = 12837654$

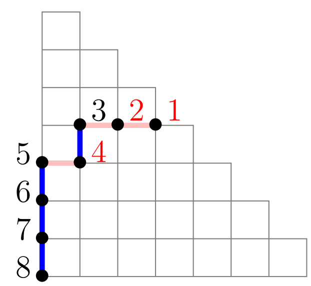


Figure 4.22 The path $NNNENEE^{*}$ $\pi^{-1}_{NNNENEE(213,231)} = 12487653$

We denote γ^{*} the path of T_n with labelled vertices, starting from the end, with the numbers 1 to $n-1$. (See Figure 4.22 for an example.) Obviously, reading the node (–•) from right to left and afterwards from bottom to top on $N\gamma^{*}$ yields the inverse permutation $\Pi_{(231,213)}(\gamma)^{-1}$. For example, the inverse of 12837654 is 12487653.

Moreover, π^{-1} is a 213, 312 avoiding permutation and one could find γ^{*} directly from π^{-1} by putting an east step in position $n-i$ when $\pi^{-1}(i)$ is an ascent and and a north step in position $n-i$ when π^{-1} is not an ascent and cutting out the first step (a north step). With this map $\Pi_{(231,213)}(\gamma)^{-1}$ is actually $\Pi_{(213,312)}(\gamma)$.

Notice $\mathrm{Id} = \Pi_{A(213,231)}(E^{n-2})$ and $w_0 = \Pi_{A(213,231)}(N^{n-2})$.

Proposition 18 : The map $\Pi_{A(213,231)}$ is a bijection from $T_{n,0}$ to $A_{n-1}(213,231)$, such that, the following equalities hold :

$$\mathrm{maj}(\Pi_{A(213,231)}(\gamma)) = \mathrm{area}(\gamma), \tag{4.2}$$

$$\mathrm{des}(\Pi_{A(213,231)}(\gamma)) = \mathrm{ht}(\gamma) = \mathrm{des}(\Pi_{A(213,231)}(\gamma)^{-1}), \tag{4.3}$$

$$\mathrm{Des}(\Pi_{A(213,231)}(\gamma)) = \mathrm{Set}(C(\gamma)), \tag{4.4}$$

$$\mathrm{Des}(\Pi_{A(213,231)}(\gamma)^{-1}) = \{n-1-\mathrm{ht}(\gamma), n-\mathrm{ht}(\gamma), \ldots, n-2\}, \tag{4.5}$$

$$\mathrm{maj}(\Pi_{A(213,231)}(\gamma)^{-1}) = \binom{\mathrm{ht}(\gamma)+1}{2} + \mathrm{ht}(\gamma)(n-1-\mathrm{ht}(\gamma)), \tag{4.6}$$

$$\mathrm{sign}(\Pi_{A(213,231)}(\gamma)) = (-1)^{\mathrm{area}(\gamma^r)}. \tag{4.7}$$

Proof. It is known from (Simon et Schmidt, 1985) that a 231 and 213 avoiding permutations, π, is a shuffle of $12\cdots d$ and $(n-1)(n-2)\cdots(d+1)$ where $d = \mathrm{asc}(\pi)$. Let π be a permutation of $A_{n-1}(213,231)$, we can build a unique path γ, such that $\Pi_{A(213,231)}(\gamma) = \pi$. Starting with the end of γ at height $n-2-\mathrm{asc}(\pi)$. Write an east step if $\pi(i) < \pi(i+1)$ and a north step if $\pi(i) > \pi(i+1)$. By construction the labels on $\gamma_{\pi(213,231)}$ are shuffles of $\{1, 2, \ldots, n-2-\mathrm{ht}(\gamma)\}$ and $\{n-1, \ldots, n-1-\mathrm{ht}(\gamma)\}$, thus Π is a bijection.

The application $\Pi_{A(213,231)}$ associates each end point of a step in $N\gamma$ to a letter of π and north steps are in the same positions as descents. Hence Equation (4.2), Equation (4.4) and the left-hand side of Equation (4.3) hold, by Lemma 34. The right-hand side of Equation (4.3) is a consequence of generating $\Pi_{A(213,231)}(\gamma)^{-1}$ with γ^* which places descents at the positions $n-1-\mathrm{ht}(\gamma), n-\mathrm{ht}(\gamma), \ldots, n-2$. This last remark also proves Equation (4.5) and Equation (4.6).

To prove Equation (4.7) we first notice that the map that changes a factor EN in γ to NE in γ' is an adjacent transposition. Let π_i and π_{i+1} be the letter associated to the EN factor of γ. Since all east steps are labelled before all north steps, then $\pi_i \in \{n-1-\mathrm{ht}(\gamma), \ldots, n-1\}$ and $\pi_{i+1} \in \{1, \ldots, n-2-\mathrm{ht}(\gamma))$. Thus when $\pi_i\pi_{i+1}$ becomes $\pi_{i+1}\pi_i$ in $\Pi_{A(213,231)}(\gamma')$ the steps switch from EN to NE. Hence inverting in such a way until we get the path $N^{\mathrm{ht}(\gamma)}E^{n-2-\mathrm{ht}(\gamma)}$ yields $\mathrm{area}(\gamma^r) -$

$\binom{\text{ht}(\gamma)+1}{2}$ adjacent transpositions. Moreover, reordering the permutation $12\dots(n-2-\text{ht}(\gamma))(n-1)(n-2)\dots(n-\text{ht}(\gamma))(n-1-\text{ht}(\gamma))$ yields $\binom{\text{ht}(\gamma)+1}{2}$ adjacent transpositions. The result holds, since $\Pi_{A(213,231)}(N^{\text{ht}(\gamma)}E^{n-2-\text{ht}(\gamma)}) = 1\dots(n-2-\text{ht}(\gamma))(n-1)\dots(n-1-\text{ht}(\gamma))$. □

This last proposition yields nice generating functions. Note that we need Proposition 22 to add a q variable into Equation (4.9).

Corollary 9 : The following equalities hold :

$$\sum_{\pi\in A_{n-1}(213,231)} q^{\text{maj}(\pi)}t^{\text{des}(\pi)} = \prod_{i=1}^{n-2}(1+tq^i) \tag{4.8}$$

$$\sum_{\pi\in A_{n-1}(213,231)} \text{sign}(\pi)t^{\text{des}(\pi)} = \prod_{i=1}^{n-2}(1+(-1)^i t) \tag{4.9}$$

$$\sum_{\pi\in A_{n-1}(213,312)} q^{\text{maj}(\pi)}t^{\text{des}(\pi)} = \sum_{k=1}^{n-2}\binom{n-2}{k} q^{\binom{k+1}{2}+k(n-1-k)}t^k \tag{4.10}$$

Proof. Equation (4.8) is true by Proposition 18 and the generating function formula found in (Wallace, 2019b) (put forth at the end of Section 4.3.4). Equation (4.9) is a consequence of Proposition 18, the generating function formula found in (Wallace, 2019b) (Chapter 3) and the equality $\text{ht}(\gamma) = \text{ht}(\gamma^r)$.

For Equation (4.10) we know by Lemma 1 of (Simon et Schmidt, 1985) that if π is a 213 and 231 avoiding permutation then π^{-1} is a 213 and 312 avoiding permuation. Moreover, there are $\binom{n-2}{k}$ paths of height k in T_n. Hence, the result hold from Proposition 18. □

The next proposition will show the P and Q tableaux of the Robinson-Schensted algorithm can be directly read of the path.

Proposition 19 : Let π be a 213, 231 avoiding permutation of $\mathbb{S}_{n-1}$ and σ be a 213, 312 avoiding permutation of $\mathbb{S}_{n-1}$ then :

1. The P-tableau of π is given by the hook-shaped standard Young tableau that has the set $\{1, \ldots, n - \mathrm{ht}(\gamma_{\pi(213,231)})$ in its first row.
2. The Q-tableau of π is the hook-shaped standard Young tableau for which the first column is filled by 1 and the numbers associated to the vertices of $\gamma^*_{\pi(213,231)}$ that are immediately followed by a north step. ($\overset{|}{\bullet}$).
3. The evacuation tableau of $P(\pi)$ has the set $\{1, \ldots, \mathrm{ht}(\gamma_{\pi(213,231)}) + 1\}$ in its first column.
4. The evacuation tableau of $Q(\pi)$ is the hook-shaped standard Young tableau such that the first column is filled by 1 and $n - j$ for all the numbers j associated to the vertices of $\gamma^*_{\pi(213,231)}$ that are immediately followed by a north step. ($\overset{|}{\bullet}$).
5. $\mathrm{maj}(Q(\pi)) = \mathrm{area}(\gamma_{\pi(213,231)})$ and
   $\mathrm{maj}(\mathrm{ev}(Q(\pi)) = n\,\mathrm{ht}(\gamma_{\pi(213,231)}) - \mathrm{area}(\gamma_{\pi(213,231)})$.
6. The P-tableau of σ is the hook-shaped standard Young tableau for which the first column is filled by 1 and the numbers associated to the vertices of $\gamma_{\sigma(213,312)}$ that are immediately followed by a north step. ($\overset{|}{\bullet}$).
7. The Q-tableau of σ is the hook-shaped standard Young tableau that has the set $\{1, \ldots, n - \mathrm{ht}(\gamma_{\sigma(213,312)})\}$ in its first row.
8. The evacuation tableau of $P(\sigma)$ is the hook-shaped standard Young tableau such that the first column is filled by 1 and $n - j$ for all the numbers j associated to the vertices of $\gamma_{\sigma(213,312)}$ that are immediately followed by a north step ($\overset{|}{\bullet}$).
9. The evacuation tableau of $Q(\sigma)$ has the set $\{1, \ldots, \mathrm{ht}(\gamma_{\sigma(213,312)}) + 1\}$ in its first column.
10. $\mathrm{maj}(P(\sigma)) = \mathrm{area}(\gamma_{\sigma(213,312)})$ and
    $\mathrm{maj}(\mathrm{ev}(P(\sigma)) = n\,\mathrm{ht}(\gamma_{\sigma(213,312)}) - \mathrm{area}(\gamma_{\sigma(213,312)})$.

For an example see Figure 4.26.

Proof. We only need to show statements (1), (2), (3), (4), (5) by Lemma 1 of (Simon et Schmidt, 1985) and the equalities $Q(\pi) = P(\pi^{-1})$ and $P(\pi) = Q(\pi^{-1})$

$$\begin{array}{|c|ccc}\cline{1-1} 8 \\ \cline{1-1} 7 \\ \cline{1-1} 6 \\ \cline{1-1} 5 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

Figure 4.23

$$\begin{array}{|c|ccc}\cline{1-1} 5 \\ \cline{1-1} 4 \\ \cline{1-1} 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

Figure 4.24

$$\begin{array}{|c|ccc}\cline{1-1} 8 \\ \cline{1-1} 7 \\ \cline{1-1} 6 \\ \cline{1-1} 4 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

Figure 4.25

$$\begin{array}{|c|ccc}\cline{1-1} 6 \\ \cline{1-1} 4 \\ \cline{1-1} 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

Figure 4.26 To the left, the P-tableau of $\pi = 12837654$. Center left is the evacuation tableau of $P(\pi)$. The Q-tableau of π is Center right and the evacuation tableau of $Q(\pi)$ is to the right.

of (Schützenberger, 1963). By Proposition 5.2 of (Reifegerste, 2004) we know $P(\pi)$ has first row $\{1, 2, \ldots, n-\mathrm{des}(\pi)\}$. Therefore, statement (1) is true by Proposition 18. In the proof of Proposition 18 we noticed that the descents for γ^* are at position $n-1-\mathrm{ht}(\gamma), n-\mathrm{ht}(\gamma), \ldots, n-2$. Hence, when building $P(\pi^{-1})$ we put the $n-1-\mathrm{ht}(\gamma)$ first letters in the first row. By construction, the first $n-1-\mathrm{ht}(\gamma)$ letters of π^{-1} are the labels of γ^* that lie in the bottom of the columns of the path. Thus, at the $n-1-\mathrm{ht}(\gamma)$ step of the Robinson Schensted algorithm all the entries of the P tableau are in the first row and they are the labels of γ^* that lie in the bottom of the columns of the path. All the labels of γ^* are smaller than all the labels that lie lower and to the left of the path and bigger than all the labels that lie higher and to the right of the path. Therefore, the Robinson-Schensted algorithm can only bump the letters of π^{-1} that are associated to a label in the same column. Since the labels associated to ( ) vertices are the smallest of their column in the path, these cannot be bumped, proving the statements (2).

In a hooked-shaped standard Young tableau, the descent set is given by $\{j - 1 | j \text{ is in the first column.}\}$. These correspond to the vertices that are followed by a north step. The labels in $\gamma^*_{\pi(213,231)}$ count the number of vertices starting at the end. Hence, the label of each vertex immediately followed by a north step is exactly the number of squares to the left of the north step minus one. Thus, statement (5) holds.

For the evacuation tableaux one only needs to notice that for hooked shaped standard Young tableau 1 is fixed and the remaining entries, i, only need to be changes for $n - i$ and reordered to obtain a standard Young tableau. $\square$

Corollary 10 : Let π, σ be in $A_{n-1}(213,231)$, then $\mathrm{ht}(\gamma_{\pi(213,231)}) = \mathrm{ht}(\gamma_{\sigma(213,231)})$ if and only if $P(\pi) = P(\sigma)$.

Similarly, for π, σ in $A_{n-1}(213,312)$, we have $\mathrm{ht}(\gamma_{\pi(213,312)}) = \mathrm{ht}(\gamma_{\sigma(213,312)})$ if and only if $Q(\pi) = Q(\sigma)$. Furthermore, σ and π have the same dual Knuth class.

Moreover, for any given hooked-shaped tableau Q there exists a unique permutation $\pi \in A_{n-1}(213,231)$ such that $Q(\pi) = Q$.

Additionally, elements of $A_{n-1}(213,231)$ in the Knuth class of π, $K(\pi)$, satisfy the equation :

$$\sum_{\sigma \in K(\pi)} q^{\mathrm{maj}(\pi)} = q^{\binom{\mathrm{des}(\pi)}{2}} \begin{bmatrix} n-2 \\ \mathrm{des}(\pi) \end{bmatrix}_q .$$

Proof. It is a classical result that $P(\pi) = P(\sigma)$ implies π and σ are in the same Knuth class and $Q(\pi) = Q(\sigma)$ implies π and σ are in the same dual Knuth class. Hence the first second and forth statements follows from Proposition 18 and Proposition 19. For the third statement, we notice that for all $\pi \in \mathbb{S}_n$ there is a unique pair $(P(\pi), Q(\pi))$ and by (2) of Proposition 19 we can construct γ such that for i in the column of Q the step in position $n-i$ is a north step and all other steps are east steps. Then by Proposition 18 such a π exists. It is unique, since for all $\pi \in A_{n-1}(213,231)$ $P(\pi)$ is determined by the length of the Ferrers diagram associated to the tableau by (1) of Proposition 19 and $Q(\pi)$ has the same shape as $P(\pi)$. $\square$

We now put forth a map, $\Pi_{A(132,312)}$, from paths of $T_{n,0}$ to $A(132,231)$.

— Starting with 1, label increasingly from bottom to top all the vertices of γE that are immediately followed by a north step (↓). Let us say there are k such vertices.
— Starting with $k+1$, label increasingly from right to left all the vertices of γE that are immediately followed by an east step (←).
— The permutation $\Pi_{A(132,312)}(\gamma)$ is obtained by reading the numbers starting at the end of the path while travelling along the path. We will write

$\gamma_{\pi(132,312)}$ for the path gamma labelled as such. (See Figure 4.27 for an example.)

Obviously, reading the vertices that are immediately followed by a north step (⬇), on $\gamma*$, from bottom to top. Then reading the remaining vertices from right to left, yields $\Pi_{A(132,312)}(\gamma)^{-1}$.

As before, π^{-1} is a 132, 231 avoiding permutation and one could find γ^* directly from π^{-1} by putting a north step in position i when $\pi^{-1}(i)$ is a descent and an east step in position i when π^{-1} is not a descent and cutting out the last step (an east step). With this map $\Pi_{(132,312)}(\gamma)^{-1}$ is actually $\Pi_{(132,231)}(\gamma)$.

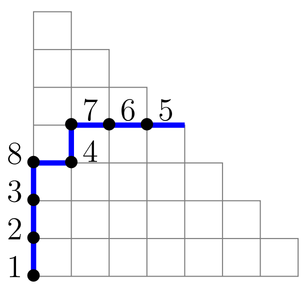


Figure 4.27 The path $NNNENEE_{\pi(132,312)}$, $\Pi_{132,312}(NNNENEE) = 56748321$

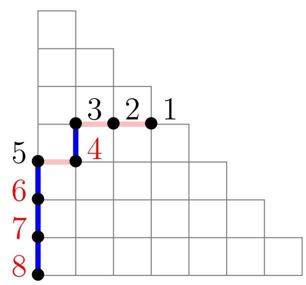


Figure 4.28 The path $NNNENEE^*$ $\pi^{-1}_{NNNENEE(132,312)} = 87641235$

Notice that as for $\Pi_{A(213,231)}$ we have $\mathrm{Id} = \Pi_{A(132,312)}(E^{n-2})$ and the longest element $w_0 = \Pi_{A(132,312)}(N^{n-2})$. Hence the missing map Γ yields $\Gamma(E^{n-2}) = E^{n-2}$ and $\Gamma(N^{n-2}) = N^{n-2}$.

Proposition 20 : The map $\Pi_{A(132,312)}$ is a bijection from $T_{n,0}$ to $A_{n-1}(132,312)$, such that, the following equalities hold :

$$\mathrm{maj}(\Pi_{A(132,312)}(\gamma)) = \mathrm{area}(\gamma), \tag{4.11}$$

$$\mathrm{des}(\Pi_{A(132,312)}(\gamma)) = \mathrm{ht}(\gamma) = \mathrm{des}(\Pi_{A(132,312)}(\gamma)^{-1}), \tag{4.12}$$

$$\mathrm{Des}(\Pi_{A(132,312)}(\gamma)) = \mathrm{Set}(C(\gamma)), \tag{4.13}$$

$$\mathrm{Des}(\Pi_{A(132,312)}(\gamma)^{-1}) = \{1, 2, \ldots, \mathrm{ht}(\gamma)\}, \tag{4.14}$$

$$\operatorname{maj}(\Pi_{A(132,312)}(\gamma)^{-1}) = \binom{\operatorname{ht}(\gamma)+1}{2}, \tag{4.15}$$

$$\operatorname{sign}(\Pi_{A(213,231)}(\gamma)) = (-1)^{\operatorname{area}(\gamma)}. \tag{4.16}$$

Proof. When noticing the following facts, the proof is similar to the proof of Proposition 18. It is known from (Simon et Schmidt, 1985) that a 132 and 312 avoiding permutations, π such that $\operatorname{des}(\pi) = d$ is a shuffle of $d \cdots 21$ and $(d+1)\cdots(n-2)(n-1)$. Moreover, generating $\Pi_{A(132,312)}(\gamma)^{-1}$ with γ^* places descents at the positions $1, 2, \ldots, \operatorname{ht}(\gamma)$. Finally, the map that changes a factor NE in γ to EN in γ' is an adjacent transposition that yields $\operatorname{area}(\gamma) - \binom{\operatorname{ht}(\gamma)+1}{2}$ adjacent transpositions to get to the path $E^{n-2-\operatorname{ht}(\gamma)}N^{\operatorname{ht}(\gamma)}$ and reordering $(n-1-\operatorname{ht}(\gamma))(n-2-\operatorname{ht}(\gamma))\ldots 1(n-\operatorname{ht}(\gamma)(n+1-\operatorname{ht}(\gamma)\ldots(n-2)(n-1)$ produces $\binom{\operatorname{ht}(\gamma)+1}{2}$ adjacent transpositions. □

This yields generating functions.

Corollary 11 : The following equalities hold :

$$\sum_{\pi \in A_{n-1}(132,312)} q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)} = \prod_{i=1}^{n-2} (1 + tq^i) \tag{4.17}$$

$$\sum_{\pi \in A_{n-1}(132,312)} \operatorname{sign}(\pi) q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)} = \prod_{i=1}^{n-2} (1 + (-q)^i t) \tag{4.18}$$

$$\sum_{\pi \in A_{n-1}(132,231)} q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)} = \sum_{k=1}^{n-2} \binom{n-2}{k} q^{\binom{k+1}{2}} t^k \tag{4.19}$$

Proof. By Proposition 20 the proof is similar than that of Corollary 9. □

Again the P and Q tableaux of the Robinson-Schensted algorithm can be directly read of the path.

Proposition 21 : Let π be a 132, 312 avoiding permutation of $\mathbb{S}_{n-1}$ and σ be a 132, 231 avoiding permutation of $\mathbb{S}_{n-1}$ then :

1. The P-tableau of π is the hook-shaped standard Young tableau that has, in its first column, the set $\{1, \ldots, \mathrm{ht}(\gamma_{\pi(132,312)} + 1)$.
2. The Q-tableau of π is the hook-shaped standard Young tableau such that the first column is filled by 1 and the numbers associated to the vertices of $\gamma^*_{\pi(132,312)}$ that are immediately followed by a north step. ($\stackrel{|}{\bullet}$).
3. The evacuation tableau of $P(\pi)$ has the set $\{1, \ldots, n-1-\mathrm{ht}(\gamma_{\pi(132,312)})\}$ in its first row.
4. The evacuation tableau of $Q(\pi)$ is the hook- standard Young tableau for which the first row is filled by 1 and $n-j$ for all the numbers j associated to the vertices of $\gamma^*_{\pi(132,312)}$ that are immediately preceded by an east step. ($-\!\bullet$).
5. $\mathrm{maj}(Q(\pi)) = \mathrm{area}(\gamma_{\pi(132,312)})$ and $\mathrm{maj}(\mathrm{ev}(Q(\pi)) = (n-1)\,\mathrm{ht}(\gamma_{\pi(132,312)}) - \mathrm{area}(\gamma_{\pi(132,312)})$.
6. The P-tableau of σ is the hook-shaped standard Young tableau where the first column is filled by 1 and the numbers associated to the vertices of $\gamma_{\sigma(132,231)}$ that are immediately followed by a north step. ($\stackrel{|}{\bullet}$).
7. The Q-tableau of σ is the hook-shaped standard Young tableau that has, in its first column, the set $\{1, \ldots, \mathrm{ht}(\gamma_{\sigma(132,231)} + 1)$.
8. The evacuation tableau of $P(\sigma)$ is the hook-shaped standard Young tableau with the first row filled by 1 and $n-j$ for all the numbers j associated to the vertices of $\gamma_{\sigma(132,231)}$ that are immediately preceded by an east step. ($-\!\bullet$).
9. The evacuation tableau of $Q(\sigma)$ is the hook-shaped standard Young tableau that has, in its first row, the set $\{1, \ldots, n-1-\mathrm{ht}(\gamma_{\sigma(132,231)})\}$.
10. $\mathrm{maj}(P(\sigma)) = \mathrm{area}(\gamma_{\sigma(132,231)})$ and $\mathrm{maj}(\mathrm{ev}(P(\sigma)) = (n-1)\,\mathrm{ht}(\gamma_{\sigma(132,231)}) - \mathrm{area}(\gamma_{\sigma(132,231)})$.

Proof. In the proof of Proposition 20 we mentioned that the descents for γ^* are at position $1, 2, \ldots, \mathrm{ht}(\gamma)$. Hence, when constructing $P(\pi^{-1})$ the Robinson-Schensted algorithm put the $\mathrm{ht}(\gamma)+1$ first letters of $\pi{-}1$ in the first column and $Q(\pi^{-1})$ as the set $\{1, \ldots, \mathrm{ht}(\gamma_{\pi(213,231)}) + 1\}$ in its first column. The labels of γ^* are smaller than all labels lying lower and to the left of the path and bigger than all labels lying higher and to the right of the path. Thus the Robinson-Schensted algorithm can only bump the first $\mathrm{ht}(\gamma)+1$ letters of π^{-1}. By construction these are the labels of γ^* that are immediately followed by a north step. The remainder of the proof is similar to Proposition 19. $\square$

Corollary 12 : Let π, σ be in $A_{n-1}(132,312)$, then $\mathrm{ht}(\gamma_{\pi(132,312)}) = \mathrm{ht}(\gamma_{\sigma(132,312)})$ if and only if $P(\pi) = P(\sigma)$.

In addition, for π, σ in $A_{n-1}(132,231)$, we have $\mathrm{ht}(\gamma_{\pi(132,231)}) = \mathrm{ht}(\gamma_{\sigma(132,231)})$ if and only if $Q(\pi) = Q(\sigma)$. Furthermore, σ and π have the same dual Knuth class.

Moreover, for any given hooked-shaped tableau Q exists a unique $\pi \in A_{n-1}(132,312)$ such that $Q(\pi) = Q$.

Additionally, elements of $A_{n-1}(132,312)$ in $K(\pi)$, the Knuth class as π, satisfy the equation :

$$\sum_{\sigma\in K(\pi)} q^{\mathrm{maj}(\pi)} = q^{\binom{\mathrm{des}(\pi)}{2}} \begin{bmatrix} n-2 \\ \mathrm{des}(\pi) \end{bmatrix}_q .$$

Proof. Follows from Proposition 20 and Proposition 21. $\square$

We have all we need to prove Theorem 3.

Proof of Theorem 3. It holds by Propositions 18, 19, 20 and 21. $\square$

For $\pi \in A(213,231)$ or $\sigma \in A(132,312)$, the composition with w_0 as effects that can be described directly on paths.

Proposition 22 : Let $\pi \in A(T)$, $T \in \{\{213,231\},\{132,312\}\}$ and S be the complement of the set T. Then :

$$\begin{aligned}
\Pi^{-1}_{A(T)}(w_0\pi) &= \overline{\Pi^{-1}_{A(T)}(\pi)}, && \text{reflects the path through line } x=y,\\
\Pi^{-1}_{A(S)}(\pi w_0) &= \overline{\Pi^{-1}_{A(T)}(\pi)}^{\,r}, && \text{reflects the path through line } x=-y\\
& && \text{and translates it into } T_{n,0},\\
\Pi^{-1}_{A(S)}(w_0\pi w_0) &= \Pi^{-1}_{A(T)}(\pi)^r, && \text{rotates the path by } 180^\circ.
\end{aligned}$$

For example, if $\pi = 615432$ then :

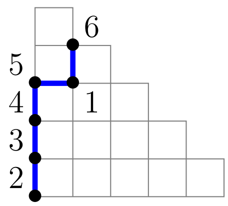


Figure 4.29 $\gamma_{\pi(213,231)}$

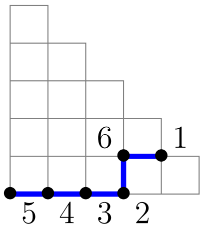


Figure 4.30 $\Pi^{-1}_{A(213,231)}(w_0\pi)$

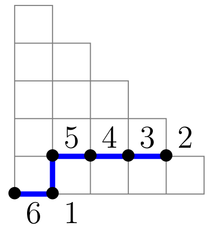


Figure 4.31 $\Pi^{-1}_{A(132,312)}(\pi w_0)$

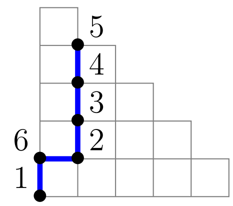


Figure 4.32 $\Pi^{-1}_{A(132,312)}(w_0\pi w_0)$

Proof. Let us first notice that the reflection the path γ through the line $x = y$ changes a north step into an east step and an east step into a north step, but the path still travels to bigger coordinates in lexicographic order. Hence, when applying this reflection γ becomes $\overline{\gamma}$.

Reflecting through the line $x = -y$ also switches north and east steps but the path travels to smaller coordinates in lexicographic order, which means that γ sent to $\overline{\gamma}^r$ up to translation.

The composition of the last two reflections is known to be a rotation. Since reversing the path and exchanging north and east step are commuting involutions then rotating the path γ yields γ^r and we only need to show the first two statements.

It is a classical result that $w_0\pi = \pi^c$ and by Lemma 1 of (Simon et Schmidt, 1985) for all $\pi \in A_{n-1}(213, 231)$ and $\sigma \in A_{n-1}(132, 312)$ we have $\pi^c \in A_{n-1}(213, 231)$ and $\sigma^c \in A_{n-1}(132, 312)$. In the proof of Proposition 18 we showed that ascents are associated to east steps and descents are associated to north steps. Since the complement of a permutation changes ascents for descents, east steps are switched into north steps and north steps are into east steps, which proves the first statement.

It is also well known that $\pi w_0 = \pi^r$ and by Lemma 1 of (Simon et Schmidt, 1985) for all $\pi \in A_{n-1}(213, 231)$ and $\sigma \in A_{n-1}(132, 312)$ we have $\pi^r \in A_{n-1}(132, 312)$ and $\sigma^r \in A_{n-1}(213, 231)$. The reverse of a permutation changes ascents for descents and reverses their positions. On the other hand, the path is in reversed order and north and east steps are switched. This proves the second statement, since the effects of ascents and descents on a path are stated in the proofs of Proposition 18 and Proposition 20. $\square$

This gives another way to compute the Major index and the descents of $w_0\pi$, πw_0 and $w_0\pi w_0$ in terms of π.

Corollary 13 : Let $\pi \in A_{n-1}(T)$, $T \in \{\{213, 231\}, \{132, 312\}\}$ then :

$$\begin{aligned}\mathrm{maj}(\pi) = &\binom{n-1}{2} - \mathrm{maj}(w_0\pi),\\ &= \mathrm{maj}(\pi w_0) - \binom{\mathrm{des}(\pi w_0)+1}{2} + \binom{n-1-\mathrm{des}(\pi w_0)}{2},\\ &= 2\binom{\mathrm{des}(w_0\pi w_0)+1}{2} + \mathrm{des}(w_0\pi w_0)(n-2-\mathrm{des}(w_0\pi w_0)) - \mathrm{maj}(w_0\pi w_0),\\ &= (n-1)\,\mathrm{des}(w_0\pi w_0) - \mathrm{maj}(w_0\pi w_0).\end{aligned}$$

and :

$$\mathrm{des}(\pi) = n-1-\mathrm{des}(w_0\pi) = n-1-\mathrm{des}(\pi w_0) = \mathrm{des}(w_0\pi w_0).$$

Proof. By Proposition 18 and Proposition 20 the Major index is the area of the associated path and the number of descents is its height. Then the proof follows by Proposition 22. $\square$

We can now add the variable q to the generating function of Equation (4.9) of Corollary 9.

Corollary 14 : The following equality holds :

$$\sum_{\pi \in A_{n-1}(213,231)} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = \prod_{i=1}^{n-2} (1 + q^i t (-1)^{(n-1)+i})$$

Proof. By Proposition 18, Proposition 22, Corollary 13, the equality $\mathrm{ht}(\gamma) = \mathrm{ht}(\gamma^r)$ and the generating function formula found in (Wallace, 2019b) (Chapter

3), we have :

$$\begin{aligned}
\sum_{\pi\in A_{n-1}(213,231)} \operatorname{sign}(\pi) q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)} &= \sum_{\pi\in A_{n-1}(213,231)} (-1)^{\operatorname{area}(\Pi^{-1}_{A(213,231)}(\pi)^r)} q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)} \\
&= \sum_{\pi\in A_{n-1}(213,231)} (-1)^{\operatorname{area}(\Pi^{-1}_{A(213,231)}(w_0\pi w_0))} q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)}, \\
&= \sum_{\pi\in A_{n-1}(213,231)} (-1)^{\operatorname{maj}(w_0\pi w_0)} q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)}, \\
&= \sum_{\pi\in A_{n-1}(213,231)} (-1)^{(n-1)\operatorname{des}(\pi)-\operatorname{maj}(\pi)} q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)}, \\
&= \sum_{\pi\in A_{n-1}(213,231)} (-q)^{\operatorname{maj}(\pi)} ((-1)^{n-1}t)^{\operatorname{des}(\pi)}, \\
&= \prod_{i=1}^{n-2} (1+q^i t(-1)^{(n-1)+i}).
\end{aligned}$$

□

4.3.6 Relations With Diagonal Harmonics

In (Wallace, 2020b) (Chapter 2) the author put forth a bijection between the set of Schröder paths of size n with area 0 and $d-1$ diagonal steps, denoted $\widetilde{\mathrm{Sch}}_{n,d-1,(0)}$, and hook-shaped Standard Young tableau with d cells in the first row, denoted $\mathrm{SYT}(d, 1^{n-d})$. It turns out this bijection yields the Q-tableau associated to the reading word when the Schröder path represented by a parking function. Before we prove this, we need to recall the bijection of (Wallace, 2020b) (Chapter 2).

The family of maps $\{\mathcal{M}_{n-1,d}\}$ are bijections, where :

$$\begin{aligned}
\mathcal{M}_{n-1,d} : \mathrm{SYT}(d, 1^{n-d-1}) &\to \mathrm{Sch}_{n-1,d-1,(0)} \\
\tau &\mapsto \gamma_1\gamma_2\cdots\gamma_n,
\end{aligned}$$

with $\gamma_{n-1} = NE$, $\gamma_{n-i-1} = NE$ if $i \in \mathrm{Des}(\tau)$ and $\gamma_{n-i-1} = D$ otherwise (see Figure 4.33 for an example).

$$\tau = \begin{array}{|c|ccc}\cline{1-1} 4 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\ \hline \end{array} \qquad \mapsto$$

$\mathrm{Des}(\tau) = \{2, 3\}$

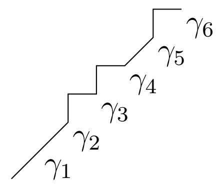


Figure 4.33 An example of the application of map $\mathcal{M}_{6,4}$.

Schröder paths of area 0 end with a north step followed by an east step and are composed of diagonal steps and consecutive north and east steps. We consider the bijection, Φ from Schröder paths with area 0 to $T_{n,0}$ that changes NE to N and D to E and take out the final north step.

Theorem 4 : Let α be a Schröder path of area 0 and d diagonal steps. If we have $\mathcal{M}_{n-1,d}(\alpha) = Q$, then the Q-tableau of $\Pi_{A(132,312)} \circ \Phi(\alpha)$ is Q and the Q-tableau of $\Pi_{A(213,231)} \circ \Phi(\alpha)$ is Q. Moreover $\mathrm{maj}(Q) = \mathrm{bounce}(\alpha)$.

Proof. If the first statement is true, then the equality $\mathrm{maj}(Q) = \mathrm{bounce}(\alpha)$ is a consequence of the bijection of (Wallace, 2020b) (Chapter 2). By definition of Φ, the vertices immediately followed by a north step in $\Phi(\alpha)$ correspond to consecutive north and east steps in α. Hence, if the i-th step is a north step in $\Phi(\alpha)$ we know γ_i in α is such that and $n-i-1$ is in the descent set of $\mathcal{M}_{n-1,d}(\alpha)$. But in hook-shaped tableaux, when $n-i-1$ is in the descent set $n-i$ is in the column of Q. Therefore, when the i step in $\Phi(\alpha)$ is a vertex immediately followed by a north step then $n-i$ is in the column of Q.

By Proposition 19 the Q-tableau associated to $(\Pi_{A(132,312)} \circ \Phi(\alpha)$ (Respectively $\Pi_{A(213,231)} \circ \Phi(\alpha))$ is the hook-shaped tableau such that the first column is filled by 1 and the numbers associated to the vertices of $\Phi(\alpha)^*_{\pi(213,231)} = \Phi(\alpha)^*_{\pi(132,312)}$ that are immediately followed by a north step. ($\downarrow$). Ergo by the definition of γ^* the Q tableau of $\Pi_{A(132,312)} \circ \Phi(\alpha)$ (respectively $\Pi_{A(213,231)} \circ \Phi(\alpha)$) as the same first column as Q. Since hook-shaped tableau are completely defined by their first column, the result holds. $\square$

Notice that $\Pi_{A(132,312)} \circ \Phi$ yields the reading word of the parking function associated to a given Schröder path but $\Pi_{A(213,231)} \circ \phi$ is related to the reading word by Γ. One could wonder if this Q-tableau is tied to the diagonal inversion statistic of parking functions of area 0 with reading words in $A(132,312)$ or $A(213,231)$. To answer this question we will first need the following lemma.

Lemma 36 : Let α be a Schröder path of area 0. Then $\mathrm{dinv}(\alpha) = \mathrm{bounce}(\alpha)$.

Proof. Let (γ, w) be the parking function associated to α. Then there is $\pi \in A_n(132,312)$ such that $w = \pi^r$ and w is a shuffle of $12\cdots d$ and $n(n-1)\cdots(d+1)$. Moreover the path $\gamma = (NE)^n$ since the area is 0. Hence all w_i's lie on the same diagonal. If $w_i \in \{1,2,\dots,d\}$ then $w_i < w_j$ for all $j > i$ and if $w_i \in \{n, n-1, \dots, d+1\}$ then $w_i > w_j$ for all $j > i$. Therefore line i contributes $n-i$ for all i such that $w_i \in \{1,2,\dots,d\}$ and $w_i < w_{i+1}$. Ergo $\pi_{n-i} > \pi n - i + 1$ and $\mathrm{dinv}((NE)^n, w) = \mathrm{maj}(\pi)$. But by Proposition 20 $\mathrm{maj}(\pi) = \mathrm{area}(\gamma_{\pi(132,312)})$, item (5) of Proposition 21 $\mathrm{area}(\gamma_{\pi(132,312)} = \mathrm{maj}(Q(\pi))$ and by Theorem 4 $\mathrm{maj}(Q(\pi)) = \mathrm{bounce}(\alpha)$. □

Lemma 37 : Let $\pi \in A_{n-1}(132,312)$, $\sigma \in A_{n-1}(213,231)$ and $\phi \in \mathbb{S}_{n-1}$ then :

$$\mathrm{dinv}((NE)^{n-1}, \pi^r) = \mathrm{area}(\gamma_\pi) = \mathrm{maj}(Q(\pi)),$$

$$\mathrm{dinv}((NE)^{n-1}, \sigma^r) = (n-1)\,\mathrm{des}(\sigma) - \mathrm{maj}(\sigma) = \mathrm{area}(\gamma_{w_0 \sigma w_0}) = \mathrm{maj}(\mathrm{ev}(Q(\sigma))).$$

$$\mathrm{dinv}((NE)^{n-1}, (\phi^r)^c) = \binom{n-1}{2} - \mathrm{dinv}((NE)^{n-1}, \phi^r)$$

$$\mathrm{dinv}((NE)^{n-1}, \phi^r) = \mathrm{dinv}((NE)^{n-1}, (\phi^{-1})^r)$$

Proof. The first equation is true by the previous lemma. For the third equation let us notice that if the area as value 0 then we can take the complement of the permutation (when the area is not 0 the complement always yields a permutation that is not compatible with the path). Then for a permutation w if $w_i < w_j$ for $i < j$ then $n - w_i + 1 > n - w_j + 1$. Thus, (i,j) is a dinv pair for w if and only if it is not a dinv pair for w^c. Since there are $\binom{n-1}{2}$ possible pairs, the result follows. The second equation follows from the first and third equations since σw_0 is in $A_{n-1}(132,312)$ by Lemma 1 of (Simon et Schmidt, 1985). Hence,

$\mathrm{dinv}((NE)^{n-1}, (\sigma w_0)^r) = \mathrm{area}(\gamma_{\sigma w_0})$. Therefore, by Corollary 13 :

$$\begin{aligned}\mathrm{dinv}((NE)^{n-1}, (w_0\sigma w_0)^r) &= \binom{n-1}{2} - \mathrm{area}(\gamma_{\sigma w_0}),\\ &= \binom{n-1}{2} - \left(\binom{n-1}{2} - \mathrm{area}(\gamma_{w_0\sigma w_0})\right),\\ &= \mathrm{area}(\gamma_{w_0\sigma w_0}).\end{aligned}$$

For the last equation one only needs to notice that the inverse swaps the positions and the labels of the permutation so if the inverse permutation is compatible with the path the condition $\phi_i < \phi_j$ for $i < j$ becomes label $i < j$ when position $\phi_i < \phi_j$. □

We can now state what effect Γ has on the diagonal inversion statistic.

Corollary 15 : Let $\pi \in A_{n-1}(132, 312)$ and $\sigma \in A_{n-1}(213, 231)$.

$$\mathrm{dinv}((NE)^{n-1}, \Gamma(\pi^r)) = \mathrm{area}(\gamma_{w_0\pi w_0}) = \mathrm{maj}(\mathrm{ev}(Q(\pi))). \tag{4.20}$$

$$\mathrm{dinv}\left((NE)^{n-1}, \Gamma(\sigma^r)\right) = \mathrm{area}(\gamma_\sigma) = \mathrm{maj}(Q(\sigma)). \tag{4.21}$$

Proof. Follows from previous lemma and Theorem 3. □

Proposition 23 : The following equalities hold :

$$\sum_{\pi\in A_{n-1}(132,312)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} t^{\mathrm{aire}(\gamma)} F_{1^{\mathrm{ht}(\gamma)},n-1-\mathrm{ht}(\gamma)}(X), \tag{4.22}$$

$$\sum_{\pi\in A_{n-1}(132,231)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} t^{\mathrm{aire}(\gamma)} F_{C(\gamma)}(X), \tag{4.23}$$

$$\sum_{\pi\in A_{n-1}(132,231)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} \prod_{i=1}^{\mathrm{ht}(\gamma)} (1 + t^{\mathrm{aireligne}_i(\gamma)}) M_{C(\gamma)}(X), \tag{4.24}$$

$$\sum_{\pi\in A_{n-1}(213,231)} q^{\mathrm{aire}((NE)^{n-1},\pi^r)} t^{\mathrm{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} t^{\mathrm{aire}(\gamma^r)} F_{n-1-\mathrm{ht}(\gamma),1^{\mathrm{ht}(\gamma)}}(X), \tag{4.25}$$

$$\sum_{\pi\in A_{n-1}(213,312)} q^{\text{aire}((NE)^{n-1},\pi^r)} t^{\text{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} t^{\text{aire}(\gamma^r)} F_{C(\gamma)}(X), \tag{4.26}$$

$$\sum_{\pi\in A_{n-1}(213,312)} q^{\text{aire}((NE)^{n-1},\pi^r)} t^{\text{dinv}((NE)^{n-1},\pi^r)} F_{ides(\pi)}(X) = \sum_{\gamma\in T_n} \prod_{i=1}^{\text{ht}(\gamma)} (1+t^{\text{aireligne}_i(\gamma)}) M_{C(\gamma^r)}(X). \tag{4.27}$$

Proof. It is obvious that $\text{area}((NE)^{n-1}) = 0$. Lemma 37 and Proposition 20 imply that the sum of Equation (4.22) can be taken over paths in T_n and that the power of t will be the area. Proposition 20 also yields the composition related to $ides(\pi)$ in terms of paths. Moreover, by Lemma 37 we know that the diagonal inversion statistic is the same for the left-hand side of Equation (4.22) and Equation (4.23), since $A(\tau^{-1}, \sigma^{-1}) = \{\pi^{-1} : \pi \in A(\tau, \sigma)\}$. Hence, Equation (4.23) can be expressed as a sum of paths in T_n such that the area is the powers of t and we use Proposition 20 to associate the proper composition to $ides(\pi)$ in terms of paths. By Corollary 15 applying Γ in Equation (4.23) yields Equation (4.26) changing $ides(\pi)$ to $ides(\Gamma(\pi))$. Lemma 37 implies that the powers of t are the same in Equation (4.25) and Equation (4.26). Proposition 18 sends $ides(\Gamma(\pi))$ to the aforementioned composition in terms of paths.

Equation (4.24) and Equation (4.27) are direct consequences of Equation (4.23), Equation (4.26), Proposition 18, Proposition 20 and Lemma 35, since for the composition c, M_c is in the decomposition of all F_d where d is a coarsening. $\square$

In (Wallace, 2020b) (Chapter 2) a map from Schröder paths with area 1 to standard Young tableau is presented. The author shows that Schröder paths of area 0 associated to a tableau Q are related to Schröder paths of area 1 associated to the tableau Q. It would be interesting to see if all Schröder paths α with parking function representation (γ, π) and statistic k can be associated to a Q-tableau such that $\text{maj}(Q) - k = \text{bounce}(\alpha)$ or $\text{maj}(Q) - k = \text{dinv}(\gamma, \pi)$. Or which parking functions make it possible ?

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CONCLUSION

4.4 Résultats

La motivation première de ces travaux est de décomposer en bicaractère irréductible les espaces de polynômes diagonaux harmoniques à deux ou plusieurs jeux de variables. Dans cette thèse, on utilise la combinatoire sous-jacente de ces espaces pour présenter des formules explicites, dans le cas où les fonctions de Schur sont indexées par une équerre. Le chapitre 2 couvre le cas à deux variables et l'on entreprend le cas multivarié au chapitre 3. Certains des résultats secondaires qui ont ressurgi pour démontrer ces formules répondent eux-mêmes à certains problèmes. En effet, Haglund recherche une bijection explicite démontrant la symétrie entre les variables q et t dans la fonction génératrice des chemins de Schröder. Puisque l'algorithme introduit au chapitre 2 regroupe les chemins de Schröder qui expriment une même fonction de Schur, il va de soi que cet algorithme sous-tend la bijection recherchée par Haglund. En outre, dans le chapitre 3, afin d'expliciter les bicaractères indexés par une équerre, on met de l'avant des conditions qui permettent de transformer une somme de polynômes dont le signe alterne en polynômes dont tous les termes sont positifs. De surcroît, on présente une nouvelle interprétation combinatoire pour toutes sommes de polynômes qui satisfont aux conditions. Par la suite, on établie une formule qui est un relèvement de toutes les fonctions de Schur du cas multivarié, indexé par un partage de forme $(a, b, 1^k)$, à partir des fonctions de Schur indexée par un partage de forme (a, b) dans le cas à deux variables. Au chapitre 4, on étudie l'objet combinatoire introduit au chapitre 3. Celui-ci, induit une bijection entre les permutations qui évitent certains motifs qui préserve certaines statistiques sur les permutations et le Q-tableau au sens de l'algorithme de Robinson-Schensted. Cette bijection composée avec le complément, l'image miroir et l'inverse, produit toutes les paires de tableaux parmi P, Q, P', Q', $\mathrm{ev}(P)$, $\mathrm{ev}(Q)$, $\mathrm{ev}(P')$ et $\mathrm{ev}(Q')$ qu'il est possible d'obtenir avec l'algorithme Robinson-Schensted. De plus, l'on peut déduire de cette bijection plusieurs formules dans le même style des fonctions de stationnements. La formule génératrice de cet objet, présenté au chapitre 3, permet de raffiner les statistiques de certains objets classiques de la combinatoire au chapitre 4. Enfin, au chapitre 2 une bijection constructive entre les tableaux de Young standard et certains chemins de Schröder est mise de l'avant. On déduit du chapitre 4 que cette bijection construit le Q-tableau de la permutation de la fonction de stationnement qui est représenté par le chemin de Schröder. De plus, la bijection entre les permutations qui évite certains motifs associe une deuxième permutation à ce Q-tableau. Il s'avère que les statistiques de cette deuxième fonction de stationnement sont également décrites par l'index Major de ce Q-tableau.

4.5 Perspectives

Pour faire suite à ces travaux, il serait intéressant de prouver la conjecture stipulée dans «Toward a Schurification of Parking Function Formulas via bijections with Young Tableaux» :

Conjecture 2 : Pour tout μ partage de n :

$$\langle \nabla(e_n), s_\mu \rangle|_{\text{hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} s_{\mathrm{maj}(\tau)}(q,t) + \sum_{i=2}^{\mathrm{des}(\tau)} s_{\mathrm{maj}(\tau)-i,1}(q,t).$$

En outre, l'extension de l'algorithme présenté au chapitre 2 permettrait à la fois de résoudre complètement le problème 3.11 de (Haglund, 2008) et de décomposer la fonction génératrice des chemins de Schröder en fonctions de Schur. Par extension, cela permettrait de donner une description de certains bicaractères irréductibles des espaces de polynômes harmoniques diagonaux multivariés et même une formule explicite décrivant explicitement la triple homologie de Khovanov-Rozansky. La recherche d'une telle extension est déjà en cours.

Une autre direction serait d'étendre les résultats aux m, n-fonctions de stationnement, aux m-chemins de Schröder. Pour ceci il faut d'abord écrire l'algorithme en vertu des statistiques dinv et aire plutôt que aire et bounce. Il est aussi possible d'utiliser les résultats du corollaire 5 et du lemme 13. Décomposer $\nabla(e_n)$ en fonction de Schur pour la variable X est un problème reconnu difficile. Il serait en théorie plus facile de décomposer d'abord en produits de fonctions symétriques et de fonctions quasisymétriques, $s_\mu(q,t)F_c(X)$, pour ensuite le décomposer en produit de fonctions symétriques, $s_\mu(q,t)s_\lambda(X)$, car on aurait moins de termes. Ceci pourrait donner lieu à des résultats similaires au lemme 13.

Une autre avenue possible est de trouver une application qui envoie les chemins de Schröder (ou les fonctions de stationnement) sur les tableaux de Young standard, tel que les statistiques pour l'aire et l'inversion diagonale sont transformées en statistiques liées à l'index Major et aux descentes du tableau associé.

Tous ces résultats pourraient être utilisés en combinaison avec les outils développés dans «Explicit combinatorial formulas for some irreducible characters of the $GL_k \times \mathbb{S}_n$-module of multivariate diagonal harmonics» pour relever les bicaractères des espaces diagonaux harmoniques en bicaractère des espaces diagonaux harmonique multivarié. Ultimement, le but est de démontrer l'égalité suivante, pour tout r et tout μ :

$$\langle \mathcal{E}_{rn,n}, s_\mu \rangle|_{\mathrm{hooks}} = \sum_{\tau \in \mathrm{SYT}(\mu)} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)}, \tag{4.28}$$

où $\mathrm{hook}(\gamma) = \left((r-1)\binom{n}{2} + \mathrm{area}(\gamma) + \mathrm{ht}(\gamma) - \mathrm{maj}(\tau') + 1, 1^{n-2-\mathrm{ht}(\gamma)}\right)$.

Une autre perspective est liée au cas, pour lequel on donne certains résultats partiels, (n, kn). Celui-ci a fait l'objet de nombreux travaux de 1996 à 2005, notamment dans (Bergeron et Garsia, 1999), (Haiman, 2002) et (Haglund *et al.*, 2005). Ce dernier énonce une conjecture prouvée dans (Carlsson et Mellit, 2018). Des extensions au cas rationnel, (kn, km), rectangulaire, (n, m), et, plus récemment, triangulaire (sous une droite allant de $(0, r)$ et $(s, 0)$, pour r et s des nombres réels positifs quelconques), ont vu le jour. Certains des outils développés dans cette thèse peuvent être utilisés ou étendus pour ces cas.

Étant donné l'action du groupe symétrique, il serait envisageable d'explorer la généralisation de ces résultats à d'autres groupes de Coxeter, comme dans les travaux de Armstrong, Reiner et Rhoades.

Enfin, une extension des résultats liée aux fonctions de stationnements pourrait s'étendre aux «labelled Dyck paths» du théorème delta. Ceux-ci pourraient ensuite être comparés avec la combinatoire des super espaces mis en place par Zabrocki et Orellana, dans (Orellana et Zabrocki, 2020). Il est espéré qu'avec les connaissances acquises sur ces structures, il soit possible de vérifier s'ils coïncident en comparant leurs combinatoires.

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