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LEFT-ORDERABILITY OF THE FUNDAMENTAL GROUP OF THE DOUBLE BRANCHED COVER OF LINKS

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L'ORDONNABILITÉ À GAUCHE DU GROUPE FONDAMENTAL DU REVÊTEMENT RAMIFIÉ DOUBLE D'ENTRELACS

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RÉSUMÉ

Cette thèse introduit une nouvelle méthode pour prouver que le groupe fondamental du revêtement ramifié double d'entrelacs dans la 3-sphère n'est pas ordonnable à gauche. Grâce à cette méthode, on trouve des familles infinies d'entrelacs non-alternés pour lesquels le groupe fondamental du revêtement ramifié double n'est pas ordonnable à gauche.

Keywords: ordonnable à gauche, groupe fondamental, revêtement double ramifié, 3-variété.

ABSTRACT

This thesis introduces a new method to prove that the fundamental group of the double branched cover of a link in the 3-sphere is not left-orderable and applies it to find new infinite families of non-alternating links with this property.

Keywords: left-orderable, fundamental group, double branched cover, 3-manifold.

INTRODUCTION

In this thesis, we study the left-orderability of the fundamental group of the double branched cover of links. Our main motivation is the *L-space conjecture*:

Conjecture 0.0.1. *(Conjecture 1 in (Boyer et al., 2013), Conjecture 5 in (Juhász, 2015)). Assume that M is a closed, connected, irreducible, orientable 3-manifold. Then the following statements are equivalent.*

1. *M is not a Heegaard Floer L-space,*

2. *M admits a co-orientable taut foliation,*

3. *$\pi_1(M)$ is left-orderable.*

The conjecture is known to be true when M has positive first Betti number ((Gabai, 1984), (Boyer et al., 2005), or is a non-hyperbolic geometric 3-manifold ((Boyer et al., 2013), (Lisca & Stipsicz, 2007), (Boyer et al., 2005)) or is a graph manifold ((Boyer & Clay, 2017), (Hanselman et al., 2015)).

The double branched covers of links are closed, connected, orientable 3-manifold. Moreover, double branched covers of prime links are irreducible and generically hyperbolic.

It is known that the fundamental group of double branched covers of alternating non-split links are not left-orderable ((Boyer et al., 2013)). Moreover, the left-orderability of the fundamental group of the double branched covers of Montesinos links has been determined ((Boyer et al., 2005)) and the same is true for arborescent knots ((Boyer & Clay, 2017)). Note that Montesinos knots are arborescent knots.

In this thesis, we determine important families of links, in particular families of non-alternating and non-arborescent links, for which the fundamental group of the double branch cover is not left-orderable.

First, we recall the following definition:

Definition 0.0.2. A group G is called *left-orderable* (LO) if $G \neq \{1\}$ and its elements can be given a (strict) total ordering $<$ which is left invariant, meaning that $g < h$ implies $fg < fh$ if $f, g, h \in G$.

Our methods build on an argument from a theorem in (Boyer et al., 2013), which we now state as a proposition.

Proposition 1.0.6. Let D be the diagram of a non-split non-trivial link L. Suppose that the Wada group $\pi(D)$ is left-orderable. If it implies that $\pi(D)$ is abelian, then $\pi_1(\Sigma(L))$ is not left-orderable.

As we will see in section 1, the *Wada group* is presented by

$$\pi(D) :=< a_1, ..., a_n : a_k^{-1} a_j a_i^{-1} a_j >$$

where the generators correspond to the arcs of D and the relations

$$a_k^{-1} a_j a_i^{-1} a_j \tag{1}$$

are in one-one correspondence with the crossings of D.

Throughout the thesis, we will find families of links for which, when we suppose that the Wada group is left-orderable, we obtain that the Wada group is abelian.

To show that the Wada group $\pi(D)$ is abelian, we will introduce a subgroup $G(D)$ called the *Graph group of D*.

Lemma 2.0.2. Let D be a diagram of a non-split link and $G(D)$ be the Graph group of D. Suppose that $\pi(D)$ is left-orderable. If the Graph group $G(D)$ is trivial, then $\pi(D)$ is abelian.

To show that the Graph group is trivial, we will simplify its generating set. From a link diagram D with m rational tangles, we will obtain successively the Wada graph Γ_0, the coarse Wada rational graph Γ' and then the Wada rational graph Γ. From the coarse Wada rational graph, we will obtain 4^m sets of generators for the Graph group. From the Wada rational graph, if we suppose that the Wada group is left-orderable, we will obtain many Wada rational groups $G(\Gamma, d)$ which are equal to the Graph group. We will suppose that the Wada group is left-orderable. From a left-order $<$, we will construct a directed Wada graph $(\Gamma, <)$. Finally, from this directed Wada graph, we will construct a group $G(\Gamma(D), <)$ which is equal to the Graph group and for which each generator is less than or equal to 1 with respect to $<$. Furthermore, we will introduce the family of *2 non-bridge links*, the only links that we will study. Note, that every knot of 10 crossings or less, except the knot 10_{161}, is a 2 non-bridge knot. There are many possible directed Wada graphs, but fewer for 2 non-bridge links. To obtain the non left-orderable property, we will show that it is sufficient that every $G(\Gamma, <)$ be trivial.

Lemma 4.0.1. Let D be a maximal two non-bridge diagram of a non-split link and suppose that $\pi(D)$ is left-orderable. If for every maximal directed Wada graph $(\Gamma, <)$, the directed Wada group $G(\Gamma, <)$ is trivial, then $\pi(D)$ is abelian.

Therefore by Proposition 1.0.6

Theorem 4.0.2. Let D be a maximal two non-bridge diagram of a non-split link

L and suppose that $\pi(D)$ is left-orderable. If for every directed Wada graph $(\Gamma, <)$, the directed Wada group $G(\Gamma, <)$ is trivial, then $\pi_1(\Sigma(L))$ is not left orderable.

Moreover, we will give a stronger result for links having a link diagram with only one maximal directed Wada graph up to reversing edges. These links are called directed. This is a large family, because the non-alternating knots of 11 crossings or less, they represent 71% of such knots. Thus, for directed links, using Theorem 4.0.2, we only have to look at the directed Wada graph.

Corollary 4.0.3. Let D be a two non-bridge directed diagram of a non-split directed link L and suppose that $\pi(D)$ is left-orderable. If a directed Wada group $G(\Gamma, <)$ is trivial, then $\pi_1(\Sigma(L))$ is not left orderable.

We will then introduce three important families of directed links: the totally monopositive links, the $(n-1)$ totally monopositive links and the steady fluid $(n-2)$ totally simple monopositive links. Those families cover 35 of the 40 non-alternating, non left-orderable and directed links of 10 crossings or less. We will prove that the directed Wada group of those directed links is trivial and thus we have, by Corollary 4.0.3, the following results.

Theorem 7.3.6. If L is a totally monopositive, 2-non-bridge and non-split link, then the fundamental group of the double branched cover of L is not left-orderable.

Theorem 7.4.3. If L is a $(n-1)$ totally monopositive, 2-non-bridge and non-split link, then the fundamental group of the double branched cover of L is not left-orderable.

Theorem 7.5.9. If L is a fluid steady $(n-2)$ totally simple monopositive link with $n \geq 3$, then the fundamental group of the double branched cover of L is not left-orderable.

We note that the knot 9_{49} is a non-alternating, non-arborescent and $(n-1)$ totally monopositive, 2-non-bridge and a non-split link. Thus, the fundamental group of the double branched cover of 9_{49} is not left-orderable. We find others non-alternating and non-arborescent knots such that the fundamental group of the double branched cover is not left-orderable.

Moreover, we will show that starting from a totally monopositive link, a $(n-1)$ totally monopositive link, or a steady fluid $(n-2)$ totally simple monopositive link, we can obtain an infinite family of links for which the fundamental group of the double branched cover is not left-orderable.

Theorem 8.3.7. Let L be a totally monopositive link and X_i be a $\frac{p}{q}$ rational tangle which is not a half-twist. If we change X_i to a similar rational tangle, then the fundamental group of the double branched cover of the new link L' is not left-orderable.

Theorem 8.3.8. Let L be a $(n-1)$ totally monopositive link and X_i be a $\frac{p}{q}$ rational tangle which is not a half-twist. If we change X_i to a similar rational tangle, then the fundamental group of the double branched cover of the new link L' is not left-orderable.

Theorem 8.3.9. Let L be a $(n-2)$ totally monopositive link and X_i be a $\frac{p}{q}$ rational tangle which is not a half-twist. If we change X_i to a rational tangle of same nature, then the fundamental group of the double branched cover of the link L' is not left-orderable.

To deal with links that are not directed, we will introduce the *triple hop link* family of links which account for 88% of non-alternating knots of less than 12 crossings. We will split this family of links into the *middle triple hop links* and the *final triple hop links.* For both subfamilies, we will find sufficient conditions for some directed Wada groups of non-directed link to be trivial.

Theorem 9.1.6. Let $(\Gamma, <)$ be a directed Wada graph of a middle triple hop link diagram. If the left directed condition (resp. right directed condition) and the left middle hop condition (resp. right middle hop condition) are fulfilled, then $G(\Gamma, <)$ is trivial.

Theorem 9.2.4. Let $(\Gamma, <)$ be a trichotomic directed Wada graph of a final triple hop link diagram. If the left graph is left graph directed (resp. the right graph is right graph directed) and the left middle hop condition and right middle hop condition are fulfilled, then $G(\Gamma, <)$ is trivial.

Thus, by Proposition 4.0.2 :

Theorem 9.1.7. Let D be a maximal two non-bridge middle triple hop diagram of a non-split link L. If for every directed Wada graph $(\Gamma, <)$, the left directed condition (or resp. right directed condition) and the left middle hop condition (or resp. right middle hop condition) are fulfilled, then the $\pi_1(\Sigma(L))$ is not left-orderable.

Theorem 9.2.5. Let D be a maximal two non-bridge final triple hop diagram of a non-split link L. If every directed Wada graph $(\Gamma, <)$ is trichotomic and for every $(\Gamma, <)$ the left graph is left graph directed or the right graph is right graph directed and both the left middle hop condition and right middle hop condition are fulfilled, then the $\pi_1(\Sigma(L))$ is not left-orderable.

We call a directed Wada graph of a middle triple hop link diagram that satisfies the hypothesis of Theorem 9.1.6, a *good middle triple hop diagram.* Similarly, from the hypothesis of Theorem 9.2.4, we define *good final triple hop diagrams.* We conclude by finding infinite families of good middle triple hop diagrams and good final triple hop diagrams.

Theorem 9.2.13. Let D be a good middle (resp. final) triple hop diagram of a link L with X the triple hop rational tangle. If we substitute X by a rational tangle X' of the same nature, then for the new link L' obtained from the new rational tangle X', $\pi_1(\Sigma(L'))$ is not left-orderable.

0.1 Overview

In the first chapter, we will cite and prove the theorem of Boyer, Gordon and Watson and introduce Proposition 1.0.6. Moreover, we will give another proof of the theorem of Boyer, Gordon and Watson. To do so, we will introduce and prove new results on *rooms* and *inhabitants* and on the *red and blue link diagram.*

In the second chapter, for a link diagram, we will first introduce the Graph group and prove Theorem 2.0.3. Then, we will construct the Wada graph Γ_0 and the coarse Wada rational graph Γ' and show how from these graphs, we can obtain new sets of generators of the Graph group. Furthermore, we will show that if the Wada group is left-orderable, then we can construct the directed Wada graphs $(\Gamma, <)$. Moreover, we will prove that the directed Wada groups $G(\Gamma, <)$ are equal to the Graph group.

In chapter three, we will narrow the number of possibilities for the directed Wada graphs. To do so, we will introduce the semi-directed Wada rational graph and show how, from the semi-directed Wada rational graph, we can obtain the directed Wada graphs. Moreover, we will introduce the *maximal Wada directed graphs* and the directed link.

In chapter four, we prove Lemma 4.0.1, Theorem 4.0.2 and Corollary 4.0.3, which underlines the importance of the directed Wada group.

In chapter five, we introduce the Hybrid Wada diagram and we relabel the vertices in a Wada directed graph. Both will be very useful in the proofs of the following chapters.

In chapter six, we prove Theorem 6.7.7, which shows that if one edge is trivial in a directed Wada graph of a directed link, then the directed Wada group is trivial. Thus, in combination with Corollary 4.0.3, we obtain Theorem 6.7.8 which plays a central role in the proofs of chapter 7. To prove Theorem 6.7.7, we will need many technical results.

In chapter seven, one of the most important chapters of this thesis, we will introduce totally monopositive links, $(n-1)$ totally monopositive links, and steady fluid $(n-2)$ totally simple monopositive links. Then, we will prove Theorem 7.3.6, Theorem 7.4.3 and Theorem 7.5.9. To prove these results, we will introduce some families of group and prove that these families have a trivial generator. Then, we will prove that the directed Wada group of the previous families of links have a trivial generator. Thus, by Theorem 6.7.7, we will obtain the desired results.

In chapter eight, from totally monopositive links, $(n-1)$ totally monopositive links, and steady fluid $(n-2)$ totally simple monopositive links, we will construct infinite families of links for which the fundamental group of the double branched cover is not left-orderable. To do so, we will prove Theorem 8.3.7, Theorem 8.3.8 and Theorem 8.3.9.

Finally, in chapter nine, we will introduce the triple hop links to investigate non directed links. We divide this family into the *middle triple hop links* and the *final triple hop links.* For both families, we will find sufficient conditions for the directed Wada group to be trivial. Using these conditions, we will prove Theorem 9.1.7 and Theorem 9.2.5. The links in these theorems will be called *good middle triple hop links* and *good final triple hop links.* We will prove that from a good middle triple hop link, we can obtain an infinite family of good middle triple hop links and similarly for good final triple hop links.

CHAPTER I

THE ALTERNATING THEOREM

To find families of links for which the fundamental group of the double branched cover is not left-orderable, we will use an argument from the following theorem of (Boyer et al., 2013).

Theorem 1.0.1. *(Boyer et al., 2013) The fundamental group of the double branched cover of a non-split alternating link is not left-orderable.*

To prove this result we need the following group presentation associated to a link diagram due to Wada (Wada, 1992).

Let L be a link in S^3 and D a diagram for L. Label the arcs of the diagram a_1 through a_n. Define the *Wada group*

$$\pi(D) :=< a_1, ..., a_n : a_k^{-1} a_j a_i^{-1} a_j >$$

where the relations

$$a_k^{-1} a_j a_i^{-1} a_j \tag{1.1}$$

are in one to one correspondence with the crossings of D as illustrated in Figure 1. Note that this relation is well-defined, as it is invariant under interchanging the indices i and k.

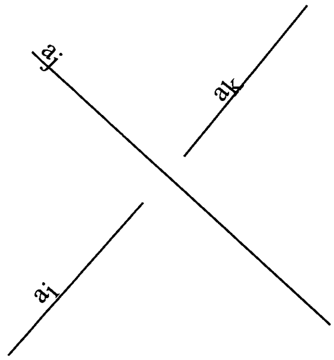


Figure 1

Definition 1.0.2. We say that an arc a_j is a *bridge* if a_j goes over at least one other pair of arcs as a_j in Figure 1. If it goes over exactly n other pair of arcs, it is called an *n-bridge*. If it does not go over any arcs, it is called a *non-bridge.*

The Wada group will be particularly useful when we suppose that it is left-orderable. If $\pi(D)$ is left-orderable, fix a crossing (a_i, a_j, a_k) where a_j is the local bridge. The Wada relations imply that exactly one of the following three possibilities occurs:

1. $a_i < a_j < a_k$
2. $a_k < a_j < a_i$
3. $a_i = a_j = a_k$.

The following result is used in the proof of Theorem 1.0.1.

Theorem 1.0.3. *(Wada, 1992) Let D be a link diagram of a link L. Then, $\pi(D) \cong \pi_1(\Sigma(L)) * \mathbb{Z}$ where $\Sigma(L)$ is the double branched cover of L.*

The following two lemmas from (Boyer et al., 2013) imply Proposition 1.0.6, which is instrumental in proving Theorem 1.0.1. Later, we will rely mostly on Proposition 1.0.6 to obtain our results.

Lemma 1.0.4. *Let D be a link diagram of a non trivial link L. Then $\pi(D)$ is non-abelian.*

Proof. The link L is not trivial, therefore $\Sigma(L)$ is a 3-manifold that isn't the 3-sphere. Thus, $\pi_1(\Sigma(L))$ is not trivial. This implies that $\pi(D) \cong \pi_1(\Sigma(L)) * \mathbb{Z}$ isn't abelian by definition of a free product. □

Lemma 1.0.5. *Let D be a link diagram of a non-trivial link L. Then, $\pi(D)$ is left-orderable if and only if $\pi_1(\Sigma(L))$ is left-orderable.*

Proof. By a result of Vinogradov (Vinogradov, 1949), the free product of two non-trivial groups is left-orderable if and only if each group is left-orderable. Therefore, $\pi_1(\Sigma(L))$ is left-orderable if and only if $\pi(D) \cong \pi_1(\Sigma(L)) * \mathbb{Z}$ is left-orderable, because $\mathbb{Z}$ is left-orderable. □

Thus, by the two previous lemmas we prove the following Proposition.

Proposition 1.0.6. *Let D be the diagram of a non-split non-trivial link L. Suppose that $\pi(D)$ is left-orderable. If it implies that $\pi(D)$ is abelian, then $\pi_1(\Sigma(L))$ is not left-orderable.*

1.1 Result for alternating non-split non-trivial links

In this section, we will prove Proposition 1.2.8, which says that if we have an alternating non-split link diagram D and we suppose the Wada group $\pi(D)$ to be left-orderable, then $\pi(D)$ is abelian. This result combined with Proposition 1.0.6 proves Theorem 1.0.1.

1.1.1 Investigation on inhabitants in a room

In this section, we will prove results on so-called inhabitants in a room that we will use in the proof of Proposition 1.2.8.

The following definitions come from Topics in Knot Theory (Rolfsen, 1993).

Definition 1.1.1. A *room* R is a region of the plane (the boundary, assumed polygonal, may be empty or disconnected), together with an even number of marked points on its boundary.

An *inhabitant* T *of a room* R is a diagram in R (part of a link diagram) whose boundary is precisely the set of marked points.

The following figure is an example of an inhabitant of a room with six marked points.

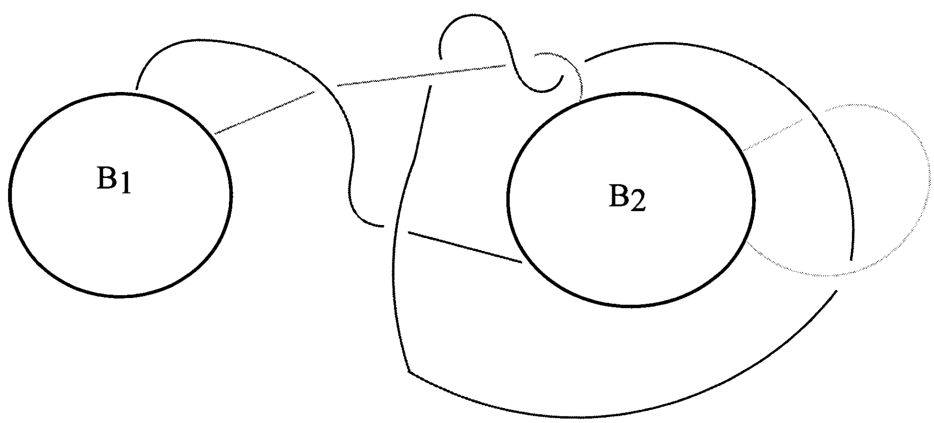


Figure 1.1 Example of an inhabitant of a room with six marked points

Remark 1.1.2. We note that an inhabitant of a room with a connected boundary and $2n$ marked points is an n-tangle.

Definition 1.1.3. Let R be a room with $2n$ marked points. Let a be a strand in an inhabitant T of the room R. We define the *height* $ht(a;T) \in \mathbb{N}$ *of* a as follows.

We start at a marked point of a and we follow the strand. Every time we go over an arc of T we add one and every time we go under an arc we subtract one. Similarly, we define the height of a knot k of T. We start at a point c on the knot and choose a direction to follow on the knot until we return to c. Every time we go over an arc of T we add one and every time we go under an arc we subtract one. Note that $ht(k;T)$ is similarly defined in a link diagram.

Lemma 1.1.4. *Let R be a room with $2n$ marked points. Let a be a strand or a knot that doesn't intersect any other knot or strand in an inhabitant T of the room R. Then $ht(a;T) = 0$.*

Proof. Suppose a is a strand. Then, for each crossing of a, by following the strand, we will pass exactly one time over a and one time under a. Thus, each crossing adds 0 to $ht(a;T)$ and so $ht(a;T) = 0$.

The proof is similar if a is a knot. □

Remark 1.1.5. Let R be a room with $2n$ marked points and a be a strand in an alternating inhabitant T of the room R. Then, the strand goes under an arc, over an arc, under an arc, etc. Thus,

$$ht(a;T) = \begin{cases} 1 & \text{if it starts and ends over an arc} \\ -1 & \text{if it starts and ends under an arc} \\ 0 & \text{if it starts over and ends under or starts under and ends over} \end{cases}$$

Let k be a knot in an alternating inhabitant T of the room R. Since T is alternating, k must go over an arc the same number of times as under an arc. Therefore, $ht(k;T) = 0$.

Let a be a strand or a knot in an inhabitant T of the room R. We say that a *is alternating in* T if it goes over an arc in T, under an arc in T, over an arc in T, and so on. Thus, if k is a knot that is alternating in T, then $ht(k;T) = 0$. Note, that when we say that a knot k is alternating in T, it doesn't imply that k is an alternating knot. Moreover, every strand and knot of an inhabitant of the room R is alternating in T if and only if T is alternating.

Let a and b be strands and/or knots in an inhabitant of the room R. We define $ht(a;b)$ by the height obtained by following a and only counting the crossings such that a intersects b.

Thus, from the definition of height and Lemma 1.1.4 we obtain the following.

Lemma 1.1.6. *Let R be a room with $2n$ marked points and a and b be some arcs or knots in an inhabitant T of the room R. Then $ht(a;b) = -ht(b;a)$ and $ht(a;a) = 0$.*

From, the previous lemma and the definition of height, we obtain the following.

Lemma 1.1.7. *Let R be a room with $2n$ marked points and $a_1, ..., a_n$ and $k_1, ..., k_m$ be the arcs and knots of an inhabitant T of the room R . Then*

$$ht(a_i;T) = \sum_{j=1}^{n} ht(a_i;a_j) + \sum_{l=1}^{m} ht(a_i;k_l).$$

We now introduce a series of technical lemmas about inhabitants T of a room R that will play an important role in proving Proposition 1.2.8.

First, knowing the that knots are alternating in T gives us information about the height of the strands.

Lemma 1.1.8. *Let R be a room with $2n$ marked points and $a_1, ..., a_n$ and $k_1, ..., k_m$ be the strands and knots of an inhabitant T of the room R. If the knots k_i are alternating in T for every $1 \leq i \leq m$, then $\sum_{i=1}^{n} ht(a_i;T) = 0$.*

Proof. To simplify the reading, we denote $ht(a_i;k_j) = E_{ij}$, $ht(k_j;k_l) = F_{jl}$ and $ht(a_i;a_k) = D_{ik}$ with $E_{ij}, F_{jl}, D_{ik} \in \mathbb{Z}$. Furthermore, by Lemma 1.1.6, $ht(k_j;a_i) = -E_{ij}$, $F_{jl} = -F_{lj}$, $D_{ik} = -D_{ki}, D_{ii} = 0$ and $F_{jj} = 0$ for all $1 \leq i,k \leq n$ and $1 \leq j,l \leq m$.

By Lemma 1.1.7, we then have

$$ht(k_i;T) = \sum_{j=1}^{n} ht(k_i;a_j) + \sum_{l=1}^{m} ht(k_i;k_l) = \sum_{j=1}^{n}(-E_{ji}) + \sum_{l=1}^{m} F_{il}. \tag{1.2}$$

However, k_i is alternating in T, therefore $ht(k_i;T) = 0$. So, $\sum_{j=1}^{n}(-E_{ji}) + \sum_{l=1}^{m} F_{il} = 0$. Therefore, we obtain

$$\begin{aligned}
0 = \sum_{i=1}^{m} ht(k_i) &= \sum_{i=1}^{m}(\sum_{j=1}^{n} ht(k_i;a_j) + \sum_{l=1}^{m} ht(k_i;k_l)) \\
&= \sum_{i=1}^{m}(-\sum_{j=1}^{n} E_{ji} + \sum_{l=1}^{m} F_{il}) \\
&= -\sum_{i=1}^{m}\sum_{j=1}^{n} E_{ji} + \sum_{i=1}^{m}\sum_{l=1}^{m} F_{il}.
\end{aligned}$$

But, $\sum_{i=1}^{m}\sum_{l=1}^{m} F_{il} = 0$, because $F_{jl} = -F_{lj}$ and $F_{jj} = 0$ for all $1 \leq j,l \leq m$. This implies that $0 = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ji}$.

Furthermore, notice that $D_{ik} = -D_{ki}$ and $D_{ii} = 0$ for all $1 \leq i,k \leq n$ implies that $\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} = 0$. We can now calculate

$$\sum_{i=1}^{n} ht(a_i;T) = \sum_{i=1}^{n}(\sum_{j=1}^{n} ht(a_i;a_j) + \sum_{l=1}^{m} ht(a_i;k_l)) \tag{1.3}$$

$$= \sum_{i=1}^{n}(\sum_{j=1}^{n} D_{ij} + \sum_{l=1}^{m} E_{ij}) \tag{1.4}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} + \sum_{i=1}^{n}\sum_{l=1}^{m} E_{ij} \tag{1.5}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} = 0. \tag{1.6}$$

□

Let R be a room with $2n$ marked points. A marked point coming from an arc that just went under another arc is called an *under marked point* and a marked point coming from an arc that just went over another arc is called an *over marked point* as is shown in the following figure.

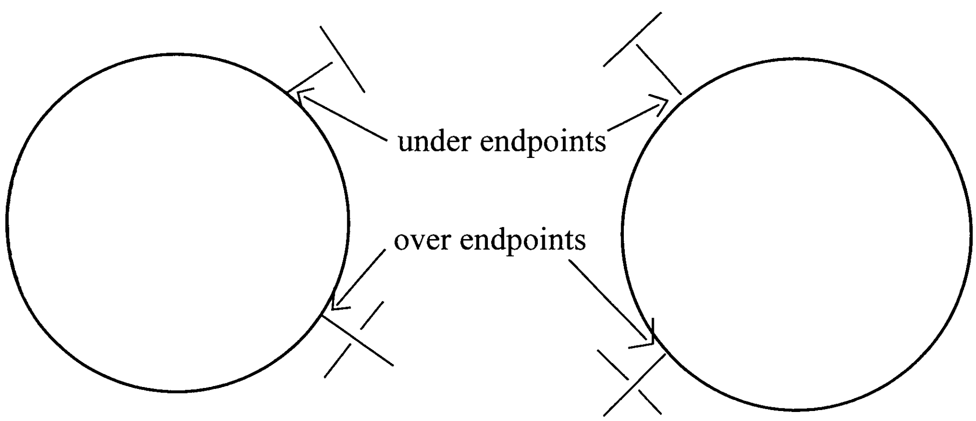


Figure 1.2 Over marked point and under marked point in an inhabitant of a room

Note that, by remark 1.1.5, for an alternating inhabitant of a room R, each marked point of a strand a_i such that $ht(a_i;T) = 1$ is an over marked point. Moreover, each marked point of a strand a_j such that $ht(a_j;T) = -1$ are under marked points. Finally, a strand a_j such that $ht(a_j;T) = 0$ has one over marked point and one under marked point.

Lemma 1.1.9. *Let R be a room with $2n$ marked points and T be an alternating inhabitant of the room R. Then T has n over marked points and n under marked points.*

Proof. Let $a_1, ..., a_n$ and $k_1, ..., k_m$ be the n strands and m knots in T. By Lemma 1.1.8, we have $\sum_{i=1}^{n} ht(a_i;T) = 0$. Moreover, because T is alternating, by remark 1.1.5, for $1 \leq i \leq n$, $ht(a_i;T) = -1, 1$ or 0. Therefore, for each a_i such that $ht(a_i;T) = 1$, there is an a_j such that $ht(a_j;T) = -1$. Thus, for each strand a_i with two over marked points, there is a strand a_j with two under marked points. For the strands a_k, such that $ht(a_k;T) = 0$, then a_k add one over marked point and one under marked point. Thus, there are as many over marked points as under marked points. □

We now show how we can relate information about the number of i bridge arcs and non-bridge arcs on a knot to the height of this knot.

Lemma 1.1.10. *Let R be a room with $2n$ marked points and T an inhabitant of the room R. If k is a knot in T such that there is at least one arc of $T \setminus k$ that goes over k, then $ht(k) = \sum_{i=0}^{l}(i-1)p_i$ where p_i is the number of i bridge arcs on k, p_0 is the number of non-bridge arcs on k and l is the maximum of the i for the i bridge arcs in T.*

Proof. By hypothesis, there is at least one arc that goes over k. Without loss of generality, we start counting the height at a point c just after k went under an arc but before k goes over or under another arc. Then, we follow k until just after the next arc over k. If we have just followed a non-bridge arc, then we went over no arc and under an arc. So, we've added -1 to $ht(k;T)$. If we have just followed an i-bridge arc, then we went over i arcs and under an arc. So, we've added $i-1$ to $ht(k;T)$. We repeat this action until we come back to c. Thus, we have

$$ht(k;T) = \sum_{i=0}^{l}(i-1)p_i.$$

□

Remark 1.1.11. Note that if l is the maximum of the i for the i bridge arcs in D, then $\sum_{i=0}^{l}(i-1)p_i = \sum_{i=0}^{q}(i-1)p_i$ for $q \geq l$, because $p_r = 0$ for every $l < r \leq q$.

We can obtain a similar result for the height of the strands in an inhabitant of a room R, when we know the number of non-bridge arcs and i bridges arcs on the knots.

Lemma 1.1.12. *Let R be a room with $2n$ marked points and $a_1, ..., a_n$ and $k_1, ..., k_m$ be the strands and knots of an inhabitant T of the room R such that for every knot k_i there is at least one arc of $T \backslash k_i$ that goes over k_i. Suppose there are a total of p_0 non-bridge and p_r r-bridge arcs on the knots. Then $\sum_{i=1}^{n} ht(a_i;T) = -\sum_{j=0}^{l}(j-1)p_j$ where l is the number of crossings in T.*

Proof. To simplify the reading, we denote $ht(a_i;k_j) = E_{ij}$, $ht(k_j;k_l) = F_{jl}$ and $ht(a_i;a_k) = D_{ik}$ with $E_{ij}, F_{jl}, D_{ik} \in \mathbb{Z}$. Furthermore, by Lemma 1.1.6, $ht(k_j;a_i) = -E_{ij}$, $F_{jl} = -F_{lj}$, $D_{ik} = -D_{ki}, D_{ii} = 0$ and $F_{jj} = 0$ for all $1 \leq i,k \leq n$ and $1 \leq j,l \leq m$. By Lemma 1.1.7,

$$ht(k_i;T) = \sum_{j=1}^{n} ht(k_i;a_j) + \sum_{l=1}^{m} ht(k_i;k_l) = \sum_{j=1}^{n}(-E_{ji}) + \sum_{l=1}^{m} F_{il}. \tag{1.7}$$

If k_i is alternating in T, then $ht(k_i;T) = \sum_{j=0}^{l}(j-1)p_{j_i} = 0p_{1_j} = 0$. So, $\sum_{j=1}^{n}(-E_{ji}) + \sum_{l=1}^{m} F_{il} = 0$.

Suppose k_i has p_{j_i} j-bridge arcs and p_{0_i} non-bridge arcs. Therefore, $ht(k_i;T) = \sum_{j=0}^{l}(j-1)p_{j_i}$ by the previous lemma and remark. Therefore,

$$ht(k_i;T) = \sum_{j=1}^{n}(-E_{ji}) + \sum_{l=1}^{m} F_{il} = \sum_{j=0}^{l}(j-1)p_{j_i}.$$

Moreover, $p_j = \sum_{i=1}^{m} p_{j_i}$ is the number of j- bridge arcs on the knots k_i and $p_0 = \sum_{i=1}^{m} p_{0_i}$ is the number of non- bridge arcs on the knots k_i. Therefore, $\sum_{i=1}^{m} ht(k_i;T) = \sum_{i=1}^{m}\sum_{j=0}^{l}(j-1)p_{j_i} = \sum_{j=0}^{l}(j-1)p_j$. Thus,

$$\begin{aligned}\sum_{j=0}^{l}(j-1)p_j = \sum_{i=1}^{m} ht(k_i) &= \sum_{i=1}^{m}(\sum_{j=1}^{n} ht(k_i;a_j) + \sum_{l=1}^{m} ht(k_i;k_l)) \\ &= \sum_{i=1}^{m}(-\sum_{j=1}^{n} E_{ji} + \sum_{l=1}^{m} F_{il}) \\ &= -\sum_{i=1}^{m}\sum_{j=1}^{n} E_{ji} + \sum_{i=1}^{m}\sum_{l=1}^{m} F_{il}.\end{aligned}$$

But, $\sum_{i=1}^{m}\sum_{l=1}^{m} F_{il} = 0$, because $F_{jl} = -F_{lj}$ and $F_{jj} = 0$ for all $1 \leq j,l \leq m$. This implies that $-\sum_{j=0}^{l}(j-1)p_j = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ji}$.

Furthermore, remark that $D_{ik} = -D_{ki}$ and $D_{ii} = 0$ for all $1 \leq i, k \leq n$ implies that $\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} = 0$.

We can now calculate

$$\sum_{i=1}^{n} ht(a_i;T) = \sum_{i=1}^{n}(\sum_{j=1}^{n} ht(a_i;a_j) + \sum_{l=1}^{m} ht(a_i;k_l)) \tag{1.8}$$

$$= \sum_{i=1}^{n}(\sum_{j=1}^{n} D_{ij} + \sum_{l=1}^{m} E_{ij}) \tag{1.9}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} + \sum_{i=1}^{n}\sum_{l=1}^{m} E_{ij} \tag{1.10}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} - \sum_{j=0}^{l}(j-1)p_j = -\sum_{j=0}^{l}(j-1)p_j. \tag{1.11}$$

□

Similarly to knots, we can know the height of a strand by knowing the number of i-bridge arcs and non-bridge arcs on it.

Lemma 1.1.13. *Let R be a room with $2n$ marked points, T be an inhabitant of the room R and let a be a strand in T. If by following a we have p_i i-bridge arcs and p_0 non-bridge arcs, then $ht(a;T) = 1 + \sum_{j=0}^{l}(j-1)p_j$ where l is the number of crossings in T.*

Proof. If the strand a doesn't go under any arc in T, then a is an i-bridge. Thus, $ht(a;T) = i = 1 + (i-1) = 1 + \sum_{j=0}^{l}(j-1)p_j$ and the proof is over.

Now, suppose that a does go under an arc in T. We start to calculate the height at a marked point until just after the strand went under an arc. If there was an i-bridge arc before the first under arc, for the height until just after the first under arc, we add $i-1$. Then, we go until just after the next under arc. Again, if there was an i-bridge arc before the under arc we add $i-1$. Similarly, for the path until just after the next under arc we add $i-1$ for each i-bridge. After the last under bridge, for the remainder of a, we add i if we finish with an i-bridge and so we add $(i-1)+1$ for the last i-bridge. Thus, we have $ht(a;T) = 1 + \sum_{j=0}^{l}(j-1)p_j$. □

We can now give an equation between non-bridge arcs and i-bridge arcs in an inhabitant of a room R.

Lemma 1.1.14. *Let R be a room with $2n$ marked points and T be an inhabitant of the room R such that for each knot k_i of T there is at least one arc of $T \setminus k_i$ that goes over k_i. If there are p_i i-bridge arcs in T and p_0 non-bridge arcs in T, then $p_0 = n + \sum_{j=1}^{l}(j-1)p_j$ where l is the number of crossings in T.*

Proof. Suppose there are a total of q_i i-bridge arcs on the knots k_j. Thus, by Lemma 1.1.12, $\sum_{i=1}^{n} ht(a_i;T) = -\sum_{j=0}^{l}(j-1)q_j$. Also, suppose there are t_{j_i} j-bridge arcs on the strands a_i for a total of r_i i-bridge on the strands.

Therefore, by Lemma 1.1.13, $ht(a_i;T) = 1 + \sum_{j=0}^{l}(j-1)t_{j_i}$.

This implies that $\sum_{i=1}^{n} ht(a_i;T) = n + \sum_{i=1}^{n}\sum_{j=0}^{l}(j-1)t_{j_i} = n + \sum_{j=0}^{l}(j-1)r_j$. Moreover, $\sum_{i=1}^{n} ht(a_i;T) = -\sum_{j=0}^{l}(j-1)q_j$. So, $-\sum_{j=0}^{l}(j-1)q_j = n + \sum_{j=0}^{l}(j-1)r_j$ and

$$0 = n + \sum_{j=0}^{l}(j-1)(q_j + r_j) = n + \sum_{j=0}^{l}(j-1)p_j.$$

Thus, $0 = n + \sum_{j=1}^{l}(j-1)p_j - p_0$. Therefore, $p_0 = n + \sum_{j=1}^{l}(j-1)p_j$. □

Suppose that T is an inhabitant of a room R and that T is a subdiagram of a link diagram D. Then, the non-bridge arcs in T can be bridge arcs in D if the prolongation of these arcs in D are bridge arcs. Moreover, if D is a non-split link diagram, then T is an inhabitant of the room R such that for every knot k_i there is at least an arc not of k_i that goes over k_i.

1.2 The red and blue link diagram

We are now going to introduce a construction which, combined with Lemma 1.1.9, will enable us to show Theorem 1.2.8. The result says that for an alternating non-split diagram D of a non-trivial link L, if we suppose $\pi(D)$ is left-orderable, then $\pi(D)$ is abelian.

Let D be a link diagram. Suppose that $\pi(D)$ is left-orderable. Moreover, suppose that a is a maximum between the arcs of D. Then, we color a in red. From the Wada relations, if a is a bridge over a_i and a_j, then we have one of the Wada inequalities :

1. $a_i < a < a_j$
2. $a_j < a < a_i$
3. $a_i = a = a_j$.

However, a is a maximum, therefore $a_i = a = a_j$. Thus, every arc that goes under a is a maximum. We color in red these arcs. Moreover, we color in red, every arc that goes under a red arc. Therefore, every red arc is a maximum. Also, if an arc a_k goes over two red arcs, then we color a_k in red. Note that from the Wada inequalities, every arc that goes over two maximum arcs is a maximum. Hence, by construction, every red arc is a maximum. We define the set of red arcs that comes from a, $R(D,a)$ and call it the *red diagram of D from a*. Note that by construction, $R(D,a)$ is connected.

Furthermore, we color every non red arc in blue and define $B(D,a)$ the set of blue arcs and call it the *blue diagram of D from a*. Notice that if b is a blue arc, then both ends of b must go under blue arcs.

We now project this colored link diagram in the plane to obtain the *blue and red graph* $G(D,a)$ where the vertices in $G(D,a)$ come from the crossings in D and the edges in $G(D,a)$ come from the arcs between crossings in D. Also, an edge is blue or red depending on whether it comes from a blue or red arc. Moreover, at each crossing, if the over arc is red, we draw a red vertex and if the over arc is blue we draw a blue vertex. The subgraph with the blue edges and blue vertices is defined as the *blue subgraph* $BG(D,a)$ of $G(D,a)$. While the subset with the red edges and red vertices is defined as the *red subset* $RG(D,a)$ of $G(D,a)$. Note that $G(D,a)$ is the disjoint union of $RG(D,a)$ and $BG(D,a)$.

In what follows, we are going to give a series of properties for $G(D,a)$, $RG(D,a)$ and $BG(D,a)$. We will eventually show that for an alternating non-split diagram D of a link L, for every a, then $G(D,a) = RG(D,a)$. Thus, every arc in D is a maximum and so $\pi(D)$ is abelian.

Remark 1.2.1. Let D be a link diagram, a an arc in D and $G(D,a)$ the blue and red graph of a. By construction, $R(D,a)$ is connected, therefore $RG(D,a)$ is connected.

Lemma 1.2.2. *Let D be a link diagram, a an arc in D and $G(D,a)$ the blue and red graph of a. Then a blue vertex in $G(D,a)$ has three or four incident blue edges, while a red vertex has exactly four red incident edges.*

Proof. The vertices of $G(D, a)$ are the crossings of the link diagram. Clearly, each vertex has a valency of 4.

Let c be a red vertex in $G(D, a)$. Then c corresponds to a crossing with a red over arc. Therefore, by definition, all the under arcs are also red. Thus, the four edges are red.

Let c be a blue vertex in $G(D, a)$. Then c corresponds to a crossing with a blue over arc. Therefore, c has at least two incident blue edges. If each other incident edges are red, then this mean that the over arc is red, which is a contradiction. Thus, at most one of the other incident edge is red. So, c has three or four incident blue edges. $\square$

We look at the complement $BG(D, a)^c$ of $BG(D, a)$ in the plane. We obtain $BG(D, a)^c = \cup_{i=1}^{n} U_i$ where each U_i is an open connected component of $BG(D, a)^c$. By the previous lemma, $RG(D, a)$ is included in a single connected component U_i. We will study this connected region U_i. Moreover, we define the closure $\overline{U_i}$ of U_i and the frontier $fr(U_i) = \overline{U_i} \setminus U_i$ of U_i. Note also, that $fr(U_i) = \overline{U_i} \cap BG(D, a)$ and therefore $fr(U_i)$ is included in $BG(D, a)$ and so is composed of blue edges and blue vertices.

Every red edge that is incident to one of the vertices of $fr(U_i)$ will be called an *end red edge.*

Now, we look at the corresponding link subdiagram $R(D, a)$ of the red subset $RG(D, a)$. Each red arc that had a corresponding end red edge is called an *end red arc* in $R(D, a)$.

Remark 1.2.3. Note that every end red arc goes under a blue arc. Therefore, if an end red arc doesn't go over an arc in $R(D, a)$, then it is a non-bridge arc. Thus, if an end red arc is a non-bridge in $R(D, a) \subset U_i$, then it is a non-bridge arc in D.

We now look at the closed set U_i^c in the plane. Clearly, $fr(U_i)$ is included in U_i^c. Also, let $U_i^c = \cup_{j=1}^k F_j$ where F_j are the connected components of U_i^c.

Lemma 1.2.4. *Let D be a link diagram, a an arc in D, $G(D,a)$ the blue and red graph of a and F_j a connected component of U_i^c. If there are no red edges incident to the vertices on F_j, then $G(D,a)$ is disconnected. Thus, D is a split diagram.*

Proof. First, suppose U_i^c is connected. Then, $U_i^c = F$. Let v be a vertex in $RG(D,a) \subset U_i$ and w a vertex in F. There are no red incident edges to vertices on F, thus, there are no paths between v and w. This implies that $G(D,a)$ is a disconnected graph.

Suppose that U_i^c is disconnected and let F_j be a connected component of U_i^c. Let v be a vertex in $RG(D,a) \subset U_i$ and w a vertex in F_j. By hypothesis, there is no path in $RG(D,a) \cup F_j$ between v and w. Moreover, because F_j is a connected component of U_i^c and $BG(D,a)$ is included in U_i^c, then $F_j \cap BG(D,a)$ is a connected component of $BG(D,a)$ and there is no blue path between w and any vertex in $BG(D,a) \setminus F_j$. Thus, there is no path between w and v and so $G(D,a)$ is disconnected. □

In a non-split link diagram, by the previous lemma, there is at least one red incident edge to a vertex in F_j.

Note that $\partial F_j \subset \partial U_i$. Moreover, $\partial U_i \subset BG(D,a)$. Hence, $\partial F_j \subset BG(D,a)$.

Let C_j be a closed curve in U_i that intersects transversely every red incident edge to a vertex in F_j but that intersects no other edges in $G(D,a)$ and such that one side of C_j contains F_j. (This curve exists because $\partial F_j \subset BG(D,a)$ and $BG(D,a)$

is a finite graph.) Let V_j be this side of C_j. Note that the red edges in V_j correspond to non-bridge arcs in V_j.

Lemma 1.2.5. *Let D be a non-split link diagram, a an arc in D, $G(D,a)$ the blue and red graph of a and F_j a connected component of U_i^c. Then C_j intersects a non-zero even number of end red edges.*

Proof. By Lemma 1.2.4, C_j intersects at least one end red edge. Thus, in the corresponding link diagram D, C_j intersects at least one end red arc. Moreover, each strand entering V_j must go out of V_j at some points, therefore C_j intersects an even number of end red arcs. □

Suppose C_j and so V_j intersects $2k$ end red edges. We will be interested in the k-tangle T_j given by the intersection between V_j and the link diagram D.

Lemma 1.2.6. *Let D be a non-split link diagram, a an arc in D, $G(D,a)$ the blue and red graph of a and F_j a connected component of U_i^c. Then, $k+p_0' = \sum_{j=1}^{l}(j-1)p_j$ where p_j is the number of j bridge arcs in T_j, p_0' is the number of non-bridge arcs in F_j and l is the number of crossings in T_j. So, there is at least an m-bridge arc with $m \geq 2$ in the k-tangle T_j.*

Proof. The k-tangle T_j has the marked points given by the end red arcs. By construction, these end red arcs are non-bridge arcs in T_j . Thus, there are at least $2k$ non-bridge arcs in T_j. Moreover, if there are other non-bridge arcs in T_j, they must be blue arcs and so in F_j. Therefore, by Lemma 1.1.14, $2k + p_0' = k + \sum_{j=1}^{l}(j-1)p_j$ where p_j is the number of j bridge arcs in T_j, p_0' is the number of non-bridge arcs in F_j and l is the number of crossings in T_j. So, $k+p_0' = \sum_{j=1}^{l}(j-1)p_j$ and there is at least an m-bridge arc with $m \geq 2$ in T_j. □

We obtain the following important result directly from the previous lemma.

Lemma 1.2.7. *Let D be an alternating non-split link diagram, a an arc in D and $G(D,a)$ the blue and red graph of a. Then, $BG(D,a)$ is empty.*

Proof. By Lemma 1.2.6, for each connected component of U_i^c there is at least an m-bridge arc with $m \geq 2$ in the k-tangle given by V_j. This is a contradiction because D is alternating. Therefore, there are no connected components for U_i^c. Which implies that U_i^c is empty. So U_i is the plane and thus $BG(D,a)$ is empty because U_i is included in $BG(D,a)^c$. □

We can now prove the desired result.

Proposition 1.2.8. *Let D be an alternating non-split diagram of a non-trivial link L. If we suppose $\pi(D)$ is left-orderable, then $\pi(D)$ is abelian.*

Proof. Let $<$ be an order on $\pi(D)$. We have a finite number of a_i, thus there is a maximum a_j. But D is alternating, therefore a_j is a bridge over some arcs a_i and a_m. Therefore it satisfies one of the Wada inequalities $a_i < a_j < a_m$, $a_m < a_j < a_i$ or $a_i = a_j = a_m$. But a_j is a maximum, thus we get $a_i = a_j = a_m$. This implies that, a_i and a_m are maxima. Thus, every arc that goes under a maximum arc, becomes a maximum arc. Moreover, if two maximum arcs a_i and a_m go under an arc a_k, then they satisfy one of the Wada inequalities $a_i < a_k < a_m$, $a_m < a_k < a_i$ or $a_i = a_k = a_m$. But a_i and a_m are maxima, thus we get $a_i = a_k = a_m$. Therefore, every arc that goes over two maxima arcs becomes a maximum.

So we can construct the blue and red graph of a_j $G(D,a_j)$. By Lemma 1.2.7, $BG(D,a_j)$ is empty. Thus, $RG(D,a_j) = G(D,a_j)$ and every arc in D is red and thus a maximum. This implies that $a_1 = ... = a_n$. So $\pi(D) =< a_1 >$ and thus $\pi(D)$ is abelian.

□

If L is the trivial knot, then its double branched cover is the 3-sphere and the fundamental group of the 3-sphere is trivial, thus not left-orderable by convention.

Hence, by combining Propositions 1.2.8 and 1.0.6, we get a new proof of Theorem 1.0.1.

In the following chapters, we will show how, for some family of links, supposing that the Wada group is left-orderable will implies that it is also abelian and thus get the desired result.

CHAPTER II

THE IMPORTANCE OF THE GRAPH GROUP AND CONSTRUCTION OF DIRECTED WADA GRAPHS

Let L be a link in S^3 and D a diagram for L. Label the arcs of the diagram a_1 through a_n. We recall that the *Wada group* is

$$\pi(D) := < a_1, ..., a_n : a_k^{-1} a_j a_i^{-1} a_j >$$

where the relations

$$a_k^{-1} a_j a_i^{-1} a_j \tag{2.1}$$

are in one to one correspondence with the crossings of D. In this chapter, we will be interested in the following subgroup of the Wada group.

Definition 2.0.1. We define $G(D)$, the *graph group of* D, as the subgroup of $\pi(D)$ generated by the $a_i^{-1} a_j$ where the arc a_i goes over the arc a_j or the arc a_j goes over the arc a_i.

The following results motivate the interest in this group when studying the fundamental group of double branched covers of links.

Lemma 2.0.2. *Let D be a diagram of a non-split link and $G(D)$ be the graph group of D. Suppose that $\pi(D)$ is left-orderable. If the graph group $G(D)$ is trivial, then $\pi(D)$ is abelian.*

Proof. Because the graph group is trivial, for every arc a_j that goes over another arc a_i, we have $a_j^{-1}a_i = 1$. Thus, $a_j = a_i$. Let a_k and a_l be two arcs in D. For every pair of arcs a_k, a_l, because D is non-split, we can follow a path of arcs of D from a_k to a_l. Hence, $a_k = a_l$. Therefore, all the generators a_i of $\pi(D)$ are equal and $\pi(D) =< a_i >$. So, $\pi(D)$ is abelian. □

Therefore, by Proposition 1.0.6 and the previous lemma.

Theorem 2.0.3. *Let D be a diagram of a non-split link L and $G(D)$ be the graph group of D. Suppose that $\pi(D)$ is left-orderable. If the graph group $G(D)$ is trivial, then $\pi_1(\Sigma(L))$ is not left-orderable.*

In this chapter, we will simplify the generating set of the Graph group. We will find a generating set of this group such that when we suppose that $\pi(D)$ is left-orderable, all the generators of $G(D)$ are less than or equal to 1 in $\pi(D)$. It will then be easier in the following chapters to find links for which the graph group is trivial and therefore links for which the fundamental group of the double branched cover is not left-orderable.

To simplify the generating set of the Graph group, we will show how from a diagram D of a link we construct the Wada graph $\Gamma_0(D)$, the coarse Wada rational graph $\Gamma_1(D)$, the Wada rational graph $\Gamma(D)$ and the Wada directed graphs $(\Gamma(D), <)$. Moreover, we will show how from these graphs, we find a simplified generating set of the Graph group.

2.1 From a link diagram to the Wada Graph

Starting from a link diagram D, we will construct a graph that will eventually enable us to simplify the generating set of the Graph group. We will call this graph the *Wada graph* $\Gamma_0(D)$. This graph will be mostly useful in constructing a more important graph, the Wada rational graph.

Twist tangles will play a central role in the construction of the Wada graph. We depict a half-twist in Figure 2.1. We won't differentiate positive half-twists from negative half-twists.

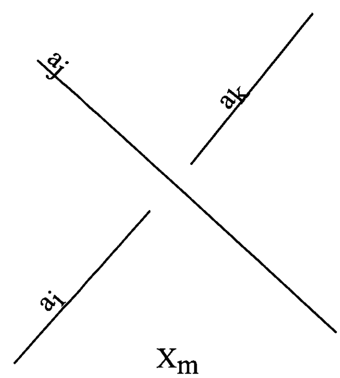


Figure 2.1 half-twist

We now depict a $\frac{3}{2}$-twist in the following figure. Again, we won't differentiate the positive $\frac{3}{2}$-twist from the negative $\frac{3}{2}$-twist.

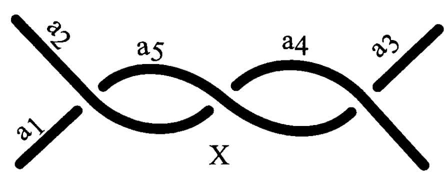


Figure 2.2 $\frac{3}{2}$-twist

Similarly, we define $\frac{k}{2}$-twist tangles. We say that a $\frac{k}{2}$-twist tangle is a *maximal twist tangle* if we can't find a $\frac{k+1}{2}$-twist tangle from the same tangle. Let X be a maximal twist tangle. We define the *non-bridge arcs of* X, as the arcs of X that do not go over any arcs in X.

From now on, let D be a link diagram of a non-split link L with the arcs labelled from a_1 to a_n. Moreover, the maximal twist tangles of D will be labelled from X_1 to X_m, as in the following figure.

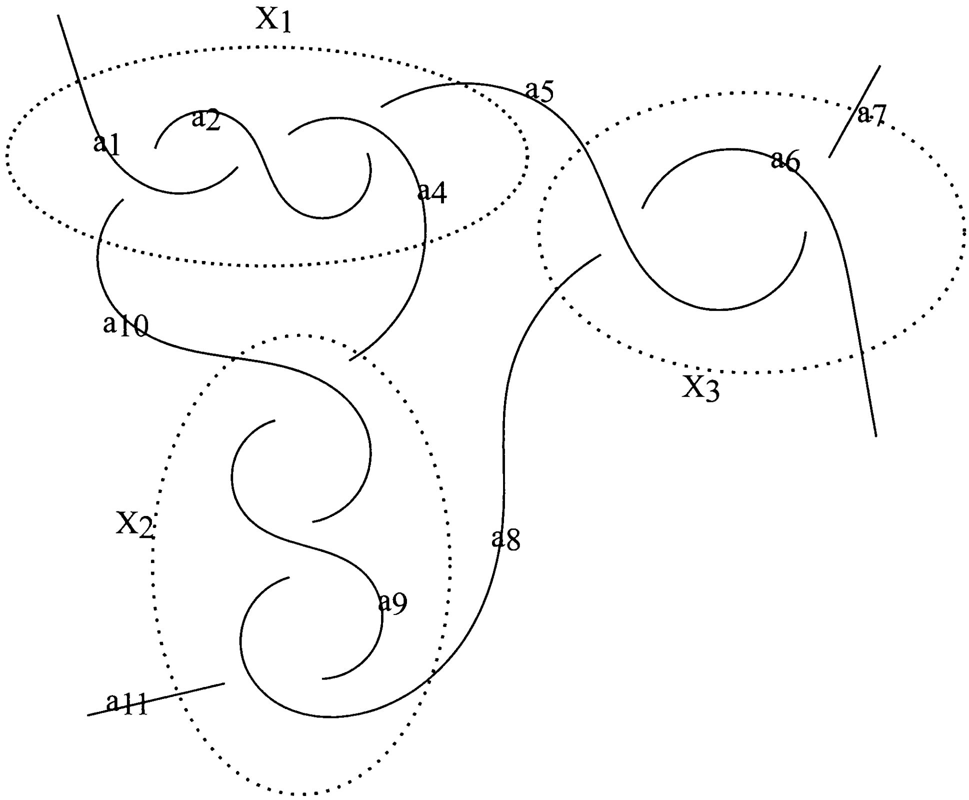


Figure 2.3 Labeling of maximal twist tangles

The vertices of the Wada graph $\Gamma_0(D)$ will be the arcs a_i's of D. For the edges, we will look at the maximal twist tangles of D. Let X_j be a maximal twist tangle. Then, we will draw edges labelled X_j between each vertices coming from arcs of the twist tangle X_j such that one arc is going over the other. Moreover, the non-bridge arcs of X_j will be called the *non-bridge vertices of* X_j. For example, for the half-twist tangle as in Figure 2.1, we obtain the following graph.

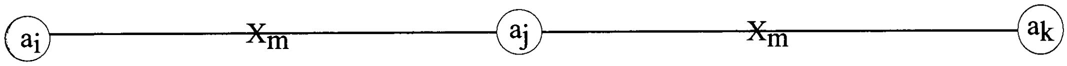


Figure 2.4 Wada graph of a half-twist

We note that there is a Wada relation $a_k^{-1}a_j = a_j^{-1}a_i$ and a Wada relation $a_j^{-1}a_k = a_i^{-1}a_j$. Moreover, a_i and a_k are called the non-bridge vertices of X_m.

Consider the case where we have a full twist :

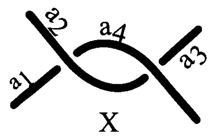


Figure 2.5 Full twist

Then, we get the following graph.

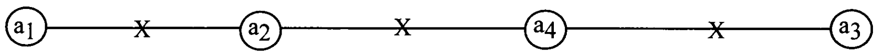


Figure 2.6 Wada graph of a full twist

We note that the Wada relations give us $a_1^{-1}a_2 = a_2^{-1}a_4 = a_4^{-1}a_3$ and $a_2^{-1}a_1 = a_4^{-1}a_2 = a_3^{-1}a_4$. Moreover, a_1 and a_3 are the non-bridge vertices of X.

Next consider the $\frac{3}{2}$-twist

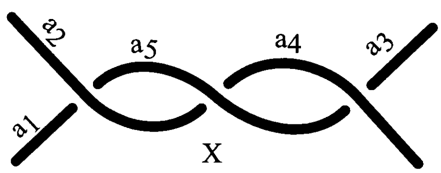


Figure 2.7 $\frac{3}{2}$-twist

Then, we obtain

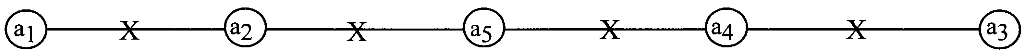


Figure 2.8 Wada graph of the $\frac{3}{2}$-twist

Again, we note that the Wada relations give us $a_1^{-1}a_2 = a_2^{-1}a_5 = a_5^{-1}a_4 = a_4^{-1}a_3$ and $a_2^{-1}a_1 = a_5^{-1}a_2 = a_4^{-1}a_5 = a_3^{-1}a_4$. Moreover, a_1 and a_3 are the non-bridge vertices of X.

Similarly, if we have an $\frac{n}{2}$-twist, we obtain the following Wada graph

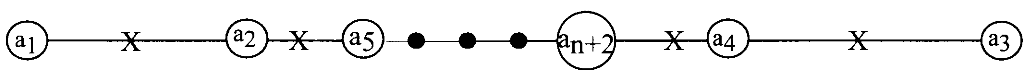


Figure 2.9 Wada graph of the $\frac{n}{2}$-twist,

We note that the Wada relations give us $a_1^{-1}a_2 = a_2^{-1}a_5 = a_5^{-1}a_6 = a_6^{-1}a_7 = ... = a_i^{-1}a_{i+1} = ... = a_{n+2}^{-1}a_4 = a_4^{-1}a_3$ and $a_2^{-1}a_1 = a_5^{-1}a_2 = a_6^{-1}a_5 = a_7^{-1}a_6 = ... = a_{i+1}^{-1}a_i = ... = a_4^{-1}a_{n+2} = a_3^{-1}a_4$. Moreover, a_1 and a_3 are the non-bridge vertices of X.

From the diagram of Figure 2.3, we obtain the following Wada graph.

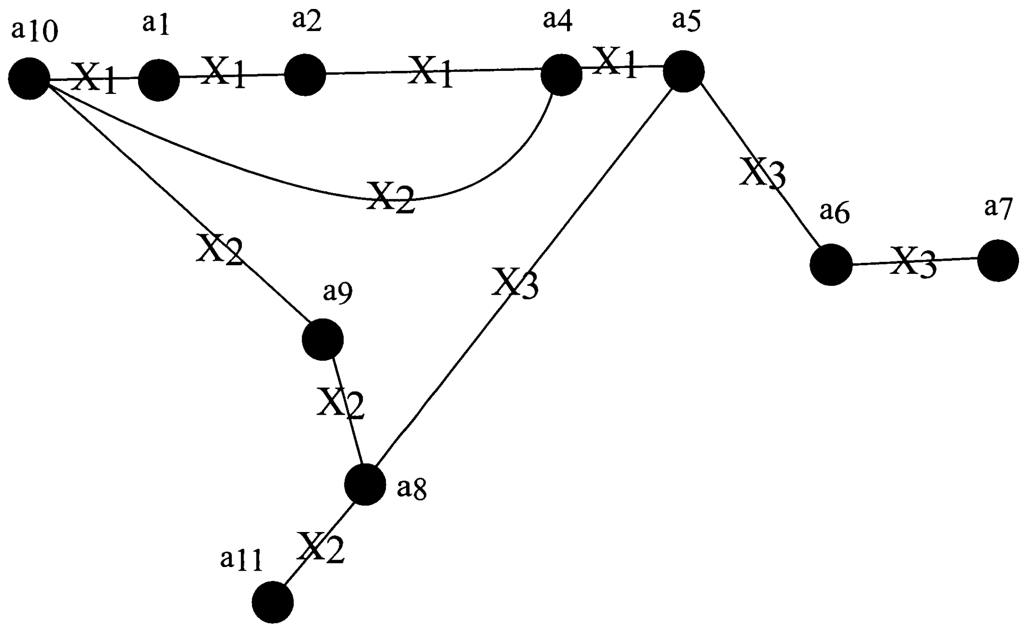


Figure 2.10 Wada graph of Figure 2.3

We are now going to give another example with the knot 8_{21}.

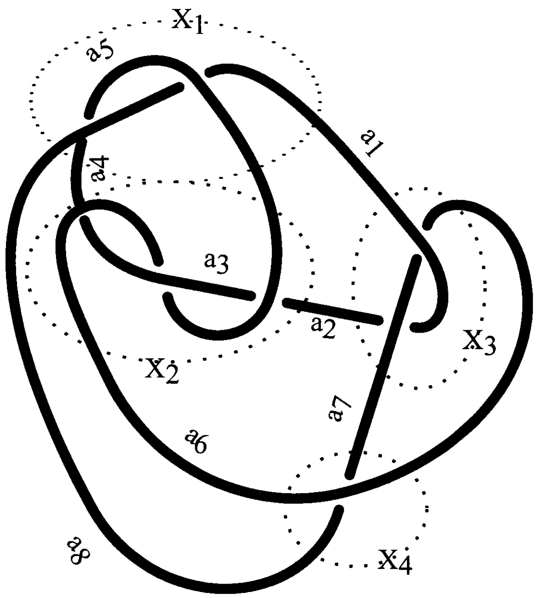


Figure 2.11 Knot diagram of the knot 8_{21}

We thus obtain the following Wada graph Γ_0.

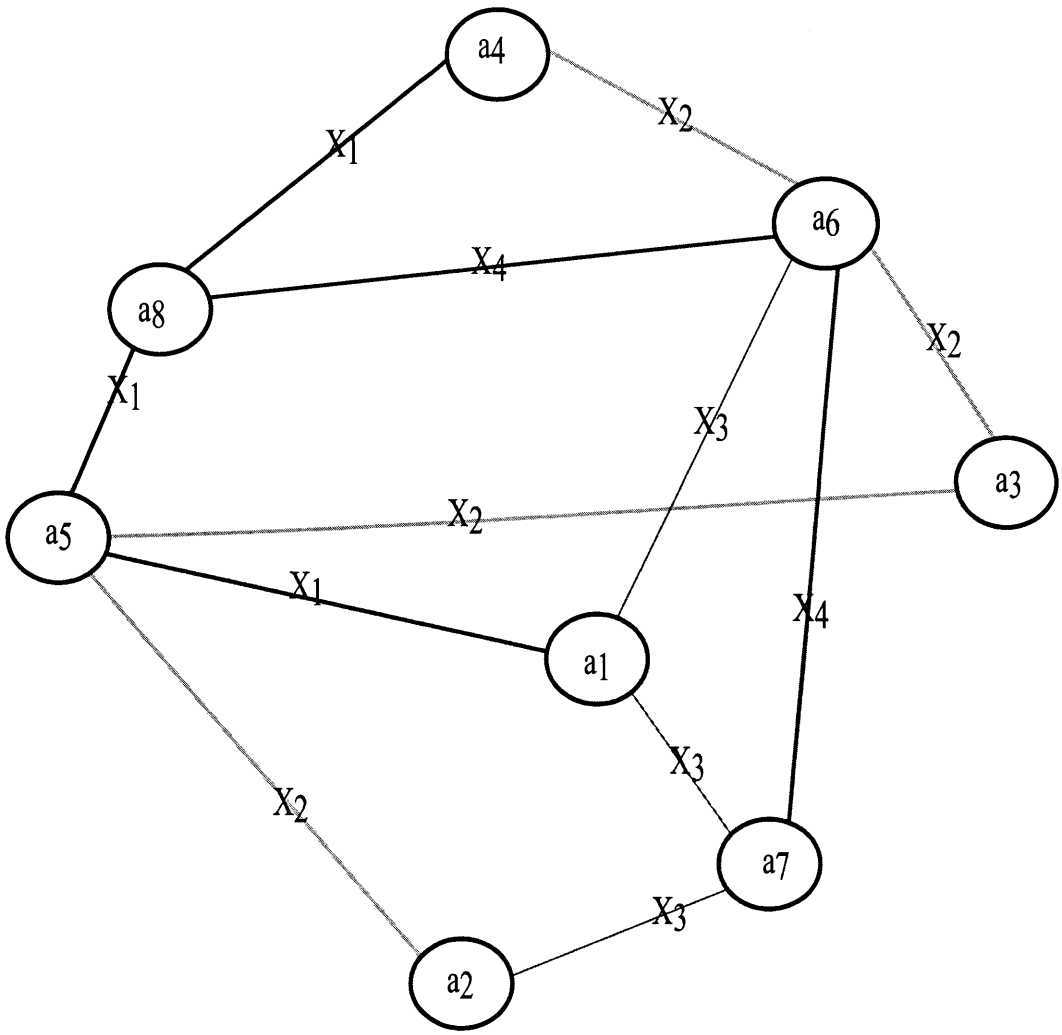


Figure 2.12 A Wada graph Γ_0 of the knot 8_{21}.

From the definition of the Wada graph and the graph group $G(D)$, we obtain the following result that correlates the Wada graph and the Wada relations.

Remark 2.1.1. Let $X_1, ..., X_m$ be the maximal twist tangles of a link diagram D, $\Gamma_0(D)$ be the Wada graph and a_i and a_j be arcs in D. Then, there is an edge X_k between a_i and a_j in $\Gamma_0(D)$ if and only if there is a Wada relation $a_i^{-1}a_j = a_j^{-1}a_l$ or a Wada relation $a_j^{-1}a_i = a_i^{-1}a_l$ for some arc a_l in D.

Now, we will show how from a Wada graph $\Gamma_0(D)$, we can obtain some simplified generating sets $S(\Gamma_0(D), d)$ of the Graph group $G(D)$ where d represents a choice between one of the possible generating sets. Recall that the Graph group is the subgroup of the Wada group generated by the $a_i^{-1}a_j$ where a_i is an arc that goes over a_j or a_j is an arc that goes over a_i.

Let $X_1, ..., X_m$ be the maximal twist tangles of a link diagram D. Let X_k be a maximal twist tangle as in the following figure.

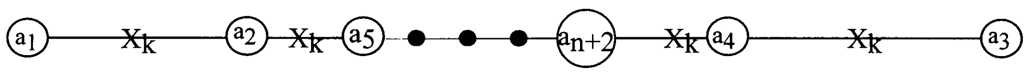


Figure 2.13 Wada graph of the $\frac{n}{2}$-twist,

Then, a_1 and a_3 are the non-bridge vertices and the Wada relations give us $a_1^{-1}a_2 = a_2^{-1}a_5 = a_5^{-1}a_6 = a_6^{-1}a_7 = ... = a_i^{-1}a_{i+1} = ... = a_{n+2}^{-1}a_4 = a_4^{-1}a_3$ and $a_2^{-1}a_1 = a_5^{-1}a_2 = a_6^{-1}a_5 = a_7^{-1}a_6 = ... = a_{i+1}^{-1}a_i = ... = a_4^{-1}a_{n+2} = a_3^{-1}a_4$. We choose either a_1 or a_3 and either $a_1^{-1}a_2 = a_2^{-1}a_5 = a_5^{-1}a_6 = a_6^{-1}a_7 = ... = a_i^{-1}a_{i+1} = ... = a_{n+2}^{-1}a_4 = a_4^{-1}a_3$ or $a_2^{-1}a_1 = a_5^{-1}a_2 = a_6^{-1}a_5 = a_7^{-1}a_6 = ... = a_{i+1}^{-1}a_i = ... = a_4^{-1}a_{n+2} = a_3^{-1}a_4$. Therefore, there are four possibilities. If we choose a_1 and $a_1^{-1}a_2 = a_2^{-1}a_5 = a_5^{-1}a_6 = a_6^{-1}a_7 = ... = a_i^{-1}a_{i+1} = ... = a_{n+2}^{-1}a_4 = a_4^{-1}a_3$, then, we define $X_k = a_1^{-1}a_2$. So, we have either $X_k = a_1^{-1}a_2$, $X_k = a_2^{-1}a_1$, $X_k = a_3^{-1}a_4$ or $X_k = a_4^{-1}a_3$. Note that from the Wada relations, $a_1^{-1}a_2 = a_3^{-1}a_4$ and $a_2^{-1}a_1 = a_4^{-1}a_3$. Moreover, $(a_1^{-1}a_2)^{-1} = a_2^{-1}a_1$ and $(a_3^{-1}a_4)^{-1} = a_4^{-1}a_3$. We do similarly for every maximal twist tangles. Then, we define $S(\Gamma_0(D), d)$ as a *Wada graph set of generators* where d represents one of the 4^m possible choice of set of generators. Moreover, we define a *Wada graph group of D* as the subgroup $G(\Gamma_0(D), d)$ of $\pi(D)$ generated by the set of generators $S(\Gamma_0(D), d)$.

Lemma 2.1.2. *Let D be a link diagram. Then, a Wada graph set of generators $S(\Gamma_0(D), d)$ is a generating set of the Graph group $G(D)$.*

Proof. Let $S(\Gamma_0(D), d)$ be a Wada graph set of generators of D and a_i and a_j be some arcs in D such that a_i goes over a_j. Then, $a_i^{-1}a_j$ is a generator of $G(D)$ and a_i and a_j are in the same maximal twist tangle X_k. Moreover, in $S(\Gamma_0(D), d)$, without loss of generality, $X_k = a_p^{-1}a_q$ where a_p and a_q are arcs in X_k such that at least one goes over the other in X_k. Therefore, by the Wada relations, we have one the following possibilities: $X_k = a_p^{-1}a_q = a_i^{-1}a_j$ or $X_k = a_p^{-1}a_q = a_i^{-1}a_j$ or $X_k^{-1} = (a_p^{-1}a_q)^{-1} = a_i^{-1}a_j$ or $X_k^{-1} = (a_p^{-1}a_q)^{-1} = a_i^{-1}a_j$. This implies that $S(\Gamma_0(D), d)$ is a generating set of the Graph group $G(D)$. □

Therefore, by the previous lemma and the definition of the Graph group and the Wada graph groups, we obtain the following result.

Proposition 2.1.3. *Let D be a link diagram. Then, a Wada graph group of D $G(\Gamma_0(D), d)$ is equal to the Graph group $G(D)$.*

Thus, the 4^m groups $G(\Gamma_0(D), d)$ are equal to the Graph group $G(D)$.

We conclude this section by a useful property of the Wada graph.

Proposition 2.1.4. *If D is a connected diagram, then the Wada graph is connected.*

Proof. Let a_i and a_j be arcs of D. Since D is connected, we can follow a path of arcs from a_i to a_j, and this path determines a path in the Wada graph. Thus, the graph is connected. □

2.2 From the Wada graph to the coarse Wada rational graph

From the Wada graph Γ_0, we will define the coarse Wada rational graph Γ_1 which will enable us to define the coarse Wada groups $G(\Gamma_1)$.

In a link diagram D, as shown in the article (Conway, 1970), we can isotope every rational tangle to obtain an $[mnp...st]$ rational tangle where $t > 1$ and $m, n, p, ...s > 0$. This rational tangle is constructed from $\frac{m}{2}$, $\frac{n}{2}$, ..., $\frac{t}{2}$ half-twists regions where each twist region is represented by an X_i. The twist region t represented by X_i is called the *leading twist region* of the rational tangle. We label each rational tangle region by the leading twist region X_i.

For Figure 2.3, we obtain

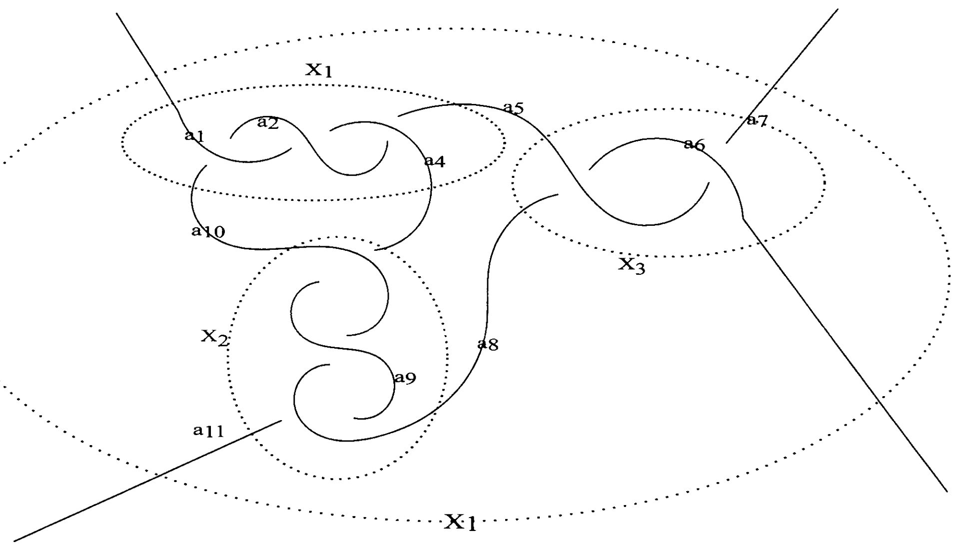


Figure 2.14 Labeling of the rational tangle of Figure 2.3

and for the knot 8_{21}

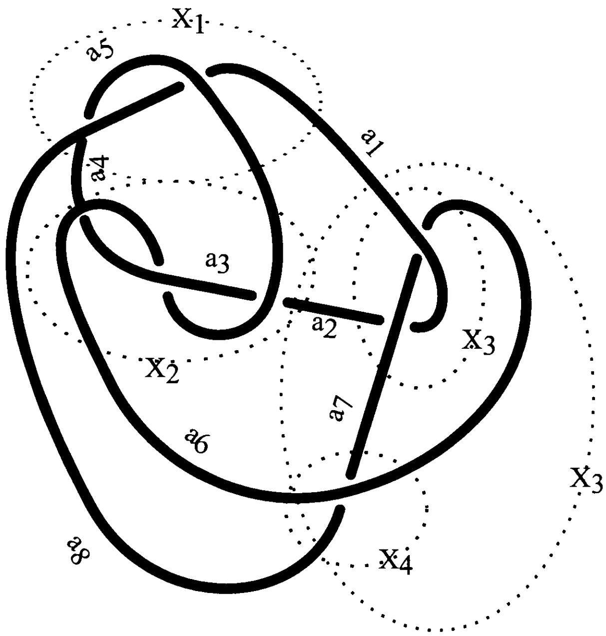


Figure 2.15 Labeling of the rational tangles for the knot 8_{21}

The $[n]$ rational tangle is the same as an $\frac{n}{2}$-twist region. Thus, as seen in the previous section, we obtain

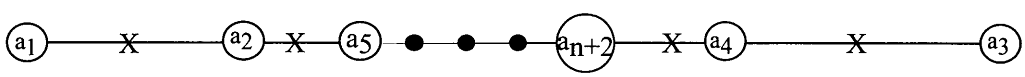


Figure 2.16 Wada graph of an $\frac{n}{2}$-twist

We embed this graph in the link diagram as follows

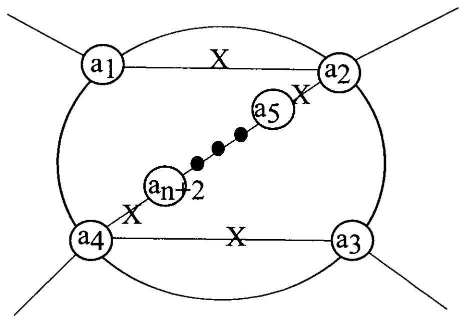
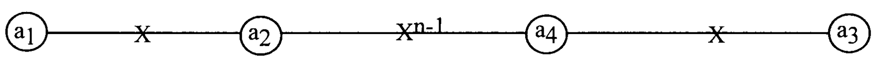


Figure 2.17 The embedding of the Wada graph of an $\frac{n}{2}$-twist in the link diagram

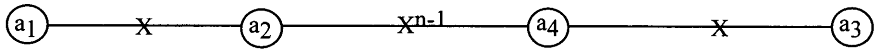
Only the four end strands a_1, a_2, a_3, a_4 will connect with other rational tangles. We call these vertices, *open vertices.* The other arcs a_i of the rational tangle will not be in contact with other rational tangles. We call these vertices *closed vertices.* Thus, for a closed vertex a_i, there is no arc a_j from another rational tangle such that there is a Wada relation $a_i^{-1}a_j = a_j^{-1}a_k$ or $a_j^{-1}a_i = a_i^{-1}a_k$ for an arc a_k. For every rational tangle which is not a half-twist there are exactly four open vertices. Because rational tangles are alternating, a_1 and a_3 are non-bridge in X and a_2 and a_4 are 1-bridge in X. In the coarse Wada rational graph, we keep only the open vertices.

Moreover for the tangle of Figure 2.17, suppose that $X = a_1^{-1}a_2$. Note that $(a_2^{-1}a_5)(a_5^{-1}a_6)(a_6^{-1}a_7)...(a_{n+2}^{-1}a_4) = a_2^{-1}a_4$. Hence, because $a_1^{-1}a_2 = a_2^{-1}a_5 = a_5^{-1}a_6 = a_6^{-1}a_7 = ... = a_{n+2}^{-1}a_4$, we have $a_2^{-1}a_4 = (a_1^{-1}a_2)^{n-1} = X^{n-1}$. We can generalize the previous example to obtain the following lemma.

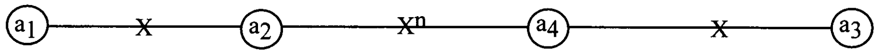
Lemma 2.2.1. *The coarse Wada rational graph* Γ *of the* $[n]$ *rational tangle is as follows.*



Proof. We will prove the lemma by induction on n. By definition of the Wada graph, the lemma is true for $n = 1$. We now suppose that for an $[n]$ rational tangle, we obtain the following coarse Wada rational graph



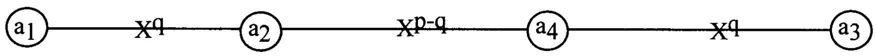
Now, to obtain an $[n+1]$ rational tangle, we only need to add one arc to the rational tangle $[n]$ and this arc corresponds to a closed vertex between a_2 and a_4. Thus we obtain the following coarse Wada rational graph



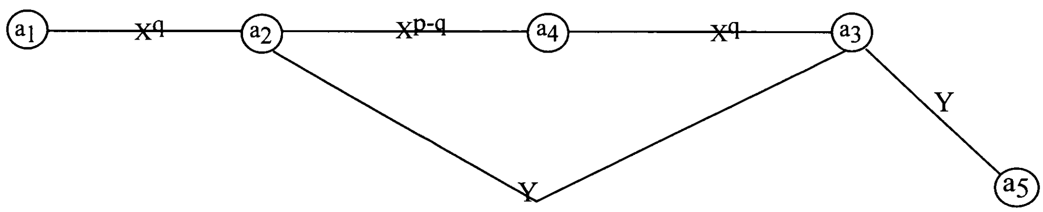
□

More generally we have the following proposition.

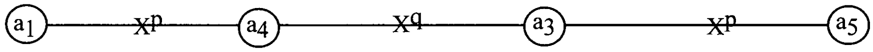
Proposition 2.2.2. *If we have an* $[n_m n_{m-1}...n_1] = \frac{p}{q}$ *tangle, then we have the following coarse Wada rational graph*



Proof. We will prove this result by induction on m. When $m = 1$, the result is true by the previous lemma. We suppose it is true for $m > 1$. We will show the result holds for $[in_m n_{m-1}...n_1]$ for every $i \in \mathbb{N}$ by induction on i. For $i = 1$, we obtain the following coarse Wada rational graph

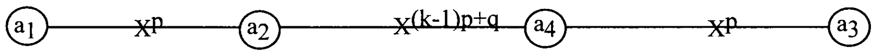


By substituting Y to $X^{p-q}X^q = X^p$ and noting that a_1, a_4, a_3, a_5 are now the open vertices we obtain the following coarse Wada rational graph

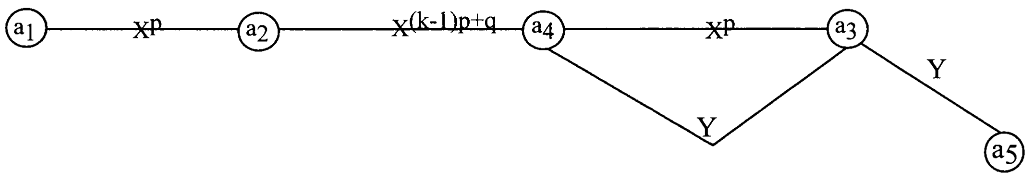


Moreover, $[1n_m n_{m-1} ... n_1] = 1 + \frac{q}{p} = \frac{p+q}{p}$ and thus the result is true when $i = 1$.

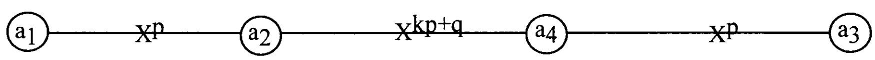
Now, we suppose it is true for $i = k$. Thus, $[kn_m n_{m-1} ... n_1] = k + \frac{q}{p} = \frac{kp+q}{p}$ and we have



When we add a twist to get the $[(k+1)n_m n_{m-1} ... n_1]$ tangle, we obtain



and thus by substituting $Y = X^p$ and noting that a_1, a_2, a_3, a_5 are now the open vertices we obtain the following coarse Wada rational graph

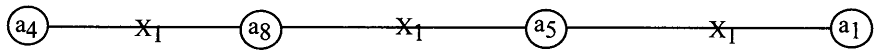


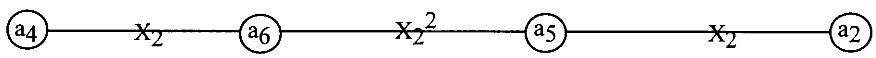
and $[(k+1)n_m n_{m-1} ... n_1] = k + 1 + \frac{q}{p} = \frac{(k+1)p+q}{p}$ and we have the desired result.

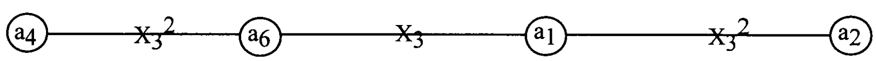
□

We are now going to continue the example of the knot 8_{21}.

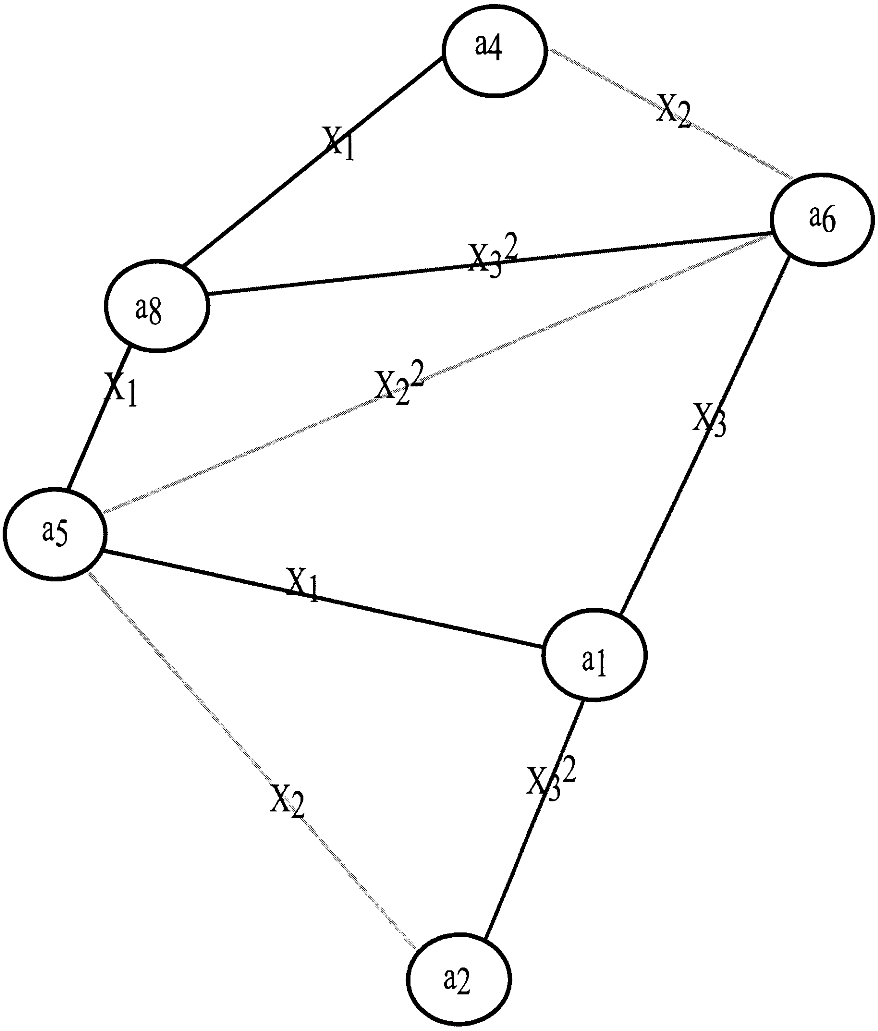
The rational tangle X_1 is a [2] rational tangle, the rational tangle X_2 is a [3] rational tangle and the rational tangle X_3 is a $[1,2] = \frac{3}{2}$ rational tangle. Hence, by the previous proposition, we obtain the following coarse Wada rational graph for each rational tangle X_1, X_2 and X_3







We thus obtain the following coarse Wada rational graph Γ_1 for the previous diagram of the knot 8_{21}.



The coarse Wada rational graph $\Gamma_1(D)$ of the knot 8_{21}.

Moreover, figure 2.3 is a $[2,3,3] = \frac{23}{10}$ rational tangle. Thus, the coarse Wada rational graph of 2.3 is

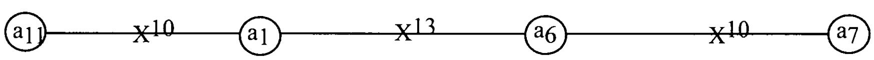


Figure 2.18 The coarse Wada rational graph of a $[2,3,3] = \frac{23}{10}$ rational tangle.

By a similar argument as in the proof of Proposition 2.2.2, we can show the following lemma.

Lemma 2.2.3. *Let D be a link diagram and $X_1, ..., X_m$ be the maximal twist tangles of a rational tangle $[n_m...n_2n_1]$ with leading twist region X_1. Then, in the coarse Wada rational graph $X_i = X_1^k$ where k is the numerator of the continuous fraction $[n_{i-1}n_{i-2}...n_2n_1]$ for $1 < i \leq m$.*

Now, we will show how from a coarse Wada rational graph $\Gamma_1(D)$, we can obtain some simplified generating sets $S(\Gamma_1(D), d)$ of the Graph group $G(D)$ where d represents a choice between one of the possible generating sets.

Let $X_1, ..., X_m$ be the rational tangles of a link diagram D. Let X_k be the leading maximal twist tangle of the rational tangle X_k. Then, similarly as for the Wada graph set of generators, with the non-bridge arcs a_i and a_l of the leading maximal twist tangle X_k and the Wada relations, we choose one of the four possibilities: $X_k = a_i^{-1}a_j$ or $X_k = a_j^{-1}a_i$ or $X_k = a_l^{-1}a_s$ or $X_k = a_s^{-1}a_s$ where a_j and a_s are arcs in X_k that goes over a_i and a_l respectively. Then, we define $S(\Gamma_1(D), d)$ as a *coarse Wada rational set of generators* where d represents one of the 4^m possible choice of set of generators. Moreover, we define a *coarse Wada rational group of D* as the subgroup $G(\Gamma_1(D), d)$ of $\pi(D)$ generated by the set of generators $S(\Gamma_1(D), d)$.

Lemma 2.2.4. *Let D be a link diagram. Then, a coarse Wada rational set of generators $S(\Gamma_1(D), d)$ is a generating set of the Graph group $G(D)$.*

Proof. Let $S(\Gamma_1(D), d)$ be a coarse Wada rational set of generators. Then, for every rational tangle X_k, we have $X_k = a_i^{-1}a_j$. There is at least one Wada graph set of generators $S(\Gamma_0(D), d_0)$ such that we also have $X_k = a_i^{-1}a_j$ for every leading maximal twist X_k.

If every rational tangle is a maximal twist, then $S(\Gamma_1(D), d) = S(\Gamma_0(D), d_0)$. Thus, because $S(\Gamma_0(D), d_0)$ is a generating set of the Graph group by Lemma 2.1.2, $S(\Gamma_1(D), d)$ is a generating set of the Graph group.

If there are rational tangles that are not maximal twists, then $S(\Gamma_1(D), d)$ is a subset of $S(\Gamma_0(D), d_0)$.

Let $X_l = a_p^{-1}a_q$ be a generator of $S(\Gamma_0(D), d_0)$ that is not a leading maximal twist of a rational tangle. Then X_l is a maximal twist in a rational tangle X_s and X_s is a generator of $S(\Gamma_1(D), d)$. By Lemma 2.2.3, in the coarse Wada rational graph $X_l = X_s^k$ where $k \in \mathbb{Z}$. Hence, in the Wada group depending on the choices made for X_l and X_s, $X_l = X_s^t$ where $t = k$ or $t = -k$. Thus, $S(\Gamma_1(D), d)$ is a generating set of $G(\Gamma_0(D), d_0)$. However, $G(\Gamma_0(D), d_0)$ is equal to the Graph group $G(D)$ by Lemma 2.1.2. This implies that $S(\Gamma_1(D), d)$ is a generating set of the Graph group $G(D)$. □

Therefore, by the previous lemma and the definition of the Graph group and the coarse Wada rational groups, we obtain the following result.

Proposition 2.2.5. *Let D be a link diagram. Then, a coarse Wada rational group of D $G(\Gamma_1(D), d)$ is equal to the Graph group $G(D)$.*

Thus, the 4^m groups $G(\Gamma_1(D), d)$ are equal to the Graph group $G(D)$. Note that here the m represents the number of rational tangles, while in the previous section, m was representing the number of maximal twist tangles.

Remark 2.2.6. The Wada graph is connected if and only if the coarse Wada rational graph is connected. Thus, from proposition 2.1.4, we obtain that if D is a connected diagram, then the coarse Wada rational graph Γ_1 will be connected.

We finish this section with a useful lemma for which the proof is similar to the proof of Proposition 2.2.2.

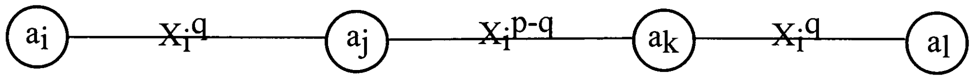


Figure 2.19 Rational tangle X_i

Lemma 2.2.7. *Let D be a link diagram, $G(\Gamma_1(D), d)$ be a coarse Wada rational group and X_i a rational tangle as in Figure 2.19. Then, $a_i^{-1}a_j = X_i^q$, $a_j^{-1}a_k = X_i^{p-q}$ and $a_k^{-1}a_l = X_i^q$ or $a_j^{-1}a_i = X_i^q$, $a_k^{-1}a_j = X_i^{p-q}$ and $a_l^{-1}a_k = X_i^q$.*

2.3 From the coarse Wada rational Graph to the Wada rational graph

In this section, we start with the coarse Wada rational graph and we will simplify it to obtain the Wada rational graph. To find families of links for which the fundamental group of the double branched cover is not left-orderable, we will use Theorem 2.0.3. To do so, for a link diagram D, we will suppose that $\pi(D)$ is left-orderable and then, we will show that $\pi(D)$ is abelian. The Wada rational graph will give us a simplified generating set of the Graph group when we suppose that $\pi(D)$ is left-orderable.

Let D be a link diagram and $\Gamma_1(D)$ the coarse Wada rational graph. If there is no 1-tangle T with only one non-bridge arc in D and such that neither marked point goes over some arcs in $D \setminus T$, then we define $\Gamma_1(D) = \Gamma(D)$.

Suppose there is a 1-tangle T with only one non-bridge arc a_i in D and such that neither marked point goes over some arcs in $D \setminus T$. Then, in the Wada rational graph, we define $a_k = a_i$ for every $a_k \in T$. Let $G(\Gamma_1(D), d)$ be a coarse Wada rational group. If there is no 1-tangle T with only one non-bridge arc in D and such that neither marked point goes over some arcs in $D \setminus T$, then $G(\Gamma(D), d) = G(\Gamma_1(D), d)$ is defined as a *Wada rational group* and by Lemma

2.2.5, it is equal to the Graph group.

Suppose there is a 1-tangle T with only one non-bridge arc in D and such that neither marked point goes over some arcs in $D\backslash T$. Then a *Wada rational set of generators* $S(\Gamma(D), d)$ is defined as the set that contains every generator of $S(\Gamma_1(D), d)$ except for the rational tangle X_i included in T. Moreover, a *Wada rational group* $G(\Gamma(D), d)$ is defined as the subgroup of $\pi(D)$ generated by $S(\Gamma(D), d)$.

In Lemma 2.3.9, at the end of this section, we will show that if $\pi(D)$ is left-orderable, then $G(\Gamma(D), d)$ is equal to $G(D)$.

2.3.1 An important result on the possible left-orders of the Wada group

In this subsection, we will prove Lemma 2.3.7, that gives us information about the possible extrema among the generator of $\pi(D)$ if we suppose that $\pi(D)$ is left-orderable. To do so, we will need the following definition and results.

Definition 2.3.1. If D has exactly n non-bridge arcs, then D is an n *non-bridge diagram.* We define a link to be n *non-bridge*, if it has an n non-bridge diagram and if all other diagrams are m non-bridge diagram with $m \geq n$.

We will principally investigate 2-non-bridge links. It is worth noting, that of the 238 non-alternating knots of 11 crossings or less, 229 are 2-non-bridge knots and 9 are 3-non-bridge knots.

We give an interesting property of 2-non-bridge diagram.

Lemma 2.3.2. *Let D be a 2-non-bridge link diagram with n crossings. Then, there are either exactly two 2-bridge arcs and $(n-4)$ 1-bridge arcs or there is one 3-bridge arc and $(n-3)$ 1-bridge arcs.*

Proof. Let a_i be a non-bridge arc in D. Then, we cut a_i into two arcs to obtain a

1-tangle T with these arcs as marked points. Both marked points are non-bridge arcs in T. Thus, by Lemma 1.1.14, $p_0 = n + \sum_{j=1}^{l}(j-1)p_j$ where p_0 is the number of non-bridge arcs in T and l is the number of crossings in T. Because D is a 2-non-bridge diagram, there is another non-bridge arc in D. Thus, with both marked points being non-bridge arcs in T, there are three non-bridge arcs in T. This implies that $3 = 1 + \sum_{j=1}^{l}(j-1)p_j$ and so $2 = \sum_{j=1}^{l}(j-1)p_j$. Therefore, either there is exactly two 2-bridge arcs or one 3-bridge arc. □

We recall that from section 1.2, we can construct the red and blue graph $G(D, a)$. We look at the complement of the blue sugbraph $BG(D, a)$ to get the connected open region U_i that contains $RG(D, a)$. Then we study the connected components $F_1, ..., F_k$ of the complement of U_i. We also define the disjoint closed curve C_j such that one side of it, the region V_j contains F_j and C_j only intersects end red edges. Moreover, we define T_j the k-tangle given by V_j.

The previous lemma and Lemma 1.2.6 give us a maximal number of connected components.

Lemma 2.3.3. *Let D be a non-split 2-non-bridge link diagram, a an arc in D and $G(D, a)$ the blue and red graph of a. Then, there are at most two connected components F_j in U_i^c.*

Proof. Suppose there are k connected components in U_i^c with $k \geq 3$. By Lemma 1.2.6 there are k m-bridge arcs with $m \geq 2$. But, by Lemma 2.3.2, there are at most two m-bridge arc with $m \geq 2$. Thus, we have a contradiction and we have at most two connected components in U_i^c. □

We now introduce the important region $W = U_i \setminus (\dot{\bigcup}_{j=1}^{k} V_j)$. Note that $\mathbb{R}^2 = U_i \cup U_i^c = U_i \cup (\dot{\bigcup}_{j=1}^{k} F_j)$. Moreover, $U_i \cup (\dot{\bigcup}_{j=1}^{k} F_j) \setminus (\dot{\bigcup}_{j=1}^{k} V_j) = U_i \setminus (\dot{\bigcup}_{j=1}^{k} V_j)$

because $F_j \subset V_j$. Thus, $W = U_i \setminus (\dot{\bigcup}_{j=1}^{k} V_j) = \mathbb{R}^2 \setminus (\dot{\bigcup}_{j=1}^{k} V_j)$ and we can view W has a room as defined in section 1.1. Let T_w be the inhabitant of W given by the intersection of W and the link diagram D.

Lemma 2.3.4. *Let D be a non-split 2-non-bridge link diagram, a an arc in D and $G(D,a)$ the blue and red graph of a. Then, either there is at most one connected components in U_i^c or the two non-bridge arcs of D are in W and both V_j adds exactly two marked points on W.*

Proof. Suppose there are two connected components F_1 and F_2 in U_i^c. Then, we observe the region $W = U_i \setminus \{V_1 \dot{\cup} V_2\}$. Let a_1, ..., a_n and $k_1, ..., k_m$ be the strands and knots of the inhabitant T_w of W. By Lemma 1.1.14, if there are p_i i-bridge arcs in T_w and p_0 non-bridge arcs in T_w, then $p_0 = n + \sum_{j=1}^{l} (j-1)p_j$ where l is the number of crossings in T_w. Moreover, by Lemma 1.2.5, each V_j adds at least two marked points on W.

Suppose that one of the V_j adds at least four marked points on W. Then, there are at least six marked points on W and so T_w has at least three strands. This implies that $n \geq 3$ and there are at least three non-bridge arcs in T_w. If the non-bridge arcs end at marked points of W, then they are end red arcs. Thus, they are non-bridge arcs in D by remark 1.2.3. Therefore, there are 3 non-bridge arcs in D. This is a contradiction and therefore each V_j adds exactly two marked points.

Suppose that both of the V_j adds exactly two marked points on W. Then, there are four marked points on W and so T_w has two strands. This implies that $n = 2$ and there are two non-bridge arcs in W. If the non-bridge arcs end at marked points of W, then they are end red arcs. Thus, they are non-bridge arcs in D by remark 1.2.3. □

In the next lemma, we show that if there is one connected component F, then there is either one or two non-bridge arc in T_w.

Lemma 2.3.5. *Let D be a non-split 2-non-bridge link diagram, a an arc in D and $G(D,a)$ the blue and red graph of a. Suppose there is exactly one connected component F in U_i^c. Then, there are two possibilities for the number of non-bridge arcs of D in W. First, there is one non-bridge arc of D in W, if V adds two marked points on W and there is no m-bridge arc with $m \geq 2$ in W. Secondly, there are two non-bridge arcs of D in W, if V adds four marked points on W or V adds two marked points on W and there is a two-bridge arcs in W.*

Proof. By Lemma 1.2.5, V adds at least 2 marked points to W. Suppose it adds $2k$ marked points to W. Then, T_w has k strands. This implies by Lemma 1.1.14, that if there are p_i i-bridge arcs in T_w and p_0 non-bridge arcs in T_w, then there is $p_0 = k + \sum_{j=1}^{l}(j-1)p_j$ non-bridge arcs in T_w, where l is the number of crossings in W. By Lemma 1.2.6, there is at least one m-bridge arc with $m \geq 2$ in V. So, by Lemma 2.3.2, either there is a two-bridge arc in T_w, or there is no m-bridge arc with $m \geq 2$ in T_w.

Suppose there are no two-bridge arcs in W. Then, there are k non-bridge arcs in W. If the non-bridge arcs ends at marked points of W, then they are end red arcs. Thus, by remark 1.2.3 every non-bridge arc in W is a non-bridge arcs in D. Moreover, because there are no two-bridge arcs in W, there are either two two-bridge arcs in V or one three bridge arc in V. Thus, by Lemma 1.2.6, we have $k + p_0' = 2$ where p_0' is the number of non-bridge arcs in F. So, there are $p_0' = 2 - k$ non-bridge arc in F. Hence, if $k = 1$, there is one non-bridge arc of D in F and so also one non-bridge arc of D in W. If $k = 2$, then both non-bridge arcs of D are in W.

Suppose there is a two-bridge arcs in W. Then, there are $k + 1$ non bridge arcs in W. Because we have a 2-non-bridge diagram, $k = 1$ and both non bridge arcs of D are in W. □

Therefore, by the previous two lemmas we have the following.

Lemma 2.3.6. *Let D be a non-split 2-non-bridge link diagram, a an arc in D and $G(D, a)$ the blue and red graph of a. Suppose there is at least one connected component F_j in U_i^c. Then, there is at least one non-bridge arc of D in U_i. Hence, there is at least one non-bridge arc of D that is red.*

Proof. By the previous two lemmas, there is at least one non-bridge arc of D in W. However, by definition of W, if an arc is in W, it will also be in U_i. Thus, there is at least one non-bridge arc of D in U_i. Hence, there is at least one non-bridge arc of D that is red. □

The following lemma shows the importance of the non-bridge arcs and will also be key in the construction of the Wada semi-directed graph.

Lemma 2.3.7. *Let D be a 2-non-bridge non-split link diagram. If $\pi(D)$ is left-orderable, then the two non-bridge arcs a_k and a_l are the extrema among the generators of $\pi(D)$ with respect to any left order on $\pi(D)$.*

Proof. Let $<$ be a left order on $\pi(D)$. If all a_i's are equals, then a_k and a_l are extrema. We now look at the case where not all a_i's are equal.

Suppose that neither a_k nor a_l are extrema. We have a finite number of a_i, thus there is a maximum a_j. But a_j is not a non-bridge by hypothesis, therefore a_j is at least a bridge over some arcs a_i and a_m. Therefore it satisfies one of the Wada inequalities $a_i < a_j < a_m$, $a_m < a_j < a_i$ or $a_i = a_j = a_m$. But a_j is a maximum, thus we obtain $a_i = a_j = a_m$. This implies that a_i and a_m are maxima. Thus, every arc that goes under a maximum arc, becomes a maximum arc. Moreover, if two maximum arcs a_i and a_m go under an arc a_p, then it satisfies one of the Wada inequalities $a_i < a_p < a_m$, $a_m < a_p < a_i$ or $a_i = a_p = a_m$. But a_i and a_m are maxima, thus we obtain $a_i = a_p = a_m$. Therefore, every arc that goes over two maxima becomes a maximum.

So, we can construct the blue and red graph of a_j, $G(D, a_j)$. By Lemma 2.3.3, there are at most two connected components (F_j, a_j) in (U_i^c, a_j).

Suppose there is no connected component (F_j, a_j) in (U_i^c, a_j). Then, every arc is red and so every arc is a maximum and the proof is over.

Now, suppose there is at least one connected component. By Lemma 2.3.6, there is at least one non-bridge arc in U_i and so at least one red non-bridge arc. So, there is at least one non-bridge arc that is a maximum.

We now suppose that at least one of a_k or a_l is an extremum. without loss of generality suppose that a_k is a minimum. We have a finite number of a_i, thus there is a maximum a_j. If $a_j = a_l$, then the proof is over. If $a_j = a_k$, then a_k is a maximum and a minimum and so all the a_i's are equals and the proof is done.

Now suppose a_j is not a non-bridge. Therefore a_j is at least a bridge over some arcs a_i and a_m. So, we can construct the blue and red graph of a_j, $G(D, a_j)$. By Lemma 2.3.3, there are at most two connected components F_j in U_i^c.

Suppose there is no connected components (F_j, a_j) in (U_i^c, a_j). Then, all the arcs are red and so all arcs are maxima and the proof is complete.

Now suppose, there is at least one connected component. By Lemma 2.3.6, there is at least one non-bridge arc in U_i and so at least one red non-bridge arc. So, there is at least one non-bridge arc that is a maximum. If a_k is also the maximum, then all the a_i are equals and the proof is over. If a_l is the maximum, then the proof is finished.

□

The following result illustrates the reason why we've constructed the Wada rational graph from the coarse Wada rational graph.

Lemma 2.3.8. *Let D be a 2-non-bridge non-split diagram with a 1-tangle T with only one non-bridge arc a_i in D and such that neither marked point goes over some arcs in $D \setminus T$. If $\pi(D)$ is left-orderable, then $a_i = a_k$ for every $a_k \in T$.*

Proof. The arc a_i is a non-bridge arc in D by hypothesis. Thus, by Lemma 2.3.7, without loss of generality, we can suppose that a_i is a maximum of D. There must be an a_j which is the minimum in T. We construct the red and blue graph of $G(D, a_j)$. The connected region $RG(D, a_j)$ must be included in T because neither marked point goes over some arcs in $D \setminus T$.

If there is no connected component F, then a_i is a maximum and a minimum and so $a_i = a_k$ for every arc in D.

Suppose there is at least one connected component F. By Lemmas 2.3.4 and 2.3.5, there is a least one non-bridge arc in U_i and so at least one red non-bridge arc in $U_i \cap RG(D, a_j)$. But, $RG(D, a_j)$ is included in T and the only non-bridge arc in T is a_i. Thus, a_i is a minimum in T and so $a_i = a_k$ for every $a_k \in T$.

□

2.3.2 Wada rational groups and the Graph group

Let $G(\Gamma_1(D), d)$ be a coarse Wada rational group. We recall, that if there is no 1-tangle T with only one non-bridge arc in D and such that neither marked point goes over some arcs in $D \backslash T$, then $G(\Gamma(D), d) = G(\Gamma_1(D), d)$ is defined as a *Wada rational group* and by Lemma 2.2.5, it is equal to the Graph group.

Suppose there is a 1-tangle T with only one non-bridge arc in D and such that neither marked point goes over some arcs in $D \backslash T$. Then a *Wada rational set of generators* $S(\Gamma(D), d)$ is defined as the set that contains every generator of $S(\Gamma_1(D), d)$ except for the rational tangle X_i included in T. Moreover, a *Wada rational group* $G(\Gamma(D), d)$ is defined as the subgroup of $\pi(D)$ generated by $S(\Gamma(D), d)$.

We can now prove Lemma 2.3.9.

Lemma 2.3.9. *Let D be a 2-non-bridge non-split link diagram. Suppose that $\pi(D)$ is left-orderable. Then, the Wada rational group $G(\Gamma(D), d)$ is equal to the Graph group $G(D)$.*

Proof. If there is no 1-tangle T with only one non-bridge arc a_i in D and such that neither marked point goes over some arcs in $D \backslash T$, then $G(\Gamma(D), d) = G(\Gamma_1(D), d)$ and by Lemma 2.2.5, $G(\Gamma(D), d)$ is equal to the Graph group.

Suppose there is a 1-tangle T with only one non-bridge arc a_i in D and such that neither marked point goes over some arcs in $D \setminus T$. Let $S(\Gamma(D), d)$ be the set of generators obtained from $S(\Gamma_1(D), d)$. By Lemma 2.3.8, $a_i = a_j$ for every a_i and a_j in T. Thus, for every a_i that goes over a_j in T, we have $a_i^{-1} a_j = 1$. Therefore, $X_i = 1$ for every rational tangle X_i in T. Thus, $G(\Gamma(D), d) = G(\Gamma_1(D), d)$. Hence, by Proposition 2.2.5, $G(\Gamma(D), d)$ is equal to the graph group $G(D)$. □

2.4 From the Wada rational Graph to the directed Wada graphs and directed Wada group

Let D be a non-split link diagram with m rational tangles. From the coarse Wada rational graph, we have 4^m groups that are equal to the Graph group. By the end of this chapter, we will find group such that all the generators are less than or equal to 1 when we suppose $\pi(D)$ left-orderable. To do so, we will construct directed Wada graphs.

2.4.1 From the Wada rational Graph to the directed Wada graphs

We now give a few lemmas linking left-orders and rational tangles. First, we need a result on twist tangles. Let X be an $\frac{n}{2}$-twist tangle, we recall that we obtain the following Wada graph.

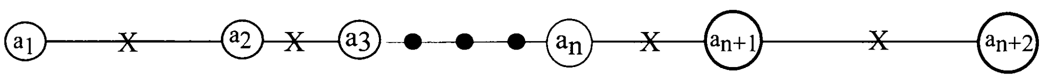


Figure 2.20 Wada graph of the $\frac{n}{2}$-twist

We note that the Wada relations give us $a_1^{-1}a_2 = a_2^{-1}a_3 = \ldots = a_i^{-1}a_{i+1} = \ldots = a_{n+1}^{-1}a_{n+2}$. Moreover, a_1 and a_{n+2} are the non-bridge vertices of X.

Lemma 2.4.1. *Let X be an $\frac{n}{2}$-twist tangle as in Figure 2.20 in a link diagram D. Suppose there is a left-order $<$ on $\pi(D)$, then either $a_i \leq a_j$ for $1 \leq i \leq j \leq n+2$ or $a_j \leq a_i$ for $1 \leq i \leq j \leq n+2$.*

Proof. Because we have a left-order on $\pi(D)$, either $a_1^{-1}a_2 \leq 1$ or $a_1^{-1}a_2 \geq 1$.

If $a_1^{-1}a_2 \leq 1$, then by the Wada equation $a_i^{-1}a_{i+1} \leq 1$ for $1 \leq i < n+2$. Thus, again because of the left-order, $a_{i+1} \leq a_i$ for $1 \leq i < n+2$.

If $a_1^{-1}a_2 \geq 1$, then by the Wada equation $a_i^{-1}a_{i+1} \geq 1$ for $1 \leq i < n+2$. Thus, again because of the left-order, $a_{i+1} \geq a_i$ for $1 \leq i < n+2$. $\square$

Let X be a rational tangle with X_1, X_2,..., X_m as maximal twists tangles as in the following figure.

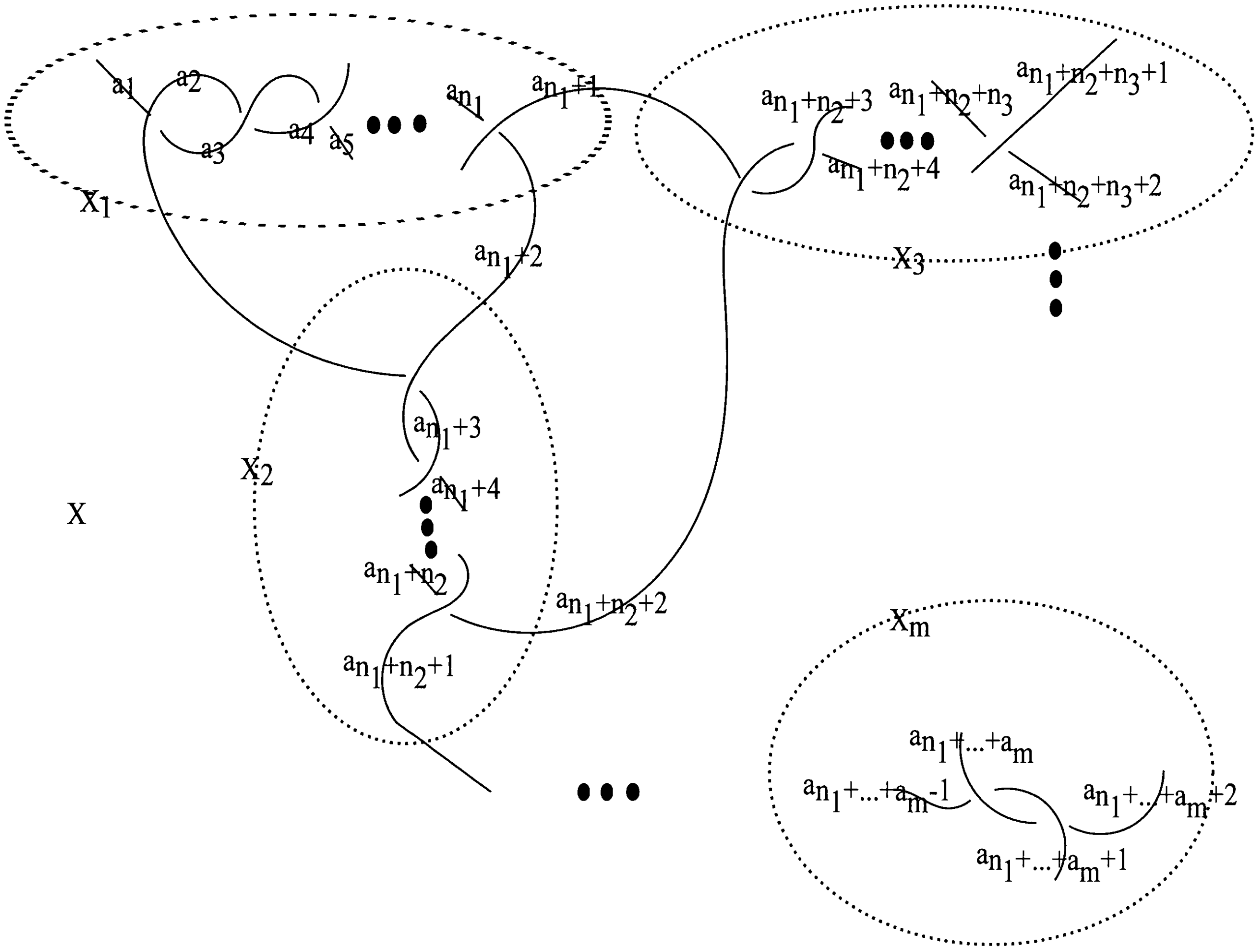


Figure 2.21 Rational tangle diagram

Lemma 2.4.2. *Let $X = [n_m, ..., n_1]$ be a rational tangle as in Figure 2.21 in a link diagran D. Suppose there is a left-order $<$ on $\pi(D)$, then either $a_i \leq a_j$ for $1 \leq i \leq j \leq (\sum_{k=1}^{m} n_k) + 2$ or $a_j \leq a_i$ for $1 \leq i \leq j \leq \sum_{k=1}^{m} n_k$.*

Proof. Let X_1 be the leading maximal twist tangle of X. Then, by Lemma 2.4.1, either $a_i \le a_j$ for $1 \le i \le j \le n_1 + 2$ or $a_j \le a_i$ for $1 \le i \le j \le n_1 + 2$. Without loss of generality, suppose that $a_i \le a_j$ for $1 \le i \le j \le n_1 + 2$. Thus, in the second maximal twist tangle X_2 of X, $a_2 \le a_{n+2}$. Therefore, $a_{n+2}^{-1} a_2 \le 1$ and by a similar argument as in Lemma 2.4.1, $a_i \le a_j$ for $1 \le i \le j \le n_1 + n_2 + 2$.

Suppose that the result is true for the first k maximal twist tangles X_k of X. Then, we have $a_{n_1+\ldots+n_{k-2}+1} \le a_{n_1+\ldots+n_{k-2}+n_{k-1}+2}$. So,

$$(a_{n_1+\ldots+n_{k-2}+n_{k-1}+2})^{-1} a_{n_1+\ldots+n_{k-2}+1} \le 1$$

and by Lemma 2.4.1, $a_i \le a_j$ for $1 \le i \le j \le (\sum_{k=1}^{m} n_k) + 2$. □

Suppose $\pi(D)$ is left orderable and $<$ is the left-order on $\pi(D)$. Let X_i be an undirected rational tangle $\frac{p}{q}$ with open vertices a_i, a_j, a_k, a_l as in figure 2.22.

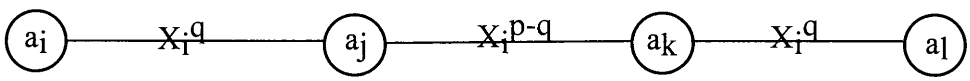


Figure 2.22 Undirected rational tangle X_i

By Lemma 2.4.2, we have the following result.

Lemma 2.4.3. *Let D be a 2-non-bridge diagram and X_i be an undirected rational tangle $\frac{p}{q}$ as in Figure 2.22. Suppose $\pi(D)$ is left orderable and $<$ is the left-order on $\pi(D)$. Then, for the order between the vertices, we either have $a_i \le a_j \le a_k \le a_l$ or $a_i \ge a_j \ge a_k \ge a_l$.*

Suppose that the left order gives us $a_i \le a_j \le a_k \le a_l$. Then, we orient the edges as follows.

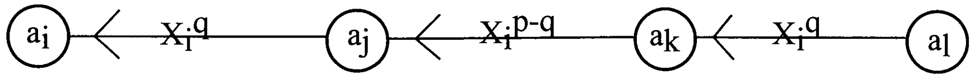


Figure 2.23 Directed rational tangle X_i

While for the left order $a_i \geq a_j \geq a_k \geq a_l$, we orient the edges as follows.

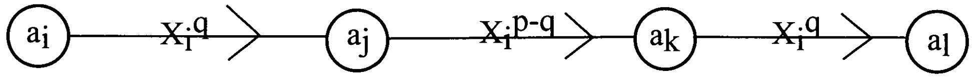


Figure 2.24 Directed rational tangle X_i

We orient every rational tangle this way following the left-order $<$ and we obtain a *directed Wada graph* $(\Gamma(D), <)$. Thus, for every left-order $<$ we obtain a directed Wada graph $(\Gamma(D), <)$.

2.4.2 From a directed Wada graph to the directed Wada group

We will now show how from $(\Gamma(D), <)$, a directed Wada graph from $<$, we can obtain a group equal to the Graph group for which each generator is less than or equal to 1 with respect to $<$. Let D be a link diagram with m rational tangles, $\Gamma(D)$ the Wada rational graph and $(\Gamma(D), <)$ a directed Wada graph. Recall that there are 4^{m-i} Wada rational groups $G(\Gamma, d)$ where i is the number of rational tangle in 1-tangle T with only one non-bridge arc in D and such that neither marked point goes over some arcs in $D \setminus T$. A directed Wada group $G(\Gamma(D), <)$ will be one of the Wada rational group $G(\Gamma, d)$ such that all the generators are less than or equal to 1.

We will first construct the *directed Wada set of generators* $S(\Gamma(D), <)$. If X_i is in a 1-tangle T with only one non-bridge arc in D and such that neither marked point goes over some arcs in $D \setminus T$, then X_i is not included in $S(\Gamma(D), <)$. Let

X_i be a rational tangle that is not in a 1-tangle T with only one non-bridge arc in D and such that neither marked point goes over some arcs in $D \setminus T$. Then, we recall that we have four possibilities for the generator X_i in a Wada rational set of generators. From the order $<$, we will choose only one. Let X_i' be the leading maximal twist tangle of X_i and a_k and a_l be the non-bridge vertices of X_i'. Then, in a Wada rational set of generators, either $X_i = a_k^{-1}a_s$, $X_i = a_s^{-1}a_k$, $X_i = a_l^{-1}a_t$ or $X_i = a_t^{-1}a_l$ for the arcs a_s and a_t that goes over the arcs a_k and a_l respectively in X_i. Because X_i' is a leading maximal twist tangle of X_i, at least one of a_k and a_l is also a non-bridge of the rational tangle X_i.

Suppose there is only one of a_k or a_l that is a non-bridge of the rational tangle X_i. Without loss of generality, suppose that a_k is a non-bridge of X_i. Then, $a_k \leq a_{k_1} \leq a_{k_2} \leq a_{k_3}$ or $a_k \geq a_{k_1} \geq a_{k_2} \geq a_{k_3}$ where $a_{k_1}, a_{k_2}, a_{k_3}$ are the other open vertices of X_i. If $a_k \leq a_{k_1} \leq a_{k_2} \leq a_{k_3}$, then, by Lemma 2.4.2, $a_k \leq a_j$ for every arc a_j in the rational tangle X_i. So, in particular, $a_k \leq a_s$. Hence, $a_s^{-1}a_k \leq 1$. Thus, we define $X_i = a_s^{-1}a_k \leq 1$. If $a_k \geq a_{k_1} \geq a_{k_2} \geq a_{k_3}$, then we define $X_i = a_k^{-1}a_s \leq 1$.

If both a_k and a_l are non-bridge of X_i, then look at the greater one with respect to $<$. For example, if $a_k \geq a_l$, then $a_k \geq a_s$ and we define $X_i = a_k^{-1}a_s \leq 1$.

Hence, $S(\Gamma, <) = S(\Gamma, d)$ for one of the possible choice d. We define the *directed Wada group* $G(\Gamma, <)$ as the subgroup of $\pi(D)$ generated by $S(\Gamma, <)$. Hence, $G(\Gamma, <) = G(\Gamma, d)$. So, by Lemma 2.3.9, $G(\Gamma(D), <)$ is equal to the group $G(D)$. Moreover, by the previous construction, the generators of $G(\Gamma(D), <)$ are less than or equal to 1 with respect to $<$. Therefore, we have the following result.

Lemma 2.4.4. *Let D be a link diagram and $\Gamma(D)$ be the Wada rational graph. Suppose that $<$ is a left order on the Wada group $\pi(D)$. Then, the directed Wada group $G(\Gamma(D), <)$ is a group equal to the Graph group $G(D)$ such that* *each generator is less than or equal to 1 with respect to* $<$.

Two different left-orders can give us the same directed Wada graph.

Definition 2.4.5. Let D be a link diagram and $\Gamma(D)$ be the Wada rational graph. Suppose that $<_A$ and $<_B$ are two left-orders on the Wada group $\pi(D)$. If $(\Gamma(D), <_A) = (\Gamma(D), <_B)$, then we say that the left-order $<_A$ and $<_B$ are *Wada equivalent.*

Two different left-orders that give the same directed Wada graph, will give the same directed Wada group, because the directed Wada group is defined from the directed Wada graph.

Lemma 2.4.6. *Let D be a link diagram and $\Gamma(D)$ be the Wada rational graph. Suppose that $<_A$ and $<_B$ are two left-orders on the Wada group $\pi(D)$. If $<_A$ and $<_B$ are Wada equivalent, then $G(\Gamma(D), <_A) = G(\Gamma(D), <_B)$.*

CHAPTER III

NARROWING THE NUMBER OF POSSIBILITIES FOR THE DIRECTED WADA GRAPHS

In this chapter, we will narrow down the number of directed Wada graphs and thus the number of possible left-orders on $\pi(D)$ that are not Wada equivalent. To do so, we will define the semi-directed Wada rational graph.

3.0.1 From the Wada rational graph to the semi-directed Wada rational graph

The semi-directed Wada rational graph will only be defined for the 2-non-bridge diagrams D. We recall that by Lemma 2.3.7, in a 2-non-bridge diagram D, if $\pi(D)$ is left-orderable, then the two non-bridge arcs a_k and a_l are extrema among the generators of $\pi(D)$. Suppose that the a_i are not all equal. Without loss of generality we suppose that a_k is a minimum and a_l is a maximum in the left-order $<$. The non-bridge arc a_l is an open vertex of two different rational tangles X_p and X_q where X_p is a $\frac{a}{b}$ rational tangle and X_q is a $\frac{c}{d}$ rational tangle. Thus, we have a Wada rational graph of the non-bridge arc a_l as in the following figure.

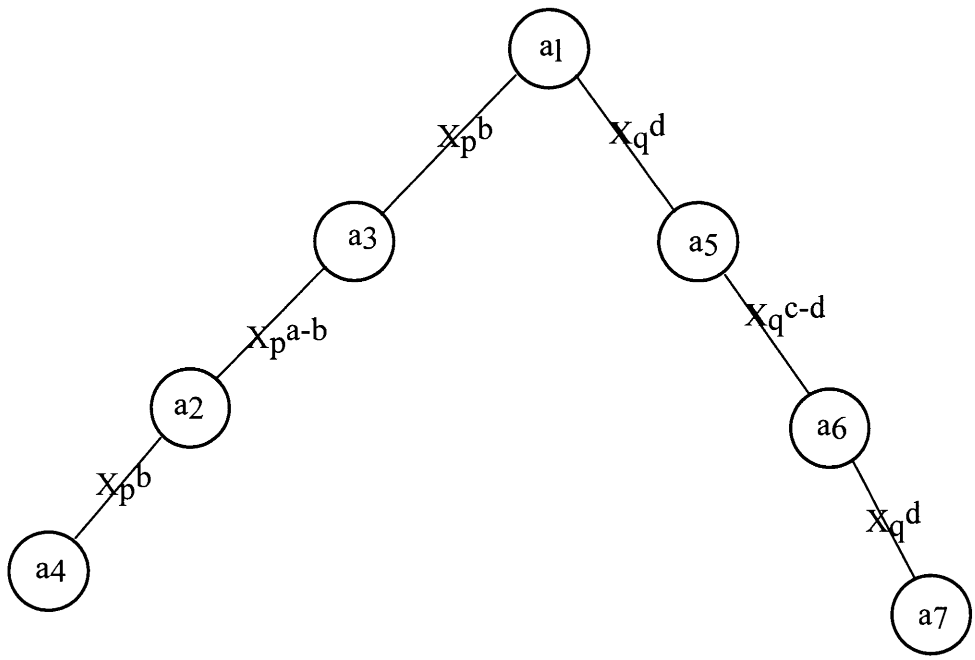


Figure 3.1 Wada rational graph of the non-bridge arc a_l

Because a_l is a maximum, from Lemma 2.4.3 and the way we orient edges, we have the following directed Wada rational graph of the non-bridge arc a_l.

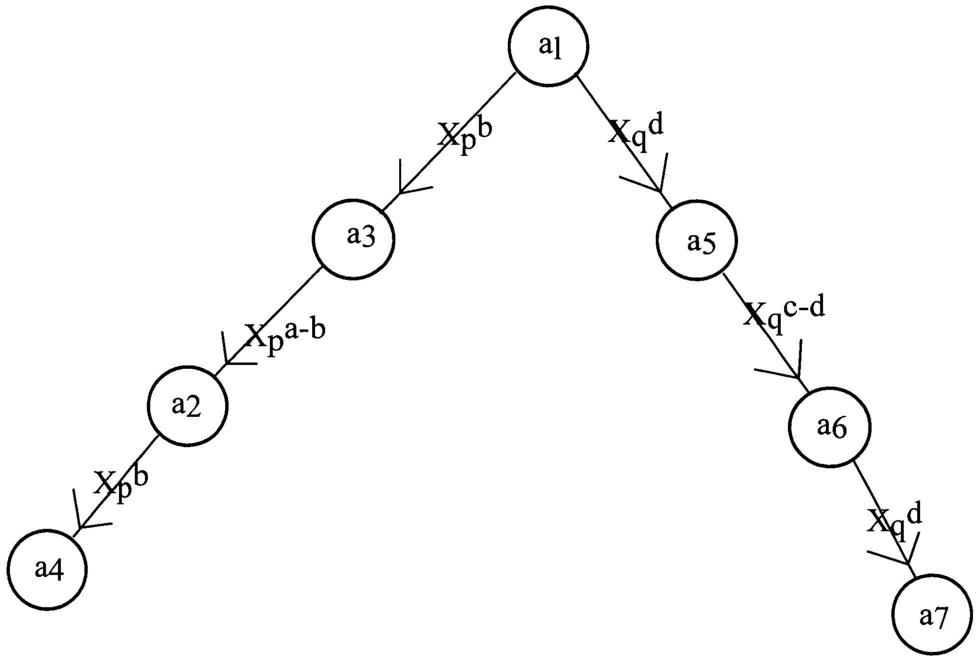


Figure 3.2 Directed Wada rational graph of the non-bridge arc a_l

Similarly for the minimum a_k, an open vertex of the two rational tangles X_r and X_s where X_r is a $\frac{e}{f}$ rational tangle and X_s is a $\frac{g}{h}$ rational tangle. We obtain the following directed Wada graph.

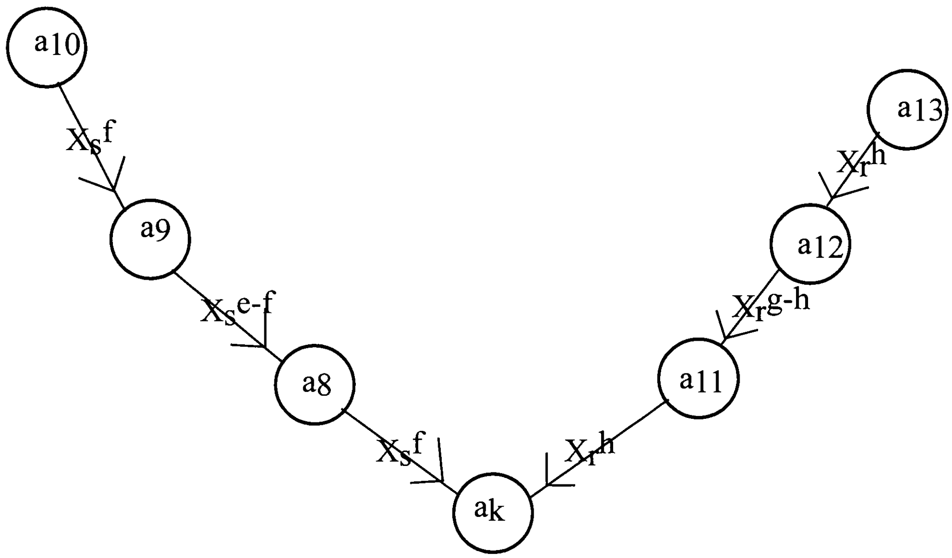


Figure 3.3 Directed Wada rational graph of the non-bridge arc a_k

When only the rational tangles of the non-bridge arcs are oriented in the Wada rational graph, we call this graph a *semi-directed Wada rational graph.* Hence, for a 2-non-bridge link diagram, there are exactly two semi-directed Wada rational graphs. Moreover, in 2-non-bridge link diagrams, left-orders that have the same maximum among the generators of $\pi(D)$ give the same semi-directed Wada rational graph. This motivates the following definition.

Definition 3.0.1. Let D be a non-split 2-non-bridge link diagram. Suppose that $<_1$ and $<_2$ are two left-orders on $\pi(D)$. If $<_1$ and $<_2$ have the same maximum among the generators of $\pi(D)$, then $<_1$ and $<_2$ are *Wada semi equivalent.*

We continue the example of the knot 8_{21}.

Firstly, the diagram of 8_{21} is a two non-bridge. We suppose that the non-bridge a_4 is the maximum and the non-bridge a_2 is the minimum. Thus, from the Wada rational graph we obtain the following semi-directed Wada graph of the knot 8_{21}.

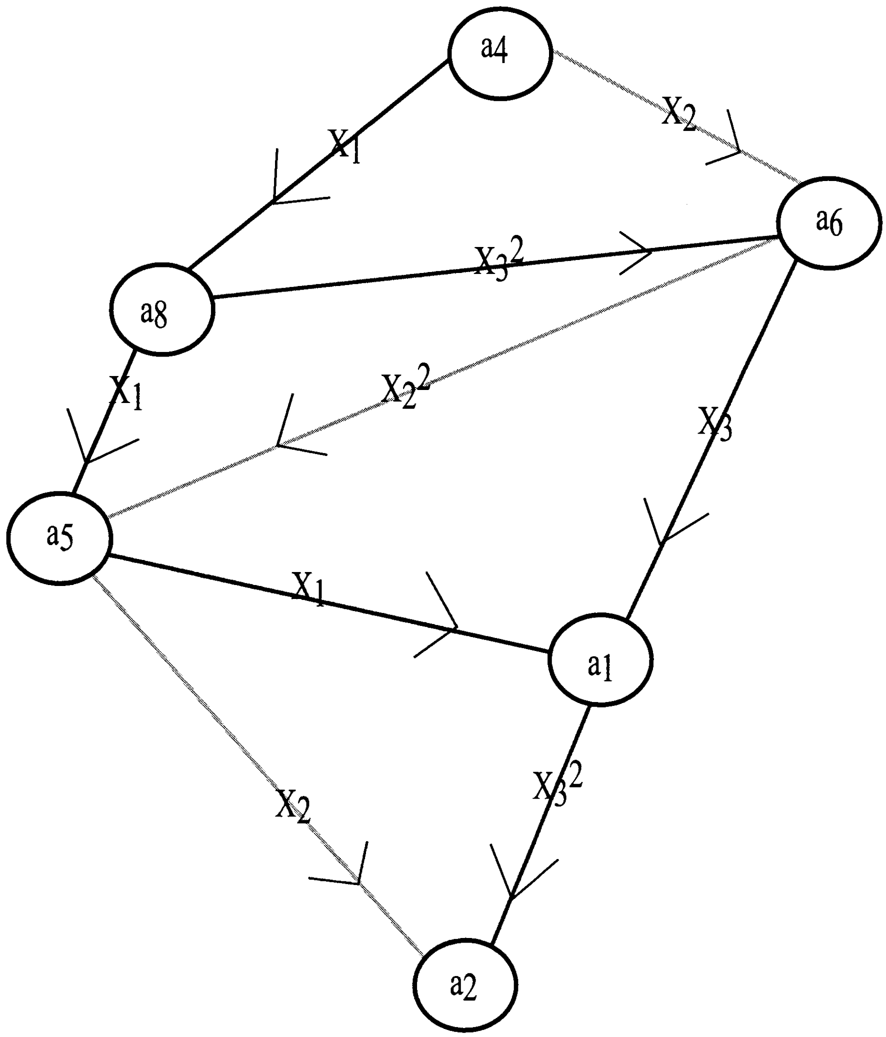


Figure 3.4 Semi-directed Wada rational graph of the knot 8_{21}

For this knot, the Wada semi-directed graph gives the direction for each edge. Thus, a directed Wada graph. Therefore, every left-order with a_4 as maximum on the Wada group of this link diagram of 8_{21} gives us this directed Wada graph. Hence, every left-order with a_4 as maximum are Wada equivalent and give a group equal to the Graph group. Thus, every Wada semi equivalent left-orders are Wada equivalent left-orders. Moreover, by Lemma 2.3.7, all the other left-orders have

a_2 as maximum and a_4 as minimum. However, all these orders can be obtained by reversing the order with a_4 as maximum.

We now introduce the following knot diagram of the knot 9_{49}. Note that this knot is not an arborescent knot (Caudron, 1987).

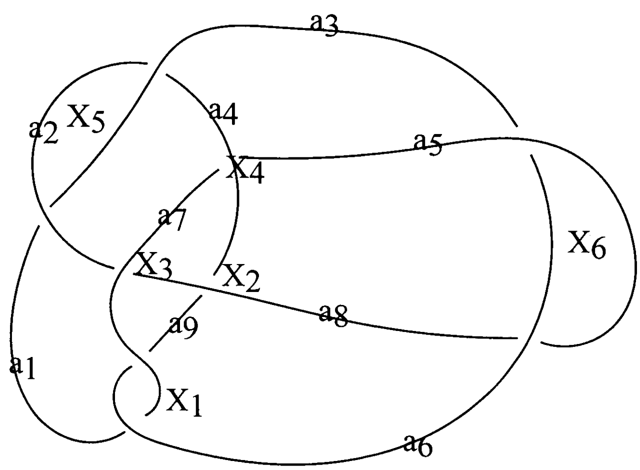


Figure 3.5 Knot diagram of the knot 9_{49}.

We give a Wada semi-directed graph of the knot 9_{49} previous knot diagram.

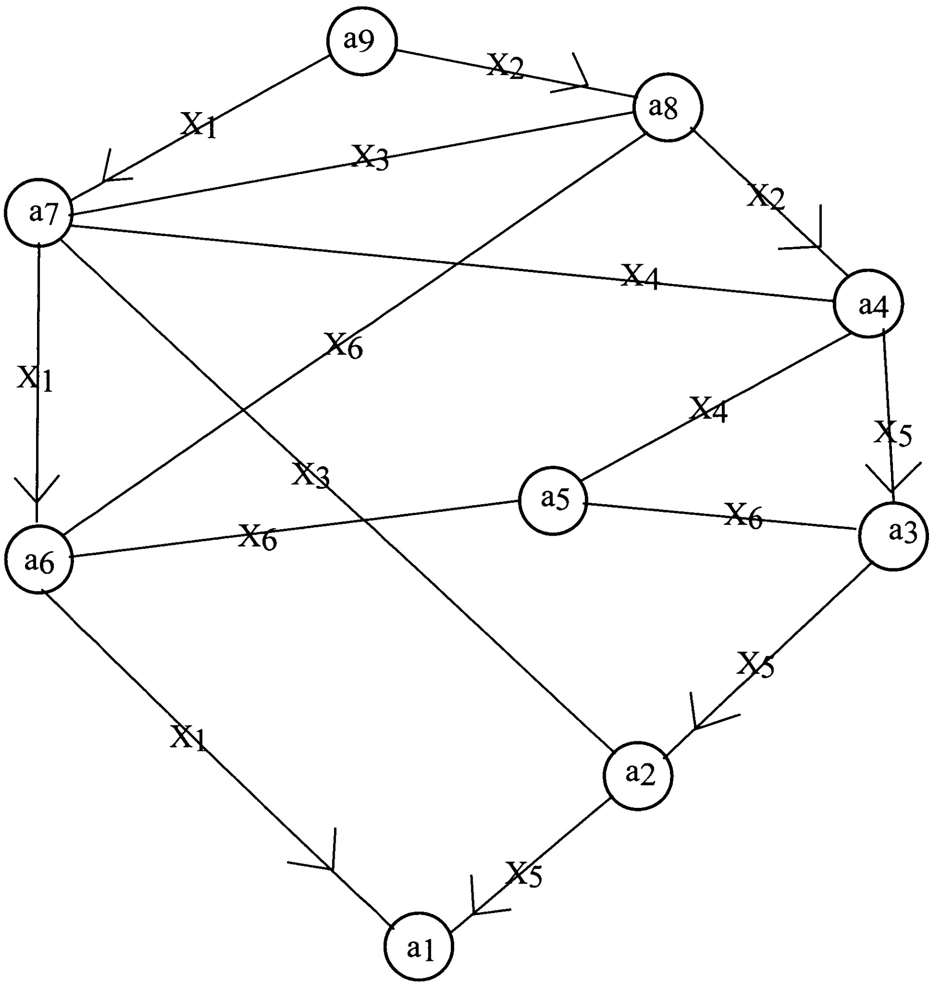


Figure 3.6 A Wada semi-directed graph of the knot 9_{49}.

For this Wada rational graph, there are some choices of orientations to make for some edges to obtain a directed graph.

3.0.2 From the Wada semi-directed graph to directed Wada graphs

From a 2-non-bridge link diagram D, we have shown how to obtain the two Wada semi-directed graphs Γ'. We now recall some definitions in graph theory. Let F

be a directed graph. We label each of the m edges from E_1 to E_m.

Definition 3.0.2. In a graph, two edges are *connected* if they share a common vertex. Two vertices are *consecutive* if there is an edge joining them. A *cycle* is a sequence of connected edges whose first and last vertices are the same and with no repeated edges or vertices (except the first and last vertices). We will either refer to a cycle by listing its edges or by listing its vertices.

Definition 3.0.3. In a directed graph, two edges are *consecutive* if the starting vertex of one is the end vertex of the other. A *directed path* in a directed graph is a sequence of consecutive directed edges. We will refer a directed path by either its edges or by its vertices.

Definition 3.0.4. A *directed cycle* is a directed path whose first and last vertices are the same and with no repeated edges or vertices (except the first and last vertices).

The next lemma shows the impact on the generators of the Graph group, of a directed cycle in a directed Wada graph.

Lemma 3.0.5. *Let D be a non-split link diagram. Suppose there is a left-order $<$ on $\pi(D)$ such that there is a directed cycle $X_1^{k_1}X_2^{k_2}...X_j^{k_j}$ in a directed Wada graph. Then, $X_1 = X_2 = ... = X_j = 1$ in $G(\Gamma, <)$.*

Proof. Without loss of generality, let $X_i^{k_i} = a_{i_1}^{-1}a_{i_2}$ and $a_{i_2} = a_{(i+1)_1}$ for $1 \leq i \leq j-1$. Moreover, because it is a directed cycle, $X_j^{k_j} = a_{j_1}^{-1}a_{i_1}$. By definition, $X_i^{k_i} = a_{i_1}^{-1}a_{i_2} \leq 1$ and $X_j^{k_j} = a_{i_j}^{-1}a_{i_1} \leq 1$ for $1 \leq i \leq j-1$. Therefore, $a_{i_1} \leq a_{i_j} \leq a_{i_{j-1}} \leq ... \leq a_{i_2} \leq a_{i_1}$. Thus, $a_{i_1} = a_{i_2} = ... = a_{i_j}$. This implies that $X_i^{k_i} = a_{i_1}^{-1}a_{i_2} = 1$ and $X_j^{k_j} = a_{i_j}^{-1}a_{i_1} = 1$ for $1 \leq i \leq j-1$. Hence, $X_1^{k_1} = X_2^{k_2} = ... = X_j^{k_j} = 1$ and so $X_1 = X_2 = ... = X_j = 1$. □

We now introduce an important family of left-orders on $\pi(D)$.

Definition 3.0.6. Let D be a non-split link diagram. Suppose that $<$ is a left order on the Wada group $\pi(D)$. If $(\Gamma, <)$, the directed Wada graph with respect to $<$, has no directed cycle, then $<$ is a *Wada maximal left-order on* $\pi(D)$ and $(\Gamma, <)$ is a *maximal Wada directed graph.*

Moreover, if there is a maximal left-order on $\pi(D)$, then we say that D is a *Wada maximal link diagram.*

If a link L has a maximal link diagram, then we say that L is a *Wada maximal link.*

Note that no alternating link is Wada maximal, while every non-alternating link of 11 crossings or less is Wada maximal. Moreover, every non-alternating link we have studied is Wada maximal.

Definition 3.0.7. Let D be a non-split two-non-bridge maximal link diagram. Suppose that $<_1$ is a left-order on $\pi(D)$ such that $(\Gamma, <_1)$ has a directed cycle $X_1^{k_1} X_2^{k_2} ... X_j^{k_j}$. Let $<$ be a Wada semi equivalent maximal order such that $(\Gamma, <_1)$ can be obtained from $(\Gamma, <)$ by reversing the order of some edges included in the directed cycle $X_1^{k_1} X_2^{k_2} ... X_j^{k_j}$. By Lemma 3.0.5, in $G(\Gamma, <_1)$, $X_1 = X_2 = ... = X_j = 1$. Thus, $G(\Gamma, <_1)$ is a subgroup of $G(\Gamma, <)$ with $X_1 = X_2 = ... = X_j = 1$. We define $<_1$ has a *Wada suborder of* $<$. Also, we define $(\Gamma, <_1)$ as a *Wada subgraph* of $(\Gamma, <)$.

If we prove that $G(\Gamma, <)$ is trivial, then $G(\Gamma, <_1)$ is trivial for every suborder $<_1$ of $<$.

We now return to the semi-directed graphs of 9_{49}.

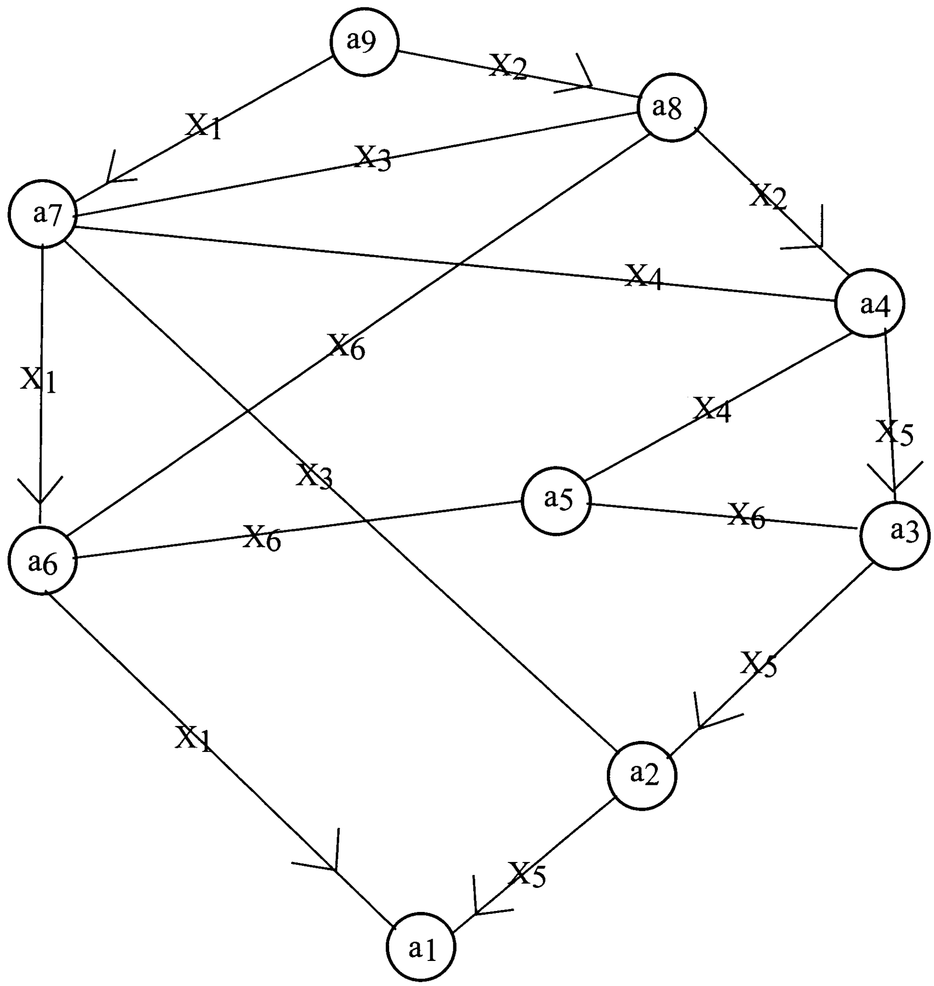


Figure 3.7 A Wada semi-directed graph of the knot 9_{49}.

Suppose that $<$ is a left-order of $\pi(D)$ with a_9 as maximum of the generators of $\pi(D)$ and such that in the directed Wada graph the edge X_3 goes from a_2 to a_7. Then, we also have the direction from a_7 to a_8. Thus, we have the directed cycle $X_3^2X_2X_5^2$ and so $X_3 = X_2 = X_5 = 1$. Moreover, $X_1X_3X_2^{-1}$ is a cycle. But $X_3 = X_1 = 1$, therefore X_1 is a directed cycle and so $X_1 = 1$. Also, $X_3 = X_2 = 1$ and so $X_4 = 1$, by the cycle passing by a_7, a_8 and a_4. Finally, $X_4 = X_5 = 1$ and so $X_6 = 1$, by the cycle passing by a_3, a_5 and a_4. Thus, $X_1 = X_2 = X_3 = X_4 =$

$X_5 = X_6 = 1$ and this implies that $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9$. Thus, for this left order of $\pi(D)$, $\pi(D)$ is abelian.

However, there is a Wada maximal left-order for this link diagram. Suppose that $<$ is a left-order of $\pi(D)$ with a_9 as maximum of the generators of $\pi(D)$ and such that in the directed Wada graph, the edge X_3 goes from a_7 to a_2 and so from a_8 to a_7. Thus, because a_8 goes to a_7 and a_7 goes to a_6, then a_8 goes to a_6 and we have a direction for the edge X_6. This implies that a_6 goes to a_5 and a_5 to a_3. Moreover, a_7 goes to a_6 that goes to a_5. Therefore, a_7 goes to a_4 and a_4 to a_5. So, the only Wada directed graph of 9_{49} with no directed cycle up to reversing every edges is the following graph.

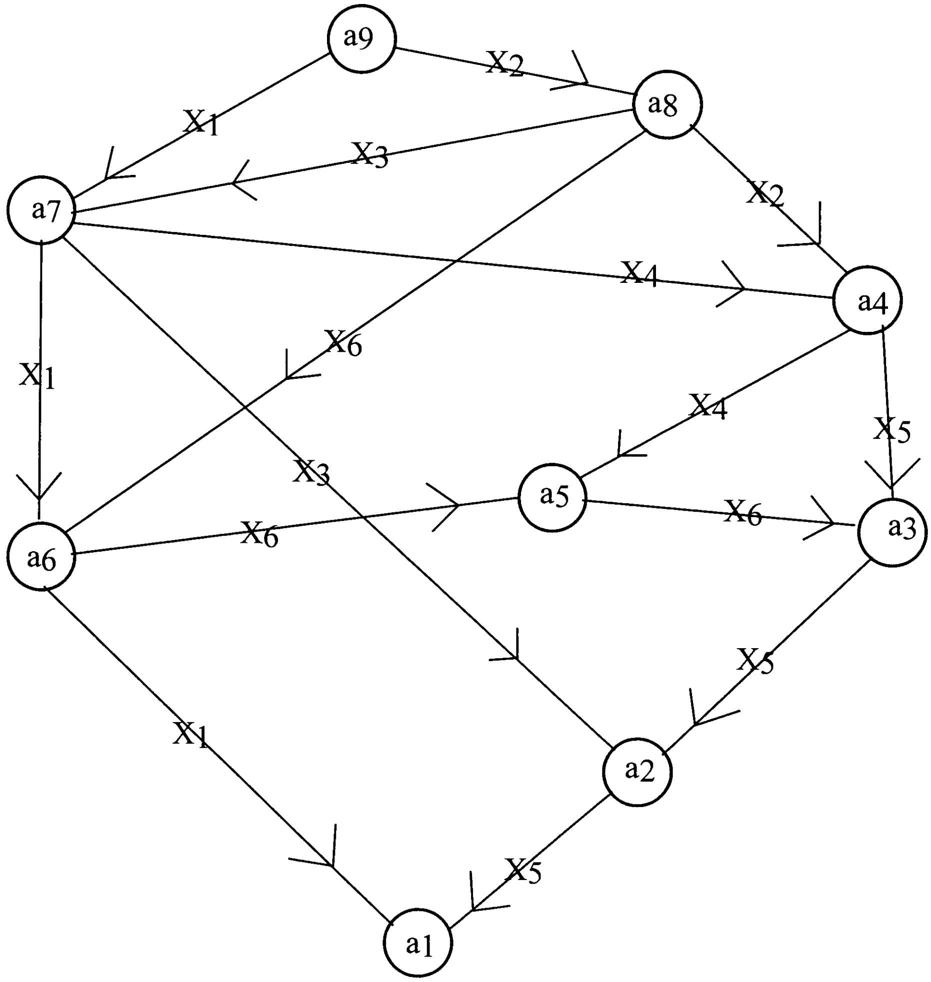


Figure 3.8 The only Wada directed graph of 9_{49} with a_9 as maximum

3.0.3 Directed link diagrams

In this thesis, we will suppose that there is a left-order on $\pi(D)$ and find the maximal Wada directed graphs. The information from the maximal Wada directed graphs on the left-order will often be enough to prove the desired results.

We now introduce an important family of link diagrams.

Definition 3.0.8. Let D be a Wada maximal 2-non-bridge link diagram. If every semi-equivalent Wada maximal left-orders are Wada equivalent, then D is *directed.*

In other words, let $<$ be a Wada maximal left-order with a_i as maximum between the generator of $\pi(D)$. If for every Wada maximal left-order $<_j$ with a_i as maximum between the generator of $\pi(D)$, we have $(\Gamma, <) = (\Gamma, <_j)$, then D is *directed.*

We will call a Wada maximal 2-non-bridge link diagram that is directed, a *directed link diagram.*

Remark 3.0.9. Thus, a directed link diagram has only one directed Wada graph without directed cycles up to reversing the direction of every edges.

Note, that the choice of the maximum between the non-bridge arcs of the link diagram in the definition does not matter. We can take the reverse order and obtain the same result with the other arc as maximum.

Definition 3.0.10. Let L be a link. If L has a directed link diagram, then L is *directed.*

Thus, the knot 8_{21} and 9_{49} are directed knot.

Remark 3.0.11. It is interesting to note that on the 220 non-alternating 2-non-bridge knots of 11 or less crossings, 158 are directed.

CHAPTER IV

DIRECTED WADA GRAPHS AND NON-LEFT-ORDERABILITY OF THE FUNDAMENTAL GROUP OF THE DOUBLE BRANCHED COVERS OF SOME LINKS

At the beginning of chapter 2, in Theorem 2.0.3, we have proved that if we suppose that the Wada group is left orderable and if we find that the Graph group is trivial, then the fundamental group of the double branched cover is not left-orderable. However, in most cases it is difficult to show that the Graph group is trivial. In this short section, we will prove that it is enough to show that for every directed Wada graph of a link diagram, the directed Wada group is trivial. In the following chapters, we will find links that satisfy this condition.

To get this result, we need the following lemma.

Lemma 4.0.1. *Let D be a maximal 2-non-bridge diagram of a non-split link and suppose that $\pi(D)$ is left-orderable. If for every maximal directed Wada graph $(\Gamma, <)$, the directed Wada group $G(\Gamma, <)$ is trivial, then $\pi(D)$ is abelian.*

Proof. Let $<_1$ be a left-order on $\pi(D)$ that is not maximal. Therefore, $(\Gamma, <_1)$ is the subgroup of $(\Gamma, <)$ for $<$ a maximal left-order. By hypothesis, $(\Gamma, <)$ is trivial, thus $(\Gamma, <_1)$ is trivial. So, for every left-order $<$, $(\Gamma, <)$ is trivial.

Let $<$ be a left-order on $\pi(D)$. Then $G(\Gamma, <)$ is trivial. So, by Lemma 2.4.4, the

Graph group $G(D)$ is trivial with respect to the left-order $<$. Thus, by Lemma 2.0.2, $\pi(D)$ is abelian. $\square$

Therefore, by Proposition 1.0.6 and the previous lemma, we have

Theorem 4.0.2. *Let D be a maximal 2-non-bridge diagram of a non-split link L and suppose that $\pi(D)$ is left-orderable. If for every directed Wada graph $(\Gamma, <)$, the directed Wada group $G(\Gamma, <)$ is trivial, then $\pi_1(\Sigma(L))$ is not left orderable.*

For directed link diagram, there is only one directed Wada graph up to reversing every edges. Moreover, directed link diagram are maximal link diagram, thus we have the following result.

Corollary 4.0.3. *Let D be a 2-non-bridge directed diagram of a non-split directed link L and suppose that $\pi(D)$ is left-orderable. If a directed Wada group $G(\Gamma, <)$ is trivial, then $\pi_1(\Sigma(L))$ is not left orderable.*

CHAPTER V

HYBRID WADA DIAGRAMS AND RELABELING OF THE VERTICES IN A DIRECTED WADA GRAPH

In this chapter, we will combine the Wada rational graph and the link diagram to obtain the hybrid Wada diagram. Moreover, we will relabel the vertices of the directed Wada graph. Both constructions will help us for the proofs in the following chapters.

5.1 The Hybrid Wada diagram

The hybrid Wada diagram $H(\Gamma)$ is obtained by combining the Wada rational graph Γ of a link diagram with the link diagram. Let D be a connected diagram of a link and Γ a Wada rational graph. To construct the hybrid Wada diagram $H(\Gamma)$, we replace each rational tangle in the link diagram by the Wada rational graph of the tangle. For example for the knot 8_{21} diagram

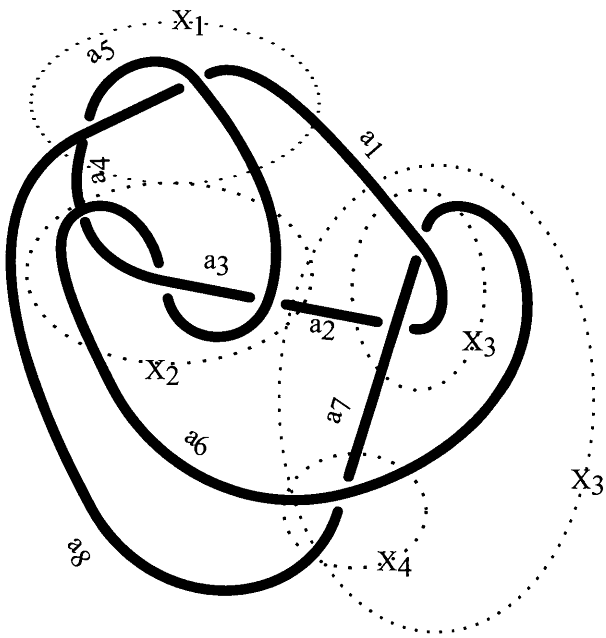
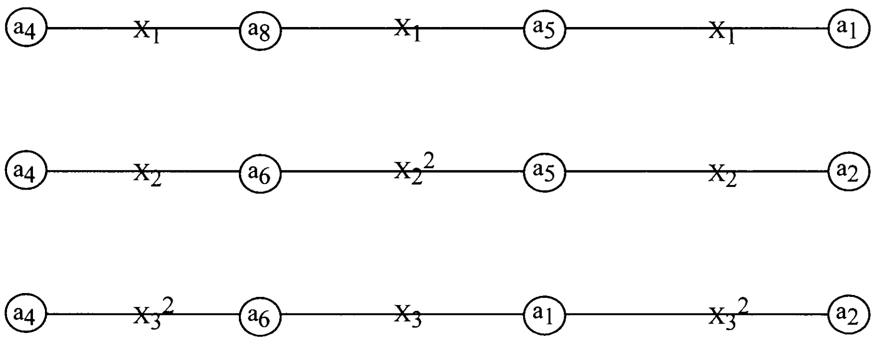


Figure 5.1 Knot diagram of 8_{21}

and the Wada rational graph of the rational tangles



we obtain the following hybrid Wada diagram where the white dots represent non-bridge arcs in the rational tangle and the black dots represent 1-bridge arcs in the rational tangle.

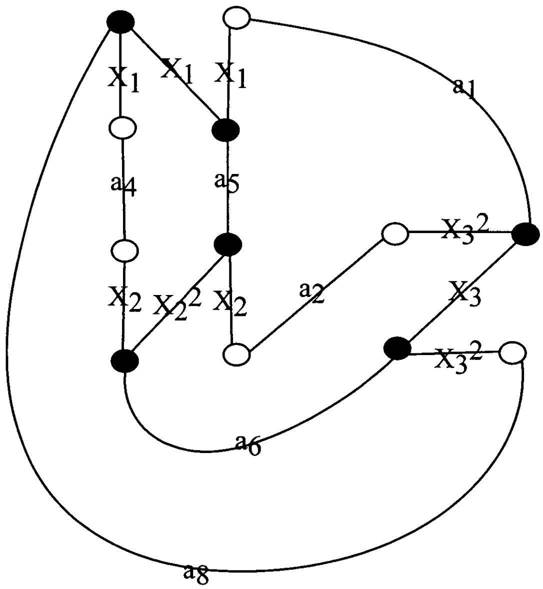


Figure 5.2 Hybrid Wada diagram of 8_{21}

Moreover, from a directed Wada graph $(\Gamma, <)$, we can obtain a directed Hybrid Wada diagram $H(\Gamma, <)$. For the direction of the rational tangles in $H(\Gamma, <)$, we choose the direction of each rational tangle in the directed Wada graph $(\Gamma, <)$. We continue the example of the knot diagram 8_{21}. Thus from the directed Wada graph $(\Gamma, <)$

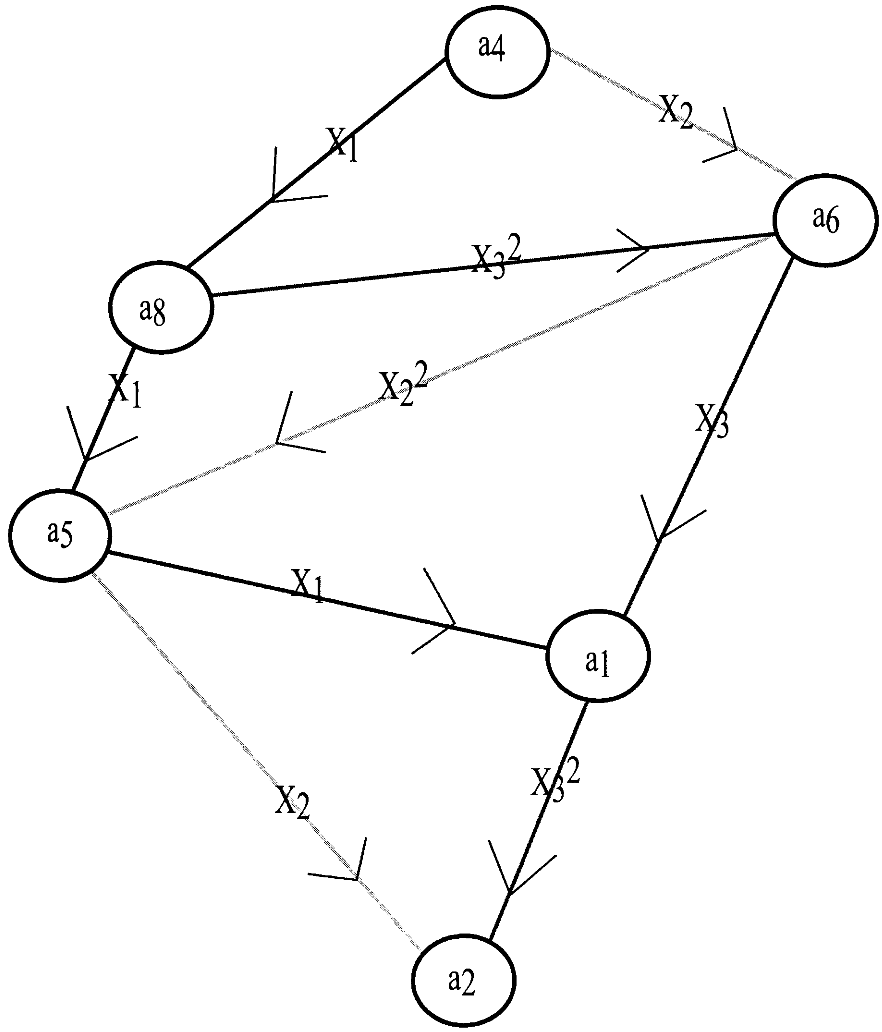


Figure 5.3 directed Wada graph of the knot 8_{21}

we obtain the following directed Hybrid Wada diagram

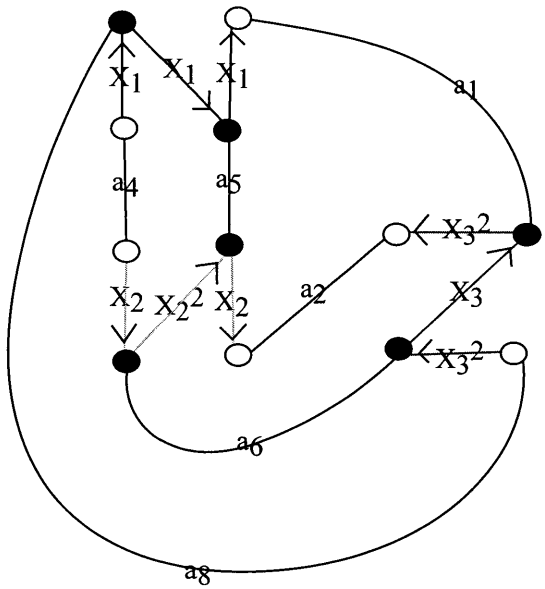


Figure 5.4 Directed Hybrid Wada diagram of 8_{21}

Also, inspired by Lemma 2.3.8, we replace every 1-tangle T with only one non-bridge arc in D such that neither marked point goes over some arcs in $D \setminus T$ by a single non-bridge arc as in the following diagram. As shown in Lemma 2.3.8, all the arcs in T are equal, therefore for the new non-bridge arc we choose an arbitrary arc in T.

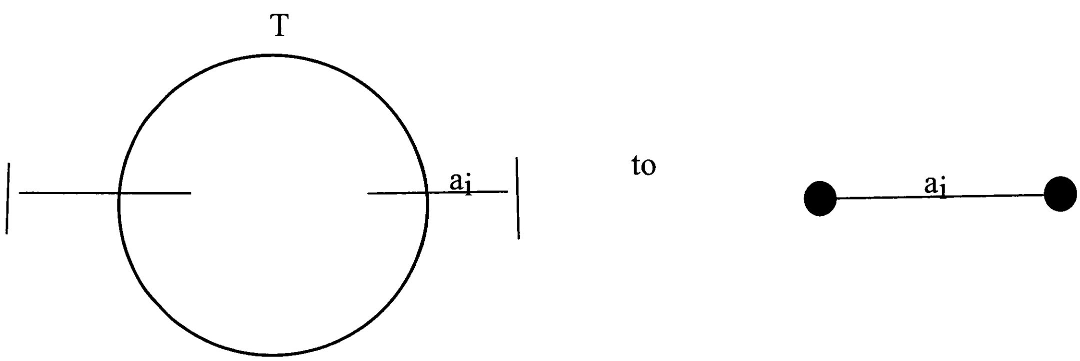


Figure 5.5 From a 1-tangle T with only one non-bridge arc in D such that neither marked point goes over some arc in $D \setminus T$ to a single non-bridge arc

5.2 Relabeling of the vertices in a directed Wada graph

In this short section, we will relabel the arcs of the link diagram to facilitate the writing of proofs in the following chapters.

First, we need to introduce the "simplest" directed path in a directed Wada graph. Recall, that two vertices are *consecutive* if there is an edge joining them.

Definition 5.2.1. Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D. If $a_i \leq a_j$ are consecutive and there is no a_k in $(\Gamma, <)$ such that there is a directed path from a_j to a_i passing by a_k, then a_i and a_j are *Wada consecutive.* Moreover, the edge joining a_i and a_j is called a *flat edge.* Let $P = \{a_j, ..., a_i\}$ be a directed path from a_j to a_i. If every consecutive vertices in P are Wada consecutive, then P is a *directed Wada path.* Equivalently, if every edge in a directed path P is a flat edge, then P is a directed Wada path.

Let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge link diagram D with n vertices. There is a partial order $<_{(\Gamma,<)}$ on the vertices induced by the directed Wada graph. A vertex a_i is greater than a_j with respect to $<_{(\Gamma,<)}$, if there is a directed Wada path from a_i to a_j. If $<_1$ is Wada equivalent to $<$, then $(\Gamma, <_1) = (\Gamma, <)$. Thus, if $a_i <_{(\Gamma,<)} a_j$, then $a_i <_1 a_j$ for every left-order $<_1$ Wada equivalent to $<$. If there is no order between a_i and a_j with respect to $<_{(\Gamma,<)}$, then the order between a_i and a_j can vary from different Wada equivalent left-orders.

With this in mind, we will relabel the vertices a_i so that the indices of the vertices will give us the partial order $<_{(\Gamma,<)}$. In other words, if $j \geq i$, then $a_j >_{(\Gamma,<)} a_i$. From now on, we will use $<$ for $<_{(\Gamma,<)}$.

Recall from Lemma 2.3.7, that the two non-bridge arcs are the maximum and the minimum between the generator of the Wada group. So, we first relabel the two non-bridge arcs a_1 and a_n where a_n is the maximum and a_1 is the minimum.

Let Q be the set of directed Wada paths from a_n to a_1 in $(\Gamma, <)$. We will be interested in the paths with the maximum number of edges.

Suppose there is a directed Wada path P with more edges than any other paths in Q. If there is more than one, choose one of the maximum ones. We call the vertices on P the *principal vertices* and the vertices not on P the *secondary vertices*. A *secondary path* is a directed Wada path that starts and ends on principal vertices, but for which all other vertices are secondary vertices. We relabel the principal vertices by recursion as follows. Let a_n still be a_n. Let a_i be a relabelled principal vertex. Then, the following Wada consecutive vertex on P will be relabelled a_{i-k} where k is the number of secondary vertices included in any secondary path that end at a_{i-k}.

For example, the knot diagram 8_{21} has no secondary vertices. Therefore, we relabel the vertices in the directed Wada graph of the knot 8_{21} as follows.

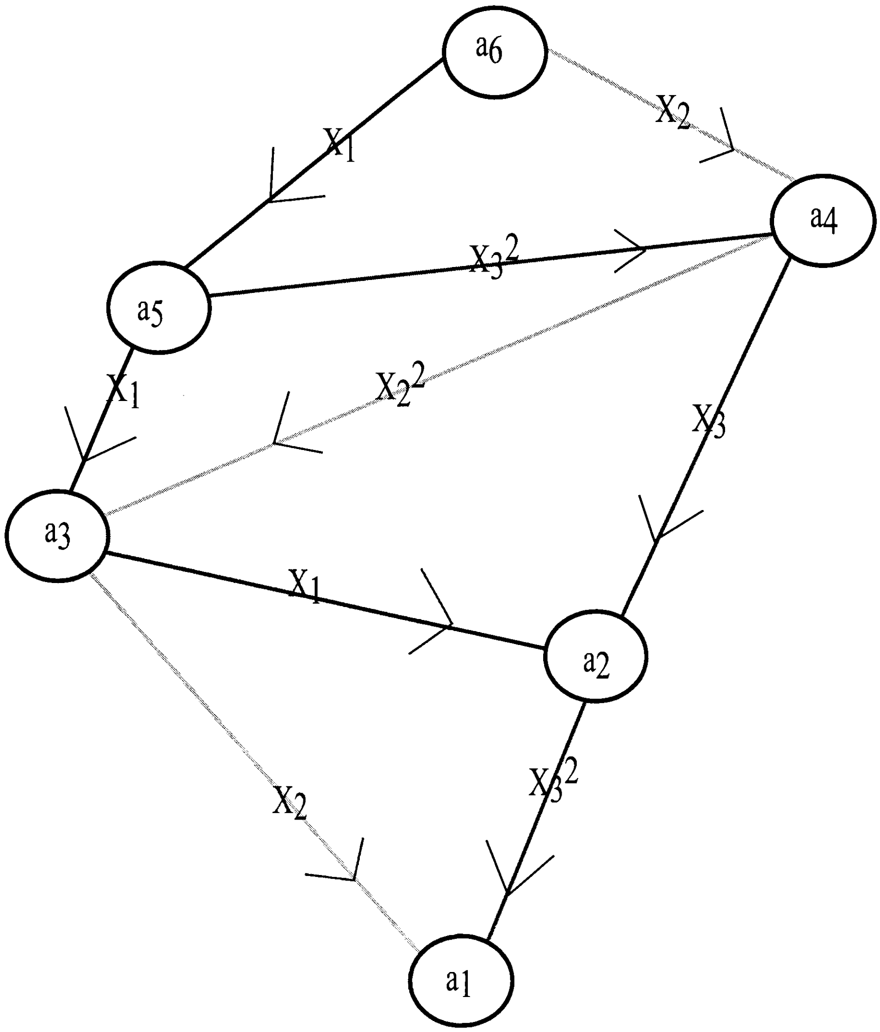


Figure 5.6 Directed Wada graph of 8_{21} with the vertices relabelled

Once we have relabelled the vertices in the directed Wada graph, we relabel the correspondant arcs in the Hybrid Wada diagram and the link diagram.

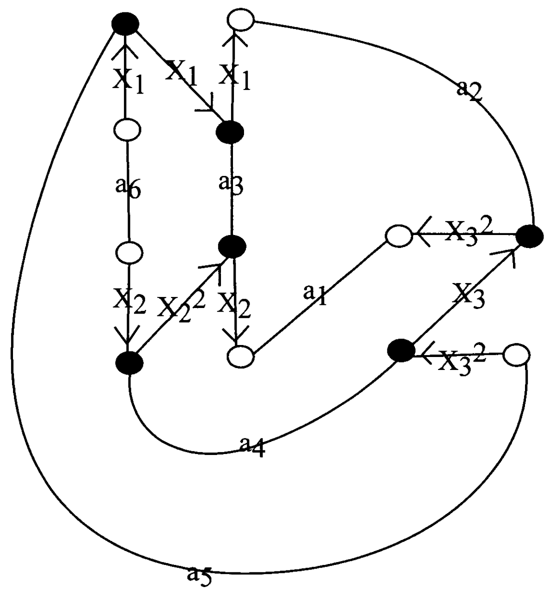


Figure 5.7 Hybrid directed Wada diagram of 8_{21} with the arcs relabelled

We will now relabel the secondary vertices for a directed Wada graph with secondary vertices. The indices of the secondary vertices will have two components (A, B). Let a_k be a secondary vertex. If a_k is such that there are no greater Wada consecutive secondary vertices, then $A = i$ where a_i is the smallest principal vertex whose greater than a_k. Similarly, if a_k is such that there are no smaller Wada consecutive secondary vertices, then $B = j$ where a_j is the greatest principal vertex whose smaller than a_k. Let a_j be a secondary vertex not already relabelled and with a greater Wada consecutive secondary vertex $a_{(A,B)}$ already labelled. Then A for a_j is equal to $(A, B)i$ where a_i is the smallest consecutive principal vertex whose greater than a_j and $B = l$ where where a_l is the greatest consecutive principal vertex whose smaller than a_j. Thus, a_j is relabelled $a_{((A,B)i,j)}$.

The diagram of the knot 9_{49} has a secondary vertex. We relabel the vertices in the directed Wada graph of the knot 9_{49} as follows.

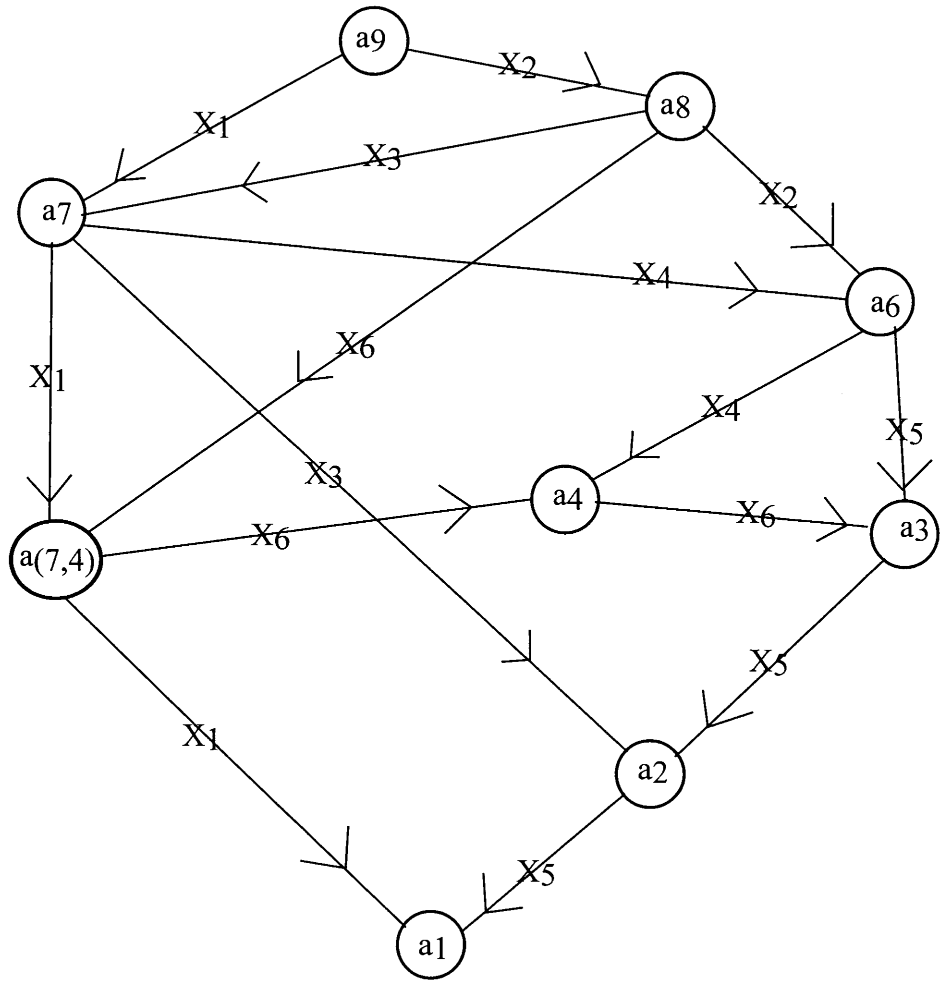


Figure 5.8 Directed Wada graph of 8_{21} with the vertices relabelled

Note that the relabeling of the vertices is not unique when there is at least one secondary vertex. We relabel the vertices this way in the directed Wada graph, because we want from the label of the vertices to know which one is greater in the order given by $(\Gamma, <)$. Thus, if we have a_i and a_j and $j \geq i$, then $a_j \geq a_i$. If we have $a_{(i,k)}$ and a_j with $j \geq i$, then $a_{(i,k)} \leq a_j$. If we have $a_{((i,j)k,l)}$, a_s and a_t where $l \geq s$ and $t \geq k$, then $a_t \geq a_{((i,j)k,l)} \geq a_s$. However, if we have $a_{((i,j)k,l)}$ and a_s where $s \geq l$ and $k \geq s$, then there is no order between $a_{((i,j)k,l)}$ and a_s.

Thus, only by the labels of two vertices a_i and a_j, we can know which one is greater than the other or if we do not have an order between them. If we do not have an order between them, we will say that a_i and a_j are *parallel vertices.*

Remark 5.2.2. From now on, we will allow a slight abuse of notation. When we will say that we have a_j and a_i with $j \geq i$, we will mean that $a_j \geq a_i$ but not necessary that a_i and a_j are principal vertices.

CHAPTER VI

THE IMPACT OF A TRIVIAL EDGE IN A DIRECTED WADA GRAPH

For the remainder of the thesis, we suppose that the links are 2-non-bridge links. In this section, we will prove Theorem 6.7.7 which states that if an edge is trivial in a directed Wada graph $(\Gamma, <)$ of a directed link diagram, then $G(\Gamma, <)$ is trivial. Thus, combining this result with Corollary 4.0.3, we will obtain Theorem 6.7.8, which plays a central role in proving that the fundamental group of the double branched cover of certain family of links is not left-orderable.

6.1 Wada natural paths and Wada directed cycles

Let D be a link diagram and $\Gamma(D)$ the Wada rational graph. In the hybrid Wada diagram $H(\Gamma)$, if there is a continuous path from an arc to another arc without passing twice over the same arc, we say that there is a *Wada path* between the two arcs. In a Wada path, if the continuous path passes from an arc a_i to an arc a_j, then we say that a_i and a_j are *consecutive.* Consecutive arcs are joined by an edge in the Wada rational graph. If there is a non-trivial Wada path from an arc a_i to itself, then we define this path to be a *Wada cycle.*

We now fix a directed Wada graph $(\Gamma, <)$ for the link diagram D. We will say that the Wada path $P = \{a_i, .., a_l, ..., a_m\}$ is a *Wada directed path* if it satisfies the following two conditions. First, $a_i \leq ... \leq a_l \leq ... \leq a_m$ or $a_i \geq ... \geq a_l \geq ... \geq a_m$.

Secondly, for every pair of consecutive arcs a_i and a_j in P such that $a_i \leq a_j$, there is no arc a_k in $H(\Gamma, <)$ such that there is a directed path from a_j to a_i passing through a_k. We call a_i and a_j *Wada consecutive arcs.* Also, we call the edge joining a_i and a_j in the directed Wada graph a *flat edge.*

Remark 6.1.1. Note that a path P in an hybrid Wada directed diagram is a Wada directed path if and only if it is a Wada directed path in the directed Wada graph as defined in Definition 5.2.1.

For example, in the Hybrid Wada directed diagram of the knot 8_{21}, $P = \{a_3, a_4, a_5\}$ is a Wada directed path.

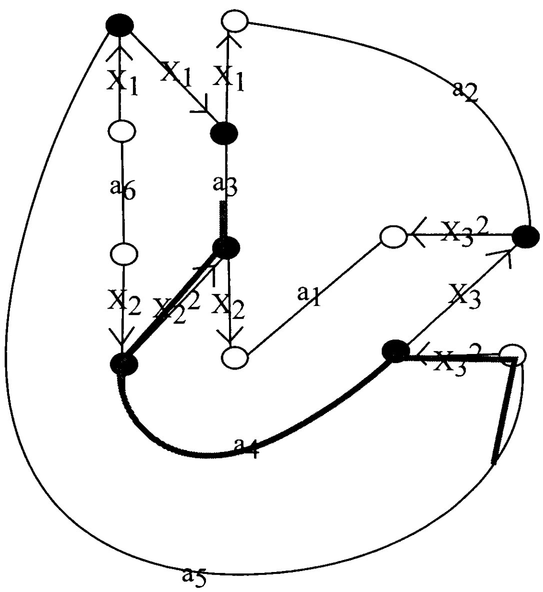
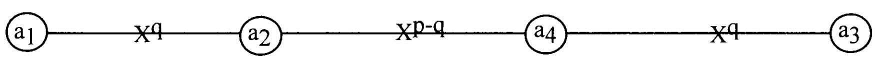


Figure 6.1 A Wada directed path in the knot 8_{21}

Definition 6.1.2. Let a rational tangle give us the following Wada rational graph



We define X to be the *edge* of the rational tangle, a_1 and a_3 to be the *final vertices* of the rational tangle X and a_2 and a_4 to be *middle vertices* of the rational tangle X. Moreover, we define the *final order* of X to be q, the *middle order* of X to be $p-q$, the *compose order* of X to be $q+p-q=p$ and the *total order* of X to be $2q+p-q=p+q$.

A word $w(Y_1, ..., Y_n)$ is defined as

$$w(Y_1, ..., Y_n) = \Pi_{j=0}^{m} Y_{i_j}^{k_{i_j}\epsilon_{i_j}}$$

where $1 \leq i_j \leq n$, $k_{i_j} \geq 0$ and $\epsilon_{i_j} = \pm 1$. We will say a word $w(Y_1, ..., Y_n)$ is *positive* if $\epsilon_{i_j} = 1$ for every i_j.

Definition 6.1.3. We use the notation $w^+(X_1, ..., X_n)$ to signify a *positive word.*

Let $(\Gamma, <)$ be a directed Wada graph. If we have a Wada directed path $P = \{a_i, .., a_l, ..., a_m\}$, where a_m is an arc of a rational tangle containing a_i, then $P' = (a_i, .., a_l, ..., a_m, a_i)$ is called a *Wada directed cycle.* Moreover, every Wada directed cycle $C = (a_i, ..., a_m, a_i)$ in $H(\Gamma, <)$ gives a relation $X_k^{m_k} = w^+(X_1, ..., X_n)$ in $G(\Gamma, <)$ where $X_k^{m_k}$ is the edge or edges from the rational tangle containing a_m and a_i and $w^+(X_1, ..., X_n)$ is a positive word coming from the Wada directed path. We say that X_k is the *cover edge* of C. Moreover, for $w^+(X_1, ..., X_n) = \Pi_{j=0}^{m} X_{i_j}^{k_{i_j}}$, X_{i_m} is defined as the *right edge of* C and X_{i_0} is defined as the *left edge of* C. Furthermore, $X_{i_{m-j}}$ is defined as the $(j+1)$-*right edge of* C and X_{i_j} is defined as the $(j+1)$-*left edge of* C.

Definition 6.1.4. We say that a cover edge X_k is a *minimal cover edge* if m_k is of minimal order, X_k is a *maximal cover edge* if m_k is of maximal order, X_k is a *total cover edge* if m_k is of total order and X_k is a *compose cover edge* if m_k is of compose order. Moreover, we say that X_k is a *simple cover edge*, if it is either a minimal or maximal cover edge.

In the example of figure 6.1, a_3 and a_5 are in the same rational tangle X_1, therefore $P' = (a_3, a_4, a_5, a_3)$ is a Wada directed cycles.

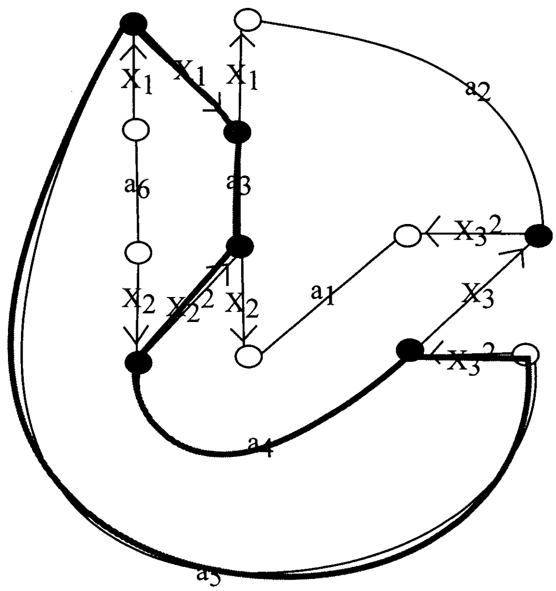


Figure 6.2 A Wada directed cycle in the knot 8_{21}

From this Wada directed cycle, we obtain the relation $X_1 = X_3^2 X_2^2$ where X_1 is the minimal cover edge of P'.

Lemma 6.1.5. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram. Then, for every vertex* a_i*, there is a Wada directed path from* a_i *to* a_1 *and from* a_n *to* a_i*.*

Proof. If a_i is a principal vertex, then by definition of principal vertices, there is a Wada directed path from a_n to a_i.

If a_i is a secondary vertex, then there is a secondary path from a principal vertex a_j to a_i. Thus, there is a Wada directed path from a_j to a_i. Also, because a_j is a principal vertex, there is a Wada directed path from a_n to a_j. Therefore, by joining these Wada directed paths, there is a Wada directed path from a_n to a_i.

Similarly, we can find a Wada directed path from a_i to a_1. □

6.2 Results obtained from the Wada directed cycles

We will show that Wada directed cycles give us important information about the directed Wada graph.

Let $C = (a_i, ..., a_l, ..., a_m, a_i)$ be a Wada cycle. First, note as in Figure 6.2 that C is a closed curve homeomorphic to S^1 in $\mathbb{R}^2$. We say that a_j *is on* C if $a_j \in C$. We will say that a_j *is inside* C, if a_j is inside C in $\mathbb{R}^2$. Similarly, we will say that a_j *is outside* C, if a_j is outside C in $\mathbb{R}^2$.

Lemma 6.2.1. *Let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge link diagram and $C = (a_i, ..., a_j, a_i)$ be a Wada directed cycle with $i < j$. If a_k is inside C and $k < i$(resp. $k > j$), then there is no Wada directed path from a_k to any arc a_m with $m < i$ (resp. $m > j$) outside of C.*

Proof. Suppose $k < i$. Suppose there is a Wada directed path P from a_k to a_m. Because a_k is inside C and a_m is outside C, P must pass by C. So, there must be an arc a_l of the cycle C with $l \geq i > k$ and $l \geq i > m$ in P. Thus, we have $P = (a_k, ..., a_l, ..., a_m)$. This is not a Wada directed path, because $a_k \leq a_l \geq a_m$.

Similarly for $k > i$. □

There is an important family of Wada directed cycles.

Definition 6.2.2. Let C be a Wada cycle in a directed Wada graph. If there is a maximum arc (resp. a minimum arc) in C, then we say that C is a *max (resp. min) primary cycle.*

If there is no maximum arc and no minimum arc inside C, then we call C a *secondary cycle.*

If there is a maximum arc (resp. minimum arc) outside of C, then we say that C is an *outside max (resp. min) primary cycle.*

If C is max (resp. min) primary and outside min (resp. max) primary, then C is called a *dichotomic cycle.*

Dichotomic Wada directed cycles will be instrumental in showing that the directed Wada group is trivial if a generator is trivial.

Using Lemma 6.2.1, we get the following useful results about Wada directed cycles.

Corollary 6.2.3. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram and* $C = (a_i, ..., a_j, a_i)$ *be a Wada directed cycle with* $i < j$*. If* a_k *is an arc inside* C *such that* $k < i$*, then* a_1 *is inside* C *and so* C *is min primary. Similarly, if* a_k *is an arc inside* C *such that* $k > j$*, then* $a_n \in C$ *and so* C *is max primary.*

Proof. By Lemma 6.1.5, there is a Wada directed path from a_k to a_1. Thus, by Lemma 6.2.1, a_1 can't be outside C. Moreover, $a_1 \leq a_k \leq a_i$, so a_1 is not on C. Therefore, a_1 is inside C.

The proof is similar for the max primary case. □

Similarly,

Corollary 6.2.4. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram and* $C = (a_i, ..., a_j, a_i)$ *be a Wada directed cycle with* $i < j$. *If* a_k *is an arc outside* C *such that* $k < i$, *then* a_1 *is outside* C *and so* C *is outside min primary. Similarly, if* a_k *is an arc outside* C *such that* $k > j$, *then* a_n *is outside* C *and so* C *is outside max primary.*

With the previous results we can obtain information about cycles in the directed Wada graph.

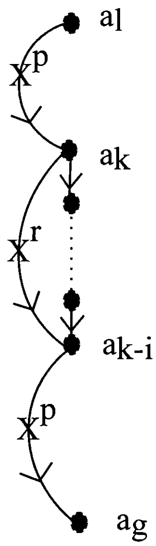
Definition 6.2.5. Let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge link diagram and $C = (a_k, ..., a_i, a_k)$ be a Wada directed cycle with $k > i$ that gives the relation $X_s^{m_s} = X_t^{m_t} ... X_u^{m_u}$. If X_s is a rational tangle that also has an edge from a_l to a_k with $l > k$, then C is a *high tail Wada directed cycle.*

Similarly, if X_s is a rational tangle, that also has an edge from a_l to a_i with $l < i$, then C is a *low tail Wada directed cycle.*

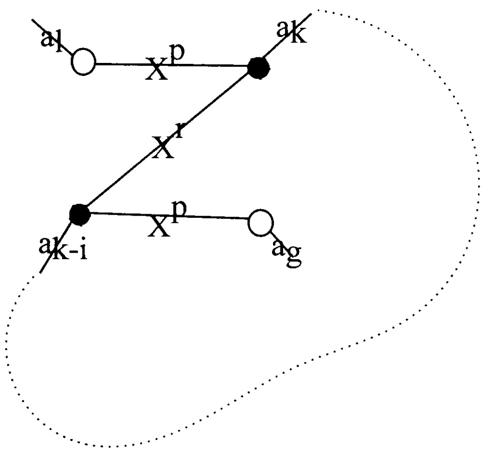
We now show that a high and low tail Wada directed cycle is dichotomic.

Lemma 6.2.6. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram* D. *If* $C = (a_k, a_j, ..., a_{k-i}, a_k)$ *with* $k > j > k - i$ *is a high and low tail Wada directed cycle with* X *the cover edge from the rational tangle with the vertices* a_g, a_{k-i}, a_k *and* a_l. *Then,* C *is a dichotomic Wada directed cycle.*

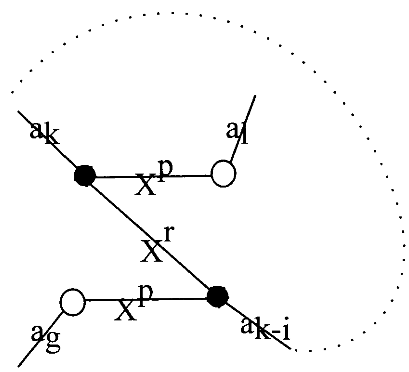
Proof. If C is a high and low tail Wada directed cycle, then we have the following subgraph in the directed Wada graph,



and thus the following two possible subdiagrams in the hybrid Wada diagram



or



In the left case, $a_g \leq a_{k-i}$ and a_g is inside C. Therefore, by Corollary 6.2.3, C is min primary. Moreover, a_l is outside C and $a_l \geq a_k$. Thus, by Corollary 6.2.4, C is max outside primary. So, C is a dichotomic Wada directed cycle.

The proof is similar for the right case. $\square$

6.3 From Rational tangles in the Wada rational graph to Rational tangles in the Hybrid Wada diagram and the introduction of Thorns

This is a technical section. We will differentiate the different part of arcs that go inside or outside a Wada cycle. Let C be a Wada cycle and X a rational tangle on C. If there is an arc a_i that is a bridge in X and that goes inside C, then the part of a_i in X is an *inside principal thorn*. Suppose there is an arc a_j that is not a bridge in X and that goes inside C. Then, if there is an inside principal thorn from X in C, then a_j is an *inside secondary thorn*. Moreover, if there is no inside principal thorn from X in C, then a_j is a *inside tertiary thorn*. We define similarly *outside principal*, *secondary* and *tertiary thorns*.

We will now look at all possibilities for rational tangles in a Wada cycle. First, we look at half-twist crossings. In a cycle of the Wada rational graph we can have the two following possibilities. First completely included in the cycle.

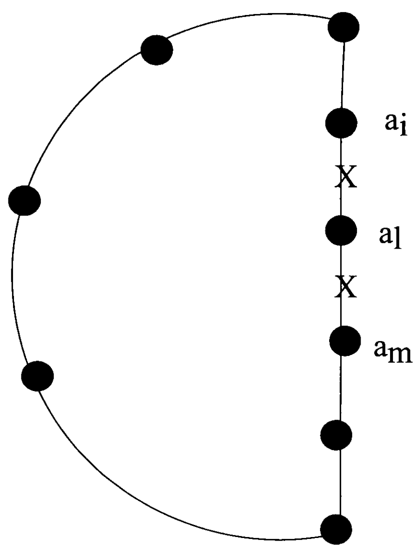


Figure 6.3 Half-twist crossing completely included in the Wada cycle in the Wada rational graph

Secondly, half included in the cycle.

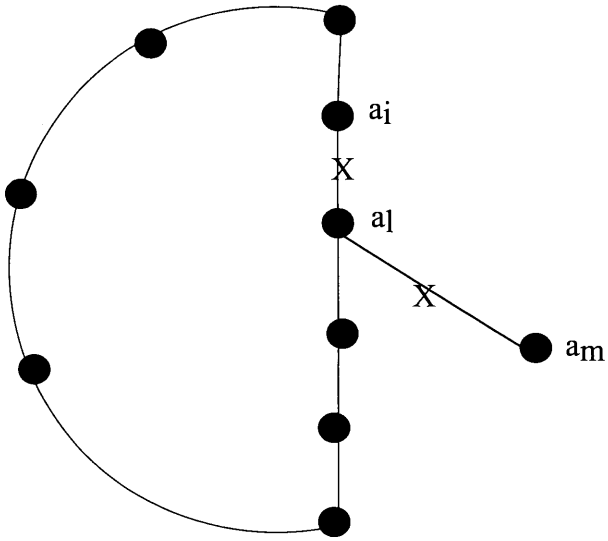


Figure 6.4 Half-twist crossing half included in the Wada cycle in the Wada rational graph

The first case will look as follows in the Hybrid Wada diagram. We define this part of a_l to be a *principal inside thorn.*

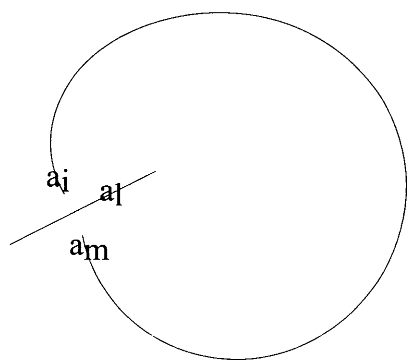


Figure 6.5 Half-twist crossing completely included in the Wada cycle in the Hybrid Wada diagram

For the second case, we have two possibilities in the Hybrid Wada diagram. The inside case where we define this part of a_l to be a *principal inside thorn* and this part of a_m to be a *secondary inside thorn.*

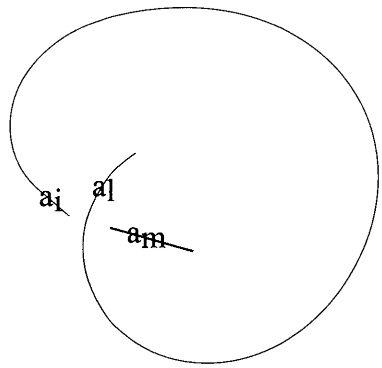


Figure 6.6 The inside case of a half-twist crossing half included in the Wada cycle in the Hybrid Wada diagram

The second possibility is the outside case where we define the following part of a_l to be a *principal outside thorn* and the following part of a_m to be a *secondary outside thorn.*

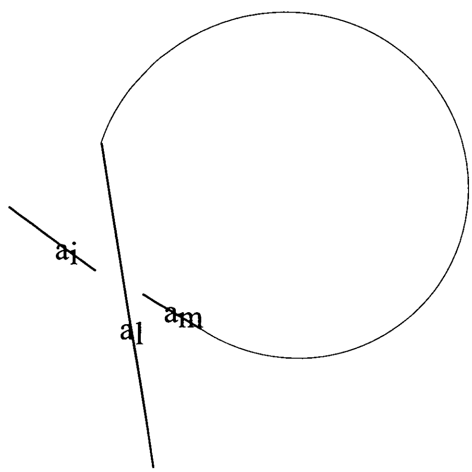


Figure 6.7 The outside case of an half-twist crossing half included in the Wada cycle in the Hybrid Wada diagram

Finally, there is the half-twist case, where no edges are on the cycle C, but a vertex is on the cycle:

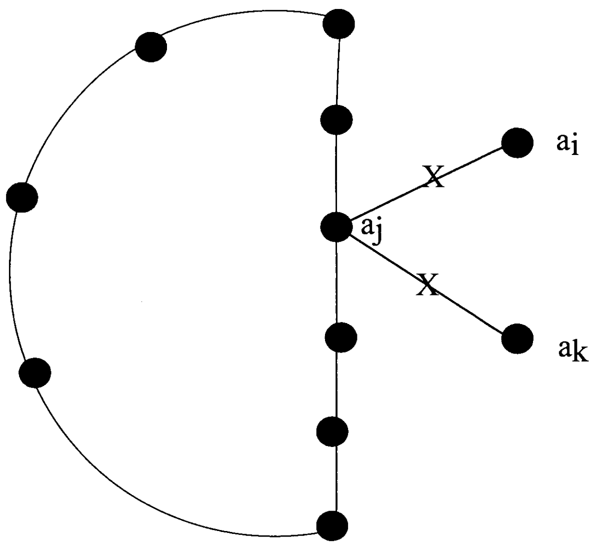


Figure 6.8 Half-twist with only a vertex included in the Wada cycle in the Wada rational graph

which gives the following Hybrid Wada diagrams

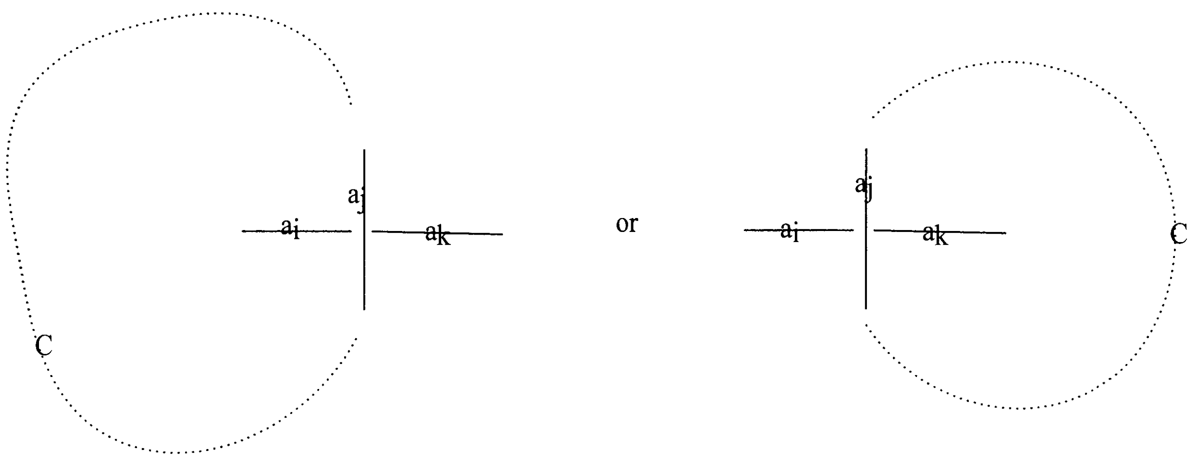


Figure 6.9 Half-twist with only a vertex included in the Wada cycle in the Hybrid Wada diagram

In the case on the left, we define this part of a_i to be a *tertiary inside thorn* and this part of a_k to be a *tertiary outside thorn*. In the other case, we define this part of a_i to be a *tertiary outside thorn* and this part of a_k to be a *tertiary inside thorn*.

We now look at rational tangles. There are three possibilities. First completely included in the cycle:

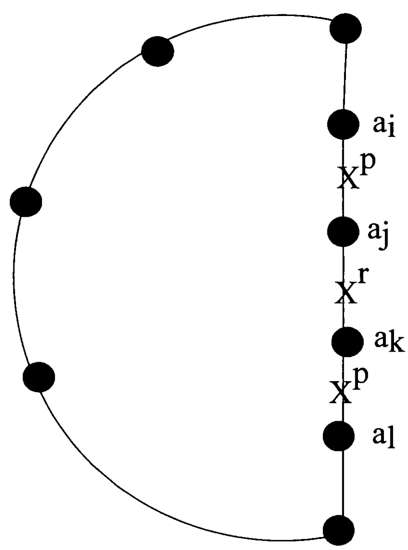


Figure 6.10 Rational tangle completely included in the Wada cycle in the Wada rational graph

which gives two possibilities for the Hybrid Wada diagram. In the case on the left, we define this part of a_k to be a *principal inside thorn* and this part of a_j to be a *principal outside thorn*. In the other case, we define this part of a_j to be a *principal inside thorn* and this part of a_k to be a *principal outside thorn*.

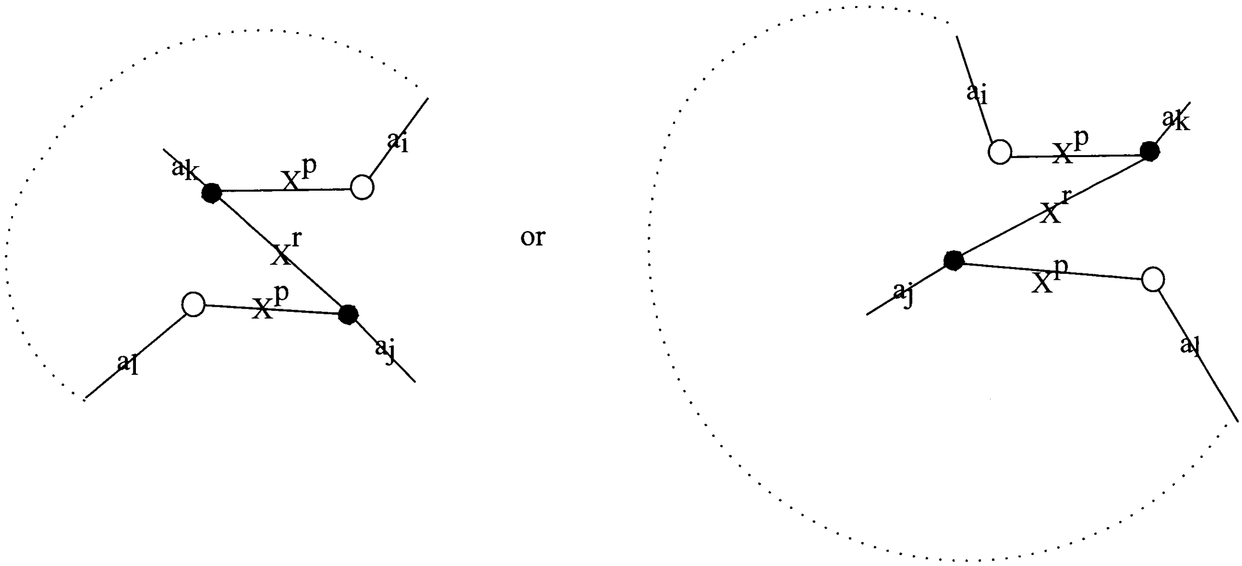


Figure 6.11 Rational tangle completely included in the Wada cycle in the Hybrid Wada diagram

Secondly, two consecutive edges included in the cycle:

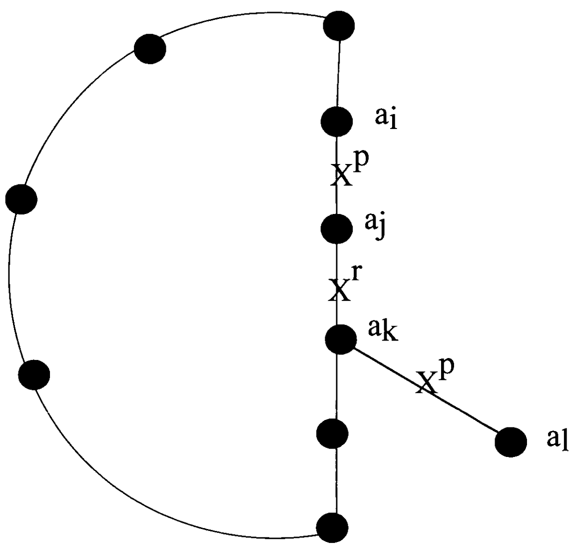


Figure 6.12 Rational tangle with two consecutive edges included in the Wada cycle in the Wada rational graph

which gives two possibilities for the Hybrid Wada diagram; the outside case to the left and the inside case to the right. In the case on the left, we define this part of a_j to be a *principal outside thorn* and this part of a_l to be a *secondary outside thorn*. In the other case, we define this part of a_j to be a *principal inside thorn* and this part of a_l to be a *secondary inside thorn*.

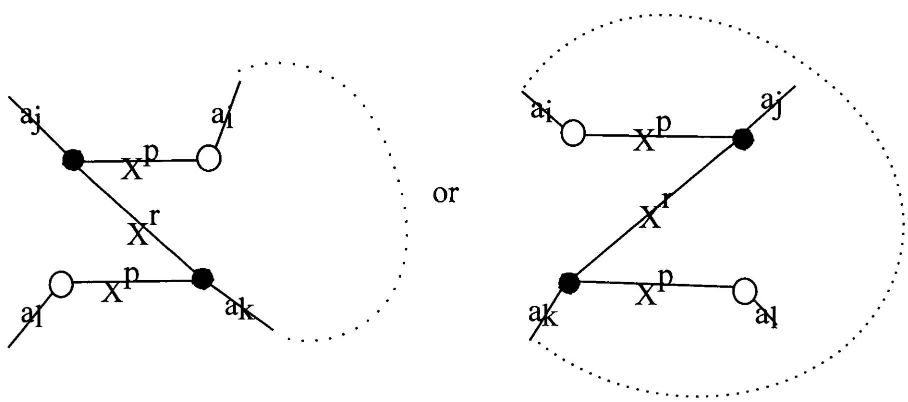


Figure 6.13 Rational tangle with two consecutive edges included in the Wada cycle in the Hybrid Wada diagram

Thirdly, there are two cases for only one edge in the Hybrid Wada diagram. First, an end edge is in the cycle:

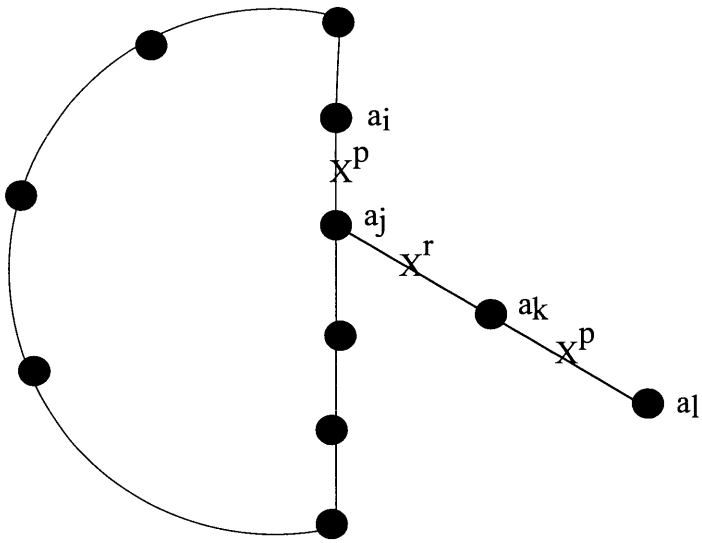


Figure 6.14 Rational tangle with only an end edge included in the Wada cycle in the Wada rational graph

which gives the following Hybrid Wada diagrams. In the case on the left, we define this part of a_k to be a *principal inside thorn* and this part of a_l to be a *secondary inside thorn*. In the other case, we define this part of a_k to be a *principal outside thorn* and this part of a_l to be a *secondary outside thorn*.

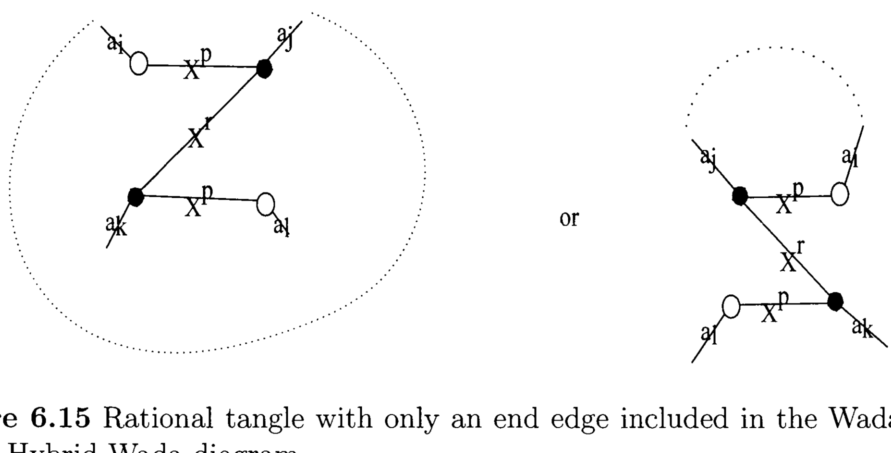


Figure 6.15 Rational tangle with only an end edge included in the Wada cycle in the Hybrid Wada diagram

Then, the second case is when it is the middle edge that is in the cycle:

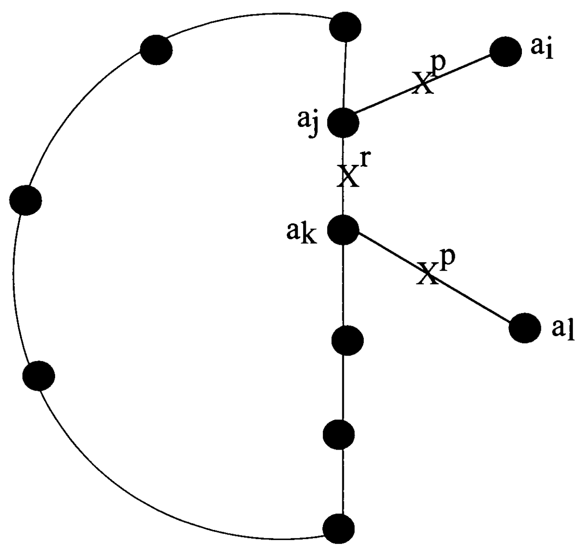


Figure 6.16 Rational tangle with only the middle edge included in the Wada cycle in the Wada rational graph

which gives the following Hybrid Wada diagrams. In the case on the left, we define this part of a_i to be a *tertiary inside thorn* and this part of a_l to be a *tertiary outside thorn*. In the other case, we define this part of a_i to be a *tertiary outside thorn* and this part of a_l to be a *tertiary inside thorn*.

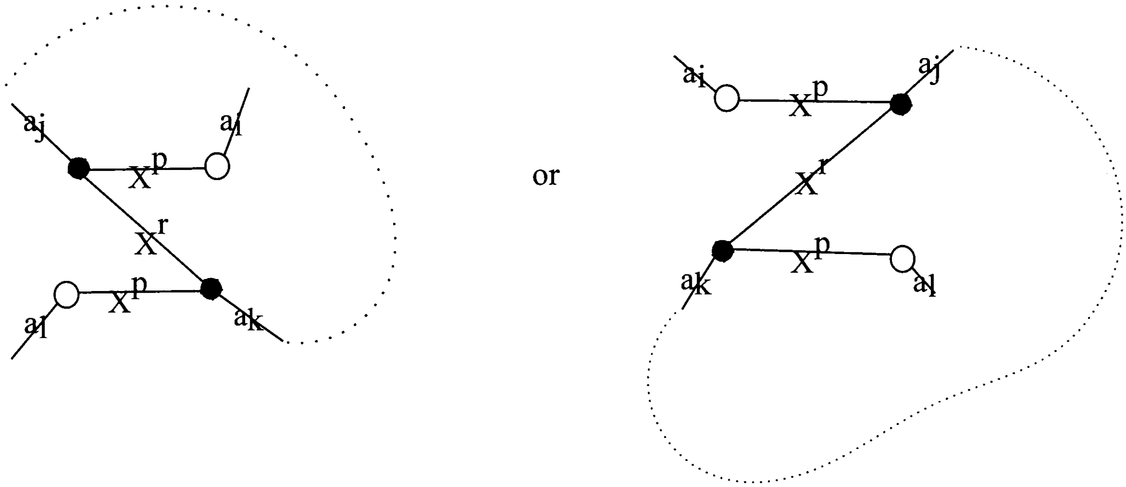


Figure 6.17 Rational tangle with only the middle edge included in the Wada cycle in the Hybrid Wada diagram

Remark 6.3.1. Note that every time we have at least one edge in the Wada cycle and the other edges not on the Wada cycle, except for the middle edge case, we have either an inside or outside case format. Every inside case format adds two arcs inside the Wada cycle in the Hybrid Wada diagram, a secondary inside thorn and a principal inside thorn, while none outside the cycle and inversely for the outside case format.

Moreover, for the rational tangles completely included in the Wada cycle, there is one arc inside the cycle, a principal inside thorn and one arc outside the cycle a principal outside thorn. We will define the family of *complementary thorns* as the family that includes the secondary and tertiary thorns. Every principal thorns comes from at least a bridge arc, while complementary thorns come from a non-bridge part of an arc.

6.4 Results on Wada cycles from thorns and tertiary tangle

Let C be a Wada cycle in a Wada rational graph. We will look at all inside thorns. They will be the marked points of an n-tangle which we will call the *inside n-tangle of C*. Similarly, we take the outside thorns of C and form the *outside n-tangle of C*. Recall from chapter 1, that a marked point coming from an arc that just went under another arc is called an *under marked point* and a marked point coming from an arc that just went over another arc is called an *over marked point*. Remark, that if a complementary thorn has an under marked point, then it is a non-bridge arc. Moreover, if a principal thorn has an over marked point, then it is at least a two-bridge arc.

We now give a series of technical results about thorns in Wada cycles.

Lemma 6.4.1. *Let D be a link diagram and C a Wada cycle in the hybrid Wada diagram. Then, the number of inside and the number of outside thorns of C must be even.*

Proof. The inside thorns form the marked points of the inside n-tangle of C. Therefore, there are $2n$ marked points. Thus, there are $2n$ inside thorns and so an even number of inside thorns.

The proof is similar for outside thorns. □

Lemma 6.4.2. *Let D be a link diagram and C a Wada cycle in the hybrid Wada diagram. If there are more inside complementary thorns than principal inside thorns in C, then there is a non-bridge arc of D inside C.*

Proof. Suppose there are $2n$ inside thorns. Thus, we have an inside n-tangle T of C. There are more inside complementary thorns than principal inside thorns, so there are $n - i$ principal inside thorns and $n + i$ complementary inside thorns for $i \geq 1$. If a complementary inside thorns is an under marked point, then it is a non-bridge arc.

Suppose there is no non-bridge arc of D inside C. This implies that every non-bridge arc in T is at least a bridge arc in D. Thus, every non-bridge arc in T is an inside principal thorn. By Lemma 1.1.14, there are at least n non-bridge arcs in T. But there are only $n-i$ principal inside thorns. Therefore, there is at least one non-bridge arc in T that is a complementary inside thorns. So, this non-bridge arc in T is a non-bridge arc in D. □

Similarly, we can obtain the following result.

Lemma 6.4.3. *Let D be a link diagram and C a Wada cycle in the hybrid Wada diagram. If there are more outside complementary thorns than principal outside thorns in C, then there is a non-bridge outside C.*

Cycles that contains tertiary thorns will be important enough that we name this family of cycles.

Definition 6.4.4. Let D be a link diagram and C a Wada cycle in the hybrid Wada diagram. If there is a tertiary thorn inside (resp. outside) C, then we say that C is *inside (resp. outside) tertiary.* Note that all tertiary inside thorn comes from tertiary tangle and tertiary tangle also gives a tertiary outside thorn. Thus, we define a Wada cycle C to be a *tertiary cycle*, if there is a tertiary tangle on C.

Now, let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge link diagram D. Recall that a flat edge from a_i to a_j with $a_i \leq a_j$ is an edge such that there is no $a_k \in (\Gamma, <)$ such that $a_i \leq_{(\Gamma,<)} a_k \leq_{(\Gamma,<)} a_j$. The rational tangles that only have flat edges will add some difficulty in proving that the directed Wada group is trivial.

Definition 6.4.5. Let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge link diagram D. If we have a half-crossing such that both edges are flat, then this crossing is a *flat crossing.* Similarly, for a rational tangle if every edge is flat, then we say that the rational tangle is a *flat rational tangle.*

Many proofs in this chapter will be obtained by finding non-bridge arcs in Wada directed cycle. By Lemma 6.4.2, if there are more complementary thorns than principal thorns inside a Wada directed cycle C, then there is a non-bridge arc inside C. Thus, principal thorns will often be obstacles in proofs. We now link principal thorns and flat rational tangles.

Lemma 6.4.6. *Let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge link diagram D and C be a Wada directed cycle in $(\Gamma, <)$. If X is a rational tangle that adds exactly one principal inside (resp. outside) thorn in C, then X is a flat rational tangle completely included on C.*

Proof. Because X adds exactly one principal inside thorn, X is completely included on C. Moreover, if X is not a flat rational tangle, then C is not a Wada directed cycle. □

The next result shows the importance of tertiary tangle and flat rational tangle.

Lemma 6.4.7. *Let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge link diagram D and C a Wada directed cycle in $(\Gamma, <)$. Suppose that C is inside (resp. outside) tertiary. If there is no non-bridge arc inside (resp. outside) C, then there is a flat rational tangle completely included on C.*

Proof. The Wada directed cycle C is inside tertiary, therefore there is a rational tangle that only adds one complementary thorn inside C. Every rational tangle on a Wada cycle adds either one principal and one complementary inside thorns, no inside thorns, one complementary thorns or one principal thorn. Moreover, by the previous lemma, a rational tangle adds exactly one principal thorn if and only if it is a flat rational tangle completely included on the Wada directed cycle. Thus, if there is no flat rational tangle completely on C, then there is more complementary inside thorns than principal inside thorns. This implies, by Lemma 6.4.2, that if there is no flat rational tangle completely included on C, then there is a non-bridge arc inside C. □

With a similar proof, we can generalize the previous result.

Lemma 6.4.8. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram* D *and* C *a Wada directed cycle in* $(\Gamma, <)$*. Suppose there are more tertiary tangle on* C *than flat rational tangle completely included on* C*. Then, there is a non-bridge arc inside* C *and a non-bridge arc outside* C*.*

Proof. By hypothesis, there are more complementary inside thorns than principal inside thorns. This implies, by Lemma 6.4.2, that there is a non-bridge arc inside C, because there is no more flat rational tangle completely included on C.

The proof for the non-bridge arc outside of C is similar. □

6.5 Supporting cycle of flat rational tangles

As already mentioned, flat rational tangles are the principal obstacles in proving that directed Wada group is trivial when a generator is trivial. In this section, we will define supporting cycle which are associated to flat rational tangles.

Lemma 6.5.1. *Let* $(\Gamma, <)$ *be the directed Wada graph of a 2-non-bridge directed link diagram* D*. If there is a flat rational tangle* X_i *in* $(\Gamma, <)$*, then there is at least one Wada directed cycle* C *with* X_i *as cover edge.*

Proof. Suppose there is no such cycle C. If we change the direction of X_i, we won't have a directed cycle. Thus, we have a contradiction because D is a directed link diagram. This implies, that there is such a cycle C. □

We call such a cycle C a *supporting cycle of* X_i. We recall from Definition 6.1.2, that the compose order is the addition of the final order and the middle order, while the total order is the addition of the order of every edges. Note that because X_i is a flat rational tangle, X_i is either a total cover edge or a compose cover edge. Moreover, if X_i is an half-twist, then the total order and the compose order are the same. Also, X_i must be a total cover edge and not a simple cover edge for it to be flat.

Lemma 6.5.2. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge directed link diagram* D *and* X_i *a flat rational tangle in* $(\Gamma, <)$. *Then, either there is a supporting cycle* C *with* X_i *as a compose cover edge or there is a supporting cycle* C *that is dichotomic. Moreover, if* X_i *is an half-twist, then there is a supporting cycle* C *that is dichotomic.*

Proof. We will show the result for rational tangle as the result for half-twist is similar. By the previous lemma, there is a supporting cycle C with X_i either as a total cover edge or as a compose cover edge. If X_i is a compose cover edge of C, then the proof is over. Suppose that X_i is a total cover edge. Then, in $(\Gamma, <)$ we have the following subgraph.

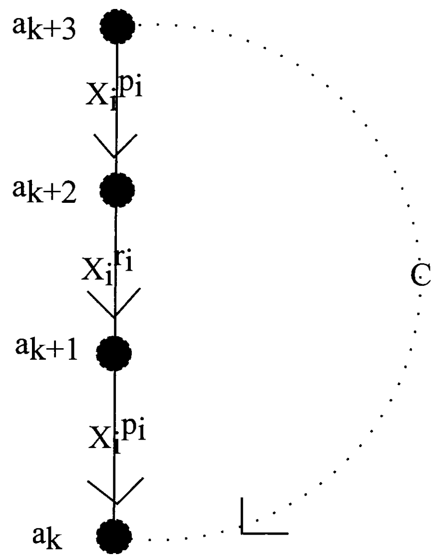


Figure 6.18 directed Wada graph of a supporting cycle C with X_i as a total cover edge

Moreover, in the hybrid Wada diagram we have the following diagram

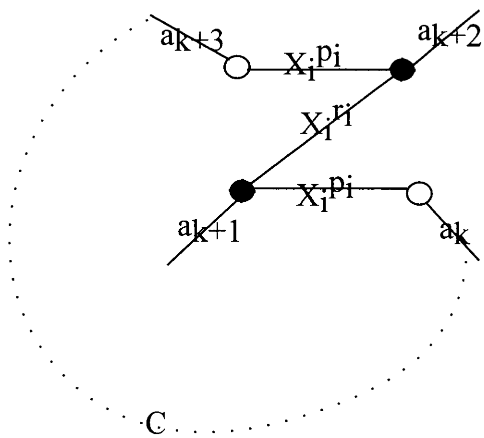


Figure 6.19 Hybrid Wada diagram of a supporting cycle C with X_i as a total cover edge

Suppose a_{k+1} or a_{k+2} are consecutive to any arc of C. Then we have a new supporting cycle B of X_i with compose cover edge X_i as shown in the following graph.

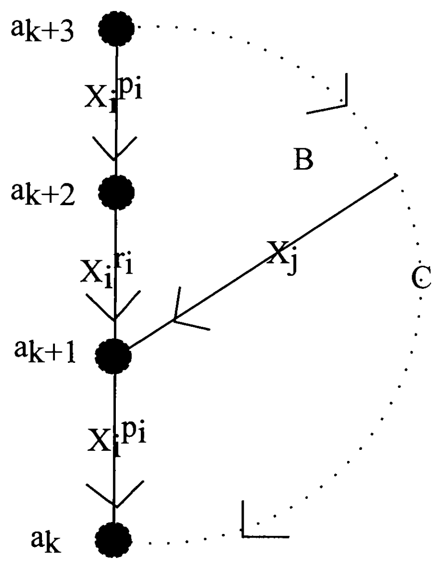


Figure 6.20 New supporting cycle B of X_i with compose cover edge

Note, that because X_i is a flat rational tangle, the edge X_j can't be in the other direction. Thus, this completes the proof if a_{k+1} or a_{k+2} are consecutive to any arcs of C.

Now, suppose that a_{k+1} is consecutive to an arc parallel to C. Let a_m be the arc parallel to C and consecutive to a_{k+1} and X_j the edge joining a_m and a_{k+1}.

First, suppose that X_j goes from a_m to a_{k+1}. By Lemma 6.1.5, there is a Wada directed path P from a_n to a_m. If one of the arcs of P is in C, then we obtain a directed Wada graph similar to Figure 6.20 and the proof is complete. If there are no arcs of P in C, then a_n must be in C. Thus, C is max primary.

Now, suppose that X_j goes from a_{k+1} to a_m. There is a Wada directed path Q from a_m to a_1. If one of the arcs of Q is in C, then we get the following directed Wada graph:

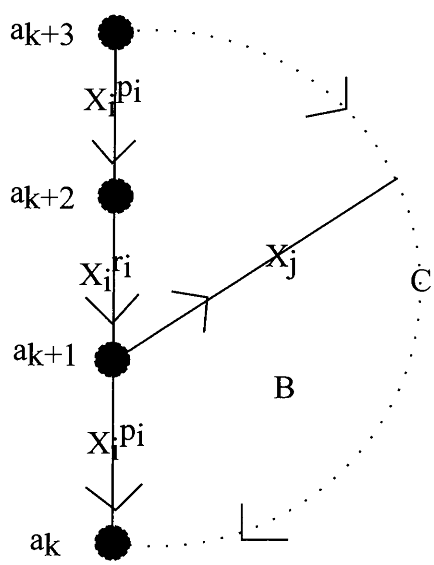


Figure 6.21 New supporting cycle B of X_i with simple cover edge

bu we can't have this case, because X_i is a flat rational tangle. Thus, there is no arcs of C in Q and so a_1 is in C, and so C is min primary.

Similarly, if a_{k+2} is consecutive to an arc parallel to C, then either we have a new supporting cycle B of X_i with compose cover edge X_i or C is either outside min primary or outside max primary.

Suppose, that both a_{k+1} and a_{k+2} are consecutive to an arc parallel to C. Because a non-bridge can't be in and out of a Wada directed cycle, either there is a new supporting cycle B of X_i with compose cover edge X_i or C is dichotomic.

Suppose that a_{k+1} is not consecutive to any arc of C or any arc parallel to C. Then, a_{k+1} must be consecutive to a a_l such that $l < k$ or $l > k+3$. If $l < k$, from Figure 6.19 and Corollary 6.2.3, C is min primary. If $l > k+3$, then C is max primary.

Similarly, suppose that a_{k+2} is not consecutive to any arc of C or any arc parallel to C. Then, C is outside max primary or outside min primary.

Therefore, if a_{k+1} and a_{k+2} are not consecutive to C, then because a non-bridge can't be in and out of a Wada directed cycle, either there is a new supporting cycle B of X_i with compose cover edge X_i or C is dichotomic.

□

6.6 Result on the triviality of the directed Wada group when a bridge arc is an extremum

We now need a lemma that shows that if a bridge arc is an extremum, then all arcs are equal.

Lemma 6.6.1. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram* D *with* a_1 *and* a_n *having non-bridge arc and let* $\pi(D)$ *be left-orderable. If there is a vertex* $a_j \in (\Gamma, <)$ *with* $j \neq n$ *such that* $a_j = a_n$*, then* $a_1 = a_2 = \ldots = a_n$.

Proof. Let $<$ be a left order on $\pi(D)$. By Lemma 2.3.7, a_1 and a_n are extrema. Without loss of generality, suppose that a_1 is the minimum and a_n is the maximum. Therefore, because a_j is equal to a_n, a_j is a maximum. But a_j is not a non-bridge by hypothesis, therefore a_j is at least a bridge over some arcs a_i and a_m. Therefore it satisfies one of the Wada inequalities $a_i < a_j < a_m$, $a_m < a_j < a_i$ or $a_i = a_j = a_m$. But a_j is a maximum, thus we obtain $a_i = a_j = a_m$. This implies that, a_i and a_m are maxima. Thus, every arc that goes under a maximum arc, becomes a maximum arc. Moreover, if two maximum arcs a_i and a_m go under an arc a_p, then it satisfies one of the Wada inequalities $a_i < a_p < a_m$, $a_m < a_p < a_i$ or $a_i = a_p = a_m$. But a_i and a_m are maxima, thus we obtain $a_i = a_p = a_m$. Therefore, every arc that goes over two maxima arcs becomes a maximum.

So we can construct the blue and red graph $G(D, a_j)$ where the property P is the maximum property. By Lemma 2.3.3, there are at most two connected components F_j in U_i^c.

Suppose there is no connected component F_j in U_i^c. Then, every arc is red and so every arc is maximum and the proof is over.

Suppose there are two connected components F_1 and F_2 in U_i^c. Then, by Lemma 2.3.4, both non-bridge arcs are in W and so both are red. Thus, $a_1 = a_n$. So the minimum is a maximum and the proof is over.

Suppose there is exactly one connected component F in U_i^c. Then, by Lemma 2.3.5, there is either one or two non-bridge arcs in W.

Suppose that both non-bridge arcs are in W. This implies that $a_1 = a_n$. So the minimum is a maximum and the proof is over.

Suppose there is exactly one non-bridge arc in W. If a_1 is in W, then a_1 is a maximum and the proof is over. Therefore, we suppose a_n is the non-bridge in W. Moreover, suppose W has $2m$ marked point with $m > 1$. Then, by Lemma 1.1.14, there are at least m non-bridge arcs in W. Thus, by remark 1.2.3, there are at least m non-bridge arcs of D in W, which is a contradiction. Therefore, there are two marked points in W. By construction of W, both arcs of the marked points do not go over some arcs in $D \setminus W$. Moreover, by hypothesis there is only one non-bridge arc of D in W. Also, by hypothesis, there is only one connected component F. Thus, W can be viewed as a 1-tangle. So, T_w is a 1-tangle with only one non-bridge arc a_i in D such that neither marked point goes over some arcs in $D \setminus T_w$. But, this is impossible by the construction of the Wada rational graph from the coarse Wada rational graph in section 2.3.

□

Similarly, we can prove

Lemma 6.6.2. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram* D *with* a_1 *and* a_n *has non-bridge arc and* $\pi(D)$ *be left-orderable. If there is a vertex* $a_i \in (\Gamma, <)$ *with* $i \neq 1$ *such that* $a_i = a_1$*, then* $a_1 = a_2 = ... = a_n$.

6.7 The Impact of a Trivial Edge in a directed Wada graph

In this section, we will prove Proposition 6.7.7 which says that if $(\Gamma, <)$ is a directed Wada graph of a directed link diagram D and there is a X_i such that $X_i = 1$, then $a_1 = ... = a_n$, and so the directed Wada group of $(\Gamma, <)$ is trivial. To do so, we will need a series of technical lemmas.

Lemma 6.7.1. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram* D *and* $C = (a_k, a_{k-1}, ..., a_{k-i}, a_k)$ *be a Wada directed cycle that gives the relation* $X_i^{m_i} = X_j^{m_j} ... X_l^{m_l}$ *and such that there is an arc* a_j *with* $j > k$ *inside* C *and an arc* a_q *with* $q < k - i$ *outside of* C*. If* $X_i = 1$*, then* $a_1 = ... = a_{k-i} = a_{k-i+1} = ... = a_n$.

Proof. First note that by Corollary 6.2.3, C is max primary and outside min primary. Moreover, by Lemma 6.2.1, if we have an arc a_t such that $t > k$, then a_t is inside C and if we have an arc a_s such that $s < k - i$, then a_s is outside C.

If $X_i = 1$, then $X_i^{m_i} = 1$. Hence, $X_j = ... = X_l = 1$ and $a_k = a_{k-1} = ... = a_{k-i}$.

Recall that every rational tangle which is a cover edge of a Wada cycle is either an high tail rational tangle, a low tail rational tangle or an high and low tail rational tangle. Note that an high and low tail rational tangle is an high tail rational tangle and a low tail rational tangle.

Now, without loss of generality, we can suppose that X_i is a high tail rational tangle. Thus, we have $X_i^{k_i} = a_p^{-1} a_k$ with $p \geq k$. Hence, a_p is inside C. Moreover,

$1 = X_i^{k_i} = a_p^{-1}a_k$, so $a_p = a_k$. If a_p is a non-bridge arc, then by Lemma 6.6.1 or Lemma 6.6.2, the proof is over. Thus, we suppose that a_p is a bridge. Therefore, there is a rational tangle $X_{i_1}^{k_{i_1}} = a_{p_1}^{-1}a_p = a_p^{-1}a_m$ with $p_1 \geq p \geq m$. So, a_{p_1} is inside C.

If $a_m \in \{a_k, ..., a_{k-1}, ..., a_{k-i}\}$, then $a_m = a_k = a_p$. Thus, $a_{p_1}^{-1}a_p = a_p^{-1}a_m = 1$ and this implies that $a_{p_1} = a_p = a_k$.

Now suppose $a_m \notin \{a_k, ..., a_{k-1}, ..., a_{k-i}\}$. Hence a_m is not on C. But a_{p_1} and a_p are inside C and not on C, thus a_m is also inside C. There is a Wada directed path from a_m to a_1. However, a_1 is outside of C. This implies that there is a Wada directed path from a_m to a_s such that $a_s \in \{a_k, a_{k-1}, ..., a_{k-i}\}$ and $a_s \leq a_m$. Because $a_s \in C$, we have $a_s = a_k = a_p$, thus $a_p = a_s \leq a_m \leq a_p$. So $a_p = a_s = a_m$. This implies that $a_{p_1}^{-1}a_p = a_p^{-1}a_m = 1$ and therefore $a_{p_1} = a_p = a_k$.

If a_{p_1} is a non-bridge arc, then by Lemma 6.6.1 or Lemma 6.6.2, the proof is complete. Now, suppose that a_{p_1} is a bridge. Therefore, there is a rational tangle $X_{i_2}^{k_{i_2}} = a_{p_2}^{-1}a_{p_1} = a_{p_1}^{-1}a_l$ with $p_2 \geq p_1 \geq l$. By the same argument as in the previous paragraph, we get $a_{p_2} = a_{p_1} = a_l$. We continue this process t times until a_{p_t} is a non-bridge arc.

□

Similarly, we have

Lemma 6.7.2. *Let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge link diagram D and $C = (a_k, a_{k-1}, ..., a_{k-i}, a_k)$ be a Wada directed cycle that gives the relation $X_i^{m_i} = X_j^{m_j} ... X_k^{m_k}$ and such that there is an arc a_j with $j \geq k$ outside C and an arc a_q with $q \leq k - i$ inside of C. If $X_i = 1$, then $a_1 = ... = a_{k-i} = a_{k-i+1} = ... = a_n$.*

Thus, directly by the definition of a dichotomic Wada directed cycle we get the following result.

Corollary 6.7.3. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram* D *and* $C = (a_k, a_{k-1}, ..., a_{k-i}, a_k)$ *be a dichotomic Wada directed cycle with cover edge* $X_i = 1$*. Then,* $a_1 = ... = a_n$*.*

Moreover, by Lemma 6.2.6 we have

Lemma 6.7.4. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge link diagram* D*. If* C *is a low and high tail Wada directed cycle with cover edge* $X_i = 1$*. Then,* $a_1 = ... = a_n$*.*

We now introduce an hybrid Wada directed diagram that will play a major role in proving Theorem 6.7.7. The following diagram will be called a *tertiary flat Wada directed cycle.*

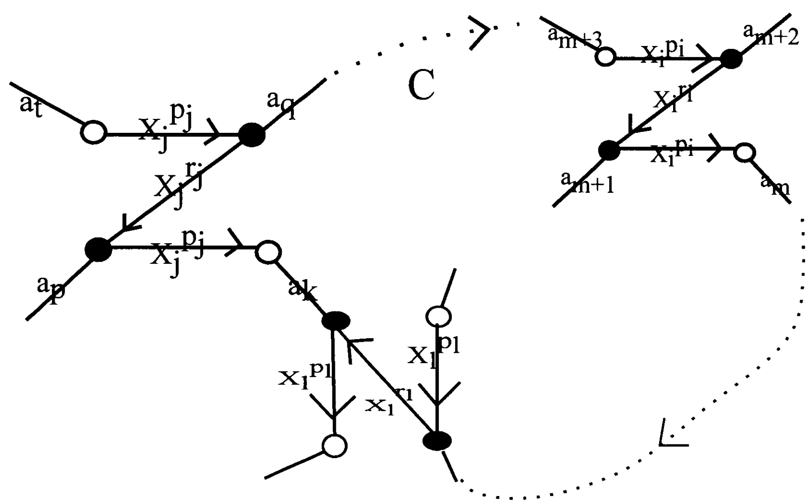


Figure 6.22 Tertiary flat Wada directed cycle

Lemma 6.7.5. *Let* $(\Gamma, <)$ *be a directed Wada graph of a 2-non-bridge directed link diagram and a tertiary flat Wada directed cycle* C *without non-bridge arc.*

Then, either we have a supporting dichotomic cycle D of the flat rational tangle X_i or we have one of the cases A, B, D or E as defined in the proof.

Proof. Because X_i is a flat rational tangle, by Lemma 6.5.2, either there is a dichotomic cycle D for X_i or there is a supporting cycle D with compose cover edge X_i. If D is dichotomic, then the proof is over by Corollary 6.7.3.

Suppose that there is a supporting cycle D with compose cover edge X_i. Thus, we have one of the following hybrid Wada directed diagram.

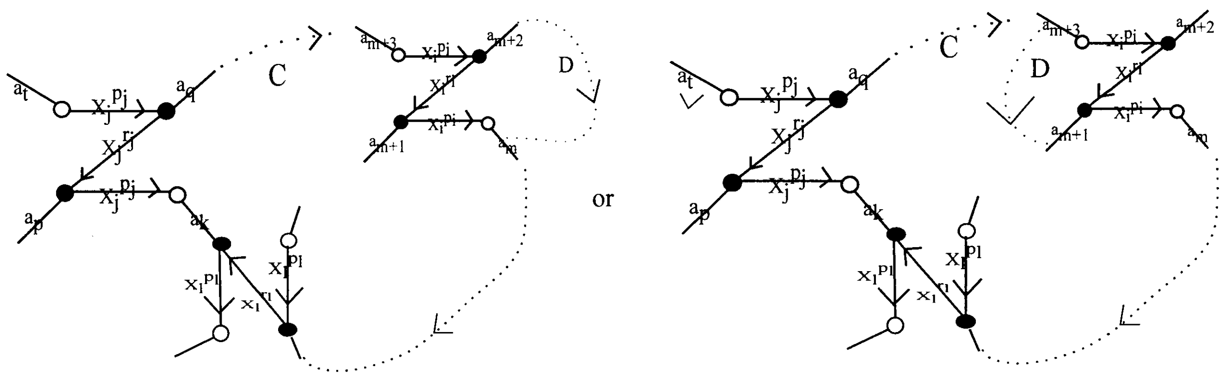


Figure 6.23 Possible supporting cycle D with compose cover edge X_i

We will give the proof for the case on the left. The proof for the case on the right is similar. We have the following possibilities.

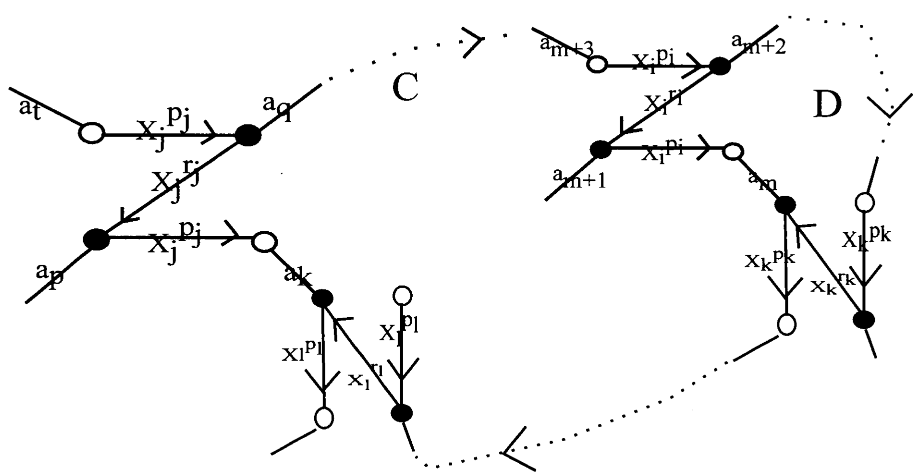


Figure 6.24 Case A for Tertiary flat Wada directed cycle

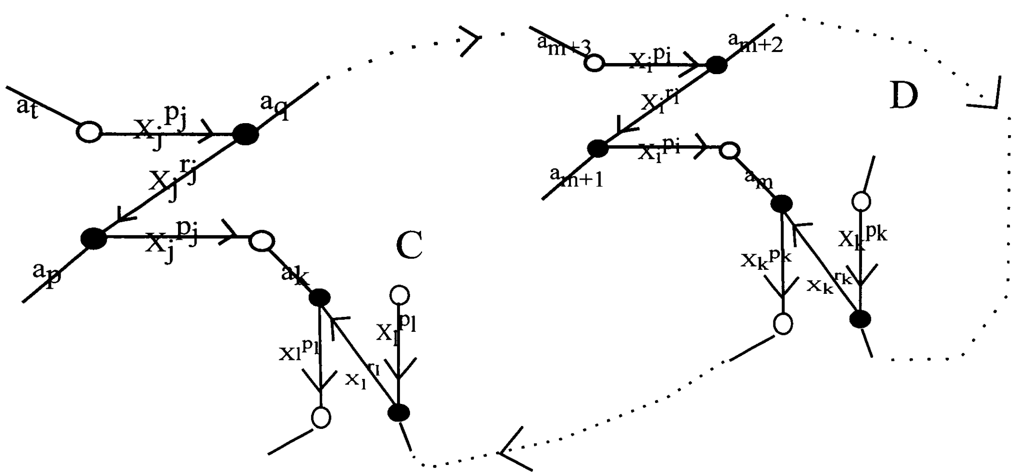


Figure 6.25 Case B for Tertiary flat Wada directed cycle

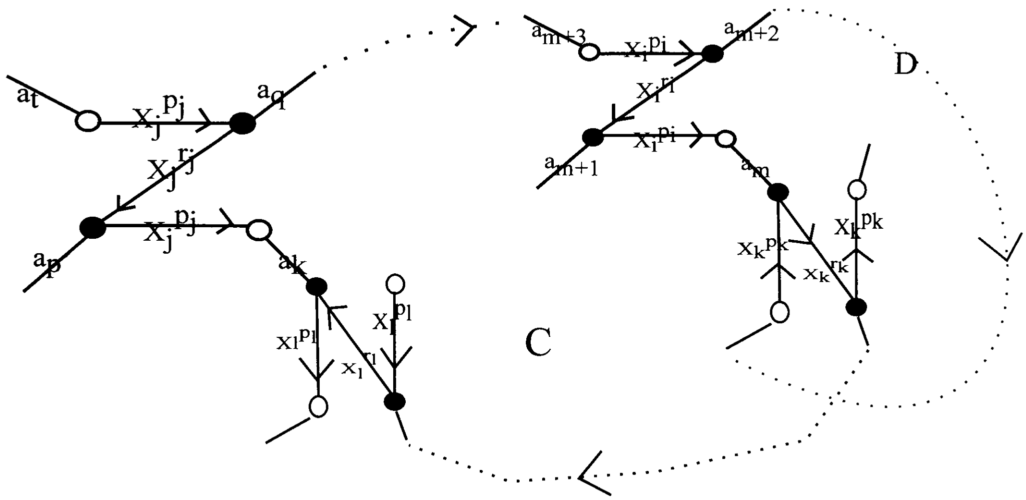


Figure 6.26 Case C for Tertiary flat Wada directed cycle

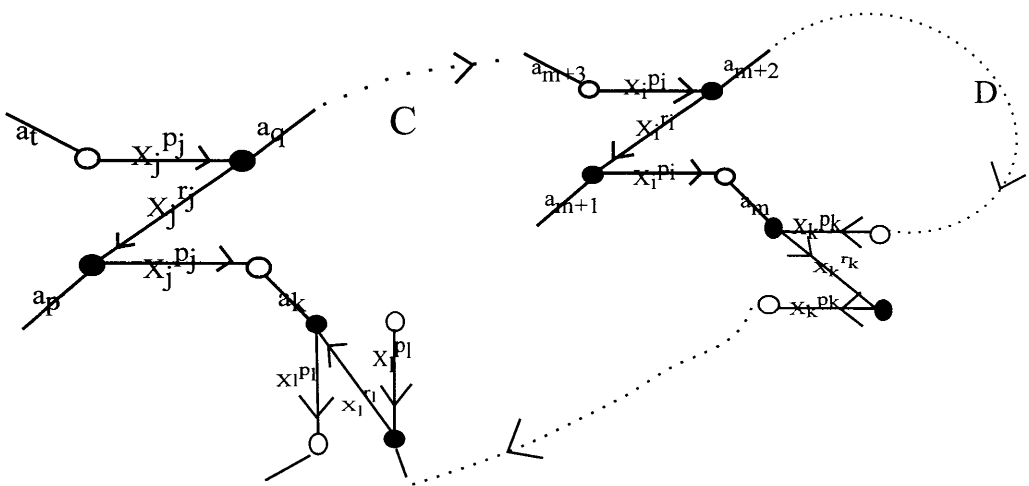


Figure 6.27 Case D for Tertiary flat Wada directed cycle

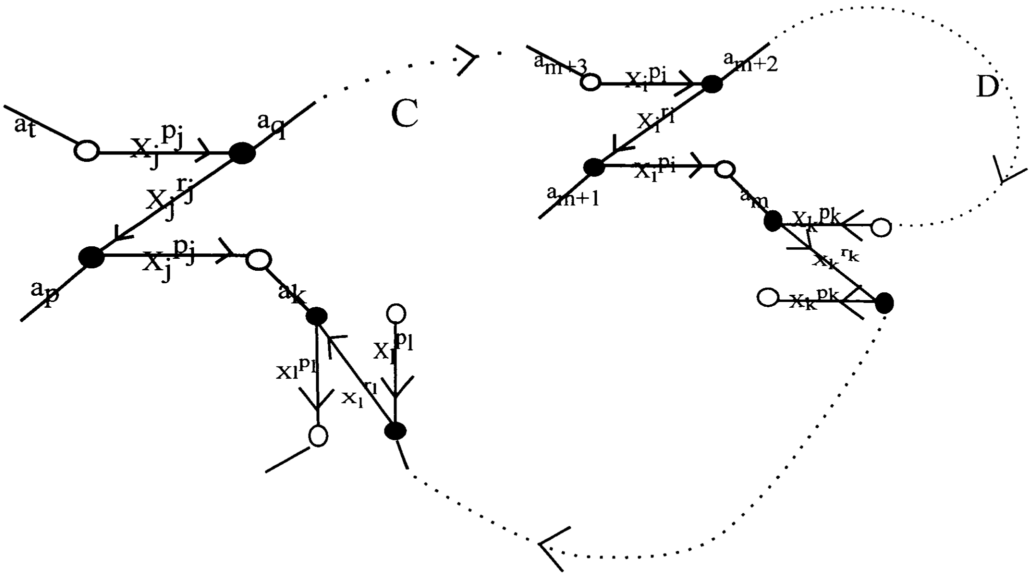


Figure 6.28 Case E for Tertiary flat Wada directed cycle

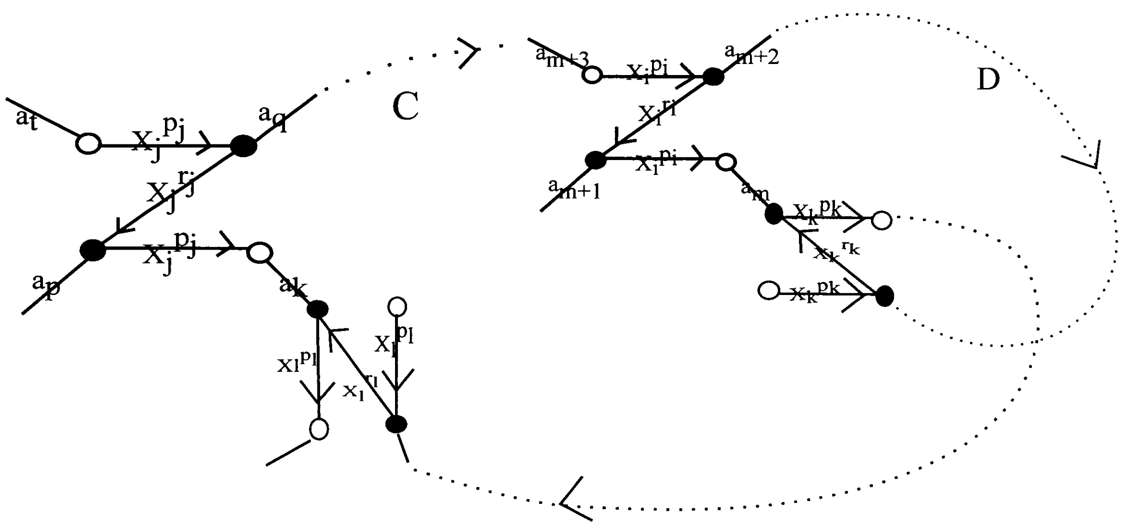


Figure 6.29 Case F for Tertiary flat Wada directed cycle

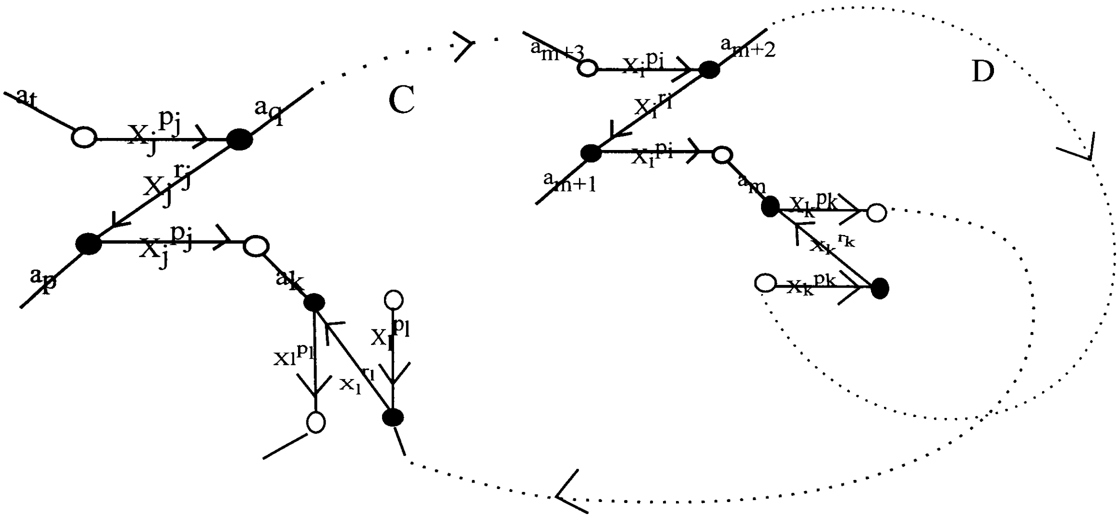
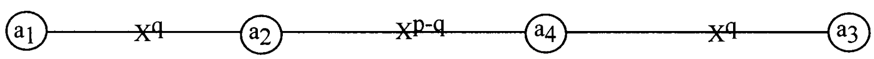


Figure 6.30 Case G for Tertiary flat Wada directed cycle

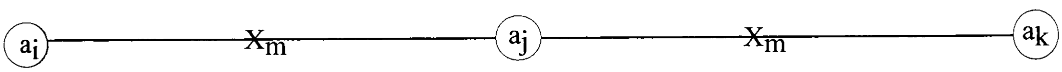
The cases C, F and G are not allowed because they give rise to directed cycle.

□

We give a final lemma before proving the desired result. First, we recall from definition 2.4.3 that for the following Wada rational graph of the rational tangle



a_1 and a_3 are the *final vertices* of the rational tangle X and a_2 and a_4 are the *middle vertices.* Moreover, for the following half-twist



a_i and a_k are the *final vertices* and a_j is the *middle vertex.*

Furthermore, let D be a link diagram and a_i be a vertex not in a 1-tangle T with only one non-bridge arc in D such that neither marked point goes over some arcs in $D \setminus T$. Then, a_i represents an arc in D. If a_i is a final or middle vertex of a rational tangle X, then one end of the arc a_i goes under an arc in the rational tangle X. Similarly, if a_i is a final vertex of the half-twist X, then one end of the arc a_i goes under an arc of the half-twist X. Thus, a vertex of a tangle goes under an arc of the tangle except for a middle vertex of an half-twist. Thus, because every arc goes exactly two times under some arcs and by the definition of the Wada rational graph we have established the following result.

Lemma 6.7.6. *Let Γ be a Wada rational graph of a link diagram and a_i a vertex in Γ. Then, except for the middle vertex of an half-twist tangle, a_i is the vertex of exactly two rational tangle.*

Theorem 6.7.7. *Let $(\Gamma, <)$ be a directed Wada graph of a 2-non-bridge directed link diagram. If there is an X_j such that $X_j = 1$, then $a_1 = ... = a_n$.*

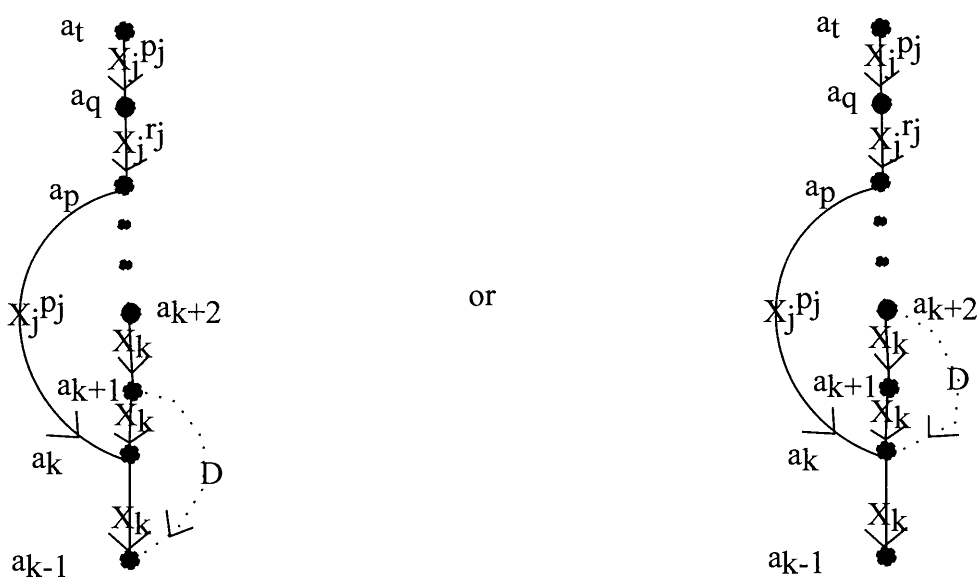
Proof. If a_1 or a_n is a vertex of one of the edges X_j, then by Lemma 6.6.1 or Lemma 6.6.2 the proof is complete. If X_j has a cover edge that gives a high and low tail cycle, then by Lemma 6.7.4 the result is obtained.

The cover edge $X_j^{k_j} = w^+(X_1, ..., X_m)$ is equal to one, thus $w^+(X_1, ..., X_m) = 1$ and every edges in $w^+(X_1, ..., X_m)$ is equal to one. If one of these edges gives a high and low tail cycle, then by Lemma 6.7.4 the result is obtained. If the right edge X_j of $w^+(X_1, ..., X_m)$ is the flat edge of a high tail cycle, then we will look at this last high tail cycle. Again, we will look at the right flat edge of the new cycle and if it is a high tail cycle, we will look at this cycle. We continue this process until there is a high tail cycle, with the right flat edge not from a high tail cycle.

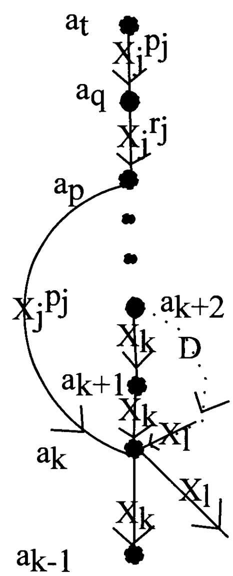
Thus, without loss of generality, we can suppose that X_j is an high tail cycle such that the right edge is not a flat edge of an high tail cycle. Suppose, without loss of generality, that the edges X_j come from a rational tangle (a_k, a_q, a_p, a_t) where $k < p < q < t$.

Suppose there is a flat rational tangle included at the end of C. If it is an half-twist, then, by Lemma 6.5.2, it has a dichotomic supporting cycle. Moreover, because it is included at the end of C, the cover edge is equal to one. Hence, by Lemma 6.7.3, the result is obtained. Again by Lemma 6.5.2, it has either a compose supporting cycle or a dichotomic supporting cycle. If it has a dichotomic supporting cycle, by Lemma 6.7.3, the result is obtained.

So, we suppose the flat rational tangle included at the end of C is not an half-twist and it has a compose supporting cycle. Therefore, we get the following directed Wada graph for C



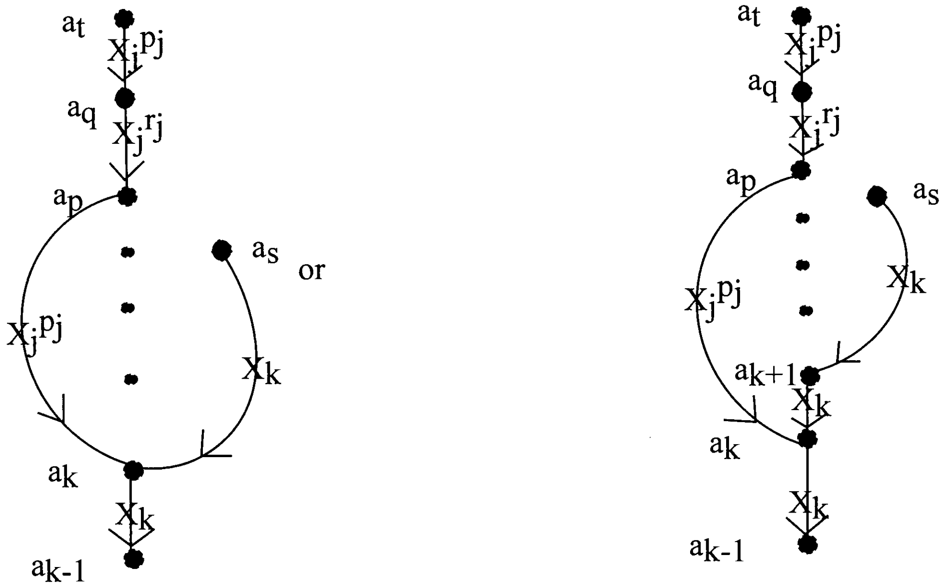
For the left case, we now look at the high tail Wada directed cycle D. For the right case, the vertex a_k is already in the rational tangles X_j and X_k. Therefore, because of Lemma 6.7.6, the rational tangle at the end of D can't end at a_k and must be an half-twist. This implies that we have the following directed Wada graph.



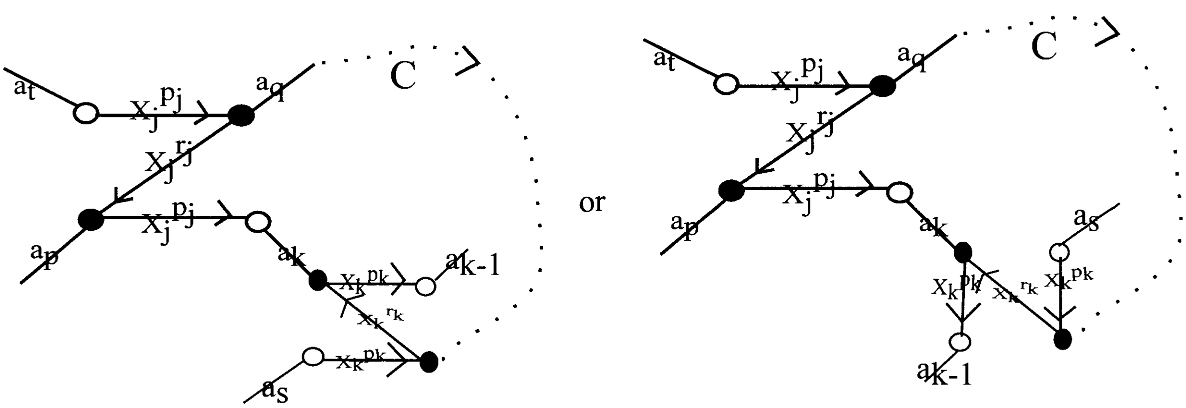
The crossing X_l is a tertiary crossing and so add a tertiary thorn inside C.

Hence, if there is a flat rational tangle included at the end of C, then either the result is obtained, we have another high tail Wada directed cycle D or we have a tertiary thorn inside C.

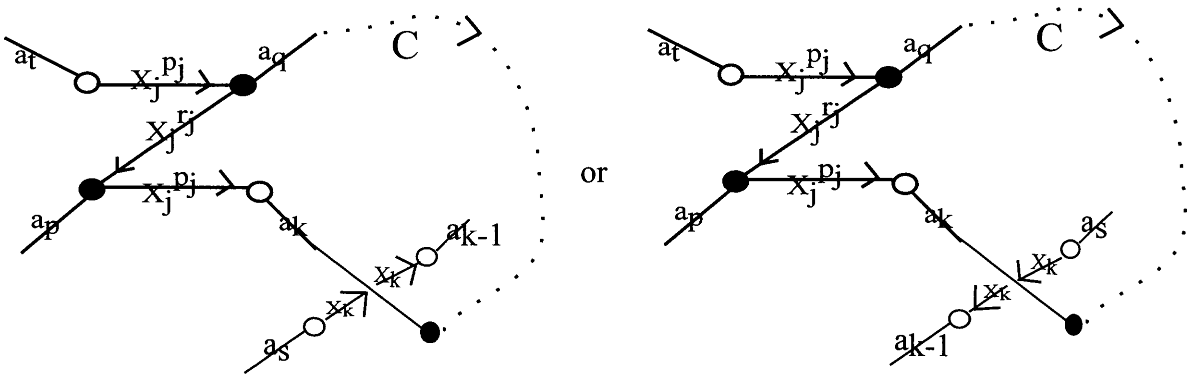
Suppose there is no flat rational tangle included at the end of C. Then, because a_k is at least a bridge and the right edge of C is not the flat edge of a high tail cycle, we get the following directed Wada graph for C :



For the right case, we get the following hybrid Wada diagrams :



For the left case, we get the following hybrid Wada diagrams :



For every possibility, we get a tertiary thorn inside C.

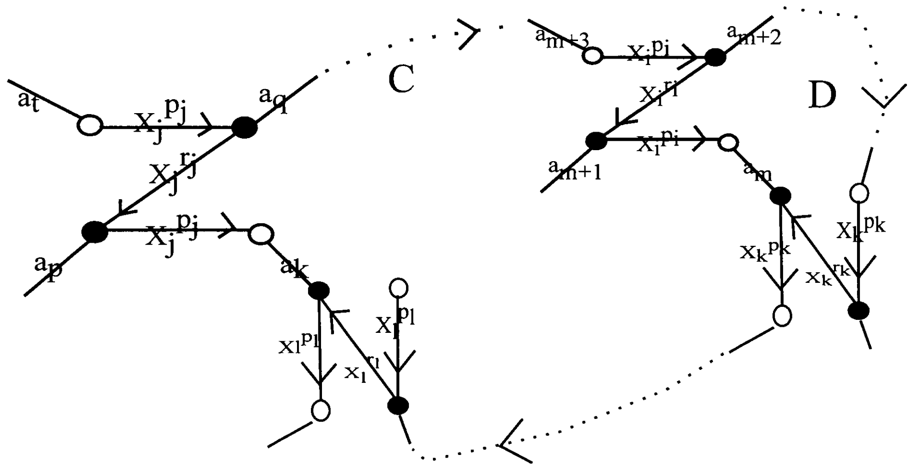
Thus, either the proof is over or we get a tertiary thorn inside C. Suppose C is inside tertiary and by Lemma 6.4.7, either there is a non-bridge arc inside C or there is a flat rational tangle completely included on C.

Suppose there is a non-bridge arc a_i inside C, then by Lemma 6.2.1 because a_t is outside C with $t > q$, a_i is not a maximum. Therefore, a_i is a minimum and C is min primary. Moreover, because a_t is outside C, by Corollary 6.2.4, C is outside max primary and so C is *dichotomic*. Thus, by Lemma 6.7.3, $a_1 = ... = a_n$.

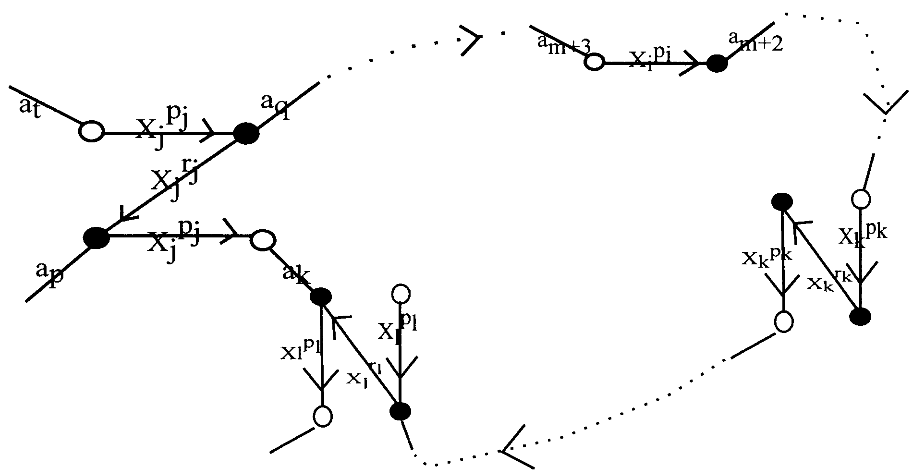
Suppose there is no non-bridge arc inside C and so there is a flat rational tangle X_i completely included on C. Therefore, we have a tertiary flat Wada directed cycle C with X_i as flat rational tangle. Thus, by Lemma 6.7.5, either we have a supporting dichotomic cycle D of the flat rational tangle X_i or we have one of the cases A, B, D or E as in the lemma.

Suppose we have a supporting dichotomic cycle D with X_i as compose cover edge. Because X_i is on C, $X_i = 1$. Thus, by Lemma 6.7.3 $a_1 = ... = a_n$.

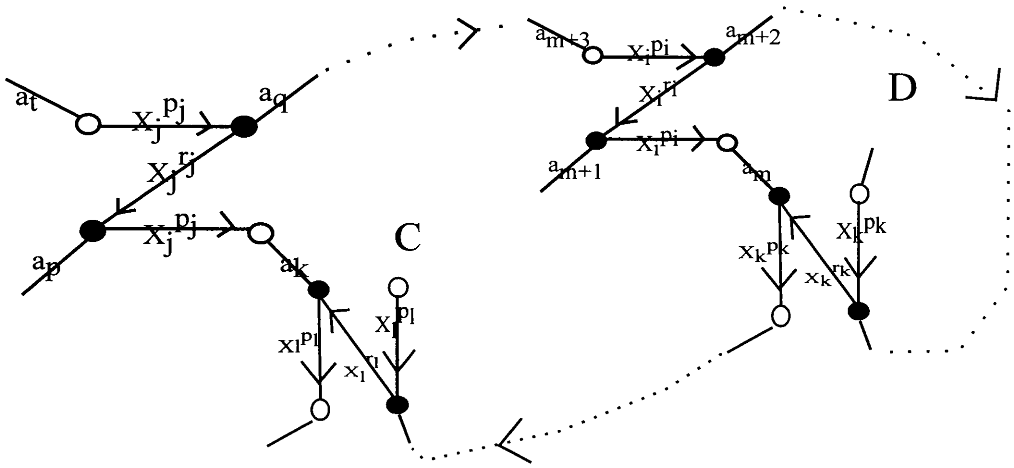
Suppose we are in case A:



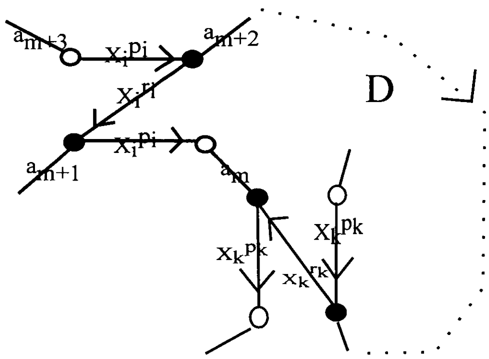
Then, we get a new tertiary flat Wada directed cycle with cover edge equal to one:



Suppose we are in case B:



Then, we get a new tertiary Wada directed cycle with cover edge equal to one:



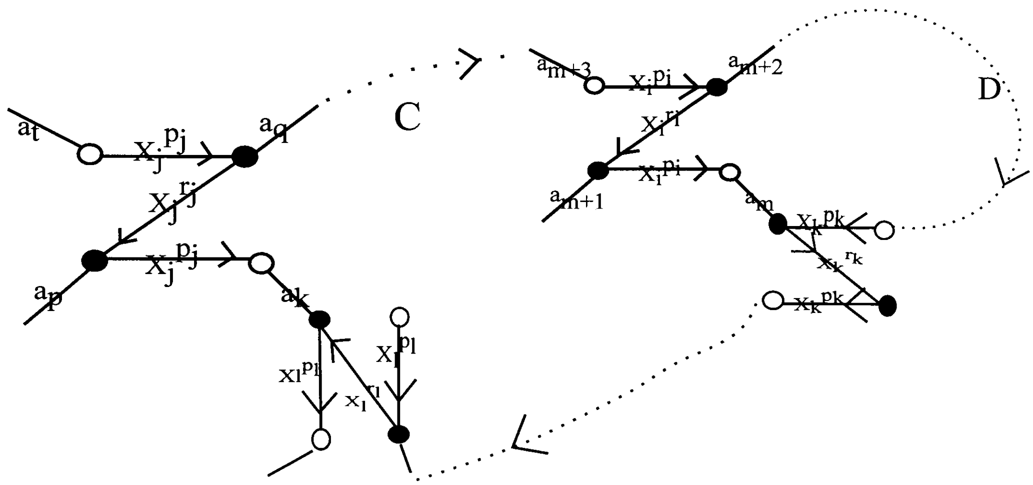
By Lemma 6.4.7, either there is a non-bridge arc inside the cycle D or there is a flat rational tangle completely included on D.

Suppose there is a non-bridge arc a_i inside D, then by Lemma 6.2.1, because a_{m+3} is outside with $a_{m+3} \geq a_{m+2}$, and a_{m+2} is on D, a_i is not a maximum. Therefore, a_i is a minimum and D is min primary. Moreover, because a_{m+3} is outside D, by Corollary 6.2.4, D is outside max primary and so D is *dichotomic*. Thus, by Lemma 6.7.3, $a_1 = ... = a_n$.

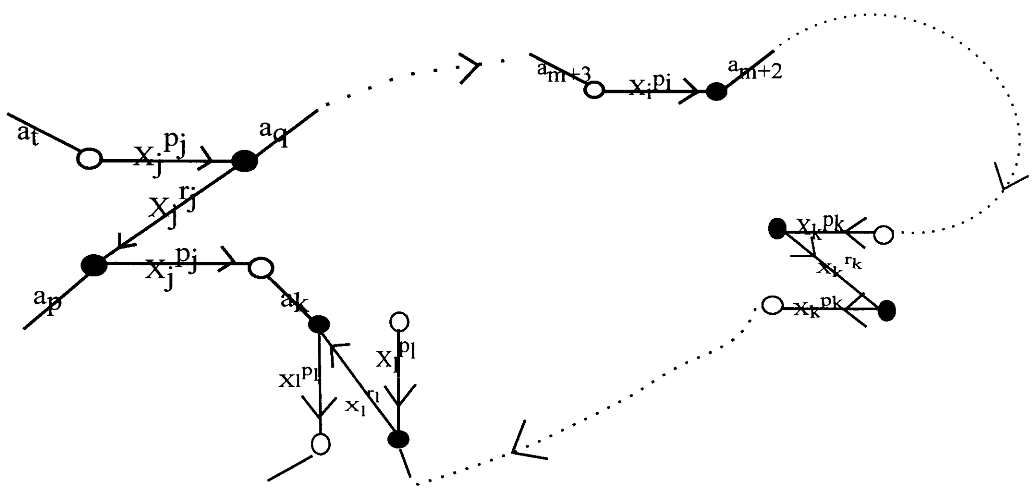
Now suppose there is a flat rational tangle completely included on the cycle D.

Then, D is a new tertiary flat Wada directed cycle with cover edge equal to one.

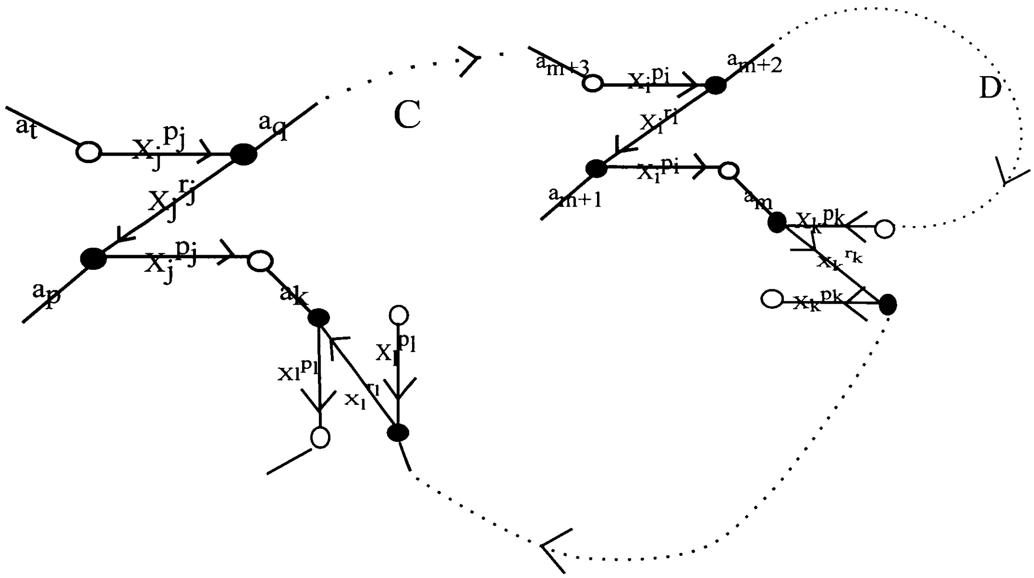
Suppose we are in case D:



Then, we get a new tertiary Wada directed cycle with cover edge equal to one:



Suppose we are in case E.



Then, we get a Wada directed cycle C with two tertiary thorns and one completely included flat rational tangle. Thus, by Lemma 6.4.8, either there is another flat rational tangle completely included on C or there is a non-bridge arc of the link diagram inside C. If there is a non-bridge arc of the link diagram inside C, then with a similar argument as case B, we get $a_1 = ... = a_n$. If there is another flat rational tangle completely included on C, we obtain a new tertiary flat Wada directed cycle with cover edge equal to one. Hence, either there is a new tertiary flat Wada directed cycle with cover edge equal to one or $a_1 = ... = a_n$.

Thus, for all cases, either $a_1 = ... = a_n$ or we have a new tertiary flat Wada directed cycle with cover edge equal to one. If we have a new tertiary flat Wada directed cycle with cover edge equal to one, by Lemma 6.7.5, either we have a supporting dichotomic cycle E of the flat rational tangle X_i or we have one of the cases A, B, D or E as in the lemma. If we have a dichotomic cycle E of the flat rational tangle X_i, then by a previous argument $a_1 = ... = a_n$. If we have one of the cases A, B, D or E as in the lemma, then by a previous argument either

$a_1 = ... = a_n$ or we have a new tertiary flat Wada directed cycle with cover edge equal to one. But, there are only finitely many flat rational tangles, thus at some point we get $a_1 = ... = a_n$.

□

Therefore, by Corollary 4.0.3

Theorem 6.7.8. *Let D be a two non-bridge directed diagram of a non-split directed link L, $(\Gamma, <)$ be a directed Wada graph and suppose that $\pi(D)$ is left-orderable. If there is an edge X_i in $G(\Gamma, <)$ such that $X_i = 1$, then $\pi_1(\Sigma(L))$ is not left orderable.*

CHAPTER VII

FAMILIES OF DIRECTED LINKS FOR WHICH THERE IS A TRIVIAL DIRECTED WADA GROUP

We recall from Corollary 4.0.3 that if a directed Wada group of a directed link diagram is trivial, then the fundamental group of the double branched of the link is not left-orderable. In this chapter, we will prove that the directed Wada group for totally monopositive, $(n-1)$ totally monopositive and fluid $(n-2)$ totally simple monopositive link diagrams are trivial.

The following table should motivate the importance of these families. The good middle triple hop links will be defined at the end of the chapter 9. We notice that the good middle triple hop links in the table below are not directed. For knots of 10 crossings or less, there are 53 non-alternating knots. Of that number, 45 are directed. Moreover, 5 are left-orderable, thus 40 are not left-orderable. Of the 40 non-alternating, directed and not left-orderable knots, 35 are either totally monopositive, $(n-1)$ totally monopositive or steady fluid $(n-2)$ totally simple monopositive.

knots	8 crossings	9 crossings	10 crossings
totally monopositive	8_{21}	9_{45}	10_{127}, 10_{131}, 10_{135}, 10_{138},10_{144},10_{149}
$(n-1)$ totally monopositive	8_{20}	9_{43}, 9_{44}, 9_{49}	10_{125}, 10_{126}, 10_{129}, 10_{130}, 10_{133}, 10_{134}, 10_{137}, 10_{141}, 10_{143}, 10_{146}, 10_{147},10_{148}, 10_{150}, 10_{151}, 10_{157}, 10_{158}, 10_{162}
steady fluid $(n-2)$ totally simple monopositive	8_{19}	9_{42}, 9_{46}, 9_{48}	10_{128}, 10_{136}, 10_{142}
good middle triple hop		9_{47}	10_{159}, 10_{163}, 10_{164}, 10_{165}
Left-orderable			10_{139}, 10_{145}, 10_{152}, 10_{153}, 10_{154}
Not left-orderable and directed			10_{124}, 10_{132}, 10_{140}, 10_{156}
Not left-orderable and non-directed			10_{155}, 10_{160}
unknown and 3-non-bridge			10_{161}

7.1 Totally monopositive links, $(n-1)$ Totally Monopositive links and $(n-2)$ Totally Simple Monopositive links

In this section, we will first introduce the totally simple positive property of links. This larger family includes the totally monopositive links, the $(n-1)$ totally monopositive links and the fluid $(n-2)$ totally simple monopositive links. But before, we recall the following definitions. Let $(\Gamma, <)$ be a directed Wada graph. Every Wada directed cycle $C = (a_i, ..., a_m, a_i)$ in $H(\Gamma, <)$ gives a relation $X_k^{m_k} = w^+(X_1, ..., X_n)$ in $G(\Gamma, <)$ where $X_k^{m_k}$ is the edge or edges from the rational tangle containing a_m and a_i and $w^+(X_1, ..., X_n)$ is a positive word coming from the Wada directed path. We say that X_k is the *cover edge* of C. Moreover, for $w^+(X_1, ..., X_n) = \Pi_{j=0}^{m} X_{i_j}^{k_{i_j}}$, X_{i_m} is defined as the *right edge of* C and X_{i_0} is defined as the *left edge of* C. Furthermore, $X_{i_{m-j}}$ is defined as the $(j+1)$-*right* *edge of* C and X_{i_j} is defined as the $(j+1)$*-left edge of* C. Also, for every X_{i_j} in $w^+(X_1, ..., X_n)$, we define k_{i_j} the *order of* X_{i_j}. Note that because the relation comes from a Wada directed cycle, every order of an edge is equal to the final order, the middle order, the compose order or the total order of the edge.

Definition 7.1.1. Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D and let a relation in $G(\Gamma, <)$ that comes from a Wada directed cycle, be of the type $X_i^{k_i} = w^+(X_1, ..., \hat{X}_i, ..., X_n)$, where $k_i \geq 0$ and there are at least two different edges X_j in $w^+(X_1, ..., \hat{X}_i, ..., X_n)$. Then this relation is called *positive.* An edge X_i for which there exists a positive relation is called a *positive edge.*

Let $X_i^{k_i} = w_i^+(X_1, ..., \hat{X}_i, ..., X_n)$ be a positive relation. If k_i is the final order or the middle order of X_i, then this relation is called *simple positive.* An edge X_i for which there exists a simple positive relation in X_i is called a *simple positive edge.* Moreover, if k_i is the minimum between the final order and the middle order, then this relation is called *monopositive.* An edge X_i for which there exists a monopositive relation in X_i is called a *monopositive edge.* A rational tangle that has a monopositive edge is called a *monopositive rational tangle.* Also, if k_i is not the minimum between the final order and the middle order, then this relation is called *pluripositive.* An edge X_i for which there exists a pluripositive relation in X_i is called a *pluripositive edge.* A rational tangle that has a pluripositive edge is called a *pluripositive rational tangle.*

Let $X_i^{k_i} = w_i^+(X_1, ..., \hat{X}_i, ..., X_n)$ be a positive relation. If the middle edge and the final edge of this relation are not simple positive relation and k_i is the sum of the final and the middle order of X_i , then this relation is called *compose positive* and X_i is a compose positive edge.

It is useful to include them in a same family of relations.

Definition 7.1.2. Let $(\Gamma, <)$ be a directed Wada graph of a link L. If a rational tangle X is either pluripositive or compose positive, then X is called *multipositive.*

Definition 7.1.3. Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D. If all the edges are positive in $G(\Gamma, <)$, then $(\Gamma, <)$ and $G(\Gamma, <)$ are called *totally positive.* If all the directed Wada graphs of D are totally positive, then D *is totally positive.*

Also, if all the edges are simple positive, then $(\Gamma, <)$ and $G(\Gamma, <)$ are called *totally simple positive.* If all the directed Wada graphs of D are totally simple positive, then D *is totally simple positive.*

Moreover, if all the edges are monopositive, then $(\Gamma, <)$ and $G(\Gamma, <)$ are called *totally monopositive.* If all the directed Wada graphs of D are totally monopositive, then D *is totally monopositive.*

We will show that the totally positive property of a directed Wada graph in fact gives the link diagram a property.

Proposition 7.1.4. *Let $(\Gamma, <)$ be a totally positive directed Wada graph of a link diagram D. Then D is directed.*

Proof. Because $(\Gamma, <)$ is totally positive, every edge has a simple positive relation or is a compose positive relation in $G(\Gamma, <)$. If we change the orientation of an edge X_i, then the positive relation $X_i^{k_i} = w_i^+(X_1, ..., X_n)$ becomes $X_i^{-k_i} = w_i^+(X_1, ..., X_n)$. Thus, we obtain the directed cycle $X_i^{k_i} w_i^+(X_1, ..., X_n)$. Therefore, $(\Gamma, <)$ is the only directed Wada graph without directed cycle. This implies that D is directed by Remark 3.0.9. □

Definition 7.1.5. Let $(\Gamma, <)$ be a totally positive directed Wada graph with n rational tangles of a link diagram D. If i edges for $i \leq n$ are multipositive and $n-i$ are monopositive edges, then $(\Gamma, <)$, $G(\Gamma, <)$ and D are *(n-i)totally monopositive.*

We define a smaller family that will give us stronger results.

Definition 7.1.6. Let $(\Gamma, <)$ be a totally simple positive directed Wada graph with n rational tangle of a link diagram D. If i edges for $i \leq n$ are pluripositive and $n-i$ are monopositive edges, then $(\Gamma, <)$, $G(\Gamma, <)$ and D are called $(n-i)$ *totally simple monopositive.*

The monopositive relations will be the key instrument in proving that the graph group is trivial.

We will now continue our example of the knot diagram 8_{21} of Figure 2.11. This knot diagram is totally monopositive. Indeed, by looking at the cycles in the directed Wada graph of 8_{21} we can find the following relations which are all monopositive relations

$$X_1 = X_3^2 X_3^2, X_2 = X_1 X_3^2, X_3 = X_2^2 X_1.$$

Moreover, we can define totally positive links.

Definition 7.1.7. Let L be a link. If L has a totally positive link diagram D, then L is called a *totally positive link.*

If L has a totally simple positive link diagram D, then L is called a *totally simple positive link.*

If L has a totally $(n-i)$ positive link diagram D, then L is called a *totally* $(n-i)$ *positive link.*

Finally, if L has a totally $(n-i)$ simple positive link diagram D, then L is called a *totally* $(n-i)$ *simple positive link.*

Thus, by Theorem 6.7.8, we have:

Theorem 7.1.8. *Let D be a totally positive link diagram of the non-split link L and $(\Gamma, <)$ a directed Wada graph. Suppose that $\pi(D)$ is left-orderable. If there is a generator X_i in $G(\Gamma, <)$ such that $X_i = 1$, then $\pi_1(\Sigma(L))$ is not left orderable.*

7.2 Totally positive groups

We will define families of groups for which left-orderability will imply that they have a trivial generator. Then we will show that directed Wada groups of totally monopositive, totally $(n-1)$ positive and totally $(n-2)$ simple positive fluid link diagram are elements of these families. Thus, by Theorem 7.1.8, we will have the result on the non-left-orderability.

Inspired by the definition in the directed Wada group, we introduce the following definition for groups.

Definition 7.2.1. Let G be a group with generators $X_1, ..., X_n$. For a relation $w^+(X_1, ..., X_n) = \Pi_{j=0}^m X_{i_j}^{k_{i_j}}$, X_{i_m} is defined as the *right generator of* $w^+(X_1, ..., X_n)$ with *order* k_{i_m} and X_{i_0} is defined as the *left generator of* $w^+(X_1, ..., X_n)$ with *order* k_{i_0}. Furthermore, $X_{i_{m-j}}$ is defined as the $(j+1)$-*right generator of* $w^+(X_1, ..., X_n)$ with *order* $k_{i_{m-j}}$ and X_{i_j} is defined as the $(j+1)$-*left generator of* $w^+(X_1, ..., X_n)$ with *order* k_{i_j}.

Definition 7.2.2. Let G be a group generated by $X_1, ..., X_n$. A relation $X_i^{h_i} = w_i^+(X_1, ..., X_n)$ is called a *positive relation of* X_i if $h_i \geq 0$ and the positive word $w_i^+(X_1, ..., X_n)$ has at least two different generators and X_i is neither the right nor the left generator of $w_i^+(X_1, ..., X_n)$. Suppose G has a positive relation for every X_i, then G is called a *totally positive group.*

Let $X_i^{h_i} = w_i^+(X_1, ..., X_n)$ for $1 \leq i \leq n$ be positive relations in a totally positive group. We look at the order r_{j_i} of the left generator X_{j_i} of every positive word

$w_i^+(X_1, ..., X_n)$. If $h_i \leq r_{j_i}$ for every $j_i = i$, then we say that the relation $X_i^{h_i} = w_i^+(X_1, ..., X_n)$ is a *left positive relation.*

Let $X_i^{h_i} = w_i^+(X_1, ..., X_n)$ for $1 \leq i \leq n$ be left positive relations in a totally positive group. We look at the order s_{j_i} of the 2-left generator X_{j_i} of every positive word $w_i^+(X_1, ..., X_n)$. If $X_i^{h_i} = w_i^+(X_1, ..., X_n)$ is a left positive relation and $h_i \leq s_{j_i}$ for every $j_i = i$, then we say that the relation $X_i^{h_i} = w_i^+(X_1, ..., X_n)$ is a *2 left positive relation.*

Similarly, we define *right positive relation*, *2 right positive relation.*

Definition 7.2.3. Let G a group generated by $X_1, ..., X_n$ be a totally positive group. If every positive relation is a left positive relation, then G is called a *totally left positive group.*

If every positive relation is a 2 left positive relation, then G is called a *totally 2 left positive group.*

Similarly, we define *totally right positive group* and *totally 2 right positive group.*

Definition 7.2.4. Let G a group generated by $X_1, ..., X_n$ be a totally positive group. If all but i positive relations are left positive relation, then G is called a *totally* $(n-i)$ *left positive group.*

If all but i positive relations are 2 left positive relation, then G is called a *totally* $(n-i)$ *2 left positive group.*

Similarly, we define *totally* $(n-i)$ *right positive group* and *totally* $(n-i)$ *2 right positive group.*

The preceding families are important, because the directed Wada group of totally positive, totally monopositive and totally $(n-i)$ monopositive link diagram will belong to these families. By the definition of positive relation, we have the following result.

Lemma 7.2.5. *Let D be a totally positive link diagram and $(\Gamma, <)$ a totally positive directed Wada graph. Then $G(\Gamma, <)$ is a totally positive group.*

Moreover,

Lemma 7.2.6. *Let D be a totally monopositive positive link diagram and $(\Gamma, <)$ be a totally positive directed Wada graph. Then $G(\Gamma, <)$ is a totally 2 left positive group and a totally 2 right positive group.*

Proof. Let $X_i^{r_i} = w_i^+(X_1, ..., X_n)$ be a monopositive relation from $G(\Gamma, <)$. By definition, there are at least two different generators in $w_i^+(X_1, ..., X_n)$. Moreover, r_i is the smallest order of the edge X_i. Thus, $X_i^{r_i} = w_i^+(X_1, ..., X_n)$ is a 2 left positive relation. Therefore, $G(\Gamma, <)$ is a totally 2 left positive group.

Similarly, we prove that $G(\Gamma, <)$ is a totally 2 right positive group. □

Similarly,

Lemma 7.2.7. *Let D be a totally $(n - i)$ monopositive positive link diagram. Then $G(\Gamma, <)$ is a totally $(n - i)$ 2 left positive group and a totally $(n - i)$ 2 right positive group.*

7.3 Totally Monopositive links and non Left-Orderability

The totally monopositive links are an important class of links, because the fundamental group of their double branched cover is not left-orderable. First, we recall that totally monopositive links are directed links. Thus, to prove the non-left-orderability, by Theorem 6.7.8 we only have to obtain a trivial generator in the Wada directed group. By Lemma 7.2.6, the Wada directed group of totally monopositive links are totally 2 left positive group. In this section, we will prove that a left-orderable totally 2 left positive group has a trivial generator.

We already know that a link with only one rational tangle is alternating. It is also known that a link with exactly two rational tangles is alternating.

Proposition 7.3.1. *(Ernst & Summers, 1990) Let L be a link with a link diagram with two rational tangles. Then, L is alternating.*

Hence, by Theorem 1.0.1, one has have the following result.

Proposition 7.3.2. *Let L be a link with a non-split link diagram with one or two rational tangles. Then, the fundamental group of the double branched cover of L is not left-orderable.*

Thus, only links with 3 or more rational tangles will be interesting to study. So for the remainder of this thesis, we will suppose that links have 3 or more rational tangles.

We now prove a lemma providing that will be used often in the following results.

Lemma 7.3.3. *Let G be a totally positive group generated by $X_1, ..., X_n$. Let $(G, <)$ be a left order on G such that $X_i \leq 1$ for every $1 \leq i \leq n$. If we can obtain a relation $X_i^k = w^+(X_1, ..., X_n)$ with $k \leq 0$ from G, then $X_i = w^+(X_1, ..., X_n) = 1$ and thus for every generator X_j in $w^+(X_1, ..., X_n)$, we conclude $X_j = 1$.*

Proof. Because $X_i \leq 1$, we have $X_i^k \geq 1$. Moreover, because every edge $X_j \leq 1$, it implies that $w^+(X_1, ..., X_n) \leq 1$. Therefore, $X_i = w^+(X_1, ..., X_n) = 1$ and thus for every edge X_j in $w^+(X_1, ..., X_n)$, we have $X_j = 1$. □

Proposition 7.3.4. *Let G be a totally positive group generated by $X_1, ..., X_n$ where $n \geq 3$ and $(G, <)$ be a left order on G such that $X_i \leq 1$ for every $1 \leq i \leq n$. If G is a totally left (resp. right) positive group, then at least one $X_i = 1$.*

Proof. Without loss of generality, we will prove the result for a totally left positive group. The proof is similar for totally right positive group.

Let $n \geq 3$ be the number of generators. We will show that if we have m left positive relations $X_i^{r_i} = X_{k_i}^{p_{k_i}} w_i^+(X_1, ..., X_n)$ where $1 \leq i \leq m \leq n$ and $1 \leq k_i \leq m$, then $X_i = 1$ for at least one $1 \leq i \leq m$. We will complete the proof of this result by induction on the number m. Let $m = 3$ and

$$X_1^{r_1} = X_i^{p_i} u^+(X_1, ..., X_n) \tag{7.1}$$

$$X_2^{r_2} = X_j^{p_j} v^+(X_1, ..., X_n) \tag{7.2}$$

$$X_3^{r_3} = X_k^{p_k} w^+(X_1, ..., X_n) \tag{7.3}$$

be left positive relations.

If $i = 1$, then $X_1^{r_1 - p_1} = u^+(X_1, ..., X_n)$. But $r_1 - p_1 \leq 0$, because it comes from a left positive relation. Thus, by Lemma 7.3.3, $X_1 = u^+(X_1, ..., X_n) = 1$. The proof is similar, if $j = 2$ or $k = 3$.

Suppose without loss of generality that $i = 2$. Then by equations 7.1 and 7.2 , $X_1^{r_1} = X_j^{p_j} v^+(X_1, ..., X_n) X_2^{p_2 - r_2} u^+(X_1, ..., X_n)$.

Moreover, suppose $j = 1$, then $X_1^{r_1 - p_1} = v^+(X_1, ..., X_n) X_2^{p_2 - r_2} u^+(X_1, ..., X_n)$. However $r_1 - p_1 \leq 0$ and $p_2 - r_2 \geq 0$, thus by Lemma 7.3.3, $X_1 = 1$ and

$v^+(X_1, ..., X_n)X_2^{p_2-r_2}u^+(X_1, ..., X_n) = 1.$

Now, suppose $j = 3$. Then by equations 7.1 and 7.3

$$\begin{aligned} X_1^{r_1} &= X_3^{p_3}v^+(X_1, ..., X_n)X_2^{p_2-r_2}u^+(X_1, ..., X_n) \\ &= X_k^{p_k}w^+(X_1, ..., X_n)X_3^{p_3-r_3}v^+(X_1, ..., X_n)X_2^{p_2-r_2}u^+(X_1, ..., X_n). \end{aligned}$$

If $k = 1$, then

$$X_1^{r_1} = X_1^{p_1}w^+(X_1, ..., X_n)X_3^{p_3-r_3}v^+(X_1, ..., X_n)X_2^{p_2-r_2}u^+(X_1, ..., X_n).$$

Thus, $X_1^{r_1-p_1} = w^+(X_1, ..., X_n)X_3^{p_3-r_3}v^+(X_1, ..., X_n)X_2^{p_2-r_2}u^+(X_1, ..., X_n)$ and by Lemma 7.3.3 $X_1 = w^+(X_1, ..., X_n)X_3^{p_3-r_3}v^+(X_1, ..., X_n)X_2^{p_2-r_2}u^+(X_1, ..., X_n) = 1$.

If $k = 2$, then by equations 7.2 and 7.3

$$\begin{aligned} X_2^{r_2} &= X_3^{p_3}v^+(X_1, ..., X_n) \\ &= X_2^{p_2}w^+(X_1, ..., X_n)X_3^{p_3-r_3}v^+(X_1, ..., X_n) \end{aligned}$$

and so $X_2^{r_2-p_2} = w^+(X_1, ..., X_n)X_3^{p_3-r_3}v^+(X_1, ..., X_n)$ and again by Lemma 7.3.3 $X_2 = 1$.

This completes the proof for $m = 3$.

Now suppose the result is true for $m = n - 1$ left positive relations $X_i^{r_i} = X_{k_i}^{p_{k_i}}w_i^+(X_1, ..., X_n)$ where $1 \leq i \leq m = n - 1$ and $1 \leq k_i \leq m = n - 1$. We investigate the case where there are $m = n$ left positive relations $X_i^{r_i} = X_{k_i}^{p_{k_i}}w_i^+(X_1, ..., X_n)$ where $1 \leq i \leq n = m$ and $1 \leq k_i \leq n = m$. We have the left positive relation $X_n^{r_n} = X_{k_n}^{p_{k_n}}w_n^+(X_1, ..., X_n)$. If $X_n^{r_n} = X_n^{p_n}w_n^+(X_1, ..., X_n)$, then $X_n^{r_n-p_n} = w_n^+(X_1, ..., X_n)$ with $r_n - p_n \leq 0$ because it comes from a left positive relation. Thus, by Lemma 7.3.3, $X_n = 1 = w_n^+(X_1, ..., X_n)$.

Suppose $X_{k_n}^{p_{k_n}} \neq X_n^{p_n}$. Recall that $X_n^{r_n} = X_{k_n}^{p_{k_n}} w_n^+(X_1, ..., X_n)$. For every $X_i^{r_i} = X_{k_i}^{p_{k_i}} w_i^+(X_1, ..., X_n)$ such that $k_i = n$, since $X_n^{p_n} = X_{k_n}^{p_{k_n}} w_n^+(X_1, ..., X_n) X_n^{p_n - r_n}$, we obtain $X_i^{r_i} = X_{k_n}^{p_{k_n}} w_n^+(X_1, ..., X_n) X_n^{p_n - r_n} w_i^+(X_1, ..., X_n)$. Also, $p_n - r_n \geq 0$ because $X_n^{r_n} = X_{k_n}^{p_{k_n}} w_n^+(X_1, ..., X_n)$ is a left positive relation. Thus, now we have $n-1$ left positive relations $X_i^{r_i} = X_{k_i}^{p_{k_i}} w_i^+(X_1, ..., X_n)$ where $1 \leq i \leq n-1$ and $1 \leq k_i \leq n-1$. Moreover, none of the $X_{k_i}^{p_{k_i}}$ is equal to $X_n^{p_n}$. This implies by induction that there is an $X_i = 1$ for $1 \leq i \leq n-1$.

The result is obtained similarly for totally right positive group. □

By construction of the Wada directed group of a directed link diagram, Lemma 7.2.6 and because totally 2 left positive group are totally left positive group, we obtain the following result:

Lemma 7.3.5. *Let D be a directed $(n-2)$ of L with at least three rational tangle and $\Gamma(D)$ be the Wada rational graph. Suppose $\pi(D)$ is left-orderable. If $(\Gamma, <)$ is a totally monopositive directed Wada graph, then $G(\Gamma, <)$ is a left-orderable totally positive group generated by $X_1, ..., X_n$ where $n \geq 3$ such that $X_i \leq 1$ for every $1 \leq i \leq n$.*

Thus, combining the previous lemma, Proposition 7.3.2, Proposition 7.3.4 and Theorem 7.1.8 we have the following result.

Theorem 7.3.6. *If L is a totally monopositive, 2-non-bridge and non-split link, then the fundamental group of the double branched cover of L is not left-orderable.*

We recall that the knot 8_{21} is totally monopositive, therefore the fundamental group of the double branched cover of the knot 8_{21} is not left-orderable.

7.4 $(n-1)$ Totally monopositive links and non Left Orderability

In this section, we will prove that the directed Wada group of $(n-1)$ totally monopositive links is trivial and thus that the fundamental group of the double branched cover of $(n-1)$ totally monopositive links is not left orderable. To prove this, we will first prove that the directed Wada group of $(n-1)$ totally monopositive links are $(n-1)$ totally 2 left positive group. Then, we will prove that a left-orderable $(n-1)$ totally 2 left positive group has at least one trivial generator.

Proposition 7.4.1. *Let G be a totally positive group generated by $X_1, ..., X_n$ where $n \geq 3$ and $(G, <)$ be a left order on G such that $X_i \leq 1$ for every $1 \leq i \leq n$. If G is a $(n-1)$ totally 2 left (resp. right) positive group, then at least one $X_i = 1$.*

Proof. We will prove the result for $(n-1)$ totally 2 left positive group. The proof for $(n-1)$ right 2 totally positive directed Wada graph is similar.

Let $n \geq 3$ be the number of generators. We will show that if we have $m-1$ 2 left positive relations $X_i^{r_i} = X_{k_i}^{p_{k_i}} X_{l_i}^{q_{l_i}} w_i^+(X_1, ..., X_n)$ where $1 \leq i \leq m \leq n$, $1 \leq k_i \leq m$ and $1 \leq l_i \leq m$ and one positive relation $X_j^{p_j} = X_{k_j}^{p_{k_j}} w_i^+(X_1, ..., X_n)$ where $1 \leq j \leq m \leq n$ and $1 \leq k_j \leq m$, then $X_i = 1$ for at least one $1 \leq i \leq m$. We will complete the proof of this result by induction on the number m. Let $m = 3$ and

$$X_1^{r_1} = X_i^{p_i} u + (X_1, ..., X_n) \tag{7.4}$$

$$X_2^{r_2} = X_j^{p_j} v^+(X_1, ..., X_n) \tag{7.5}$$

be 2 left positive relation and

$$X_3^{p_3} = X_k^{p_k} w^+(X_1, ..., X_n) \tag{7.6}$$

be a positive relation.

Suppose that $i = 1$. Then, because Equation 7.4 is a 2 left positive relation, we obtain $r_1 - p_1 \leq 0$ and by Lemma 7.3.3, we get $X_1 = 1 = u^+(X_1, ..., X_n)$.

Similarly, the result is obtained if $j = 2$.

If we have $X_1^{r_1} = X_2^{p_2} u_1^+(X_1, ..., X_n)$ and $X_2^{r_2} = X_1^{p_1} u_2^+(X_1, ..., X_n)$, then

$$X_1^{r_1 - p_1} = u_2^+(X_1, ..., X_n) X_2^{p_2 - r_2} u_1^+(X_1, ..., X_n).$$

Thus, because $u_2^+(X_1, ..., X_n)$ and $u_1^+(X_1, ..., X_n)$ are positive words and $r_1 - p_1 \leq 0$ and $p_2 - r_2 \geq 0$ because both come from a 2 left positive relation, by Lemma 7.3.3, we have $X_1 = X_2 = 1$.

Therefore, we only have to look at the following cases:

A: $X_1^{r_1} = X_2^{p_2} u_1^+(X_1, ..., X_n)$, $X_2^{r_2} = X_3^{r_3} u_2^+(X_1, ..., X_n)$ and $X_3^{p_3} = X_1^{p_1} u_3^+(X_1, ..., X_n)$;

B: $X_1^{r_1} = X_2^{p_2} u_1^+(X_1, ..., X_n)$, $X_2^{r_2} = X_3^{r_3} u_2^+(X_1, ..., X_n)$ and $X_3^{p_3} = X_2^{p_2} u_3^+(X_1, ..., X_n)$;

$C:$ $X_1^{r_1} = X_3^{r_3} u_1^+(X_1, ..., X_n)$, $X_2^{r_2} = X_3^{r_3} u_2^+(X_1, ..., X_n)$ and $X_3^{p_3} = X_1^{p_1} u_3^+(X_1, ..., X_n)$;

D: $X_1^{r_1} = X_3^{r_3} u_1^+(X_1, ..., X_n)$, $X_2^{r_2} = X_3^{r_3} u_2^+(X_1, ..., X_n)$ and $X_3^{p_3} = X_2^{p_2} u_3^+(X_1, ..., X_n)$;

E: $X_1^{r_1} = X_3^{r_3} u_1^+(X_1, ..., X_n)$, $X_2^{r_2} = X_1^{r_1} u_2^+(X_1, ..., X_n)$ and $X_3^{p_3} = X_2^{p_2} u_3^+(X_1, ..., X_n)$;

F: $X_1^{r_1} = X_3^{r_3} u_1^+(X_1, ..., X_n)$, $X_2^{r_2} = X_1^{r_1} u_2^+(X_1, ..., X_n)$ and $X_3^{p_3} = X_1^{p_1} u_3^+(X_1, ..., X_n)$;

The proof of E is similar to the proof of A, the one for F is similar to the one for B and the proof of D is similar to the proof of C. Therefore, we will only give the proofs for the cases A, B and C

A) Let

$$X_1^{r_1} = X_2^{p_2} u_1^+(X_1, ..., X_n) = X_2^{p_2} X_i^{p_i} u_1'^+(X_1, ..., X_n) \tag{7.7}$$

$$X_2^{r_2} = X_3^{r_3} u_2^+(X_1, ..., X_n) = X_3^{r_3} X_j^{p_j} u_2'^+(X_1, ..., X_n) \tag{7.8}$$

be 2 left positive relations and

$$X_3^{p_3} = X_1^{p_1} u_3^+(X_1, ..., X_n) \tag{7.9}$$

be a positive relation.

Then by the previous equations,

$$X_3^{p_3 - r_3} = X_j^{p_j} u_2'^+(X_1, ..., X_n) X_2^{p_2 - r_2} u_1^+(X_1, ..., X_n) X_1^{p_1 - r_1} u_3^+(X_1, ..., X_n). \tag{7.10}$$

Suppose $j = 1$. Then by equation 7.7 and the previous equation,

$$X_3^{p_3 - r_3} = X_2^{p_2} u_1^+(X_1, ..., X_n) X_1^{p_1 - r_1} u_2'^+(X_1, ..., X_n) X_2^{p_2 - r_2} u_1^+(X_1, ..., X_n) X_1^{p_1 - r_1} u_3^+(X_1, ..., X_n).$$

Moreover, by equation 7.8

$$\begin{aligned} X_3^{p_3-r_3} =& X_3^{r_3}u_2^+(X_1,...,X_n)X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_2'^+(X_1,...,X_n) \\ & X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_3^+(X_1,...,X_n). \end{aligned}$$

Therefore,

$$\begin{aligned} X_3^{p_3-2r_3} =& X_1^{p_1}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_2'^+(X_1,...,X_n) \\ & X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_3^+(X_1,...,X_n). \end{aligned}$$

There exists a $k \in \mathbb{N}$ such that $p_3 \leq kr_3$. By the same argument done k times, we obtain

$$X_3^{p_3-kr_3} = u_2^+(X_1,...,X_n)w'^+(X_1,...,X_n)$$

where $w'^+(X_1,...,X_n)$ is a positive word. Thus, $p_3 - kr_3 \leq 0$ and so by Lemma 7.3.3 $X_3 = 1 = u_2^+(X_1,...,X_n) = w'^+(X_1,...,X_n)$.

Now suppose $j = 2$. Then by equations 7.8 and 7.10

$$X_3^{p_3-r_3} = X_2^{p_2}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_3^+(X_1,...,X_n).$$

Moreover, by equation 7.7

$$X_3^{p_3-r_3} = X_3^{r_3}u_2^+(X_1,...,X_n)X_2^{p_2-r_2}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_3^+(X_1,...,X_n).$$

Therefore,

$$X_3^{p_3-2r_3} = u_2^+(X_1,...,X_n)X_2^{p_2-r_2}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_3^+(X_1,...,X_n).$$

There exists a $k \in \mathbb{N}$ such that $p_3 \leq kr_3$. By the same argument done k times, we

obtain

$$X_3^{p_3-kr_3} = u_2^+(X_1,...,X_n)w'^+(X_1,...,X_n)$$

where $u'^+(X_1,...,X_n)$ is a positive word. Thus, $p_3 - kr_3 \leq 0$ and so by Lemma 7.3.3 $X_3 = 1 = u_2^+(X_1,...,X_n) = w'^+(X_1,...,X_n)$.

B) Let

$$X_1^{r_1} = X_2^{p_2}u_1^+(X_1,...,X_n) = X_2^{p_2}X_i^{p_i}u_1'^+(X_1,...,X_n) \tag{7.11}$$

$$X_2^{r_2} = X_3^{r_3}u_2^+(X_1,...,X_n) = X_3^{r_3}X_j^{p_j}u_2'^+(X_1,...,X_n) \tag{7.12}$$

be 2 left positive relations and

$$X_3^{p_3} = X_2^{p_2}u_3^+(X_1,...,X_n) \tag{7.13}$$

be a 2 left positive relation.

Then, by the previous equations

$$X_3^{p_3-r_3} = X_j^{p_j}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n). \tag{7.14}$$

Suppose $j = 1$. Then, by equations 7.11 and 7.14,

$$X_3^{p_3-r_3} = X_2^{p_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n).$$

Moreover by equation 7.12,

$$X_3^{p_3-r_3} = X_3^{r_3}u_2^+(X_1,...,X_n)X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n).$$

Therefore,

$$X_3^{p_3-2r_3} = X_1^{p_1}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_1^+(X_1,...,X_n)X_1^{p_1-r_1}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n).$$

There exists a $k \in \mathbb{N}$ such that $p_3 \leq kr_3$. By the same argument done k times, we obtain

$$X_3^{p_3-kr_3} = u_2^+(X_1,...,X_n)v'^+(X_1,...,X_n)$$

where $v'^+(X_1,...,X_n)$ is a positive word. Thus, $p_3 - kr_3 \leq 0$ and so by Lemma 7.3.3, $X_3 = 1 = u_2^+(X_1,...,X_n) = v'^+(X_1,...,X_n)$.

Suppose $j = 2$. Then by equations 7.12 and 7.14,

$$X_3^{p_3-r_3} = X_2^{p_2}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n).$$

Moreover, again by equation 7.12,

$$X_3^{p_3-r_3} = X_3^{r_3}u_2^+(X_1,...,X_n)X_2^{p_2-r_2}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n).$$

Therefore,

$$X_3^{p_3-2r_3} = X_2^{p_2}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n).$$

There exists a $k \in \mathbb{N}$ such that $p_3 \leq kr_3$. By the same argument done k times, we

obtain

$$X_3^{p_3-kr_3} = u_2^+(X_1,...,X_n)q^+(X_1,...,X_n)$$

where $q^+(X_1,...,X_n)$ is a positive word. Thus, $p_3 - kr_3 \leq 0$ and so by Lemma 7.3.3, $X_3 = 1 = u_2^+(X_1,...,X_n) = q^+(X_1,...,X_n)$.

C) Let

$$X_1^{r_1} = X_3^{p_3}u_1^+(X_1,...,X_n) = X_3^{p_3}X_i^{p_i}u_1'^+(X_1,...,X_n) \tag{7.15}$$

$$X_2^{r_2} = X_3^{r_3}u_2^+(X_1,...,X_n) = X_3^{r_3}X_j^{p_j}u_2'^+(X_1,...,X_n) \tag{7.16}$$

be 2 left positive relations and

$$X_3^{p_3} = X_2^{p_2}u_3^+(X_1,...,X_n) \tag{7.17}$$

be a 2 left positive relation. Then,

$$X_3^{p_3-r_3} = X_j^{p_j}u_2'^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n). \tag{7.18}$$

1) Suppose $j = 1$. Then, by equations 7.15 and 7.18,

$$X_3^{p_3-2r_3} = X_i^{p_i}u_1'^+(X_1,...,X_n)X_1^{p_1-r_1}u_2^+(X_1,...,X_n)X_2^{p_2-r_2}u_3^+(X_1,...,X_n). \tag{7.19}$$

a) Now suppose $i = 1$. Then,

$$X_3^{p_3-2r_3} = X_3^{p_3} u_1'^+(X_1, ..., X_n) X_1^{p_1-r_1} u_2'^+(X_1, ..., X_n) X_2^{p_2-r_2} u_3^+(X_1, ..., X_n).$$

Thus,

$$X_3^{p_3-3r_3} = X_1^{p_1} u_1'^+(X_1, ..., X_n) X_1^{p_1-r_1} u_2'^+(X_1, ..., X_n) X_2^{p_2-r_2} u_3^+(X_1, ..., X_n).$$

There exists a $k \in \mathbb{N}$ such that $p_3 \leq kr_3$. By the same argument done k times, we obtain

$$X_3^{p_3-kr_3} = X_1^{p_1} w^+(X_1, ..., X_n)$$

where $w^+(X_1, ..., X_n)$ is a positive word. Thus, $p_3 - kr_3 \leq 0$ and so by Lemma 7.3.3, $X_1 = X_3 = 1 = w^+(X_1, ..., X_n)$.

b) Now suppose $i = 2$. Then by equation 7.19,

$$X_3^{p_3-2r_3} = X_1^{p_1} u_2'^+(X_1, ..., X_n) X_2^{p_2-r_2} u_1'^+(X_1, ..., X_n) X_1^{p_1-r_1} u_2^+(X_1, ..., X_n) X_2^{p_2-r_2} u_3^+(X_1, ..., X_n)$$

and so

$$X_3^{p_3-3r_3} = X_3^{p_3} X_2^{p_2} u_1'^+(X_1, ..., X_n) X_1^{p_1-r_1} u_2'^+(X_1, ..., X_n) X_2^{p_2-r_2} u_1'^+(X_1, ..., X_n) X_1^{p_1-r_1} u_2^+(X_1, ..., X_n) X_2^{p_2-r_2} u_3^+(X_1, ..., X_n)$$

which implies that

$$X_3^{p_3-4r_3} = X_2^{p_2} u_1'^{+}(X_1, ..., X_n) X_1^{p_1-r_1} u_2'^{+}(X_1, ..., X_n) X_2^{p_2-r_2} u_1'^{+}(X_1, ..., X_n) X_1^{p_1-r_1} u_2^{+}(X_1, ..., X_n) X_2^{p_2-r_2} u_3^{+}(X_1, ..., X_n)$$

There exists a $k \in \mathbb{N}$ such that $p_3 \leq kr_3$. By the same argument done k times, we obtain

$$X_3^{p_3-kr_3} = w^{+}(X_1, ..., X_n)$$

where $w^{+}(X_1, ..., X_n)$ is a positive word. Thus, $p_3 - kr_3 \leq 0$ and so by Lemma 7.3.3, $X_3 = 1 = w^{+}(X_1, ..., X_n)$.

2) Now suppose that $j = 2$. Then by equations 7.16 and 7.18,

$$X_3^{p_3-2r_3} = X_2^{p_2} u_2'^{+}(X_1, ..., X_n) X_2^{p_2-r_2} u_2^{+}(X_1, ..., X_n) X_2^{p_2-r_2} u_3^{+}(X_1, ..., X_n). \quad (7.20)$$

There exists a $k \in \mathbb{N}$ such that $p_3 \leq kr_3$. By the same argument done k times, we obtain

$$X_3^{p_3-kr_3} = X_2^{p_2} u_2'^{+}(X_1, ..., X_n) w^{+}(X_1, ..., X_n)$$

where $w^{+}(X_1, ..., X_n)$ is a positive word. Thus, $p_3 - kr_3 \leq 0$ and so by Lemma 7.3.3, $X_2 = X_3 = 1 = u_2'^{+}(X_1, ..., X_n) = w^{+}(X_1, ..., X_n)$.

Thus, the result is true for $m = 3$.

Now suppose the result is true for $m = n-1$. Thus, if we have $(n-2)$ 2 left positive relations $X_i^{r_i} = X_{k_i}^{p_{k_i}} X_{l_i}^{q_{l_i}} w_i^{+}(X_1, ..., X_n)$ where $1 \leq i \leq n-2$, $1 \leq k_i \leq n-1$ and

$1 \leq l_i \leq n-1$ and one positive relation $X_{n-1}^{p_{n-1}} = X_{k_{n-1}}^{p_{k_{n-1}}} w_{n-1}^+(X_1, ..., X_n)$ where $1 \leq k_{n-1} \leq n-2$, then $X_i = 1$ for at least one $1 \leq i \leq n-1$. We will look at the case with $m = n$. Therefore, without loss of generality we have $(n-2)$ 2 left positive relations $X_i^{r_i} = X_{k_i}^{p_{k_i}} X_{l_i}^{q_{l_i}} w_i^+(X_1, ..., X_n)$ where $1 \leq i \leq n-1$, $1 \leq k_i \leq n$ and $1 \leq l_i \leq n$ and one positive relation $X_n^{r_n} = X_{k_n}^{p_{k_n}} w_n^+(X_1, ..., X_n)$ where $1 \leq k_n < n$ and $r_n > p_n$. Thus, for the generator X_1, there is a 2 left positive relation $X_1^{r_1} = X_{k_1}^{p_{k_1}} X_{l_1}^{q_{l_1}} w_1^+(X_1, ..., X_n)$.

A) If $k_1 = 1$, then $X_1^{r_1 - p_1} = X_{l_1}^{q_{l_1}} w_1^+(X_1, ..., X_n)$ with $r_1 - p_1 \leq 0$ because it comes from a 2 left positive relation. This implies by Lemma 7.3.3 that $X_1 = 1 = u^+(X_1, X_2, ..., X_n)$ and thus the proof is over.

B) Suppose that $k_1 \neq 1$. If $l_1 \neq 1$, then $X_1^{r_1} = X_{k_1}^{p_{k_1}} X_{l_1}^{q_{l_1}} w_1^+(X_1, ..., X_n)$ with k_1 and l_1 not equal to one.

If $l_1 = 1$, then we have

$$X_1^{r_1} = X_{k_1}^{p_{k_1}} X_1^{q_1} w_1^+(X_1, ..., X_n) \tag{7.21}$$

$$= X_{k_1}^{p_{k_1}} X_{k_1}^{p_{k_1}} X_1^{q_1} w_1^+(X_1, ..., X_n) X_1^{q_1 - r_1} w_1^+(X_1, ..., X_n). \tag{7.22}$$

a) If $k_1 = n$, then

$$X_1^{r_1} = X_n^{2p_n} X_1^{q_1} w_1^+(X_1, ..., X_n) X_1^{q_1 - r_1} w_1^+(X_1, ..., X_n).$$

There exists a $s \in \mathbb{N}$ such that $r_n \leq s p_n$. By the same argument done s times, we

obtain

$$X_1^{r_1} = X_n^{sp_n} X_1^{q_1} w_1^+(X_1, ..., X_n) X_1^{q_1 - r_1} w_1^+(X_1, ..., X_n) \tag{7.23}$$

$$= X_{k_n}^{p_{k_n}} w_n^+(X_1, ..., X_n) X_n^{sp_n - r_n} X_1^{q_1} w_1^+(X_1, ..., X_n) X_1^{q_1 - r_1} w_1^+(X_1, ..., X_n). \tag{7.24}$$

i) If $k_n = 1$, then

$$X_1^{r_1} = X_1^{p_1} w_n^+(X_1, ..., X_n) X_n^{sp_n - r_n} X_1^{q_1} w_1^+(X_1, ..., X_n) X_1^{q_1 - r_1} w_1^+(X_1, ..., X_n).$$

Thus,

$$X_1^{r_1 - p_1} = w_n^+(X_1, ..., X_n) X_n^{sp_n - r_n} X_1^{q_1} w_1^+(X_1, ..., X_n) X_1^{q_1 - r_1} w_1^+(X_1, ..., X_n)$$

with $r_1 - p_1 \leq 0$ because it comes from a 2 left positive relation. This implies by Lemma 7.3.3 that $X_1 = 1$ and thus the proof is over.

ii) If $k_n \neq 1$, then

$$X_1^{r_1} = X_{k_1}^{p_{k_1}} X_{k_n}^{p_{k_n}} A$$

with k_1 and k_n not equal to one and where

$$A = w_n^+(X_1, ..., X_n) X_n^{sp_n - r_n} X_1^{q_1} w_1^+(X_1, ..., X_n) X_1^{q_1 - r_1} w_1^+(X_1, ..., X_n) X_1^{q_1 - r_1} w_1^+(X_1, ..., X_n).$$

b) We recall that $X_1^{r_1} = X_{k_1}^{p_{k_1}} X_1^{q_1} w_1^+(X_1, ..., X_n)$. If $1 < k_1 < n$, then there is a 2 left positive relation $X_{k_1}^{r_{k_1}} = X_{k_{k_1}}^{p_{k_{k_1}}} X_{l_{k_1}}^{q_{l_{k_1}}} w_{k_1}^+(X_1, ..., X_n)$.

So, $X_1^{r_1} = X_{k_{k_1}}^{p_{k_{k_1}}} X_{l_{k_1}}^{q_{l_{k_1}}} w_{k_1}^+(X_1, ..., X_n) X_{k_1}^{p_{k_1} - r_{k_1}} X_1^{q_1} w_1^+(X_1, ..., X_n)$

i) If $k_{k_1} = 1$, then

$$X_1^{r_1} = X_1^{p_1} X_{l_{k_1}}^{q_{l_{k_1}}} w_{k_1}^+(X_1, ..., X_n) X_{k_1}^{p_{k_1} - r_{k_1}} X_1^{q_1} w_1^+(X_1, ..., X_n)$$

and so

$$X_1^{r_1 - p_1} = X_{l_{k_1}}^{q_{l_{k_1}}} w_{k_1}^+(X_1, ..., X_n) X_{k_1}^{p_{k_1} - r_{k_1}} X_1^{q_1} w_1^+(X_1, ..., X_n)$$

with $r_1 - p_1 \leq 0$ because it comes from a 2 left positive relation. This implies by Lemma 7.3.3 that $X_1 = 1$ and thus the proof is over.

ii) If $k_{k_1} \neq 1$, then we have

$$X_1^{r_1} = X_{k_1}^{p_{k_1}} X_{k_{k_1}}^{p_{k_{k_1}}} X_{l_{k_1}}^{q_{l_{k_1}}} w_{k_1}^+(X_1, ..., X_n) X_{k_1}^{q_{k_1} - r_{k_1}} w_1^+(X_1, ..., X_n)$$

with k_1 and k_{k_1} not equal to one. We rewrite the last equation

$$X_1^{r_1} = X_{k_1}^{p_{k_1}} X_{k_{k_1}}^{p_{k_{k_1}}} u^+(X_1, ..., X_n)$$

where $u^+(X_1, ..., X_n) = X_{l_{k_1}}^{q_{l_{k_1}}} w_{k_1}^+(X_1, ..., X_n) X_{k_1}^{q_{k_1} - r_{k_1}} w_1^+(X_1, ..., X_n)$.

Thus, either the proof is over or we have the relation $X_1^{r_1} = X_k^{p_k} X_q^{p_q} v^+(X_1, ..., X_n)$ where k and q are not equal to one.

Suppose that we have the relation $X_1^{r_1} = X_k^{p_k} X_q^{p_q} v^+(X_1, ..., X_n)$ where k and q are not equal to one. Then in every other 2 left positive relations

$$X_i^{r_i} = X_{k_i}^{p_{k_i}} X_{l_i}^{q_{l_i}} w_i^+(X_1, ..., X_n)$$

if k_i or l_i is equal to one, we replace $X_1^{p_1}$ by

$$X_k^{p_k} X_q^{p_q} v^+(X_1, ..., X_n) X_1^{p_1 - r_1}.$$

Similarly, for the positive relation $X_n^{r_n} = X_{k_n}^{p_{k_n}} w_n^+(X_1, ..., X_n)$, if $k_n = 1$, we replace $X_1^{p_1}$ by $X_k^{p_k} X_q^{p_q} v^+(X_1, ..., X_n) X_1^{p_1 - r_1}$. Thus, we now have $(n-2)$ 2 left positive relations $X_i^{r_i} = X_{k_i}^{p_{k_i}} X_{l_i}^{q_{l_i}} w_i^+(X_1, ..., X_n)$ where $1 < i \leq n-1$, $1 < k_i \leq n$ and $1 < l_i \leq n$ and one positive relation $X_n^{p_n} = X_{k_n}^{p_{k_n}} w_n^+(X_1, ..., X_n)$ where $k_n \neq 1$. Therefore, by the induction hypothesis, $X_i = 1$ for some $1 < i \leq n$.

□

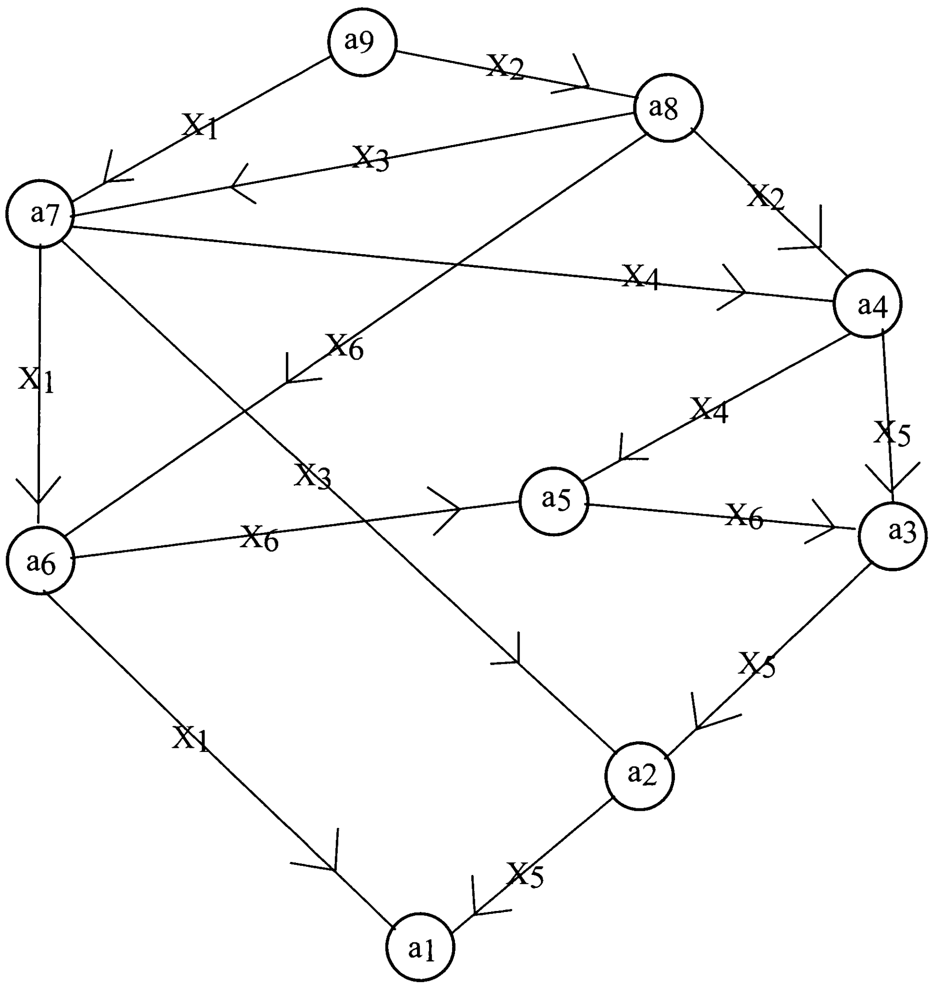
By construction of the Wada directed group of a directed link diagram and Lemma 7.2.7 we obtain the following result.

Lemma 7.4.2. *Let D be a directed $(n-2)$ of L with at least three rational tangle and $\Gamma(D)$ be the Wada rational graph. Suppose $\pi(D)$ is left-orderable. If $(\Gamma, <)$ is a $(n-1)$ totally monopositive directed Wada graph, then $G(\Gamma, <)$ is a left-orderable $(n-1)$ totally 2-left positive group generated by $X_1, ..., X_n$ where $n \geq 3$ such that $X_i \leq 1$ for every $1 \leq i \leq n$.*

Thus, combining the previous lemma, Proposition 7.3.2, Proposition 7.4.1 and Theorem 7.1.8 we have the following result.

Theorem 7.4.3. *If L is a $(n-1)$ totally monopositive, 2-non-bridge and non-split link, then the fundamental group of the double branched cover of L is not left-orderable.*

It is interesting to recall that 9_{49} is a non-alternating and non-arborescent knot.



The only Wada directed graph of 9_{49} with a_9 as maximum

From this Wada directed graph, we have the following positive relation.

$X_1 = X_2X_3$

$X_2 = X_3X_4$

$X_3 = X_4X_5^2$

$X_5 = X_4X_6$

$X_6 = X_3X_1$

$X_4^2 = X_1X_6$

Thus, 9_{49} is $(n-1)$ totally monopositive. Hence, the fundamental group of the double branched cover of 9_{49} is not left-orderable.

7.5 $(n-2)$ fluid Totally Simple Monopositive Links and non Left-Orderability

In this section, we will prove that for a fluid steady $n-2$ totally simple monopositive directed Wada graph, the directed Wada group is trivial. To do so, we will show that left-orderable fluid steady $n-2$ left and $n-2$ right totally positive group have a trivial generator. Moreover, the directed Wada group of fluid steady $n-2$ totally simple monopositive directed Wada graph are fluid steady $n-2$ left and $n-2$ right totally positive group. Therefore by Theorem 4.0.3, the fundamental group of the double branched cover of fluid steady $n-2$ totally simple monopositive links is not left-orderable.

We start by giving the definition of fluid totally positive group and steady totally positive group.

Remember that for a $\frac{p}{q}$ tangle we obtain final edges of order q and middle edge of order $p-q$. For edges that are not monopositive, the property we will want the tangle to have is that the order of the final edge is not more than twice the order of the middle edge and the order of the middle edge is not more than twice the order of the final edge. Large rational tangles will fail to satisfy this property. Note that for a rational tangle, we always have $\frac{p}{q} \geq 1$. If $\frac{p}{q} = 1$, then we have the half-twist. Moreover, we either have $p-q \geq q$ or $p-q \leq p$. Firstly, $p-q \geq q$ implies that $\frac{p}{q} \geq 2$. We want that $2q \geq p-q$, thus $\frac{p}{q} \leq 3$. If we had $p-q \leq p$, then this implies that $\frac{p}{q} \leq 2$. In this case, we want that $2(p-q) \geq q$, thus $\frac{p}{q} \geq \frac{3}{2}$.

However, if the edge is monopositive, we will want that the order of the final edge is more than twice the order of the middle edge or the order of the middle edge is not more than twice the order of the final edge.

Definition 7.5.1. Let X be a $\frac{p}{q}$ rational tangle. If $\frac{p}{q} > 3$ or $\frac{p}{q} < \frac{3}{2}$, then we say that X is a *large rational tangle.*

Monopositive relations from large rational tangles will be helpful. However, if we have either a pluripositive relation or a compose positive relation, it will be extremely difficult to transform it into a monopositive relation. That is why we consider it as an obstacle. With this in mind we define a new family of directed graphs for which large rational tangles won't be an obstacle.

Definition 7.5.2. Let D be a link diagram and $(\Gamma, <)$ be a totally simple positive directed Wada graph of D. If D contains no large pluripositive rational tangles and every monopositive rational tangle is a half-twist, full twist or large rational tangle, then we say that $(\Gamma, <)$ is *fluid.*

Moreover, if D is directed and has a fluid directed Wada graph, then we say that D is *fluid.*

Let D be a fluid link diagram and $(\Gamma, <)$ be a fluid directed Wada graph of D. Let $X_k^{h_k} = w_k^+(X_1, ..., X_n)$ be a pluripositive relation in $G(\Gamma, <)$. Then, by definition of fluidity, X_k is a rational tangle $\frac{p_k}{q_k}$ that is not large. Thus, $\frac{3}{2} \leq \frac{p_k}{q_k} \leq 3$. So, $\frac{3q_k}{2} \leq p_k \leq 3q_k$. This implies that $q_k \leq 2(p_k - q_k)$ and $p_k - q_k \leq 2q_k$. The middle order is $p_k - q_k$ and the final order is q_k. Because X_k is pluripositive, h_k is equal to the maximum between $p_k - q_k$ and q_k. Suppose $h_k = p_k - q_k$. Then, $(p_k - q_k) - q_k \leq q_k$ and $(p_k - q_k) - 2q_k \leq 0$. Suppose that $h_k = q_k$. Then, $q_k - (p_k - q_k) \leq (p_k - q_k)$ and $q_k - 2(p_k - q_k) \leq 0$.

Let $X_k^{h_k} = w_k^+(X_1, ..., X_n)$ be a monopositive relation in $G(\Gamma, <)$. Then, by definition of fluidity, X_k is either a large rational tangle $\frac{p_k}{q_k}$, an half-twist or a full twist. Suppose it is a large rational tangle. Then, either $\frac{3}{2} > \frac{p_k}{q_k}$ or $\frac{p_k}{q_k} > 3$. So, $\frac{3q_k}{2} > p_k$ or $p_k > 3q_k$. This implies that $q_k > 2(p_k - q_k)$ or $p_k - q_k > 2q_k$. The middle order is $p_k - q_k$ and the final order is q_k. Because X_k is monopositive, h_k is equal to the minimum between $p_k - q_k$ and q_k. Suppose $h_k = p_k - q_k \leq q_k$. Then, $p_k \leq 2q_k \leq 3q_k$ and so $\frac{3}{2} > \frac{p_k}{q_k}$. Thus, $q_k > 2(p_k - q_k)$ and $q_k - 2(p_k - q_k) > 0$. Suppose that $h_k = q_k \leq p_k - q_k$. Then, $p_k \geq 2q_k \geq \frac{3}{2}q_k$ and so $\frac{p_k}{q_k} > 3$. Therefore, $p_k - q_k > 2q_k$ and $p_k - q_k - 2q_k > 0$. Suppose that X_k is a half twist or a full twist. Then, $h_k = 1$ and the order of X_k in any monopositive relation $X_i^{h_i} = w_i^+(X_1, ..., X_n)$ is one.

Inspired by the previous comments, we will define fluidity for groups. But before, we need the following definitions.

Definition 7.5.3. Let G be a totally positive group. Let $X_k^{h_k} = w_k^+(X_1, ..., X_n)$ be the positive relations for $1 \leq k \leq n$ of the group and let X_i be a generator for $1 \leq i \leq n$. If every time that X_i is a left or right generator of a positive relation it has the same order, then X_i is *1-steady.* If every generator is 1-steady, then G is *1-steady.*

Similarly, if every time that X_i is a j-left or j-right generator for $1 \leq j \leq m$ of a positive relation, it has the same order, then X_i is *m-steady.* If every generator is m-steady, then G is *m-steady.* Moreover, if a generator has the same order in every positive relation, then X_i is *steady.* If every generator is steady, then G is *steady.*

Let D be a directed link diagram and $(\Gamma, <)$ be a directed Wada graph of D. If $G(\Gamma, <)$ is steady, then we say that $(\Gamma, <)$ and D are *steady.*

Definition 7.5.4. Let G be a steady totally positive group and $X_k^{h_k} = w_k^+(X_1, ..., X_n)$

be the positive relations for $1 \leq k \leq n$ of the group. Let $X_i^{h_i} = w_i^+(X_1, ..., X_n)$ be a positive relation with $1 \leq i \leq n$ and p_i be the order of X_i in all the positive relation. If $h_i \leq p_i$ and either $h_i \leq 2p_i$ or $h_i = p_i$, then X_i is called *ultra positive.* If $h_i \geq p_i$ and $h_i \leq 2p_i$, then X_i is called *small.*

Definition 7.5.5. Let G a group generated by $X_1, ..., X_n$ be a steady totally positive group. If every generator is either small or ultra positive, then G is *fluid.*

By construction of the Wada directed group, and of the fluid and steady diagram and Lemma 7.2.7, we obtain the following result.

Lemma 7.5.6. *Let D be a directed $(n-2)$ of L with at least three rational tangle and $\Gamma(D)$ be the Wada rational graph. Suppose $\pi(D)$ is left-orderable. If $(\Gamma, <)$ is a fluid steady $(n-i)$ totally simple monopositive directed Wada graph, then $G(\Gamma, <)$ is a left-orderable fluid steady $(n-i)$ totally 2-left and 2-right positive group generated by $X_1, ..., X_n$ where $n \geq 3$ such that $X_i \leq 1$ for every $1 \leq i \leq n$.*

Before we prove the desired result, we need the following lemma.

Lemma 7.5.7. *Let G be a fluid steady $(n-2)$ totally 2-left and 2-right positive group generated by $X_1, ..., X_n$ where $n \geq 3$ and $(G, <)$ be a left order on G such that $X_i \leq 1$ for every $1 \leq i \leq n$. Let X_i be a 2-left and 2-right positive generator. Then, either we can simplify its relation to obtain either $X_i^{r_i} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}} u^+(X_1, ..., X_n) X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$, $X_i^{r_i} = X_{i_k}^{p_{i_k}} X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$ or $X_i^{r_i} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}}$ where j_k, l_k, i_k, s_k are not 2-left and 2-right positive or we obtain a trivial generator. Moreover, let X_i not be a 2-left and 2-right positive generator. Then, either we can simplify its relation to obtain either $X_i^{r_i} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}} u^+(X_1, ..., X_n) X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$ or $X_i^{r_i} = X_{i_k}^{p_{i_k}} X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$ where j_k, l_k, i_k, s_k are not 2-left and 2-right positive or we obtain a trivial generator.*

Proof. Without loss of generality, we suppose that X_j are 2-left and 2-right positive for $1 \leq j \leq n-2$ and not 2-left and 2-right positive for $n-1 \leq j \leq n$. Because there is 2-left and 2-right positive, all the relations are of one of the following form $X_k^{r_k} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}} u_k^+(X_1, ..., X_n) X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$ or $X_k^{r_k} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}} X_{l_k}^{p_{l_k}}$ or $X_k^{r_k} = X_{j_k}^{p_{j_k}} X_{l_k}^{p_{l_k}}$. We will do the proof for $X_k^{r_k} = X_{j_k}^{p_{j_k}} X_{l_k}^{p_{l_k}}$, because for the other cases the proof is similar and even simpler.

Without loss of generality, we will prove the result for X_1. We have $X_1^{r_1} = X_{j_1}^{p_{j_1}} X_{l_1}^{p_{l_1}}$. Without loss of generality, let $X_{j_1}^{r_{j_1}} = X_{j_2}^{p_{j_2}} X_{l_2}^{p_{l_2}}$.

First, we suppose that $j_1 \leq n-2$. Hence, $X_1^{r_1} = X_{j_2}^{p_{j_2}} X_{l_2}^{p_{l_2}} X_{j_1}^{p_{j_1}-r_{j_1}} X_{l_1}^{p_{l_1}}$.

If $j_2 > n-2$, then we will continue this case later.

Suppose that $j_2 \leq n-2$. Then, without loss of generality let $X_{j_2}^{r_{j_2}} = X_{j_3}^{p_{j_3}} X_{l_3}^{p_{l_3}}$. Thus,

$$X_1^{r_1} = X_{j_3}^{p_{j_3}} X_{l_3}^{p_{l_3}} X_{j_2}^{p_{j_2}-r_{j_2}} X_{l_2}^{p_{l_2}} X_{j_1}^{p_{j_1}-r_{j_1}} X_{l_1}^{p_{l_1}}.$$

We continue this process until there is a k and an m both in $\mathbb{N}$ such either $j_{k+i} > n-2$ for $1 \leq i \leq m$ or $j_{k+m} = j_k \leq n-2$. We must have one of these two cases because there are only finitely many generators.

If $j_{k+i} > n-2$ for $1 \leq i \leq m$, then we will continue this case later.

If $j_{k+m} = j_k \leq n-2$ with $m \in \mathbb{N}$, then, without loss of generality, $X_{j_k}^{r_{j_k}} = X_{j_{k+1}}^{p_{j_{k+1}}} X_{l_{k+1}}^{l_{j_{k+1}}}$. Moreover,

$$X_{j_k}^{r_{j_k}} = X_{j_{k+m}}^{p_{j_{k+m}}} X_{l_{k+m}}^{p_{l_{k+m}}} X_{j_{k+m-1}}^{(p_{j_{k+m-1}}-r_{j_{k+m-1}})} X_{l_{k+m-1}}^{p_{l_{k+m}}} v^+(X_1, ..., X_m)$$

where $v^+(X_1, ..., X_n)$ is a positive word, because we have $p_{j_i} - r_{j_i} \geq 0$ since $j_i \leq n-2$ for $1 \leq i \leq k+m$ and so X_{j_i} is 2 left positive. But $j_{k+m} = j_k$, so $X_{j_k}^{r_{j_k}} = X_{j_k}^{p_{j_k}} w^+(X_1, ..., X_n)$ and $X_{j_k}^{r_{j_k}-p_{j_k}} = w^+(X_1, ..., X_n)$. However, $r_{j_k} - p_{j_k} \leq 0$

because $j_k \leq n-2$ and so X_{j_k} is 2 left positive. This implies by Lemma 7.3.3, that $w^+(X_1,...,X_n) = 1$ and $X_i = 1$ for every X_i in $w^+(X_1,...,X_n)$. Because $w^+(X_1,...,X_n)$ is not empty, this completes the proof.

We now come back to the case where there exists an $m \in \mathbb{N}$ such that $j_m > n-2$. Let k be the smallest natural number such that $j_k > n-2$. Then, we have

$$X_1^{r_1} = X_{j_k}^{p_{j_k}} X_{l_k}^{p_{l_k}} v^+(X_1,...,X_n) X_{l_2}^{p_{l_2}} X_{j_1}^{p_{j_1}-r_{j_1}} X_{l_1}^{p_{l_1}}$$

where $v^+(X_1,...,X_n)$ is a positive word, because we have $p_{j_i} - r_{j_i} \geq 0$, since $j_i \leq n-2$ and so X_{j_i} is 2 left positive.

We will now look at X_{l_k}. If $l_k > n-2$, then the proof is over for the left case.

If $l_k \leq n-2$, then by a similar argument as for j_k, either there is a $X_i = 1$ or

$$X_1^{r_1} = X_{j_k}^{p_{j_k}} X_{l_m}^{p_{l_m}} w^+(X_1,...,X_n) X_{l_2}^{p_{l_2}} X_{j_1}^{p_{j_1}-r_{j_1}} X_{l_1}^{p_{l_1}}$$

with $l_m \geq n-1$.

We now return to $X_1^{r_1} = X_{j_2}^{p_{j_2}} X_{l_2}^{p_{l_2}} X_{j_1}^{p_{j_1}-r_{j_1}} X_{l_1}^{p_{l_1}}$ and look at $X_{l_1}^{p_{l_1}}$.

A) If $l_1 \geq n-1$, then we look at $X_{j_1}^{p_{j_1}-r_{j_1}}$. Because G is fluid and we have already supposed that X_{j_1} is 2 left and 2 right positive, either $p_{j_1} - r_{j_1} = 0$ or $p_{j_1} - r_{j_1} \geq r_{j_1}$.

a) If $p_{j_1} - r_{j_1} = 0$, then

$$X_1^{r_1} = X_{j_k}^{p_{j_k}} X_{l_m}^{p_{l_m}} v^+(X_1,...,X_n) X_{l_2}^{p_{l_2}} X_{l_1}^{p_{l_1}}$$

and we look at X_{l_2}.

i) If $l_2 \geq n-1$, then the proof is over.

ii) If $l_2 < n-1$, then by a similar argument as before, we obtain

$$X_1^{r_1} = X_{j_k}^{p_{j_k}} X_{l_k}^{p_{l_k}} v^+(X_1, ..., X_n) X_{l_2}^{p_{l_2} - r_{l_2}} w^+(X_1, ..., X_n) X_{l_s}^{p_{l_s}} X_{l_1}^{p_{l_1}}$$

with $l_s \geq n-1$. So the proof is over.

b) If $p_{j_1} - r_{j_1} \geq r_{j_1}$, then by a similar argument as before, we obtain

$$X_1^{r_1} = X_{j_k}^{p_{j_k}} X_{l_k}^{p_{l_k}} v^+(X_1, ..., X_n) X_{l_2}^{p_{l_2}} X_{j_1}^{p_{j_1} - 2r_{j_1}} w^+(X_1, ..., X_n) X_{l_s}^{p_{l_s}} X_{l_1}^{p_{l_1}}$$

with $l_s \geq n-1$. So the proof is over.

B) If $l_1 \leq n-2$, then by a similar argument as before done twice, we obtain

$$X_1^{r_1} = X_{j_k}^{p_{j_k}} X_{l_k}^{p_{l_k}} v^+(X_1, ..., X_n) X_{l_2}^{p_{l_2}} X_{l_1}^{p_{l_1} - r_{l_1}} w^+(X_1, ..., X_n) X_{l_s}^{p_{l_s}} X_{l_t}^{p_{l_t}}$$

with $l_s \geq n-1$ and $l_t \geq n-1$. So the proof is over.

Therefore, either ${X_1}^{r_1} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}} u^+(X_1, ..., X_n) X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$, $X_1^{r_1} = X_{i_k}^{p_{i_k}} X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$ or $X_1^{r_1} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}}$ where $j_k, l_k, i_k, s_k \geq n-1$ or there is a trivial generator.

Moreover, let X_n not be a 2 left and 2 right positive edge. Then, as a relation of the group of G, the left edge and the right edge of X_n can't be X_n. Thus, we can't have $X_n^{r_n} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}}$ where $j_k, i_k \geq n-1$ with $j_k \neq i_k$. Hence, for X_n, either $X_n^{r_n} = X_{i_k}^{p_{i_k}} X_{j_k}^{p_{j_k}} u^+(X_1, ..., X_n) X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$ or $X_n^{r_n} = X_{i_k}^{p_{i_k}} X_{s_k}^{p_{s_k}} X_{l_k}^{p_{l_k}}$ where $j_k, l_k, i_k, s_k \geq n-1$ or there is a trivial generator. □

We now prove the main result of this section.

Proposition 7.5.8. *Let G be a totally positive group generated by $X_1, ..., X_n$ where $n \geq 3$ and $(G, <)$ be a left order on G such that $X_i \leq 1$ for every $1 \leq i \leq n$. If G is a fluid steady $(n-2)$ totally 2 left and 2 right positive group, then at least one* *generator X_i is trivial.*

Proof. Let $n \geq 3$ be the number of generators. Without loss of generality, let X_i with $1 \leq i \leq n-2$ be the generators with a 2 left and 2 right positive relations and X_{n-1} and X_n be the generators that are not 2 left and 2 right positive. We first investigate X_n. By Lemma 7.5.7, then, either we can simplify its relation to obtain either $X_n^{r_n} = X_{i_n}^{p_{i_n}} X_{j_n}^{p_{j_n}} u^+(X_1, ..., X_n) X_{s_n}^{p_{s_n}} X_{l_n}^{p_{l_n}}$ or $X_n^{r_n} = X_{i_n}^{p_{i_n}} X_{s_n}^{p_{s_n}} X_{l_n}^{p_{l_n}}$ where j_n, l_n, i_n, s_n are equal to $n-1$ or n or we obtain a trivial generator. If we obtain a trivial generator, then the proof is over. We will prove the case when $X_n^{r_n} = X_{i_n}^{p_{i_n}} X_{s_n}^{p_{s_n}} X_{l_n}^{p_{l_n}}$ as the other cases are similar and even easier to prove.

Suppose $X_n^{r_n} = X_{i_n}^{p_{i_n}} X_{s_n}^{p_{s_n}} X_{l_n}^{p_{l_n}}$ where l_n, i_n and s_n are equal to $n-1$ or n.

a) Suppose that i_n and l_n are equal to n. Then, $X_n^{r_n} = X_n^{p_n} X_{s_n}^{p_{s_n}} X_n^{p_n}$. So, $X_n^{r_n - 2p_n} = X_{s_n}^{p_{s_n}}$. Because X_n is fluid, $r_n - 2p_n \leq 0$ and by Lemma 7.3.3, $X_{s_n} = X_n = 1$ and the proof is over.

b) Suppose that either i_n or l_n is equal to n and that $s_n = n$. Then, either $X_n^{r_n - 2p_n} = X_{l_n}^{p_n}$ or $X_n^{r_n - 2p_n} = X_{i_n}^{p_{i_n}}$. Because X_n is fluid, $r_n - 2p_n \leq 0$ and by Lemma 7.3.3, $X_{l_n} = X_n = 1$ or $X_{i_n} = X_n = 1$ and the proof is over.

c) Suppose that either i_n or l_n is equal to n and that $s_n = n-1$. Then, $X_n^{r_n - p_n} = X_{n-1}^{2p_{n-1}}$. We will complete this case later.

d) Suppose that $l_n = i_n = s_n = n-1$. Then, $X_n^{r_n} = X_{n-1}^{3p_{n-1}}$. We will complete this case later.

e) Suppose that i_n and l_n is equal to $n-1$ and that $s_n = n$. Then, $X_n^{r_n} = X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{p_{n-1}}$. We will complete this case later.

Similarly, for X_{n-1}, by Lemma 7.5.7, then, either we can simplify its relation to obtain either $X_{n-1}^{r_{n-1}} = X_{i_{n-1}}^{p_{i_{n-1}}} X_{j_{n-1}}^{p_{j_{n-1}}} u^+(X_1, ..., X_n) X_{s_{n-1}}^{p_{s_{n-1}}} X_{l_{n-1}}^{p_{l_{n-1}}}$ or $X_{n-1}^{r_{n-1}} =$

$X_{i_{n-1}}^{p_{i_{n-1}}} X_{s_{n-1}}^{p_{s_{n-1}}} X_{l_{n-1}}^{p_{l_{n-1}}}$ where $l_{n-1}, i_{n-1}, s_{n-1}$ are equal to $n-1$ or n or we obtain a trivial generator. If we obtain a trivial generator, then the proof is over. We will prove the case when $X_{n-1}^{r_{n-1}} = X_{i_{n-1}}^{p_{i_{n-1}}} X_{s_{n-1}}^{p_{s_{n-1}}} X_{l_{n-1}}^{p_{l_{n-1}}}$ as the other cases are similar and even easier to prove.

By a similar argument as for X_n, either the proof is over, or we have the following possibilities:

1) $X_{n-1}^{r_{n-1}-p_{n-1}} = X_n^{2p_n}$;

2) $X_{n-1}^{r_{n-1}} = X_n^{3p_n}$;

3) $X_{n-1}^{r_{n-1}} = X_n^{p_n} X_{n-1}^{p_{n-1}} X_n^{p_n}$.

We must deal with all the possible combinations for X_n and X_{n-1}. The proof of combination c)2) is similar to d)1), combination c)3) is similar to e)1) and combination d)3) is similar to e)2). Therefore, we will only prove the cases c)1), c)2), c)3), d)2), d)3 and e3).

Suppose we have the cases c) and 1). Then, by substituting 1) in c), $X_n^{r_n-p_n} = X_n^{2p_n} X_{n-1}^{3p_{n-1}-r_{n-1}}$. Hence, $X_n^{r_n-3p_n} = X_{n-1}^{3p_{n-1}-r_{n-1}}$. Because X_n and X_{n-1} are fluid, $r_n - 3p_n \leq 0$ and $3p_{n-1} - r_{n-1} \geq 0$. Thus, by Lemma 7.3.3, $X_{n-1} = X_n = 1$ and the proof is over.

Suppose we have the cases c) and 2). Then, by substituting 2) in c), $X_n^{r_n-p_n} = X_n^{3p_n} X_{n-1}^{2p_{n-1}-r_{n-1}}$. Hence, $X_n^{r_n-4p_n} = X_{n-1}^{2p_{n-1}-r_{n-1}}$. Because X_n and X_{n-1} are fluid, $r_n - 4p_n \leq 0$ and $2p_{n-1} - r_{n-1} \geq 0$. Thus, by Lemma 7.3.3, $X_{n-1} = X_n = 1$ and the proof is over.

Suppose we have the cases c) and 3). Then, by substituting 3) in c), $X_n^{r_n-p_n} = X_n^{p_n} X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{2p_{n-1}-r_{n-1}}$. Hence, $X_n^{r_n-2p_n} = X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{2p_{n-1}-r_{n-1}}$. Because X_n and X_{n-1} are fluid, $r_n - 2p_n \leq 0$ and $2p_{n-1} - r_{n-1} \geq 0$. Thus, by Lemma 7.3.3,

$X_{n-1} = X_n = 1$ and the proof is over.

Suppose we have the cases d) and 2). Then, by substituting 2) in d), $X_n^{r_n} = X_n^{3p_n} X_{n-1}^{3p_{n-1}-r_{n-1}}$. So, $X_n^{r_n-3p_n} = X_{n-1}^{3p_{n-1}-r_{n-1}}$. Because X_n and X_{n-1} are fluid, $r_n - 3p_n \leq 0$ and $3p_{n-1} - r_{n-1} \geq 0$. Thus, by Lemma 7.3.3, $X_{n-1} = X_n = 1$ and the proof is over.

Suppose we have the cases d) and 3). Then, by substituting 3) in d), $X_n^{r_n} = X_n^{p_n} X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{3p_{n-1}-r_{n-1}}$. So,

$$X_n^{r_n-p_n} = X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{3p_{n-1}-r_{n-1}}. \tag{7.25}$$

By substituting the last equation in 3) from the left, we obtain

$$X_{n-1}^{r_{n-1}} = X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{3p_{n-1}-r_{n-1}} X_n^{2p_n-r_n} X_{n-1}^{p_{n-1}} X_n^{p_n}.$$

This implies that

$$X_{n-1}^{r_{n-1}-p_{n-1}} = X_n^{p_n} X_{n-1}^{3p_{n-1}-r_{n-1}} X_n^{2p_n-r_n} X_{n-1}^{p_{n-1}} X_n^{p_n}.$$

We substitute the last equation in Equation 7.25 and obtain

$$X_n^{r_n-p_n} = X_n^{p_n} X_{n-1}^{3p_{n-1}-r_{n-1}} X_n^{2p_n-r_n} X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{2p_{n-1}-r_{n-1}} X_n^{p_n} X_{n-1}^{3p_{n-1}-r_{n-1}}.$$

So,

$$X_n^{r_n-2p_n} = X_{n-1}^{3p_{n-1}-r_{n-1}} X_n^{2p_n-r_n} X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{2p_{n-1}-r_{n-1}} X_n^{p_n} X_{n-1}^{3p_{n-1}-r_{n-1}}.$$

Because X_n and X_{n-1} are fluid, $r_n - 2p_n \leq 0$, $3p_{n-1} - r_{n-1} \geq 2p_n - r_n \geq 0$. Thus, by Lemma 7.3.3, $X_{n-1} = X_n = 1$ and the proof is over.

Suppose we have the cases e) and 3). We substitute e) in 3) to obtain

$$X_n^{r_n} = X_n^{p_n} X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{p_{n-1}-r_{n-1}} X_n^{p_n} X_{n-1}^{p_{n-1}}.$$

So,

$$X_n^{r_n - p_n} = X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{p_{n-1}-r_{n-1}} X_n^{p_n} X_{n-1}^{p_{n-1}}. \tag{7.26}$$

Note that $p_{n-1} - r_{n-1} \leq 0$. By 3), $X_{n-1}^{r_{n-1}} = X_n^{p_n} X_{n-1}^{p_{n-1}} X_n^{p_n}$. This implies that $X_{n-1}^{p_{n-1}-r_{n-1}} X_{n-1}^{r_{n-1}} X_n^{-p_n} = X_{n-1}^{p_{n-1}-r_{n-1}} X_n^{p_n} X_{n-1}^{p_{n-1}}$. Therefore,

$$X_{n-1}^{p_{n-1}} X_n^{-p_n} = X_{n-1}^{p_{n-1}-r_{n-1}} X_n^{p_n} X_{n-1}^{p_{n-1}}.$$

We define $C = X_{n-1}^{p_{n-1}} X_n^{-p_n} = X_{n-1}^{p_{n-1}-r_{n-1}} X_n^{p_n} X_{n-1}^{p_{n-1}}$.

i) If $C \leq 1$, then we obtain from Equation 7.26 $X_n^{r_n - p_n} = X_{n-1}^{p_{n-1}} X_n^{p_n} C$. So, because X_n is fluid, we obtain a 2 left positive relation for X_n. Thus, we have an $(n-1)$ totally 2 left positive group. Thus, by Proposition 7.4.1, there is an X_i such that $X_i = 1$.

ii) If $C \geq 1$, then $X_{n-1}^{p_{n-1}} X_n^{-p_n} \geq 1$. Thus, $X_n^{p_n} X_{n-1}^{-p_{n-1}} \leq 1$. Recall that by 3), $X_{n-1}^{r_{n-1}} = X_n^{p_n} X_{n-1}^{p_{n-1}} X_n^{p_n}$. So, $X_{n-1}^{r_{n-1}-p_{n-1}} = X_n^{p_n} X_{n-1}^{p_{n-1}} X_n^{p_n} X_{n-1}^{-p_{n-1}}$. Because $X_n^{p_n} X_{n-1}^{-p_{n-1}} \leq 1$ and X_{n-1} is fluid, we obtain a 2 left positive relation for X_{n-1}. Thus, we have a $(n-1)$ totally 2 left positive group. Thus, by Proposition 7.4.1, there is an X_i such that $X_i = 1$.

(n-2) 2

□

Thus, by Theorem 7.1.8, the previous proposition and Lemma 7.5.6 we have the following important result.

Theorem 7.5.9. *If L is a fluid steady $(n-2)$ totally simple monopositive link with $n \geq 3$, then the fundamental group of the double branched cover of L is not left-orderable.*

For example, the knot $11n_{129}$ is a non-alternating and non-arborescent knot (Caudron, 1987).

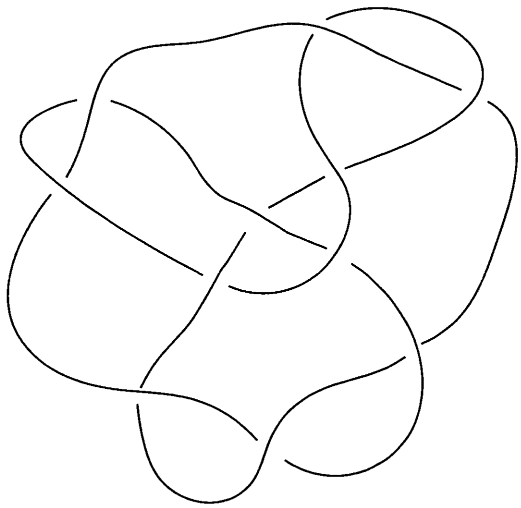


Figure 7.1 A knot $11n_{129}$ diagram

From this knot diagram, we obtain the following directed Wada graph.

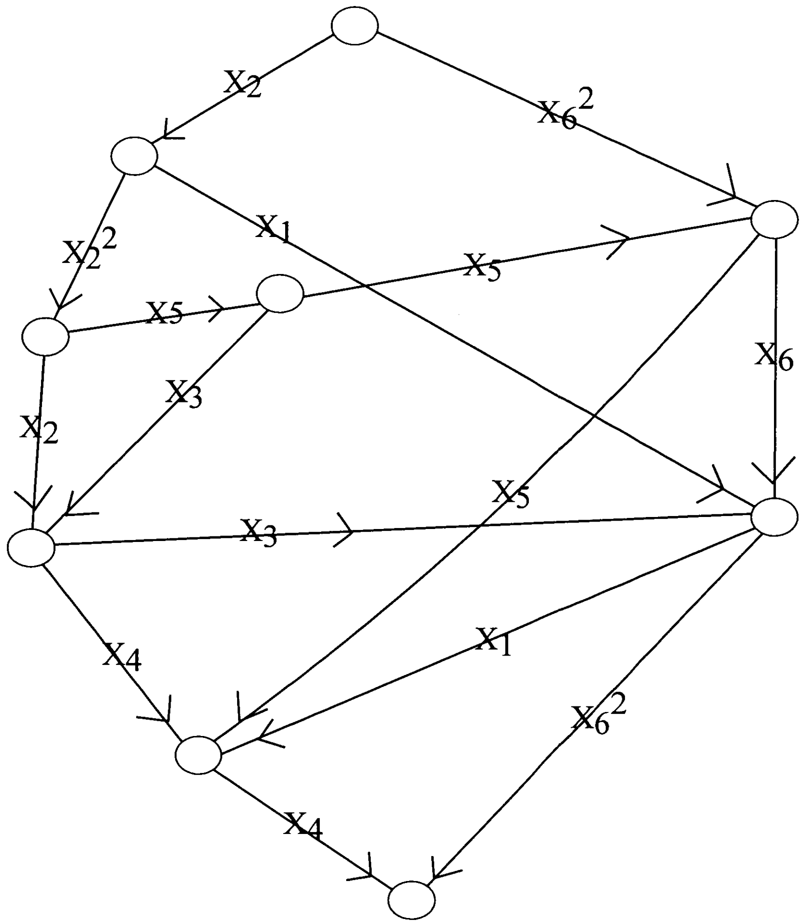


Figure 7.2 A directed Wada graph of the knot $11n_{129}$

From this Wada directed graph, we have the following positive relation.

$X_1 = X_2^3 X_3$

$X_2 = X_5 X_3$

$X_3^2 = X_5 X_6$

$X_4 = X_3 X_1$

$X_5 = X_6 X_1$

$X_6^2 = X_1 X_4$

Thus, $11n_{129}$ is steady fluid $(n-2)$ totally monopositive. Hence, the fundamental group of the double branched cover of $11n_{129}$ is not left-orderable.

CHAPTER VIII

INFINITE FAMILIES OF LINKS FOR WHICH THE FUNDAMENTAL GROUP OF THE DOUBLE BRANCHED COVER IS NOT LEFT-ORDERABLE

In this section, we will find infinite families of links for which the fundamental group of the double branched cover is not left-orderable. To do so, we will start from a "good" link diagram and substitute a rational tangle by another rational tangle without losing the desired property. In other words, we start with a link for which the fundamental group of the double branched cover is not left-orderable and by replacing a rational tangle by others rational tangles, we will find an infinite family of links for which the fundamental group of the double branched cover is not left-orderable. First, we will look at the hybrid Wada diagram and investigate how a change of rational tangle affects the directed Wada graph. The building blocks will be the totally monopositive, the $(n-1)$ totally monopositive and the steady fluid $(n-2)$ simple monopositive links.

8.1 Properties of rational tangles in a hybrid Wada diagram

Let D be a link diagram and $H(\Gamma)$ be the hybrid Wada diagram. Let X be a rational tangle that is not an half-twist in the hybrid Wada diagram. Without loss of generality, suppose that a_1 and a_3 are the non-bridge arcs in X. Then, we have the two following possibilities for the non-bridge arcs in X, that we define as the *negative diagonal non-bridge* and *positive diagonal non-bridge.* Recall that the non-bridge arcs of X are the white dots.

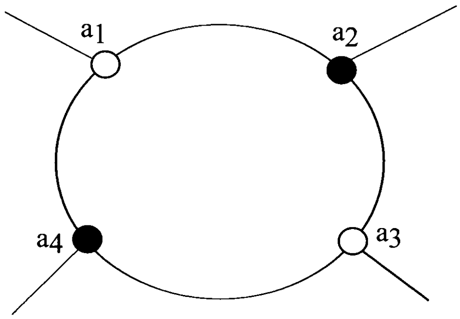


Figure 8.1 A negative diagonal non-bridge

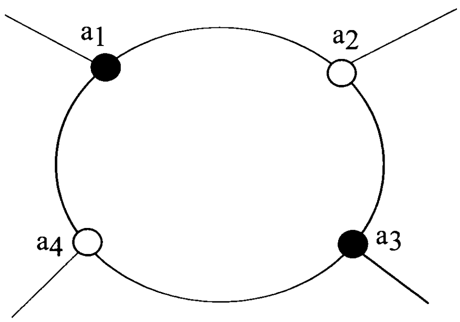


Figure 8.2 A positive diagonal non-bridge

If two rational tangles are negative diagonal non-bridge or the two are positive diagonal non-bridge, then we say that the two rational tangles are *diagonal equivalent.*

Let X be a negative diagonal non-bridge rational tangle. Then, the hybrid Wada diagram of X can be of two type. Firstly,

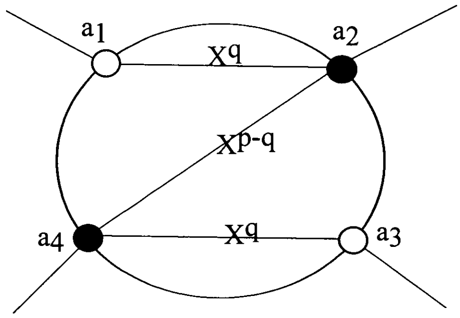


Figure 8.3 A negative diagonal non-bridge horizontal rational tangle

We call these rational tangles in the link diagram *horizontal rational tangle.* We can also have the following rational tangle in the link diagram:

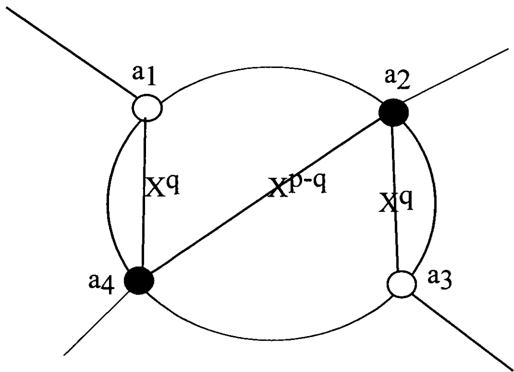


Figure 8.4 A negative diagonal non-bridge vertical rational tangle

We call these rational tangles in the link diagram *vertical rational tangle.* We define similarly, horizontal and vertical rational tangles with positive diagonal non-bridge. If we have two diagonal equivalent rational tangles that are both horizontal or both vertical, then we say they have the *same direction* or the two rational tangles are *direction equivalent.*

Moreover, to differentiate the $[n_m n_{m-1}...n_1] = \frac{p}{q}$ rational tangles, we can look at the last half-twist region. Then, $n_m + \frac{1}{k} = \frac{p}{q}$ with $k > 1$ and $k \in \mathbb{R}$ because k is a continuous fraction. If $n_m = 1$, then $1 + \frac{1}{k} = \frac{p}{q}$ with $k > 1$. This implies that $\frac{p-q}{q} = \frac{1}{k} < 1$, thus $p - q < q$ and the middle order of X is smaller than the final order of X. We call these *final rational tangles.*

If $n_m > 1$, then $n_m \geq 2$ because $n_m \in \mathbb{N}$. Hence, $1 < n_m - 1 + \frac{1}{k} = \frac{p}{q} - 1 = \frac{p-q}{q}$, thus $p - q > q$ and the middle order of X is greater than the final order of X. We call these *middle rational tangles.* If both rational tangles are final or both are middle, then we say they have the *same shape* or they are *shape equivalent.*

Finally, if two rational tangles are large or if both are not large, then we say that they have the *same strength* or they are *strength equivalent.*

Definition 8.1.1. If two rational tangles are diagonal equivalent and they have the same direction, the same shape and the same strength, then we say that the two rational tangles are of the *same nature.*

Definition 8.1.2. If two rational tangles are diagonal equivalent and they have the same direction and the same shape, then we say that the two rational tangles are *similar.*

Remark 8.1.3. Thus, if we substitute a rational tangle X for a similar rational tangle, we will get the same Hybrid Wada diagram except for the middle and the final order of X. Thus, we will have the same Wada rational graph at the except for the middle and the final order of X. However, if we change a rational tangle X for a rational tangle of the same nature, we will get the same Hybrid Wada diagram at the except for the middle and the final order of X. Hence, we will get the same Wada rational graph at the except for the middle and the final order of X, but the new tangle will have the same shape and strength property.

8.2 Impact of Changing a Rational Tangle on the directed Wada Group

Let D be a link diagram, $H(\Gamma)$ be the hybrid Wada diagram, $(\Gamma, <)$ be a directed Wada graph and $G(\Gamma, <)$ the directed Wada group of $(\Gamma, <)$. Recall from chapter 5 that a Wada directed cycle C in the directed Hybrid Wada diagram gives a relation r in the directed Wada group. If the change of a rational tangle in the link diagram changes the Hybrid Wada diagram in a way that C is unchanged, then the relation r will stay unchanged in the directed Wada group of the new directed Wada graph. If the change of a rational tangle in the link diagram changes the directed hybrid Wada diagram in a way that C is changed into C', then we will obtain the relation r' in the directed Wada group of the new directed Wada graph of the new link, where r' is the relation obtained from the cycle C'.

We will first give a series of technical lemmas that come directly from the Remark 8.1.3 and the previous comment.

Lemma 8.2.1. *Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D, C be a Wada directed cycle in the directed hybrid Wada diagram and X_i be a rational tangle which is not a half-twist. If there is no X_i edges in C and we replace X_i by a similar rational tangle which is not a half-twist, then the cycle C will stay unchanged in the new directed hybrid Wada diagram.*

Lemma 8.2.2. *Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D, C be a Wada directed cycle in the directed hybrid Wada diagram, X_i be a $\frac{p}{q}$ rational tangle which is not a half-twist and X_i^{p-q} be an edge in C. If we change X_i to a similar $\frac{s}{t}$ rational tangle, then the cycle C will change into C' where X_i^{p-q} will become X_i^{s-t} where $s-t$ is the middle order of the new X_i in the new directed hybrid Wada diagram.*

Lemma 8.2.3. *Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D, C be a Wada directed cycle in the directed hybrid Wada diagram, X_i be a $\frac{p}{q}$ rational tangle* *which is not a half-twist and X_i^q be an edge in C. If we change X_i to a similar $\frac{s}{t}$ rational tangle, then the cycle C will change into C' where X_i^q will become X_i^t where t is the final order of the new X_i in the new directed hybrid Wada diagram.*

Lemma 8.2.4. *Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D, C be a Wada directed cycle in the directed hybrid Wada diagram, X_i be a $\frac{p}{q}$ rational tangle which is not a half-twist and X_i^q (resp. X_i^{p-q}) be an edge in C. If we change X_i to a similar $\frac{s}{t}$ rational tangle with different strength, then the cycle C will change into C' where X_i^q (resp. X_i^{p-q}) will become X_i^t (resp. X_i^{s-t}) where t (resp. $s-t$) is the final order (resp. middle order) of the new X_i in the new directed hybrid Wada diagram.*

8.3 Infinite Families of Totally Simple Positive Links

Recall that the fundamental group of the double branched cover of a totally monopositive link, a (n-1) totally monopositive and a steady fluid (n-2) simple monopositive link is not left-orderable. From the lemmas 8.2.1, 8.2.2 and 8.2.3, we get the following results on relation in the directed Wada group.

Proposition 8.3.1. *Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D, X_i be a rational tangle which is not a half-twist and which is monopositive. If we change X_i to a similar rational tangle X_i', then X_i' will still be monopositive in the new directed Wada group of the new directed Wada graph.*

Proposition 8.3.2. *Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D, X_i be a rational tangle which is not a half-twist and X_j a monopositive rational tangle. If we change X_i to a similar rational tangle X_i', then X_j will still be monopositive in the new directed Wada group of the new directed Wada graph.*

Thus, we get

Proposition 8.3.3. *Let $(\Gamma, <)$ be a steady fluid directed Wada graph of a link diagram D and X_i be a rational tangle which is not a half-twist. If we change X_i to a rational tangle of the same nature, then the new directed Wada group of the new directed Wada graph will still be steady and fluid.*

Moreover,

Theorem 8.3.4. *Let $(\Gamma, <)$ be a totally monopositive directed Wada graph of a link L and X_i be a rational tangle which is not a half-twist. If we change X_i to a similar rational tangle, then the new directed Wada graph will still be totally monopositive.*

Theorem 8.3.5. *Let $(\Gamma, <)$ be a $(n-1)$ totally monopositive directed Wada graph and X_i be a rational tangle which is not a half-twist. If we change X_i to a similar rational tangle, then the new directed Wada graph will still be $(n-1)$ totally monopositive.*

Theorem 8.3.6. *Let $(\Gamma, <)$ be a steady fluid $(n-2)$ totally monopositive directed Wada graph and X_i be a rational tangle which is not a half-twist. If we change X_i to a rational tangle of the same nature, then the new directed Wada graph will still be steady fluid $(n-2)$ totally simple monopositive.*

Therefore, by the Theorem 7.3.6, Theorem 7.4.3 and Theorem 7.5.9 we get the following results

Theorem 8.3.7. *Let L be a totally monopositive link and X_i be a rational tangle which is not a half-twist. If we change X_i to a similar rational tangle, then the fundamental group of the double branched cover of the new link L' is not left-orderable.*

Theorem 8.3.8. *Let L be a $(n-1)$ totally monopositive link and X_i be a rational tangle which is not a half-twist. If we change X_i to a similar rational tangle,* *then the fundamental group of the double branched cover of the new link L' is not left-orderable.*

Theorem 8.3.9. *Let L be a fluid steady $(n-2)$ totally monopositive link and X_i be a rational tangle which is not a half-twist. If we change X_i to a rational tangle of same nature, then the fundamental group of the double branched cover of the new link L' is not left-orderable.*

The knot 8_{21} is a totally monopositive link.

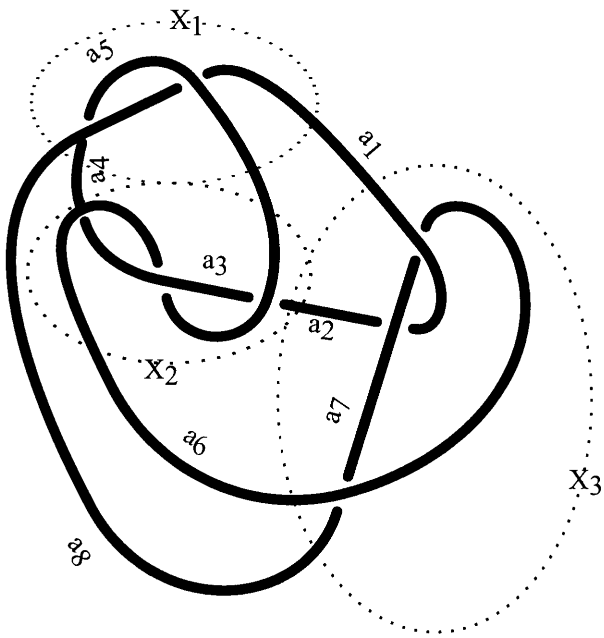


Figure 8.5 A knot diagram of the knot 8_{21}

By Theorem 8.3.7, if we replace the rational tangles X_1, X_2 or X_3 by similar rational tangles, then the fundamental group of the double branched cover of the new link is not left-orderable.

The knot 9_{49} is a $(n-1)$ totally monopositive link.

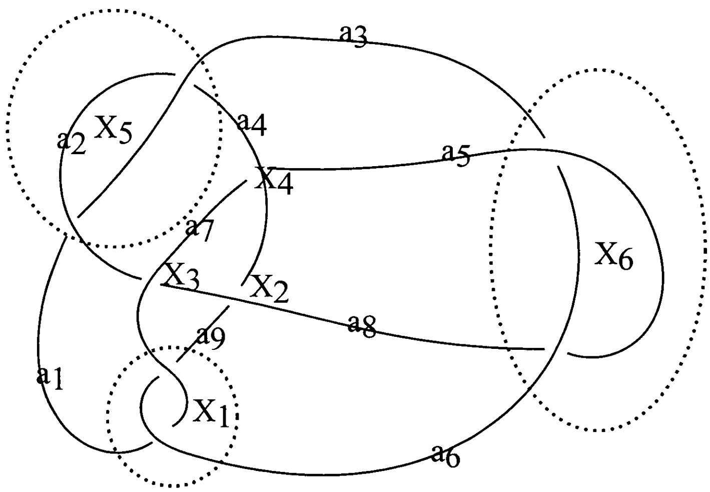


Figure 8.6 A knot diagram of the knot 9_{49}

By Theorem 8.3.8, if we replace the rational tangles X_1, X_5 or X_6 by similar rational tangles, then the fundamental group of the double branched cover of the new link is not left-orderable.

The knot $11n_{129}$ is a fluid steady $(n-2)$ totally monopositive.

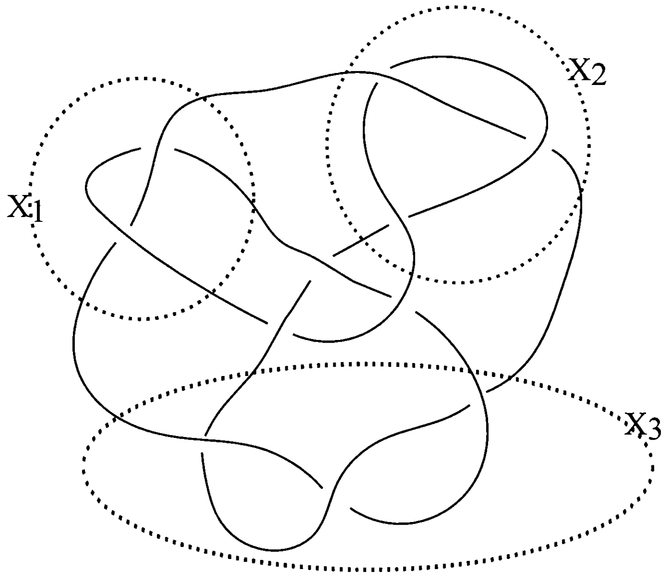


Figure 8.7 A knot diagram of the knot $11n_{129}$

By Theorem 8.3.9, if we replace the rational tangles X_1, X_2 or X_3 by similar rational tangles, then the fundamental group of the double branched cover of the new link is not left-orderable.

CHAPTER IX

TRIPLE HOP LINKS

In the previous sections, we have looked mostly at directed links. In this section, we will introduce a different method to show that the directed Wada group presentation of some directed Wada graph is trivial. We will use this method for links that are not directed.

In this section, we will study a particular type of links called the *Triple Hop links.* It is worth noting that, for non-alternating knot of 10 or less crossing, every knot that is not a three non-bridge knot, is a triple hop knot. We will show that some subfamilies of triple hop links have non-left orderable fundamental group. To do so, for middle triple hop links, we will introduce two sufficient conditions for the directed Wada group presentation to be trivial, the left and right directed condition and the left and right middle hop condition. For final triple hop links, we will introduce the left graph directed and right graph directed conditions. On the 7 non-alternating, non-directed and not left orderable knots of 10 or less crossing, the theorems in this section will cover 5 of them. It is also worth noting that the triple hop links argument works for a lot of the directed links.

Definition 9.0.1. A link L is called a *Triple Hop link* if it has a minimal diagram with exactly two non-brige arcs which moreover are the final vertices of the same rational tangle. We call this rational tangle the *triple hop tangle*

The knot 9_{47} is a triple hop knot. Note that the knot 9_{47} is not a directed knot and not an arborescent knot (Caudron, 1987).

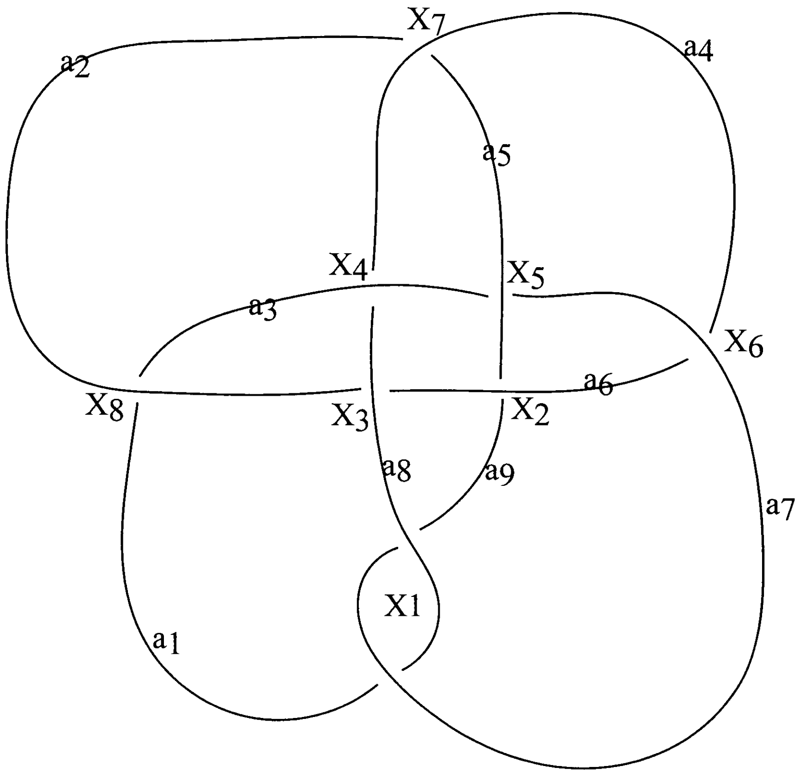


Figure 9.1 The knot 9_{47} is a triple hop knot with X_1 as triple hop tangle. The non-bridge arcs a_9 and a_1 are final vertices of X_1.

We will give some definitions about rational tangle that will help us study the triple hop links. Before, we recall that for two vertices a_i and a_j in a directed Wada graph $(\Gamma, <)$, we say that $a_i <_{(\Gamma,<)} a_j$ if there is a directed Wada path from a_j to a_i.

Definition 9.0.2. Let $(\Gamma, <)$ be a directed Wada graph of a link diagram D and X an edge from a_j to a_i in $(\Gamma, <)$ with $a_i <_{(\Gamma,<)} a_j$. If there is a vertex a_k such that $a_i <_{(\Gamma,<)} a_k <_{(\Gamma,<)} a_j$, then we say that X is a *cover edge*. If there are m vertices a_{k_m} such that $a_i <_{(\Gamma,<)} a_{k_m} <_{(\Gamma,<)} a_j$ but no other vertices satisfy this condition, then we say that X is an *m-cover edge.* If there is no such a_k, then we say that X is a *flat edge.* Recall that such vertices are called Wada consecutive. For a crossing $a_l >_{(\Gamma,<)} a_k >_{(\Gamma,<)} a_j >_{(\Gamma,<)} a_i$, we say the *left edge* is the edge from a_l to a_k, the *middle edge* is the edge from a_k to a_j and the *right edge* is the edge from a_j to a_i. We define a rational tangle as a $C - F - F$ rational tangle if the left edge is a cover edge and the middle and right edges are flat edges. If we want to specify that we have an m cover edge, we write $mC - F - F$. Similarly, for the other rational tangles.

Thus a flat crossing will be a $F - F$ crossing and a flat rational tangle will be a $F - F - F$ rational tangle. Note that each cover edge X gives a simple positive relation. If this simple positive relation is pluripositive, then we say that X is a *pluripositive cover edge.* Moreover, we add an apostrophe after the C from this edge X in the naming of the rational tangle. For example, for a $C-F-F$ rational tangle with the cover edge giving a pluripositive relation, we write $C' - F - F$ rational tangle. Note that the orders of the final edges are the same, thus if both of them are cover edges, they will be both pluripositive or both monopositive. Moreover, a $C-C-F$ rational tangle will be either a $C'-C-F$ rational tangle, a $C - C' - F$ rational tangle or a $C - C - F$ rational tangle, because either the order of the finals edges is minimal, the order of the middle edge is minimal or we have a full twist and both are minimal. Similarly, for the other edges with more than one cover edge.

When we look at a directed Wada graph, we say that a directed path go *right* when we go from a vertex a_j to a vertex a_i such that $a_j > a_i$ and *left* when the path go from a_i to a_j. We now define important rational tangle defined from there cover edges.

Definition 9.0.3. A $F - C$ or $F - F - C$ rational tangle will be called a *left* *rational tangle* and a $C-F$ or $C-F-F$ rational tangle will be called an *right rational tangle.* Moreover a $F-C-F$ rational tangle will be called a *middle rational tangle.* Note that we want the cover edge to be a monopositive edge. We recall that C' in the labeling of the rational tangle means that C is a pluripositive edge. For rational tangles with two cover edges, $C'-C-F$ rational tangle will be called *double right rational tangle* and $C-C'-F$ rational tangle will be called *pluripositive double right rational tangle.* Similarly, $F-C'-C$ will be called *pluripositive double left rational tangle,* $F-C-C'$ *double left rational tangle,* $C-F-C$ *double end rational tangle,* $C'-F-C'$ *pluripositive double end rational tangle* and $C-C$ *double rational tangle* and finally $C'-C-C'$ or $C-C'-C$ *triple rational tangle.*

9.1 Results on left-orderability for middle triple hop links

Recall that a rational tangle is a *middle rational tangle*, if the order of the final edges is less than or equal than the order of the middle edge. We say that a triple hop link is a *middle triple hop links*, if the triple hop rational tangle is a middle rational tangle. In this section, we will find two sufficient conditions to prove that middle triple hop links have a non left-orderable fundamental group of their double branched cover. For example, the knot 9_{47}, introduced at the beginning of the chapter, is a middle rational tangle.

The next definition will be used in the definition of one of the most important properties of triple hop links.

Definition 9.1.1. Let P be a directed path in a directed Wada graph. Let C be the set of directed paths that includes P. Then, the *prolongement* of P, is the set of edges and vertices included in C.

We will now prove a lemma that will enable us to introduce the first important property of directed Wada graph in triple hop links.

Lemma 9.1.2. *Let* $(\Gamma, <)$ *be a directed Wada graph of a triple hop link diagram,* C *be a Wada directed path in* $(\Gamma, <)$ *and* X_i *be the left edge of* C*. If all the edges of* C *come from a left, double left or double end rational tangles for which the monopositive left cover edge ends on the prolongement of the path* C*, then* $X_i^{p_i} = X_j^{k_j} A_j$ *for every flat edge* X_j *in* C *where* A_j *is a positive word of edges.*

Proof. Without loss of generality we suppose that $C = \{X_1^{k_1}, X_2^{k_2}, ..., X_m^{k_m}\}$ where X_i is the i-left edge of C. We will prove the result by induction on the number k of flat edges in C. If $k = 2$, then $C = \{X_1^{k_1}, X_2^{k_2}\}$. Moreover, by hypothesis, X_1 comes from a left, double left or double end rational tangles for which the monopositive left cover edge end on the prolongement of C. Because X_1 is a left, double left or double end rational tangle, there is a cover edge X_1 that starts at the beginning of X_2 and that ends on the prolongement of C. Therefore, $X_1^{p_k} = X_2^{k_2} w^+(X_1, ..., X_n)$. Thus, the proof is over for $k = 2$.

Let the result be true for $m - 1$ flat edges in C. This implies that $X_1^{p_1} = X_j^{k_j} A_j$ for every flat edge X_j in C where A_j is a positive word of edges. We will now prove the case when $k = m$ and so for the Wada directed path $C = \{X_1^{k_1}, X_2^{k_2}, ..., X_m^{k_m}\}$. To do so, we will first look at $C_1 = \{X_1^{k_1}, X_2^{k_2}, ..., X_m^{k_{m-1}}\}$. Hence, $X_1^{p_1} = X_j^{k_j} A_j$ for every flat edge X_j in C_1 where A_j is a positive word of edges. Moreover, X_{m-1} comes from a left, double left or double end rational tangles for which the monopositive left cover edge end on the prolongement of C. Because X_{m-1} is a left, double left or double end rational tangle, there is a cover edge X_{m-1} that starts at the beginning of X_m and that ends on the prolongement of C. Therefore, $X_{m-1}^{p_{m-1}} = X_m^{k_m} w^+(X_1, ..., X_n)$. Furthermore, we already have $X_1^{p_1} = X_{m-1}^{k_{m-1}} A_{m-1}$. So, because the cover edge X_{m-1} is monopositive, we have $X_1^{p_1} = X_m^{k_m} w^+(X_1, ..., X_n) X_{m-1}^{k_{m-1} - p_{m-1}} A_{m-1}$ where $k_{m-1} - p_{m-1} \geq 0$. Thus,

$X_1^{p_1} = X_m^{k_m} A_m$ where $A_m = w^+(X_1, ..., X_n) X_{m-1}^{k_{m-1}-p_{m-1}} A_{m-1}$ is a positive word and the proof is over. □

Similarly,

Lemma 9.1.3. *Let $(\Gamma, <)$ be a directed Wada graph of a triple hop link diagram, C be a Wada directed path in $(\Gamma, <)$ and X_i be the right edge of C. If all the edges of C comes from a right, double right or double end rational tangles for which the monopositive right cover edge end on the prolongement of the path C, then $X_i^{p_i} = A_j X_j^{k_j}$ for every flat edge X_j in C where A_j is a positive word of edges.*

Definition 9.1.4. Let $C = \{X_1^{k_1}, X_2^{k_2}, ..., X_m^{k_m}\}$ be a directed path of flat edges in a directed Wada graph and let each flat edge comes from a left, double left or double end rational tangles that end on the prolongement of the path C. Thus, by Lemma 9.1.2, $X_1^{p_1} = X_i^{k_i} A_i$ for $2 \leq i \leq m$ where A_i is a positive word of edges. We say that X_i is *left included* in X_1 for $2 \leq i \leq m$. We call this path a *left directed path.*

We define similarly with Lemma 9.1.3 a *right directed path* with right, double right and double end rational tangles.

We will now introduce two conditions that will be sufficient to show that for a directed Wada graph the directed Wada group presentation is trivial.

Definition 9.1.5. Let $(\Gamma, <)$ be a directed Wada graph with triple hop (X^k, X^p, X^k) and let the left edge of X be a cover edge. Then $X^k = w^+(X_1, ..., X_n)$.

If there is a left directed path from the left edge of $w^+(X_1, ..., X_n)$ to the right edge of $w^+(X_1, ..., X_n)$, we call this condition, the *left directed condition.*

Let $X_i^{k_i}$ be the right edge of $w^+(X_1, ..., X_n)$ and $X_i^{p_i}$ be the consecutive edge to the right of $X_i^{k_i}$. Then, either $X_i^{p_i}$ is also a left edge of the middle edge of X or it is not a left edge of the middle edge of X.

Let $X_i^{p_i}$ be also a left edge of the middle edge of X. If X^p is left included in $X_i^{m_i+p_i}$ where $m_i + p_i$ is the compose order of X_i , then we say that the *left middle hop condition* is satisfied.

Let $X_i^{p_i}$ not be a left edge of the middle edge X. If X^p is left included in $X_i^{p_i}$, then we say that the *left middle hop condition* is satisfied.

We define similarly, the *right directed condition* and the *right middle hop condition.*

Theorem 9.1.6. *Let* $(\Gamma, <)$ *be a directed Wada graph of a middle triple hop link diagram. If the left directed condition (resp. right directed condition) and the left middle hop condition (resp. right middle hop condition) are respected, then* $G(\Gamma, <)$ *is trivial.*

Proof. Suppose that the left directed condition and the left middle hop condition are satisfied. Let $X = (X^k, X^p, X^k)$ be the edge of the triple hop rational tangle (a_n, a_m, a_k, a_1) and without loss of generality $X_1^{k_1}, X_2^{k_2}, ..., X_l^{k_l}$ be the edges of the left directed path from a_n to a_m. This left directed path exists because of the left directed condition. Thus, by Lemma 9.1.2, $X_1^{p_1} = X_i^{k_i} A_i$ for $2 \leq i \leq l$ where A_i is a positive word of edges. In particular, $X_1^{p_1} = X_l^{k_l} A_l$ where A_l is a positive word of edges. Moreover,

$$X^k = X_1^{k_1} A \tag{9.1}$$

where A is a positive word of edges, because $X_1^{k_1}$ is the left edge of $X^k = w^+(X_1, ..., X_n)$. Hence,

$$X^k = X_l^{k_l} A_l X_1^{k_1 - p_1} A \tag{9.2}$$

where $k_1 - p_1 \geq 0$, because $X_1^{k_1}$ is in a left directed path.

Suppose that X_i is not a left edge of the middle edge X. Then, X^p is left included in $X_l^{p_l}$ by the left middle hop condition. So,

$$X_l^{p_l} = X^p B \tag{9.3}$$

where B is a positive word of edges. Thus, by the previous equation and equation 9.2, $X^k = X^p B X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A$ where $k_l - p_l \geq 0$ because $X_l^{k_l}$ is in a left directed path. This implies that $X^{k-p} = B X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A$ where $k - p \leq 0$, because X is a middle rational tangle.

If $k \neq p$, then by Lemma 7.3.3, $X = 1$ and so $a_n = a_1$. This implies that $G(\Gamma, <)$ is trivial.

If $k = p$, then $1 = B X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A$. So, $1 = B = X_l^{k_l - p_l} = A_l = X_1^{k_1 - p_1} = A$. If $k_l \neq p_l$, then $X_l = 1$. Therefore, by Equation 9.3, $X = 1$ and so $a_n = a_1$. Thus, $G(\Gamma, <)$ is trivial. If $k_l = p_l$ and $k_1 \neq p_1$, then $X_1 = 1$. From Equation 9.1, this implies that $X = 1$. So, $a_n = a_1$ and $G(\Gamma, <)$ is trivial.

Suppose that $k_l = p_l$ and $k_1 = p_1$ and recall Equation 9.1. Because X^k is a cover edge, A is not empty. Moreover, since $X_1^{k_1}, X_2^{k_2}, ..., X_l^{k_l}$ are the edges of the left directed path from a_n to a_m, $A = X_2^{k_2} A'$. Thus, because $A = 1$, $X_2 = 1$. From the left directed condition, $X_2^{p_2} = X_3^{k_3} A_3$, so $X_3 = 1$. By a similar argument, $X_i = 1$ for $2 \leq i \leq l$. Therefore, $X_l = 1$ and by Equation 9.3, $X = 1$. This implies that $a_n = a_1$ and $G(\Gamma, <)$ is trivial.

Now, suppose that X_i is a left edge of the middle edge X. Then, X^p is left included in $X_l^{p_l + k_l}$ by the left middle hop condition. So,

$$X_l^{p_l + k_l} = X^p D \tag{9.4}$$

where D is a positive word of edges. Moreover, because a_m is in X and in X_l

and both rational tangles are not half-twist, by Lemma 6.7.6, a_m is in no other rational tangle. Also, because X_l is a left edge of the middle edge X and the right edge of the left edge X, then $X_1^{p_1} = X_l^{k_l} A_l = X_l^{k_l+p_l} A_l'$. Therefore,

$$X^k = X_l^{k_l+p_l} A_l' X_1^{k_1-p_1} A \tag{9.5}$$

and by a similar argument as the previous case, $G(\Gamma, <)$ is trivial.

The proof is similar when the right directed condition and the right middle hop condition are satisfied. □

By Theorem 9.1.6 and Lemma 4.0.2, we have

Theorem 9.1.7. *Let D be a maximal two non-bridge middle triple hop diagram of a non-split link L. If for every directed Wada graph $(\Gamma, <)$, the left directed condition (or resp. right directed condition) and the left middle hop condition (or resp. right middle hop condition) are respected, then the $\pi_1(\Sigma(L))$ is not left-orderable.*

The knot diagram 9_{47} of figure 9.1 has the two following directed Wada graph.

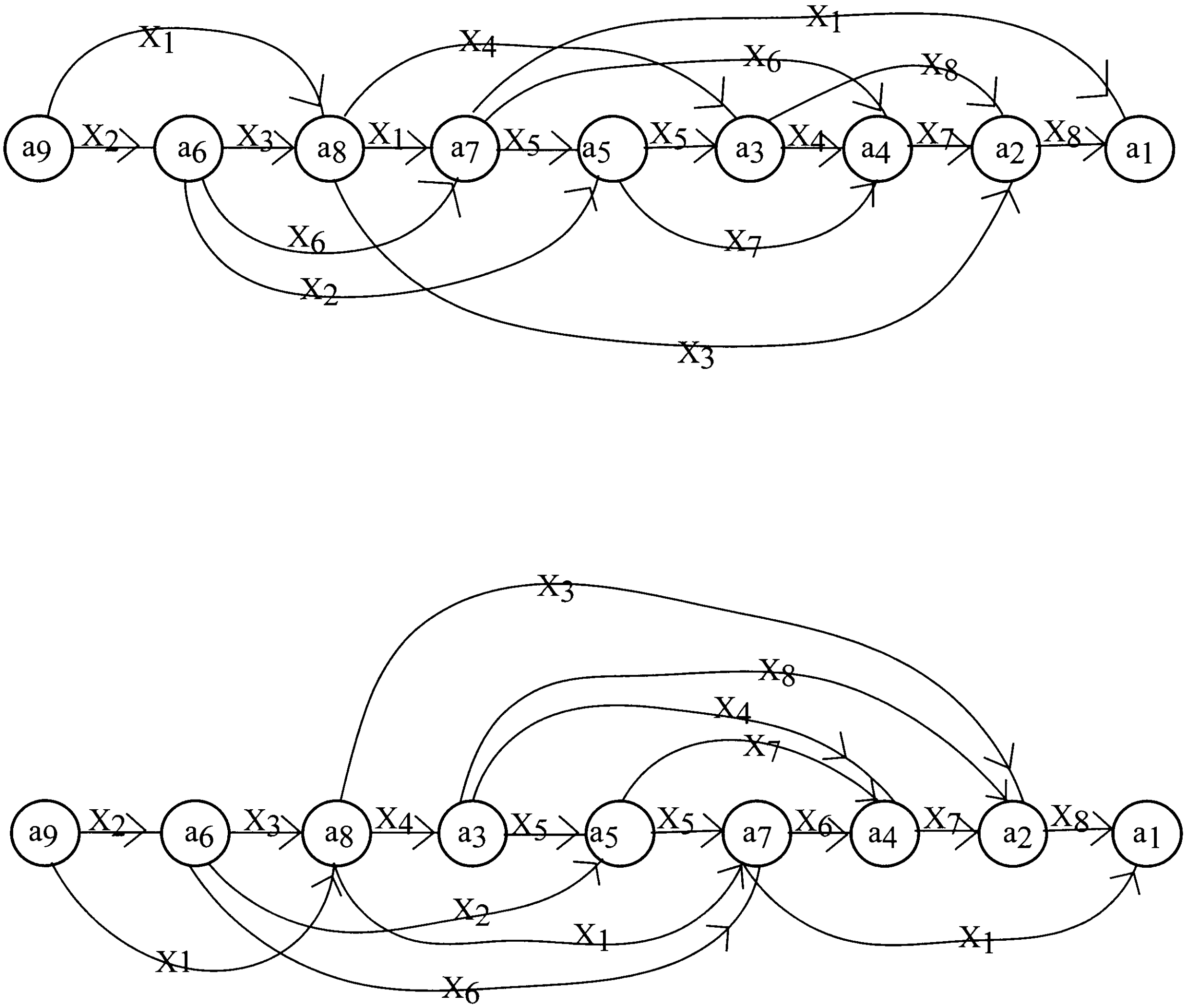


Figure 9.2 The two directed Wada graphs of the knot diagram 9_{47} of figure 9.1

For both directed Wada graphs, the left directed condition, the right directed condition, the left middle hop condition and the right middle hop condition are respected. Thus, the fundamental group of the double branched cover of the knot 9_{47} is not left-orderable.

9.2 Results on left-orderability for final triple hop links

Recall that a rational tangle is a *final rational tangle*, if the order of the final edges is greater than the order of the middle edge. We say that a triple hop link is a *final triple hop links*, if the triple hop rational tangle is a final rational tangle. In this section, we will find three sufficient conditions to prove that final triple hop links have a non left-orderable fundamental group of their double branched cover.

The non-alternating knot $11n_{47}$ is a final triple hop knot. The final triple hop links are more complicated to deal with than middle triple hop links. We will need more definitions.

For some triple hop link diagrams, we will be able to split the triple hop link diagram into three parts.

Definition 9.2.1. Let $(\Gamma, <)$ be a directed Wada graph of a triple hop diagram of a link L and $X = (a_n, a_l, a_k, a_1)$ be the triple hop rational tangle. All the a_i such that $a_n \geq_{(\Gamma,<)} a_i \geq_{(\Gamma,<)} a_l$ form the vertices of the *left graph* of $(\Gamma, <)$. All edges between vertices of the left part form the edges of the left graph.

All the a_i such that $a_l \geq_{(\Gamma,<)} a_i \geq_{(\Gamma,<)} a_k$ forms the vertices of the *middle graph* of $(\Gamma, <)$. All edges between vertices of the middle graph form the edges of the middle graph.

All the a_i such that $a_k \geq_{(\Gamma,<)} a_i \geq_{(\Gamma,<)} a_1$ forms the vertices of the *right graph* of $(\Gamma, <)$. All edges between vertices of the lower graph form the edges of the right graph.

If every vertex is at least in one of the left, middle or right graph, then we say that $(\Gamma, <)$ is *trichotomic.*

We now define a property of left and right graph that will be useful for the final triple hop links.

Definition 9.2.2. Let $(\Gamma, <)$ be a directed Wada graph of a trichotomic triple hop diagram.

If every directed path in the left graph is left directed, then we say that the left graph is *left graph directed.*

If every directed path in the right graph is right directed, then we say that the right graph is *right graph directed.*

Remark 9.2.3. Clearly, if the left graph is left graph directed, then the left directed condition is satisfied. Similarly, if the right graph is right graph directed, then the right directed condition is satisfied.

We can now prove a result for final triple hop links.

Theorem 9.2.4. *Let $(\Gamma, <)$ be a trichotomic directed Wada graph of a final triple hop link diagram. If the left graph is left graph directed (resp. the right graph is right graph directed) and the left middle hop condition and right middle hop condition are respected, then $G(\Gamma, <)$ is trivial.*

Proof. Without loss of generality suppose that the left graph is left graph directed. Then, the left directed condition and the left middle hop condition are satisfied. Let $X = (X^k, X^p, X^k)$ be the triple hop rational tangle with vertices (a_n, a_m, a_k, a_1) and without loss of generality let $X_1^{k_1}, X_2^{k_2}, ..., X_l^{k_l}$ be the edges of a left directed path from a_n to a_m. This left directed path exists because of the left graph directed condition. Thus, by Lemma 9.1.2, $X_1^{p_1} = X_i^{k_i} A_i$ for $2 \leq i \leq l$ where A_i is a positive word of edges. In particular, $X_1^{p_1} = X_l^{k_l} A_l$ where A_l is a positive word of edges. However,

$$X^k = X_1^{k_1} A \tag{9.6}$$

where A is a positive word of edges, because $X_1^{k_1}$ is the left edge of $X^k = w^+(X_1, ..., X_n)$. Hence,

$$X^k = X_l^{k_l} A_l X_1^{k_1 - p_1} A \tag{9.7}$$

where $k_1 - p_1 \geq 0$, because $X_1^{k_1}$ is in a left directed path.

Suppose that X_i is not a left edge of the middle edge X. Then, X^p is left included in $X_l^{p_l}$ by the left middle hop condition. So,

$$X_l^{p_l} = X^p B \tag{9.8}$$

where B is a positive word of edges. Thus, by the previous equation and equation 9.6, $X^k = X^p B X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A$ where $k_l - p_l \geq 0$ because $X_l^{k_l}$ is in a left directed path. This implies that $X^{k-p} = B X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A$ where $k - p \geq 0$, because X is a final rational tangle.

We now investigate $B = X_s^{k_s} B'$ where $X_s^{k_s}$ is a flat edge and B' is a positive word. We know that $X_l^{p_l} = X^p B = X^p X_s^{k_s} B'$, but X^p goes from a_m to a_k, so $X_s^{k_s}$ goes from a_k to a_t such that $t \leq k$. So, $a_t \leq_{(\Gamma,<)} a_k$ and a_t is in the right graph. Because of the right middle hop condition, $X_s^{p_s} = C X^p$ where $C = c^+(X_1, ..., X_n)$. However, X^p goes from a_m to a_k, so the edges in C are in the left graph. This implies that $C = X_u^{k_u} C'$ where $X_u^{k_u}$ is a flat edge in the left graph and $C' = c'^+(X_1, ..., X_n)$. There is a directed Wada path from X_u to $X_a^{k_a}$ where $X_a^{k_a}$ is consecutive to X^p. Thus, by the left graph directed condition $X_u^{p_u} = X_a^{k_a} D$ where $D = d^+(X_1, ..., X_n)$. Moreover, by the left middle hop condition $X_a^{p_a} = X^p E$ where $E = e^+(X_1, ..., X_n)$. So, combining the previous equations we obtain

$$C = X_u^{k_u} C' = X_a^{k_a} D X_u^{k_u - p_u} C' = X^p E X_a^{k_a - p_a} D X_u^{k_u - p_u} C'.$$

This implies that

$$X_s^{p_s} = X^p E X_a^{k_a - p_a} D X_u^{k_u - p_u} C' X^p.$$

Therefore,

$$B = X^p E X_a^{k_a - p_a} D X_u^{k_u - p_u} C' X^p X_s^{k_s - p_s} B'.$$

So, because $X^{k-p} = B X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A$, we get

$$X^{k-p} = B X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A = X^p E X_a^{k_a - p_a} D X_u^{k_u - p_u} C' X^p X_s^{k_s - p_s} B' X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A$$

Thus,

$$X^{k-2p} = E X_a^{k_a - p_a} D X_u^{k_u - p_u} C' X^p X_s^{k_s - p_s} B' X_l^{k_l - p_l} A_l X_1^{k_1 - p_1} A.$$

We repeat a similar argument v time until $k - vp \leq 0$ and so by Lemma 7.3.3 the proof is over.

The proof when X_i is a left edge of the middle edge X is similar.

The proof is similar when the right graph is right graph directed and the right and left middle hop condition are satisfied. □

By Theorem 9.2.4 and Lemma 4.0.2, we have

Theorem 9.2.5. *Let D be a maximal two non-bridge final triple hop diagram of a non-split link L. If every directed Wada graph $(\Gamma, <)$ is trichotomic and for every $(\Gamma, <)$ the left graph is left graph directed or the right graph is right graph directed and both the left middle hop condition and right middle hop condition are respected, then the $\pi_1(\Sigma(L))$ is not left-orderable.*

For the knot $11n_{47}$ both directed Wada graph $(\Gamma, <)$ are trichotomic and for both $(\Gamma, <)$ the left graph is left graph directed and both the left middle hop condition and right middle hop condition are respected. Thus, the fundamental group of the double branched cover of $11n_{47}$ is not left-orderable.

9.2.1 Infinite families of good middle triple hop link and good final middle triple hop link

To simplify the classification of links, we will define families of links that satisfies the hypothesis of the two theorems of the previous sections.

Definition 9.2.6. Let D be a maximal two non-bridge middle triple hop diagram of a non-split link L. If for every directed Wada graph $(\Gamma, <)$, the left directed condition (or resp. right directed condition) and the left middle hop condition (or resp. right middle hop condition) are respected, then D is a *good middle triple hop link diagram* and L is a *good middle triple hop link.*

Definition 9.2.7. Let D be a maximal two non-bridge final triple hop diagram of a non-split link L. If every directed Wada graph $(\Gamma, <)$ is trichotomic and for every $(\Gamma, <)$ the left graph is left graph directed or the right graph is right graph directed and both the left middle hop condition and right middle hop condition are respected, then D is a *good final triple hop link diagram* and L is a *good final triple hop link.*

From a good middle triple hop link diagram, we can obtain an infinite family of good middle triple hop link diagrams. Similarly, from a good middle triple hop link diagram, we can obtain an infinite family of good middle triple hop link diagrams. In both cases, we will obtain the infinite family by substituting the triple hop rational by a rational tangle of the same nature.

By definition of rational tangles of same nature and of triple hop link diagram, we have the following result.

Lemma 9.2.8. *Let D be a middle (resp. final) triple hop diagram with X the triple hop rational tangle. If we substitute X by a rational tangle X' of the same nature, then the new link diagram D' is a middle (resp. final) triple hop diagram with X' the triple hop rational tangle.*

Furthermore, by Propositions 8.3.1 and 8.3.2, we obtain the following series of lemmas.

Lemma 9.2.9. *Let $(\Gamma(D), <)$ be a directed Wada graph of a triple hop link diagram D with triple hop X such that the left directed condition (or resp. right directed condition) is respected. If we substitute X by a rational tangle X' of same nature, then the new directed Wada graph $(\Gamma(D'), <)$ of the new triple hop link diagram D' with triple hop X' satisfies the left directed condition (or resp. right directed condition) .*

Lemma 9.2.10. *Let $(\Gamma(D), <)$ be a directed Wada graph of a triple hop link diagram D with triple hop X such that the the left middle hop condition (or resp. right middle hop condition) is respected. If we substitute X by a rational tangle X' of same nature, then the new directed Wada graph $(\Gamma(D'), <)$ of the new triple hop link diagram D' with triple hop X' satisfies the left middle hop condition (or resp. right middle hop condition).*

Lemma 9.2.11. *Let $(\Gamma(D), <)$ be a trichotomic directed Wada graph of a triple hop link diagram D with triple hop X such that the left graph is left graph directed (or resp. the right graph is right directed). If we substitute X by a rational tangle X' of same nature, then the new directed Wada graph $(\Gamma(D'), <)$ of the new triple hop link diagram D' with triple hop X' is trichotomic and the left graph is left graph directed (or resp. the right graph is right directed).*

Thus, directly by the previous lemmas, a good triple hop link stays a good triple hop link, if we change the triple hop rational tangle by a rational tangle of the same nature.

Proposition 9.2.12. *Let D be a good middle (resp. final) triple hop diagram with X the triple hop rational tangle. If we substitute X by a rational tangle X' of the same nature, then the new link diagram D' is a good middle (resp. final) triple hop diagram with X' the triple hop rational tangle.*

Hence, by Theorems 9.1.7 and 9.2.5 we have the final result of this chapter.

Theorem 9.2.13. *Let D be a good middle (resp. final) triple hop diagram of a link L with X the triple hop rational tangle. If we substitute X by a rational tangle X' of the same nature, then for the new link L' obtained from the new rational tangle X', $\pi_1(\Sigma(L'))$ is not left-orderable.*

The knot 9_{47} is a good middle triple hop diagram.

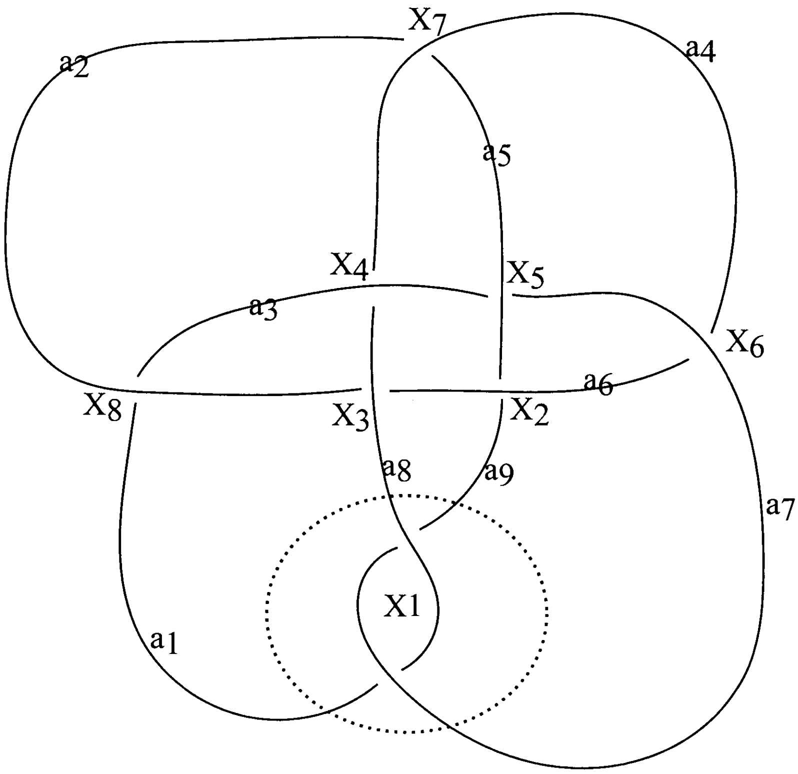


Figure 9.3 A knot diagram of the knot 9_{47}.

By Theorem 9.2.13, if we replace the rational tangle X_1 by a similar rational tangle, then the fundamental group of the double branched cover of the new link is not left-orderable.

CONCLUSION

We have obtained results about the non-left-orderability of directed and non-directed 2-non-bridge links. In further work, more results could be obtained for the non-left-orderability of directed links, non-directed 2-non-bridge links and non-directed k-non-bridge links with $k \geq 3$. Moreover, with the machinery developed in this thesis we should be able to obtain results on the left-orderability of links.

For directed links, we have shown that the totally monopositive, $(n-1)$ totally monopositive and $(n-2)$ steady fluid totally monopositive links have double branched covers with non-left-orderable fundamental groups. We are confident that we could either enlarge these family or find new ones. For example, the knots 10_{156} is almost an $(n-2)$ steady fluid totally monopositive knots but not quite. However, we can easily prove by hand that it was not left-orderable. We should be able to define a family that includes 10_{156} and prove non-left-orderability for this family of links.

For non-directed links, we have used the triple hop machinery. However, the left or right directed conditions can be blocked by pluripositive rational tangles. For example, the middle triple hop knot 10_{155} is not a good middle triple hop knot because it is not left directed. However, we can prove the non-left-orderability by hand using an argument similar to the $(n-1)$ totally monopositive argument so it becomes a good middle triple hop knot. We could generalize this kind of argument to obtain $(n-1)$ good middle triple hop links and $(n-1)$ good final triple hop links.

More investigations could be done for k-non-bridge links with $k \geq 3$. For example, the 3-non-bridge knot 10_{161} is more difficult to study, a priori because we have three possibilities for the maximum and after two possibilities for the minimum. Therefore, there are many different directed Wada graphs and different directed Wada group to study. For now, we do not even know if the fundamental group of the double branched cover of 10_{161} is left-orderable or not.

In this thesis, we didn't find results for left-orderability, however we think that the machinery developed could also be useful to obtain results in this direction. In fact, empirically, the Graph group is the fundamental group of the double branched cover. It would be interesting to prove this result. Moreover, empirically, we can find a group presentation of the Graph group. If both results are true, then for every directed link, we obtain a possible group presentation for the fundamental group of the double branched cover. Therefore, if it is left-orderable, we have a presentation of the left-orderable group. Furthermore, we have a presentation of the left-orderable group, such that each generator is less than or equal to 1.

If the Graph group is the fundamental group of the double branched cover, then it would also be useful in the study of quasi-alternating links. We recall the definition of quasi-alternating links.

Definition 9.2.14. A link is called *quasi-alternating* if it belongs to the set $\mathcal{Q}$ that is the smallest set of links characterized by the following two properties:

1. The unknot is in $\mathcal{Q}$.

2. If L has a diagram D with a crossing c such that

   (a) the two smoothings D_0 and D_∞ at c represent links L_0, L_∞ both of which belong to $\mathcal{Q}$,

   (b)$\det(L_0)+\det(L_\infty)=\det(L)$,

   then L belongs to $\mathcal{Q}$.

Such a crossing c is called a *quasi-alternating crossing.*

If the Graph group is the fundamental group of the double branched cover, then we can find the determinant of the link from the abelianization of the fundamental group. From the machinery of Wada rational graphs, we know relatively well the effects of changing a rational tangle, in the link diagram, on the directed Wada group, and therefore on the Graph group. Thus, we could know the effect of changing a rational tangle in the link diagram on the determinant of the link. This knowledge would prove very useful for the property 2 b) of quasi-alternating links.

BIBLIOGRAPHY

Boyer, S. & Clay, A. (2017). Foliations, orders, representations, l-spaces and graph manifolds. *Adv. Math., 310*, *310*, 159–234.

Boyer, S., Gordon, C.-M. & Watson, L. (2013). On l-spaces and left-orderable fundamental groups. *Mathematische Annalen*, *356*, 1213–1245.

Boyer, S., Rolfsen, D. & Wiest, B. (2005). Orderable 3-manifold groups. *Annales de l'institut Fourier 55*, *55*, 1255–1296.

Caudron, A. (1987). *Classification des noeuds et des enlacements.* (PhD thesis). University of Paris XI.

Conway, J. H. (1970). An enumeration of knots and links, and some of their algebraic properties. *Computational problems in Abstract Algebra*, pp. 329–358.

Ernst, C. & Summers, D. W. (1990). A calculus for rational tangles: applications to dna recombination. *Mathematical Proceedings of the Cambridge Philosophical Society*, *108*, 489–515.

Gabai, D. (1984). Foliations and genera of links. *Topology*, *23*, 381–394.

Hanselman, J., Rasmussen, J., Rasmussen, S. & Watson, L. (2015). Taut foliations on graph manifolds. *Math. GT.*

Juhász, A. (2015). A survey of heegaard floer homology. *New ideas in low dimensional topology*, pp. 237–296.

Lisca, P. & Stipsicz, A. (2007). Ozsváth-szabó invariants and tight contact 3-manifolds. *J. Symplectic Geom. 5*, *3*(5), 357–384.

Rolfsen, D. (1993). The quest for a knot with trivial jones polynomial: Diagram surgery and the temperley-lieb algebra. In M. Bozhüyük (ed.), *Topics in Knot Theory* pp. 195–210. Springer Netherlands.

Vinogradov, A. (1949). On the free product of ordered groups. *Math. Sbornik*, *25*, 163–168.

Wada, M. (1992). Group invariants of links. *Topology*, *31*, 399–406.