UNIVERSITÉ DU QUÉBEC À MONTRÉAL

THE COXETER TRANSFORMATION ON COMINUSCULE POSETS

THESIS

PRESENTED

AS PARTIAL REQUIREMENT

TO THE PH.D IN MATHEMATICS

BY

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MAY 2018

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UNIVERSITÉ DU QUÉBEC À MONTRÉAL

LA TRANSFORMATION DE COXETER SUR LES ENSEMBLES ORDONNÉS COMINUSCULES

THÈSE

PRÉSENTÉE

COMME EXIGENCE PARTIELLE

DU DOCTORAT EN MATHÉMATIQUES

PAR

EMINE YILDIRIM

MAI 2018

ACKNOWLEDGMENTS

I would like to thank my supervisor Hugh Thomas for his generous support and for his careful reading of this thesis. His insight and enthusiasm guide me through whole my program. I always admired his intuition, working with him was a great experience.

I would like to thank Thomas Brüstle, Ralf Schiffler, Franco Saliola for reading the thesis and giving me their valuable suggestions.

I would like to thank Institut des Sciences Mathématiques (ISM) for the scholarships I received throughout my program. I am grateful for my friends and mentors Véronique Bazier-Matte, Nathan Chapelier, Aram Dermenjian, Guillaume Douville, Alexander Garver, Patrick Labelle, Kaveh Mousavand, Rebecca Patrias whom I worked with in working goups at LaCIM. I learned a lot from you! My office mates Aram Dermenjian and Kaveh Mousavand are awesome friends. I valued their unlimited support, they were there everytime I needed them. I would like to thank Aram for frequently leaving me encouraging notes around my desk to cheer me up.

I thank my mom Hatice, my father Mühendis, my siblings Fatma, Zeynep, Elif, Harun, and my nephew Eymen for being an amazing and supportive family. Everything becomes fantastic when I share it with them.

Finally, I would like to thank my precious Atabey for his never-ending kindness, support, and his unconditional love. He never ceases to amaze me with his unique point of view on life.

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RÉSUMÉ

Soit $\mathsf{J}(\mathsf{C})$ l'ensemble ordonné de parties commençantes dans un ensemble ordonné cominuscule C, où C est membre de deux des trois familles infinies d'ensembles ordonnés cominuscules, ou est un des deux ensembles cominuscules exceptionels. Nous démontrons que la translation de Auslander-Reiten τ sur le groupe de Grothendieck de la catégorie dérivée bornée pour l'algèbre d'incidence de l'ensemble ordonné $\mathsf{J}(\mathsf{C})$, qui s'appelle la transformation de Coxeter dans ce cas, a un ordre fini. Spécifiquement, nous démontrons que $\tau^{(h+1)} = \pm id$ où h est le nombre de Coxeter pour le système de racines pertinent. Écrivant Pan pour l'application de Panyushev, c'est connu que $\text{Pan}^h = id$ sur un ensemble ordonné cominuscule (Rush & Shi, 2013). Nous étudions aussi la relation entre la transformation de Coxeter et l'application de Panyushev.

Mots clés: Théorie des représentations, algèbre d'incidence, la transformation de Coxeter, la translation de Auslander-Reiten, ensemble ordonné cominuscule.

ABSTRACT

Let $\mathsf{J}(\mathsf{C})$ be the poset of order ideals of a cominuscule poset C where C comes from two of the three infinite families of cominuscule posets or the exceptional cases. We show that the Auslander-Reiten translation τ on the Grothendieck group of the bounded derived category for the incidence algebra of the poset $\mathsf{J}(\mathsf{C})$, which is called the *Coxeter transformation* in this context, has finite order. Specifically, we show that $\tau^{(h+1)} = \pm id$ where h is the Coxeter number for the relevant root system. Let Pan denote the Panyushev map. It is known that $\text{Pan}^h = id$ on cominuscule posets (Rush & Shi, 2013). We also investigate the relation between Coxeter transformation and Panyushev map.

Keywords: Representation theory, incidence algebra, Coxeter transformation, Auslander-Reiten translation, cominuscule poset.

INTRODUCTION

Let A be the incidence algebra of a poset P over a base field $\Bbbk$. If the poset P is finite, then A is a finite dimensional algebra with finite global dimension. We are interested in incidence algebras coming from cominuscule posets. A cominuscule poset can be thought of as a parabolic analogue of the poset of positive roots of a finite root system. Cominuscule posets (also called minuscule posets) appear in the study of representation theory and algebraic geometry, especially in Lie theory and Schubert calculus (Billey & Lakshmibai, 2000), (Green, 2013).

Let $\mathsf{J}(\mathsf{C})$ be the poset of order ideals of a given cominuscule poset C. The poset $\mathsf{J}(\mathsf{C})$ is an interesting object in its own right. For instance, there is a correspondence between the elements of $\mathsf{J}(\mathsf{C})$ and the minimal coset representatives of the corresponding Weyl group (Billey & Lakshmibai, 2000), (Rush & Shi, 2013). Many combinatorial properties of order ideals of cominuscule posets are explained in (Thomas & Yong, 2009). However, our main motivation for this thesis comes from a conjecture by Chapoton which can be stated as follows.

Let $\mathsf{J}(\Phi^+)$ be the poset of order ideals of Φ^+ where Φ^+ is the poset of positive roots of a finite root system Φ. Let H be a hereditary algebra of type Φ and let $\mathcal{T}_H$ be the poset of torsion classes. Consider the incidence algebras of $\mathsf{J}(\Phi^+)$ and $\mathcal{T}_H$. Chapoton conjectures that there is a triangulated equivalence between the bounded derived categories $\mathcal{D}^b(\operatorname{mod} \mathsf{J}(\Phi^+))$ and $\mathcal{D}^b(\operatorname{mod} \mathcal{T}_H)$ and that $\mathcal{D}^b(\operatorname{mod} \mathsf{J}(\Phi^+))$ is fractionally Calabi-Yau, i.e. some non-zero power of the Auslander-Reiten translation τ equals some power of the shift functor.

In (Chapoton, 2007), Chapoton proved that the Auslander-Reiten translation

τ on the Grothendieck group of the bounded derived category (which is called *Coxeter transformation* in this context) for Tamari posets is periodic. There is an abundance of studies on Coxeter transformation in the literature. We refer to Ladkani (Ladkani, 2008) for references on the topic and recent results on the periodicity of Coxeter transformation. Kussin, Lenzing, Meltzer (Kussin et al., 2013) showed that the bounded derived category is fractionally Calabi-Yau for certain posets by using singularity theory. Recently, Diveris, Purin, Webb (Diveris et al., 2017) introduced a new method to determine whether the bounded derived category of a poset is fractionally Calabi-Yau.

In this thesis, we consider Chapoton's Conjecture on the level of Grothendieck groups for cominuscule posets instead of root posets. We calculate the action of Auslander-Reiten translation τ on the bounded derived category $\mathcal{D}^b(\mathrm{mod}\,\mathsf{J}(\mathsf{C}))$ of the incidence algebra of the poset $\mathsf{J}(\mathsf{C})$ of order ideals of a cominuscule poset C, and then we write the action on the Grothendieck group. We show that the Auslander-Reiten translation τ acting on the corresponding Grothendieck group has finite order for two of the three infinite families of cominuscule posets, and for the exceptional cases. One can do this by considering the action of τ on any convenient set of generators. The obvious choices are the set of the isomorphism classes of simple modules, and the set of the isomorphism classes of projective modules. However, the periodicity of τ is not evident on these sets of generators. Instead, we consider a special spanning set of the Grothendieck group on which we see the periodicity combinatorially.

The plan of the thesis is as follows. In the first chapter, we give necessary background material we use throughout the thesis. In the second chapter, we give a short survey on derived categories, and we prove certain results which are important in our study. In the third chapter, we give the definition of the Grothendieck group in our setting. We explain the machinery of the Grothendieck group which we will apply in our study. In the fourth chapter, we introduce a special collection of projective resolutions for *grid posets* which are the first infinite family of cominuscule posets coming from the type A. Then, we also show the corresponding combinatorics for the homology of these projective resolutions. These projective resolutions provide a spanning set for the Grothendieck group, and in the same section we show that the Coxeter transformation τ permutes this collection. In the fifth section, we study the generalization of our main results for cominuscule posets. The sixth chapter is devoted to developing certain combinatorial tools and using them to show a connection between Coxeter transformation and the Panyushev map. Finally, in the conclusion, we further discuss the second infinite family of cominuscule posets with some examples. We also give a brief discussion about the Panyushev map in this context.

CHAPTER I

PRELIMINARIES

In this preliminary section, we will recall basic definitions to fix notation in the thesis.

1.1 Posets and order ideals

A partially ordered set P is a set together with a binary relation $\leq$ which is reflexive, antisymmetric, and transitive. Any partially ordered set P is going to be called a *poset* throughout this thesis. Two elements a, b are comparable in P if $a \leq b$ or $b \leq a$, otherwise we say they are incomparable. We say b covers a in P if $a < b$ and there is no element c such that $a < c < b$. An interval $[a, b]$ in P consists of all elements x such that $a \leq x \leq b$. A poset P is called locally finite when every interval in P is finite. When P is locally finite, we will represent it by its *Hasse diagram*, i.e. we represent every element in the poset P by a vertex and we put an edge between two vertices if one covers the other one. We use the convention that the arrows in the Hasse diagram go downwards in the poset even though we simply draw edges in the figures.

A chain is a subset of P in which every pair of elements is comparable. An antichain is a subset of P such that every pair of different elements is incomparable.

Definition 1.1.1. A *grid poset* $\mathsf{P}_{\mathsf{m,n}}$ is the product of two chains of length m and of length n. Explicitly, the elements of $\mathsf{P}_{\mathsf{m,n}}$ are the pairs (i, j) with $i \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, n\}$. We compare elements *entry-wise* in $\mathsf{P}_{\mathsf{m,n}}$, i.e. $(a, b) \leq (c, d)$ when $a \leq c$ and $b \leq d$.

A lattice L is a poset such that every pair of elements in L has a unique least upper bound, also called the *join*, and a unique greatest lower bound, also called the *meet*. A lattice is distributive if the join and meet operations distribute over each other.

Definition 1.1.2. An *order ideal* of a poset P is a subset $\mathsf{I} \subseteq \mathsf{P}$ such that if $b \in \mathsf{I}$ and $a \leq b$, then $a \in \mathsf{I}$. We write $\mathsf{J(P)}$ for the set of all order ideals of P.

Note that $\mathsf{J(P)}$ is always a distributive lattice. Recall that the fundamental theorem for finite distributive lattices states that if L is a finite distributive lattice, then there is a unique finite poset P such that $\mathsf{L} \cong \mathsf{J(P)}$ (Stanley, 2012).

Definition 1.1.3. A non-decreasing finite sequence of positive integers $(a_1, \ldots, a_m)$ is called *a partition*. We are going to represent such sequences by $\alpha = (\lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r})$ where $0 \leq \lambda_1 < \cdots < \lambda_r \leq n$ denote the distinct integers in the partition α while α_i's denote the number of repetitions of each λ_i.

For each $k \in \{1, \ldots, m\}$, by the *k-th row* of $\mathsf{P}_{\mathsf{m,n}}$ we mean elements of the form (k, b) in $\mathsf{P}_{\mathsf{m,n}}$. We represent an order ideal I in $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$ by its corresponding partition, i.e. the list of the number of elements of I in each row from top to bottom.

Example 1.1.4. Here is an example of a grid poset and its poset of order ideals with the corresponding partitions:

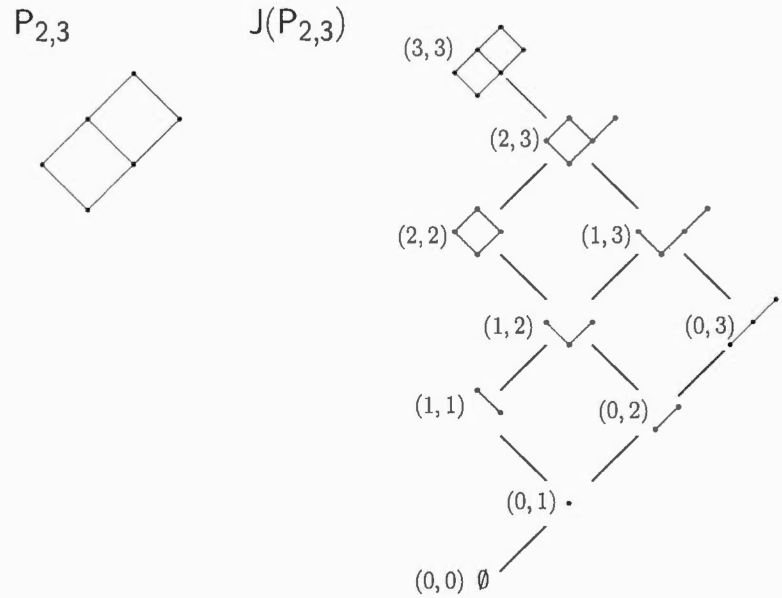


Figure 1.1 The Hasse diagrams of the grid poset $\mathsf{P}_{2,3}$ and the poset of order ideals $\mathsf{J}(\mathsf{P}_{2,3})$.

For instance, take the order ideal I represented by the partition $(2,3)$ in $\mathsf{J}(\mathsf{P}_{2,3})$. The order ideal I consists of two rows: there are two elements in the first row, and we have three elements in the second row.

Example 1.1.5. This example shows the poset of order ideals in $\mathsf{P}_{3,3}$. We draw some of the edges with dotted lines for the sake of presentation.

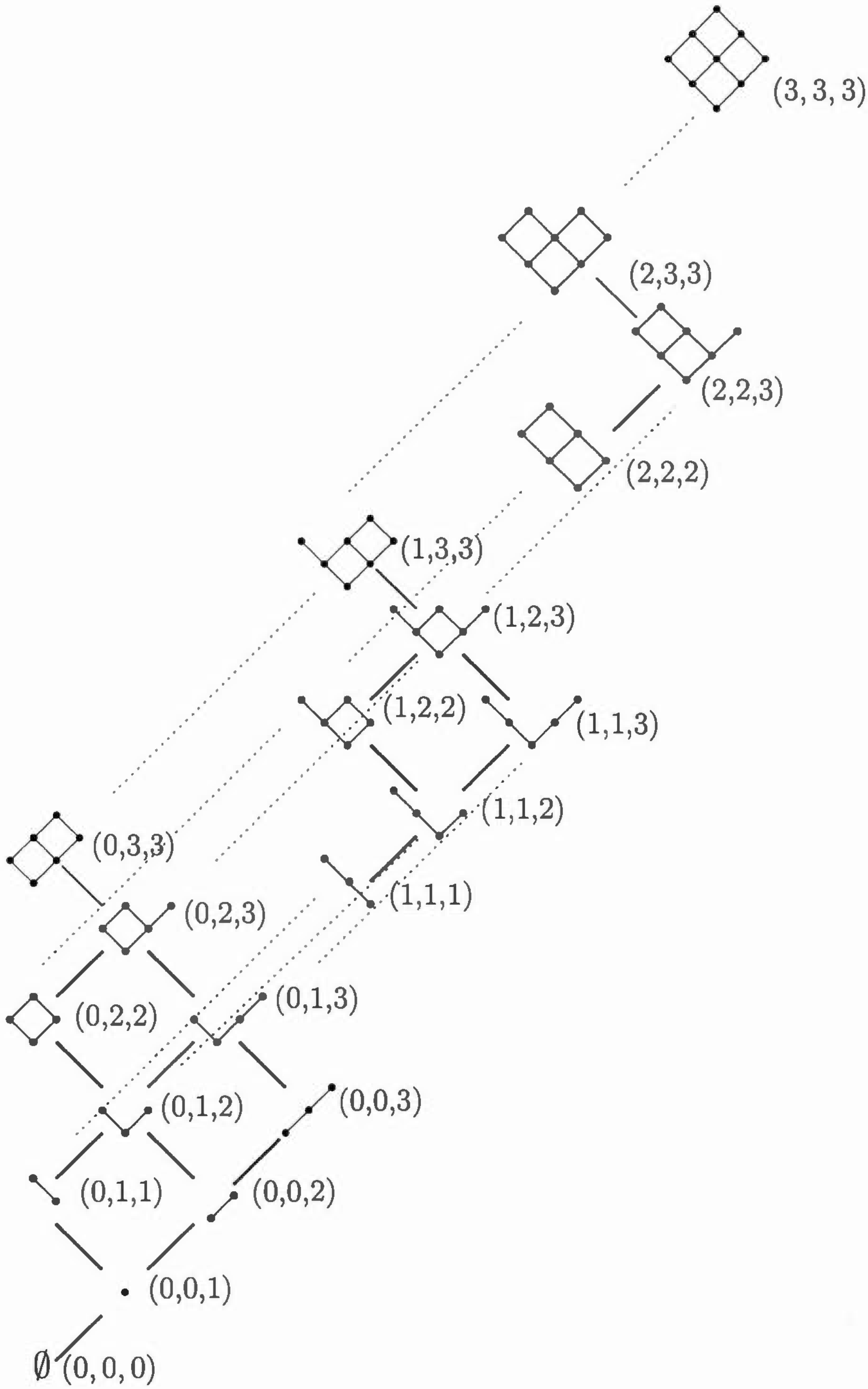


Figure 1.2 The Hasse diagram of the poset of order ideals $J(P_{3,3})$.

1.2 A quiver and representations of a quiver

A *quiver* consists of four pieces of information (Q_0, Q_1, s, t): Q_0 is the set of vertices, Q_1 is the set of arrows between vertices, and $s,\ t : Q_1 \to Q_0$ are two maps on the arrows where s sends an arrow to its starting vertex, call it *source* and t sends an arrow to its ending vertex, call it *target.*

A quiver is said to be finite if Q_0 and Q_1 are finite sets. Two arrows α and β are *composable* if $t(\alpha) = s(\beta)$, and we write the composition as *concatenation* of α and β, $\alpha\beta$. A *path* in Q is a sequence of composable arrows. A path of length greater than or equal to one is called a *cycle* if its source and target coincide. A cycle of length one is called a *loop.* A quiver is *acyclic* if it contains no cycles. For every vertex v in Q_0, we define the *lazy path* at v to be the path of length 0 at a vertex v. Let $\Bbbk$ be a ground field. A *relation* in Q is a $\Bbbk$-linear combination of path of length at least two, with the same source and the target.

All vector spaces we mention are over $\Bbbk$. A quiver representation $M = (V_i, f_a)$ is a collection of vector spaces V_i for each vertex $i \in Q_0$ and linear maps $f_a : V_i \to V_j$ for every arrow $a : i \to j \in Q_1$. If each vector space V_i is finite-dimensional, then M is called finite-dimensional. In this case, we define *the dimension vector* of M as $\dim(M) = (\dim V_i)_{i \in Q_0}$ where $\dim V_i$ denotes the dimension of the vector space at vertex $i \in Q_0$.

Let $M = (V_i, f_a)$ and $N = (W_i, g_a)$ be two representations of a quiver Q. We define a morphism of representations $\phi : M \to N$ as a collection of linear maps $(\phi_i)_{i \in Q_0}$ such that for every arrow $a \in Q_1$ the following diagram commutes:

$$\begin{array}{ccc} V_i & \xrightarrow{f_a} & V_j \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_j} \\ W_i & \xrightarrow[g_a]{} & W_j \end{array}$$

We define $Rep(Q)$ to be the category of $\Bbbk$-linear representations of Q and their morphisms and $rep(Q)$ as the full subcategory of $Rep(Q)$ consisting of finite dimensional representations. Both $Rep(Q)$ and $rep(Q)$ are abelian categories.

A quiver representation M is said to be indecomposable if M is nonzero and M has no direct sum decomposition $M \cong N \oplus L$, where L and N are nonzero representations of the quiver.

We recall here a fundamental result, the Krull-Schmidt theorem, in representation theory.

Theorem 1.2.1. *(Etingof et al., 2011, Theorem 2.19) Any finite dimensional representation can be uniquely decomposed into a direct sum of indecomposable representations up to an isomorphism and order of summands.*

Example 1.2.2. (Etingof et al., 2011) Let us now discuss the indecomposable representations of the A_3 quiver with the following orientation.

$$A_3 : \bullet \longrightarrow \bullet \longrightarrow \bullet$$

A representation of this quiver looks like

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$$

In total, for A_3 quiver, we have six indecomposable representations as listed below:

$$\Bbbk \longrightarrow 0 \longrightarrow 0\,,\ 0 \longrightarrow 0 \longrightarrow \Bbbk\,,$$
$$\Bbbk \longrightarrow \Bbbk \longrightarrow 0\,,\ \Bbbk \longrightarrow \Bbbk \longrightarrow \Bbbk\,,$$
$$0 \longrightarrow \Bbbk \longrightarrow 0\,,\ 0 \longrightarrow \Bbbk \longrightarrow \Bbbk\,.$$

Example 1.2.3. Let Q be the quiver $\bullet \longleftarrow \bullet \longrightarrow \bullet$ and X and Y be the representations; $\Bbbk \xleftarrow{1} \Bbbk \xrightarrow{1} \Bbbk$ and $\Bbbk \xleftarrow{1} \Bbbk \xrightarrow{0} 0$. Let us show that $\mathrm{Hom}(X, Y)$ is the one-dimensional and $\mathrm{Hom}(Y, X) = 0$.

$$\begin{array}{ccccc} \Bbbk & \xleftarrow{1} & \Bbbk & \xrightarrow{1} & \Bbbk \\ {\scriptstyle\lambda}\downarrow & & {\scriptstyle\lambda}\downarrow & & {\scriptstyle 0}\downarrow \\ \Bbbk & \xleftarrow{1} & \Bbbk & \xrightarrow{0} & 0 \end{array}$$

The only morphisms are as shown. This shows that $\mathrm{Hom}(X, Y) = \Bbbk$

$$\begin{array}{ccccc} \Bbbk & \xleftarrow{1} & \Bbbk & \xrightarrow{0} & 0 \\ {\scriptstyle\lambda}\downarrow & & {\scriptstyle\mu}\downarrow & & {\scriptstyle 0}\downarrow \\ \Bbbk & \xleftarrow{1} & \Bbbk & \xrightarrow{1} & \Bbbk \end{array}$$

The only values for λ and μ which make the diagram commute are $\lambda = \mu = 0$. From this, we can easily see that $\mathrm{Hom}(Y, X) = 0$.

1.3 Path algebras

Let $P(Q)$ be the set of all paths over a quiver Q. We define the path algebra of the quiver Q as $A = \Bbbk Q := span_{\Bbbk}(P(Q))$ with a $\Bbbk$-linear multiplication $\mu : A \times A \to A$ as follows

$$\mu(p, q) = \begin{cases} pq & \text{if } s(q) = t(p) \\ 0 & \text{otherwise} \end{cases}$$

where $p,\ q \in P(Q)$. This multiplication is unital (unit being the sum of lazy paths) if Q_0 is finite, and it is associative.

Lemma 1.3.1. *(Assem et al., 2006) The path algebra $A = \Bbbk Q$ is finite dimensional if and only if Q is finite and acyclic.*

Assume that A is a finite dimensional $\mathbb{k}$-algebra. An element $e \in A$ is called an idempotent if $e^2 = e$. A finite list of idempotents $\{e_1, \cdots, e_n\}$ is called a complete set of primitive pairwise orthogonal idempotents if

1. $e_i^2 = e_i$,
2. $e_i e_j = 0$ if $i \neq j$,
3. e_i cannot be written as a sum of two nontrivial orthogonal idempotents,
4. $1_A = \sum_{i=1}^{n} e_i$.

The algebra A is called basic if $e_i A \ncong e_j A$ for all $i \neq j$. For instance, the path algebra $\mathbb{k}Q$ satisfies these conditions; e_i's are the vertices, in other words paths of lenght zero.

Let J be the two-sided ideal of the path algebra $A = \mathbb{k}Q$ generated by the arrows of Q. A two-sided ideal I of $A = \mathbb{k}Q$ is called admissible if there exists $m \geq 2$ such that $J^m \subseteq I \subseteq J^2$ where J^i, $2 \leq i \leq m$ is the two-sided ideal generated by paths of length greater or equal to i.

Example 1.3.2. Let the quiver Q be defined as

$$\begin{array}{ccc} e & \xrightarrow{x} & f \\ {\scriptstyle z}\downarrow & & {\scriptstyle y}\downarrow \\ g & \xrightarrow{t} & h \end{array}$$

In this case, the set of paths is $P(Q) = \{e, f, g, h, x, y, z, t, xy, zt\}$. Some examples of admissible ideals are $span_{\mathbb{k}}\{xy\}$, $span_{\mathbb{k}}\{zt\}$, $span_{\mathbb{k}}\{xy, zt\}$, $span_{\mathbb{k}}\{xy - zt\}$. The vector space $span_{\mathbb{k}}\{x, xy, zt\}$ is an example of a non-admissible ideal.

We recall the following results:

Proposition 1.3.3. *(Assem et al., 2006) Let Q be a finite quiver with admissible ideal I in $\Bbbk Q$, then $\Bbbk Q/I$ is finite dimensional.*

Let (Q, R) be the quiver with a set of relations R which consists of some linear combination of paths of at least length two, with the same source and the target. We call (Q, R) a bound quiver. Let $rep(Q, R)$ be the full subcategory of $rep(Q)$ consisting of the representations of Q bound by the relations in R.

Theorem 1.3.4. *(Assem et al., 2006) Let Q be a finite connected quiver and I be an admissible ideal of $\Bbbk Q$. There is an equivalence between the category* $\operatorname{mod} A$ *of right modules over the algebra $A = \Bbbk Q/I$ and the category $rep(Q, I)$ of representations of a quiver Q bound by I.*

The equivalence of these categories is given by the following functors:

$$F : \operatorname{mod} A \to rep(Q, I) \text{ and } G : rep(Q, I) \to \operatorname{mod} A$$

For a module $M \in \operatorname{mod} A$, $F(M) = (V_i, f_a)_{i \in Q_0, a \in Q_1}$ where $V_i = Me_i$ which is the vector space consisting of all ve_i for all $v \in M$ and $f_a : V_i \to V_j$ is defined as follows: for every arrow $a : i \to j$

$$f_a(ve_i) = v(e_i a) = \begin{cases} va \text{ if } s(a) = i \\ 0 \text{ otherwise.} \end{cases}$$

For every map $\phi : M \to M'$ in $\operatorname{mod} A$, $F(\phi)$ is the morphism defined as $\phi_i : M_i \to M'_i$ for the vertex i which sends $ve_i \mapsto \phi(v)e_i$.

For a quiver representation $V = (V_i, f_a)_{i \in Q_0, a \in Q_1}$, $G(V) = M$ where $M = \bigoplus_{i \in Q_0} V_i$ with the action of A is defined as follows: for $v = (v_i)_{i \in Q_0} \in M$, and $s =$

$\sum \delta_c c + I \in A$ where c runs over all paths in Q, $vs = \sum \delta_c f_c(v)$ where $f_c(v) = (0, \cdots, 0, f_c(v_{s(c)}), 0, \cdots, 0)$ with the only nonzero entry in $t(c)$.

For every map $\psi = (\psi_i)_{i \in Q_0} : (V_i, f_a) \to (V_i', f_a')$, $G(\psi) : V \to V'$ defined by $G(f)(m) = (\psi_i(v_i))_{i \in Q_0}$.

1.4 The indecomposable projective modules

We first recall the following.

Definition 1.4.1. An A-module X is called *decomposable* if it can be written as a direct sum of two non-trivial A-modules as $X = Y \oplus Z$. Otherwise, it is called *indecomposable*. A module X is called *simple* if it has no non-trivial submodules. Note that each simple module is indecomposable, but the converse need not be true.

Let i be a vertex of the quiver Q. We define P_i to be the indecomposable projective module at the vertex i which is spanned by the set of all paths starting at i. Let us use $x_{i \geq j}$ to denote the unique path that starts at vertex i and ends at vertex j. Then, the morphisms between two indecomposable projective modules P_i and P_j are

$$\mathrm{Hom}_A(P_i, P_j) = \mathrm{Hom}_A(x_{i \geq i} A, x_{j \geq j} A) = x_{j \geq j} A x_{i \geq i}$$

By definition, P_i is a subspace of $\Bbbk Q$. But, it is also a submodule, and therefore, it is a right ideal. We also have a decomposition of $\Bbbk Q$ as a direct sum of indecomposable modules as

$$\Bbbk Q = \bigoplus_{i \in V} P_i = \bigoplus_{i \in V} e_i \Bbbk Q$$

Let $A = \Bbbk Q / I$ be a quotient of a quiver algebra with $\Bbbk Q$ an admissible ideal I. If the algebra A is finite dimensional, the algebra A viewed as a right module over itself, then it admits a direct sum decomposition

$$A = P_1 \oplus \cdots \oplus P_n$$

where $P_i = e_iA$ are projective indecomposable modules which are also ideals and the e_i's are a set of primitive pairwise orthogonal idempotents of A such that $1 = \sum_{i=1}^{n} e_i$. This is a decomposition of A as a sum of indecomposable summands.

Example 1.4.2. Let Q be the quiver depicted as follows:

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3$$

Then the indecomposable projective modules are

$$P_1 : \Bbbk \longrightarrow \Bbbk \longrightarrow \Bbbk$$

$$P_2 : 0 \longrightarrow \Bbbk \longrightarrow \Bbbk$$

$$P_3 : 0 \longrightarrow 0 \longrightarrow \Bbbk$$

Example 1.4.3. Let Q be the quiver depicted as follows:

$$\bullet_1 \rightarrow \bullet_2 \rightarrow \bullet_3, \quad \bullet_1 \rightarrow \bullet_3$$

Then the indecomposable projective modules are

$$P_1 : \Bbbk \rightarrow \Bbbk \xrightarrow{\binom{0}{1}} \Bbbk^2, \quad \Bbbk \xrightarrow{\binom{1}{0}} \Bbbk^2$$

$$P_2 : 0 \rightarrow \Bbbk \rightarrow \Bbbk, \quad 0 \rightarrow \Bbbk$$

$$P_3 : 0 \rightarrow 0 \rightarrow \Bbbk, \quad 0 \rightarrow \Bbbk$$

We define I_i to be the indecomposable injective module at the vertex i which is spanned by the set of all paths ending at i. The simple module S_i at vertex i is spanned by the lazy path at i. For a more detailed description of indecomposable projective, injective and simple modules over a quotient of a path algebra, see (Assem et al., 2006), (Schiffler, 2014).

1.5 Nakayama functor

Let $A = \Bbbk Q/I$ be a quotient of a quiver algebra $\Bbbk Q$ with an admissible ideal I.

Definition 1.5.1. The endofunctor $\nu = D\mathrm{Hom}_A(-, A)$ on $\mathrm{mod}\, A$ is called the *Nakayama functor* where $D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk)$ is a duality between left and right A-modules.

Lemma 1.5.2. *The Nakayama functor ν is right exact and is functorially isomorphic to $- \otimes_A DA$.*

Proof. The functor D is contravariant exact, and $\mathrm{Hom}_A(-, A)$ is a contravariant left exact functor. So, their composition ν is right exact. The isomorphism between ν and $- \otimes_A DA$ is given by the following map.

$$\phi : X \otimes_A DA \to D\mathrm{Hom}_A(X, A)$$

with $x \otimes f \mapsto (\psi \mapsto f(\psi(x)))$ where $x \in X$, $f \in DA$, and $\psi \in \mathrm{Hom}_A(X, A)$. □

Let $\mathrm{proj}\, A$ be the full subcategory of $\mathrm{mod}\, A$ whose objects are the projective modules and $\mathrm{inj}\, A$ be the full subcategory of $\mathrm{mod}\, A$ whose objects are the injective modules.

Lemma 1.5.3. *The Nakayama functor ν induces an equivalence between* $\mathrm{proj}\, A$ *and* $\mathrm{inj}\, A$. *The quasi-inverse of this equivalence is $\nu^{-1} = Hom_A(DA, -)$ from* $\mathrm{inj}\, A$ *to* $\mathrm{proj}\, A$.

Proof. Let P_x be the indecomposable projective module e_xA where e_x is the idempotent at vertex x. Then, we have $\nu P_x = D\mathrm{Hom}_A(e_xA, A) \cong D(Ae_x) = I_x$ where I_x is the injective module at vertex x. Similarly, we can show that $\mathrm{Hom}_A(DA, I_x) = \mathrm{Hom}_A(DA, D(Ae_x)) \cong \mathrm{Hom}_{A^{op}}(Ae_x, A) \cong e_xA = P_x$. □

1.6 Coxeter transformation

Definition 1.6.1. Let A be a basic finite dimensional algebra with a complete set $\{e_1, e_2, \cdots, e_n\}$ of primitive orthogonal idempotents. The *Cartan matrix* C_A of A is the $n \times n$ matrix

$$C_A = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix} \in \mathbb{M}_n(\mathbb{Z})$$

where $c_{ji} = \dim_{\Bbbk} e_iAe_j$, for $i, j = 1, \cdots, n$.

Example 1.6.2. Let Q be the same quiver as in 1.3.2. We consider the path algebra $A = \Bbbk Q/I$ where I is the ideal generated by the relation $< xy - zt >$. Then, the Cartan matrix is

$$C_A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The following lemma states that since we have the isomorphisms

$$e_yAe_x \cong \mathrm{Hom}_A(P_x, P_y) \cong \mathrm{Hom}_A(I_x, I_y)$$

the Cartan matrix C_A records the number of linearly independent homomorphisms between the indecomposable projective modules and the number of linearly independent homomorphisms between the indecomposable injective modules.

Lemma 1.6.3. *We have the following properties for the Cartan matrix C_A;*

1. *The i-th column of C_A is the dimension vector* $\dim P_i$ *of the indecomposable projective P_i,*

2. *The i-th row of C_A is the transpose of the dimension vector* $\dim I_i$ *of the indecomposable projective I_i.*

Definition 1.6.4. The *Coxeter matrix* of A is defined as $\Phi_A = -C_A^t C_A^{-1}$. The corresponding linear transformation $\Phi_A \colon \mathbb{Z}^n \to \mathbb{Z}^n$ defined by matrix multiplication $\Phi_A(x) = \Phi_A x$ for all $x \in \mathbb{Z}^n$ is called the *Coxeter transformation.*

Notice that since dimension vectors are elements of $\mathbb{Z}^n$, we can apply the Coxeter transformation on dimension vectors.

Example 1.6.5. We continue with Example 1.6.2. In this case, the Coxeter matrix is the following

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

Lemma 1.6.6. $\Phi_A \dim P_i = -\dim I_i$.

Proof. Let $\dim S_i$ be the dimension vector of the simple module at vertex i. So, we have the following by Lemma 1.6.3

$$\Phi_A \dim P_i = -C_A^t C_A^{-1} \dim P_i = -C_A^t \dim S_i = -\dim I_i$$

□

Observe that we have $\Phi_A \dim P_i = -\dim \nu P_i$ for every projective module P_i since $\nu P_i = I_i$.

1.7 Incidence algebra of a poset

For a given locally finite poset P, we define the incidence algebra of P as the path algebra of the Hasse diagram of P modulo the relation that any two paths are equal if their starting and ending points are the same. Recall that we draw edges in the Hasse diagram oriented downwards.

Throughout this thesis, let $\mathcal{A}$ be the incidence algebra of $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$. The algebra $\mathcal{A}$ has finite global dimension since we do not have any oriented cycles in the quiver of $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$. Let $\operatorname{mod}\mathcal{A}$ be the category of finitely generated right modules over $\mathcal{A}$. Let P_α be an indecomposable projective module over the algebra $\mathcal{A}$ for a vertex α in $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$ and denote $x_{\alpha\geq\beta}$ the unique path that starts at vertex α and ends at vertex β. Such elements in P_α form a $\Bbbk$-basis for P_α.

CHAPTER II

DERIVED CATEGORIES

Our treatment of the subject follows Krause's Chicago Lectures on derived categories (Krause, 2007). Let $\Bbbk$ be a ground field, and let A be a finite dimensional unital algebra over $\Bbbk$. We will use $\operatorname{mod} A$ to denote the category of finite dimensional right A-modules.

2.1 Category of complexes

Throughout this thesis, we will use the cohomological convention for complexes: all differentials increase the degree by one and we use superscripts to denote the degree at which the module is placed in the complex.

Definition 2.1.1. A *chain complex* C over an algebra A is defined as a diagram of A-modules and morphisms of A-modules

$$\cdots \to C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \to \cdots$$

such that $d_{n+1}d_n = 0$ for all $n \in \mathbb{Z}$.

Definition 2.1.2. The *n-th homology* $H^n(C)$ of a complex C is defined as follows:

$$H^n(C) = \operatorname{Ker} d_n / \operatorname{Im} d_{n-1}$$

A complex C is called *acyclic* if $H^n(C)$ is the trivial A-module 0 for every $n \in \mathbb{Z}$.

Definition 2.1.3. A *chain map* between two complexes C and D is a sequence of maps $f = (f^n \colon C^n \to D^n)_{n \in \mathbb{Z}}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{n-1} & \xrightarrow{d_{n-1}} & C^n & \xrightarrow{d_n} & C^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & D^{n-1} & \xrightarrow[d_{n-1}]{} & D^n & \xrightarrow[d_n]{} & D^{n+1} & \longrightarrow & \cdots \end{array}$$

The $\Bbbk$-vector space of chain maps between two complexes C and D is denoted by $\operatorname{Hom}(C, D)$, and the category of chain complexes of A-modules is denoted by $Ch(A)$.

Definition 2.1.4. A complex C in $Ch(A)$ is called *bounded below* if there is an index $N \in \mathbb{Z}$ such that $C_n = 0$ for every $n \leq N$. Similarly, C is called *bounded above* if there is an $N \in \mathbb{Z}$ such that $C_n = 0$ for every $n \geq N$. Finally, a complex C is called *bounded* if it is both bounded below and bounded above. The subcategory of bounded above complexes is denoted by $Ch_-(A)$, the subcategory of bounded below complexes is denoted by $Ch_+(A)$, and finally the subcategory of bounded complexes is denoted by $Ch_b(A)$.

Proposition 2.1.5. *The n-th homology is a functor of the form $H^n \colon Ch(A) \to$ $\operatorname{mod} A$.*

Proof. Assume (C, d^C) and (D, d^D) are complexes in $Ch(A)$, and let $f \colon C \to D$ be a map of complexes. By definition,

$$f^n(\operatorname{Ker}(d_n^C)) \subseteq \operatorname{Ker}(d_n^D) \quad \text{and} \quad f^n(\operatorname{Im}(d_n^C)) \subseteq \operatorname{Im}(d_n^D)$$

for every $n \in \mathbb{Z}$. Then there is a well-defined map $H^n(f) \colon H^n(C) \to H^n(D)$. The fact that $H^n(id_C)$ is the identity map, and that $H^n(gf) = H^n(g)H^n(f)$ for a pair of composable map of complexes, $f \in \operatorname{Hom}(C, D)$ and $g \in \operatorname{Hom}(D, E)$, and for every $n \in \mathbb{Z}$ follows from the definitions. □

2.2 Homotopy category

Definition 2.2.1. We say a chain map $f : C \to D$ is *null-homotopic* if there is a sequence of maps $h = (h_n : C^n \to D^{n-1})$ such that for the following diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{n-1} & \xrightarrow{d_{n-1}} & C^{n} & \xrightarrow{d_n} & C^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & \swarrow{\scriptstyle h^n} & \downarrow{\scriptstyle f^{n}} & \swarrow{\scriptstyle h^{n+1}} & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & D^{n-1} & \xrightarrow[d_{n-1}]{} & D^{n} & \xrightarrow[d_n]{} & D^{n+1} & \longrightarrow & \cdots
\end{array}$$

we have $f^n = d_{n-1}h^n + h^{n+1}d_n$ for all $n \in \mathbb{Z}$. If we have two chain maps f, g, then we say f is *homotopic* to g if $f - g$ is null-homotopic. The chain map h is called *a homotopy.*

Definition 2.2.2. A collection of chain maps $\mathcal{I}$ in the category $Ch(A)$ is called a *right ideal* if for every morphism $f \in \mathrm{Hom}(C, D)$ in $Ch(A)$ and every morphism $g \in \mathrm{Hom}(D, E) \cap \mathcal{I}$ the composition gf is also in $\mathcal{I}$. Left ideals and two sided ideals are defined similarly.

Lemma 2.2.3. *Let $\mathcal{I}$ be a two sided ideal of $Ch(A)$. The quotient category $Ch(A)/\mathcal{I}$ which shares the same set of objects with morphisms are defined as quotients*

$$Hom(C, D)/\mathcal{I}(C, D)$$

is a $\Bbbk$-linear category.

Proof. The fact that each Hom object for the quotient is a $\Bbbk$-vector space and compositions are $\Bbbk$-bilinear is obvious. The fact that the compositions are well-defined follows from the fact that $\mathcal{I}$ is a two sided ideal. □

Proposition 2.2.4. *The subcategory $Null(A)$ of null homotopic maps is a two sided ideal in $Ch(A)$. The quotient category $Ch(A)/Null(A)$ is called* the homotopy category of chain complexes of A-modules, *and is denoted by $K(A)$.*

The corresponding subcategories of bounded above, bounded below and bounded complexes are denoted by $K^-(A)$, $K^+(A)$, and $K^b(A)$, respectively.

2.3 Multiplicative systems

Definition 2.3.1. A subcategory S of $K(A)$ is called a *a multiplicative system* if

(i) For every $s \in S(C, D)$ and $f \in \mathrm{Hom}_{K(A)}(C, C')$, there are morphisms $s' \in S(C', D')$ and $f' \in \mathrm{Hom}_{K(A)}(D, D')$ such that the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{s} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ C' & \xrightarrow[s']{} & D' \end{array}$$

(ii) For every $s \in S(C, D)$ and $g \in \mathrm{Hom}_{K(A)}(D', D)$, there are morphisms $s' \in S(C', D')$ and $g' \in \mathrm{Hom}_{K(A)}(C', C)$ such that the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{s} & D \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle g} \\ C' & \xrightarrow[s']{} & D' \end{array}$$

(iii) For every $s, t \in S(C, D)$, there is a $f \in \mathrm{Hom}_{K(A)}(D, D')$ with $fs = ft$ if and only if there is a $g \in \mathrm{Hom}_{K(A)}(C', C)$ such that $sg = tg$.

Theorem 2.3.2. *(Gabriel & Zisman, 1967) Assume S is a multiplicative system in $K(A)$. Then, there is a category $S^{-1}K(A)$ in which morphisms of S are all invertible.*

2.4 Derived category

Definition 2.4.1. A map of complexes f is a *quasi-isomorphism* if f induces isomorphisms in all homology groups, i.e. $H^n(f)\colon H^n(C) \to H^n(D)$ is an isomorphism for every $n \in \mathbb{Z}$.

Proposition 2.4.2. *(Krause, 2007, Sect. 3.1) The subcategory of quasi-isomorphisms $Q(A)$ of $K(A)$ forms a multiplicative system. The resulting localization $Q(A)^{-1}K(A)$ is called the (unbounded) derived category of A-modules, and is denoted by $\mathcal{D}(A)$.*

The corresponding derived subcategories of bounded above, bounded below and bounded complexes are denoted by $\mathcal{D}^-(A)$, $\mathcal{D}^+(A)$, and $\mathcal{D}^b(A)$, respectively.

2.5 Exact sequences and exact functors

Definition 2.5.1. A composable pair of morphisms of A-modules $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called *an exact sequence* if $\mathrm{Ker}(g) = \mathrm{Im}(f)$. Note that for such an exact sequence $X = 0$ if and only if g is injective, and $Z = 0$ if and only if f is surjective. Similarly,

$$0 \to X \xrightarrow{f} Y \to 0$$

is an exact sequence if and only if f is an isomorphism.

A composable pair of morphisms of A-modules of the form $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is called *a short exact sequence* if it is exact and f is injective, g is surjective.

With this definition in hand, we see that a complex C is acyclic when C viewed as a sequence of morphisms

$$\cdots \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \xrightarrow{d_{n+1}} \cdots$$

is a long exact sequence, i.e. $\mathrm{Im}(d_n) = \mathrm{Ker}(d_{n+1})$ for every $n \in \mathbb{Z}$.

Definition 2.5.2. Assume $F\colon \operatorname{mod} A \to \operatorname{mod} B$ is a functor. Then F is called *right exact* if every short exact sequence of A-modules $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is mapped to an exact sequence of the form

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0,$$

and similarly, F is called *left exact* if we have an exact sequence

$$0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z).$$

Such a functor is called *exact* if it is both left and right exact.

Example 2.5.3. Assume M is a left A-module and a right B-module such that

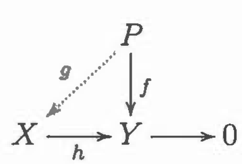
$$(a \cdot m) \cdot b = a \cdot (m \cdot b)$$

for every $m \in M$, $a \in A$ and $b \in B$. The functor $F\colon \operatorname{mod} A \to \operatorname{mod} B$ from the category of right A-modules to the category of right B-modules defined on the objects as $F(X) = X \otimes_A M$ is right exact. On the other hand, $G(X) = \operatorname{Hom}_A(X, M)$ defines a left exact contravariant functor from the category of left A-modules to the category of right B-modules.

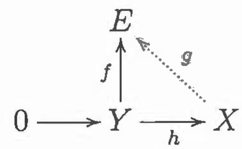
2.6 Resolutions

We start recalling some definitions.

Definition 2.6.1. An A-module P is called *projective* if for every epimorphism of A-modules $h\colon X \to Y$, every morphism of A-modules $f\colon P \to Y$ lifts to a morphism $g\colon P \to X$ such that the following diagram commutes:



Similarly, we call an A-module E *injective* if for every monomorphism of A-modules $h\colon Y \to X$, every morphism of A-modules $f\colon Y \to E$ lifts to a morphism $g\colon X \to E$ such that the following diagram commutes:



Definition 2.6.2. For a module M from $\operatorname{mod} A$, by the *stalk complex* of M we mean the complex which consists of just M at one degree and 0 everywhere else.

Definition 2.6.3. For a module M from $\operatorname{mod} A$, by a *resolution* of M we mean a complex $\mathcal{X} = (X^i)_{i\in\mathbb{Z}}$ where $X^i = 0$ for all $i > 0$ or for all $i < 0$ with the property that the homology $H^0(\mathcal{X})$ is M while $H^n(\mathcal{X})$ is zero for all $n \neq 0$.

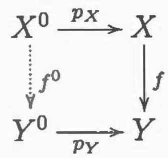
If $\mathcal{X}$ is bounded from above and each X^i is projective, then $\mathcal{X}$ is called a *projective resolution.* Similarly, if $\mathcal{X}$ is bounded from below and each X^i is injective, then $\mathcal{X}$ is called an *injective resolution.*

Notice that if $\mathcal{X}$ is a projective resolution of a module X, then the natural surjection $\mathcal{X} \to X$ of complexes is a quasi-isomorphism. Similarly, if $\mathcal{E}$ is an injective resolution of X, then the natural injection $X \to \mathcal{E}$ is also a quasi-isomorphism.

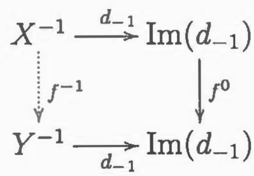
Lemma 2.6.4. *Assume $f\colon X \to Y$ is a morphism of A-modules. Assume $\mathcal{X}$ is a projective (respectively, injective) resolution of X, and $\mathcal{Y}$ is a projective (respectively, injective) resolution of Y. Then there is a unique (up to a homotopy) chain map of complexes of the form $f^*\colon \mathcal{X} \to \mathcal{Y}$ such that $H^0(f^*) = f$.*

Proof. We give the proof below for the projective case only. The proof for the injective case is similar, and we omit it. Since $\mathcal{X}$ is a resolution of X and $\mathcal{Y}$ is a resolution of Y, there are epimorphisms of A-modules $p_X\colon X^0 \to X$ and

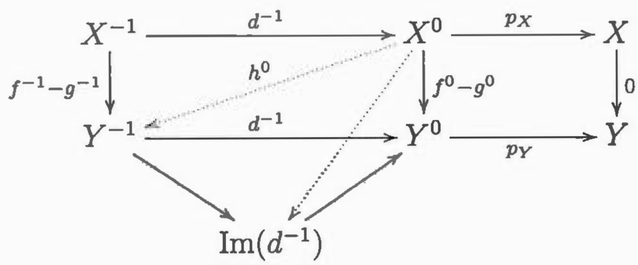
$p_Y\colon Y^0 \to Y$. Then there is a commutative square



Here f^0 lifts the morphism $f \circ p_X$. Since the square above is commutative, f^0 restricts to a morphism of the form $f^0\colon \operatorname{Ker}(p_X) \to \operatorname{Ker}(p_Y)$. Since both $\mathcal{X}$ and $\mathcal{Y}$ are resolutions, we must have $\operatorname{Ker}(p_X) = \operatorname{Im}(d_{-1})$ and $\operatorname{Ker}(p_Y) = \operatorname{Im}(d_{-1})$. So, we now have



where this time f^{-1} lifts the composition $f^0 d_{-1}$. Now, we proceed by induction to get the chain map f^*. As for our second claim, assume we have two lifts f^* and g^* for the same morphism of A-modules $f\colon X \to Y$. Then, we get a commutative diagram of the form



where

1. the lift $X^0 \to \operatorname{Im}(d^{-1})$ exists because $\operatorname{Im}(d^{-1}) = \operatorname{Ker}(p_Y)$ and $p_Y \circ (f^0 - g^0) = 0$,

2. the lift $h^0\colon X^0 \to Y^{-1}$ exists because X^0 is projective and $Y^{-1} \to \operatorname{Im}(d^{-1})$ is an epimorphism,

3. and we have $d^{-1}h^0 = f^0 - g^0$.

Note that we do not necessarily have $h^0 d^{-1} = f^{-1} - g^{-1}$, but we have

$$d^{-1}(f^{-1} - g^{-1} - h^0 d^{-1}) = d^{-1}f^{-1} - d^{-1}g^{-1} - d^{-1}h^0 d^{-1} = 0$$

Thus, we get a lift of $f^{-1} - g^{-1} - h^0 d^{-1} \colon X^{-1} \to Y^{-1}$ first to $X^{-1} \to \mathrm{Im}(d^{-2})$ then to $h^{-1} \colon X^{-1} \to Y^{-2}$ that satisfies

$$d^{-2}h^{-1} = f^{-1} - g^{-1} - h^0 d^{-1}$$

Proceeding by induction, we get maps $h^i \colon X^i \to Y^{i-1}$ such that

$$d^{i-1}h^i = f^i - g^i - h^{i-1}d^i$$

for every $i \in \mathbb{N}$ which gives us the desired null homotopy. □

2.7 Derived functors

Definition 2.7.1. Assume $F\colon \operatorname{mod} A \to \operatorname{mod} B$ is a right exact functor. For every $p \in \mathbb{N}$ we define the left derived functor $L^pF\colon \operatorname{mod} A \to \operatorname{mod} B$ as $L^pF(X) := H^p(F(\mathcal{X}))$ where $\mathcal{X}$ is any projective resolution of X. Note that L^pF is well-defined because of Lemma 2.6.4, and that $L^0F = F$ since F is right exact.

Example 2.7.2. Recall that for a A-B-bimodule M, the functor $\_ \otimes_A M$ defines a right exact covariant functor from the category of right A-modules to the category of right B-modules. The left derived functors of $\_ \otimes_A M$ are denoted by $\mathrm{Tor}_*^A(\_, M)$.

Definition 2.7.3. Assume $G\colon \operatorname{mod} A \to \operatorname{mod} B$ is a left exact functor. For every $p \in \mathbb{N}$ we define the right derived functor $R^pG\colon \operatorname{mod} A \to \operatorname{mod} B$ as $R^pG(X) := H^p(G(\mathcal{X}))$ where $\mathcal{X}$ is any injective resolution of X. Note that, again, R^pG is well-defined because of Lemma 2.6.4, and that $R^0G = G$ since G is left exact.

Example 2.7.4. Recall that for an A-B-bimodule M, the functor $\mathrm{Hom}_A(\_, M)$ defines a left exact contravariant functor from the category of left A-modules to the category of right B-modules. The right derived functors of $\mathrm{Hom}_A(\_, M)$ are denoted by $\mathrm{Ext}^*_A(\_, M)$.

CHAPTER III

GROTHENDIECK GROUP AND THE EULER CHARACTERISTIC

Throughout this chapter, we assume A and B are finite dimensional $\Bbbk$-algebras, and we use $\operatorname{mod} A$ and $\operatorname{mod} B$ to denote the categories of finite dimensional right A and B-modules. We also assume $\mathcal{C}$ is a subcategory of $\operatorname{mod} A$.

3.1 The definition

Definition 3.1.1. The *Grothendieck group* $K_0(\mathcal{C})$ of $\mathcal{C}$ is defined to be the abelian group generated by isomorphism classes of objects in $\mathcal{C}$ divided by the relation

$$[X] - [Y] + [Z] \text{ for every short exact sequence } 0 \to X \to Y \to Z \to 0 \text{ in } \mathcal{C}.$$

Theorem 3.1.2. *Assume $F\colon \operatorname{mod} A \to \operatorname{mod} B$ is an exact functor. Then F induces a map of Grothendieck groups of the form $K_0(F)\colon K_0(\operatorname{mod} A) \to K_0(\operatorname{mod} B)$.*

Proof. Any functor sends an isomorphism to another isomorphism. Moreover, F sends short exact sequences to short exact sequences since F is also exact. Then $K_0(F)[X] = [F(X)]$ is well-defined for every A-module X. □

3.2 Composition series

Definition 3.2.1. We say that an object X in mod A admits a composition series in $\mathcal{C}$ if there is a sequence of submodules

$$X_0 \subset \cdots \subset X_n = X$$

such that $X_0 \in \mathcal{C}$ and each $X_{i+1}/X_i \in \mathcal{C}$. We are going to use *Comp* $\mathcal{C}$ to denote the subcategory of modules that admit a composition series over $\mathcal{C}$. Notice that $\mathcal{C}$ is contained in *Comp* $\mathcal{C}$ as a subcategory.

Lemma 3.2.2. *The embedding $\mathcal{C} \to Comp\ \mathcal{C}$ induces an epimorphism of abelian groups $K_0(\mathcal{C}) \to K_0(Comp\ \mathcal{C})$. In other words, $K_0(Comp\ \mathcal{C})$ is generated by isomorphism classes of modules from $\mathcal{C}$.*

Proof. We prove the surjectivity by mathematical induction on the length of a composition series. If X admits a composition series of length one then we already have $X \in \mathcal{C}$. Now assume that if X has a composition series of length n then $[X]$ is in the image of $K_0(\mathcal{C}) \to K_0(Comp\ \mathcal{C})$. Assume X' is a module which has a composition series of length $n+1$

$$X_0 \subset \cdots \subset X_{n+1} = X'$$

Then $X = X'/X_0$ has a composition series of length n

$$X_1/X_0 \subset \cdots \subset X_{n+1}/X_0$$

So, $[X_{n+1}/X_0] = [X_{n+1}] - [X_0]$ is in the image of the morphism $K_0(\mathcal{C}) \to K_0(Comp\mathcal{C})$. Then

$$[X_{n+1}] = ([X_{n+1}] - [X_0]) + [X_0]$$

is also in the image. □

Proposition 3.2.3. *The Grothendieck group $K_0(\operatorname{mod} A)$ is generated by the set of isomorphism classes of simple modules.*

Proof. Assume X is a finite dimensional A-module X, and consider the poset of all its submodules. Since X is finite dimensional, every decreasing chain of submodules must terminate. Then, X has simple submodules, say X_0. The same is also true for X/X_0, and therefore, we get a simple module S_1 of X/X_0. By the Fourth Isomorphism Theorem, we have a submodule $X_1 \subseteq X$ such that $X_1/X_0 = S_1$. Proceeding by induction we get a series

$$X_0 \subset \cdots \subset X_n \subset \cdots \subset X$$

But X is finite dimensional, and therefore, the series must terminate. Thus we get a composition series for X where each consecutive quotient is simple. □

3.3 Resolutions and the Euler characteristic

Definition 3.3.1. We say that an object X in $\operatorname{mod} A$ admits a resolution in $\mathcal{C}$ if there is a finite resolution $\mathcal{X}$

$$0 \to X^0 \xrightarrow{d_0} \cdots \xrightarrow{d_n} X^{n+1} \to 0$$

such that each $X^i \in \mathcal{C}$. We are going to use *Res* $\mathcal{C}$ to denote the subcategory of modules that admit a finite resolution over $\mathcal{C}$. Notice that $\mathcal{C}$ is a subcategory of *Res* $\mathcal{C}$.

Definition 3.3.2. We say that A has projective dimension $n \in \mathbb{N}$ if every object admits a resolution of length less than or equal to n over the subcategory of finitely generated projective A modules. Similarly, we say that A has injective dimension $n \in \mathbb{N}$ if every object admits a resolution of length less than or equal to n over the subcategory of finitely generated injective A modules. The *global dimension* of A is the supremum of the projective dimensions of all finite A-modules.

Theorem 3.3.3. *(Lam, 1999, Section 5C) A has finite projective dimension if and only if A has finite injective dimension.*

Definition 3.3.4. Assume $\mathcal{C}$ is a subcategory of $\operatorname{mod} A$, and let C be a finite complex in $\mathcal{C}$, i.e. $C^i \in \mathcal{C}$ for every $i \in \mathbb{Z}$, and $C^i = 0$ for all but finitely many $i \in \mathbb{Z}$. The *Euler characteristic* of C is an element $\chi(C) \in K_0(\mathcal{C})$ which is defined as

$$\chi(C) = \sum_i (-1)^i [C^i]$$

Proposition 3.3.5. *Let $Res\ \mathcal{C}$ be an abelian subcategory of $\operatorname{mod} A$. Then the embedding $\mathcal{C} \to Res\ \mathcal{C}$ induces an epimorphism of abelian groups $K_0(\mathcal{C}) \to K_0(Res\ \mathcal{C})$. In other words, $K_0(Res\ \mathcal{C})$ is generated by isomorphism classes of modules in $\mathcal{C}$.*

Proof. Assume X admits a resolution $\mathcal{X}$ in $\mathcal{C}$

$$0 \to X^0 \xrightarrow{d_0} \cdots \xrightarrow{d_n} X^{n+1} \to 0$$

such that $H^0(\mathcal{X}) = X$, and $H^i(\mathcal{X}) = 0$ for every $i \neq 0$. In $\operatorname{mod} A$ we have short exact sequences

$$0 \to \operatorname{Ker}(d_i) \to X^i \to \operatorname{Im}(d_i) \to 0,$$

and therefore, $[X^i] = [\operatorname{Im}(d_i)] + [\operatorname{Ker}(d_i)]$ in $K_0(Res\ \mathcal{C})$ for every i since $Res\ \mathcal{C}$ is closed under kernels and cokernels. Now,

$$\chi(\mathcal{X}) = \sum_i (-1)^i [X^i] = \sum_i (-1)^i ([\operatorname{Ker}(d_i)] + [\operatorname{Im}(d_i)]) \tag{3.1}$$

$$= \sum_i (-1)^i ([\operatorname{Ker}(d_i)] - [\operatorname{Im}(d_{i-1})]) \tag{3.2}$$

$$= \sum_i (-1)^i [H^i(\mathcal{X})] = [X] \tag{3.3}$$

Since each $X^i \in \mathcal{C}$, we see that $\chi(\mathcal{X})$ is in the image of $K_0(\mathcal{C}) \to K_0(Res\ \mathcal{C})$. So, we also see that $[X]$ is in the image. □

Theorem 3.3.6. *If A has a finite global dimension then $K_0(\operatorname{mod} A)$ is generated by isomorphism classes of finitely generated projective modules, and by isomorphism classes of finitely generated injective modules.*

Proof. If A has finite global dimension then every module has a finite projective resolution, and equivalently, a finite injective resolution. Then we use Proposition 3.3.5 to get the equality we wanted to prove. □

3.4 The Grothendieck group of a derived category

Note that we defined the Euler characteristic of a finite complex $\mathcal{X}$ by

$$\chi(\mathcal{X}) = \sum_i (-1)^i [X^i]$$

and we saw in the proof of Proposition 3.3.5 that even when $\mathcal{X}$ is not a resolution, we still have

$$\chi(\mathcal{X}) = \sum_i (-1)^i [H^i(\mathcal{X})]$$

Thus, we have

Proposition 3.4.1. *The Euler characteristic of a complex is a well-defined function on the set of objects of $\mathcal{D}^b(A)$.*

This result suggests that we should be able to define the Grothendieck group of $\mathcal{D}^b(A)$.

Definition 3.4.2. A sequence of morphisms $\mathcal{X}_1 \xrightarrow{f_1} \mathcal{X}_2 \xrightarrow{f_2} \mathcal{X}_3$ is called a *triangle* if there is a short exact sequence of complexes

$$0 \to \mathcal{X}_1' \xrightarrow{f_1'} \mathcal{X}_2' \xrightarrow{f_2'} \mathcal{X}_3' \to 0$$

such that there are quasi-isomorphisms $\alpha_i\colon \mathcal{X}_i \to \mathcal{X}'_i$ that fit into commutative squares of the form

$$\begin{array}{ccc} \mathcal{X}_i & \xrightarrow{f_i} & \mathcal{X}_{i+1} \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha_{i+1}} \\ \mathcal{X}'_i & \xrightarrow{f'_i} & \mathcal{X}'_{i+1} \end{array}$$

Now, we define the *Grothendieck group* $K_0(\mathcal{D}^b(A))$ as the free abelian group on the set of isomorphism classes of objects in $\mathcal{D}^b(A)$ divided by the relations $[\mathcal{X}_1] - [\mathcal{X}_2] + [\mathcal{X}_3]$ for every triangle $\mathcal{X}_1 \xrightarrow{f_1} \mathcal{X}_2 \xrightarrow{f_2} \mathcal{X}_3$.

Theorem 3.4.3. *The natural map $K_0(\operatorname{mod} A) \to K_0(\mathcal{D}^b(A))$ that sends the isomorphism class $[X]$ of a module X in $K_0(\operatorname{mod} A)$ to the isomorphism class of its stalk complex $[X]$ in $K_0(\mathcal{D}^b(A))$ induces an isomorphism between the corresponding Grothendieck groups.*

Proof. Isomorphisms in $\operatorname{mod} A$ are sent to isomorphisms in $Ch(A)$. So, for a A-module X the quasi-isomorphism class of the stalk complex $[X]$ is well-defined. Moreover, since short exact sequences of A-modules are sent to short exact sequences of stalk complexes, any relation $[X]-[Y]+[Z]$ in $K_0(A)$ is sent to another relation in $K_0(\mathcal{D}^b(A))$. So, the natural map $K_0(A) \to K_0(\mathcal{D}^b(A))$ is well-defined. Next, we must show that the image of the natural map covers everything in $K_0(\mathcal{D}^b(A))$. We are going to do this by induction on the length of a complex. If an isomorphism class $[\mathcal{X}]$ in $K_0(\mathcal{D}^b(A))$ contains a stalk complex, then it is already in the image of the natural map. So, by our induction hypothesis, let us assume that if $[\mathcal{X}]$ is an isomorphism class such that all shortest complexes in $[\mathcal{X}]$ have length n or less, then it is in the image of the natural map. Let us take a generator $[\mathcal{X}]$ such that a shortest complex $\mathcal{U}$ in $[\mathcal{X}]$ has length $n+1$. Without loss of generality, we can assume $\mathcal{U}$ looks like

$$0 \to U^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{n-1}} U^n \to 0$$

Now, consider the complex $\mathcal{U}'$

$$0 \to U^0 \to im(d_0) \to 0$$

where U^0 is placed at degree 0, and the complex $\mathcal{U}''$

$$0 \to U^1/im(d_0) \to U^2 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} U^n \to 0$$

where $U^1/im(d_0)$ is placed at degree one. Since $\mathcal{U}'$ is quasi-isomorphic to the stalk complex of $H^0(\mathcal{U})$, its class $[\mathcal{U}']$ is already in the image of the natural map. On the other hand, the length of the complex $\mathcal{U}''$ is n. So, $[\mathcal{U}'']$ is also in the image of the natural map too. Finally, there is a short exact sequence of complexes of the form

$$0 \to \mathcal{U}' \to \mathcal{U} \to \mathcal{U}'' \to 0$$

This means $[\mathcal{U}] = [\mathcal{U}'] + [\mathcal{U}'']$ is also in the image of the natural map. □

CHAPTER IV

MAIN RESULT

We first recall some definitions from Chapter 1, Section 1.1 and Section 1.7. Let $\mathsf{P}_{\mathsf{m},\mathsf{n}}$ be a grid poset and $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$ be the poset of order ideals in $\mathsf{P}_{\mathsf{m},\mathsf{n}}$. Recall that we orient the edges in the Hasse diagram of $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$ downwards in the poset and we label the vertices in $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$ with the corresponding partitions to the order ideals. We then consider the incidence algebra $\mathcal{A}$ of the poset $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$. We also recall that P_α is the indecomposable projective module at α and the elements $x_{\alpha\geq\beta}$ form a $\Bbbk$-basis for P_α.

In this chapter, we will describe a special collection of projective resolutions in $\mathcal{D}^b(\operatorname{mod}\mathcal{A})$ that will span the Grothendieck group. In order to prove the periodicity of τ, we are going to need these resolutions. One key thing about these projective resolutions is that when we apply τ to these projective resolutions, we will prove that the resulting complexes are injective resolutions (up to a shift). So, these resolutions are going to have the homology in only one place. This allows us to study these resolutions from a combinatorial point of view. We define two functions f and g to give a combinatorial description of the homologies of these projective and injective resolutions. Then, with the help of this combinatorial description we will able to keep track of the elements in τ-orbits with another combinatorial function $\tilde{f}$. We also define the notion of *configurations* associated to the elements in τ-orbits to show that the action τ corresponds to a cyclic action on these configurations. Finally, we show that τ-orbits which come from these projective resolutions are enough to generate the Grothendieck group K_0. This establishes that the Auslander-Reiten translation τ in K_0, i.e. the Coxeter transformation, is periodic for $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$.

4.1 Projective resolutions

We will define *enhanced partitions* to write a special collection of resolutions.

Definition 4.1.1. We call a tuple of non-negative integers of the following form an *enhanced partition*:

$$(0^{\alpha_0}|\lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}})$$

with $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_r \leq n$ and $\sum_{i=0}^{r+1} \alpha_i = m$. Each exponent α_i refers to the number of times λ_i is repeated. We will refer to the 0's and the n's separated by vertical bars as *fixed entries*. Note that α_0 and α_{r+1} can be zero.

Let E be the set of enhanced partitions, and let F be the set of partitions. We now define a function ρ from E to F, which sends an enhanced partition to a partition by forgetting the bars. This means there are no fixed entries anymore. Formally,

$$\rho[(0^{\alpha_0}|\lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}})] = (0^{\alpha_0}, \lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r}, n^{\alpha_{r+1}}) \tag{4.1}$$

The function ρ allows us to treat enhanced partitions as usual partitions.

Let E_L be the set of enhanced partitions of the form $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}})$ where $\lambda_1 \neq 0$ and E_R be to the set of enhanced partitions of the form $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}})$ where $\lambda_r \neq n$.

Definition 4.1.2. For a given enhanced partition $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}})$, let R_α be the set of indices of the nonzero entries λ_i in α.

If $\alpha \in E_L$, then $R_\alpha = \{1, 2, \cdots, r\}$, and if $\alpha \notin E_L$, then $R_\alpha = \{2, \cdots, r\}$.

Let δ_i be the sequence of 0's and 1's where 1 is placed at those places λ_i's appear in α, i.e. $\delta_i = (0^{\alpha_0}, 0^{\alpha_1}, \cdots, 0^{\alpha_{i-1}}, 1^{\alpha_i}, 0^{\alpha_{i+1}}, \cdots, 0^{\alpha_r}, 0^{\alpha_{r+1}})$. For a given subset $X \subseteq R_\alpha$, let $\delta_X = \sum_{i \in X} \delta_i$. Notice that we define δ_i's only for non-fixed and nonzero entries in α.

Definition 4.1.3. For an enhanced partition $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}) \in E_L$, we define a complex of projective modules $\mathcal{P}_\alpha$ as follows:

$$\mathcal{P}_\alpha : 0 \to P^{-r}_{\alpha-\delta_{R_\alpha}} \xrightarrow{\partial^{-r}} \bigoplus_{\substack{J \subseteq R_\alpha, \\ |J|=r-1}} P^{-r+1}_{\alpha-\delta_J} \to \cdots \to \bigoplus_{\substack{J \subseteq R_\alpha, \\ |J|=1}} P^{-1}_{\alpha-\delta_J} \xrightarrow{\partial^{-1}} P^0_\alpha \to 0 \qquad (4.2)$$

with the maps

$$\partial^{-k} : \bigoplus_{\substack{J \subseteq R_\alpha, \\ |J|=k}} P_{\alpha-\delta_J} \to \bigoplus_{\substack{J \subseteq R_\alpha, \\ |J|=k-1}} P_{\alpha-\delta_J}, \qquad x_{(\alpha-\delta_{\{i_1,\cdots,i_k\}} \geq \beta)} \mapsto \sum_t (-1)^t x_{(\alpha-\delta_{\{i_1,\cdots,\widehat{i_t},\cdots,i_k\}} \geq \beta)}$$

for each $J = \{i_1, \cdots, i_k\} \subseteq R_\alpha$ and $x_{\alpha-\delta_J \geq \beta} \in P_{\alpha-\delta_J}$.

Remark 4.1.4. The grading of the complex $\mathcal{P}_\alpha$ comes from the cardinality of $J \subseteq R_\alpha$, and $\alpha - \delta_J$ is just a vector subtraction.

Proposition 4.1.5. *The complex $\mathcal{P}_\alpha$ in Equation* (4.2) *defines a projective resolution.*

Proof. Notice that for any $\beta \leq \beta'$ there is a unique embedding of P_β into $P_{\beta'}$ by left multiplication with $x_{\beta' \geq \beta}$ sending $x_{\beta \geq \gamma} \mapsto x_{\beta' \geq \gamma}$ for each $\gamma \leq \beta$. Thus, the maps ∂^{-k} are all right $\mathcal{A}$-module maps. Therefore, it is enough to show that we have a complex in the category of $\Bbbk$-vector spaces.

For every $\beta \leq \alpha$, the graded $\Bbbk$-subspace $\mathcal{P}_\alpha \cdot x_{\beta \geq \beta}$ is actually a $\Bbbk$-subcomplex of $\mathcal{P}_\alpha$ since the differentials preserve the grading. Therefore, it is enough to prove the exactness of $\mathcal{P}_\alpha$ by showing we have an exact complex at each vertex in the support of P_α. Note that P_α has support over the vertices $\beta \leq \alpha$.

For a given $\beta \leq \alpha$ we find the maximal $J_\beta \subseteq R_\alpha$ so that the inequality $\beta \leq \alpha - \delta_{J_\beta}$ holds. Let $k = |J_\beta|$. When we multiply the complex by $x_{\beta \geq \beta}$ on the right, each of the projective modules in $\mathcal{P}_\alpha$ reduces to the ground field $\Bbbk$. Moreover, after the reduction, the face maps in the differentials are all identity maps. Now, J_β determines which summands of $\mathcal{P}_\alpha$ have S_β in their composition series. Then we have the following subcomplex of $\Bbbk$-vector spaces:

$$\mathcal{S} : 0 \to \Bbbk^{-k} \to \bigoplus_{\binom{k}{k-1}} \Bbbk^{-k+1} \to \cdots \to \bigoplus_{\binom{k}{2}} \Bbbk^{-2} \to \bigoplus_{\binom{k}{1}} \Bbbk^{-1} \to \Bbbk^{0} \to 0$$

with the differential as defined above. This is the face complex of the standard $(k-1)$-simplex. We show this fact as follows. Consider the subset $J \subseteq R$ such that $|J| = k$. Any such subset J is linearly ordered since $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_r \leq n$, say without loss of generality $J = \{\lambda_1, \cdots, \lambda_k\}$, and we define $\binom{J}{u}$ as all subset of J of size u. Let us define a graded set $C_k = \bigsqcup_{u=0}^{k} C_u$ where $C_u = \binom{J}{u}$. Now, we are going to define an abstract simplicial set structure on C_k. This requires the definitions of the face maps $d_j : C_u \to C_{u-1}$ for $0 \leq j \leq k$ defined by $\{\lambda_1, \lambda_2, \cdots, \lambda_u\} \mapsto \{\lambda_1, \cdots, \widehat{\lambda_j}, \cdots, \lambda_u\}$. Note that since λ_j's are linearly ordered, d_u is well-defined. Let us now define a simplicial vector space $\Bbbk[C_u] := span_{\Bbbk}(C_u)$ and we extend the face maps to $d_j : \Bbbk[C_u] \to \Bbbk[C_{u-1}]$ linearly. From this simplicial vector space, we define its face complex by defining the differentials as $\partial_k =$

$$\sum_{j=0}^{k}(-1)^j d_j.$$

Since the standard $(k-1)$-simplex is contractible, its reduced homology is zero provided $k-1 \geq 0$ (Hatcher, 2002). This means that there is a homology if and only if $k = |J_\beta| = 0$.

$$H^n(\mathcal{S}) = \begin{cases} 0 & \text{if } 0 < k \leq r, n \neq 0, \\ \Bbbk & \text{if } k = 0,\ n = 0. \end{cases}$$

Then, we have

$$H^n(\mathcal{P}_\alpha) \cdot x_{\beta \geq \beta} = \begin{cases} 0 & \text{if } 0 < k \leq r, n \neq 0, \\ \Bbbk & \text{if } k = 0,\ n = 0. \end{cases}$$

and this implies that $\mathcal{P}_\alpha$ is a projective resolution.

□

Remark 4.1.6. Notice that the homology of $\mathcal{P}_\alpha$ is supported only over vertices β such that $J_\beta = \emptyset$. This is equivalent to $\beta \leq \alpha$ and $\beta \not\leq \alpha - \delta_i$ for each $i \in R_\alpha$. We will further investigate the homology of $\mathcal{P}_\alpha$ in Section 4.4.

Example 4.1.7. Let $\mathcal{A}$ be the incidence algebra for the poset $\mathsf{J}(\mathsf{P}_{5,7})$. Let us consider $\alpha = (0|2,2,3,7|)$. Then $R_\alpha = \{1,2,3\}$ and

$$\mathcal{P}_\alpha : 0 \to P^{-3}_{(0,1,1,2,6)} \to P^{-2}_{(0,1,1,2,7)} \oplus P^{-2}_{(0,2,2,2,6)} \oplus P^{-2}_{(0,1,1,3,6)} \to P^{-1}_{(0,1,1,3,7)} \oplus P^{-1}_{(0,2,2,2,7)} \oplus P^{-1}_{(0,2,2,3,6)} \to P^{0}_{(0|2,2,3,7|)} \to 0.$$

Now let $\alpha' = (0|2,2,3|7)$. Note that α' is the same as α except that 7 is now a fixed entry. Then we have $R_{\alpha'} = \{1,2\}$ and

$$\mathcal{P}_{\alpha'} : 0 \to P^{-2}_{(0,1,1,2,7)} \to P^{-1}_{(0,1,1,3,7)} \oplus P^{-1}_{(0,2,2,2,7)} \to P^{0}_{(0|2,2,3|7)} \to 0$$

4.2 Action of the Auslander-Reiten translation on the projective resolutions

In this section, we are going to look at the action of the Auslander-Reiten translation on the projective resolutions $\mathcal{P}_\alpha$ and discuss the homology of the resulting complex.

Proposition 4.2.1. *Let $\mathcal{P}_\alpha$ be the projective resolution defined in Equation* (4.2)*. If we apply the Auslander-Reiten translation τ to the complex $\mathcal{P}_\alpha$, the resulting complex is an injective resolution up to a shift.*

Proof. After applying $- \otimes D\mathcal{A}$ on $\mathcal{P}_\alpha$, we get the following injective complex:

$$\mathcal{I}_\alpha : 0 \to I^{-r}_{\alpha-\delta_{R_\alpha}} \to \bigoplus_{|J|=r-1} I^{-r+1}_{\alpha-\delta_J} \to \cdots \to \bigoplus_{|J|=1} I^{-1}_{\alpha-\delta_J} \to I^0_\alpha \to 0 \tag{4.3}$$

The proof of Proposition 4.1.5 with some modifications can be applied here. Firstly, notice that $I_{\alpha-\delta_J}$ has support over the vertices $\gamma \geq \alpha - \delta_J$. Then we write the subcomplexes as follows: for a given γ we find the minimal J_γ such that the inequality $\gamma \geq \alpha - \delta_{J_\gamma}$ holds. Let $k = |J_\gamma|$, then the vertices γ will appear as in the following:

$$0 \to \mathbb{k}^{-r}_\gamma \to \bigoplus_{\binom{r-k}{1}} \mathbb{k}^{-r+1}_\gamma \to \cdots \to \bigoplus_{\binom{r-k}{r-k-2}} \mathbb{k}^{-k-2}_\gamma \to \bigoplus_{\binom{r-k}{r-k-1}} \mathbb{k}^{-k-1}_\gamma \to \mathbb{k}^{-k}_\gamma \to 0$$

This is another face complex which only has homology when $k = r$. Consequently, the complex (4.3) is an injective resolution up to a shift.

□

Remark 4.2.2. As in the projective case, the homology of $\mathcal{I}_\alpha$ has support over vertices γ only when $J_\gamma = R_\alpha$. This means that $\alpha - \delta_{R_\alpha} \leq \gamma$ and $\alpha - \delta_J \nleq \gamma$ where $|J| = r - 1$. We will further investigate the homology of $\mathcal{I}_\alpha$ in Section 4.4.

4.3 Intervals in the poset $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$

In this section, we are going to define two functions: f from E_L to E_R, and g from E_R to E_L. These functions will help us to describe the homology of the complexes $\mathcal{P}_\alpha$ and $\mathcal{I}_\alpha$ defined in Sections 4.1 and 4.2 combinatorially.

1. Let $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}) \in E_L$ and then $R_\alpha = \{1, 2, \cdots, r\}$. The function $f(\alpha)$ is defined as follows: We first apply ρ which is defined in (4.1) to α. Then for each $i \in R_\alpha$, we leave the last occurrence of λ_i in α unchanged while we minimize the rest of the occurrences including the fixed n's at the end if there are any, thus making the partition as small as possible. Finally, we enhance the result as follows: the first bar is placed in the same position as in α and if there is no n in $f(\alpha)$ the second bar obviously goes at the end, while if there are n's we put the second bar before all of them.

   Formally,

$$f((0^{\alpha_0}|\lambda_1^{\alpha_1},\lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}))$$
$$= \begin{cases} (0^{\alpha_0}|0^{\alpha_1-1}, \lambda_1^{\alpha_2}, \lambda_2^{\alpha_3}, \cdots, \lambda_{r-1}^{\alpha_r}|\lambda_r^{1+\alpha_{r+1}}) & \text{if } \lambda_r = n, \\ (0^{\alpha_0}|0^{\alpha_1-1}, \lambda_1^{\alpha_2}, \lambda_2^{\alpha_3}, \cdots, \lambda_{r-1}^{\alpha_r}, \lambda_r^{1+\alpha_{r+1}}|) & \text{if } \lambda_r \neq n. \end{cases}$$

2. Let $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}) \in E_R$. Note that here $R_\alpha = \{1, 2, \cdots, r\}$ if $\lambda_1 \neq 0$ and $R_\alpha = \{2, \cdots, r\}$ if $\lambda_1 = 0$. The function $g(\alpha)$ is defined as follows: We first apply ρ to α. For each $i \in R_\alpha$, we leave the first occurrence of λ_i in α unchanged while maximizing the rest of the occurrences, thus making the partition as large as possible. Notice that we do not change 0's which were fixed in α, but we do maximize the unfixed 0's. Then we place the first bar in the same place as in α; the position of second bar can be seen in the following formal definition.

Formally,

$$g((0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}))$$

$$= \begin{cases} (0^{\alpha_0}|\lambda_2^{\alpha_1+1}, \lambda_3^{\alpha_2}, \cdots, \lambda_r^{\alpha_{r-1}}|n^{\alpha_r - 1}) & \text{if } \alpha_{r+1} = 0 \text{ and } \lambda_1 = 0, \\ (0^{\alpha_0}|\lambda_2^{\alpha_1+1}, \lambda_3^{\alpha_2}, \cdots, \lambda_r^{\alpha_{r-1}}, n^{\alpha_r}|n^{\alpha_{r+1}-1}) & \text{if } \alpha_{r+1} \neq 0 \text{ and } \lambda_1 = 0, \\ (0^{\alpha_0}|\lambda_1, \lambda_2^{\alpha_1}, \lambda_3^{\alpha_2}, \cdots, \lambda_r^{\alpha_{r-1}}|n^{\alpha_r - 1}) & \text{if } \alpha_{r+1} = 0 \text{ and } \lambda_1 \neq 0, \\ (0^{\alpha_0}|\lambda_1, \lambda_2^{\alpha_1}, \lambda_3^{\alpha_2}, \cdots, \lambda_r^{\alpha_{r-1}}, n^{\alpha_r}|n^{\alpha_{r+1}-1}) & \text{if } \alpha_{r+1} \neq 0 \text{ and } \lambda_1 \neq 0. \end{cases}$$

Example 4.3.1. Let $\alpha = (0^2|2^3, 5, 6^4, 9^2|13^2) = (0, 0|2, 2, 2, 5, 6, 6, 6, 6, 9, 9|13, 13)$. Then $R_\alpha = \{1, 2, 3, 4\}$. To find $f(\alpha)$ we first apply ρ. Now, we fix the last occurrences of each λ_i, $i \in R_\alpha$ while minimizing the rest as shown in the following:

$$(0, 0, 2, 2, 2, 5, 6, 6, 6, 6, 9, 9, 13, 13)$$

We leave the last occurrences of each λ_i unchanged:

$$(0, 0, 2, 2, 2, 5, 6, 6, 6, 6, 9, 9, 13, 13)$$

$$\text{We minimize: } (0, 0, 0, 0, 2, 5, 5, 5, 5, 6, 6, 9, 9, 9)$$

$$\text{Then we enhance: } (0, 0|0, 0, 2, 5, 5, 5, 5, 6, 6, 9, 9, 9|)$$

The result is $f(\alpha) = (0^2|0^2, 2, 5^4, 6^2, 9^3|)$.

Similarly, we can get $g(\alpha) = g((0^2|2^3, 5, 6^4, 9^2|13^2)) = (0^2|2, 5^3, 6, 9^4, 13^2|13)$.

Lemma 4.3.2. *The functions f and g are inverses of each other.*

Proof. Let $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}) \in E_L$. Then

$$g \circ f(\alpha) = \begin{cases} g((0^{\alpha_0}|0^{\alpha_1-1}, \lambda_1^{\alpha_2}, \lambda_2^{\alpha_3}, \cdots, \lambda_{r-1}^{\alpha_r}|\lambda_r^{\alpha_{r+1}+1})) & \text{if } \lambda_r = n \\ g((0^{\alpha_0}|0^{\alpha_1-1}, \lambda_1^{\alpha_2}, \lambda_2^{\alpha_3}, \cdots, \lambda_{r-1}^{\alpha_r}, \lambda_r^{\alpha_{r+1}+1}|)) & \text{if } \lambda_r \neq n \end{cases}$$

We can easily conclude that the result is α. Similarly, one can show that $f \circ g(\beta) = \beta$ for $\beta \in E_R$. $\square$

4.4 Homologies and intervals

In this section, we will discuss the homology of $\mathcal{P}_\alpha$, and the homology of $\mathcal{I}_\alpha$ in relation to intervals in the poset $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$.

Definition 4.4.1. Let $[\gamma, \gamma']$ be an interval in $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$. We define the corresponding element in the Grothendieck group K_0 for the interval $[\gamma, \gamma']$ as $[[\gamma, \gamma']] := \sum\limits_{\gamma \leq x \leq \gamma'} [S_x]$.

Proposition 4.4.2. *The class of the projective resolution $\mathcal{P}_\alpha$ in K_0 is $[[f(\alpha), \alpha]]$ for every $\alpha \in E_L$.*

Proof. Let $I = \{x \in \mathsf{P}_{\mathsf{m,n}} \mid x \leq \alpha \text{ and } x \not\leq \alpha - \delta_i \text{ for any } i \in R_\alpha\}$. Firstly, notice that the class in K_0 for the homology of the projective resolution $\mathcal{P}_\alpha$ is supported over the vertices in I by the result in Proposition 4.1.5. Recall also Remark 4.1.6. We will prove that I forms an interval in the poset $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$. Clearly, α is the maximum element in I.

Let $\alpha = (0^{\alpha_0} | \lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r} | n^{\alpha_{r+1}})$. We also write $\alpha = (a_1, a_2, \cdots, a_m)$. Then, we write $\alpha - \delta_i = (b_1, b_2, \cdots, b_m)$ where

$$b_j = \begin{cases} a_j & \text{when } a_j \neq \lambda_i, \\ a_j - 1 & \text{otherwise.} \end{cases}$$

for $1 \leq j \leq m$. Let $x = (c_1, c_2, \cdots, c_m) \in I$. In order for $x \leq \alpha$ but $x \not\leq \alpha - \delta_i$ for all $i \in R_\alpha$, for each i we must have $c_i \leq a_i$, and at least one of $c_{\alpha_0 + \cdots + \alpha_{i-1} + 1}, \cdots, c_{\alpha_0 + \cdots + \alpha_i}$ must be greater than $b_{\alpha_0 + \cdots + \alpha_{i-1} + 1}, \cdots, b_{\alpha_0 + \cdots + \alpha_i}$, i.e. must equal λ_i. Since $c_1 \leq c_2 \leq \cdots \leq c_m$, it must be that $c_{\alpha_0 + \cdots + \alpha_i} = \lambda_i$. Now, it is clear that

$$I = \{x \in \mathsf{P}_{\mathsf{m,n}} \mid c_{\alpha_0 + \cdots + \alpha_i} = \lambda_i \text{ for each } i \in R_\alpha\}.$$

This is an interval having minimum $f(\alpha)$. This completes the proof. $\square$

The following is an illustration of the proof with an example. For this example, let us assume $\alpha = (|2,2,6,6,6,6,9,9,9,9,9|)$. We illustrate the corresponding order ideal with the black contour in Figure 4.1. Then we have $R_\alpha = \{1,2,3\}$. For instance, $\alpha - \delta_2 = (2,2,5,5,5,5,9,9,9,9,9)$ which is shown with the red contour in Figure 4.1. The gray box shows the row where $c_{\alpha_0+\alpha_1+\alpha_2} = c_6 = \lambda_2 = 6$. Finally, $f(\alpha)$ is illustrated with the blue dotted contour in Figure 4.1.

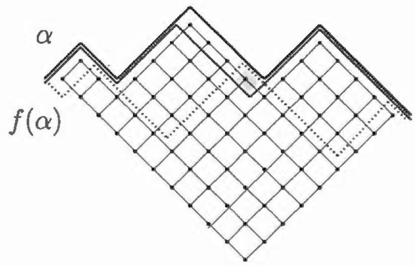


Figure 4.1 An illustration of the function $f(\alpha)$

Example 4.4.3. As in Example 4.1.7, let $\alpha = (0|2,2,3,7|)$ and consider its projective resolution. The corresponding element in K_0 for the homology of this projective resolution is $[[f(\alpha),\alpha]] = [(0,0,2,3,7),(0,2,2,3,7)]$. Now, assume $\alpha' = (0|2,2,3|7)$, then $[[f(\alpha'),\alpha']] = [(0,0,2,3,3),(0,2,2,3,7)]$.

In the following, we would like to analyze the homology of the injective resolution $\mathcal{I}_\alpha$ after the action of τ on the projective resolution $\mathcal{P}_\alpha$.

Proposition 4.4.4. *The class of the injective resolution $\mathcal{I}_\alpha$ in K_0 is $\pm[[\alpha - \delta_{R_\alpha}, g(\alpha - \delta_{R_\alpha})]]$ for every $\alpha \in E_L$.*

Remark 4.4.5. Before proving Proposition 4.4.4, we need to discuss the rule that enhances the partition $\alpha - \delta_{R_\alpha}$ so that we can apply the function g. Let us assume $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}) \in E_L$. Then the enhanced partition $\alpha - \delta_{R_\alpha}$ is defined as follows: The position of the second bar is the same as in α.

This implies that we fix all of the n's in $\alpha - \delta_{R_\alpha}$. Now, if we have 0's in $\alpha - \delta_{R_\alpha}$, we have to determine which of them are fixed. To do so, we will look at α. Recall that $\lambda_1 \neq 0$ in α for $\mathcal{P}_\alpha$. If $\lambda_1 \neq 1$, then we do not fix any 0's in $\alpha - \delta_{R_\alpha}$, i.e we put the first bar before all of the entries. If $\lambda_1 = 1$, then we look at the location of first appearance of λ_1 in α, say in k-th position from the beginning. Then we put the first bar after the k-th 0 in $\alpha - \delta_{R_\alpha}$. Formally,

$$\alpha - \delta_{R_\alpha} = \begin{cases} (|0^{\alpha_0}, (\lambda_1 - 1)^{\alpha_1}, (\lambda_2 - 1)^{\alpha_2}, \cdots, (\lambda_r - 1)^{\alpha_r} | n^{\alpha_{r+1}}) & \text{if } \lambda_1 \neq 1 \\ (0^{\alpha_0+1} | 0^{\alpha_1 - 1}, (\lambda_2 - 1)^{\alpha_2}, \cdots, (\lambda_r - 1)^{\alpha_r} | n^{\alpha_{r+1}}) & \text{if } \lambda_1 = 1. \end{cases}$$

Proof of Proposition 4.4.4. Here finding $g(\alpha - \delta_{R_\alpha})$ is the dual of finding $f(\alpha)$.

Let $I' = \{x \in \mathsf{P}_{\mathsf{m,n}} \mid \alpha - \delta_{R_\alpha} \leq x \text{ and } x \not\geq \alpha - \delta_J \text{ with } |J| = r - 1\}$. In this case, we know the minimum element of I' is $\alpha - \delta_{R_\alpha}$.

Let $\alpha = (0^{\alpha_0} | \lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r} | n^{\alpha_{r+1}})$, and let $x = (c_1, c_2, \cdots, c_m) \in I'$. We can deduce the following as in the proof of Proposition 4.4.2: In order for $\alpha - \delta_{R_\alpha} \leq x$ and $x \not\geq \alpha - \delta_J$ where $|J| = r - 1$, the entries $c_{\alpha_0 + \cdots + \alpha_{i-1} + 1}$ must equal $\lambda_i - 1$ for each i. Now, we can conclude that

$$I' = \{x \in \mathsf{P}_{\mathsf{m,n}} \mid c_{\alpha_0 + \cdots + \alpha_{i-1} + 1} = \lambda_i - 1 \text{ for each } i \in R_\alpha\}.$$

This is an interval with the maximum $g(\alpha - \delta_{R_\alpha})$. This finishes the proof. □

We will illustrate the idea of the proof by an example as we study in the previous case. Assume $n = 9$ and $\alpha = (|2, 2, 6, 6, 6, 6, 9, 9, 9, 9, 9|)$. Then $R_\alpha = \{1, 2, 3\}$ and $\alpha - \delta_{R_\alpha} = (|1, 1, 5, 5, 5, 5, 8, 8, 8, 8, 8|)$. The corresponding order ideal is illustrated with the black contour in Figure 4.2. Also, we can calculate that $g(\alpha - \delta_{R_\alpha}) = (|1, 5, 5, 8, 8, 8, 8|9, 9, 9, 9)$ as illustrated with the blue contour. The red contour shows $\alpha - \delta_{\{1,2\}} = (1, 1, 5, 5, 5, 5, 9, 9, 9, 9, 9)$. The gray box shows the row $\alpha_0 + \alpha_1 + \alpha_2 + 1 = 7$ where $c_{\alpha_0 + \alpha_1 + \alpha_2 + 1} = \lambda_3 - 1 = 8$.

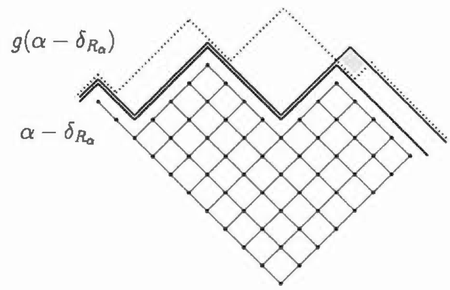


Figure 4.2 An illustration of the function $g(\alpha)$

Proposition 4.4.2 and 4.4.4 in combination with Proposition 4.2.1 show that Auslander-Reiten translation sends $[[f(\alpha), \alpha]]$ to $\pm[[\alpha - \delta_{R_\alpha}, g(\alpha - \delta_{R_\alpha})]]$. As we have seen, the function $\alpha \mapsto g(\alpha - \delta_{R_\alpha})$ is important, and it will be useful to calculate it more directly. So, we will define a new function $\tilde{f}$, and then later prove that $\tilde{f}(\alpha) = g(\alpha - \delta_{R_\alpha})$ in Lemma 4.4.7.

We define the function $\tilde{f}$ from E_L to E_L as follows: Let $\alpha \in E_L$. First apply ρ, then deduct one from the first occurrence of each λ_i, $i \in R_\alpha$ while maximizing all of the other indices, i.e. make the partition as large as possible. Then we fix all of the 0's, i.e. we put the first bar at the end of the $0's$ in the result. If we have n fixed k times in α, we make n fixed $k-1$ times in $\tilde{f}(\alpha)$. If we do not have any fixed n's in α, then we fix all n's in $\tilde{f}(\alpha)$.

Formally,

$$\tilde{f}((0^{\alpha_0}|\lambda_1^{\alpha_1},\lambda_2^{\alpha_2},\cdots,\lambda_r^{\alpha_r}|n^{\alpha_{r+1}})) =$$
$$\begin{cases} (0^{\alpha_0+1}|(\lambda_2-1)^{\alpha_1},\cdots,(\lambda_r-1)^{\alpha_{r-1}}|n^{\alpha_r-1}) & \text{if } \alpha_{r+1}=0, \lambda_1=1 \\ (0^{\alpha_0+1}|(\lambda_2-1)^{\alpha_1},\cdots,(\lambda_r-1)^{\alpha_{r-1}},n^{\alpha_r}|n^{\alpha_{r+1}-1}) & \text{if } \alpha_{r+1}\neq 0, \lambda_1=1 \\ (|(\lambda_1-1)^{\alpha_0+1},(\lambda_2-1)^{\alpha_1},\cdots,(\lambda_r-1)^{\alpha_{r-1}}|n^{\alpha_r-1}) & \text{if } \alpha_{r+1}=0, \lambda_1\neq 1 \\ (|(\lambda_1-1)^{\alpha_0+1},(\lambda_2-1)^{\alpha_1},\cdots,(\lambda_r-1)^{\alpha_{r-1}},n^{\alpha_r}|n^{\alpha_{r+1}-1}) & \text{if } \alpha_{r+1}\neq 0, \lambda_1\neq 1. \end{cases}$$

Example 4.4.6. Consider the same α as in Example 4.3.1. Then

$$\tilde{f}(\alpha) = \tilde{f}((0^2|2^3,5,6^4,9^2|13^2)) = (1^3,4^3,5,8^4,13^2|13)$$

Lemma 4.4.7. *We have $\tilde{f}(\alpha) = g(\alpha-\delta_{R_\alpha})$ for every $\alpha \in E_L$.*

Proof. Let $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1},\lambda_2^{\alpha_2},\cdots,\lambda_r^{\alpha_r}|n^{\alpha_{r+1}}) \in E_L$. Recall that in Remark 4.4.5 we explained how we get the enhanced partition $\alpha-\delta_{R_\alpha}$. So, we have the 0's fixed in $\alpha-\delta_{R_\alpha}$ only when $\lambda_1 = 1$ in α. Recall also that since $\alpha \in E_L$, we have $\lambda_1 \neq 0$. Firstly assume $\lambda_1 \neq 1$, i.e. there is no 0 fixed in $\alpha-\delta_{R_\alpha}$. Then we get $\alpha-\delta_{R_\alpha} = (|0^{\alpha_0},(\lambda_1-1)^{\alpha_1},(\lambda_2-1)^{\alpha_2},\cdots,(\lambda_r-1)^{\alpha_r}|n^{\alpha_{r+1}})$. Now, we apply the map g. We get the following which is the desired result.

$$g(\alpha-\delta_{R_\alpha}) =$$
$$\begin{cases} (|(\lambda_1-1)^{\alpha_0+1},(\lambda_2-1)^{\alpha_1},\cdots,(\lambda_r-1)^{\alpha_{r-1}}|n^{\alpha_r-1}) & \text{if } \alpha_{r+1}=0 \\ (|(\lambda_1-1)^{\alpha_0+1},(\lambda_2-1)^{\alpha_1},\cdots,(\lambda_r-1)^{\alpha_{r-1}},n^{\alpha_r}|n^{\alpha_{r+1}-1}) & \text{if } \alpha_{r+1}\neq 0. \end{cases}$$

The case $\lambda_1 = 1$ can be calculated similarly. This finishes the proof. □

Proposition 4.4.8. $\tau([\mathcal{P}_\alpha]) = \pm[\mathcal{P}_{g(\alpha-\delta_{R_\alpha})}]$, *or equivalently* $\tau([\mathcal{P}_\alpha]) = \pm[\mathcal{P}_{\tilde{f}(\alpha)}]$.

Proof. Since we proved that f and g are inverses of each other in Lemma 4.3.2, it is easy to see that the class of the projective resolution $\mathcal{P}_{g(\alpha-\delta_{R_\alpha})}$ of the enhanced partition $g(\alpha-\delta_{R_\alpha})$ in K_0 is $\pm[[\alpha-\delta_{R_\alpha}, g(\alpha-\delta_{R_\alpha})]]$. □

To sum up, for any enhanced partition $\alpha \in E_L$, one can write the projective resolution $\mathcal{P}_\alpha$ and find the class of $\mathcal{P}_\alpha$ in K_0 which is $[[f(\alpha), \alpha]]$. The class of $\tau\mathcal{P}_\alpha$ in K_0 is $[[\alpha - \delta_{R_\alpha}, g(\alpha - \delta_{R_\alpha})]]$. Moreover, we can determine $g(\alpha - \delta_{R_\alpha})$ directly from α by using the map $\tilde{f}$. We are now in a good position to iterate the application of τ.

4.5 Configurations and enhanced partitions

The goal of this section is to define a bijection from E_L to a set $\mathcal{D}_{m,n}$ which we define below and has a natural action of $\mathbb{Z}/(m+n+1)$.

Consider $\mathcal{Z} := \{-m, \cdots, -1, 0, 1, \cdots, n\}$ for the elements of $\mathbb{Z}/(m+n+1)$.

Definition 4.5.1. A *configuration* D is an increasing sequence of m elements from $\mathcal{Z}$. We write $D = \{i_1 < i_2 < \cdots < i_m\}$ for a configuration. The set $\mathcal{D}_{m,n}$ denotes all configurations. Notice that the cardinality of $\mathcal{D}_{m,n}$ is $\binom{m+n+1}{m}$.

Consider a configuration D. By $D\{i\}$ we mean the i times shifted version of D, i.e. $D\{i\}$ is the set of elements $\{i_1 - i, \cdots, i_m - i\}$ which are sorted into increasing order, and we write *sorted*$\{i_1 - i, \cdots, i_m - i\}$. Call this operation $\{i\}$ a shift. Clearly, $D\{m+n+1\} = D$. The set of all configurations $D\{i\}$ for all $0 \leq i \leq m+n$ is called *the full orbit of* D.

Recall that E_L is the set of enhanced partitions of the form

$$\alpha = (0^{\alpha_0} | \lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r} | n^{\alpha_{r+1}})$$

where $\lambda_1 \neq 0$. We also write it as a sequence $\alpha = (a_1, a_2, \cdots, a_m)$. We are going to define a function ψ from E_L to $\mathcal{D}_{m,n}$ as follows.

Let us first define a function $\mu_\alpha : \{1, \cdots, m\} \to \mathcal{Z}$

$$\mu_\alpha(j) = \begin{cases} a_j & \text{if } j = \sum_{i=0}^{k} \alpha_i \text{ for some } k \in \{0, \ldots, r\}, \\ -j & \text{otherwise.} \end{cases}$$

It is easy to see that it is well-defined. Now, we define

$$\psi : E_L \to \mathcal{D}_{m,n}$$

as follows:

$$\psi(\alpha) = sorted\{\mu_\alpha(j) \mid \text{ for } j \in \{1, \cdots, m\}\}.$$

Example 4.5.2. We continue in the setting of Example 4.1.7. Consider the enhanced partition $\alpha = (0|2, 2, 3, 7|)$, then the corresponding configuration is $\psi(\alpha) = \{-2 < 0 < 2 < 3 < 7\}$. Now, consider the enhanced partition $\alpha' = (0|2, 2, 3|7)$. Then the corresponding configuration is $\psi(\alpha') = \{-5 < -2 < 0 < 2 < 3\}$.

Lemma 4.5.3. *The map ψ is a bijection.*

Proof. We can think of an element in E_L as a multiset on $\{0, 1, \cdots, n, n^*\}$ where we use n^* to represent the fixed n's. Notice that the cardinalities of E_L and $\mathcal{D}_{m,n}$ are the same. Then, it is enough to show that the map ψ is injective.

Let $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}})$, $\beta = (0^{\beta_0}|\xi_1^{\beta_1}, \xi_2^{\beta_2}, \cdots, \xi_s^{\beta_s}|n^{\beta_{s+1}}) \in E_L$. To simplify the exposition, we assume $\alpha_i > 1$ for all $i \in \{0, 1, \cdots, r\}$ and $\beta_j > 1$ for all $j \in \{0, 1, \cdots, s\}$. The proof for the general case is similar.

We also write

$$\alpha = (a_1, \ldots, a_{\alpha_0 - 1}, a_{\alpha_0} | a_{\alpha_0+1}, \ldots, a_{\alpha_0+\alpha_1-1}, a_{\alpha_0+\alpha_1}, \ldots,$$
$$a_{\alpha_0+\alpha_1+\cdots+\alpha_{r-1}+1}, \ldots, a_{\alpha_0+\alpha_1+\cdots+\alpha_r-1}, a_{\alpha_0+\alpha_1+\cdots+\alpha_r} | n^{\alpha_{r+1}})$$

$$\beta = (b_1, \ldots, b_{\beta_0-1}, b_{\beta_0}, b_{\beta_0+1}, \ldots, b_{\beta_0+\beta_1-1}, b_{\beta_0+\beta_1}, \ldots,$$
$$b_{\beta_0+\beta_1+\cdots+\beta_{s-1}+1}, \ldots, b_{\beta_0+\beta_1+\cdots+\beta_s-1}, b_{\beta_0+\beta_1+\cdots+\beta_s} | n^{\beta_{s+1}})$$

Assume $\psi(\alpha) = \psi(\beta)$, then we have

$$\{\underbrace{-m < -m+1 < \cdots < -m+\alpha_{r+1}-1}_{\alpha_{r+1}} <$$

$$\underbrace{-x_0 < -x_0+1 < \cdots < -x_0+\alpha_r-2}_{\alpha_r-1} < \cdots <$$

$$\underbrace{-x_r < -x_r+1 < \cdots < -x_r+\alpha_0-2}_{\alpha_0-1} < 0 < \lambda_1 < \cdots < \lambda_r\}$$

$$= \{\underbrace{-m < -m+1 < \cdots < -m+\beta_{s+1}-1}_{\beta_{s+1}} <$$

$$\underbrace{-y_0 < -y_0+1 < \cdots < -y_0+\beta_s-2}_{\beta_s-1} < \cdots <$$

$$\underbrace{-y_s < -y_s+1 < \cdots < -y_s+\beta_0-2}_{\beta_0-1} < 0 < \xi_1 < \cdots < \xi_s\}$$

where $x_k = \alpha_0 + \alpha_1 + \cdots + \alpha_{r-k} - 1$ for $k \in \{0, 1, \cdots, r\}$ and $y_k = \beta_0 + \beta_1 + \cdots + \beta_{s-k} - 1$ for $k \in \{0, 1, \cdots, s\}$.

Since λ_i's and ξ_i's are all positive and linearly ordered, then for each $i > 0$, $\lambda_i = \xi_i$ and $r = s$. Now, assume $\alpha_{r+1} \neq \beta_{s+1}$. Without loss of generality, say $\beta_{s+1} < \alpha_{r+1}$. Then $-m + \beta_{r+1} - 1 < -m + \alpha_{r+1} - 1$ which implies $-m + \beta_{r+1} \leq -m + \alpha_{r+1} - 1$. Thus, $-m + \beta_{r+1} \in \psi(\alpha)$. But this is a contradiction, because $-m + \beta_{r+1}$ cannot be in $\psi(\beta)$. By the same argument, we can prove that for each j, $\alpha_j = \beta_j$. This proves that $\alpha = \beta$. □

Now, let $\mathcal{F} := \{0, 1, \cdots, n, n^*\}$ where n^* is a formal element distinct from n. We also define a map φ from $\mathcal{D}_{m,n}$ to E_L. Let $D = \{i_1 < i_2 < \cdots < i_m\}$. First of all, let us define a function $\eta_D : \{1, \cdots, m\} \to \mathcal{F}$ as follows. For each j,

$$\eta_D(j) = \begin{cases} i_{|i_j|+j} & \text{if } i_j < 0 \text{ and } |i_j| + j < m+1, \\ n^* & \text{if } i_j < 0 \text{ and } |i_j| + j = m+1, \\ i_j & \text{otherwise.} \end{cases} \tag{4.4}$$

Lemma 4.5.4. *The map η_D is well-defined.*

Proof. Let $D = \{i_1 < i_2 < \cdots < i_m\}$. The only case we need to check is that if $|i_j| + j < m + 1$ and $i_j < 0$, then $i_{|i_j|+j} \in \mathcal{F}$. Assume $i_l, \cdots, i_m$ are nonnegative. Then we have the following,

$$m > |i_1| > |i_2| > |i_3| > \cdots > |i_{l-1}| > 0$$

which implies

$$m \geq |i_1| + 1 \geq |i_2| + 2 \geq |i_3| + 3 \geq \cdots \geq |i_{l-1}| + l - 1 \geq l.$$

Therefore, for $1 \leq j \leq l-1$, we have $m \geq |i_j| + j \geq l$. This shows that $i_{|i_j|+j} \in \mathcal{F}$. $\square$

Now, we define a map

$$\varphi : \mathcal{D}_{m,n} \to E_L$$

as follows:

$$\varphi(D) = enhanced(sorted\{\eta_D(j) \mid j \in \{1, \cdots, m\}\})$$

where we first sort into a non-decreasing order with n^* after n and then enhance the result as follows: fix all of the 0's and fix all of the n^*'s. This map is the inverse map of ψ. But we will not prove it here since we do not need this fact.

4.6 The main result

We consider the Grothendieck group of the bounded derived category of the incidence algebra $\mathcal{A}$ of the poset of order ideals in a grid $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$. In this section, we are going to prove the following:

Theorem 4.6.1. *The Auslander-Reiten translation τ has finite order on $K_0(\mathcal{D}^b(\mathcal{A}))$ for the incidence algebra $\mathcal{A}$ of the poset of order ideals $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$ of a grid poset $\mathsf{P}_{\mathsf{m,n}}$. Specifically, $\tau^{m+n+1} = \pm id$.*

Our main result follows from two auxiliary Propositions.

(1) In Proposition 4.6.2 we show that Auslander-Reiten translation satisfies $\tau^{2(m+n+1)} = id$ on the elements $[\mathcal{P}_\alpha]$ in K_0.

(2) In Proposition 4.6.4 we show that the Grothendieck group K_0 is generated by the elements of the form $[\mathcal{P}_\alpha]$.

Proposition 4.6.2. *$\tau^{2(m+n+1)} = id$ on the elements $[\mathcal{P}_\alpha]$ in K_0.*

Proof. First, we are going to prove that the following diagram commutes since the shift $\{1\}$ has finite order of $(m+n+1)$ on $\mathcal{D}_{m,n}$.

$$\begin{array}{ccc} E_L & \xrightarrow{\psi} & \mathcal{D}_{m,n} \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle \{1\}} \\ E_L & \xrightarrow[\psi]{} & \mathcal{D}_{m,n} \end{array}$$

Let $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}) \in E_L$ be an enhanced partition. To simplify the exposition, assume $\alpha_i > 1$ for all i and assume $\lambda_1 \neq 1$. The general case is similar. Then, $\tilde{f}(\alpha) = ((\lambda_1 - 1)^{\alpha_0+1}, (\lambda_2 - 1)^{\alpha_1}, \cdots, (\lambda_r - 1)^{\alpha_{r-1}}, n^{\alpha_r}|n^{\alpha_{r+1}-1})$

$$\psi(\alpha) = \{\underbrace{-m < -m+1 < \cdots < -m+\alpha_{r+1}-1}_{\alpha_{r+1}} <$$
$$\underbrace{-x_0 < -x_0+1 < \cdots < -x_0+\alpha_r-2}_{\alpha_r-1} < \cdots <$$
$$\underbrace{-x_r < -x_r+1 < \cdots < -x_r+\alpha_0-2}_{\alpha_0-1} < 0 < \lambda_1 < \cdots < \lambda_r\}$$

$$\psi(\tilde{f}(\alpha)) = \{\underbrace{-m < -m+1 < \cdots < -m+\alpha_{r+1}-2}_{\alpha_{r+1}-1} <$$
$$\underbrace{-x_0-1 < -x_0 < \cdots < -x_0+\alpha_r-3}_{\alpha_r-1} < \cdots <$$
$$\underbrace{-x_{r-1}-1 < -x_{r-1} < \cdots < -x_{r-1}+\alpha_r-3}_{\alpha_1-1} <$$
$$\underbrace{-x_r-1 < -x_r < \cdots < -x_r+\alpha_0-2}_{\alpha_0} < \lambda_1-1 < \cdots < \lambda_r-1 < n\}$$

where the x_i's are defined as in the proof of Lemma 4.5.3.

From these calculations, it is easy to see that $\psi(\alpha)\{1\} = \psi(\tilde{f}(\alpha))$. This shows that $\tau^{m+n+1}[\mathcal{P}_\alpha] = \pm[\mathcal{P}_\alpha]$. Therefore, $\tau^{2(m+n+1)}[\mathcal{P}_\alpha] = [\mathcal{P}_\alpha]$. The proof of the case $\lambda_1 = 1$ is similar. Also, observe that the order of τ cannot be less than $(m+n+1)$ because this is obviously true for the action of $\{1\}$ on $\mathcal{D}_{m,n}$. This finishes the proof. □

Example 4.6.3. In this example, we will write the action of τ algebraically and combinatorially. Assume $m = 5$ and $n = 3$.

Let $\alpha = (|1,1,2,3,3|)$. Write the projective resolution as follows:

$$\mathcal{P}_\alpha : 0 \to P^{-3}_{(0,0,1,2,2)} \to P^{-2}_{(0,0,1,3,3)} \oplus P^{-2}_{(1,1,1,2,2)} \oplus P^{-2}_{(0,0,2,2,2)} \to$$
$$P^{-1}_{(0,0,2,3,3)} \oplus P^{-1}_{(1,1,1,3,3)} \oplus P^{-1}_{(1,1,2,2,2)} \to P^{0}_{(|1,1,2,3,3|)} \to 0$$

with the homology $[[f(\alpha), \alpha]] = [[(0,1,2,2,3),(1,1,2,3,3)]]$.

Apply τ to $\mathcal{P}_\alpha$:

$$\mathcal{I}_\alpha : 0 \to I^{-3}_{(0,0,1,2,2)} \to I^{-2}_{(0,0,1,3,3)} \oplus I^{-2}_{(1,1,1,2,2)} \oplus I^{-2}_{(0,0,2,2,2)} \to$$
$$I^{-1}_{(0,0,2,3,3)} \oplus I^{-1}_{(1,1,1,3,3)} \oplus I^{-1}_{(1,1,2,2,2)} \to I^{0}_{(1,1,2,3,3)} \to 0$$

with the homology $[[(0,0,1,2,2),(0,1,1,2,3)]]$.

Note that $\tilde{f}((|1,1,2,3,3|)) = (0|1,1,2|3)$. So, $\tau\mathcal{P}_{(|1,1,2,3,3|)} \cong \mathcal{P}_{(0|1,1,2|3)}[-2]$. In the Grothendieck group, we have the following

$$\tau[[(0,1,2,2,3),(1,1,2,3,3)]] = [[(0,0,1,2,2),(0,1,1,2,3)]]$$

Now, let us look at the action of τ combinatorially. Firstly, we find the corresponding configuration D of α which is $D = \psi(\alpha) = \{-4 < -1 < 1 < 2 < 3\}$.

We now compute that $\psi\tilde{f}(\alpha) = \psi((0|1,1,2|3)) = \{-5 < -2 < 0 < 1 < 2\}$ which equals $D\{1\}$.

Proposition 4.6.4. *The set $\{[\mathcal{P}_\alpha] \mid \alpha$ is a partition$\}$ generates the Grothendieck group K_0 of the incidence algebra $\mathcal{A}$ of the poset* $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$.

Proof. The Grothendieck group $\mathcal{K}_0$ is generated by all the isomorphism classes of indecomposable projective modules $[P_\alpha]$, $\alpha \in \mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$. Now, we will think of α as an enhanced partition with the first bar is placed after 0's and the second bar at the very end. Let us define $L_x = [[f(x), x]]$ where x is an enhanced partition. We will show that each $[P_\alpha]$ can be written as a linear combination of elements of the form $[L_x]$. We will proceed by strong induction on partitions ordered lexicographically. The base case is $\alpha = (0, \cdots, 0)$. Then $[P_\alpha] = [L_\alpha]$, and we are done.

Recall that we get the element $[\mathcal{P}_\alpha]$ in $\mathcal{K}_0$ by taking the Euler characteristic of the projective resolution $\mathcal{P}_\alpha$.

Recall also the notation from Subsection 4.1. Let us write each partition $\alpha - \delta_J$ where $J \subseteq R_\alpha$ and $J \neq \emptyset$. Notice that each $\alpha - \delta_J$ comes before α in the lexicographical order.

Now, we write $[P_\alpha] = [\mathcal{P}_\alpha] + [\bigoplus_{\substack{J\subseteq R_\alpha,\\ |J|=1}} P_{\alpha-\delta_J}] - [\bigoplus_{\substack{J\subseteq R_\alpha,\\ |J|=2}} P_{\alpha-\delta_J}] + \cdots + (-1)^{|J|}[P_{\alpha-\delta_{R_\alpha}}]$. Therefore, by the induction hypothesis, each $[P_{\alpha-\delta_J}]$ can be written as a linear combination of elements of the form $[L_x]$. So, we have

$$\sum_{J\neq\emptyset}(-1)^{(|J|+1)}[P_{\alpha-\delta_J}] = \sum_x a_x[L_x]$$

And we know that $[\mathcal{P}_\alpha] = [L_\alpha]$. Now, we have the desired result:

$$[P_\alpha] = [L_\alpha] + \sum_x a_x[L_x]$$

□

CHAPTER V

A GENERAL FRAMEWORK : COMINUSCULE POSETS

In Chapter 4, we proved the finiteness of the Coxeter transformation on grid posets. In this chapter, we are going to study the generalization of our result to the incidence algebra of the poset of order ideals $\mathsf{J}(\mathsf{C})$ of a cominuscule poset C. We start by recalling some basic facts on root systems and Coxeter groups. Then, we are going to give the definition of a cominuscule poset C in the poset of positive roots of a given root system Φ. Next, we are going to investigate the action of the Coxeter transformation τ for the poset of order ideals $\mathsf{J}(\mathsf{C})$. We are going to show that $\tau^{h+1} = \pm id$ for two infinite families of cominuscule posets and exceptional cases where h is the Coxeter number for the relevant root system.

5.1 Reflection along hyperplanes

The material in this section is drawn from (Humphreys, 1978).

Let V be a finite dimensional real vector space together with a positive definite symmetric bilinear form (x, y) for $x,\ y \in V$. A reflection is defined as an invertible linear map from V to itself fixing a hyperplane and sending its normal vector to its negative. For a given hyperplane H with a normal vector $\sigma \in V$, we can define

s_σ the reflection along H as follows:

$$s_\sigma(v) = v - 2\frac{(v,\sigma)}{(\sigma,\sigma)}\sigma$$

for every vector $v \in V$.

A subset Φ of V is *a root system* if the following conditions hold:

R1) Φ is finite, spans V and does not contain 0.

R2) For $\sigma \in \Phi$, the only multiples of σ in Φ are exactly $\pm\sigma$.

R3) If $\sigma \in \Phi$, then the reflection s_σ leaves Φ invariant.

R4) If $\sigma, \sigma' \in \Phi$, then $\frac{2(\sigma',\sigma)}{(\sigma,\sigma)} \in \mathbb{Z}$.

The elements of Φ are called *roots*. A subset Δ of Φ is called a *base* of Φ if:

B1) it is a basis for V,

B2) each root in Φ can be written as a linear combination of roots in Δ with integral coefficients which are either all nonnegative or all non-positive.

Every root system has a base. The elements in Δ are called *simple roots*. The subset Φ^+ consists of roots which are the combinations of elements in Δ, and Φ^- consists of roots which are the combination of the negative of elements in Δ. One can easily see that $\Phi^- = -\Phi^+$ and $\Phi = \Phi^+ \sqcup \Phi^-$. Note that the decomposition of Φ into Φ^+ and Φ^- is not unique.

One can define a partial order on Φ^+ naturally as follows: For σ, $\sigma' \in \Phi^+$, we say that $\sigma' \prec \sigma$ if and only if $\sigma - \sigma'$ is a sum of positive roots. This poset is called *the poset of positive roots* of the root system Φ. The partial order on Φ^+ is well-defined, and it does not depend on the choice of Δ up to isomorphism.

We say that Φ is irreducible if it cannot be partitioned into two proper, orthogonal subsets. Also, we can state this as follows (Humphreys, 1978, Section 10.4): Φ is irreducible if Δ cannot be partitioned into two proper, orthogonal subsets.

The height of a root σ is defined as the sum of the coefficients in the expression of σ as a linear combination of simple roots. Assume Φ is irreducible, then the poset Φ^+ has always a highest root, say η. For a detailed discussion of the subject, see (Humphreys, 1978).

Example 5.1.1. The following is the poset of positive roots of A_4 with the set of simple roots $\Delta = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $\eta = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$ is the highest root.

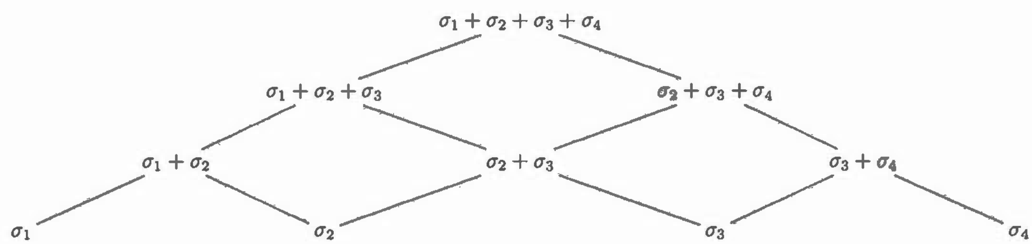


Figure 5.1 Root poset of A_4

5.2 Coxeter groups

Coxeter groups provide a good abstraction of the geometric setting we described in Section 5.1. They were introduced and classified by H. S. M. Coxeter (Coxeter, 1934), (Coxeter, 1936).

A finitely generated Coxeter group has a presentation of the form

$$W = < s_1, \cdots, s_n | s_i^2, (s_i s_j)^{m_{ij}},\ 1 \leq i, j \leq n >$$

where $m_{ii} = 1$ and $2 \leq m_{ij} = m_{ji} \leq \infty$, $m_{ij} = \infty$ means no corresponding relation at all. We can encode this information by a graph called *the Coxeter diagram*, In the Coxeter diagram the vertices are labeled with reflections $1, \cdots, n$. We connect two vertices with an edge only when $m_{ij} > 3$ if $i \neq j$ and we label the edge with the corresponding m_{ij}.

We know that if Φ is irreducible, its Coxeter diagram is connected, and vice versa. It is sufficient to classify irreducible root systems. The classification of irreducible root systems is done by using the *Dynkin diagrams.* We construct Dynkin diagrams from Δ as follows: We put a vertex for each element in Δ. We connect two vertices with respect to the following rule. We put

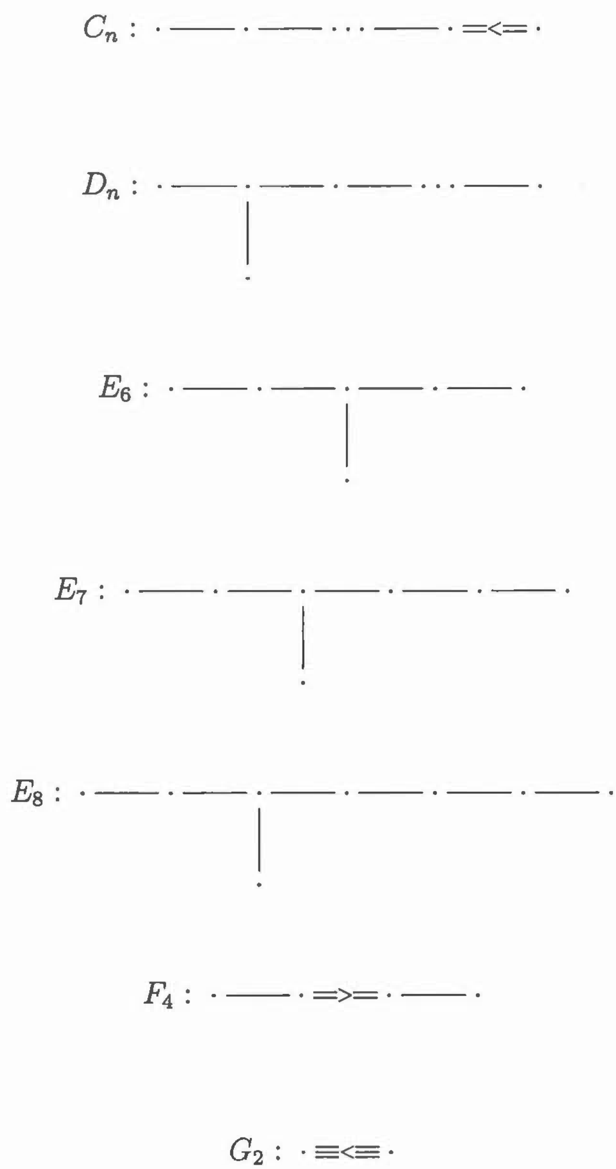
- no edge if the corresponding roots are orthogonal.
- one edge if the angle between the corresponding roots is 120 degrees.
- two edges if the angle between the corresponding roots is 135 degrees.
- three edges if the angle between the corresponding roots is 150 degrees.

We direct an edge from the vertex corresponding to the longer root to the vertex corresponding to the shorter one. If the roots have the same length, the edges are undirected.

Here is the list of Dynkin diagrams for finite root systems:

A_n : · —— · —— · · · —— · —— ·

B_n : · —— · —— · · · —— · =>= ·



5.3 Cominuscule posets

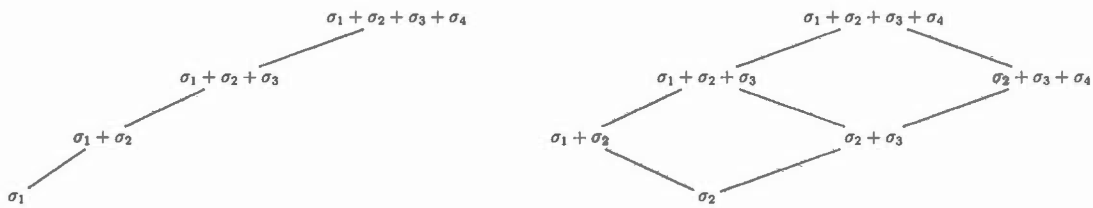
Throughout this section, assume that Φ is a finite irreducible root system and let η be its highest root.

Definition 5.3.1. A simple root σ is called a *cominuscule root* if the multiplicity of σ in the simple root expansion of η is 1.

Definition 5.3.2. An interval C of the form $[\sigma, \eta] \subset \Phi^+$ is called a *cominuscule poset* if σ is a cominuscule root.

Cominuscule posets appear in representation theory of Lie groups, Schubert calculus, and combinatorics. For more details on root systems see (Humphreys, 1978), and for cominuscule posets (Billey & Lakshmibai, 2000), (Green, 2013), (Rush & Shi, 2013) and (Thomas & Yong, 2009).

Example 5.3.3. In Example 5.1.1, all simple roots are cominuscule roots. The following are cominuscule posets for the simple roots σ_1 and σ_2, respectively.



The following shows an illustration of the possible shapes of cominuscule posets except the exceptional cases. The exceptional cominuscule posets will be discussed at the end of this section.

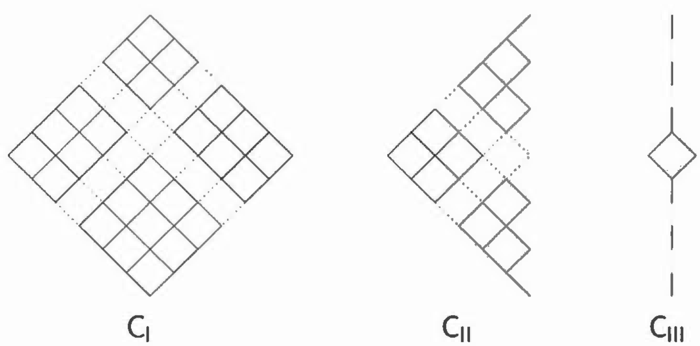


Figure 5.2 Cominuscule posets of type I, II, III

Conjecture 5.3.4. *Auslander-Reiten translation τ has finite order on the Grothendieck group of the bounded derived category for the incidence algebra of the poset* $\mathsf{J}(\mathsf{C})$. *Specifically, $\tau^{2(h+1)} = id$ where h is the Coxeter number for the relevant root systems (A, B, C, D, E_6 or E_7).*

In this chapter, we prove that the Conjecture 5.3.4 is true for cominuscule posets $\mathsf{J}(\mathsf{C}_{\mathsf{I}})$, $\mathsf{J}(\mathsf{C}_{\mathsf{III}})$, $\mathsf{J}(\mathsf{C}_{\mathsf{E_6}})$, $\mathsf{J}(\mathsf{C}_{\mathsf{E_7}})$.

Remark 5.3.5. If the same cominuscule poset comes from two different root systems, since the order of the Coxeter transformation τ obviously agree, Conjecture 5.3.4 is self-consistent.

We will proceed with a case by case analysis and prove some results.

5.4 Type A

All simple roots in the root poset of A_n are cominuscule roots. So, they all give rise to cominuscule posets.

The cominuscule poset over any simple root in A_n is of type I in Figure 5.2. In other words, they are the grid posets $\mathsf{P}_{\mathsf{m,k}}$ of size $m + k = n + 1$.

Example 5.4.1. The following figure is an illustration of root poset of A_n. The shaded area shows the cominuscule poset C_{I} over the simple root σ_k.

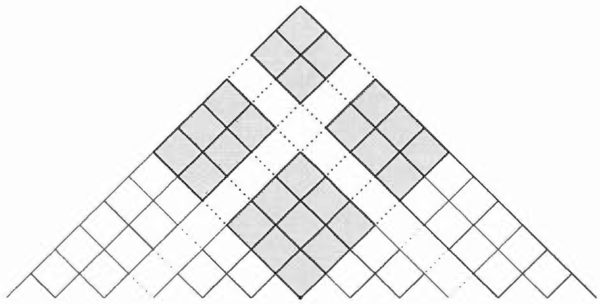


Figure 5.3 The root poset of A_n and the cominuscule poset $\mathsf{C_I}$ over the simple root σ_k.

In this case, we have the following results.

Theorem 5.4.2 (Theorem 4.6.1). *The Coxeter transformation τ acts finitely on the poset $\mathsf{J(P_{m,k})}$ of order ideals in $\mathsf{P_{m,k}}$.*

Recall that the Coxeter number h is $n+1$ in type A_n. So, the Conjecture 5.3.4 holds for the type A_n, i.e. $\tau^{2(h+1)} = id$ in $K_0(\mathcal{D}^b(\mathcal{A}))$ for the incidence algebra $\mathcal{A}$ of the poset of order ideals $\mathsf{J(C_I)}$.

Now, we will prove the order of τ explicitly.

Proposition 5.4.3. *Let $\mathsf{J(P_{m,n})}$ be the poset of order ideal of a grid poset $\mathsf{P_{m,n}}$. If m, n are both even, then the order of τ is $2(m+n+1)$. Otherwise, the order of τ is $m+n+1$.*

Proof. First, we observe that if $|R_\alpha|$ is odd, then $[\tau\mathcal{P}_\alpha]$ is a positive sum of simples in $\mathcal{K}_0$; if $|R_\alpha|$ is even, then $[\tau\mathcal{P}_\alpha]$ is a negative sum of simples in $\mathcal{K}_0$.

Let us state this fact in terms of configurations as follows. We will work with sign configurations which are just configurations with a sign attached. Let D

be the corresponding configuration to α with a sign attached. In this case, $|R_\alpha|$ is the number of positive entries in D. We define the action of $\{1\}$ on signed configurations as follows: if $|R_\alpha|$ is odd, then $D\{1\}$ will have the same sign; if $|R_\alpha|$ is even, then $D\{1\}$ will have the opposite sign. We will write this fact in an explicit way as follows. Let $D = \{a_1 < \cdots < a_m\}$ for $\alpha = (a_1, \cdots, a_m)$.

We first define $r : D \to \{1, -1\}$ by sending $a_i \mapsto r(a_i)$ where

$$r(a_i) = \begin{cases} -1 & \text{if} \quad 0 < a_i \leq n \\ 1 & \text{if} \ -m \leq a_i \leq 0. \end{cases}$$

Then $\text{sign}(D\{1\})$ is as follows:

$$\text{sign}(D\{1\}) = (-1)\left(\prod_{j=1}^{m} r(a_j)\right)\text{sign}(D)$$

We know that each a_i will be non-negative n times in the full orbit of D. So, we have

$$\prod_{j=0}^{m+n} r(a_i - j) = (-1)^n$$

Now, let us determine the sign of $D\{m + n + 1\}$.

$$\begin{aligned}
&\operatorname{sign}(D\{m+n+1\}) \\
&\quad= \left((-1)\prod_{j=1}^{m} r(a_j - m - n)\right) \operatorname{sign}(D\{m+n\}) \\
&\quad= \left((-1)\prod_{j=1}^{m} r(a_j - m - n)\right)\left((-1)\prod_{j=1}^{m} r(a_j - m - n - 1)\right) \\
&\quad\cdots \left((-1)\prod_{j=1}^{m} r(a_j)\right) \operatorname{sign}(D) \\
&\quad= (-1)^{m+n+1}\left(\prod_{j=0}^{m+n} r(a_1 - j)\right)\cdots\left(\prod_{j=0}^{m+n} r(a_m - j)\right) \operatorname{sign}(D) \\
&\quad= (-1)^{m+n+1}(-1)^{nm} \operatorname{sign}(D) = (-1)^{(m+1)(n+1)} \operatorname{sign}(D)
\end{aligned}$$

From this, we see that when m and n are both even, the order is $2(m+n+1)$. Otherwise, it is $m+n+1$. □

5.5 Type B

In type B_n, the only simple root which is a cominuscule root is σ_1 and the cominuscule poset is the grid poset $\mathsf{P}_{1,2n-1}$. The Coxeter number h is $2n$ in this case. Therefore, we have the desired result, i.e. $\tau^{2(1+2n-1+1)} = \tau^{2(h+1)} = id$.

5.6 Type C

In type C_n, the only simple root which gives rise to the cominuscule poset is σ_n and the cominuscule poset is the type of II in Figure 5.2.

Example 5.6.1. Here we illustrate the root poset of C_n and the shaded area shows the cominuscule poset over the simple root σ_n.

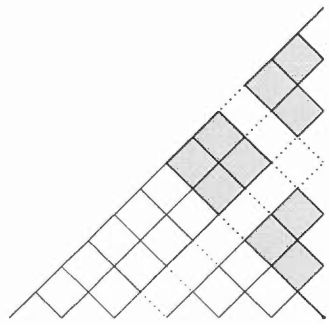


Figure 5.4 The root poset of C_n and the cominuscule poset C_{II} over the simple root σ_n.

This case is still open. We give the detailed discussion of this case in the conclusion of the thesis.

5.7 Type D

In type D_n, $n > 3$, there are three simple roots σ_1, σ_{n-1}, and σ_n which give rise to cominuscule posets. The simple roots σ_{n-1} and σ_n give rise to cominuscule posets of type II which is the same as the cominuscule poset for C_{n-1}.

The simple root σ_1 gives rise to the cominuscule poset of type III. The cominuscule poset $\mathsf{C}_{\mathsf{III}}$ and the poset of order ideals $\mathsf{J}(\mathsf{C}_{\mathsf{III}})$ is shown as follows:

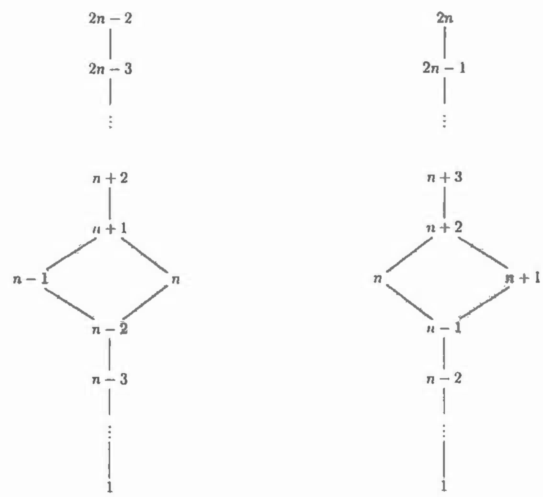


Figure 5.5 The cominuscule poset $\mathsf{C}_{\mathsf{III}}$ and the order ideal poset $\mathsf{J}(\mathsf{C}_{\mathsf{III}})$ for the simple root σ_1 in the root system for D_n.

Now, we are going to prove Conjecture 5.3.4 for the cominuscule poset $\mathsf{C}_{\mathsf{III}}$.

Theorem 5.7.1. *Let $\mathcal{A}_{(D_n,\sigma_1)}$ be the incidence algebra of the poset $\mathsf{J}(\mathsf{C}_{\mathsf{III}})$ of size $2n$ for the corresponding root system D_n. The Auslander-Reiten translation τ on $K_0(\mathcal{D}^b(\operatorname{mod}\mathcal{A}_{(D_n,\sigma_1)}))$ has finite order of $2(h+1)$ where h is the Coxeter number for type D_n.*

Proof. We will first write some orbits of τ explicitly to see the order. In this poset, we distinguish four cases for the action of τ on the stalk complexes of the corresponding simple modules:

1. Let S_j be the simple module supported over the vertex j where $j \neq n, n+1, n+2$ or 1. Consider the stalk complex $0 \to S_j^0 \to 0$, then the action of τ is calculated as follows:

- Take the projective resolution $0 \to P_{j-1}^{-1} \to P_j^0 \to 0$ of S_j.
- After applying the functor $(- \otimes D\mathcal{A}_{(D_n,\sigma_1)})$ we get the injective resolution $0 \to I_{j-1}^{-1} \to I_j^0 \to 0$ which is quasi-isomorphic to $0 \to S_{j-1}^{-1} \to 0 \to 0$.
- Finally the shift $[-1]$, $0 \to S_{j-1}^0 \to 0$.

We conclude that $\tau(0 \to S_j^0 \to 0) \cong 0 \to S_{j-1}^0 \to 0$.

2. Let S_1 be the simple module supported over the vertex 1. Consider $0 \to S_1^0 \to 0$, then we compute τS_1 as follows:

   - Write the projective resolution $0 \to P_1^0 \to 0$.
   - Apply the functor, then we have $0 \to I_1^0 \to 0$.
   - Now apply the shift $[-1]$.

   Thus, we get $\tau(0 \to S_1^0 \to 0) \cong 0 \to I_1^1 \to 0$.

3. Let us now consider the simple module S_{n+2}, then the action of τ on the stalk complex $0 \to S_{n+2}^0 \to 0$ can be computed as follows:

   - $0 \to P_{n-1}^{-2} \to P_n^{-1} \oplus P_{n+1}^{-1} \to P_{n+2}^0 \to 0$ is the projective resolution.
   - After applying the functor we get $0 \to I_{n-1}^{-2} \to I_n^{-1} \oplus I_{n+1}^{-1} \to I_{n+2}^0 \to 0$ which is quasi-isomorphic to $0 \to S_{n-1}^{-2} \to 0$.
   - and now apply the shift functor $[-1]$, then we get $0 \to S_{n-1}^{-1} \to 0$.

   So, we have $\tau(0 \to S_{n+2}^0 \to 0) \cong 0 \to S_{n-1}^{-1} \to 0$.

4. Finally we will look at the simple S_n or S_{n+1}. We write the proof only for the vertex n since the proof of the other case is identical. Consider $0 \to S_n^0 \to 0$.

   - Write the projective resolution $0 \to P_{n-1}^{-1} \to P_n^0 \to 0$.

- Next, we get the injective resolution $0 \to I_{n-1}^{-1} \to I_n^0 \to 0$ and the corresponding element in K_0 is $-[S_{n-1}] - [S_{n+1}]$.
- The shift functor is $-id$ in K_0, thus $\tau([S_n]) = [S_{n-1}] + [S_{n+1}]$ in K_0.

Note that $\tau([S_{n+1}]) = [S_{n-1}] + [S_n]$ in K_0.

Now, we will write two different τ-orbits in K_0. Assume $j \geq n+2$. First τ-orbit is as follows:

$$[S_j] \xrightarrow{\tau} [S_{j-1}] \xrightarrow{\tau} \cdots \xrightarrow{\tau} [S_{n+3}] \xrightarrow{\tau} [S_{n+2}] \xrightarrow{\tau} -[S_{n-1}] \xrightarrow{\tau}$$
$$-[S_{n-2}] \xrightarrow{\tau} \cdots \xrightarrow{\tau} -[S_1] \xrightarrow{\tau} [I_1] \xrightarrow{\tau} -[S_{2n}] \xrightarrow{\tau} \cdots \xrightarrow{\tau} -[S_j]$$

Therefore, we conclude that for $j \in \{1, 2, \cdots, n-1, n+2, n+3, \cdots, 2n\}$,

$$\tau^{2n-1}([S_j]) = -[S_j]$$

The second τ-orbit is calculated as follows. For simplicity, we assume n is odd since the case n is even is not significantly different.

$$\begin{aligned}
[S_n] \xrightarrow{\tau} & [S_{n-1} \oplus S_{n+1}] \xrightarrow{\tau} [S_{n-2} \oplus S_{n-1} \oplus S_n] \\
& \xrightarrow{\tau} [S_{n-3} \oplus S_{n-2} \oplus S_{n-1} \oplus S_{n+1}] \xrightarrow{\tau} \cdots \\
& \xrightarrow{\tau} [S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1} \oplus S_n] \\
& \xrightarrow{\tau} -[I_n] \xrightarrow{\tau} -[S_{n+1} \oplus S_{n+2} \oplus S_{n+3} \oplus \cdots \oplus S_{2n-1}] \\
& \xrightarrow{\tau} -[S_n \oplus S_{n+2} \oplus S_{n+3} \oplus \cdots \oplus S_{2n-2}] \\
& \xrightarrow{\tau} \cdots \xrightarrow{\tau} -[S_{n+1} \oplus S_{n+2}] \\
& \xrightarrow{\tau} -[S_n]
\end{aligned}$$

Thus,

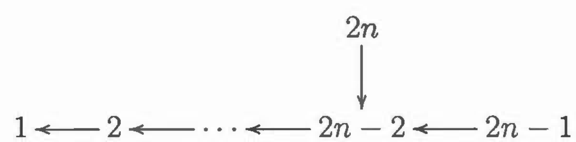
$$\tau^{2n-1}([S_n]) = -[S_n]$$

To sum up, in both orbits we see that the order of τ is $2(2n-1)$. We can see from these orbits that τ acts finitely on every simple. So, $\tau^{2n-1} = -id$ in the Grothendieck group K_0. Also, recall that the Coxeter number in type D_n is $2n-2$. This finishes the proof. □

We can give a second proof of the previous result which in fact establishes more.

Theorem 5.7.2. $\mathcal{D}^b(\operatorname{mod} \mathcal{A}_{(D_n,\sigma_1)})$ *is fractionally Calabi-Yau.*

Proof. Let D_{2n} be the path algebra with the following orientation:



Let M_0 be the module over the algebra D_{2n} with dimension vector $(1,1,\cdots,1,1)$, and M_i be the module with dimension vector $(1,1,\cdots,1,2,\cdots,2,1)$ where we have the dimension 2 appears i times for $i > 0$. Let us now consider the following module: $T = P_1 \oplus P_2 \oplus \cdots \oplus P_{n-1} \oplus P_{2n-1} \oplus P_{2n} \oplus M_0 \oplus M_1 \oplus \cdots \oplus M_{n-2}$ which is illustrated in Figure 5.6.

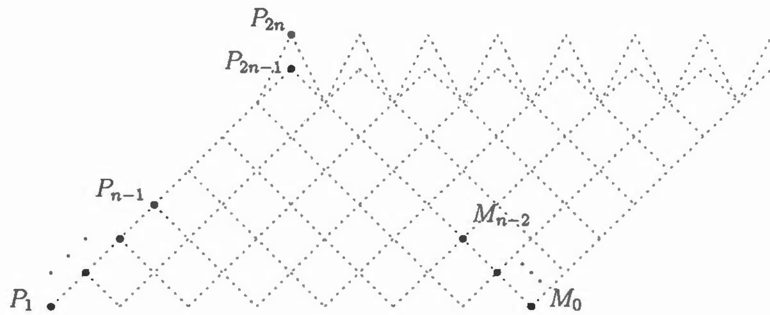


Figure 5.6 An illustration of the module category of D_{2n} and the tilting module T.

It is not difficult to see T is a tilting module. So, we consider the endomorphism algebra End$T$ which is the algebra $\mathcal{A}_{(D_n,\sigma_1)}$. Therefore, we have that $\mathcal{D}^b(\text{mod}\,\mathcal{A}_{(D_n,\sigma_1)})$ is derived equivalent to $\mathcal{D}^b(\text{mod}\,D_{2n})$. We know by (Keller, 2005, Example 8.3(2)), $\mathcal{D}^b(\text{mod}\,D_{2n})$ is fractionally Calabi-Yau, and therefore $\mathcal{D}^b(\text{mod}\,\mathcal{A}_{(D_n,\sigma_1)})$ is also fractionally Calabi-Yau. Thus, we have the desired result. □

Remark 5.7.3. Theorem 5.7.2 can also be proved by using the technique of flip-flops of Ladkani (Ladkani, 2007)[Theorem 1.1]. Let P be a finite poset, P^1 be the poset with a unique maximum element added to P, and P_0 be the poset with a unique minimum element added to P. Ladkani shows that P^1 and P_0 are derived equivalent. Using this fact, we can show that the poset $\mathsf{J}(\mathsf{C}_{\mathsf{III}})$ of size $2n$ is derived equivalent to D_{2n}.

Example 5.7.4. In this example, we use Ladkani's technique showing that $\mathsf{J}(\mathsf{C}_{\mathsf{III}})$ of size 8 is derived equivalent to D_8.

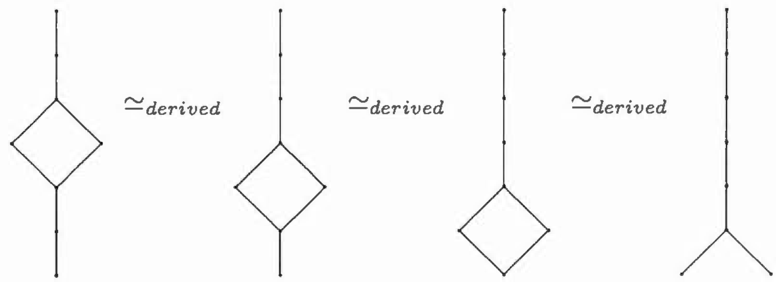


Figure 5.7 The derived equivalent posets

5.8 Exceptional cases

There are two cominuscule roots which give rise to the same cominuscule poset for type E_6 and there is only one cominuscule root which gives a cominuscule poset for type E_7.

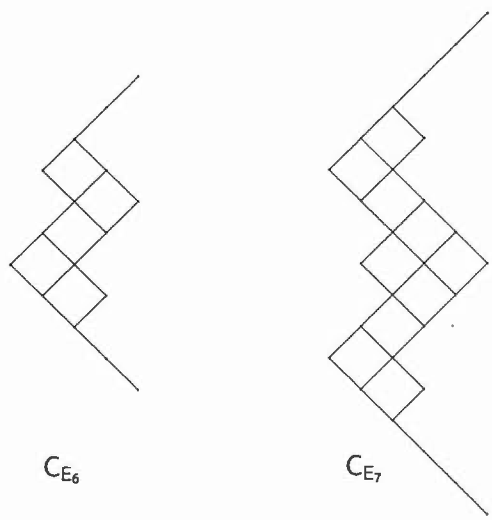


Figure 5.8 The cominuscule poset C_{E_6} and the cominuscule poset C_{E_7}

We checked that Conjecture 5.3.4 also holds for these two exceptional cases by using the mathematical software *SageMath* (Stein et al., 2017).

CHAPTER VI

MOVIES AND THE PANYUSHEV MAP

In this chapter, we will define a *domino configuration* $\mathcal{D}$ associated to a configuration D. Recall that in Section 4.5, we proved the periodicity of τ for $\mathsf{J}(\mathsf{P}_{\mathsf{m},\mathsf{n}})$ with a corresponding cyclic action on the configurations. Now, we will interpret the periodicity of τ from a different perspective using the combinatorics of domino configurations.

We will also study a remarkable map which is defined on order ideals in a poset (Panyushev, 2009). In the literature, there are different names for this map. We will call it by Panyushev map Pan. Moreover, Pan, or some variations of it, is extensively studied by many mathematicians in various setting (Brouwer & Schrijver, 1974), (Fon-Der-Flaass, 1993), (Cameron & Fon-Der-Flaass, 1995), (Reiner et al., 2004), (Panyushev, 2009), (Stanley, 2009), (Striker & Williams, 2012) (Armstrong et al., 2013), (Rush & Shi, 2013), (Grinberg & Roby, 2014). We investigate a connection between the Coxeter transformation τ and the Panyushev map Pan. This connection requires us to work with a variation of domino configurations called *a configuration of singletons* S associated to a configuration D. We will see that the behavior of Pan is obtained by the rotation of the configuration of singletons. We remark that this amounts to a proof of the order of Pan on a grid which goes back to Brouwer and Schrijver (Brouwer & Schrijver, 1974) in only slightly different form.

6.1 Configurations of dominos

We start this section by recalling that we write $\mathcal{Z} := \{-m, \cdots, -1, 0, 1, \cdots, n\}$ for the elements of $\mathbb{Z}/(m+n+1)$. Let $D = \{i_1 < i_2 < \cdots < i_m\}$ be a configuration where $i_j \in \mathcal{Z}$, $1 \leq j \leq m$.

We consider a frame Fr_{m+n+1} which consists of $m+n+1$ columns. We divide the frame into two different regions: the dark zone and the light zone. We label the columns with respect to our starting poset $\mathsf{P}_{\mathsf{m,n}}$: we let the width of the light zone be $n+1$, and we label the columns $0, \ldots, n$. The width of the dark zone is m; we label the columns of the dark zone from $-m$ to -1.

Definition 6.1.1. The *domino configuration* $\mathcal{D}_D$ associated to a configuration $D = \{i_1 < i_2 < \cdots < i_m\}$ consists of m dominos placed in Fr_{m+n+1} so as to occupy the columns $i_j - 1$, i_j for each $1 \leq j \leq m$.

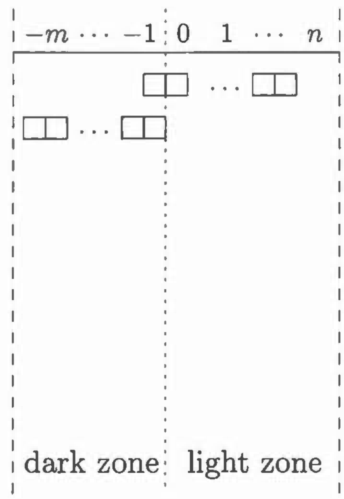


Figure 6.1 An illustration of a domino configuration.

For convenience, we do not necessarily draw all dominos in the same row, but their vertical position does not carry any significance. Notice that no domino configuration is going to fit into the dark zone completely since we have m distinct dominos. This means we are going to have at least one domino or part of it in the light zone. Since we assume our frame is cylindrical, i.e. we identified both ends, it is possible to have a domino half of which sits in the light zone while the other half of which sits in the dark zone. There are two possible positions for this case: being in columns $-m$ and n, as well as being in columns -1 and 0.

Let $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}}) \in E_L$ be an enhanced partition. In the following, we will define a *marked light domino configuration* $\mathcal{D}_\alpha^*$ corresponding to the enhanced partition $\alpha \in E_L$.

Definition 6.1.2. In the frame Fr_{m+n+1}, we place

- for each $i \in R_\alpha$, one domino marked on both sides with $\alpha_i - 1$ dots and the right half of it sitting in the column λ_i.
- one domino marked on the right half with $\alpha_0 - 1$ dots and the right half of it sitting in the column 0.
- one domino marked on the left half with $\alpha_{r+1} - 1$ dots and the left half of it sitting in the position n.

  We call this configuration of $r + 2$ dominos with the marks on the dominos as a *marked light domino configuration* $\mathcal{D}_\alpha^*$.

In the following Figure 6.2, an example of a marked light domino configuration is shown.

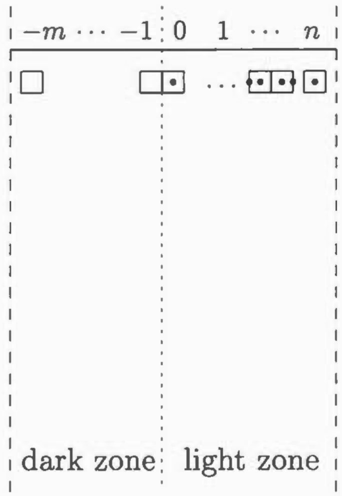


Figure 6.2 An illustration of a marked light domino configuration

Let $\mathfrak{D}$ denote the set of domino configurations and $\mathfrak{D}^*$ denote the set of marked light domino configurations. Note that there are m dominos in a domino configuration associated to $D = \{i_1 < i_2 < \cdots < i_m\}$ and there are $r+2$ dominos and a total of $m-r-2$ marks in a marked light domino configuration associated to $\alpha = (0^{\alpha_0}|\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \cdots, \lambda_r^{\alpha_r}|n^{\alpha_{r+1}})$.

We defined a map φ in Section 4.5 from the set of configurations $\mathcal{D}_{m,n}$ to the set of enhanced partitions E_L. Now, we can translate this map to a map φ_* from $\mathfrak{D}$ to $\mathfrak{D}^*$. Let

$$\varphi_* : \mathfrak{D} \to \mathfrak{D}^*$$

Let $\mathcal{D}_D \in \mathfrak{D}$. Then $\varphi_*(\mathcal{D}_D)$ is defined as follows: For every domino in $\mathcal{D}_D$ whose right half sits in the column k in the dark zone we count k dominos to the right, and mark the corresponding domino on both sides, and then delete the domino in the dark zone.

The map φ_* is obviously well-defined. The following lemma shows the reason why we call them *'marked light'*.

Lemma 6.1.3. *For each domino whose right half sits in the column k in the dark zone, the k-th domino to its right sits, partially or fully, in the light zone.*

Proof. Here the argument is that for a domino, the right half of which sits in the column k in the dark zone, there are only $k-1$ dominos that can fit entirely in the dark zone to its right. □

Therefore, a marked light domino configuration is a domino configuration at least partly in the light zone with some marks. Notice that the number of dominos and the number of times it is marked in total for $\mathcal{D}^*_\alpha$ sums up to m with respect to the description above.

Lemma 6.1.4. *The map φ_* is a bijection.*

Proof. Let $\mathcal{D}_D, \mathcal{D}'_{D'} \in \mathfrak{D}$. Assume $\varphi_*(\mathcal{D}_D) = \varphi_*(\mathcal{D}'_{D'})$. This means that $\mathcal{D}_D$ and $\mathcal{D}'_{D'}$ has exactly same domino configurations in the light zone. Then, the injectivity of φ_* follows from the fact that dominos do not coincide. We cannot have two dominos sitting in the same place in the dark zone, and the marks in the light zone are certainly determined by the positions in the dark zone. For an $m \times n$ grid poset, recall that the cardinality of the set $\mathfrak{D}$ of domino configurations is $\binom{m+n+1}{m}$ which equals to the cardinality of the set $\mathfrak{D}^*$ of marked light domino configurations, because the set $\mathfrak{D}^*$ can be thought of as the collection of multisets of size m on $n+2$ elements. This argument gives us the desired bijection. □

Example 6.1.5. Here are examples of a domino configuration and the corresponding marked light domino configuration for the grid poset $\mathsf{P}_{5,7}$. Since m is 5, we have five dominos placed in the frame Fr_{13}. We label the light zone from 0 to 7 and dark zone from -5 to -1.

Since we have two dominos placed in the dark zone, we are going to mark two dominos in the light zone.

Consider the first domino in the dark zone. Since its right half sits in the column -1, we mark the first domino to its right which is the one sitting in columns -1, 0. Notice that we marked only right half of the domino which sits in the column 0.

For the other domino the right half of which sits in the column -3, we mark the third domino on both sides.

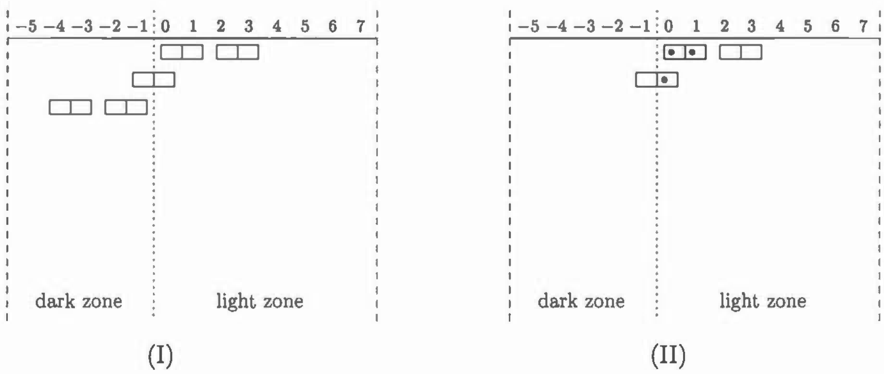
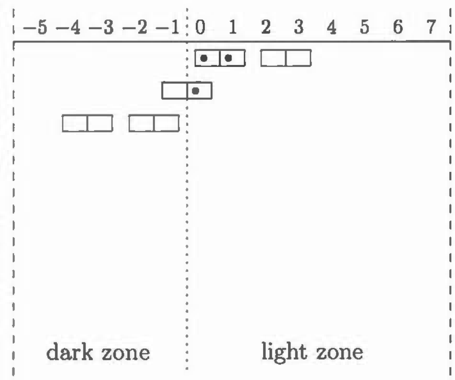


Figure 6.3 A domino configuration (I) and the corresponding marked light domino configuration (II)

We sometimes present domino configurations and marked light domino configurations in one frame as shown below.



Remark 6.1.6. Recall that for a given enhanced partition $\alpha \in E_L$, we can write the projective resolution $\mathcal{P}_\alpha$. Assume that $|R_\alpha| = k$, then $\mathcal{P}_\alpha$ has 2^k modules in total. Notice that the corresponding marked light domino configurations $\mathcal{D}^*_\alpha$ records the labeling of every module in $\mathcal{P}_\alpha$ as follows. Dominos in $\mathcal{D}^*_\alpha$ are labeled by the columns in which they sit in the frame. We read one label from each domino in $\mathcal{D}^*_\alpha$ at a time and form a partition from these labels. If a domino lies in columns -1 and 0, we only read a zero. If a domino has marks, we repeat the label in the partition that many times. By the description above, we produce 2^k partitions where k is the number of dominos in $\mathcal{D}^*_\alpha$. Notice also that α is the partition which is obtained by reading the rightmost part of the dominos. We enhanced α from $\mathcal{D}^*_\alpha$ as follows: (*i*) If the label 0 comes from the right half of a domino and if that domino has l marks, then we put the first bar after the l-th 0 in α; (*ii*) If the label n comes from the left half of a domino and if that domino has l marks, then we put the second bar before the l-th n in α.

Example 6.1.7. We write the corresponding projective resolution of the dominos in Example 6.1.5 as follows:

$$\mathcal{P}_{\alpha=(0,0|1,1,3|)} : 0 \to P_{(0,0,0,0,2)} \to P_{(0,0,1,1,2)} \oplus P_{(0,0,0,0,3)} \to P_{(0,0|1,1,3|)} \to 0$$

6.2 The movie for a domino configuration

Let $\mathcal{D}_D$ be a domino configuration and $\mathcal{D}^*_\alpha$ be the marked light domino configuration. As we define a shift operation $\{1\}$ on configurations, we will do the same for domino configurations. Consider a domino configuration $\mathcal{D}_D$ in a frame Fr_{m+n+1}, and start shifting this configuration by one to the left in the frame at each step. We call this operation on domino configurations also a shift $\{1\}$. Therefore, $\mathcal{D}_D\{i\}$ is the i-th times shifted version of $\mathcal{D}_D$ in the frame. After $m+n+1$ steps, clearly we will come back to the domino combination we started with in Fr_{m+n+1}. We will have a full orbit of these domino configurations in the frame. By the map φ_* we can consider the full orbit of the corresponding marked light domino configurations as well.

Remark 6.2.1. When we apply τ on the projective resolution P_α, we get a projective resolution starting with $P_{\tilde{f}(\alpha)}$. Therefore, as it happened in the case of configurations, $\tilde{f}$ corresponds to shift operation $\{1\}$ on $\mathcal{D}_D$.

Example 6.2.2. For 5×7 grid, consider the following projective resolution and the corresponding combinatorics:

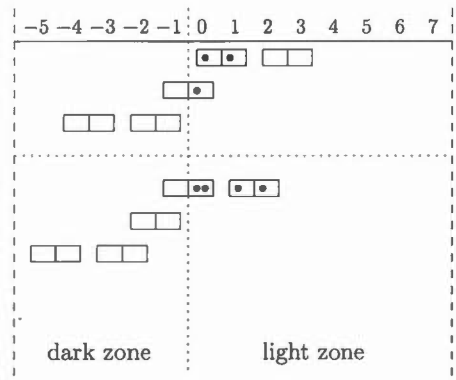
$$\mathcal{P}_\alpha\colon 0 \to P_{(0,0,0,0,2)} \to P_{(0,0,1,1,2)} \oplus P_{(0,0,0,0,3)} \to P_{(0,0|1,1,3|)} \to 0$$

$$M_1 = \{f(\alpha) = (0,0,0,1,3) < \cdots < (0,0,1,1,3)\}$$

We can write the projective resolution after the action of τ to $\mathcal{P}_\alpha$, and show the corresponding combinatorics as follows:

$$\tau\mathcal{P}_\alpha\colon 0 \to P_{(0,0,0,1,1)} \to P_{(0,0,0|2,2|)} \to 0$$

$$M_2 = \{(0,0,0,0,2) < \cdots < g(\alpha) = (0,0,0,2,2)\}$$



Definition 6.2.3. We call the orbit of marked light domino configurations in a frame a *movie.*

Example 6.2.4. We continue with Example 4.6.3. Let $\alpha = (|1, 1, 2, 3, 3|)$. We will write the orbit of projective resolutions with the homology at the top of the resolutions.

$$[[(0,1,2,2,3),(1,1,2,3,3)]]$$

$$0 \to P^{-3}_{(0,0,1,2,2)} \to P^{-2}_{(0,0,1,3,3)} \oplus P^{-2}_{(1,1,1,2,2)} \oplus P^{-2}_{(0,0,2,2,2)} \to P^{-1}_{(0,0,2,3,3)} \oplus P^{-1}_{(1,1,1,3,3)} \oplus P^{-1}_{(1,1,2,2,2)} \to P^{0}_{(|1,1,2,3,3|)} \to 0$$

$$[[(0,0,1,2,2),(0,1,1,2,3)]]$$

$$0 \to P^{-2}_{(0,0,0,1,3)} \to P^{-1}_{(0,0,0,2,3)} \oplus P^{-1}_{(0,1,1,1,3)} \to P^{0}_{(0|1,1,2|3)}$$

$$[[(0,0,0,1,3),(0,0,1,1,3)]]$$

$$0 \to P^{-2}_{(0,0,0,0,2)} \to P^{-1}_{(0,0,1,1,2)} \oplus P^{-1}_{(0,0,0,0,3)} \to P^{0}_{(0,0|1,1,3|)}$$

$$[[(0,0,0,0,2),(0,0,0,2,2)]]$$

$$0 \to P^{-1}_{(0,0,0,1,1)} \to P^{0}_{(0,0,0|2,2|)}$$

$$[[(0,0,0,1,1),(1,1,1,1,3)]]$$

$$0 \to P^{-1}_{(0,0,0,0,3)} \to P^{0}_{(|1,1,1,1|3)}$$

$$[[(0,0,0,0,3),(0,3,3,3,3)]]$$

$$0 \to P^{-1}_{(0,2,2,2,2)} \to P^{0}_{(0|3,3,3,3|)}$$

$$[[(0,2,2,2,2),(2,2,3,3,3)]]$$

$$0 \to P^{-1}_{(1,1,3,3,3)} \to P^{0}_{(|2,2|3,3,3)}$$

$$[[(1,1,3,3,3),(1,3,3,3,3)]]$$

$$0 \to P^{-2}_{(0,2,2,3,3)} \to P^{-1}_{(1,2,2,3,3)} \oplus P^{-1}_{(0,3,3,3,3)} \to P^{0}_{(|1,3,3|3,3)}$$

$$[[(0,2,2,3,3),(0,2,3,3,3)]]$$

$$0 \to P^{-2}_{(0,1,2,2,3)} \to P^{-1}_{(0,2,2,2,3)} \oplus P^{-1}_{(0,1,3,3,3)} \to P^{0}_{(0|2,3,3|3)}$$

Here is the corresponding movie for this orbit:

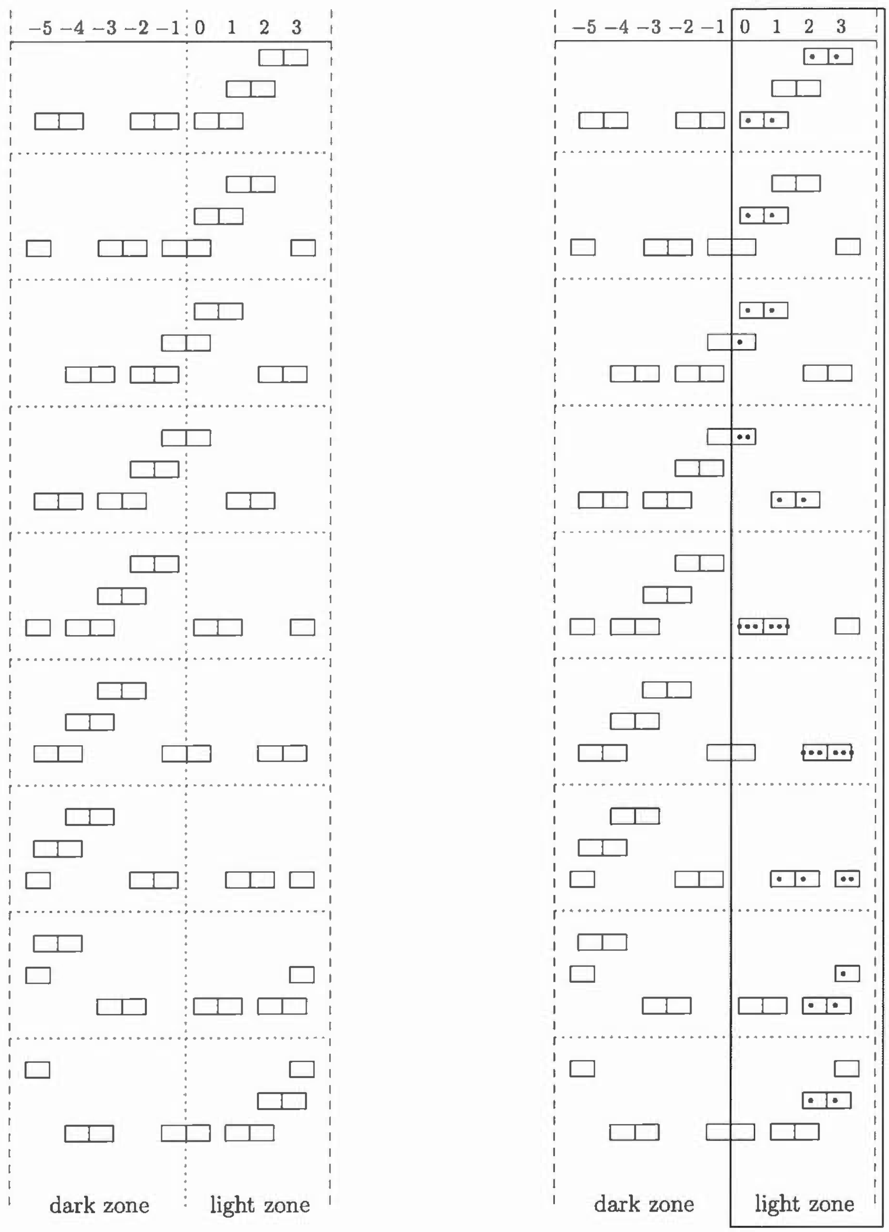
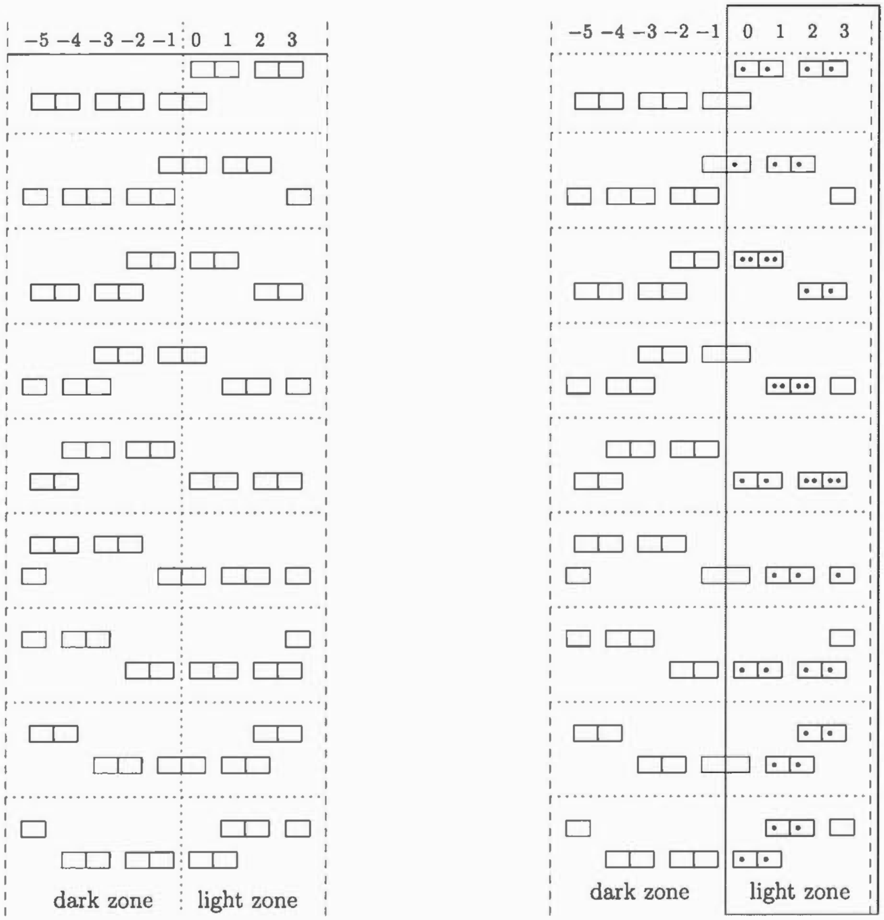


Figure 6.4 The corresponding movie for $\alpha = (|1, 1, 2, 3, 3|)$.

Example 6.2.5. Here is another example of a movie for the corresponding τ-orbit of $\mathcal{P}_{(0|1,1,3,3|)}$.



6.3 The Panyushev map

In this section, we will discuss a remarkable map Pan defined on the order ideals of a poset in (Panyushev, 2009). By (Rush & Shi, 2013), we know that the Panyushev map $\text{Pan}^h = id$ on the order ideals coming from cominuscule posets. In Chapter 5, we proved that $\tau^{(h+1)} = -id$ for the two of three infinite families of cominuscule posets and exceptional cases. Ignoring the sign, we will investigate why is the order of τ one bigger than the order of Pan. We will use the combinatorics of domino configurations and *configurations of singletons* to show the similarities of τ with the Panyushev map Pan.

Definition 6.3.1. Let I be an order ideal of a poset P. Then Pan(I) is defined to be the order ideal generated by the minimal elements not in I.

Note that the order ideals are down-closed sets in a poset. We define *order filters* as up-closed sets in a poset. Then, the inverse of the Panyushev map Pan^{-1} is the complement of the order filter generated by the maximal elements of I.

Theorem 6.3.2. *(Rush & Shi, 2013, Theorem 1.4) The Panyushev map has finite order on the cominuscule posets, specifically* $\text{Pan}^h = id$ *where h is the relevant Coxeter number.*

Example 6.3.3. In Figure 6.5 we show an example of one orbit of Pan on the order ideals of $\mathsf{P}_{2,3}$.

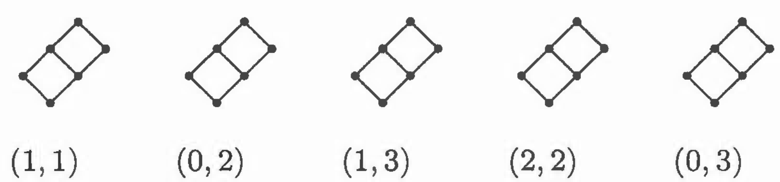
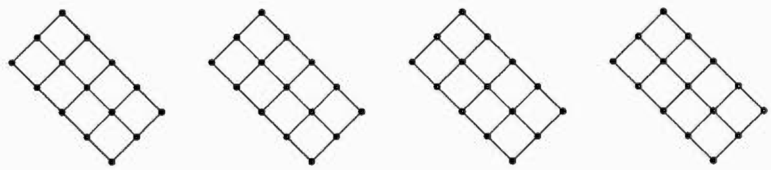
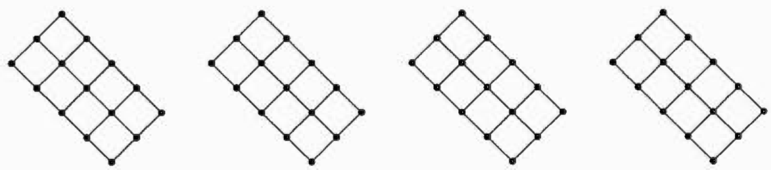


Figure 6.5 An orbit of Panyushev map Pan

Example 6.3.4. Here is another example of a Pan-orbit for $\mathsf{P}_{5,3}$.



(1,1,2,3,3) (0,2,3,3,3) (1,3,3,3,3) (2,2,2,2,2)



(0,0,0,0,3) (0,0,0,1,1) (0,0,1,1,2) (0,1,1,2,3)

6.4 The short movie for the orbit of the Panyushev map

In this section, we are going to describe a movie of orbits of Pan for the grid posets $\mathsf{P}_{\mathsf{m,n}}$. We recall that $\text{Pan}^{m+n} = id$ for grid posets which come from the first infinite family of the cominuscule posets. In this section, we write $\mathcal{Z}' := \{-m+1, \cdots, -1, 0, 1, \cdots, n\}$ for the elements of $\mathbb{Z}/(m+n)$ and $D = \{i_1 < i_2 < \cdots < i_m\}$ for a configuration where $i_j \in \mathcal{Z}'$, $0 \leq j \leq m$.

Similar to the previous section, we consider a frame Fr_{m+n} of size $m+n$ and we divide the frame into two different regions: the dark zone and the light zone. We label the columns with respect to our starting poset $\mathsf{P}_{\mathsf{m,n}}$. We let the width of the light zone be $n+1$, and we label the columns $0, \ldots, n$. The width of the dark zone is m; in this case we label the columns of the dark zone from $(-m+1)$ to -1.

We first define a *configuration of singletons* S_D for a configuration D as follows.

Definition 6.4.1. The *configuration of singletons* S_D associated to a configuration D consists of m distinct boxes in the frame Fr_{m+n} each of which is placed in the column $i_j \in D$. We write $\mathfrak{S}$ for the collection of configurations of singletons.

Secondly, we give the following definition. Let $\alpha = (\lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r})$, $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_r \leq n$ be a partition.

Definition 6.4.2. The *marked light configuration of singletons* S_α^* associated to a partition α consists of r distinct boxes in the frame Fr_{m+n} each of which occupy the column $\lambda_i \in \alpha$ and each of which has $\alpha_i - 1$ marks. We denote by $\mathfrak{S}^*$ the collection of marked light configurations of singletons.

The map φ_* we defined from $\mathfrak{D}$ to $\mathfrak{D}^*$ can be modified to this setting.

$$\varphi_*^s : \mathfrak{S} \to \mathfrak{S}^*$$

where $\mathfrak{S}$ is the set of configurations of singletons with m boxes and $\mathfrak{S}^*$ is the set of marked light configurations of singletons with r boxes and $m - r$ marks.

Let $S_D \in \mathfrak{S}$. $\varphi_*^s(S_D)$ is defined as follows: For every singleton in S_D in the column k in the dark zone we count k dominos to the right, and mark the corresponding singleton, and then delete the singleton in the dark zone. Similar to the proof of Lemma 6.1.4, it can be shown that this gives us a bijective map.

Definition 6.4.3. We call the orbit of marked light configurations singletons in a frame Fr_{m+n} a *short movie.*

Example 6.4.4. Let us see the corresponding short movie for the partition $(0, 1, 1, 3, 3)$.

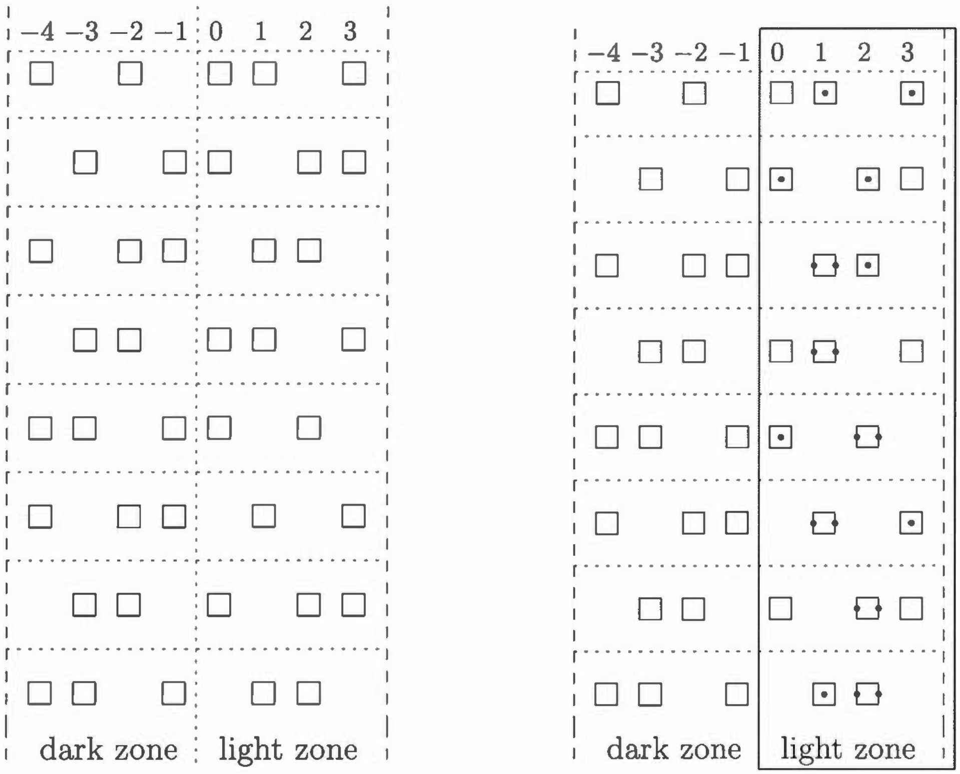


Figure 6.6 The orbit of $\{-4,-2,0,1,3\}$ under the shift $\{1\}$ and the corresponding short movie.

Let $\alpha = (\lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r}) \in F$. We reformulate the action of Pan^{-1} on the partitions as follows: $\mathrm{Pan}^{-1}(\alpha) =$

$$\begin{cases} ((\lambda_1 - 1), (\lambda_2 - 1)^{\alpha_1}, \cdots, (\lambda_r - 1)^{\alpha_{r-1}}, n^{\alpha_r - 1}) & \text{if } \lambda_1 \neq 0 \\ ((\lambda_2 - 1)^{\alpha_1 + 1}, (\lambda_3 - 1)^{\alpha_2}, \cdots, (\lambda_r - 1)^{\alpha_{r-1}}, n^{\alpha_r - 1}) & \text{otherwise.} \end{cases}$$

Note that the set $\mathfrak{S}^*$ is a combinatorial reformulation of the partition set F.

Lemma 6.4.5. *The action of* Pan *on the set partitions* $\mathfrak{S}^*$ *corresponds to the shift* $\{1\}$ *on* $\mathfrak{S}$.

Proof. We claim the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\{1\}} & \mathcal{S} \\ {\scriptstyle \varphi_*^s}\downarrow & & {\scriptstyle \varphi_*^s}\downarrow \\ \mathfrak{S}^* & \xrightarrow[\text{Pan}]{} & \mathfrak{S}^* \end{array}$$

This is just a reformulation of the proof we gave in Lemma 6.1.4. □

Notice that we can easily see that $\text{Pan}^{m+n} = id$ within this combinatorics since the order of $\{1\}$ is obviously $m+n$ on $\mathcal{S}$.

6.5 The Coxeter transformation and The Panyushev map

In this section, we define a map Ω from $\mathfrak{D}$ to $\mathfrak{S}$ as follows: for every $\mathcal{D}_D \in \mathfrak{D}$, the map $\Omega(\mathcal{D}_D)$ is just deleting the left side of the domino so that it becomes a box.

Remark 6.5.1. When we start shifting the configuration $\mathcal{D}_D$ in a frame Fr_{m+n+1}, we come back in the same position after $m+n+1$ times. Now, we consider the configuration of singletons $S_D = \Omega(\mathcal{D}_D)$ in a frame Fr_{m+n}. We come back in the same position after $m+n$ steps. The movie associated to $\mathcal{D}_D$ exhibits the τ-orbit for $\mathcal{P}_\alpha$ and the short movie associated to $S_D = \Omega(\mathcal{D}_D)$ exhibits the Pan-orbit. This explains why is the order of Panyushev Pan is one less than the order of Coxeter transformation τ.

Example 6.5.2. In Example 6.2.5, if we apply Ω to the domino configuration $\mathcal{D}_D$ for the configuration $\{-4 < -2 < 0 < 1 < 3\}$, then we obtain the orbit of Panyushev for the corresponding order ideal to $(0, 1, 1, 3, 3)$.

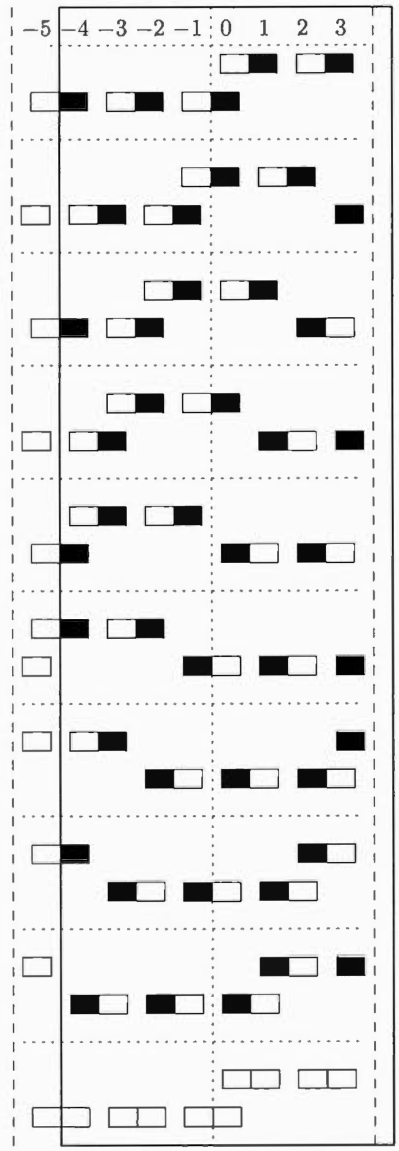


Figure 6.7 Movie and short movie for $(0, 1, 1, 3, 3)$

6.6 Explicit description of orbits of τ and Pan for $m = 2$

In this section, we will show the τ-orbits and Pan-orbits for order ideals of $\mathsf{P}_{(2,n)}$ explicitly. First, we recall the order ideal poset of $\mathsf{P}_{(2,n)}$ as follows.

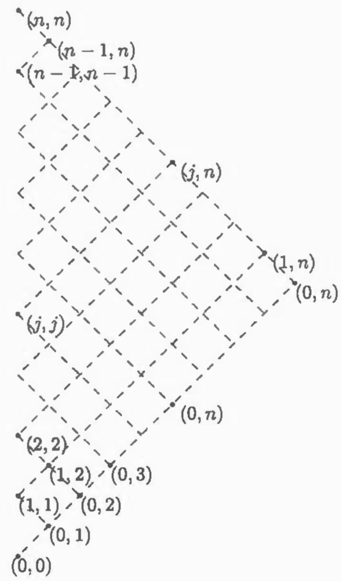


Figure 6.8 An illustration of the order ideal poset $\mathsf{J}(\mathsf{P}_{(2,n)})$

We will first write down the orbits of τ for the order ideal poset $\mathsf{J}(\mathsf{P}_{(2,n)})$. Denote the simple module at a vertex (i, j) in the quiver of $\mathsf{J}(\mathsf{P}_{(2,n)})$ simply by (i, j). Now, let us look at the orbits of simples. We can write the action of τ on the simples in the Grothendieck group as follows: see Section 4.3.

1. $\tau(0,0) = -\sum_{0 \le i \le j \le n} (i, j)$,

2. $\tau(0, j) = \sum_{k=0}^{j-1} (k, j-1)$ where $1 \le j \le n$,

3. $\tau(i, i) = \sum_{k=i}^{n} (i-1, k)$ where $1 \le i \le n$,

4. $\tau(i,j) = -(i-1, j-1)$ where $0 < i < j \leq n$.

We know that

$$\tau^{n+3}(n,n) = \begin{cases} (n,n) & \text{if } n \text{ is odd,} \\ -(n,n) & \text{if } n \text{ is even.} \end{cases} \quad (6.1)$$

For the purpose of analyzing the combinatorics, we can ignore the sign. The orbits of τ are shown in Figure 6.9. We number the orbits with respect to our starting point and then iteration of τ. Intervals are drawn as triangles or rectangles.

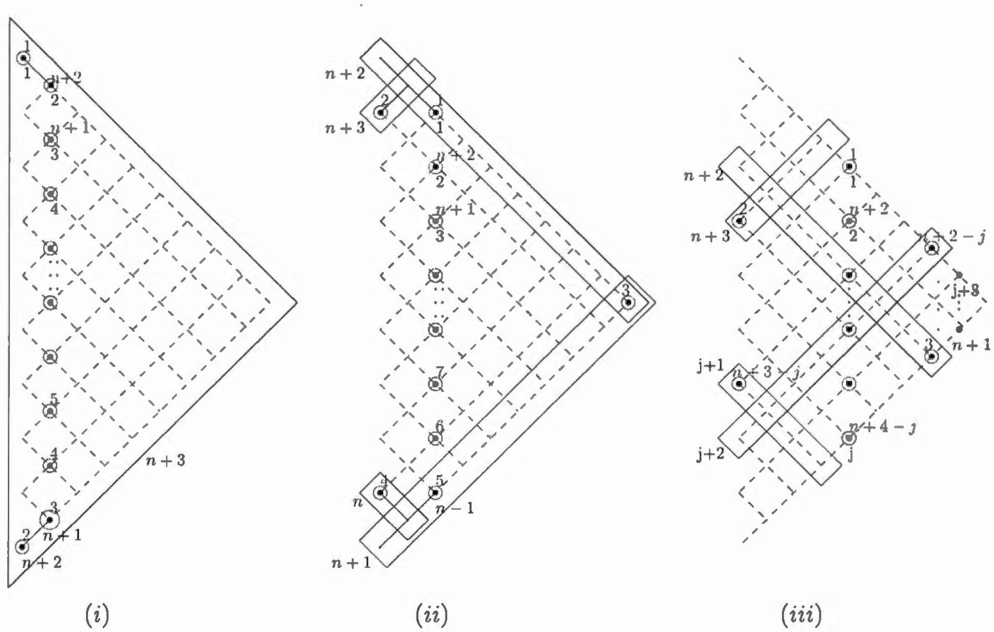


Figure 6.9 An illustration of some orbits of τ and Pan in red for $\mathsf{J}(\mathsf{P}_{(2,n)})$

The only simples which we did not consider are the (i,i) for $1 \leq i \leq n$. But, we know that $\tau(i,i) = \sum_{k=i}^{n}(i-1,k)$ and since we already analyzed the latter, we are done.

Example 6.6.1. In this example, we show the movie and the short movie in blue for $\mathcal{P}_{(5,9)}$ in the order ideal poset $\mathsf{J}(\mathsf{P}_{(2,9)})$.

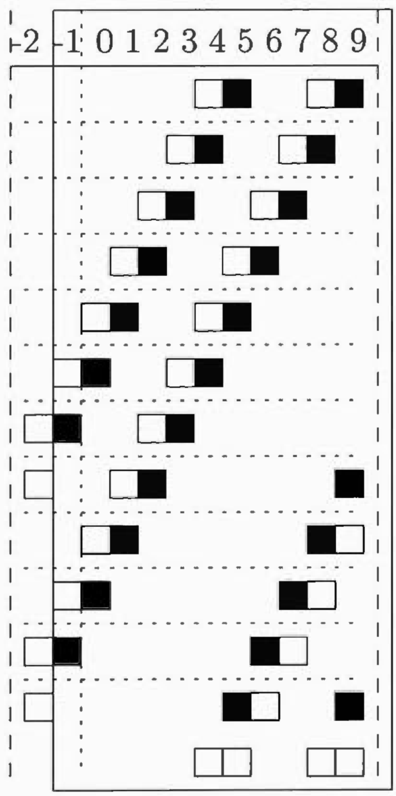


Figure 6.10 τ-orbit and Pan-orbit in blue for $\mathsf{J}(\mathsf{P}_{(2,9)})$

CONCLUSION

Our aim in this thesis was to investigate the following conjecture by Chapoton:

> Given a root system Φ, consider the distributive lattice $\mathsf{J}(\Phi^+)$ of order ideals of Φ^+. The bounded derived category $\mathcal{D}^b(\text{mod}\,\mathsf{J}(\Phi^+))$ of the incidence algebra of $\mathsf{J}(\Phi^+)$ is fractionally Calabi-Yau, i.e. some non-zero power of the Auslander-Reiten translation τ equals some power of the shift functor.

One can relax the conjecture in various directions. In one direction, since the Auslander-Reiten translation τ on the bounded derived category $\mathcal{D}^b(\text{mod}\,\mathsf{J}(\Phi^+))$ naturally defines an endomorphism on the Grothendieck group of the incidence algebra $\mathsf{J}(\Phi^+)$ which we call Coxeter transformation, one can ask if the Coxeter transformation has finite order on the Grothendieck group for $\mathsf{J}(\Phi^+)$. This is still a very difficult problem to solve.

In this thesis, we investigated a version of this conjecture relaxed in two directions. First of all, we considered a variation of Chapoton conjecture where $\mathsf{J}(\Phi^+)$ is replaced with the poset of order ideals $\mathsf{J}(\mathsf{C})$ of a cominuscule poset C. Then, instead of working with the bounded derived category we worked on the level of Grothendieck group of the bounded derived category. This group happens to be the same as the Grothendieck group of the module category (see Theorem 3.4.3).

Let $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$ be the poset of order ideals in the grid poset $\mathsf{P}_{\mathsf{m,n}}$ and $\mathcal{A}$ be the incidence algebra of $\mathsf{J}(\mathsf{P}_{\mathsf{m,n}})$. There are two obvious choices of generators for the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$-modules: (i) The isomorphism classes of simple A-modules, and (ii) the isomorphism classes of finitely generated projective A-modules. The action of the Coxeter transformation (the Auslander-Reiten translation τ on $K_0(A)$) written in terms of these generators does not yield an easy-to-follow pattern. However, there are other sets of generators one can use. In Chapter 4, we develop such a finite set of generators. We started with discussing the periodicity of the Coxeter transformation τ for the algebra $\mathcal{A}$. Using some powerful combinatorial tools, we showed that $\tau^{2(m+n+1)} = id$ in this case. A key idea was to consider the orbits of τ consisting of elements which correspond to certain projective resolutions. These projective resolutions helped us to understand the structures of the orbits combinatorially. After explaining the corresponding combinatorics, we showed that the Coxeter transformation acts by a cyclic permutation on the classes corresponding to these projective resolutions, i.e. there is a cyclic order of these generators and the Coxeter transformation sends one generator to the next. Finally, by showing that the elements corresponding to these projective resolutions generate the Grothendieck group, we succeed to prove our claim in Chapter 4 for grid posets $\mathsf{P}_{\mathsf{m},\mathsf{n}}$ which come from the first infinite family of cominuscule posets.

We ultimately would like to address Chapoton's conjecture in the most general setting, and therefore, we need to expand the validity of our combinatorial result to a more general setting. So, we consider the incidence algebra of the poset of order ideals $\mathsf{J}(\mathsf{C})$ of a cominuscule poset C in Chapter 5.

Cominuscule posets are special type of subposets of the poset of order ideals $\mathsf{J}(\Phi^+)$. There are three infinite families of cominuscule posets and the two exceptional cases. We succeeded to extend our result to two of the three infinite families of cominuscule posets, and for the exceptional cases. Moreover, we showed that in each case $\tau^{2(h+1)} = id$ where h is the relevant Coxeter number.

We also managed to go further and verified that the $2(h+1)$-th power of the Auslander-Reiten translation is indeed is a power of the ordinary shift functor [1] on the bounded derived category for the cominuscule poset of the third infinite family, i.e. $\mathcal{D}^b(\operatorname{mod} \mathsf{J}(\mathsf{C}_{\mathsf{III}}))$ is fractionally Calabi-Yau.

For the exceptional cases, we verified the finiteness of the order of the Coxeter transformation with the help of a computer algebra system SageMath (Stein et al., 2017).

The remaining open case is the case of the cominuscule poset of second infinite family shown in Figure 5.2. We conjecture that the order is again $2(h+1)$ where h is the relevant Coxeter number. It turns out that the combinatorics of this case is really different from the other families. The difficulty one has to overcome is finding a nice basis for the Grothendieck group and showing the action is by permutations, cyclic or otherwise, in this remaining case too. We provided some examples and found promising tactics to attack this case below.

7.1 The second infinite family of cominuscule posets

In this section, we will further investigate the cominuscule posets of type C_{II}, and we will show some orbits of Auslander-Reiten translation τ.

If every element in a τ-orbit forms an interval over the poset, we call this orbit a *good orbit* of τ. We ask the following question: Are there good orbits of τ for $\mathsf{J}(\mathsf{C}_{\mathsf{II}})$? We will first give some examples.

Example 7.1.1. Let $\mathsf{J}(\mathsf{C}_{C_4})$ be the order ideal poset of the cominuscule poset C_{C_4} coming from type C_4. We show here an example of one of the good τ-orbits for $\mathsf{J}(\mathsf{C}_{C_4})$ in Figure 7.11.

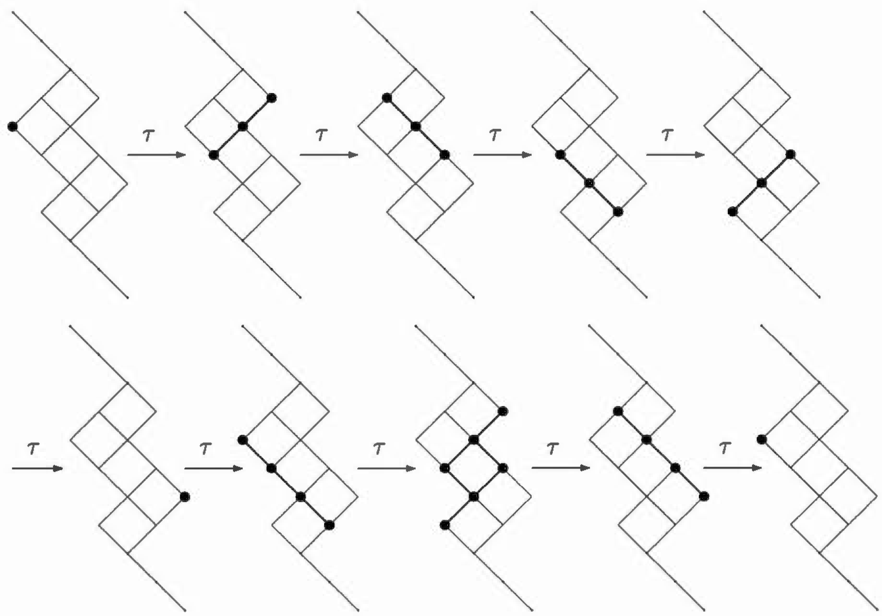


Figure 7.11 An example of a good orbit of τ for the poset $\mathsf{J}(\mathsf{C}_{\mathsf{C}_4})$

Example 7.1.2. Here is another example for $\mathsf{J}(\mathsf{C}_{\mathsf{C}_5})$ in Figure 7.12.

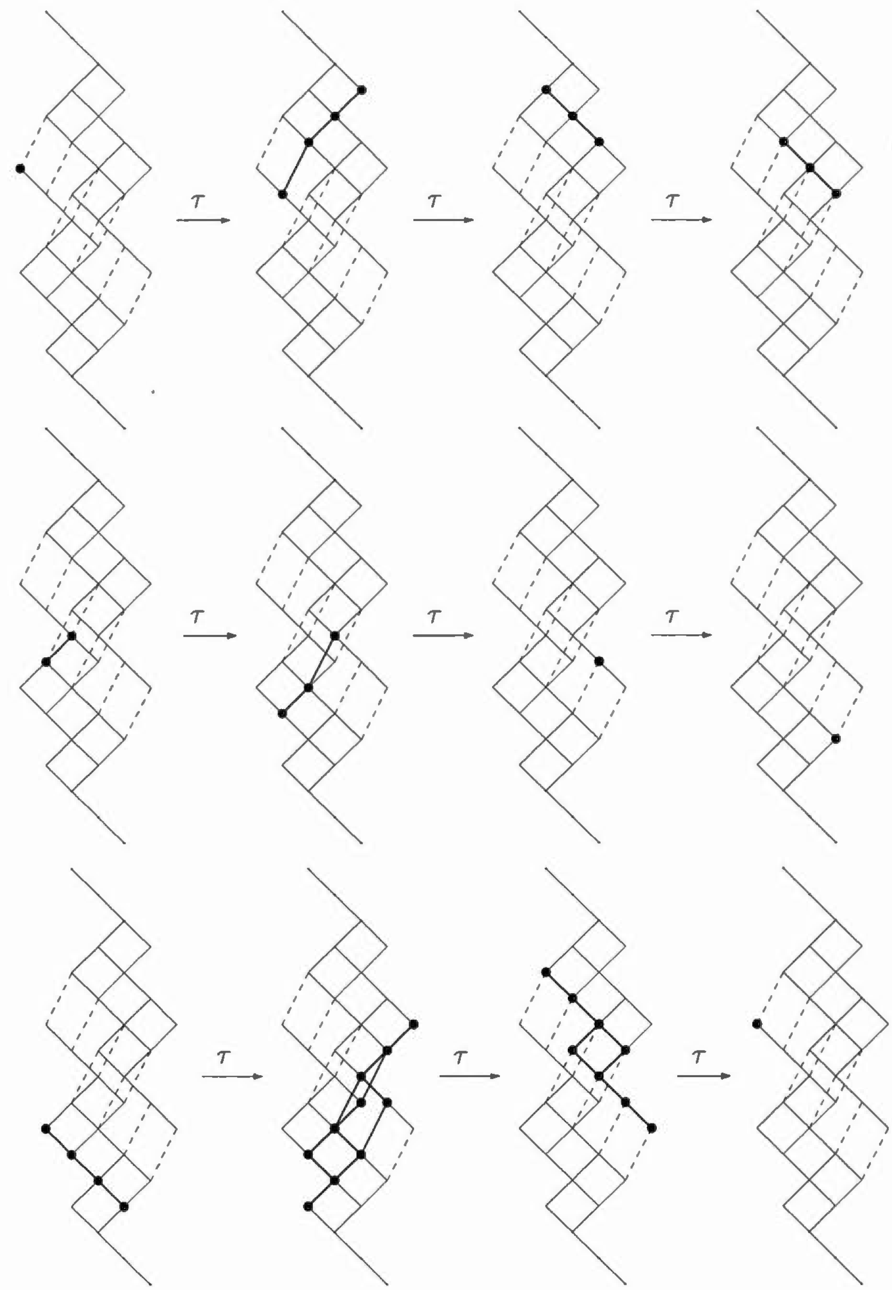


Figure 7.12 An example of a good orbit of τ for the poset $\mathsf{J}(\mathsf{C}_{\mathsf{C}_5})$

Now, the question is how many good τ-orbits we have for $\mathsf{J}(\mathsf{C}_{\mathsf{II}})$. More specifically, we can ask the following question: For which vertices does the iteration of τ gives rise to intervals over the poset. We will call the vertices which do not give rise to intervals as *red vertices.*

Example 7.1.3. Figure 7.13 shows the red vertices for $\mathsf{J}(\mathsf{C}_{\mathsf{C}_4})$ and $\mathsf{J}(\mathsf{C}_{\mathsf{C}_5})$.

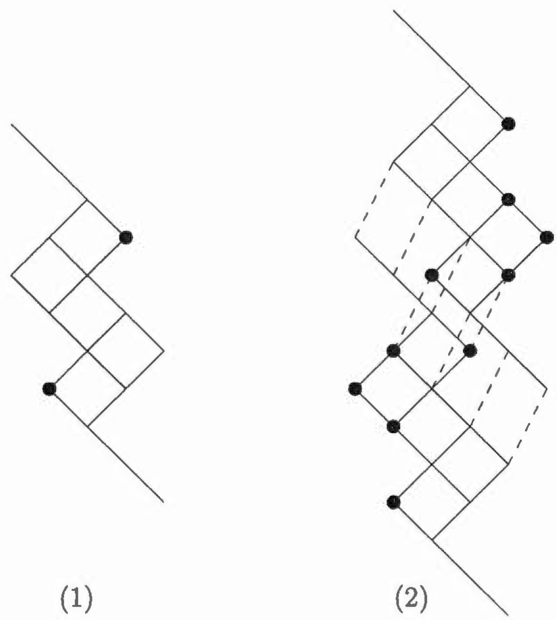


Figure 7.13 (1) for $\mathsf{J}(\mathsf{C}_{\mathsf{C}_4})$, (2) for $\mathsf{J}(\mathsf{C}_{\mathsf{C}_5})$

We observe that the number of red vertices goes as follows: 2, 10, 32, 84 .... The (OEIS Foundation Inc., 2018) suggests that the formula for these numbers may be $a(n) = 2(2^n - 1 - n(n+1)/2)$.

7.2 The Coxeter transformation and the Panyushev map on cominuscule posets

In the course of constructing a combinatorial setup for our proof of the finiteness of the Coxeter transformation, we observed that the orbits of the Coxeter transformation and the Panhusyev map are very similar. The Panyushev map is a combinatorial function defined on posets of order ideals. In Chapter 6, we show that there is a structural similarity between the Coxeter transformation and the Panhusyev map for the grid posets, i.e. the first infinite family of cominuscule posets. Now, we need to investigate the other cases.

For the third infinite family of cominuscule posets, it is not difficult to verify a similarity thanks to the simple structure of its order ideal poset. We give an example in the following.

Example 7.2.1. In this example, we will look at the τ-orbits and Pan orbits for the cominuscule poset $\mathsf{C}_{\mathsf{D}_4}$ coming from the third infinite family.

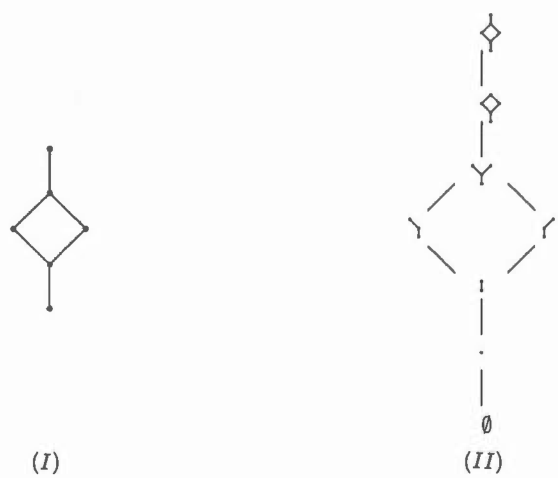


Figure 7.14 (I) for $\mathsf{C}_{\mathsf{D}_4}$, (II) for $\mathsf{J}(\mathsf{C}_{\mathsf{D}_4})$.

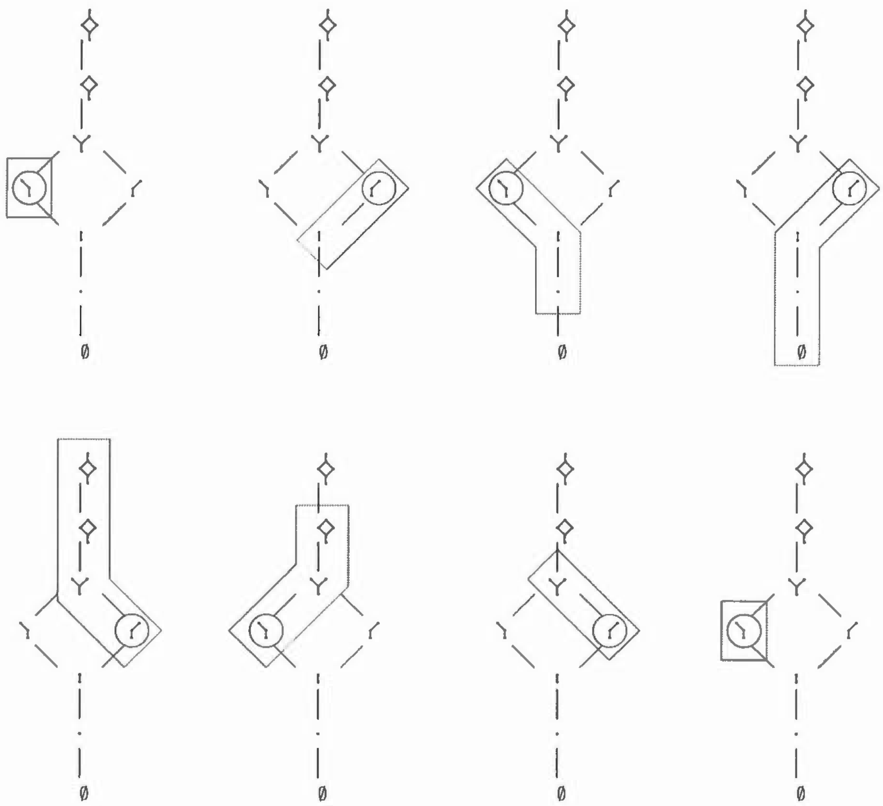


Figure 7.15 τ-orbits in red and Pan-orbits in blue.

The difficulty with the second infinite family of cominuscule posets is that they are neither as simple as the third family of cominuscule posets, nor do we have nice combinatorial tools such as movies and short movies we developed for the first infinite family of cominuscule posets. So, it becomes difficult to see the reason for the similarity between the orders of the Coxeter transformation and the Panyushev map in this case. We conjecture that for the second infinite family of cominuscule poset, $\tau^{2(h+1)} = id$. Recall that Panyushev map $\mathrm{Pan}^{2h} = id$ on order ideal poset of cominuscule posets. We expect to see a similar connection between τ and Pan in this case as well. But, for now this still remains as an open problem.

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