ELEMENTARY EQUIVALENCE AND CODIMENSION
IN p-ADIC FIELDS

L. Bélair
L. van den Dries
A. Macintyre

We give examples of fields elementarily equivalent to a given finite extension of the p-adic numbers but not containing a subfield of finite codimension elementarily equivalent to the p-adics.

SECTION 0. INTRODUCTION.

It is well known that an algebraically closed field of characteristic zero contains a real closed field of codimension 2. From the point of view of the model theory of real closed fields, this means that a field elementarily equivalent to a finite extension of the real numbers \( \mathbb{R} \) contains a subfield of the right codimension elementarily equivalent to \( \mathbb{R} \). In this note we show that this is not the case for any field \( K \)
finite-dimensional over $\mathbb{Q}_p$. Namely, for any finite extension of $\mathbb{Q}_p$ of a given degree $d > 1$, there exists an elementary equivalent field which does not contain a subfield of finite codimension elementarily equivalent to $\mathbb{Q}_p$. So this is a point in which the model theory of $\mathbb{R}$ and $\mathbb{Q}_p$ differ. It is worth noticing that the examples below have the simplest possible value group, namely $\mathbb{Z}$.

From the work of Ax-Kochen and Ershov we know that the elementary theory of $\mathbb{Q}_p$ is the theory of henselian valued fields with residue field $\mathbb{F}_p$, and discretely valued in a $\mathbb{Z}$-group so that $v(p) = 1$. Let $pCF$ be the above theory. Its models are called $p$-adically closed fields. The model theory of finite extension fields of $\mathbb{Q}_p$ was studied in [PR]. The only model-theoretic fact we shall need is that if $K$ is a finite extension of $\mathbb{Q}_p$ and $K_1 \subseteq K$ a subfield relatively algebraically closed in $K$ then $K_1 \preceq K$, i.e. the inclusion is elementary.

Our arguments rely on basic algebraic-geometric facts, together with the completeness of $\mathbb{Q}_p$ via Baire's theorem. We can and shall assume everything to take place in a fixed algebraic closure of $\mathbb{Q}_p$. If $F$ is a field then $\widetilde{F}$ denotes its algebraic closure. We denote by $\mathbb{A}_p^n$ the field of algebraic $p$-adic numbers, i.e. the $p$-adic numbers algebraic over the rationals, and by $\mathbb{A}_p^n$ the affine $n$-space. If $I$ is a polynomial ideal then $V(I)$ denotes its zero set.
SECTION 1. THE EXAMPLES.

Let \( K \) be a finite extension field of \( \mathbb{F}_p \) of degree \( d > 1 \). We know that \( K = \mathbb{F}_p(\alpha) \) for some algebraic number \( \alpha \). Let \( t_0, \ldots, t_{d-1} \in \mathbb{F}_p \) be algebraically independent over \( \mathbb{F}_p \) and let \( K_1 \) be the relative algebraic closure of
\[
A_\mathbb{F}_p(\alpha, t_0 + t_1 \alpha + \cdots + t_{d-1} \alpha^{d-1}) \text{ in } K.
\]
Then \( K_1 \) is elementarily equivalent to \( K \) and the transcendence degree of \( K_1 \) over \( A_\mathbb{F}_p \) is 1. If \( K_0 \subseteq K_1 \) is a subfield of \( K_1 \) elementarily equivalent to \( \mathbb{F}_p \) and of codimension \( d \) then \( \text{tr deg} \, K_0 \mid A_\mathbb{F}_p = 1 \).

Moreover:

**LEMMA 1.** \( K_0 = K_1 \cap \mathbb{F}_p \).

**PROOF.** \( K_0 \) has a unique Henselian valuation, which is the restriction of the unique Henselian valuation on \( K_1 \). Since \( K_1 \) is a finite extension of \( K_0 \), the valuegroup of \( K_0 \) is of finite index in that of \( K_1 \). Also, \( v(p) \) is the least positive value in \( K_0 \), hence \( K_0 \) is an immediate extension of \( A_\mathbb{F}_p \). Therefore \( A_\mathbb{F}_p \) is dense in \( K_0 \), so \( A_\mathbb{F}_p \) and \( K_0 \) have the same (topological) closure in the algebraic closure of \( \mathbb{F}_p \). It follows that \( K_0 \subseteq \mathbb{F}_p \). On the other hand \( K_1 \cap \mathbb{F}_p \) is relatively algebraically closed in \( \mathbb{F}_p \) and so is a model of pCF. Since the extension \( K_1 \cap \mathbb{F}_p \mid K_0 \) is algebraic the equality follows. \( \Box \)

We show below that there exists \( t_0, \ldots, t_{d-1} \) for which \( K_1 \cap \mathbb{F}_p = A_\mathbb{F}_p \). Such \( t_i \) prevent the existence of a suitable \( K_0 \) and thus yield our example for \( K \).
Let us point out that an analogous construction for \( \mathbb{R} \) does not force \( K_0 = K_1 \cap \mathbb{R} \), the reason being that the field \( K_0 \) would not necessarily be archimedean and hence not necessarily embeddable in \( \mathbb{R} \).

**Proposition.** There exist \( t_0, \ldots, t_{d-1} \in \phi_p \) which are algebraically independent over \( A_p \) and for which
\[
K_1 \cap \phi_p = A_p.
\]

**Proof.** It suffices to find algebraically independent \( t_i \) such that for every irreducible polynomial
\[
f(X,Y) \in A_\mathbb{P}(\alpha)[X,Y]\backslash A_\mathbb{P}(\alpha)[X]
\]
and every \( x \in \phi_p \), if
\[
f(x, t_0^1 \alpha + \ldots + t_{d-1} \alpha^{d-1}) = 0
\]
then \( x \in A_\mathbb{P} \). In fact we find \( t_i \) such that
\[
f(x, t_0^1 \alpha + \ldots + t_{d-1} \alpha^{d-1}) \neq 0
\]
for all \( x \in \phi_p \), so a fortiori fulfilling the requirement with respect to \( A_\mathbb{P} \). Let
\[
f(X,Y)
\]
be such a polynomial and \( C \) be the affine curve it defines. Set
\[
x = x_0^1 + x_1 \alpha + \ldots + x_{d-1} \alpha^{d-1}
\]
and
\[
y = y_0^1 + y_1 \alpha + \ldots + y_{d-1} \alpha^{d-1}
\]
and let
\[
W = R_{A_\mathbb{P}(\alpha)}|A_\mathbb{P}(\alpha)\text{ extension } A_\mathbb{P}(\alpha)|A_\mathbb{P}\text{ (see [W]).}
\]
Let \( \sigma_1, \ldots, \sigma_d \) be the \( d \) distinct \( A_\mathbb{P} \)-embeddings of \( A_\mathbb{P}(\alpha) \) in \( \tilde{A} \). The affine variety \( W \) is isomorphic to the product \( C^{\sigma_1} \times \ldots \times C^{\sigma_d} \) over the Galois closure of \( A_\mathbb{P}(\alpha) \) over \( A \) via the following isomorphism (ibid.)
\[
\phi(x_0^1, \ldots, x_{d-1}^1, y_0^1, \ldots, y_{d-1}^1) = (X_1, Y_1, \ldots, X_d, Y_d)
\]
where
\[
x_i^1 = \Sigma_j \sigma_i(\alpha^j)
\]
and
\[
y_i^1 = \Sigma_j \sigma_i(\alpha^j),
\]
\( j = 0, \ldots, d - 1 \). Let \( M = (a_{ij}) \) be the \( d \times d \) matrix with \( (i,j) \)-th entry \( a_{ij} = \sigma_i(\alpha^{j-1}) \) and let
\( x, y, X, Y \) be the column vectors obtained from the
components $x_j', y_j', x_i', y_i'$, respectively. Then $M$ is invertible and $X = Mx, Y = My$. The ideal $(f^{(i)}(x_i', y_i'), i = 1, \ldots, d)$ is an ideal of definition for $C^0 \times \cdots \times C^0$. Let $W'$ be the intersection of $W$ with $x_1 = 0, \ldots, x_{d-1} = 0$ and let $J$ be the ideal generated by the $f^{(i)}(x_i', y_i')$ and $x_1 - x_j'$ for $j = 2, \ldots, d$.

**Lemma 2.** We have $\phi W' = V(J)$.

**Proof.** The inclusion $\subseteq$ is clear. On the other hand if $P = (X_1, Y_1, \ldots, X_d, Y_d)$ lies in $V(J)$ then it lies in $C^0 \times \cdots \times C^0$ and $X_1 = X_2 = \ldots = X_d$. Consider the linear system $X = Mx$ with the $x_i$ as unknowns.

If $X_1 = 0$ then $X_0 = 0 = x_1 = \ldots = x_{d-1}'$ and $\phi^{-1}P$ is in $W'$. Otherwise, setting $z_i = X_1^{-1}X_i'$, we get the equivalent system $1 = Mz$, where $1$ denotes the column vector whose entries are all equal to $1$. Now the first column of $M$ is equal to $1$, so by Cramer $z_0 = 1$ and $z_j = 0$ for $j \geq 1$, whence $\phi^{-1}P$ lies in $W'$. \qed

Using this lemma a straightforward computation of transcendence degree shows that $\dim \phi W' \leq 1$ and we conclude, via $\phi$, that $\dim W' \leq 1$.

Let $\pi W'$ be the set theoretic projection of $W'$ on the last $d$ components, i.e. the set of $(y_0', \ldots, y_{d-1}')$ for which there exists an $x_0$ such that $(x_0', 0, \ldots, 0, y_0', \ldots, y_{d-1}')$ is in $W'$. Since $\dim W' \leq 1$, it follows that the Zariski closure $\overline{\pi W'}$ of $\pi W'$ has also dimension $\leq 1$. In order to use a Baire argument for $\mathbb{Q}^d_p$ to get the $\mathbb{Q}^d_p$, we isolate the following fact.
**Lemma 3.** Let $V \subseteq \mathbb{A}^d$ be an affine variety of dimension $n < d$ defined over $\mathbb{A}_p$. Then $V \cap \mathbb{Q}_p^d$ has empty interior in $\mathbb{Q}_p^d$ for the $p$-adic topology.

**Proof.** For cardinality reasons every ball in $\mathbb{Q}_p^d$ contains a point with components algebraically independent over $\mathbb{A}_p$. This can be seen by proving by induction on $d$ that for any $\beta_1, \ldots, \beta_r$ in $\mathbb{Q}_p$ and any ball $B$ in $\mathbb{Q}_p^d$ there is a point $P$ of $B$ whose coordinates are algebraically independent over $\mathbb{Q}(\beta_1, \ldots, \beta_r)$. For $d = 1$ this is a simple cardinality argument. For $d = c + 1$, first choose last coordinate $\beta_{r+1}$ independent of $\beta_1, \ldots, \beta_r$, and then work in $\mathbb{Q}_p^c$ with $\beta_1, \ldots, \beta_r, \beta_{r+1}$ to get the first $c$ coordinates. \[\Box\]

It follows that $\prod_{i \in \mathbb{Z}} \cap \mathbb{Q}_p^d$ is a nowhere dense subset of $\mathbb{Q}_p^d$ in the $p$-adic topology, as is $V(g) \cap \mathbb{Q}_p^d$ for any $g \in \mathbb{A}_p[X_1, \ldots, X_d]$. Considering all the $\prod_{i \in \mathbb{Z}}$ thus obtained and all $V(g)$, we conclude by Baire's Theorem that there exists $(t_0, \ldots, t_{d-1}) \in \mathbb{Q}_p^d$ in the complement of all those sets. These are the required $t_i$ and this concludes the proof of the Proposition. \[\Box\]

**Section 2. Concluding Remarks.**

It is clear that in the above discussion we can replace $\mathbb{Q}_p$ by any of its finite extensions and adjust the arguments accordingly. Let us refer to a field $K$ as having the "codimension property" if any field
elementarily equivalent to a finite extension of $K$ contains a subfield elementarily equivalent to $K$ and of the same codimension. The field of rational numbers $\mathbb{Q}$ has, like $\mathbb{R}$, the codimension property, but this time it is related to undecidability. Indeed by Julia Robinson's result, $\mathbb{Q}$ is definable in any fixed finite extension field of itself. This is to be contrasted with the situation of the reals, where both $\mathbb{R}$ and $\mathbb{C}$ are decidable, and the field of the $p$-adics which, while not having the codimension property, is decidable and has every finite extension decidable.

REFERENCES


ADDRESSES

L. Bélair,
Department of Mathematics,
Université de Montréal,
Montréal,
Quebec 83C 3J7,
Canada

A. Macintyre,
Mathematical Institute,
University of Oxford,
24-29 St. Giles',
Oxford,
OX1 3LB,
England

L. van den Dries,
Department of Mathematics,
University of Illinois at Urbana-Champaign,
273 Altgeld Hall,
Urbana,
IL 61801,
U.S.A.

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