

Kobayashi pseudometric on hyperkähler manifolds

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Abstract

The Kobayashi pseudometric on a complex manifold M is the maximal pseudometric such that any holomorphic map from the Poincaré disk to M is distance-decreasing. Kobayashi has conjectured that this pseudometric vanishes on Calabi-Yau manifolds. Using ergodicity of complex structures, we prove this result for any hyperkähler manifold if it admits a deformation with a Lagrangian fibration, and its Picard rank is not maximal. The SYZ conjecture claims that any parabolic nef line bundle on a deformation of a given hyperkähler manifold is semi-ample. We prove that the Kobayashi pseudometric vanishes for all hyperkähler manifolds satisfying the SYZ property. This proves the Kobayashi conjecture for K3 surfaces and their Hilbert schemes.

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1 Introduction

The Kobayashi pseudometric on a complex manifold M is the maximal pseudometric such that any holomorphic map from the Poincaré disk to

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M is distance-decreasing (see Section 1.3 for more details and references). Kobayashi conjectured that the Kobayashi pseudometric vanishes for all projective varieties with trivial canonical bundle (see Problems C.1 and F.3 in [Ko1]). The conjecture was proved for projective K3 surfaces via the nontrivial theorem in [MM] that all projective K3 surfaces are swept out by elliptic curves (see Lemma 1.51 in [Vo]). We prove the conjecture for all K3 surfaces as well as for many classes of hyperkähler manifolds. For an extensive survey on problems of Kobayashi and Lang we recommend the beautiful survey papers [Vo] by Voisin and [D] by Demailly.

Using density arguments and the existence of Lagrangian fibrations, it was proved in [KV] that all known hyperkähler manifolds are Kobayashi non-hyperbolic. Then in [V3] this result was generalized further to all hyperkähler manifolds with $b_2 > 3$. All known examples of hyperkähler manifolds have $b_2 > 5$, and this has been conjectured to be true in general.

We introduce the basics of hyperkähler geometry and Teichmüller spaces in Subsection 1.1. Upper semicontinuity of the Kobayashi pseudometric is discussed in Subsection 1.4. Our main results are in Sections 2 and 3.

For a compact complex manifold M , the Teichmüller space Teich is the space of complex structures up to isotopies. The mapping class group Γ , or the group of “diffeotopies”, acts naturally on Teich . Complex structures with dense Γ -orbits are called **ergodic** (see Definition 1.17). If the Kobayashi pseudometric of M vanishes, then the Kobayashi pseudometric vanishes for all ergodic complex structures on M in the same deformation class (Theorem 2.1). As a corollary, the Kobayashi metric on all K3 surfaces vanish (Corollary 2.2), and the Kobayashi metric on a hyperkähler manifold vanishes for all ergodic complex structures (Theorem 2.3). The SYZ conjecture predicts that all hyperkähler manifolds admit Lagrangian fibrations. Assuming this conjecture to be true, we show the vanishing of the Kobayashi pseudometric for all hyperkähler manifolds with $b_2 \geq 7$ (which is expected for all hyperkähler manifolds). When, in addition, the complex structure is non-ergodic we prove that the infinitesimal Kobayashi pseudometric defined by Royden vanishes on a Zariski dense open subset of the manifold.

We summarize the main results of this article in the following theorems; please see the main body of the paper for details of the definitions and of the proofs.

Theorem 1.1: Let M be a compact simple hyperkähler manifold. Assume that a deformation of M admits a holomorphic Lagrangian fibration and the Picard rank of M is not maximal. Then the Kobayashi pseudometric on M vanishes.

Proof: See Corollary 2.14. ■

Remark 1.2: All known examples of hyperkähler manifolds can be deformed to one which admits a Lagrangian fibration ([KV, Claim 1.20]). By the above result, the Kobayashi pseudometric on known manifolds vanishes, unless their Picard rank is maximal.

Theorem 1.3: Let M be a compact simple hyperkähler manifold with $b_2(M) \geq 7$. Assume that any nef bundle on any deformation of M is semiample or that M is projective and admits a birational holomorphic Lagrangian fibration with a smooth base and no multiple fibres in codimension one. Then the Kobayashi pseudometric on M vanishes and the infinitesimal pseudometric vanishes on a Zariski dense open subset of M .

Proof: See Corollary 3.4 and Theorem 3.1. ■

1.1 Teichmüller spaces and hyperkähler geometry

We summarize the definition of the Teichmüller space of hyperkähler manifolds, following [V2].

Definition 1.4: Let M be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M , equipped with a structure of Fréchet manifold, and let $\text{Teich} := \text{Comp} / \text{Diff}_0(M)$. We call it **the Teichmüller space** of M .

Remark 1.5: In many important cases, such as for Calabi-Yau manifolds ([Cat]), Teich is a finite-dimensional complex space; usually it is non-Hausdorff.

Definition 1.6: Let $\text{Diff}(M)$ be the group of orientable diffeomorphisms of a complex manifold M . Consider the **mapping class group**

$$\Gamma := \text{Diff}(M) / \text{Diff}_0(M)$$

acting on Teich . The quotient $\text{Comp} / \text{Diff} = \text{Teich} / \Gamma$ is called **the moduli space** of complex structures on M . Typically, it is very non-Hausdorff. The set $\text{Comp} / \text{Diff}$ corresponds bijectively to the set of isomorphism classes of complex structures.

Definition 1.7: A **hyperkähler manifold** is a compact Kähler holomorphically symplectic manifold.

Definition 1.8: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$. In the literature, such manifolds are often called **irreducible holomorphic symplectic**, or **irreducible symplectic varieties**.

The equivalence between these two notions is based on Yau’s solution of Calabi’s problem ([Bes]), as is the following theorem of Bogomolov that motivated the last definition.

Theorem 1.9: ([Bo1]) Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds. ■

Remark 1.10: Further on, all hyperkähler manifolds are assumed to be simple, Comp is the space of all complex structures of hyperkähler type on M , and Teich its quotient by $\text{Diff}_0(M)$.

A simple hyperkähler manifold admits a primitive integral quadratic form on its second cohomology group known as the Beauville-Bogomolov-Fujiki form. We define it using the Fujiki identity given in the theorem below; see [F]. For a more detailed description of the form we refer the reader to [Bea] and [Bo2].

Theorem 1.11: (Fujiki, [F]) Let M be a simple hyperkähler manifold of dimension $2n$ and $\alpha \in H^2(M, \mathbb{Z})$. Then $\int_M \alpha^{2n} = cq(\alpha, \alpha)^n$, where q is a primitive integral quadratic form on $H^2(M, \mathbb{Z})$ of index $(3, b_2(M) - 3)$, and $c > 0$ is a rational number. ■

Definition 1.12: From Theorem 1.11, the form q is defined uniquely up to a sign, except the case of even n and $b_2 \neq 6$. To fix the sign, we make the additional assumption that $q(\omega, \omega) > 0$ for every Kähler form ω . Such a form q is called **the Bogomolov-Beauville-Fujiki form** (or the **BBF form**) of M . It exists and is unique on any simple hyperkähler manifold.

Remark 1.13: The BBF form is remarkably similar to the intersection form on second cohomology of a complex surface. In particular, for any two Kähler classes $\omega, \omega' \in H^2(M, \mathbb{R})$, one has $q(\omega, \omega') > 0$ (see e. g. [H1] or [Bou]).

The mapping class group of a hyperkähler manifold can be described in terms of the BBF form as follows.

Theorem 1.14: ([V2]) Let M be a simple hyperkähler manifold, Γ its mapping class group, and $\Gamma \xrightarrow{\varphi} O(H^*(M, \mathbb{Z}), q)$ the natural map. Then φ has finite kernel and its image has finite index in $O(H^*(M, \mathbb{Z}), q)$. ■

Definition 1.15: Let Teich^I be a connected component of the Teichmüller space containing $I \in \text{Teich}$, and Γ^I the subgroup of the mapping class group preserving Teich^I . The group Γ^I is called **the mondromy group** of (M, I) ([Mar1]).

Remark 1.16: In [V2] it was shown that Γ^I is a finite index subgroup in $O(H^*(M, \mathbb{Z}), q)$ independent of I .

1.2 Ergodic complex structures

Definition 1.17: Let M be a complex manifold, Teich its Teichmüller space, and $I \in \text{Teich}$ a point. Consider the set $Z_I \subset \text{Teich}$ of all $I' \in \text{Teich}$ such that (M, I) is biholomorphic to (M, I') . Clearly, $Z_I = \Gamma \cdot I$ is the orbit of I . A complex structure is called **ergodic** if the corresponding orbit Z_I is dense in Teich .

Theorem 1.18: Let M be a simple hyperkähler manifold or a compact complex torus of dimension ≥ 2 , and I a complex structure on M . Then I is non-ergodic iff the Neron-Severi lattice of (M, I) has maximal possible rank. This means that $\text{rk } NS(M, I) = b_2(M) - 2$ for M hyperkähler, and $\text{rk } NS(M, I) = (\dim_{\mathbb{C}} M)^2$ for M a torus.

Proof: See [V3]. ■

1.3 Kobayashi pseudometric/pseudodistance

Let M be a complex manifold. Recall that a pseudometric on M is a function d on $M \times M$ that satisfies all the properties of a metric (or distance function) except for the non degeneracy condition: $d(x, y) = 0$ only if $x = y$. The Kobayashi pseudometric (a.k.a. pseudodistance) d_M on M is defined as the supremum of all pseudometrics d on M that satisfy the distance decreasing property with respect to holomorphic maps f from the Poincaré disk (\mathbb{D}, ρ) to M :

$$f^*d \leq \rho \quad \text{or equivalently} \quad d(f(x), f(y)) \leq \rho(x, y) \quad \forall x, y \in \mathbb{D}.$$

Here ρ denotes the Poincaré metric on \mathbb{D} .

The following is S. Kobayashi's standard construction of d_M . Let

$$\delta_M(p, q) = \inf \{ \rho(x, y) \mid f : \mathbb{D} \rightarrow M \text{ holomorphic, } f(x) = p, f(y) = q \}.$$

Although it does not satisfy the triangle inequality in general, this is a very useful invariant of the complex structure on M . For an ordered subset $S = \{p_1, \dots, p_l\}$ of M , let

$$\delta_M^S(p, q) = \delta_M(p, p_1) + \delta_M(p_1, p_2) + \dots + \delta_M(p_l, q).$$

Then the triangle inequality is attained by setting

$$d_M(p, q) = \inf \delta_M^S(p, q)$$

where the infimum is taken over all finite ordered subsets S in M .

Royden introduced an infinitesimal version of d_M as follows. The Kobayashi-Royden Finsler norm on TM is given, for $v \in TM$, by

$$|v|_M = \inf \left\{ \frac{1}{R} \mid f : \mathbb{D} \rightarrow M \text{ holomorphic, } R > 0, f'(0) = Rv \right\}.$$

It is the largest ‘‘Finsler’’ pseudonorm on TM that satisfies the distance decreasing property with respect to holomorphic maps from the Poincaré disk and therefore it is automatically ‘‘distance decreasing’’ with respect to holomorphic maps. Royden showed that $| \cdot |_M$ is upper semicontinuous and that d_M is the integrated version of $| \cdot |_M$, see [Roy]. In particular, this implies the well known fact that d_M is a continuous function for a complex manifold M .

We recall that both the pseudometric and its infinitesimal version are insensitive to removing complex codimension two subsets of M .

Theorem 1.19: Let M be a complex manifold and $Z \subset M$ be a complex analytic subvariety of codimension at least 2.¹ Then $d_{M \setminus Z} = d_X|_{M \setminus Z}$ and $| \cdot |_{M \setminus Z} = (| \cdot |_X)|_{M \setminus Z}$.

Proof: Theorems 3.2.19 and 3.5.35 in [Ko2]. ■

Corollary 1.20: Let $\tau : M \dashrightarrow M'$ be a birational equivalence of Calabi-Yau manifolds. Suppose that the Kobayashi pseudometric on M vanishes. Then it vanishes on M' .

Proof: It is easy to check (see subsection 4.4 in [H1]) that the exceptional set of τ is a subvariety of codimension at least 2. Then Theorem 1.19 can be applied to obtain that the Kobayashi pseudometric vanishes on M and M' (by the distance decreasing property) whenever it vanishes on the smooth locus of τ . ■

¹In fact, the same proof would work for any subset $Z \subset M$ of Hausdorff codimension at least 3.

1.4 Upper semi-continuity

Recall that a function F on a topological space X with values in $\mathbb{R} \cup \{\infty\}$ is upper semi-continuous if and only if $\{x \in X \mid F(x) < \alpha\}$ is an open set for every $\alpha \in \mathbb{R}$. It is upper semi-continuous at a point $x_0 \in X$ if for all $\varepsilon > 0$ there is a neighbourhood of x_0 containing $\{x \in X \mid F(x) < F(x_0) + \varepsilon\}$. If X is a metric space, this is equivalent to

$$\limsup_{t_i \rightarrow t_0} F(t_i) \leq F(t_0),$$

for all sequence (t_i) converging to t_0 . From its very definition, the infimum of a collection of upper semicontinuous functions is again upper semicontinuous.

We will be interested in the upper semicontinuity of d_{M_t} and $|\cdot|_{M_t}$ in the variable t for a proper smooth fibration $\pi : \mathcal{M} \rightarrow T$, i.e., π is holomorphic, surjective, having everywhere of maximal rank and connected fibers $M_t = \pi^{-1}(t)$. This follows in the standard way as is for the case of $|\cdot|_M$ by the following result of Siu.

Theorem 1.21: ([Siu]) Let $f : D \rightarrow M$ be a holomorphic immersion of a Stein manifold D to a complex manifold M . Identify D as the zero section of the pullback $X = f^*TM$. Then there is a holomorphic immersion of a neighbourhood of D in X which extends f . ■

Since π is locally differentiably trivial, we may assume that \mathcal{M} is differentiably a product $T \times M$ and π its projection to the first factor. One easily deduce from the above theorem of Siu applied to the graph of a holomorphic map from $D = \mathbb{D}$ that $\delta_{J(t)}(p, q)$ and $|v|_{J(t)}$ are upper semicontinuous with respect to $p, q \in M$, $v \in TM$ and $t \in T$, where we have replaced the subscript M_t by its associated complex structure $J(t)$. It follows then that $\delta_{J(t)}^S(p, q)$ is upper semicontinuous with respect to p, q and t and hence so is $d_{J(t)}(p, q)$. We have established the following proposition, c.f. [Zai].

Proposition 1.22: Let $\pi : \mathcal{M} \rightarrow T$ be a proper holomorphic and surjective map having everywhere of maximal rank and connected fibers $M_t = \pi^{-1}(t)$. Then d_{M_t} and $|\cdot|_{M_t}$ are upper semicontinuous with respect to all variables involved, including t . ■

Although we will not need this, a little reflection will show that one can relax many of the conditions on π . An immediate consequence of the above proposition is the following.

Corollary 1.23: For M a compact complex manifold, let $\text{diam}(M)$ be the diameter of M with respect to d_M . Then $\text{diam}(M)$ is upper semicontinuous with respect to the variation of the complex structure on M .

Proof: We need to show that $\text{diam}(M_t)$ is upper semicontinuous with respect to t for a family as given above, i.e. for all $t_0 \in T$ and sequences (t_i) converging to t_0 ,

$$\limsup_{t_i \rightarrow t_0} \text{diam}(M_{t_i}) \leq \text{diam}(M_{t_0}).$$

If the inequality is false, then after replacing the sequence (t_i) by a subsequence there is an $\varepsilon > 0$ such that $\text{diam}(M_{t_i}) > \text{diam}(M_{t_0}) + \varepsilon$ for all i . By compactness and the continuity of the pseudometric on each M_t , there exist p_i, q_i such that $\text{diam}(M_{t_i}) = d_{M_{t_i}}(p_i, q_i)$. Replacing by a further subsequence if necessary, we may assume that the sequences (p_i) and (q_i) are convergent. Let $p, q \in M_{t_0}$ be their respective limit. Then by upper semicontinuity, we have

$$\text{diam}(M_{t_0}) + \varepsilon \leq \limsup_{i \rightarrow \infty} d_{M_{t_i}}(p_i, q_i) \leq d_{M_{t_0}}(p, q) \leq \text{diam}(M_{t_0}).$$

This is a contradiction. ■

2 Vanishing of the Kobayashi pseudometric

2.1 Kobayashi pseudometric and ergodicity

The main technical result of this paper is the following theorem. Recall that an ergodic complex structure I on M is one which has a dense $\text{Diff}(M)$ -orbit in the deformation space of complex structures.

Theorem 2.1: Let M be a complex manifold with vanishing Kobayashi pseudometric. Then the Kobayashi pseudometric vanishes for all ergodic complex structures in the same deformation class.

Proof: Let $\text{diam} : \text{Teich} \rightarrow \mathbb{R}^{\geq 0}$ map a complex structure I to the diameter of the Kobayashi pseudodistance on (M, I) . By Corollary 1.23, this function is upper semi-continuous. Let I be an ergodic complex structure. The set of points $I' \in \text{Teich}$ such that (M, I') is biholomorphic to (M, I) is dense, because I is ergodic. By upper semi-continuity, $0 = \text{diam}(I) \geq \inf_{I' \in \text{Teich}} \text{diam}(I')$. ■

Corollary 2.2: Let M be a K3 surface. Then the Kobayashi pseudometric on M vanishes.

Proof: Notice that any non-ergodic complex structure on a hyperkähler manifold is projective. Indeed, if the rank of the Picard group is maximal, the set of rational $(1, 1)$ -classes is dense in $H^{1,1}(M)$, hence the Kähler cone contains a rational class and M is projective. For all projective M , one has $\text{diam}(M) = 0$ (see Lemma 1.51 in [Vo] or Corollary 4.5 in [Lu0]). Therefore Theorem 2.1 implies that $\text{diam}(M) = 0$ for non-projective complex structures as well. ■

The same argument leads to the following result.

Theorem 2.3: Let M be a hyperkähler manifold admitting a complex structure with vanishing Kobayashi pseudometric and $b_2(M) \geq 4$. Then the Kobayashi pseudometric vanishes for all complex structures I in the same deformation class.

Proof: The diameter of the Kobayashi pseudometric is upper semicontinuous, by Corollary 1.23. Choose any ergodic complex structure J on M (such J exists because $b_2(M) > 3$). By definition of ergodic complex structures, in any neighbourhood of I one has a complex manifold isomorphic to (M, J) . By upper semicontinuity, one has $\text{diam}(M, J) \leq \text{diam}(M, I) = 0$. Now vanishing of the Kobayashi pseudometric follows from Theorem 2.1. ■

2.2 Lagrangian fibrations in hyperkähler geometry

The theory of Lagrangian fibrations on hyperkähler manifolds is based on the following remarkable theorem of D. Matsushita

Theorem 2.4: ([Mat1]) Let M be a simple hyperkähler manifold, and $\varphi : M \rightarrow X$ a surjective holomorphic map, with $0 < \dim X < \dim M$. Then the fibers of φ are Lagrangian subvarieties on M , and the general fibers of φ are complex tori.¹ ■

Remark 2.5: Such a map is called a **Lagrangian fibration**. All the known examples of hyperkähler manifolds admit Lagrangian fibrations ([KV, Claim 1.20]).

Definition 2.6: A cohomology class $\eta \in H^2(M, \mathbb{R})$ is called **nef** if it lies in the closure of the Kähler cone; a line bundle L is nef if $c_1(L)$ is nef. A nef line bundle L is **big** if $\int_M c_1(L)^{\dim_{\mathbb{C}} M} \neq 0$. A non-trivial nef line bundle L on a

¹These fibers are known to be abelian varieties, even if torus is non-algebraic; see [AC].

hyperkähler manifold is called **parabolic** if it is not big. From the definition of the BBF form, this is equivalent to $q(c_1(L), c_1(L)) = 0$. Lagrangian fibrations are in bijective correspondence with semiample parabolic bundles, as follows from Matsushita's theorem.

Claim 2.7: Let M be a simple hyperkähler manifold, and L a non-trivial semiample bundle on M . Assume that L is not ample. Consider the holomorphic map $\pi : M \rightarrow \text{Proj}(\bigoplus_N H^0(M, L^N))$. Then π is a Lagrangian fibration. Moreover, every Lagrangian fibration is uniquely determined by a parabolic nef line bundle.

Proof: The first statement of Claim 2.7 is a corollary of Theorem 2.4. Let $M \xrightarrow{\pi} X$ be a Lagrangian fibration. By Matsushita's results ([Mat1]), X is projective and $H^*(X) \cong H^*(\mathbb{C}P^n)$. Denote by $\eta \in H^2(M, \mathbb{Z})$ the ample generator of $\text{Pic}(X)$. Then $\pi^*\eta = c_1(L)$, where $L = \pi^*\mathcal{O}_X(1)$ is a parabolic nef bundle on M . ■

The **SYZ conjecture** ([Saw], [V1]) claims that any parabolic nef line bundle on a hyperkähler manifold is semiample, that is, it is associated with a Lagrangian fibration. This is true for K3 surfaces (as it follows from the Riemann-Roch formula) and for all deformations of Hilbert schemes of K3 surfaces ([Mar4] and [BM]).

Further on, we shall need a birational version of Matsushita's theorem on Lagrangian fibrations, which is due to Matsushita-Zhang.

Theorem 2.8: ([MZ, Theorem 1.4]) Let X be a projective hyperkähler manifold, and P an effective \mathbb{R} -divisor on X . Then there exists a birational modification $\tau : X' \dashrightarrow X$, where X' is a projective hyperkähler manifold such that τ^*P is nef. ■

Theorem 2.9: Let M be a projective hyperkähler manifold, and L a line bundle of Kodaira dimension $\frac{1}{2} \dim_{\mathbb{C}} M$. Then there exists a birational modification $\tau : M' \dashrightarrow M$ from a projective hyperkähler manifold such that τ^*L is semiample, and induces a Lagrangian fibration as in Claim 2.7.

Proof: Let L be a nef bundle on a Kähler manifold. Recall that the **numerical Kodaira dimension** of L is the maximal k such that $c_1(L)^k \neq 0$. The **Kodaira dimension** of L is the Krull dimension of the ring $\bigoplus_N H^0(M, L^N)$.

Consider the modification $\tau : M' \dashrightarrow M$ produced by the Matsushita-Zhang theorem. Then the numerical dimension of τ^*L is equal to $\frac{1}{2} \dim_{\mathbb{C}} M$,

by [V0], and the Kodaira dimension stays the same. As shown in [Kaw, Theorem 1] (see also [BCEKPRSW, Proposition 2.8]), whenever the numerical dimension of a nef bundle is equal to its Kodaira dimension, the bundle is semiample. Then Theorem 2.9 follows from Claim 2.7. ■

This result motivates the following definition.

Definition 2.10: Let $\tau : M' \dashrightarrow M$ be a birational map of hyperkähler manifolds, and \mathcal{L} a Lagrangian fibration on M . Then $\tau^*\mathcal{L}$ is called a **birational Lagrangian fibration on M'** . Its fibers are proper preimages of those fibers of \mathcal{L} which are not contained in the exceptional locus of τ .

2.3 Kobayashi pseudometrics and Lagrangian fibrations

The idea to use Theorem 2.11 is suggested by Claire Voisin. We are very grateful to Prof. Voisin for her invaluable help.

Theorem 2.11: Let M be a simple hyperkähler manifold admitting two Lagrangian fibrations associated with two non-proportional parabolic classes. Then the Kobayashi pseudometric on M vanishes.

Proof: Let $\pi_i : M \rightarrow X_i$, $i = 1, 2$, be the Lagrangian fibration maps. Since the general fibers of π_i are tori, the Kobayashi metric vanishes on each fiber of π_i . To prove that the Kobayashi pseudometric vanishes on M , it would suffice to show that a general fiber of π_1 intersects all fibers of π_2 .

Let now ω_i be an ample class of X_i lifted to M , and $2n = \dim_{\mathbb{C}} M$. Since the ω_i 's are not proportional, the standard linear-algebra argument, often called the Hodge index formula, implies that $q(\omega_1, \omega_2) > 0$. Indeed, $q(\omega_1, \omega_2) \neq 0$ or else the space $(H^{1,1}(M, \mathbb{R}), q)$ would contain a 2-dimensional isotropic plane while its signature is $(1, b_2 - 3)$. The number $q(\omega_1, \omega_2)$ is not negative, because every neighbourhood of (ω_1, ω_2) contains pairs of Kähler forms (ω'_1, ω'_2) on M , and $q(\omega'_1, \omega'_2) > 0$ for any pair of Kähler forms (Remark 1.13).

Clearly, the fundamental class $[Z_i]$ of a fiber of π_i is proportional to ω_i^n . Fix the constant multiplier in such a way that $[Z_i] = \omega_i^n$. The fibers of π_i intersect if $\int_M [Z_1] \wedge [Z_2] > 0$. However, Fujiki's formula (Theorem 1.11) implies that $\int_M [Z_1] \wedge [Z_2] = Cq(\omega_1, \omega_2)^n > 0$. This means that Z_1 and Z_2 always intersect. ■

Further on, we shall use a birational version of this statement.

Theorem 2.12: Let M be a simple hyperkähler manifold admitting a Lagrangian fibration A and a birational Lagrangian fibration B associated to two non-proportional parabolic classes. Then the Kobayashi pseudometric on M vanishes.

Proof: Let $\tau : M \dashrightarrow M'$ be a birational modification such that B is the pullback of a Lagrangian fibration on M' . Since M and M' have trivial canonical bundle, the exceptional locus of τ has codimension at least two, hence a general fiber L of B is birationally equivalent to a torus outside of this exceptional locus. By Theorem 1.19, the Kobayashi pseudometric on L vanishes. The same argument as used in Theorem 2.11 shows that L meets all general fibers of A and thus the Kobayashi pseudodistance between any two general points x, y in M vanishes. Indeed, take a general fiber L of B . Let x', y' be the points where the fibers of A associated with x, y intersect L . The Kobayashi pseudodistance $d(x', y')$ vanishes, because it vanishes on L , and $d(x, x') = d(y, y') = 0$, because these are points in the same complex tori. ■

Theorem 2.13: Let M be a simple hyperkähler manifold with a Lagrangian fibration $\varphi : M \rightarrow X$. Then M has a deformation admitting both a Lagrangian fibration and a birational Lagrangian fibration that correspond to different classes $\eta, \eta' \in H^2(M, \mathbb{Z})$ respectively.

Proof: Let $\eta \in H^{1,1}(M)$ be a parabolic nef class associated with φ as in Claim 2.7. Denote by Teich_η the divisor parametrizing deformations of M for which η is of type (1,1). As shown in [KV], for a dense subset D_0 of $J \in \text{Teich}_\eta$ where η is nef, the corresponding bundle L with $c_1(L) = \eta$ is semiample, giving a Lagrangian fibration $(M, J) \rightarrow \text{Proj}(\bigoplus_N H^0(L^N))$. When $J \notin D_0$, the Kodaira dimension of L is at least $\dim \bigoplus_N H^0(L^N)$, by upper semicontinuity. Since η is nef whenever $\text{Pic}(M)$ is generated by η , we may assume that for all $J \notin D_0$, $\text{Pic}(M)$ contains a positive vector. As shown by Huybrechts, [H1, Theorem 3.11], (M, J) is then projective.

Therefore, by Theorem 2.9, these manifolds admit a birational Lagrangian fibration

Consider now the action of the monodromy group Γ^I on $H^2(M, \mathbb{Z})$. As follows from Remark 1.16, Γ^I is an arithmetic subgroup in $O(H^2((M, \mathbb{Z}), q))$. Therefore, Γ^I contains an element γ such that $\gamma(\eta) \neq \eta$. It is easy to see that the divisors Teich_η and $\text{Teich}_{\gamma(\eta)}$ intersect transversally. Their intersection corresponds to a manifold M with two birational Lagrangian fibrations A and B . Replacing M by a birationally equivalent hyperkähler manifold where A is holomorphic, we obtain the statement of Theorem 2.13. ■

Corollary 2.14: Let M be a simple hyperkähler manifold with a Lagrangian fibration. Then the Kobayashi pseudometric vanishes for all ergodic complex structures on M .

Proof: Consider a deformation (M, I') of M admitting two birational Lagrangian fibrations. Then the Kobayashi pseudometric of (M, I') vanishes by Theorem 2.12. For an ergodic complex structure I , we obtain

$$\text{diam}(I) \leq \inf_{I' \in \text{Teich}} \text{diam}(I') = 0$$

by upper semi-continuity. ■

3 Vanishing of the infinitesimal pseudometric

In this section, we are interested in conditions that guarantee the vanishing of the infinitesimal Kobayashi pseudometric $| \cdot |_M$ on a Zariski dense open subset of M . Recall that the SYZ conjecture predicts the existence of a Lagrangian fibration for every hyperkähler M , $\dim_{\mathbb{C}} M = 2n$. Furthermore, if the base of the fibration is smooth (this is conjectured, see [Hw0]), then the base is isomorphic to $\mathbb{C}P^n$, as shown by Hwang (see [Hw0, GL]). If M is projective and admits an abelian fibration, then we have the following two results.

Theorem 3.1: Let M be a projective manifold with an equidimensional abelian fibration $f : M \rightarrow B$ (holomorphic surjective with all fibres of the same dimension and general fibres isomorphic to abelian varieties) where B is a complex projective space of lower dimension. If f has no multiple fibres in codimension one, then $| \cdot |_M$ vanishes everywhere on M . In particular, if M is a projective hyperkähler manifold with a birational Lagrangian fibration over a nonsingular base without multiple fibres in codimension one, then $| \cdot |_M$ vanishes everywhere.

Proof: Let $v \in T_x M$. Then v can be regarded as the first order part of some non vertical k -jet ν , which in turn push forward to a non-trivial jet prescription μ at $b = f(x) \in B$. This jet prescription μ is clearly satisfied by an algebraic holomorphic map $h : \mathbb{C} \rightarrow B$. Since this map can be chosen to avoid any subset of codimension two or more, we see by so doing that the pull back fibration $M_h \rightarrow \mathbb{C}$ has no multiple fibres. Hence all higher order jet infinitesimal pseudometric vanishes on M_h by Theorem 4.3. Since ν is in the image of a k -jet on M_h , it also has zero k -jet infinitesimal pseudometric by the distance decreasing property and therefore $|v|_M = 0$. ■

Theorem 3.2: Let M be a projective manifold. Let $f : M \rightarrow B$ define an abelian fibration. Assume that there is a subvariety $Z \subset M$ that dominates B and is birational to an abelian variety. Then $| \cdot |_M$ vanishes everywhere above a Zariski dense open subset U in B . In particular, this holds for hyperkähler manifolds with $b_2 \geq 5$ having two birational Lagrangian fibrations.

Proof: By hypothesis and the resolution of singularity theorem, Z is the holomorphic image of a nonsingular projective variety A obtained from an abelian variety by blowing up smooth centres. By construction, any vector in A is in the tangent space of an entire holomorphic curve. Let $g : A \rightarrow B$ be the composition with the projection to B and $\text{disc}(g)$ its discriminant locus. Let $v \in TM$ be a nonzero vector above the complement U in B of $\text{disc}(f) \cup \text{disc}(g)$. If v is vertical, then it is a vector on the fibre A through p , which is an abelian variety and clearly $|v|_M \leq |v|_A = 0$ in this case. If v is horizontal, then there is a vector v' in TA by construction such that $f_*v = g_*v'$. Let $h : \mathbb{C} \rightarrow A$ be such that $h'(0) = v'$ and $\pi : M_h \rightarrow \mathbb{C}$ be the pull back fibration via the base change by $g \circ h$. Then π has no multiple fibres and v lies in the image of TM_h by construction. Theorem 4.2 from Appendix now applies to show that $| \cdot |_{M_h}$ vanishes and so the distance decreasing property yields $|v|_M = 0$. The last statement follows from the projectivity of M by the proof of Corollary 2.2. ■

We also have the following result modulo the SYZ conjecture.

Theorem 3.3: Let M be a simple hyperkähler manifold with $b_2 \geq 7$. Assume that the SYZ conjecture is true for any deformation of M , and the Picard rank of M is maximal. Then M admits two birational Lagrangian fibrations.

Proof. Consider a non-zero integral vector $z \in \text{Pic}(M)$ such that $q(z, z) = 0$, where q denotes the Beauville-Bogomolov-Fujiki form. Since $\text{rk Pic}(M) \geq 5$, such a vector exists by Meyer's Theorem (see [Cas], page 75). As shown in the proof of Theorem 2.13, z is associated with a birational Lagrangian fibration. Denote by Γ_1 the group of automorphisms of the lattice $\text{Pic}(M)$. Since this group is arithmetic, it contains an element γ which does not preserve z . Then $\gamma(z)$ is another vector associated with a birational Lagrangian fibration. ■

The above theorems together imply the following corollary.

Corollary 3.4: Let M be a simple hyperkähler manifold with $b_2 \geq 7$. Assume that the SYZ conjecture is true for any deformation of M . Then d_M

is identically zero. If, further, the rank of $\text{Pic}(M)$ is at least 5, then $|\cdot|_M$ vanishes on a dense Zariski open subset of M . ■

Proof: By the proof of Corollary 2.2, M is projective and therefore Theorem 3.2 applies. ■

We remark again the expectation that the above assumption on the rank of the Picard group and on $b_2(M)$ should always hold for hyperkähler manifolds of maximal Picard rank, and hence for nonergodic hyperkähler manifolds.

4 Appendix on abelian fibrations

The following are some relevant basic results concerning abelian fibrations found in [Lu0], which was cited and used in [Camp], [Lu1] and [Lu2]. Recall that a fibration is a proper surjective map with connected fibres. All fibrations are assumed to be projective in this section and abelian fibrations are those whose general fibres are abelian varieties.

Proposition 4.1: Let $e : P \rightarrow \mathbb{D}$ define an abelian fibration which, outside $0 \in \mathbb{D}$, is smooth with abelian varieties as fibers. Let n_0 be the multiplicity of the central fiber P_0 . Then there is a component of multiplicity n_0 in P_0 .

Proof We may reduce the problem to the case of $n_0 = 1$ by the usual base change $z \mapsto z^{n_0}$ so that the resulting object (after normalization) is again such a fibration with an unramified cover to the original P . Let $\{m_1, m_2, \dots, m_k\}$ be the set of multiplicities of the components of P_0 . By assumption, there exists integers l_i such that $l_1 m_1 + l_2 m_2 + \dots + l_k m_k = 1$.

As fibrations are assumed to be projective in this paper, we may assume that f is algebraic. By restricting to $\mathbb{D}_r = \{t : |t| < r\}$ for an $r < 1$ if necessary, we can construct an algebraic multi-section s_i with multiplicity m_i above \mathbb{D}_r by simply taking a one dimensional algebraic slice transversal to the i -th component for each i . Above each point t outside 0, s_i consists of m_i points $s_i^j(t)$, $j = 1, 2, \dots, m_i$. Then it is easy to verify that

$$(l_1 \sum_j s_1^j(t)) + (l_2 \sum_j s_2^j(t)) + \dots + (l_k \sum_j s_k^j(t))$$

is independent of the choice of an origin in the abelian variety P_t . This gives a section s of the fibration outside 0 and we now show that s must be algebraic, giving a section of f and establishing our proposition.

This can be accomplished by looking at the base change via $z \mapsto t = z^m$ where m is the least common multiple of the m_i 's. Then each s_i lifts to

m_i sections which the cyclic Galois action permutes. Hence, the Galois action of Z_m acts transitively on the m sections constructed by replacing the i th term in the above expressing with each of the m_i sections and so this set of sections descends to a section of the original fibration as desired. ■

We remark that (even non projective) elliptic fibrations are all locally projective so that the above proposition applies to them. Also the above proposition is really a special case of a result of Lang and Tate found in [LT].

This proposition allows us to do exactly the same analysis as in the case for elliptic fibrations done in [BL1] to obtain the following theorem. We refer the reader there or to [Lu0] for the detail of the proof.

Theorem 4.2: Let $f : X \rightarrow C$ define an abelian fibration over a complex curve C . Then, for each $s \in C$, the multiplicity of the fibre X_s at s is the same as the minimum multiplicity m_s of the components of X_s . Let the \mathbb{Q} -divisor $A = \sum_s (1 - 1/m_s)s$ be the resulting orbifold structure on C . Then the three conditions $d_X = 0$ on X , $|\cdot|_X = 0$ on X and (C, A) is nonhyperbolic (that is, C is quasiprojective and $e(C) - \deg A \geq 0$) are equivalent for such a fibration. In the case C is quasiprojective, these three conditions are equivalent to the absence of non-commutative free subgroups in $\pi_1(X)$ and to $\pi_1(X)$ being solvable up to a finite extension.

Proof: In the case (C, A) is uniformizable, we may pull back the fibration to the universal cover U of (C, A) with resulting fibration $\tilde{f} : Y \rightarrow X$. This is the case when C is not quasi projective and otherwise when $e(C) \leq \deg A$, with equality if and only if $U = \mathbb{C}$, and when $e(C) > \deg A$ in which case either $C = U = \mathbb{C}$ and A is supported at one point, or $C = \mathbb{P}^1$ and A is supported at more than two points, see for example [FK]. In all these cases U is non hyperbolic if and only if (C, A) is. By construction Y has no multiple fibres over U and is unramified over X (in codimension one) so that all holomorphic curves in X lifts to Y . Hence the Kobayashi pseudometrics and norms vanish on X if and only if it is so on Y and so we only need to show the vanishing of $|\cdot|_Y$ in this case since the fundamental group characterization in the quasi projective case follows from the same characterization of the Galois group of the uniformization $\tilde{f} : U \rightarrow C$ and the exact sequence of fundamental groups of a fibration without multiple fibres. Note that in the case $U = \mathbb{P}^1$, to show that $|\cdot|_Y$ vanish at a point above $z \in U$ we may replace U by $\mathbb{C} = U \setminus \{z\}$ since $|\cdot|_Y \leq |\cdot|_{Y'}$ where $Y' := Y \setminus \tilde{f}^{-1}(z) \subset Y$.

In the case (C, A) is not uniformizable, then $C = \mathbb{P}^1$ and A is supported at one or two points and the exact sequence of orbifold fundamental groups shows that $\pi_1(X)$ is a quotient of $\pi_1(X_s)$ for a general fibre X_s , hence

abelian. Thus it suffice to show that for a point $p \in X$, $|\cdot|_X$ vanishes there in this case and for this it is sufficient to replace X by the complement of a fibre X_z different from the fibre X_w containing p and C by $C \setminus \{z\}$, where in the case w lies in the support of A , we choose z to be the other point in this support if one exists. Then (C, A) is uniformizable by \mathbb{C} .

Hence it remains to show that $|\cdot|_X$ vanishes at a point p for the case X has no multiple fibres and $C = \mathbb{C}$. In fact, given a finite jet prescription at p , we can find an entire holomorphic curve through p satisfying the jet prescription as follows. The jet prescription gives rise to a jet prescription at $f(p) \in C$ which we assume without loss of generality to be the origin of $C = \mathbb{C}$. Let l be the first non vanishing order of the latter jet and let $\tilde{f} : Y \rightarrow \mathbb{C}$ be the pull back fibration by the base change $z \mapsto z^l$. Then the inverse function theorem allows us to translate the jet prescription at p to a section jet prescription on Y over $0 \in \mathbb{C}$. As there are no multiple fibres for \tilde{f} , proposition Proposition 4.1 yield the existence of local sections of \tilde{f} through any point of \mathbb{C} . The Cousin principles apply in this situation (i.e., an analogue of Weierstrass' theorem can be worked out, see [BL1, Lu0]) so that we can patch up a minimal covering family of such sections, including the one with the jet prescription, to give a global section of \tilde{f} with the jet prescription and this gives the required entire holomorphic curve. ■

Instead of restricting our attention to just the first order jets for the infinitesimal pseudometric, one can generalize the definition of $|\cdot|_X$ to jets of arbitrary finite order, see [Lu0]. By their very definition, these infinitesimal pseudometrics dominates $|\cdot|_X$ by truncating the jets to first order. The exact same proof as above yields the following generalization, see [Lu0].

Theorem 4.3: The above theorem holds if $|\cdot|_X$ is replaced by its more general k -th order jet version, for all integer $k > 0$.

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