

# LONGEST INCREASING SUBSEQUENCES, PLANCHEREL-TYPE MEASURE AND THE HECKE INSERTION ALGORITHM

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ABSTRACT. We define and study the *Plancherel-Hecke probability measure* on Young diagrams; the *Hecke algorithm* of [Buch-Kresch-Shimozono-Tamvakis-Yong '06] is interpreted as a polynomial-time exact sampling algorithm for this measure. Using the results of [Thomas-Yong '07] on *jeu de taquin for increasing tableaux*, a symmetry property of the Hecke algorithm is proved, in terms of longest strictly increasing/decreasing subsequences of words. This parallels classical theorems of [Schensted '61] and of [Knuth '70], respectively, on the *Schensted* and *Robinson-Schensted-Knuth* algorithms. We investigate, and conjecture about, the limit typical shape of the measure, in analogy with work of [Vershik-Kerov '77], [Logan-Shepp '77] and others on the “longest increasing subsequence problem” for permutations. We also include a related extension of [Aldous-Diaconis '99] on *patience sorting*. Together, these results provide a new rationale for the study of increasing tableau combinatorics, distinct from the original algebraic-geometric ones concerning K-theoretic Schubert calculus.

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## 1. INTRODUCTION AND MAIN RESULTS

1.1. **Overview.** Let  $W_{n,q}$  denote the set of words of length  $n$  generated using the alphabet  $\{1, 2, \dots, q\}$ . Let  $\text{LIS}(w)$  denote the **length of the longest strictly increasing subsequence** of  $w = w_1 w_2 \cdots w_n$ , i.e., the largest  $\ell$  with a subsequence  $i_1 < i_2 < \dots < i_\ell$  such that  $w_{i_1} < w_{i_2} < \dots < w_{i_\ell}$ . Similarly, we consider the **length of the longest strictly decreasing subsequence**  $\text{LDS}(w)$  of  $w$ . Our main goal is to introduce and study a discrete probability measure on Young diagrams, in connection with the study of the distributions of LIS and LDS on uniform random words. An additional goal is to provide a novel motivation for the K-theoretic Schubert calculus combinatorics of [BuKrShTaYo06, ThYo07].

There are analogies with the study of LIS and LDS in the **permutation case**, i.e., when  $w$  is chosen uniformly at random from the symmetric group  $S_n$ . The latter topic has attracted considerable attention; we refer the reader to the surveys [AIDi99, St06] and the references therein. In the permutation case, random Young diagrams are distributed according to the *Plancherel measure* (on irreducible representations) of  $S_n$ . This discrete probability measure is the push-forward of the uniform distribution on  $S_n$ , under the *Robinson-Schensted correspondence*. Schensted [Sc61] established that this correspondence encodes  $\text{LIS}(w)$  and  $\text{LDS}(w)$  symmetrically in the shape  $\lambda$  associated to  $w$ . In [VeKe77, LoSh77], these ideas are applied to determine the asymptotics of the expectation of LIS over  $S_n$  (solving the old “longest increasing sequences problem”), via a study of the “limit typical shape” under the Plancherel measure.

As a continuation of this theme, we apply the Hecke (insertion) algorithm of [BuKrShTaYo06] to define the Young diagram  $\text{Heckeshape}(w)$  for each  $w \in W_{n,q}$ ; using this we define *Plancherel-Hecke measure*. Our belief that this measure should actually be worthy of analysis was initially guided by our theorem that Hecke symmetrically encodes  $\text{LIS}(w)$  and  $\text{LDS}(w)$  for  $w \in W_{n,q}$ , a generalization of Schensted’s theorem. During the course of our investigation, we found that many other aspects of the Plancherel-Hecke measure (conjecturally) also resemble those of the Plancherel measure. This paper records these results, both theoretical and computational, as a justification for further study.

Briefly, this is how the two aforementioned measures compare: Let  $q = \Theta(n^\alpha)$ . Sending  $q, n \rightarrow \infty$ , we conjecture that for  $\alpha > \frac{1}{2}$ , our measure is concentrated around the limit typical shape under Plancherel measure. This *Plancherel curve* plays an important role in [VeKe77, LoSh77]. On the other hand, for  $\alpha < \frac{1}{2}$  we conjecture the measure is concentrated near the “staircase shape”. In particular, a “phase transition” is suggested at  $\alpha = \frac{1}{2}$ . As we tune  $\alpha$ , a symmetric deformation of the Plancherel curve occurs. In view of the above mentioned result on the Hecke algorithm, this transition phenomenon is further evidenced by computations (with contributions by O. Zeitouni) of the expectation of LIS and LDS as  $\alpha$  varies; see Section 5 and the Appendix.

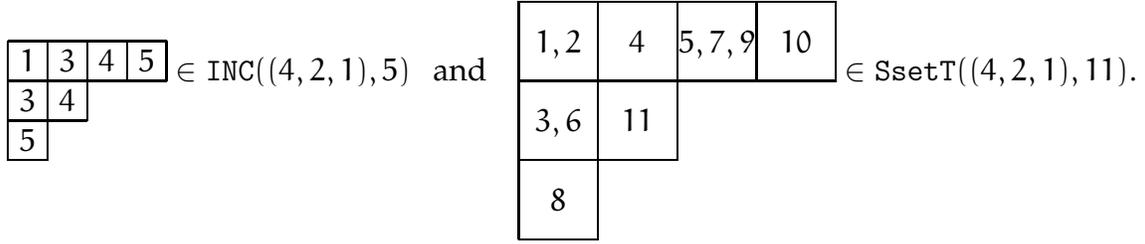


FIGURE 1. An increasing tableau and a set-valued standard Young tableau

There have been earlier extensions of the permutation case to  $W_{n,q}$ . The limit distribution of the length of the longest *weakly* increasing/decreasing subsequence (LwIS/LwDS) on  $W_{n,q}$  was found in work of [TrWi01], following the breakthrough [BaDeJo99] on the limit distribution of LIS on  $S_n$ . See also the more recent work [HoLi06]. However, analogous understanding of the distribution of LIS and LDS on  $W_{n,q}$  appears to be less developed; see, e.g., [Bi01, BoOl07, TrWi01] for contributions.

As a point of comparison and contrast with our approach, previous work on LIS, LDS and  $W_{n,q}$  utilizes the combinatorics of the *Robinson-Schensted-Knuth correspondence*, which asymmetrically encodes LwIS and LDS. We offer an alternative viewpoint on the relationship between Young diagrams and LIS, LDS. New questions and conjectures are raised, stemming from the Coxeter-theoretic viewpoint of [BuKrShTaYo06] (which in turn generalizes ideas of [EdGr87]).

This text expresses our desire to point out a natural link between the probabilistic combinatorics of LIS, LDS and the combinatorial algebraic geometry of K-theoretic Schubert calculus. In particular, we apply and further develop the *jeu de taquin* for *increasing tableaux* from [ThYo07], thereby giving another perspective on that work, distinct from the original one. In summary, we believe that the availability of these two disparate interpretations for [BuKrShTaYo06, ThYo07] provides something atypical to recommend K-theoretic tableau combinatorics, among the large array of interesting generalizations of the classical Young tableau and symmetric function theories known today.

**1.2. Plancherel-Hecke measure.** We identify a partition  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0)$  with its Young diagram (in English notation); set  $|\lambda| := \sum_i \lambda_i$ . Let  $\mathbb{Y}$  denote the set of all Young diagrams. A filling of a shape  $\lambda$  with a subset of the labels  $\{1, 2, \dots, q\}$  is an **increasing tableau** if it is strictly increasing in both rows and columns. Let  $\text{INC}(\lambda, q)$  be the set of all increasing tableaux of shape  $\lambda$ .

We also need **set-valued tableaux** [Bu02a], which are fillings of  $\lambda$  assigning to each box a nonempty subset of  $\{1, 2, \dots, n\}$  such that the largest entry of a box is smaller than the smallest entry in the boxes directly to the right of it, and directly below it. We call a set-valued tableau **standard** if each label is used precisely once. Let  $\text{SsetT}(\lambda, n)$  denote the set of all standard set-valued tableaux. See Figure 1.

The **Plancherel measure** on  $\mathbb{Y}$  assigns to  $\lambda$  the probability

$$\frac{(f^\lambda)^2}{n!}, \text{ where } f^\lambda := e^\lambda(|\lambda|) \text{ is the number of standard Young tableaux of shape } \lambda.$$

Let  $d^\lambda(q) := \#\text{INC}(\lambda, q)$  and  $e^\lambda(n) = \#\text{SsetT}(\lambda, n)$ .

**Definition 1.1.** The **Plancherel-Hecke probability measure**  $\mu_{n,q}$  on  $\mathbb{Y}$  is defined by letting  $\lambda_{n,q}$  be a random (non-uniform) Young diagram with distribution

$$\text{Prob}(\lambda_{n,q} = \lambda) := \frac{1}{q^n} d^\lambda(q) e^\lambda(n).$$

**Proposition 1.2.** The Plancherel-Hecke measure is well-defined as a probability distribution; i.e., the following identity holds:

$$(1) \quad q^n = \sum_{\lambda} d^\lambda(q) e^\lambda(n),$$

where

$$(2) \quad |\lambda| \leq \min \left( n, \binom{q+1}{2} \right) \text{ and } \lambda \subseteq (q, q-1, q-2, \dots, 3, 2, 1).$$

There is an exact polynomial-time sampling algorithm

$$\text{Heckeshape} : W_{n,q} \rightarrow \mathbb{Y},$$

terminating in  $O(nq^2)$  operations, that induces  $\mu_{n,q}$  from the uniform distribution on  $W_{n,q}$ .

The core technical result of this paper is a generalization of the aforementioned theorem of Schensted [Sc61]:

**Theorem 1.3.** Heckeshape simultaneously and symmetrically encodes  $\text{LIS}(w)$  and  $\text{LDS}(w)$  as the size of the first row and column of  $\text{Heckeshape}(w)$ , respectively.

Theorem 1.3 is obtained by establishing another new result, connecting Heckeshape to the “K-infusion” operation defined in [ThYo07]. This latter result is an analogue of the classical fact that connects the Robinson-Schensted correspondence to the (ordinary) *jeu de taquin* rectification procedure.

We prove of Proposition 1.2 in Section 2, after recalling the Hecke algorithm of Buch-Kresch-Shimozono-Tamvakis-Yong [BuKrShTaYo06] (originally constructed to study degeneracy loci of vector bundles).  $\text{Heckeshape}(w)$  is the Young diagram associated to  $w$  under Hecke. The proof of Theorem 1.3 is given in Section 4.

*Example 1.4.* We illustrate the identity (1) for  $n = 4$  and  $q = 3$ . There are nine partitions  $\lambda$  satisfying (2). These are

$$(1), (2), (1, 1), (2, 1), (3), (1, 1, 1), (3, 1), (2, 1, 1), (2, 2).$$

Then (1) reads

$$81 = 3^4 = 3 \cdot 1 + 3 \cdot 3 + 3 \cdot 3 + 5 \cdot 8 + 1 \cdot 3 + 1 \cdot 3 + 2 \cdot 3 + 2 \cdot 3 + 1 \cdot 2,$$

where the products on the righthand side of the equality are listed in order corresponding to the above partitions. Thus, the “typical shape” is  $(2, 1)$ , possessing nearly 50% of the distribution. The remainder of the above results will be illustrated in Section 2, after we define Heckeshape.

Theorem 1.3 has some immediate consequences, familiar from the permutation case.

**Corollary 1.5.** *Under the uniform measure on  $W_{n,q}$ , we have*

$$(3) \quad \mathbb{E}(\text{LIS}) = \frac{1}{q^n} \sum_{\lambda} \lambda_1 d^\lambda(q) e^\lambda(n).$$

*In addition,*

$$\text{Prob}(\text{LIS} = \ell) = \frac{1}{q^n} \sum_{\lambda, \lambda_1 = \ell} d^\lambda(q) e^\lambda(n).$$

Two other consequences will be given in Section 3. One gives a ‘‘Coxeter-theoretic’’ generalization of the widely known Erdős-Szekeres theorem [ErSz35]. Another expands upon the discussion of patience sorting given in [AlDi99].

**1.3. Remarks on Proposition 1.2 and Theorem 1.3.** In our experiments, Heckeshape was reasonably efficient as a sampling algorithm.<sup>1</sup> For example, when  $n \leq 10,000$  sampling one Young diagram takes on the order of seconds to minutes on current technology. For larger  $n$ , we could sample one Young diagram when  $n = 50,000$  in several hours on the same technology. A sample when  $n = 100,000$  took about one and a half days. The memory demands were modest. In view of the apparent ‘‘concentration’’ suggested below, one sample was enough to be of interest for our purposes, when  $n$  is large.

There are classical antecedents of Theorem 1.3. As stated earlier, Schensted [Sc61] proved the analogous conclusion about the shape coming from the Robinson-Schensted correspondence for a permutation  $w$ . In contrast, Knuth [Kn70] proved that the first row of the Robinson-Schensted-Knuth algorithm (RSK) encodes the length of the longest weakly increasing subsequence (LwIS) of a word  $w$ .

What is perhaps less well-known is that RSK also encodes  $\text{LDS}(w)$  as the length of the first column of the shape it associates to  $w$ . However, unlike Heckeshape, it is asymmetric:  $\text{LIS}(w)$  is not encoded by the length of the first row (as  $\text{LIS}(w) \neq \text{LwIS}(w)$  in general). Thus, the symmetry of Hecke(shape) makes it natural to analyze, as it seems desirable to simultaneously capture the statistics of LIS and LDS.

However, we do not have handy formulas for  $\text{Prob}(\lambda)$ ,  $d^\lambda(q)$  or  $e^\lambda(n)$ , such as the *hook-length formula* for  $f^\lambda$ . (In the permutation case, the hook-length formula plays a crucial role, see [VeKe77, LoSh77].) In small examples large prime factors appear, showing that such a formula is unlikely. For instance,  $d^{(4,2,1)}(7) = 1337 = 7 \cdot 191$  and  $e^{(4,2,1)}(8) = 452 = 2^2 \cdot 113$ . This issue is closely related to the open question of finding ‘‘good’’ determinantal expressions for *Grothendieck polynomials* [LaSc82]. That being said, special cases exhibit connections to work of Stanley [St96] on polygonal dissections, and of [FoGr98] on generalized Littlewood-Richardson rules (further discussion may appear elsewhere). Thus, the enumerative combinatorics of these numbers might of interest in their own right.

It is not difficult to give recursions to calculate  $d^\lambda(q)$  and  $e^\lambda(n)$  that are useful in moderately large cases. Are there efficient (possibly randomized or approximate) counting algorithms?

Objectively, the lack of simple formulas to compute  $\text{Prob}(\lambda)$  makes it trickier to apply standard approaches directly; this is an admitted defect of our setting. Nevertheless, we believe the framework of problems described here is tractable. In addition to the results

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<sup>1</sup>Software available at the authors’ websites.

below, in the Appendix one gains useful and nontrivial information about the Plancherel-Hecke measure by exploiting the related work of [Bi01]. In this way, the techniques of [LoSh77, VeKe77] can be applied to the present context.

**1.4. Analysis of  $\mu_{n,q}$  and the limit typical shape.** We organize our analysis by first setting

$$(4) \quad q = f(n) \in \Theta(n^\alpha), \quad \text{where } 0 < \alpha \leq 1,$$

and considering the limit behavior of  $\mu_{n,q}$  when  $q \rightarrow \infty$ , as  $n \rightarrow \infty$ . (The case  $\alpha = 0$  is trivial.) As is explained below, we conjecture that there is a **critical value** of  $\alpha$ , denoted  $\alpha_{\text{critical}} := \frac{1}{2}$ : the behavior of  $\mu_{n,q}$  is qualitatively different in the intervals  $\alpha \in (\alpha_{\text{critical}}, 1]$  and  $\alpha \in (0, \alpha_{\text{critical}})$ . At  $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$ , further refinement of the analysis is needed, as we transition from one state to the other.

In the permutation case, to study the Plancherel measure, it is useful to consider the most likely, or “typical” shape. There, three facts are true. First, in the large limit (and after rescaling), a well-defined typical shape exists. Second, the expectation of the LIS and LDS of a large random permutation is encoded respectively in the length of the first row and column of the limit shape. Third, the Plancherel measure is concentrated near the typical shape.

We conjecture that analogues of all three of the aforementioned features also hold for the Plancherel-Hecke measure.

To be more precise, let the **typical shape**  $\Lambda_{n,q}$  be the shape  $\lambda$  (contained in  $(q, q - 1, \dots, 2, 1)$ ) maximizing  $\text{Prob}(\lambda)$ . This Young diagram  $\Lambda_{n,q}$  can be interpreted as a step-function. It can furthermore be associated to a piecewise linear approximation  $f_{n,q} : [0, \infty) \rightarrow \mathbb{R}_{\geq 0}$ . Finally, rescale by

$$\hat{f}_{n,q}(x) := \begin{cases} \frac{1}{2\sqrt{n}} f_{n,q}(x \cdot 2\sqrt{n}) & \text{if } \alpha \geq \alpha_{\text{critical}} = \frac{1}{2} \\ \frac{1}{q} f_{n,q}(x \cdot q) & \text{otherwise.} \end{cases}$$

**Conjecture 1.6.** (I) For any  $0 \leq \alpha \leq 1$ , there is a unique continuous function

$$\Lambda \in C([0, \infty) \rightarrow \mathbb{R}_{\geq 0})$$

such that for any  $\epsilon > 0$ ,

$$\text{Prob} \left( \sup_{x \in \mathbb{R}_{\geq 0}} |\hat{f}_{n,q} - \Lambda| > \epsilon \right) \rightarrow 0$$

as  $n \rightarrow \infty$ . We call this  $\Lambda$  the **limit typical shape**.

(II) A “phase transition” occurs at  $\alpha_{\text{critical}} = \frac{1}{2}$ :

For  $0 < \alpha < \alpha_{\text{critical}} = \frac{1}{2}$ ,  $\Lambda$  is the line

$$(5) \quad y = 1 - x \quad \text{for } 0 \leq x \leq 1.$$

For  $\alpha_{\text{critical}} = \frac{1}{2} < \alpha \leq 1$ ,  $\Lambda$  is the **Plancherel curve**, which is parametrically given by

$$(6) \quad x = y + \cos \theta, \quad y = \frac{1}{\pi} (\sin \theta - \theta \cos \theta), \quad \text{for } 0 \leq \theta \leq \pi.$$

(The curves defined by (5) and (6) are declared to be identically 0 for  $1 \leq x < \infty$ .)

(III) For  $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$ : there is a constant  $C > 0$  such that if

$$q = kn^{\frac{1}{2}} + \text{lower order terms},$$

then if  $k < C$ ,  $\Lambda$  is given by (5). Otherwise,  $\Lambda$  is given by a deformation of (6) which is symmetric across the line  $y = x$ . In either case, the  $x$  and  $y$  intercepts are at

$$0 \leq \beta(k) \leq 1$$

where

$$E(\text{LIS}) \approx \beta(k)2\sqrt{n}$$

and explicitly,

$$\beta(k) = \begin{cases} \frac{k}{2} & \text{if } 0 < k \leq 1 \\ \frac{2-k^{-1}}{2} & \text{if } k > 1 \end{cases}.$$

We have reasonable support for the cases (I) and (II) of Conjecture 1.6. *Heuristically*, part (II) of the conjecture says that when  $\alpha$  is large, and thus  $q$  is “close” to  $n$ , a random word is “close to” being a random permutation. For a permutation, Schensted and Hecke behave the same. Hence the Plancherel and Plancherel-Hecke measures ought to be maximized on the same shape. When  $\alpha$  is small, the limit typical shape is a rescaling of the “staircase shape” which plays a distinguished role in the Edelman-Greene algorithm [EdGr87] and the Hecke algorithm; see Theorem 1.8 and its proof.

Conjecture 1.6(III) is more speculative, since we did not have as much computational evidence for the shape of  $\Lambda$ . There appears to be a continuous “flattening” of the Plancherel curve to a line as we tune  $k$  from  $\infty$  to 0. Our data was insufficient to rule out the possibility that  $\Lambda$  is simply a rescaling of the Plancherel curve by a factor of  $\beta(k)$ .

**Problem 1.7.** Explicitly describe the deformation of  $\Lambda$  when  $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$ , as  $k$  varies.

Our best estimate is that  $\frac{1}{2} \leq C \leq 1$  (probably just  $C = 1$ ). However, the values of  $\beta(k)$  for  $k$  relatively small can be experimentally estimated. The table below was based on Monte Carlo estimates of  $E(\text{LIS})$  for  $n = 50,000, 100,000, 200,000$  and  $300,000$  and the estimates were stable throughout this range. They closely agree with the conjecture for  $\beta(k)$  given above.

$k$	0.5	1	2	4	10
$\beta(k)$ estimate	0.25	0.50	0.74	0.86	0.94

TABLE 1. Estimates of  $\beta(k)$  for the  $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$  case

Notice that since  $\text{Prob}(\lambda) = \text{Prob}(\lambda')$  where  $\lambda'$  is the conjugate shape of  $\lambda$  we know that if  $\Lambda$  exists and is unique, then  $\Lambda$  is symmetric. This is consistent with the limit curves we predict.

We prove in Section 5 that:

**Theorem 1.8.** Conjecture 1.6 is true for  $0 \leq \alpha < \frac{1}{3}$ . More precisely, in this range, a random shape  $\lambda$  under  $\mu_{n,q}$  satisfies  $\lambda_i = q - i + 1$  almost surely, as  $n, q \rightarrow \infty$ .

The proof of this theorem depends on the analysis of a certain random walk on the symmetric group. Our analysis is not sharp enough to extend to the range  $\frac{1}{3} \leq \alpha \leq \alpha_{\text{critical}} = \frac{1}{2}$ , although a refinement might be possible.

Empirically, one finds that the first row and column of  $\Lambda_{n,q}$  are approximations of  $E(\text{LIS})$  and  $E(\text{LDS})$  that improve as  $n \rightarrow \infty$ . Therefore, it makes sense to study the asymptotics of  $E(\text{LIS})$  and  $E(\text{LDS})$  as a means to understand the characteristics of  $\Lambda_{n,q}$ . From this point of view, the following result supports the phase transition phenomena asserted in Conjecture 1.6.

**Theorem 1.9** (With O. Zeitouni). *If  $0 \leq \alpha < \alpha_{\text{critical}} = \frac{1}{2}$  then  $\lim_{n \rightarrow \infty} E(\text{LIS}) = q$ , whereas if  $\alpha_{\text{critical}} = \frac{1}{2} < \alpha \leq 1$  then  $E(\text{LIS}) \approx 2\sqrt{n}$ . The same statements hold when  $E(\text{LDS})$  replaces  $E(\text{LIS})$ .*

The proof for  $\alpha < \alpha_{\text{critical}} = \frac{1}{2}$  is a variation on the approach we use to prove Theorem 1.8. We also conjectured the answer for  $\alpha > \alpha_{\text{critical}} = \frac{1}{2}$ ; after showing O. Zeitouni our guess during an early stage in the project, he communicated to us a proof, and kindly allowed us to reproduce his argument here.

In private communication, E. Rains offered a proof that  $E(\text{LIS}) = q$  in the  $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$  and  $k \leq 1$  case. Afterward, in the appendix for this paper, O. Zeitouni and the second author present a simple proof that  $E(\text{LIS}) \approx \beta(k)2\sqrt{n}$ , for all  $k$ , in the  $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$  case, thereby closing the gap in Theorem 1.9. The proof builds on work of [Bi01] (see further discussion in Section 1.5). These results further support the belief that  $C = 1$  in Conjecture 1.6(III).

In addition, we have the following conjecture about the fluctuation of LIS and LDS.

**Conjecture 1.10.** *Let  $\sigma(\text{LIS})$  denote the standard deviation of LIS. For  $0 < \alpha < \alpha_{\text{critical}} = \frac{1}{2}$  then*

$$\lim_{n \rightarrow \infty} \sigma(\text{LIS}) = 0,$$

*whereas if  $\alpha_{\text{critical}} = \frac{1}{2} < \alpha \leq 1$  then*

$$\lim_{n \rightarrow \infty} \sigma(\text{LIS}) = O(n^{\frac{1}{\epsilon}}).$$

*The same statements hold for LDS.*

Note that Theorem 1.8 implies Conjecture 1.10 holds for  $0 \leq \alpha < \frac{1}{3}$ . Tables 2 and 3 give numerical evidence for Conjecture 1.10 and are consistent with Theorem 1.9.

TABLE 2.  $n = 50,000$  with 1,000 Monte Carlo trials

estimate \ $\alpha$	0.45	0.50	0.55	0.60	0.75	1.00
$E(\text{LIS})$	130.00	222.50	311.06	368.38	422.48	436.36
$\sigma$	0.00	0.63	2.86	4.01	5.07	5.09

TABLE 3.  $n = 100,000$  with 500 Monte Carlo trials

estimate \ $\alpha$	0.45	0.50	0.55	0.60	0.75	1.00
$E(\text{LIS})$	177.00	315.43	448.2	523.63	603.78	619.64
$\sigma$	0.00	0.67	3.52	4.62	5.90	6.29

The bulk of  $\mu_{n,q}$  appears “concentrated” near  $\Lambda_{n,q}$ , i.e., the probability of sampling a random shape differing, in the sup-norm, from  $\Lambda$  after rescaling, by some fixed  $\epsilon > 0$ , goes to 0 as  $n, q \rightarrow \infty$ . See Figure 2: already at  $n = 5,000$  we see that two random

samples are visibly “close” to one another, and are similar in shape to the third curve which is an approximation of the Plancherel curve. By  $n = 100,000$  the curves appear undeniably to be rescalings of one another, with a rescaling factor of about 0.95 in our experiments. Naturally, as  $\alpha$  gets larger, the empirical convergence of the curves occurs faster.

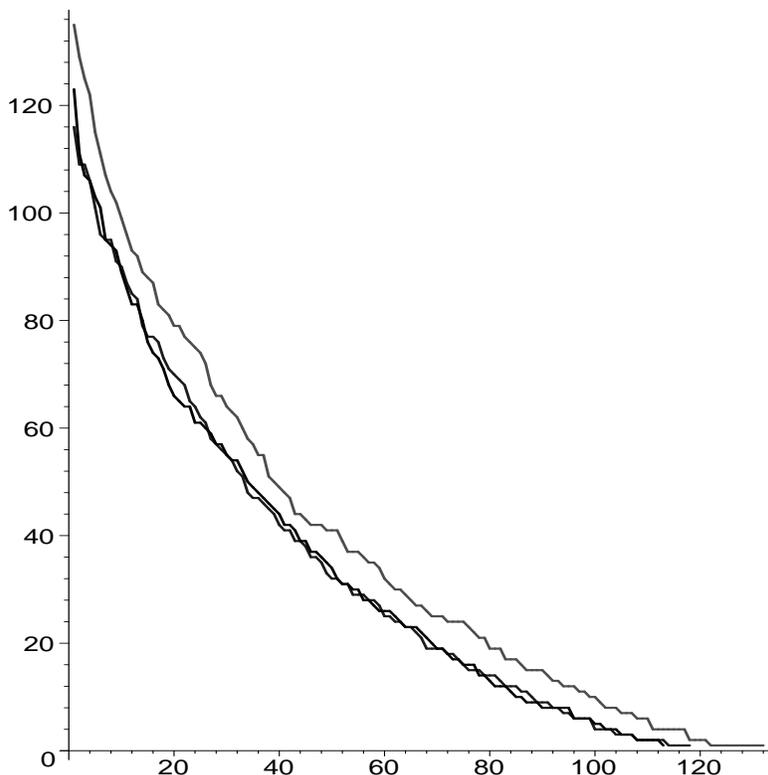


FIGURE 2. Two samples at  $n = 5000$ ,  $\alpha = \frac{2}{3}$  compared with an empirical approximation of the Plancherel curve; conjecturally as  $n \rightarrow \infty$ , the sample curves converge to one another

**1.5. Further comparisons with the literature.** As mentioned earlier, the limit distribution of LIS on permutations, and that of  $LwIS$  on words is well understood.

The study of LIS on  $W_{n,q}$  was considered, e.g., in [TrWi01]. In addition, the study of the distribution of LIS, in the critical case  $\alpha_{\text{critical}} = \frac{1}{2}$  is implicit in [Bi01, BoOl07]. In [Bi01], an alternative measure on Young diagrams is studied: **Schur-Weyl duality** implies that one has the decomposition

$$(\mathbb{C}^q)^{\otimes n} \cong \bigoplus_{\lambda} S^{\lambda} \otimes V_{\lambda},$$

where here  $S^{\lambda}$  is the  $S_n$  irreducible *Specht module* and  $V_{\lambda}$  is the  $GL_q(\mathbb{C})$  irreducible *Schur module*. Now taking dimensions one defines a probability measure that assigns to  $\lambda$  the likelihood  $(\dim S^{\lambda} \cdot \dim V_{\lambda})/q^n$ . Biane explicitly determines the rescaled limit typical shape in this context. Combinatorially, Biane’s measure arises from the RSK algorithm. Since we know that  $RSK(w)$  encodes the  $LIS(w)$  in the first column (by reading  $w$  backwards), one expects, by analogy with [LoSh77, VeKe77] that a certain rescaling of the first column of Biane’s limit shape is the  $\beta(k)$  of Conjecture 1.6. However, to justify this conclusion

rigorously one needs more work. Further, the fluctuations around Biane’s curve have been studied, in the  $k = 1$  case, by Borodin-Olshanski [BoOl07].

Hecke was originally developed in [BuKrShTaYo06] as a generalization of the *Edelman-Greene correspondence* which bijects Coxeter reduced words in the symmetric group to pairs of tableaux [EdGr87]. Our proof of Theorem 1.3 implies that this algorithm encodes the LIS of such words, although the study of LIS of reduced words appears unmotivated. On the other hand, the Coxeter-theoretic viewpoint on words will be useful in our analysis of LIS, LDS and  $\Lambda$ .

**1.6. Summary and organization.** In Section 2 we recall the Hecke algorithm and give an additional example of the results of Section 1.2. We then prove Proposition 1.2. In Section 3, we include two consequences of Theorem 1.3. We split our remaining proofs according to the main flavor of technique used: in Section 4 we explain the increasing tableau theory we need from [ThYo07] and prove Theorem 1.3. In Section 5, we utilize probabilistic-combinatorial techniques, combined with our main results, to prove Theorems 1.8 and 1.9.

## 2. THE HECKE ALGORITHM

**2.1. The 0-Hecke monoid.** We need to recall some notions used in [BuKrShTaYo06]. The **0-Hecke monoid**  $\mathcal{H}_{0,q}$  is the quotient of the free monoid of all finite words in the alphabet  $\{1, 2, \dots, q\}$  by the relations

$$\begin{aligned} (7) \quad & i i \equiv i && \text{for all } i \\ (8) \quad & i j i \equiv j i j && \text{for all } i, j \\ (9) \quad & i j \equiv j i && \text{for } |i - j| \geq 2. \end{aligned}$$

There is a bijection between  $\mathcal{H}_{0,q}$  and the symmetric group  $S_{q+1}$ . Given any word  $a \in \mathcal{H}_{0,q}$  there is a unique permutation  $\pi \in S_{q+1}$  such that  $a \equiv b$  for any reduced word  $b$  of  $\pi$ ; see, e.g., the textbook [BjBr05] for basic Coxeter theory for the symmetric group. In this case, we write  $W(a) = \pi$  and say that  $a$  is a **Hecke word** for  $\pi$ . Indeed, the reduced words for  $\pi$  are precisely the Hecke words for  $\pi$  that are of the minimum length  $\ell(\pi)$ , the **Coxeter length** of  $\pi$  (after identifying the label  $i$  with the simple reflection  $s_i = (i \ i + 1)$ ). Given an additional permutation  $\rho$  with Hecke word  $b$ , the **Hecke product** of  $\pi$  and  $\rho$  is defined as the permutation  $\pi \cdot \rho = W(ab)$ .

The (row reading) word of a tableau  $T$ , denoted  $\text{word}(T)$ , is obtained by reading the rows of the tableau from left to right, starting with the bottom row, followed by the row above it, etc. We also define  $W(T) := W(\text{word}(T))$ . So for example, if  $T$  is the increasing tableau of Figure 1, then  $\text{word}(T) = 5\ 3\ 4\ 1\ 3\ 4\ 5$ .

The Hecke algorithm defined in [BuKrShTaYo06] identifies pairs  $(w, i)$  of words

$$w = w_1 w_2 \cdots w_n, \quad i = i_1 i_2 \cdots i_n,$$

where  $w$  is a **Hecke word** and  $i$  satisfies

$$i_1 \leq i_2 \leq \dots \leq i_n \quad \text{and} \quad i_j < i_{j+1} \quad \text{whenever} \quad w_j \leq w_{j+1},$$

with pairs of tableaux  $(P, Q)$  of the same shape, where  $P$  is an increasing tableau such that  $\text{word}(P) \equiv w$  and the **content** (i.e., multiset of labels) of  $Q$  matches the content of  $i$ . We refer to the  $P$ -tableau as the **insertion tableau** and the  $Q$ -tableau as the **recording tableau**.

(We point out that the “column” convention in [BuKrShTaYo06] differs slightly from the “row” one used here.)

**2.2. Description of Hecke and Heckeshape.** The following description of Hecke was originally given in [BuKrShTaYo06]:

*Description of Hecke:* In this algorithm, one inserts an integer  $x$  into an increasing tableau  $T$ . We denote this by  $T \leftarrow x$ . The output is a triple  $(U, c, \alpha)$  where  $U$  is a modification of  $T$  (possibly  $T = U$ ),  $c$  is a corner of  $U$  and  $\alpha \in \{0, 1\}$  is a parameter. Initially, we attempt to insert  $x$  into the first row of  $T$ , and an output integer is possibly created which is inserted into the next row and so on, until no output integer is created. We refer to this final insertion as the **terminating step**, and the previous insertions as **bumping steps**.

Suppose  $R$  is a row that we are attempting to insert  $x$  into. If  $x$  is larger than or equal to all the entries of  $R$ , then no output integer is generated and the algorithm terminates: if adjoining  $x$  to the end of  $R$  results in an increasing tableau  $U$ , then set  $\alpha = 1$  and  $c$  to be the new corner added. Otherwise end with the present  $U$ , without modification;  $\alpha = 0$  and  $c$  is the corner that is at the end of the column containing the rightmost box of  $R$ . On the other hand, if  $R$  contains boxes strictly larger than  $x$ , let  $y$  be the smallest such box. If replacing  $y$  with  $x$  results in an increasing tableau, then do so. In either case,  $y$  is the output integer to be inserted into the next row.

Inserting a word  $w$  using this algorithm terminates with an increasing tableau

$$P = (((\emptyset \leftarrow w_1) \leftarrow w_2) \leftarrow \dots) \leftarrow w_n).$$

The  $Q$  tableau is obtained by placing each  $i_j$  in the  $c$ -corner resulting from the insertion of  $w_j$ .  $\square$

We also have the following reverse insertion algorithm  $\text{Hecke}^{-1}$ .

*Description of  $\text{Hecke}^{-1}$ :* Let  $Z$  be an increasing tableau,  $c$  a corner of  $Z$ , and  $\alpha \in \{0, 1\}$ . Reverse insertion applied to the triple  $(Z, c, \alpha)$  produces a pair  $(Y, x)$  of an increasing tableau  $Y$  and a positive integer  $x$  as follows. Let  $y$  be the integer in the cell  $c$  of  $Z$ . If  $\alpha = 1$ , remove  $y$ . In any case, reverse insert  $y$  into the row above the corner  $c$ .

Whenever a value  $y$  is reverse inserted into a row  $R$ , let  $x$  be the largest entry of  $R$  such that  $x < y$ . If replacing  $y$  with  $x$  results in an increasing tableau, then this is done. In any case, the integer  $x$  is passed up. If  $R$  is not the top row, this means that  $x$  is reverse inserted into the row above of  $R$ ; otherwise  $x$  becomes the final output value, along with the modified tableau.

We now complete the description of Hecke. Locate the bottom-most corner with the largest label, in the  $Q$  tableau, and remove the label. If it was the only entry in its corner, remove the corner, set  $\alpha = 1$ . Otherwise set  $\alpha = 0$ . Set  $c$  to be this corner. Then reverse insert  $(P, c, \alpha)$ . Repeat until all the entries of  $Q$  (and  $P$ ) have been removed.  $\square$

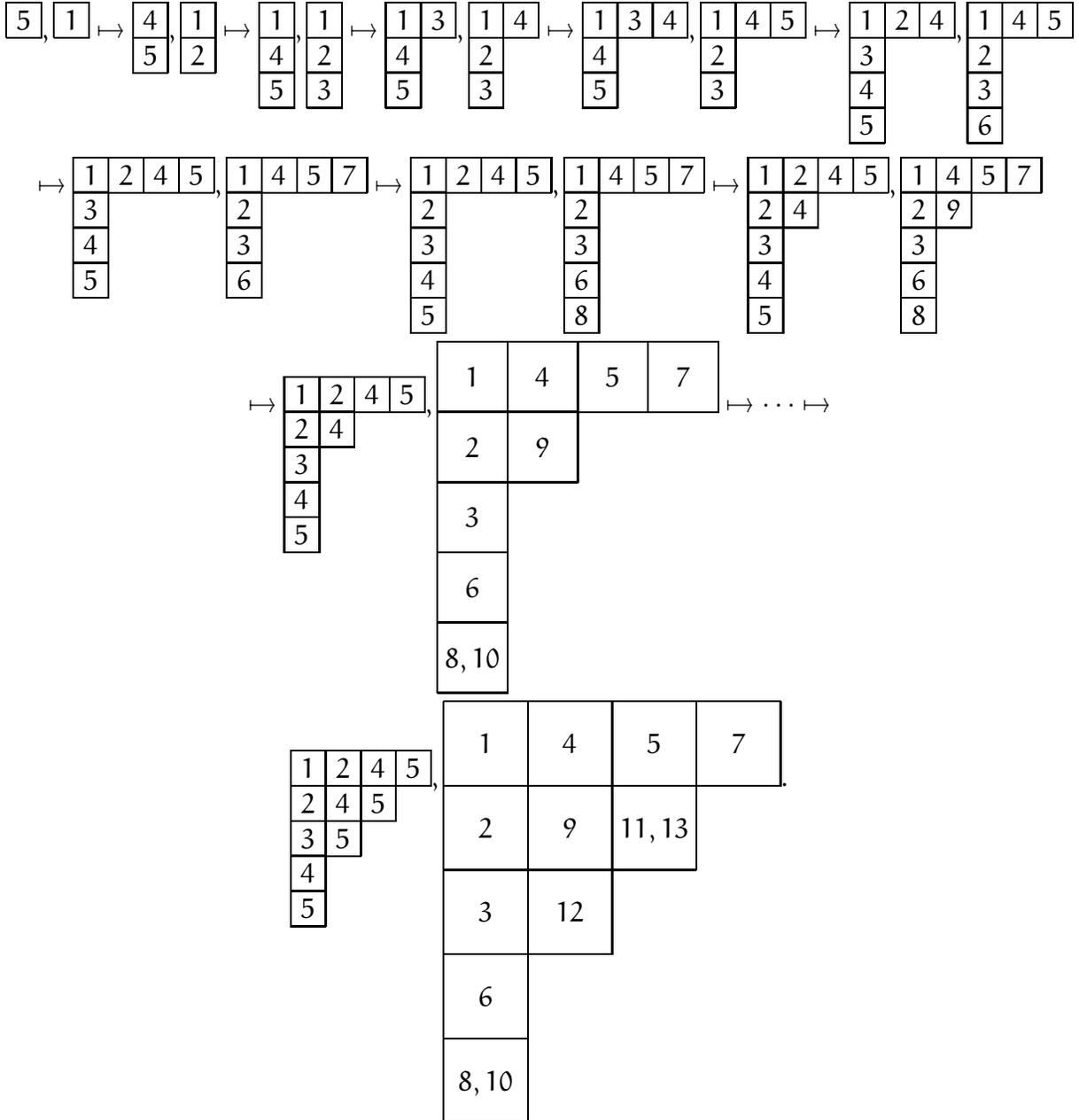
Hecke is a generalization of the Robinson-Schensted correspondence in the sense that it agrees with that correspondence whenever  $w$  is a permutation in  $S_n$ . In that case the  $P$  and  $Q$  tableaux are both standard Young tableaux.

In this paper, we are only concerned with the case  $i_j = j$ . Therefore, we also set  $\text{Hecke}(w) := \text{Hecke}(w, 123 \cdots n)$  and define

$$\text{Heckeshape} : W_{n,q} \rightarrow \mathbb{Y}$$

by setting  $\text{Heckeshape}(w)$  to be the common shape of  $P$  and  $Q$  under  $\text{Hecke}(w)$ . (An alternative description of this map is given in Theorem 4.2 in Section 4.)

*Example 2.1.* Let  $w = 5\ 4\ 1\ 3\ 4\ 2\ 5\ 1\ 2\ 1\ 4\ 2\ 4 \in W_{13,5}$ . Then the reader can check that  $\text{Hecke}$  produces the following steps:



Here  $\text{Heckeshape}(w) = (4, 3, 2, 1, 1)$  and indeed the length of the first row of this shape equals  $\text{LIS}(w) = 4$ , whereas the length of the first column equals  $\text{LDS}(w) = 5$ .

**2.3. Proof of Proposition 1.2.** The claim that  $\mu_{n,q}$  is a probability distribution follows if  $\text{Hecke}$  extends to provide a bijection between:

$$W_{n,q} \quad \text{and} \quad \Gamma_{n,q} := \bigcup_{\lambda} \text{INC}(\lambda, q) \times \text{SsetVT}(\lambda, n),$$

where  $\lambda$  satisfies (2).

Associate to each word  $w \in W_{n,q}$  the pair  $(w, 123 \cdots n)$ . Clearly Hecke injectively maps these pairs into  $\Gamma_{n,q}$ .

To prove surjectivity, let  $(P, Q) \in \Gamma_{n,q}$ . Then under  $\text{Hecke}^{-1}$ ,  $(P, Q)$  corresponds to some pair  $(w, i)$ . Now  $i = 123 \cdots n$  since that is the only possible sequence that can arise from a standard tableau  $Q$ . Also, since  $W(w) = W(\text{word}(P))$ ,  $w$  must use some subset of  $\{1, 2, \dots, q\}$ . Thus  $w \in W_{n,q}$ . Hence  $W_{n,q} \rightarrow \Gamma_{n,q}$ . The claim (2) is then clear from the properties of Hecke.

Finally, from the above discussion it is immediate that Heckeshape is a sampling algorithm for  $\mu_{n,q}$ . The bottleneck of the algorithm is the insertion process (a random uniform word  $w \in W_{n,q}$  can be generated in  $O(n \log(q))$  time). By (2) we know that each of the  $n$  insertions demand at most  $\binom{q+1}{2}$  operations. Hence  $O(nq^2)$  operations are needed. This completes the proof of Proposition 1.2.  $\square$

### 3. SOME FURTHER CONSEQUENCES OF THEOREM 1.3

**3.1. A generalization of the Erdős-Szekeres theorem.** The following classic result is due to Erdős-Szekeres [ErSz35]:

**Theorem 3.1.** *Let  $a, b \geq 1$ . If  $w \in S_{ab+1}$  then  $\text{LIS}(w) > a$  or  $\text{LDS}(w) > b$ .*

It is known that this result can be readily deduced from Schensted's results, see, e.g., [St06, Section 2]. Theorem 1.3 similarly leads to an extension of Theorem 3.1 that relates LIS and LDS to Coxeter length.

**Proposition 3.2.** *Let  $w \in W_{n,q}$ . Suppose  $1 \leq a, b < q$  and*

$$(10) \quad \ell(W(w)) > \sum_{i=1}^a \min(b, q - i), \quad \text{or equivalently} \quad \ell(W(w)) > \sum_{j=1}^b \min(a, q - j),$$

*then  $\text{LIS}(w) > a$  or  $\text{LDS}(w) > b$ ; recall  $W(w)$  is the permutation identified with  $w$ .*

*Proof.* If  $\text{LIS}(w) \leq a$  and  $\text{LDS}(w) \leq b$  then by Proposition 1.2 and Theorem 1.3, we have:

$$\text{Heckeshape}(w) \subseteq (a \times b) \cap (q, q - 1, \dots, 3, 2, 1).$$

Thus

$$|\text{Heckeshape}(w)| \leq \sum_{i=1}^a \min(b, q - i) = \sum_{j=1}^b \min(a, q - j).$$

Since  $\ell(W(w)) \leq |\text{Heckeshape}(w)|$ , the result then follows.  $\square$

*Example 3.3.* If  $q = 4$  and  $a = b = 3$  then if  $\ell(W(w)) > 8$  then  $\text{LIS}(w) > 3$  or  $\text{LDS}(w) > 3$ . This inequality is tight in the sense that the bound 8 cannot be reduced: consider  $w = 2\ 1\ 3\ 4\ 2\ 3\ 1\ 2$ . This is already a reduced word of Coxeter length 8, viewed as an element of  $S_5$ , and  $\text{LIS}(w) = \text{LDS}(w) = 3$ .

Proposition 3.2 generalizes Theorem 3.1 because if  $w \in S_{ab+1}$  is viewed as a Hecke word, we have  $\ell(w) = ab + 1$  (any word where all letters are distinct is automatically reduced). Then set  $q = ab + 1$  and thus (10) is satisfied.



*Proof.* The proof of (I) is an easy induction, comparing the description of Hecke with the “tied allowed” rules of patience sorting.

For (II), by (I) and Theorem 1.3 we know the “greedy strategy” terminates with  $\text{LIS}(w)$  piles. The proof of optimality is the same *mutatis mutandis* as the one for the original variant, see [AlDi99, Lemma 1].  $\square$

Briefly, probabilistic and statistical analysis on the “tied allowed” case of patience sorting is possible, in analogy to the work of [AlDi99]. Below we have tabulated the results of a Monte Carlo simulation with 100,000 trials on a standard 52-card deck. Typically, the number of piles is between 8 and 11. The average number of piles is 9.2 which, naturally, is less than the average number of piles when the deck is totally ordered, which is 11.6 as reported by Aldous-Diaconis. So a deck ordering is “lucky” if the number of piles is less than 7, which occurs only about 3% of the time.

number of piles	6	7	8	9	10	11	12	13
frequency	82	2993	20336	39039	27843	8489	1166	52

TABLE 4. Monte Carlo simulation for standard 52-card deck with 100,000 trials. The average number of piles is 9.2.

Figure 3 shows that there is a definite shape describing the mean pile sizes as  $\alpha$  varies. Questions about the structure of this shape may be interpreted as enriched questions related to LIS. One can analyze such questions using the dichotomy of Section 1.4; we do not pursue this here.

#### 4. INCREASING TABLEAU THEORY AND THE PROOF OF THEOREM 1.3

Now we show that the size of the first row of  $\lambda = \text{Heckeshape}(w)$  computes  $\text{LIS}(w)$ . Let  $r(w, t)$  be the largest index such that the longest strictly increasing subsequence ending at  $w_{r(w,t)}$  has length  $t$ . For example, if  $w = 2\ 3\ 4\ 1\ 5\ 2$  then  $r(w, 1) = 4$ ,  $r(w, 2) = 6$  and  $r(w, 3) = 3$ . We now in fact prove the following claim, which is stronger than the LIS assertion of Theorem 1.3 (cf. [St99, Prop. 7.23.10]):

**Proposition 4.1.** *Suppose  $w \in \mathcal{W}_{n,q}$ ,  $s = \text{LIS}(w)$  and  $P$  is the insertion tableau of Hecke( $w$ ). Then the first row of  $P$  is given by  $w_{r(w,1)}, w_{r(w,2)}, \dots, w_{r(w,s)}$ .*

*Proof.* By induction on  $n$ . The base case  $n = 1$  is trivial.

Suppose that the claim holds for  $w^\circ := w_1 w_2 \dots w_{n-1}$ . Thus if  $P^\circ$  is the insertion tableau of Hecke( $w^\circ$ ) then by induction, the first row is given by

$$w_{r(w^\circ,1)}^\circ, w_{r(w^\circ,2)}^\circ, \dots, w_{r(w^\circ,s^\circ)}^\circ,$$

where  $s^\circ = \text{LIS}(w^\circ)$ .

We consider the possibilities of what happens as we insert  $w_n$  into the first row of  $P^\circ$ . We prove the desired conclusion holds for the case that  $w_{r(w^\circ,t)}^\circ < w_n$  for some maximally chosen  $t < s^\circ$ ; other cases are similar. So after inserting  $w_n$ , the first row of  $P$  is

$$w_{r(w^\circ,1)}^\circ, w_{r(w^\circ,2)}^\circ, \dots, w_{r(w^\circ,t)}^\circ, w_n, w_{r(w^\circ,t+2)}^\circ, \dots, w_{r(w^\circ,s^\circ)}^\circ.$$

The assumption shows that the longest increasing subsequence  $\omega$  in  $w$  using  $w_n$  is of length at least  $t + 1$ , since we can adjoin  $w_n$  to the length  $t$  subsequence ending at  $w_{r(w^\circ,t)}^\circ$ .

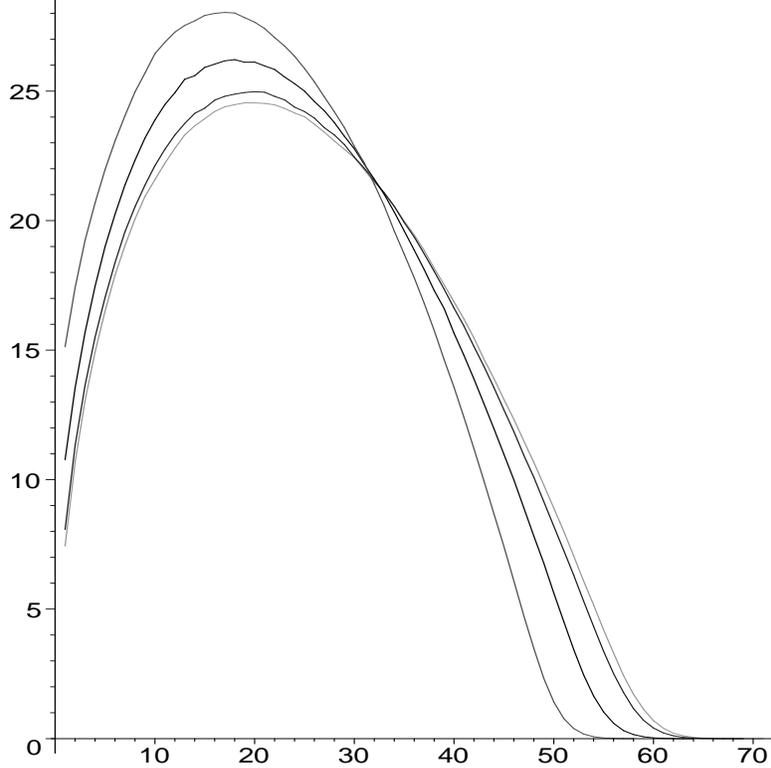


FIGURE 3. Simulation of mean pile sizes for  $n = 1000$  and  $q = 100, 200, 600, 1000$  with  $10^4$  Monte Carlo trials. The x-axis indicates the position of the pile, counting from the left.

On the other hand,  $w$  is of length at most  $t + 1$  since if it were, say, of length  $t + 2$ , there must be a length  $t + 1$  increasing subsequence in  $w^\circ$  ending at  $w_r$  with  $w_r < w_n$  and  $r < r(w^\circ, t + 1)$ . But this is a contradiction of the definition of  $r(w^\circ, t + 1)$  since we would then have a length  $t + 2$  increasing subsequence ending at  $w_{r(w^\circ, t+1)}^\circ (\geq w_n)$ .

Since we have just shown  $w_n = w_{r(w, t+1)}$ , it is now clear that  $w_{r(w, h)} = w_{r(w^\circ, h)}^\circ$  for  $h \neq t + 1$ . Thus, the first row of  $P$  satisfies the desired claim, and the induction follows.  $\square$

In order to prove that the first column of  $\lambda$  computes  $\text{LDS}(w)$ , we need to draw a connection to [ThYo07], where we developed a K-theoretic *jeu de taquin* theory. Rather than repeat the setup in full here, for brevity, we refer the reader to that paper for the complete background on K-rectification and K-infusion used below.

Although what follows also constitutes a proof of our LIS claims, we felt that including the direct proof via the stronger claim of Proposition 4.1 was worthwhile. However, a similarly direct proof of our LDS claim seems harder.

Let

$$\gamma_n = (n, n - 1, n - 2, \dots, 3, 2, 1)$$

be the staircase shape. Also, let  $\lambda_{\text{perm}(n)} = \gamma_n / \gamma_{n-1}$  be the **permutation shape** consisting of  $n$  single boxes arranged along an antidiagonal. Given  $w \in W_{n, q}$ , define  $T_w \in \text{INC}(\lambda_{\text{perm}(n)})$  to be the tableau where  $w_1, w_2, \dots, w_n$  is arranged from southwest to northeast. Also let  $S \in \text{SYT}(\gamma_{n-1})$  be the **superstandard** Young tableau, i.e., the one whose first row is labeled  $1, 2, \dots, n - 1$ , the second row is labeled by  $n, n + 1, n + 2, \dots, 2n - 3$  etc. This latter tableau

determines a *particular* K-rectification of  $T_w$ , which by definition is  $K\text{-infusion}_1(S, T_w)$ , see [ThYo07, Section 3]. (An important subtlety in K-theoretic jeu de taquin is that K-rectification depends on the order in which it is performed, unlike the rectification of classical jeu de taquin. However, the order defined by  $S$  is particularly nice.)

The following result is an analogue of a classical result linking the Robinson-Schensted algorithm to the (ordinary) rectification of  $T_w$ :

**Theorem 4.2.** *Let  $w \in W_{n,q}$ . Then  $K\text{-infusion}_1(S, T_w)$  is the insertion tableau of  $\text{Hecke}(w)$ .*

*Proof.* We induct on  $n$ . The base cases  $n = 1, 2$  are easy. We may assume that the steps of the  $K\text{-infusion}_1$  that are defined by the “inner” labels  $\underline{n}, \underline{n+1}, \underline{n+2}, \dots, \underline{\binom{n}{2}}$  results in a skew shape of the form  $P^\circ \star \boxed{w_n}$ , as depicted below:

$$(11) \quad \begin{array}{|c|c|c|c|c|} \hline \underline{1} & \underline{2} & \cdots & \underline{n-1} & w_n \\ \hline P_{i,1}^\circ & P_{i,2}^\circ & \cdots & P_{i,n-1}^\circ & \\ \hline P_{2,1}^\circ & \cdots & P_{2,n-2}^\circ & & \\ \hline \cdots & P_{n-2,2}^\circ & & & \\ \hline P_{n-1,1}^\circ & & & & \\ \hline \end{array}$$

The induction hypothesis is that  $P^\circ$  is the insertion tableau obtained by Hecke inserting  $w_1 w_2 \cdots w_{n-1}$ . (In the depiction of  $P^\circ$  from (11), some of the boxes with labels  $P_{i,j}^\circ$  may be empty.) The non-underlined labels occupy the boxes of  $P^\circ \star \boxed{w_n}$ , whereas the underlined labels dictate the remaining steps to perform to complete the  $K\text{-infusion}_1$  computation (these steps are recalled below).

Hence it remains to show that the tableau obtained by the Hecke-insertion  $P^\circ \leftarrow w_n$  is the same as carrying out the  $K\text{-infusion}_1$  indicated by (11), i.e., the operation

$$(12) \quad K\text{-infusion}_1 \left( \begin{array}{|c|c|c|c|} \hline \underline{1} & \underline{2} & \cdots & \underline{n-1} \\ \hline \end{array}, P^\circ \star \boxed{w_n} \right).$$

To do this, we first develop a technical fact. In [ThYo07, Section 1.1], we defined the procedure *switch*, which we restate now (in a more convenient form). Let  $\text{Mixedtab}(\alpha, p, q)$  be the set of **mixed tableaux**, which, by definition, are tableaux of shape  $\alpha$ , each of whose boxes is filled with an entry from one of two alphabets,  $\{\underline{1}, \dots, \underline{p}\}$  and  $\{1, \dots, q\}$ , such that, within each row or column, the entries for each alphabet appear at most once. (No increasingness condition is demanded.) We also include the **null tableau**  $\emptyset$ , as a special element of  $\text{Mixedtab}(\alpha, p, q)$ .

Define an operator

$$\text{switch}(\underline{i}, j) : \text{Mixedtab}(\alpha, p, q) \rightarrow \text{Mixedtab}(\alpha, p, q)$$

as follows. Given  $\emptyset \neq T \in \text{Mixedtab}(\alpha, p, q)$ , consider the subshape  $S$  of  $T$  consisting of boxes whose entry is either  $\underline{i}$  or  $j$ . For each non-singleton connected component of  $S$ ,

interchange the  $\underline{i}$ 's and the  $\underline{j}$ 's. If this results in a (non null) mixed tableau, then the result is that tableau. Otherwise the result is  $\emptyset$ . By definition  $\text{switch}(\underline{i}, \underline{j})(\emptyset) = \emptyset$ .

*Example 4.3.* Let  $\alpha = (4, 3, 1)$  and  $p = q = 3$ . Then  $T \in \text{Mixedtab}(\alpha, p, q)$  is given below, together with two different switch computations applied to it:

$$T \in \begin{array}{|c|c|c|c|} \hline \underline{2} & \underline{1} & \underline{3} & 1 \\ \hline \underline{1} & 3 & 1 & \\ \hline 2 & & & \\ \hline \end{array}, \quad \text{switch}(\underline{1}, \underline{2})(T) = \begin{array}{|c|c|c|c|} \hline \underline{2} & \underline{1} & \underline{3} & 1 \\ \hline 2 & 3 & 1 & \\ \hline \underline{1} & & & \\ \hline \end{array}, \quad \text{switch}(\underline{3}, \underline{1})(T) = \begin{array}{|c|c|c|c|} \hline \underline{2} & \underline{1} & 1 & \underline{3} \\ \hline \underline{1} & 3 & \underline{3} & \\ \hline 2 & & & \\ \hline \end{array}.$$

On the other hand,

$$\text{switch}(\underline{1}, \underline{2}) \left( \begin{array}{|c|c|} \hline \underline{1} & \underline{2} \\ \hline \underline{1} & \\ \hline \underline{2} & \\ \hline \end{array} \right) = \emptyset.$$

The following lemma is easy to verify from the definitions:

**Lemma 4.4.** *If  $i \neq j$  and  $r \neq s$  then the operators  $\text{switch}(\underline{i}, r)$  and  $\text{switch}(\underline{j}, s)$  commute, i.e.,*

$$\text{switch}(\underline{i}, r) \text{switch}(\underline{j}, s) \equiv \text{switch}(\underline{j}, s) \text{switch}(\underline{i}, r)$$

*is a relation in the algebra generated by switch operators on  $\text{Mixedtab}(\alpha, p, q)$ .*

The procedure described in [ThYo07, Section 3] for computing  $K\text{-infusion}_1(A, B)$  is to consider the entries of  $A$  as being underlined, where the maximum entry of  $A$  is  $\underline{p}$ , say, and the entries of  $B$  as not underlined, where the maximum entry of  $B$  is  $q$ . Now perform the following sequence of switch operations, from left to right, as indexed by:

$$(13) \quad (\underline{p}, 1), (\underline{p}, 2), \dots, (\underline{p}, q), (\underline{p-1}, 1), \dots, (\underline{p-1}, q), \dots, \dots, (\underline{1}, 1), \dots, (\underline{1}, q),$$

We refer to this sequence of pairs (interchangeably, the corresponding sequence of switch operators) as the **standard switch sequence**.

The technical fact we need is that  $K\text{-infusion}$  can in fact be computed differently: a switch sequence is called **viable** if it is a “shuffling” of (13), in the following sense:

- every  $(\underline{i}, \underline{j})$  occurs exactly once, for  $1 \leq i \leq p$  and  $1 \leq j \leq q$ ;
- for any  $1 \leq i \leq p$ , the pairs  $(\underline{i}, 1), \dots, (\underline{i}, q)$  occur in that relative order; and
- for any  $1 \leq j \leq q$  the pairs  $(\underline{p}, j), \dots, (\underline{1}, j)$  occur in that relative order.

This definition is explained by the proof of the following proposition:

**Proposition 4.5.** *Any viable switch sequence can be used to calculate  $K\text{-infusion}$ .*

*Proof.* It is straightforward to show that one can obtain any viable switch sequence from the standard switch sequence (13) by repeated applications of the commutation relation of Lemma 4.4.  $\square$

Thus, in view of Proposition 4.5, to complete the induction it suffices to construct a viable switch sequence whose result is the same as  $P^\circ \leftarrow w_n$ . (A caution: in [ThYo07] it was shown that the standard switch sequence necessarily maintains increasingness along rows and columns of the members of each alphabet. We will not prove that a viable switch sequence also achieves this during the intermediate steps of a  $K\text{-infusion}$ . However, this does not play a logical role in how we apply Proposition 4.5 below.)

Let  $y_1 := w_n$ , and for  $i > 1$ , let  $y_i$  be the number which is inserted in row  $i$  according to Hecke insertion of  $w_n$  into  $P^\circ$ . During a bumping step of Hecke insertion, let  $z_i$  be the smallest number already in row  $i$  which is greater than  $y_i$ .

We say that a mixed tableau, obtained after some number of switch operations applied to  $P^\circ \star \boxed{w_n}$ , is in **row  $i$  normal form** if:

- the  $i$ -th row is of the form

<u>1</u>	<u>2</u>	...	<u>k-1</u>	$y_i$	<u>k</u>	...	<u>t</u>
----------	----------	-----	------------	-------	----------	-----	----------

where  $y_i$  has not yet moved from its initial position in column  $k$ , or

<u>1</u>	<u>2</u>	...	<u>k-1</u>	<u>k</u>	<u>k+1</u>	...	<u>t</u>
----------	----------	-----	------------	----------	------------	-----	----------

depending, respectively, on whether the Hecke insertion  $P^\circ \leftarrow w_n$  terminates at row  $i$  or after, or strictly earlier. Here the  $i$ -th row of  $P^\circ$  has length  $t$ ; and

- all non-underlined symbols in rows  $i + 1$  and below have not moved from their initial positions.

Having explained our general strategy, what remains is some tedious but straightforward case analysis to describe the viable switch sequence we use: Our initial mixed tableau is in row 1 normal form. Now, suppose we have arrived, after some sequence of applications of (A), (B) and (C) below, at a mixed tableau in row  $i$  normal form. There are three possibilities for a bumping step of Hecke insertion of  $y_i$ .

(A)  $y_i$  is inserted, bumping  $z_i$ : Consider the switch sequence that moves  $y_i$  to the left along row  $i$ : specifically, using

$$(14) \quad \text{switch}(\underline{k-1}, y_i), \text{switch}(\underline{k-2}, y_i), \text{switch}(\underline{k-3}, y_i), \dots,$$

until it is directly above the  $z_i$  in row  $i + 1$ , and then, starting from the right, swaps each box in row  $i$  having an underlined label with the one directly below, which has a non-underlined label (this can always be done since no label numerically equal to  $y_i$  appears among the latter boxes, by assumption). The result is therefore in row  $i + 1$  normal form since  $z_i$  doesn't move in this process, it is the unique box with a non-underlined label in row  $i + 1$ , and  $y_{i+1} = z_i$ , as demanded.

Note that the non-underlined labels of the  $i$ -th row of the mixed tableau we obtain after this process is the same as the  $i$ -th row of  $P^\circ$ , with  $z_i$  replaced by  $y_i$ , as desired.

*Example 4.6.* If  $i = 1$  and we started with

<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	3
1	2	4	5	
2	3	5	6	

so that  $y_i = 3$  and  $z_i = 4$ , then we begin by moving  $y_i$  above  $z_i = y_{i+1}$ :

<u>1</u>	<u>2</u>	3	<u>3</u>	<u>4</u>
1	2	4	5	
2	3	5	6	

We conclude with  $\text{switch}(\underline{3}, 5)$ ,  $\text{switch}(\underline{2}, 2)$  and then finally  $\text{switch}(\underline{1}, 1)$ , resulting in

1	2	3	5	<u>4</u>
<u>1</u>	<u>2</u>	4	<u>3</u>	
2	3	5	6	

which is in row  $i + 1 (= 2)$  normal form. Moreover, the non-underlined labels in row  $i = 1$  of this mixed tableau, namely  $\boxed{1 \ 2 \ 3 \ 5}$  agrees with the first row of  $\boxed{1 \ 2 \ 4 \ 5} \leftarrow y_i$ , as desired.

(B)  $y_i$  is not inserted because of a horizontal violation: (That is, a label numerically equal to  $y_i$  already appears in the row that we are Hecke inserting into.) Proceed as in (A) by moving  $y_i$  to the left, until it is directly above  $z_i$ , i.e., apply (14). Now, “locally” the situation in the column containing  $z_i$  and the column to its left is:

$$\dots \begin{array}{|c|c|} \hline \underline{t} & y_i \\ \hline y_i & z_i \\ \hline \end{array} \dots \text{ or } \dots \begin{array}{|c|c|} \hline & \underline{t} \\ \hline \underline{t} & y_i \\ \hline y_i & z_i \\ \hline \end{array} \dots,$$

for some  $t$ . The former case shows rows  $i$  and  $i + 1$  when  $\underline{t}$  does not appear in row  $i - 1$ , while the latter case also includes row  $i - 1$  if it does. To the right of the column containing  $z_i$ , we swap boxes with underlined and non-underlined labels, as in (A). Then we perform the transformation

$$(15) \quad \dots \begin{array}{|c|c|} \hline \underline{t} & y_i \\ \hline y_i & z_i \\ \hline \end{array} \dots \mapsto \dots \begin{array}{|c|c|} \hline y_i & \underline{t} \\ \hline \underline{t} & z_i \\ \hline \end{array} \dots \text{ or } \dots \begin{array}{|c|c|} \hline & \underline{t} \\ \hline \underline{t} & y_i \\ \hline y_i & z_i \\ \hline \end{array} \dots \mapsto \dots \begin{array}{|c|c|} \hline & y_i \\ \hline y_i & \underline{t} \\ \hline \underline{t} & z_i \\ \hline \end{array} \dots$$

respectively. After this transformation, we complete by swapping, right to left, the boxes to the left of the  $y_i$ , as in (A). The result is in the demanded row  $i + 1$  normal form.

Note that unlike (A), row  $i$  still has a box  $c$  with an underlined label in it. However, when we work on the row  $i + 1$  normal form, the reader can check that our descriptions will force that box  $c$  to be filled by  $z_i$ : e.g., if we apply case (A) next, this will occur when we execute the switches (14), if we apply case (B) next, this will occur when we execute either the switches (14) or (15), whereas if we apply (C) next, the replacement will occur during the switches (16) below. Hence, in the end, we find that the non-underlined labels in row  $i$  are the same as the  $i$ -th row of the Hecke insertion of  $w_n$  into  $P^\circ$ , as desired.

*Example 4.7.* If  $i = 1$  and we started with

<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	2
1	2	3	4	
4				

then moving the “2” in row 1 to the left, as in (A), gives

<u>1</u>	<u>2</u>	2	<u>3</u>	<u>4</u>
1	2	3	4	
4				

The remaining swaps give

$$\begin{array}{|c|c|c|c|c|} \hline \underline{1} & \underline{2} & 2 & \underline{3} & \underline{4} \\ \hline 1 & 2 & 3 & 4 & \\ \hline 4 & & & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|c|} \hline \underline{1} & \underline{2} & 2 & 4 & \underline{4} \\ \hline 1 & 2 & 3 & \underline{3} & \\ \hline 4 & & & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|c|} \hline \underline{1} & 2 & \underline{2} & 4 & \underline{4} \\ \hline 1 & \underline{2} & 3 & \underline{3} & \\ \hline 4 & & & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \underline{2} & 4 & \underline{4} \\ \hline \underline{1} & \underline{2} & 3 & \underline{3} & \\ \hline 4 & & & & \\ \hline \end{array},$$

and the latter is in row  $i + 1 = 2$  normal form. The mild complication of this case, as suggested above, is that row 1 is not yet the same as the first row of the insertion tableau of Hecke.

However, as we begin to work on this row 2 normal form, we start by using  $\text{switch}(\underline{2}, 3)$  (beginning to move the 3 in row 2 to the left), giving:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & \underline{4} \\ \hline \underline{1} & 3 & \underline{2} & \underline{3} & \\ \hline 4 & & & & \\ \hline \end{array}$$

Hence row 1 now does agree with Hecke insertion.

(C)  $y_i$  will not be inserted because of a vertical violation (and a horizontal violation does not occur): While Hecke inserting  $y_i$  into row  $i$  of  $P^\circ$ ,  $z_i$  is directly below a label (numerically equal to)  $y_i$  that is in row  $i - 1$ . This prevents one from replacing  $z_i$  by the  $y_i$  we are inserting. (Note that this case can only occur if  $i > 1$ .) Moreover, since there is no horizontal violation, the number immediately to the left of  $z_i$ , say  $b$ , satisfies  $b < y_i$ . Notice, in order for a vertical violation to occur, the previous step of Hecke inserting  $y_{i-1}$  must have been an instance of case (B) or (C) (since we are Hecke inserting a number into row  $i$  which also appears in row  $i - 1$ ). Hence the box directly above and to the left of the  $y_i$  contains an underlined label  $\underline{t}$ .

Now we begin by switching all the boxes with underlined labels in row  $i$  to the right of  $y_i$ , with the non-underlined labels directly below them. The remaining switches, which involve the rows  $i - 1$  to  $i + 1$ , in the column of  $y_i$  and the column to its immediate left, are as given as follows:

$$(16) \quad \dots \begin{array}{|c|c|} \hline \underline{t} & \\ \hline \underline{t} & y_i \\ \hline b & z_i \\ \hline \end{array} \dots \mapsto \dots \begin{array}{|c|c|} \hline \underline{t} & \\ \hline b & y_i \\ \hline \underline{t} & z_i \\ \hline \end{array} \dots \mapsto \dots \begin{array}{|c|c|} \hline & y_i \\ \hline b & \underline{t} \\ \hline \underline{t} & z_i \\ \hline \end{array} \dots$$

As in (B), we finish by completing a sequence of swaps involving the columns to the left of the  $b$ . The result of this process is a tableau in row  $i + 1$  normal form.

As at the conclusion of (B), row  $i$  of the resulting row  $i + 1$  normal form mixed tableau, still contains an underlined label. As in that case, this label will be switched with  $z_i$ , as desired, during the forthcoming switches.

*Example 4.8.* To give an example of (C), the previous insertion must have been of type (B) or (C), so consider the following example:

$$\begin{array}{|c|c|c|c|} \hline \underline{1} & \underline{2} & \underline{3} & 3 \\ \hline 1 & 3 & 5 & \\ \hline 2 & 4 & 6 & \\ \hline \end{array}$$

After the first step, an insertion of type (B), we reach row 2 normal form:

1	3	<u>2</u>	<u>3</u>
<u>1</u>	<u>2</u>	5	
2	4	6	

As compelled by the conditions of a viable switch sequence, we switch 2 with 4 before we switch 2 with 5, and as a result, as described above, we get to row 3 normal form:

1	3	5	<u>3</u>
2	4	<u>2</u>	
<u>1</u>	<u>2</u>	6	

The final result is:

1	3	5	<u>3</u>
2	4	6	
6	<u>1</u>	<u>2</u>	

Again, this tableau agrees with  $P^\circ \leftarrow 3$ .

Note that after each of (A), (B) and (C), when we reach row  $i + 1$  normal form,  $z_{i+1}$  is necessarily in a column weakly to the left of  $y_{i+1}$ , because  $P^\circ$  is an increasing tableau and  $y_{i+1}$  together with the rows strictly below row  $i + 1$  are, at this point, still unaltered. This observation guarantees that the switches as described above can actually be executed, i.e., the above descriptions are well-defined.

Using similar analysis one can give switch sequences for the terminating steps of Hecke insertion, such that one maintains row  $i$  normal form for all  $i \leq \ell + 1$ , where  $\ell$  is the number of rows of  $P^\circ$ . We leave the straightforward details to the reader.

These constructions then show, by induction on the number of rows  $\ell$ , that we have a sequence of switch operations transforming  $P^\circ \star \boxed{w_n}$  into  $P^\circ \leftarrow w_n$ .

*Conclusion of the proof of Theorem 4.2:* by the fact that  $P^\circ$  is an increasing tableau, and by the definition of normal form, it is easy to see that the sequence of switch operations used forms a viable sequence, after suitable insertions of any trivial  $\text{switch}(\underline{i}, r)$  operators (that is to say, switch operations which do not have any effect on the tableau). We can therefore apply Proposition 4.5 as we claimed earlier.  $\square$

*Example 4.9.* Continuing Example 4.8, the switch sequence we obtain, by following the descriptions of the cases (B) and (C) that are needed is:

$$(\underline{3}, 2), (\underline{2}, 3), (\underline{2}, 4), (\underline{2}, 5), (\underline{2}, 6), (\underline{1}, 6).$$

This is not quite a viable sequence: although our constructions guarantee that it satisfies the second and third conditions to be a viable sequence, it fails the first, since, e.g.,  $(\underline{3}, 1)$  doesn't appear in the sequence, since this switch is never needed. However, clearly we can simply insert this trivial switch, along with the others that are missing, giving the viable sequence:

$$(\underline{3}, 1), (\underline{3}, 2), (\underline{3}, 3), (\underline{3}, 4), (\underline{3}, 5), (\underline{3}, 6), (\underline{2}, 1), (\underline{2}, 2), (\underline{2}, 3), (\underline{2}, 4), (\underline{2}, 5), (\underline{2}, 6),$$

$$(\underline{1}, 1), (\underline{1}, 2), (\underline{1}, 3), (\underline{1}, 4), (\underline{1}, 5), (\underline{1}, 6).$$

The action of this viable sequence on the original mixed tableau is therefore the same as the original switch sequence, which we highlight in boldface. This viable sequence also happens to be the standard switch sequence, although it needn't be in general. Hence

$$\text{K-infusion}_1 \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline 1 & 3 & 5 & \\ 2 & 4 & 6 & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array} \leftarrow 3,$$

in agreement with Theorem 4.2.

In [ThYo07, Theorem 6.1] we showed that the first row of  $\text{K-infusion}_1(R, T_w)$  has length  $\text{LIS}(w)$ , for any increasing tableau  $R$  of shape  $\gamma_{n-1}$ . So by Theorem 4.2, the first row of  $\text{Heckeshape}(w) = \text{K-infusion}_1(S, T_w)$  has length  $\text{LIS}(w)$ . By symmetry, [ThYo07, Theorem 6.1] also implies that the first column of  $\text{K-infusion}_1(R, T_w)$  has length  $\text{LDS}(w)$ . Hence the LDS claim follows. This completes the proof of Theorem 1.3.  $\square$

Given  $w = w_1 w_2 \cdots w_n$ , define  $\text{rev}(w) = w_n w_{n-1} \cdots w_1$ . The following is symmetry statement is immediate from Theorem 1.9, since  $\text{LIS}(w) = \text{LDS}(\text{rev}(w))$ :

**Corollary 4.10.** *Let  $\lambda = \text{Heckeshape}(w)$  and  $\mu = \text{Heckeshape}(\text{rev}(w))$ . Then  $\lambda_1 = \mu'_1$  and  $\mu_1 = \lambda'_1$  where  $\lambda'$  and  $\mu'$  are the conjugate shapes of  $\lambda$  and  $\mu$  respectively.*

A warning is needed: unlike Robinson-Schensted correspondence setting, with Hecke, one *cannot* conclude that the insertion tableaux associated to  $w = w_1 w_2 \cdots w_n$  and  $\text{rev}(w) = w_n w_{n-1} \cdots w_1$  differ only by a reflection across the main diagonal. A counterexample is  $w = 1\ 3\ 4\ 2\ 2$ . (In [Sc61], the symmetry property of the Robinson-Schensted correspondence, was applied to prove the LDS claim in the classical version of Theorem 1.3.)

**Problem 4.11.** *Give an explicit description of  $\text{Hecke}(\text{rev}(w))$  in terms of  $\text{Hecke}(w)$ .*

Finally, Greene [Gr74] has given an explanation of the other rows of the shape  $\lambda$  associated to a permutation  $w \in S_n$  under the Robinson-Schensted correspondence:  $\lambda_1 + \dots + \lambda_i$  equals the maximal size of a union of  $i$  disjoint increasing subsequences of  $w$ .

However, we could not find any extension of Greene's theorem in the Hecke context. The naive tries do not work: Since  $|\lambda| \leq n$ , the simplest case to analyze is when  $|\lambda| = n$ . The example  $w = 2\ 1\ 2\ 3\ 2$  corresponds to  $\lambda = (3, 2)$ ; this shows that it is not valid to merely replace "increasing" by "strictly increasing" in Greene's theorem, since that would predict  $\lambda = (3, 1, 1)$ .

## 5. PROBABILISTIC COMBINATORICS AND PROOFS OF THEOREMS 1.8 AND 1.9

*Proof of Theorem 1.8:* Let  $w_0 = \begin{pmatrix} 1 & 2 & 3 & \dots & q-1 & q & q+1 \\ q+1 & q & q-1 & \dots & 3 & 2 & 1 \end{pmatrix}$  be the word in  $S_{q+1}$  of maximal Coxeter length. Hence  $\ell(w_0) = \binom{q+1}{2}$ . This is the unique permutation in  $S_{q+1}$  with this length.

We need the following lemma, which characterizes when  $\text{Heckeshape}(w)$  is maximized.

**Lemma 5.1.** *For  $w \in W_{n,q}$ ,  $\text{Heckeshape}(w) = (q, q-1, \dots, 3, 2, 1)$  if and only if  $W(w) = w_0$ .*

*Proof.* First suppose  $W(w) = w_0$ . Under  $\text{Hecke}(w)$ , the insertion tableau  $P$  satisfies  $W(\text{word}(P)) = w_0$ . Hence the shape of  $P$  has at least  $\binom{q+1}{2}$  boxes and so by Proposition 1.2 it must be  $(q, q-1, \dots, 3, 2, 1)$ .

Conversely, if  $\text{Heckeshape}(w) = (q, q-1, \dots, 3, 2, 1)$ , then note that there is a unique increasing filling  $P$  of that shape (using  $1, 2, \dots, q$  in the first row,  $2, 3, \dots, q$  in the second row, etc). Then it is well-known that  $W(\text{word}(P)) = w_0$ .  $\square$

In view of the Lemma 5.1, the Theorem will follow if we can show that

$$(17) \quad W(w) = w_0 \text{ almost surely, as } n \rightarrow \infty.$$

(We conjecture this to be true whenever  $0 \leq \alpha < \alpha_{\text{critical}} = \frac{1}{2}$ . This would imply Conjecture 1.6 for this entire range.)

Set  $w(k) := w_1 \cdots w_k$ . Then either

$$\ell(W(w(k) w_{k+1})) = \ell(W(w(k))) + 1 \text{ or } \ell(W(w(k)))$$

depending on whether the simple reflection  $W(w_{k+1})$  (say equal to  $s_t = (t \ t+1)$ ) is an **ascent** of  $W(w(k))$  or not. (An **ascent** occurs at a position  $t$  for a permutation  $\pi$  if  $\pi(t) < \pi(t+1)$ .)

Provided that  $\pi \neq w_0$ ,  $\pi$  has at least one ascent. Thus when  $W(w_k) \neq w_0$ , the probability that  $w_{k+1}$ 's introduction increases the Coxeter length is at least  $\frac{1}{q}$ .

Let

$$E_k = \text{the event that } \ell(W(w(k))) < \binom{q+1}{2}.$$

Related to this, let  $Y_i \in \{0, 1\}$  be Bernoulli distributed with parameter  $\frac{1}{q}$ . Set

$$Z_k := Y_1 + \cdots + Y_k.$$

Clearly,

$$(18) \quad \text{Prob}(E_k) \leq \text{Prob}\left(Z_k < \binom{q+1}{2}\right).$$

We now show that when

$$k = O(q^{3+\epsilon}), \text{ for } 0 < \epsilon,$$

the righthand side of the inequality (18) goes to zero as  $q \rightarrow \infty$ .

This is a simple application of (a special case of) Bennet's large deviation inequality, see, e.g., [DeZe02, Cor. 2.4.7]: suppose  $X_i$  are independent, mean zero random variables with  $|X_i| \leq 1$ . Set  $S_k = \sum_{i=1}^k X_i$ . Then for  $y \geq 0$  we have

$$(19) \quad \text{Prob}(k^{-\frac{1}{2}} S_k \geq y) \leq e^{-\frac{y^2}{2}}.$$

To apply this to our setting, let  $X_i = -Y_i + \frac{1}{q}$ . Hence  $S_k = -Z_k + \frac{k}{q}$ . Then with  $a = o(q^2)$

$$\begin{aligned} \text{Prob}(Z_k < a) &= \text{Prob}\left(-S_k < a - \frac{k}{q}\right) = \text{Prob}\left(S_k \geq \frac{k}{q} - a\right) \\ &\leq \text{Prob}\left(S_k > \frac{k}{2q}\right) \text{ (for } q \text{ large, since } a = o(q^2)\text{)} \\ &= \text{Prob}\left(k^{-\frac{1}{2}} S_k \geq \frac{\sqrt{k}}{2q}\right) \leq e^{-\frac{k}{8q^2}} \rightarrow 0, \text{ as } q \rightarrow \infty. \end{aligned}$$

The result then follows. □

In the above argument, we interpreted  $w \in W_{n,q}$  as a random walk in  $S_{q+1}$  that begins at the identity and works its way up in the weak Bruhat order to  $w_0$ . At each step the probability of going up is *at least*  $\frac{1}{q}$  (as we have used), but is larger in general. However, since this probability varies, even for permutations with the same Coxeter length, a more refined analysis is needed to push the argument we have used further, up towards  $\alpha_{\text{critical}} = \frac{1}{2}$ .

*Proof of Theorem 1.9:* For  $0 \leq \alpha < \alpha_{\text{critical}} = \frac{1}{2}$  we will apply an argument similar to that for Theorem 1.8.

Given  $u \in W_{n,q}$  let

$$m(u) = \max_{t \geq 1} 1, 2, \dots, t \text{ is a subsequence of } u.$$

Let  $w(k) = w_1 \cdots w_k$  and set

$$E_k = \text{the event that } m(w(k)) < q.$$

Provided  $E_k$  occurs, then

$$m(w(k)w_{k+1}) = m(w(k)) + 1$$

with probability  $\frac{1}{q}$ , and is equal to  $m(w(k))$  otherwise.

Let  $\{Y_i\}$  and  $Z_k = Y_1 + \cdots + Y_k$  be discrete random variables, where  $Y_i$  is Bernoulli distributed with parameter  $\frac{1}{q}$ . Now,

$$\text{Prob}(E_k) = \text{Prob}(Z_k < q).$$

Thus it will be enough to show that when  $k = O(q^{2+\epsilon})$  for  $\epsilon > 0$  then

$$\text{Prob}(Z_k < O(q^{1+\epsilon})) \rightarrow 0 \text{ as } q \rightarrow \infty.$$

This is another application of the large deviation inequality (19).

For  $\alpha_{\text{critical}} = \frac{1}{2} < \alpha \leq 1$  we use a proof provided for us by O. Zeitouni:  $E(\text{LwIS})$ , the expected length of the longest weakly increasing subsequence of  $w \in W_{n,q}$  (with  $\alpha > \alpha_{\text{critical}} = \frac{1}{2}$ ) is known to satisfy

$$E(\text{LwIS}) \approx 2\sqrt{n};$$

see [Joh01, Theorem 1.7]. The argument shows that the difference between the LIS and LwIS of  $w$  is typically small.

Let  $\text{LwIS}_{a,b}$  be the random variable for the value of  $\text{LwIS}(w)$  of a random uniform word  $w \in W_{[a],[b]}$  where  $[a]$  is the integer part of  $a$ , etc. Similarly define  $\text{LIS}_{a,b}$  where LIS replaces LwIS.

Fix  $\epsilon > 0$  and let  $L_0 = L_0(\epsilon)$  be large enough such that

$$(20) \quad \inf_{L > L_0} \lim_{n \rightarrow \infty} \text{Prob} \left( \text{LwIS}_{L^2(1-\epsilon), \frac{qL}{\sqrt{n}}} > 2(1-4\epsilon)L \right) > 1 - \epsilon.$$

We need a ‘‘graphical’’ representation of a word in  $W_{n,q}$ : consider a  $q \times n$  rectangle subdivided into unit squares. In each of the  $n$  columns, one places a single ‘‘dot’’ in one of the  $q$  rows. The set of such configurations is in obvious bijection with words in  $W_{n,q}$ .

Given  $L$ , draw  $\sqrt{n}/L$  smaller rectangles of dimension  $\frac{qL}{\sqrt{n}} \times L\sqrt{n}$  along an antidiagonal inside the  $q \times n$  rectangle, as depicted in Figure 4 below.

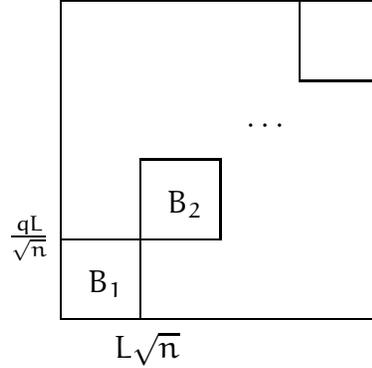


FIGURE 4.  $\alpha > \alpha_{\text{critical}} = \frac{1}{2}$  case of proof of Theorem 1.9

Label the  $i$ -th southwest most box  $B_i$ . Let  $N_i$  be the random variable giving the number of dots in  $B_i$ . Notice that the  $N_i$ 's are independent. Also, the dots inside  $B_i$  define a word, and we can speak of  $\text{LIS}(B_i)$  and  $\text{LwIS}(B_i)$ , the length of the longest strictly (respectively, weakly) increasing subsequence of that word.

Say that  $B_i$  is **good** if the following conditions simultaneously hold:

- (a)  $N_i \geq L^2(1 - \epsilon)$ ;
- (b)  $\text{LwIS}(B_i) \geq 2(1 - 4\epsilon)L$ ; and
- (c) no two dots in  $B_i$  have the same height (hence  $\text{LwIS}(B_i) = \text{LIS}(B_i)$ ).

Now, we have

$$E(N_i) = L\sqrt{n} \times \left( \frac{qL}{\sqrt{n}} \times \frac{1}{q} \right) = L^2$$

and we claim for an  $L_1$  sufficiently large, for  $L \geq L_1$ , and for all  $n$  large, we have:

$$\text{Prob}(N_i \leq L^2(1 - \epsilon)) \leq \epsilon.$$

The proof is a standard argument: let  $Y_k$  be the indicator random variable which evaluates to 1 if column  $k$  has a dot that lies in the box  $B_i$  that occupies that column (hence with probability  $L/\sqrt{n}$ ), and evaluates to 0 otherwise. Hence

$$\begin{aligned} \text{Prob}(N_i \leq L^2(1 - \epsilon)) &= \text{Prob}\left(\sum_{k=1}^{L\sqrt{n}} Y_k \leq L^2(1 - \epsilon)\right) \\ &= \text{Prob}\left(\sum_{k=1}^{L\sqrt{n}} (Y_k - E(Y_k)) < L^2(1 - \epsilon) - L^2\right) \\ &= \text{Prob}\left(\sum_{k=1}^{L\sqrt{n}} (Y_k - E(Y_k)) < -L^2\epsilon\right) \\ &\leq \frac{L\sqrt{n} \cdot (L/\sqrt{n})}{L^4\epsilon^2} = \frac{1}{L^2\epsilon^2}, \end{aligned}$$

where the previous line is an application of Chebyshev's inequality. Now take  $L$  sufficiently large (bigger than some  $L_1$ ) so  $\frac{1}{L^2\epsilon^2} \leq \epsilon$ .

Assuming  $L \geq L_0$ , if the event (a) occurs, then with probability at least  $1 - \epsilon$ , when  $n$  is large, (b) holds, because of the definition of  $L_0$  using (20).

The probability of the event (c) not occurring is bounded above (using a union bound) by

$$\begin{aligned} \#\text{rows} \times \text{Prob}(\text{two dots share a given height}) &\leq \frac{qL}{\sqrt{n}} \times (L\sqrt{n})^2 \times (1/q)^2 \\ &= L^3 \frac{\sqrt{n}}{q} \rightarrow 0, \end{aligned}$$

because  $n = o(q^2)$ .

So,

$$\text{Prob}(B_i \text{ is good}) \geq 1 - 3\epsilon$$

for  $L > \max(L_0, L_1)$  and  $n$  large.

Another standard argument with Chebyshev's inequality shows with high probability, say at least  $1 - \epsilon$ , and for  $n$  large, the number of good boxes is

$$\frac{\sqrt{n}}{L}(1 - \epsilon)(1 - 3\epsilon) \geq \frac{\sqrt{n}}{L}(1 - 4\epsilon).$$

Hence, with that probability, for  $w \in W_{n,q}$

$$\text{LIS}_{n,q} \geq \frac{\sqrt{n}}{L}(1 - 4\epsilon) \times 2(1 - 4\epsilon)L \geq 2\sqrt{n}(1 - 8\epsilon).$$

Since

$$2\sqrt{n}(1 - 8\epsilon) \leq E(\text{LIS}_{n,q}) \leq E(Lw\text{IS}_{n,q}) \approx 2\sqrt{n}$$

the  $\alpha > \alpha_{\text{critical}} = \frac{1}{2}$  case follows by taking  $\epsilon \rightarrow 0$ , completing the proof of the theorem.  $\square$

## 6. APPENDIX (BY A. YONG AND O. ZEITOUNI)

The goal of this appendix is to present a proof of the following result:

**Theorem 6.1.** *Let  $q = k\sqrt{n} + \text{lower order terms}$ . Then*

$$E(\text{LIS}) \approx \beta(k)2\sqrt{n}$$

where

$$\beta(k) = \begin{cases} \frac{k}{2} & \text{if } 0 < k \leq 1 \\ \frac{2-k}{2} & \text{if } k > 1. \end{cases}$$

However, in order to prove this statement, we need to work with another variant of Plancherel measure, utilized, e.g., by [Bi01] and alluded to in Section 1.5 of the main text. Our approach parallels the one developed in [LoSh77, VeKe77] to prove  $E(\text{LIS}) = 2\sqrt{n}$  in the permutation case, by utilizing work of [Bi01].

**6.1. Preliminaries.** A **semistandard Young tableau** of shape  $\lambda \in \mathbb{Y}$  with labels from  $\{1, 2, \dots, q\}$  is a filling of the Young shape  $\lambda$  with these labels so that the entries weakly increase along rows, and strictly increase along columns. For example, if  $\lambda = (2, 1)$  and  $q = 2$  there are two such tableaux:  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$  and  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$ . Let  $g^\lambda(q)$  denote the number of such tableaux.

Define the **Plancherel-RSK measure**  $\nu_{n,q}$  on the set  $\mathbb{Y}_n$  of Young diagrams  $\lambda$  with  $n$  boxes, by declaring that a random Young shape  $\lambda_{n,q}$  occurs with probability

$$\text{Prob}(\lambda_{n,q} = \lambda) = \frac{1}{q^n} f^\lambda g^\lambda(q).$$

We make no claims of originality in this definition. Indeed, this is the same measure studied in, e.g., [Bi01]; although there the measure is defined in terms of dimensions of irreducible  $S_n$  and  $GL_n(\mathbb{C})$  modules associated to  $\lambda$ ; the equivalence is well-known. The fact that  $\nu_{n,q}$  is in fact a probability distribution follows from either Schur-Weyl duality, as in Section 1.5, or by the RSK algorithm, see, e.g., [St99, Section 7.11].

A crucial advantage of  $\nu_{n,q}$  for the purposes of understanding  $E(\text{LIS})$ , in comparison to Plancherel-Hecke measure, is that both  $f^\lambda$  and  $g^\lambda(q)$  have simple multiplicative formulas. This makes it more readily analyzed using ideas of [LoSh77, VeKe77], which we modify to the present setting.

Given a box  $u \in \lambda$ , define the **hook-length** associated to  $u$  to be  $H(u) := A(u) + L(u) + 1$  where  $A(u)$  is the number of boxes strictly to the right of  $u$ , and in the same row, and  $L(u)$  is the number of cells strictly below  $u$  and in the same column. Then we have [St99, Chapter 7], the **hook-length formula** and **hook-content formulas**, respectively:

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} H(u)} \quad \text{and} \quad g^\lambda(q) = \prod_{u \in \lambda} \frac{q + C(u)}{H(u)},$$

where in the second formula  $C(u)$  is the **content** of  $u$ , the column index of  $u$  minus the row index of  $u$ . So for example, if  $\lambda = (4, 3, 2)$ , the contents are given by

0	1	2	3
-1	0	1	
-2	-1		

**6.2. Plancherel-RSK as a Markov measure.** **Young's lattice** is the poset structure on  $\mathbb{Y}$  where  $\lambda \leq \mu$  if the shape of  $\lambda$  is contained in the shape of  $\mu$ . We write  $\lambda \rightarrow \mu$  to denote a covering relation in this poset, i.e., where  $\mu$  is obtained from  $\lambda$  by adding a single box at a corner.

Define a Markov process on  $\mathbb{Y}$  with the transition probabilities

$$\text{Prob}(\lambda \rightarrow \mu) = \frac{g_\mu(q)}{q g_\lambda(q)}.$$

We need the following lemma, that in particular shows that Plancherel-RSK measure is a Markov measure with the above transition probabilities.

- Lemma 6.2.** (I)  $\sum_{\lambda: \lambda \rightarrow \mu} \text{Prob}(\lambda \rightarrow \mu) = 1$   
 (II)  $\sum_{\lambda: \lambda \rightarrow \mu} \text{Prob}(\lambda \rightarrow \mu) \nu_{n,q}(\lambda) = \nu_{n+1,q}(\mu)$

*Proof.* The claim (I) is equivalent to

$$\sum_{\lambda: \lambda \rightarrow \mu} g^\mu(q) = q g^\lambda(q).$$

This follows from the following *Pieri rule* for Schur polynomials

$$\sum_{\Lambda: \lambda \rightarrow \mu} s_{\mu}(x_1, \dots, x_q) = s_{(1)}(x_1, \dots, x_q) \cdot s_{\lambda}(x_1, \dots, x_q).$$

See [St99, Theorem 7.15.7]. Here

$$s_{\lambda}(x_1, \dots, x_q) = \sum_{T} \mathbf{x}^T$$

is the **Schur polynomial**, where the sum is over all semistandard Young tableaux of shape  $\lambda$  with entries from  $\{1, 2, \dots, q\}$ ,  $\mathbf{x}^T = x_1^{i_1} x_2^{i_2} \dots x_q^{i_q}$ , and  $i_j$  is the number  $j$ 's used in  $T$ . In particular, (I) is immediate from  $g^{\lambda}(q) = s_{\lambda}(1, 1, \dots, 1)$ .

For (II), the claim is

$$\sum_{\lambda: \lambda \rightarrow \mu} \left( \frac{g^{\mu}}{q g^{\lambda}} \right) \left( \frac{f^{\lambda} g^{\lambda}}{q^n} \right) = \frac{f^{\mu} g^{\mu}}{q^{n+1}},$$

that is,  $\sum_{\lambda: \lambda \rightarrow \mu} f^{\lambda} = f^{\mu}$ , which is well-known (and straightforward from the definitions).  $\square$

**6.3. Conclusion of Proof of Theorem 6.1.** Work of Biane [Bi01, Theorem 3] describes the typical shape under Plancherel-RSK after the rescaling  $f \mapsto \frac{1}{2\sqrt{n}} f(2\sqrt{n})$ . Biane's theorem implies that

$$(21) \quad E(\text{LIS}) \geq \beta(k)$$

but not

$$(22) \quad E(\text{LIS}) \leq \beta(k).$$

Briefly, we explicate how his work applies to our situation (the reader is directed to the original source for details): Biane works with the coordinate axes rotated 45-degrees counterclockwise, as in, e.g., [VeKe77, VeKe85]. His aforementioned theorem states that if  $\{f_n\}_{n=1}^{\infty}$  is any sequence of (rescaled and rotated) Young diagrams, then for any  $\epsilon > 0$  we have

$$(23) \quad \lim_{n \rightarrow \infty} \text{Prob} \left( \sup_{u \in \mathbb{R}} |f_n(u) - P_{\frac{1}{k}}(u)| > \epsilon \right) \rightarrow 0,$$

where the probability is computed with respect to  $\nu_{n,q}$ . Also,  $P_{\frac{1}{k}}$  is Biane's limit shape, which has the property that it meets the line  $y = x$  at a distance  $\beta(k)$  from the origin. In other words, the "first column"  $C$  of  $P_{\frac{1}{k}}$  satisfies

$$(24) \quad C = \beta(k).$$

For each  $n$ , let  $C_n$  be the length of the first column of  $f_n$ . From (23) it follows that for any  $\epsilon > 0$ ,

$$(25) \quad \lim_{n \rightarrow \infty} \text{Prob}(C_n < C - \epsilon) \rightarrow 0.$$

Moreover, since it is known that for any  $w \in W_{n,q}$  the first column of the Young diagram associated to  $\text{RSK}(w)$  equals  $\text{LDS}(w)$ , (21) follows immediately from (24) and (25) combined.

Note that the above argument does not also prove (22) since (23) does not rule out the possibility that  $\{f_n\}_{n=1}^\infty$  consists of Young diagrams with “tails” along the  $y = x$  axis that both “lengthen” and “thin out” as  $n \rightarrow \infty$ . Therefore, it remains to verify (22).

To do this, we modify an argument found in [VeKe85], which establishes the analogous assertion in the permutation case: consider the set  $\mathbb{Y}^\infty$  of all sequences of Young diagrams

$$\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \dots, \lambda^{(i)}, \dots)$$

where  $\lambda^{(i)} \rightarrow \lambda^{(i+1)}$  for  $i \geq 1$ .

For a Young diagram  $\lambda$ , let  $\lambda^\downarrow$  denote the diagram obtained by adding a single box to  $\lambda$ , in the first column. For each integer  $i \geq 1$ , define the indicator function  $\psi_i : \mathbb{Y}^\infty \rightarrow \{0, 1\}$  by setting  $\psi_i(\lambda) = 1$  if  $\lambda^{(i)} = (\lambda^{(i-1)})^\downarrow$ , and setting  $\psi_i(\lambda) = 0$  otherwise.

Studying the expectation of  $\psi_i$  we have:

$$\begin{aligned} E(\psi_i)^2 &= \left( \sum_{\lambda} \nu_{i-1,q}(\lambda) \cdot \text{Prob}(\lambda \rightarrow \lambda^\downarrow) \right)^2 \\ &\leq \sum_{\lambda} \nu_{i-1,q}(\lambda) \cdot \text{Prob}(\lambda \rightarrow \lambda^\downarrow)^2 \quad (\text{Cauchy-Schwarz inequality}) \\ &= \sum_{\lambda} \nu_{i-1,q}(\lambda) \frac{f^{\lambda^\downarrow} g^{\lambda^\downarrow}(q)}{f^\lambda g^\lambda(q)} \cdot \frac{1}{q} \cdot \frac{1}{q} \cdot \frac{g^{\lambda^\downarrow}(q)/g^\lambda(q)}{f^{\lambda^\downarrow}/f^\lambda} \\ &= \frac{1}{q} \sum_{\lambda} \nu_{i,q}(\lambda^\downarrow) \frac{g^{\lambda^\downarrow}(q)/f^{\lambda^\downarrow}}{g^\lambda(q)/f^\lambda}, \end{aligned}$$

where we have just used

$$\nu_{i,q}(\lambda^\downarrow) = \nu_{i-1,q}(\lambda) \frac{f^{\lambda^\downarrow} g^{\lambda^\downarrow}(q)}{f^\lambda g^\lambda(q)} \cdot \frac{1}{q}.$$

Let

$$L(\lambda) := g^\lambda(q)/f^\lambda.$$

Note that by the hook-content formula we have

$$L(\lambda^\downarrow)/L(\lambda) = \frac{\prod_{u \in \lambda^\downarrow} \frac{q+c(u)}{i!}}{\prod_{u \in \lambda} \frac{q+c(u)}{(i-1)!}} = \frac{q - \lambda'_1}{i}$$

where  $\lambda'_1$  is the length of the first column of  $\lambda$ .

Summarizing, we have

$$(26) \quad E(\psi_i)^2 \leq \frac{1}{q} \sum_{\lambda} \nu_{i,q}(\lambda^\downarrow) \frac{q - \lambda'_1}{i} = \frac{1}{qi} (q - \gamma_i),$$

where  $\gamma_i$  denotes the expectation of  $(\lambda_1^{(i)})'$ , i.e., the expected length of the first column of a random shape with  $i$  boxes, drawn under the Plancherel-RSK measure.

Notice also that since  $\psi_i$  is an indicator random variable, we have

$$(27) \quad E(\psi_i^2) = E(\psi_i) = \gamma_i - \gamma_{i-1}.$$

Therefore, combining (26) and (27) we obtain, by the Cauchy-Schwarz inequality, the following difference inequality:

$$(28) \quad \gamma_i - \gamma_{i-1} \leq \sqrt{\frac{1}{qi}} \sqrt{q - \gamma_i}.$$

We claim that  $\gamma_i \leq \beta(i)2\sqrt{n}$ .

To prove this, note the following facts about  $\gamma_i$ :

- (a)  $\gamma_{i+1} \geq \gamma_i$ ; and
- (b)  $\gamma_i \leq q$ .

Now define a linear interpolation: for  $t \in [i/q, (i+1)/q]$ , set

$$(29) \quad \beta_t = \frac{\gamma_i}{q} + q \left( t - \frac{i}{q} \right) \left( \frac{\gamma_{i+1}}{q} - \frac{\gamma_i}{q} \right).$$

Note that for such  $t$ ,  $\sqrt{1 - \gamma_i/q} \leq \sqrt{1 - \beta_t}$ . Therefore, combining equations (28) and (29) we obtain

$$(30) \quad \frac{d}{dt} \beta_t = \gamma_{i+1} - \gamma_i \leq \frac{1}{\sqrt{qt}} \sqrt{1 - \beta_t}, \quad \beta_{1/q} = 1/q.$$

Since  $\gamma_i, \gamma_{i+1} \leq q$  we have  $\beta_t \leq 1$ , hence the above differential inequality is equivalent to

$$-\frac{d}{dt} \sqrt{1 - \beta_t} \leq 1/(2\sqrt{qt}).$$

Hence it follows that

$$\sqrt{1 - \beta_t} \geq \sqrt{1 - 1/q} - \sqrt{t/q} + 1/q \geq \sqrt{1 - 1/q} - \sqrt{t/q}.$$

That is,

$$\beta_t \leq 2\sqrt{t/q} - (t - 1)/q.$$

Now we care about  $t = n/q = \sqrt{n}/k$ . We always have  $\beta_{n/q} \leq 1$  (trivially), but for  $k > 1$  we have the better inequality  $\beta_{n/q} \leq 2/k - 1/k^2 + 1/q$ . Therefore,

$$\gamma_n = q\beta_{n/q} \leq q,$$

for  $k \leq 1$  and

$$\gamma_n = q\beta_{n/q} \leq (2 - 1/k)\sqrt{n} + 1,$$

for  $k > 1$ . The result then follows. □

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