LATTICE STRUCTURE OF TORSION CLASSES FOR PATH ALGEBRAS

OSAMU IYAMA, IDUN REITEN, HUGH THOMAS, GORDANA TODOROV

Abstract. We consider module categories of path algebras of connected acyclic quivers. It is shown in this paper that the set of functorially finite torsion classes form a lattice if and only if the quiver is either Dynkin quiver of type A, D, E, or the quiver has exactly two vertices.

0. Introduction

Let Λ be a finite dimensional algebra over an algebraically closed field k, and $\mathsf{mod}\,\Lambda$ the category of finite dimensional Λ-modules. In this setup a subcategory $\mathcal{T}$ is a *torsion class* if it is closed under factor modules, isomorphisms and extensions. The set $\operatorname{tors}\Lambda$ of torsion classes is a partially ordered set by inclusion, and it is easy to see that it is always a lattice (see Definition 1.2). There is however an important subset $\text{f-tors}\Lambda$ of $\operatorname{tors}\Lambda$, where $\text{f-tors}\Lambda$ denotes the set of torsion classes which are functorially finite in $\mathsf{mod}\,\Lambda$. In this setting a torsion class is *functorially finite* precisely when it is of the form $\mathsf{Fac}X$ for some X in $\mathsf{mod}\,\Lambda$ [AS]. The set $\text{f-tors}\Lambda$ is of special interest since the elements are in bijection with the support τ-tilting modules (see Definition 1.6), which were introduced in [AIR]. This bijection also induces a structure of partially ordered set on the support τ-tilting modules. A related partial order has been studied in classical tilting theory by many authors (e.g. [RS, HU, AI, K]). There is also a connection with the weak order on finite Coxeter groups [M].

The aim of this paper is to study the following questions.

Question 0.1. *Let Λ be a finite dimensional k-algebra.*

(a) *When is* $\text{f-tors}\Lambda$ *a complete lattice?*
(b) *When is* $\text{f-tors}\Lambda$ *a lattice?*

A simple answer to Question 0.1(a) is given in terms of the τ-rigid finiteness (see Definition 1.8 for details):

Theorem 0.2. *Let Λ be a finite dimensional k-algebra. Then the following conditions are equivalent.*

(a) $\text{f-tors}\Lambda$ *forms a complete lattice.*
(b) $\text{f-tors}\Lambda$ *forms a complete join-semilattice.*
(c) $\text{f-tors}\Lambda$ *forms a complete meet-semilattice.*
(d) $\text{f-tors}\Lambda = \operatorname{tors}\Lambda$ *holds (i.e. any torsion class in* $\mathsf{mod}\,\Lambda$ *is functorially finite).*
(e) Λ *is τ-rigid finite.*

On the other hand, Question 0.1(b) for an arbitrary algebra Λ above does not seem to have a simple answer. Hence we are mainly concerned with $\text{f-tors}(kQ)$ where kQ is the path algebra of a finite connected acyclic quiver Q. Our main theorem is the following.

Theorem 0.3. *Let Q be a finite connected quiver with no oriented cycles. Then the following conditions are equivalent.*

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(a) f-tors(kQ) *forms a lattice.*
(b) f-tors(kQ) *forms a join-semilattice (see Definition 1.2).*
(c) f-tors(kQ) *forms a meet-semilattice.*
(d) Q *is either a Dynkin quiver or has at most 2 vertices.*

We remark that condition (d) is equivalent to the property that all the rigid indecomposable kQ-modules are preprojective or preinjective.

We also shows the following result.

Theorem 0.4. *Let Λ be a concealed canonical algebra (in particular a canonical algebra) or a tubular algebra. Then the following conditions are equivalent.*
(a) f-torsΛ *forms a lattice.*
(b) f-torsΛ *forms a join-semilattice.*
(c) f-torsΛ *forms a meet-semilattice.*
(d) Λ *has at most 2 simple modules up to isomorphism.*

The paper is organized as follows. In section 1 we give a proof of Theorem 0.2 and give an important criterion for deciding if f-torsΛ is a lattice, together with some preliminary results. In subsection 2.1 we show our sufficient conditions for f-tors(kQ) to be a lattice. In subsection 2.2 we show that f-tors(kQ) is not a lattice for a path algebra kQ of an extended Dynkin quiver Q with at least 3 vertices. In subsection 2.3 we deal with a path algebra kQ of a wild quiver Q with 3 vertices, and show that f-tors(kQ) is not a lattice. In subsection 2.4 we put things together to prove Theorem 0.3. In subsection 2.5 we prove Theorem 0.4.

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1. Lattice structure of torsion classes for finite dimensional algebras

1.1. **General results.** Let Λ be a finite dimensional k-algebra. A full subcategory $\mathcal{F}$ of $\mathsf{mod}\Lambda$ is a *torsionfree class* if it is closed under submodules, isomorphisms and extensions. We denote by torfΛ the set of all torsionfree classes in $\mathsf{mod}\Lambda$, and by f-torfΛ the set of all *functorially finite* torsionfree classes (i.e. torsionfree classes of the form $\mathsf{Sub}X$ for some $X \in \mathsf{mod}\Lambda$). The following observation is classical.

Proposition 1.1. (a) *We have a bijection*

$$\mathrm{tors}\Lambda \to \mathrm{torf}\Lambda \quad \mathcal{T} \mapsto \mathcal{T}^{\perp} := \{X \in \mathsf{mod}\Lambda \mid \mathrm{Hom}_{\Lambda}(\mathcal{T}, X) = 0\}$$

whose inverse is given by

$$\mathrm{torf}\Lambda \to \mathrm{tors}\Lambda \quad \mathcal{F} \mapsto {}^{\perp}\mathcal{F} := \{X \in \mathsf{mod}\Lambda \mid \mathrm{Hom}_{\Lambda}(X, \mathcal{F}) = 0\}.$$

(b) [S] *They induce bijections between* f-torsΛ *and* f-torfΛ.

Clearly torsΛ and f-torsΛ have a structure of partially ordered sets with respect to the inclusion relation.

Definition 1.2. Let P be a partially ordered set and x_i $(i \in I)$ be elements in P. If there exists a unique maximal element in the subposet $\{y \in P \mid y \leq x_i, \ \forall i \in I\}$ of P, we call it a *meet* of x_i $(i \in I)$ and denote it by $\bigwedge_{i\in I} x_i$. Dually we define a *join* $\bigvee_{i\in I} x_i$. We say that P is a *meet-semilattice* (respectively, *join-semilattice*) if any finite subset of P has a meet (respectively, join). We say that P is a *lattice* if it is a join-semilattice and a meet-semilattice. More strongly, we say that P is a *complete lattice* (respectively, *complete join-semilattice*, *complete meet-semilattice*) if any subset of P has a meet and a join (respectively, a join, a meet).

If a map $f : P \to P'$ between lattices preserves a join and a meet of any finite subset (respectively, any subset), we call f a *morphism* of lattices (respectively, complete lattices).

We have the following statement.

Proposition 1.3. (a) $\mathrm{tors}\Lambda$ *and* $\mathrm{torf}\Lambda$ *are complete lattices, and we have an isomorphism* $\mathrm{tors}\Lambda \to (\mathrm{torf}\Lambda)^{\mathrm{op}}$, $\mathcal{T} \mapsto \mathcal{T}^{\perp}$ *of complete lattices.*

(b) *For torsion classes $\mathcal{T}_i$ $(i \in I)$ in* $\mathsf{mod}\Lambda$, *we have*

$$\bigwedge_{i\in I} \mathcal{T}_i = \bigcap_{i\in I} \mathcal{T}_i \quad \text{and} \quad \bigvee_{i\in I} \mathcal{T}_i = {}^{\perp}(\bigcap_{i\in I} \mathcal{T}_i^{\perp}).$$

(c) *For torsionfree classes $\mathcal{F}_j$ $(j \in J)$ in* $\mathsf{mod}\Lambda$, *we have*

$$\bigwedge_{j\in J} \mathcal{F}_j = \bigcap_{j\in J} \mathcal{F}_j \quad \text{and} \quad \bigvee_{j\in J} \mathcal{F}_j = (\bigcap_{j\in J} {}^{\perp}\mathcal{F}_j)^{\perp}.$$

Proof. It is clear that a meet of torsion classes $\mathcal{T}_i$ $(i \in I)$ is given by $\bigcap_{i\in I} \mathcal{T}_i$. Dually a meet of torsionfree classes $\mathcal{F}_j$ $(j \in J)$ is clearly given by $\bigcap_{j\in J} \mathcal{F}_j$.

It is also clear that the bijection in Proposition 1.1 gives an isomorphism $\mathrm{tors}\Lambda \to (\mathrm{torf}\Lambda)^{\mathrm{op}}$ of partially ordered sets. Hence ${}^{\perp}(\bigcap_{i\in I} \mathcal{T}_i^{\perp})$ gives a join of $\mathcal{T}_i$ $(i \in I)$, and $\bigvee_{j\in J} \mathcal{F}_j = (\bigcap_{j\in J} {}^{\perp}\mathcal{F}_j)^{\perp}$ gives a join of $\mathcal{F}_j$ $(j \in J)$. □

Proposition 1.4. *Let Λ be a finite dimensional k-algebra. Then*

(a) *We have an isomorphism of complete lattices:*

$$\text{f-tors}\Lambda \to (\text{f-tors}(\Lambda^{\mathrm{op}}))^{\mathrm{op}}, \quad \mathcal{T} \mapsto D(\mathcal{T}^{\perp}).$$

(b) *The map in (a) induces a bijection* $\text{f-tors}\Lambda \to \text{f-tors}(\Lambda^{\mathrm{op}})$. *In particular,* $\text{f-tors}\Lambda$ *forms a meet-semilattice if and only if* $\text{f-tors}(\Lambda^{\mathrm{op}})$ *forms a join-semilattice.*

Proof. (a) We have an isomorphism $\mathrm{torf}\Lambda \to \mathrm{tors}(\Lambda^{\mathrm{op}})$, $\mathcal{F} \mapsto D(\mathcal{F})$ of complete lattices. Thus the assertion follows from Proposition 1.3.

(b) This follows from Proposition 1.1 since $\mathcal{F}$ is functorially finite if and only if so is $D(\mathcal{F})$. □

We now consider arbitrary finite dimensional algebras Λ, and show that $\text{f-tors}\Lambda$ being a lattice is preserved by factoring by ideals $\langle e\rangle$, where e is an idempotent element in Λ.

Proposition 1.5. *Let Λ be a finite dimensional k-algebra, and e an idempotent in Λ.*

(a) $\text{f-tors}(\Lambda/\langle e\rangle)$ *is the interval* $\{\mathcal{T} \in \text{f-tors}\Lambda \mid 0 \subseteq \mathcal{T} \subseteq \mathsf{mod}(\Lambda/\langle e\rangle)\}$ *in* $\text{f-tors}\Lambda$.

(b) *If* $\text{f-tors}\Lambda$ *is a lattice, then* $\text{f-tors}(\Lambda/\langle e\rangle)$ *is a lattice.*

Proof. (a) This is shown in [AIR, Theorem 2.7] and [AIR, Proposition 2.27].

(b) This is a consequence of (a), using that an interval of a lattice is again a lattice. □

1.2. **Proof of Theorem 0.2.** We denote by τ the Auslander-Reiten translation of Λ.

Definition 1.6. (a) We call $M \in \mathsf{mod}\Lambda$ *τ-rigid* if $\mathrm{Hom}_{\Lambda}(M, \tau M) = 0$. We call $M \in \mathsf{mod}\Lambda$ *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$ holds, where $|M|$ is the number of non-isomorphic indecomposable direct summands of M.

(b) We call $M \in \mathsf{mod}\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

We denote by $\mathrm{s}\tau\text{-tilt}\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules. Then we have the following result.

Proposition 1.7. [AIR, Theorem 2.7] *There exists a bijection* $\mathrm{s}\tau\text{-tilt}\Lambda \to \text{f-tors}\Lambda$ *given by* $M \mapsto \mathsf{Fac}M$.

Using the bijection in Proposition 1.7, we regard $\mathrm{s}\tau\text{-tilt}\Lambda$ as a partially ordered set which is isomorphic to $\text{f-tors}\Lambda$.

Definition 1.8. [DIJ] We say that Λ is *τ-rigid finite* if there are only finitely many indecomposable τ-rigid Λ-modules. This is equivalent to $|\mathrm{s}\tau\text{-tilt}\Lambda| < \infty$, and to $|\text{f-tors}\Lambda| < \infty$.

For example, any local algebra is τ-rigid finite. In fact $\mathrm{s}\tau\text{-tilt}\,\Lambda = \{\Lambda, 0\}$ holds in this case. A path algebra kQ of an acyclic quiver Q is τ-rigid finite if and only if Q is a Dynkin quiver. On the other hand, any preprojective algebra of Dynkin type is τ-rigid finite [M].

We say that two non-isomorphic basic support τ-tilting Λ-modules M and N are *mutations* of each other if $M = X \oplus U$, $N = Y \oplus U$ and X and Y are either 0 or indecomposable. Then any support τ-tilting Λ-module has exactly n mutations.

The following results play a crucial role.

Proposition 1.9. *Let Λ be a finite dimensional k-algebra.*

(a) [AIR, Theorem 2.35] *If M and N are support τ-tilting Λ-modules such that $M > N$, then there exists a mutation L of N such that $M \geq L > N$.*
(b) [DIJ] *Assume that Λ is not τ-rigid finite. Then there exists an infinite descending chain of mutations $\Lambda = M_0 > M_1 > M_2 > \cdots$.*

Now we are ready to prove Theorem 0.2.

(b)$\Leftrightarrow$(c) This was shown in [DIJ].

(c)$\Rightarrow$(a) This is immediate from 1.3.

(a)$\Rightarrow$(b) We assume that $\text{f-tors}\Lambda$ is a complete lattice and that Λ is not τ-rigid finite. Take an infinite descending chain in Proposition 1.9(b). Since $\mathrm{s}\tau\text{-tilt}\,\Lambda \simeq \text{f-tors}\Lambda$ is a complete lattice by our assumption, there exists a meet M of M_i $(i \geq 0)$ in $\mathrm{s}\tau\text{-tilt}\,\Lambda$. Let $N_1, \dots, N_n$ be all mutations of M. Since $\mathsf{Fac} M_i \supsetneq \mathsf{Fac} M$, the set $I_i := \{1 \leq k \leq n \mid M_i \geq N_k > M\}$ is non-empty by Proposition 1.9(a). Since we have a descending chain

$$I_0 \supset I_1 \supset I_2 \supset \cdots$$

of finite non-empty sets, their intersection $I := \bigcap_{i \geq 0} I_i$ is also non-empty. Then any $k \in I$ satisfies $M_i \geq N_k > M$ for all i. This is a contradiction since M is a meet of M_i $(i \geq 0)$. □

1.3. **A criterion for the existence of joins and meets.** In this subsection, we need the following result, which improves Proposition 1.9(a).

Proposition 1.10. [DIJ] *Let M be a support τ-tilting Λ-module and $\mathcal{T}$ a torsion class in* $\mathsf{mod}\Lambda$.

(a) *If $\mathsf{Fac} M \supsetneq \mathcal{T}$, then there exists a mutation N of M satisfying $\mathsf{Fac} M \supsetneq \mathsf{Fac} N \supset \mathcal{T}$.*
(b) *If $\mathsf{Fac} M \subsetneq \mathcal{T}$, then there exists a mutation N of M satisfying $\mathsf{Fac} M \subsetneq \mathsf{Fac} N \subset \mathcal{T}$.*

Immediately we have the following property of non-functorially finite torsion classes.

Proposition 1.11. *Let Λ be a finite dimensional k-algebra, and $\mathcal{T}$ a torsion class in* $\mathsf{mod}\Lambda$ *which is not functorially finite.*

(a) *For any $\mathcal{T}' \in \text{f-tors}\Lambda$ satisfying $\mathcal{T}' \supsetneq \mathcal{T}$, there exists $\mathcal{T}'' \in \text{f-tors}\Lambda$ satisfying $\mathcal{T}' \supsetneq \mathcal{T}'' \supset \mathcal{T}$.*
(b) *For any $\mathcal{T}' \in \text{f-tors}\Lambda$ satisfying $\mathcal{T}' \subsetneq \mathcal{T}$, there exists $\mathcal{T}'' \in \text{f-tors}\Lambda$ satisfying $\mathcal{T}' \subsetneq \mathcal{T}'' \subset \mathcal{T}$.*

Proof. The statement follows immediately from Propositions 1.7 and 1.10. □

We give a more explicit criterion for existence of a meet and a join.

Theorem 1.12. *Let Λ be a finite dimensional k-algebra.*

(a) *A subset $\{\mathcal{T}_i \mid i \in I\}$ of $\text{f-tors}\Lambda$ has a meet if and only if $\bigcap_{i \in I} \mathcal{T}_i$ is functorially finite.*
(b) *A subset $\{\mathcal{T}_i \mid i \in I\}$ of $\text{f-tors}\Lambda$ has a join if and only if ${}^{\perp}(\bigcap_{i \in I} \mathcal{T}_i^{\perp})$ is functorially finite.*

Proof. We only have to prove (a) since (b) is a dual.

If $\bigcap_{i \in I} \mathcal{T}_i$ is functorially finite, then it is a meet of $\mathcal{T}_i$ $(i \in I)$ in $\text{f-tors}\Lambda$, by Proposition 1.3. Thus we only have to prove the 'only if' part.

Assume that $\mathcal{T}_i$ $(i \in I)$ has a meet $\mathcal{S}$ in $\text{f-tors}\Lambda$ and that $\mathcal{T} := \bigcap_{i \in I} \mathcal{T}_i$ is not functorially finite. Since $\mathcal{S} \subset \mathcal{T}_i$ for all $i \in I$, we have $\mathcal{S} \subset \mathcal{T}$. Since $\mathcal{T}$ is not functorially finite, we have $\mathcal{S} \subsetneq \mathcal{T}$. Applying Proposition 1.11, there exists $\mathcal{S}' \in \text{f-tors}\Lambda$ such that

$$\mathcal{S} \subsetneq \mathcal{S}' \subset \mathcal{T}.$$

Thus $\mathcal{S}' \subset \mathcal{T}_i$ holds for any $i \in I$. This is a contradiction since $\mathcal{S}$ is a meet of $\mathcal{T}_i$ $(i \in I)$. □

Remark 1.13. The statements in the above theorem mean that a meet (respectively, join) in f-torsΛ has to be the same as a meet (respectively, join) in the complete lattice torsΛ.

2. Lattice structure of torsion classes for path algebras

2.1. **Sufficient conditions for** f-tors(kQ) **to be a lattice.** Let Q be a finite connected acyclic quiver. In this section we give two sufficient conditions for f-tors(kQ) to be a lattice. Since for an artin algebra of finite representation type any subcategory is functorially finite, the first result is a direct consequence of the fact that tors(kQ) is a lattice.

Proposition 2.1. *If Q is a Dynkin diagram, then* f-tors(kQ) *is a lattice.*

When Q is a Dynkin diagram, the lattice f-tors(kQ) was shown in [IT, Theorem 4.3] to be a Cambrian lattice in the sense of Reading [Re] .

The second sufficient condition is the following.

Proposition 2.2. *Assume that Q has at most two vertices. Then* f-tors(kQ) *is a lattice.*

Proof. If Q has one vertex, then $kQ \cong k$, hence the claim is obvious. Assume then that we have two vertices. Then our quiver Q is $1 \xrightarrow{(n)} 2$, with $n \geq 2$ arrows. We can assume $n \geq 2$ since otherwise Q is Dynkin. The AR-quiver is then of the form:

$$\begin{tikzcd}[row sep=small]
 & 2 \arrow[dr, "(n)"] & & 4 & & & & & 3' \arrow[dr, "(n)"] & & 1' \\
1 \arrow[ur, "(n)"] & & 3 \arrow[ur, "(n)"] & & \cdots & \Big\vert\ \mathbf{R}\ \Big\vert & \cdots & 4' \arrow[ur, "(n)"] & & 2' \arrow[ur, "(n)"] &
\end{tikzcd}$$

Here $\mathbf{R}$ consists of tubes when $n = 2$, and of $\mathbb{Z}A_\infty$-components when $n > 2$. It is known that no indecomposable rigid module lies in $\mathbf{R}$. The tilting modules are given by two consecutive vertices in the preprojective or preinjective component. Denote by A_i the indecomposable modules at the vertex i in the preprojective component, and by $B_{i'}$ the indecomposable modules at the vertex i' in the preinjective component. So for $i \geq 2$ we have the tilting module $A_{i-1} \oplus A_i$, with associated torsion class $\mathcal{T}_i = \mathsf{Fac}(A_{i-1} \oplus A_i)$ which is equal to $\mathsf{Fac}A_{i-1}$ when $i \geq 3$. For $i' \geq 2$ we have the tilting modules $B_i \oplus B_{i-1}$ with associated torsion class $\mathcal{T}'_i = \mathsf{Fac}(B_i \oplus B_{i-1}) = \mathsf{Fac}B_i$. There are no other tilting modules. The additional support tilting modules are the simple modules A_1 and B_1, and hence we have the additional torsion classes $\mathcal{T}_1 = \mathsf{Fac}A_1$ and $\mathcal{T}'_1 = \mathsf{Fac}B_1$.

We have the inclusions $\{0\} \subset \mathcal{T}_1 \subset \mathcal{T}_2 \supset \mathcal{T}_3 \supset \cdots \supset \mathcal{T}_i \supset \cdots \supset \mathcal{T}'_j \supset \cdots \supset \mathcal{T}'_1$ for all elements of f-tors(kQ). It is clear that if neither $\mathcal{T}$ nor $\mathcal{T}'$ is $\mathcal{T}_1$, then $\mathcal{T} \vee \mathcal{T}'$ is the larger one and $\mathcal{T} \wedge \mathcal{T}'$ is the smaller one. Further, $\mathcal{T}_1 \vee \mathcal{T} = \mathcal{T}_2 (= \mathsf{mod}kQ)$ for $\mathcal{T} \neq \mathcal{T}_1$, and $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1$, $\mathcal{T}_1 \wedge \mathcal{T} = \{0\}$ for $\mathcal{T} \neq \mathcal{T}_2$. $\square$

2.2. **Tame algebras.** In this section we deal with path algebras kQ of extended Dynkin quivers with at least 3 vertices, and show that in that case the f-tors(kQ) do not form lattices.

Proposition 2.3. *Let Q be an acyclic extended Dynkin quiver with at least 3 vertices. Then* f-tors(kQ) *is neither a join-semilattice nor a meet-semilattice.*

Proof. Since kQ is extended Dynkin with at least 3 vertices, there is a tube $\mathbf{C}$ of rank $r \geq 2$ and there are r quasi-simple modules $S_1, \ldots, S_r$ in $\mathbf{C}$. Since $S_1, \ldots, S_r$ are τ-rigid, we have that $\mathcal{T}_1 = \mathsf{Fac}S_1, \ldots, \mathcal{T}_r = \mathsf{Fac}S_r$ are in f-tors(kQ). By Theorem 1.12 there is a join of these $\mathcal{T}_i$ in f-tors(kQ) if and only if ${}^\perp(\bigcap_{i \in I} \mathcal{T}_i^\perp)$ is functorially finite, where $I = \{1, \ldots, n\}$. However ${}^\perp(\bigcap_{i \in I} \mathcal{T}_i^\perp) = \mathsf{add}(\mathbf{C} \cup \{\text{preinjectives}\})$ which is not functorially finite, since it clearly cannot be written as $\mathsf{Fac}Y$ for any Y. Therefore there is no join in f-tors(kQ), and hence f-tors(kQ) is not a join-semilattice.

Since Q^{op} is an acyclic extended Dynkin quiver with at least 3 vertices, f-tors(kQ^{op}) is not a join-semilattice. By Proposition 1.4, f-tors(kQ) is not a meet-semilattice. $\square$

2.3. **Wild algebras.** In this section we show that f-tors(kQ) is not a lattice for the quiver Q is connected wild, with 3 vertices.

For a finite dimensional algebra Λ and a set S of Λ-modules, we denote by $\mathsf{Filt} S$ the full subcategory of $\mathsf{mod}\Lambda$ whose objects are the Λ-modules which have a finite filtration with factors in S.

Proposition 2.4. *Let Q be an acyclic quiver, and let M and N be indecomposable rigid kQ-modules such that $\mathrm{Hom}_{kQ}(M,N) = 0 = \mathrm{Hom}_{kQ}(N,M)$, $\mathrm{Ext}^1_{kQ}(M,N) \neq 0$ and $\mathrm{Ext}^1_{kQ}(N,M) \neq 0$.*

(a) [Ri] *The category $\mathcal{A} := \mathsf{Filt}(M,N)$ is an exact abelian subcategory of $\mathsf{mod} kQ$ with two simple objects M and N.*
(b) $\mathrm{End}_{kQ}(M) \cong k \cong \mathrm{End}_{kQ}(N)$ *holds, and M and N are regular.*
(c) *For any $\ell \geq 0$, there exists an object X_ℓ in $\mathcal{A}$ which is uniserial of length ℓ in $\mathcal{A}$.*

Proof. (a) This is shown in [Ri, Theorem 1.2].

(b) Since M and N are rigid, we have the first assertion. Since $\mathrm{Ext}^1_{kQ}(M,N) \neq 0$ and $\mathrm{Ext}^1_{kQ}(N,M) \neq 0$ hold, M and N are in a cycle. Hence they are regular.

(c) The assertion is clear for $\ell = 1$. Assume that we have a uniserial object X_ℓ of length ℓ in $\mathcal{A}$. Without loss of generality, let M be the top of X_ℓ in $\mathcal{A}$. Then there exists an exact sequence $0 \to \mathrm{rad}_{\mathcal{A}} X_\ell \to X_\ell \to M \to 0$. Since $\mathrm{Ext}^1_{kQ}(N,M) \neq 0$, there exists a non-split exact sequence $0 \to M \to E \to N \to 0$. Since kQ is hereditary, we have a commutative diagram of exact sequences:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & M \arrow[u] \arrow[r] & E \arrow[u] \arrow[r] & N \arrow[r] & 0 \\
0 \arrow[r] & X_\ell \arrow[u] \arrow[r] & Y \arrow[u] \arrow[r] & N \arrow[u, equal] \arrow[r] & 0 \\
 & \mathrm{rad}_{\mathcal{A}} X_\ell \arrow[u] \arrow[r, equal] & \mathrm{rad}_{\mathcal{A}} X_\ell \arrow[u] & & \\
 & 0 \arrow[u] & 0. \arrow[u] & &
\end{tikzcd}$$

Clearly Y belongs to the category $\mathcal{A}$. We show that Y is uniserial of length $\ell+1$ in $\mathcal{A}$. It is enough to show $\mathrm{rad}_{\mathcal{A}} Y = X_\ell$. Otherwise $\mathrm{rad}_{\mathcal{A}} Y$ is strictly contained in X_ℓ, and hence $\mathrm{rad}_{\mathcal{A}} Y = \mathrm{rad}_{\mathcal{A}} X_\ell$ holds since X_ℓ is uniserial. Then $Y/\mathrm{rad}_{\mathcal{A}} Y = E$ holds, a contradiction since E is not semisimple in the category $\mathcal{A}$. Thus the assertion follows. $\square$

We shall also need the following.

Lemma 2.5. *Let $\mathcal{C}$ be a full subcategory of $\mathsf{mod} kQ$ closed under extensions. Then $\mathsf{Fac}\mathcal{C}$ is also closed under extensions.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathsf{mod} kQ$, where X and Z are in $\mathsf{Fac}\mathcal{C}$. Then we have surjections $f : C_0 \to X$ and $g : C_1 \to Z$, where C_0 and C_1 are in $\mathcal{C}$. This gives rise to exact sequences:

$$\mathrm{Ext}^1_{kQ}(C_1, C_0) \to \mathrm{Ext}^1_{kQ}(C_1, X) \to \mathrm{Ext}^2_{kQ}(C_1, \mathrm{Ker} f) = 0, \tag{1}$$
$$\mathrm{Ext}^1_{kQ}(Z, X) \to \mathrm{Ext}^1_{kQ}(C_1, X) \to \mathrm{Ext}^2_{kQ}(\mathrm{Ker}\, g, X) = 0. \tag{2}$$

From (2) and (1) we get the exact commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & C_0 \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & Y' \arrow[r] & C_1 \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{C}$ is extension closed, then Y'' is in $\mathcal{C}$, and we have surjections $Y'' \to Y' \to Y$, so that Y is in $\mathsf{Fac}\mathcal{C}$, as desired. $\square$

Combining the above results, we get the following.

Proposition 2.6. *Let kQ be an acyclic quiver, and M and N be kQ-modules satisfying the assumptions in Proposition 2.4. Let $\mathcal{A} := \mathsf{Filt}(M, N)$ and $\mathcal{T} := \mathsf{Fac}\mathcal{A}$. Then:*

(a) *The subcategory $\mathcal{T}$ is a torsion class which is not functorially finite in $\mathsf{mod}(kQ)$.*
(b) f-tors(kQ) *is neither a join-semilattice nor a meet-semilattice.*

Proof. (a) It follows from Lemma 2.5 that $\mathcal{T}$ is a torsion class. Let $\mathcal{T}_1 := \mathsf{Fac}M$ and $\mathcal{T}_2 := \mathsf{Fac}N$. Since M and N are rigid, the subcategories $\mathcal{T}_1$ and $\mathcal{T}_2$ are in f-tors(kQ).

Assume that $\mathcal{T}$ is functorially finite. Then there exists a module X in $\mathcal{T}$ so that $\mathcal{T} = \mathsf{Fac}X$. By the definition of $\mathcal{T}$, there is a module C in $\mathcal{A}$ and an epimorphism $C \to X$ in $\mathsf{mod}kQ$, and hence $\mathcal{T} = \mathsf{Fac}C$. Now let ℓ be the Loewy length of C in $\mathcal{A}$. Since the modules M and N satisfy the conditions of Proposition 2.4, there is a uniserial object $X_{\ell+1}$ of length $\ell + 1$ in $\mathcal{A}$. Since $X_{\ell+1} \in \mathsf{Fac}C$, there is an epimorphism $C^m \to X_{\ell+1}$ in $\mathsf{mod}kQ$ (and hence in $\mathcal{A}$) for some $m \geq 0$. This is a contradiction since the Loewy length of $X_{\ell+1}$ is bigger than that of C.

(b) If f-tors(kQ) is a lattice, we know from section 1 that the join of $\mathsf{Fac}M$ and $\mathsf{Fac}N$ must be the smallest torsion class containing $\mathsf{Fac}M$ and $\mathsf{Fac}N$, which is clearly $\mathcal{T}$. But since we have seen that this is not a functorially finite subcategory of $\mathsf{mod}kQ$ by (a), it follows that f-tors(kQ) is not a join-semilattice.

Since the kQ^{op}-modules DM and DN satisfy the conditions of Proposition 2.4, we have that f-tors(kQ^{op}) is not a join-semilattice. By Proposition 1.4, f-tors(kQ) is not a meet-semilattice. $\square$

Now we are able to show the following result.

Lemma 2.7. *Let* $Q = \begin{tikzcd} 1 \arrow[r, "(a)"] \arrow[rr, bend right, "(c)"'] & 2 \arrow[r, "(b)"] & 3 \end{tikzcd}$ *be a quiver with $a \geq 2$, $b \geq 1$ and $c \geq 0$. Then there exist M and N satisfying the conditions in Proposition 2.4.*

Proof. Let $Q' := (1 \xrightarrow{(a)} 2)$ be a full subquiver of Q. We regard the projective kQ'-module corresponding to the vertex 1 as a kQ-module M, and let $N := \tau_{kQ}M$. We show that M and N satisfy the conditions in Proposition 2.4 with $p > 0$ and $q > 0$. We have $\underline{\dim} M = (1, a, 0)^t$. Since the Cartan matrix of kQ (see [ASS]) is $C = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ab+c & b & 1 \end{bmatrix}$ and the Coxeter matrix of kQ (see [ASS]) is given by

$$\Phi = -C^t \cdot C^{-1} = -\begin{bmatrix} 1 & a & ab+c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -c & -b & 1 \end{bmatrix} = \begin{bmatrix} a^2+abc+c^2-1 & ab^2+bc-a & -ab-c \\ a+bc & b^2-1 & -b \\ c & b & -1 \end{bmatrix},$$

we have

$$\underline{\dim} N = \Phi \cdot \underline{\dim} M = (a^2b^2 + 2abc + c^2 - 1, ab^2 + bc, ab + c)^t.$$

(Step 1) Since M is a rigid kQ'-module and Q' is a full subquiver of Q, it is a rigid kQ-module. Hence N is also a rigid kQ-module since τ preserves the rigidity of kQ-modules.

Since M is rigid, we have $\mathrm{Hom}_{kQ}(M, N) = \mathrm{Hom}_{kQ}(M, \tau M) = 0$. We have $\mathrm{Ext}^1_{kQ}(M, N) = \mathrm{Ext}^1_{kQ}(M, \tau M) \simeq D\underline{\mathrm{End}}_{kQ}(M) \neq 0$ by AR duality. It remains to show that $\mathrm{Hom}_{kQ}(N, M) = 0$ and $\mathrm{Ext}^1_{kQ}(N, M) \neq 0$.

(Step 2) To prove $\mathrm{Hom}_{kQ}(N, M) = 0$, it is enough to show $\mathrm{Hom}_{kQ}(M, \tau^{-1}M) = 0$. Since M does not have S_3 as a composition factor, it is enough to show that $\mathrm{soc}\,\tau^{-1}M$ is a direct sum of copies of S_3. Since S_1 is injective, it does not appear in $\mathrm{soc}\,\tau^{-1}M$ by the indecomposability of $\tau^{-1}M$.

Assume that S_2 appears in $\mathrm{soc}\,\tau^{-1}M$. Then we have an exact sequence $0 \to S_2 \to \tau^{-1}M \to L \to 0$. Applying $\mathrm{Hom}_{kQ}(-, S_3)$, we have an exact sequence

$$\mathrm{Ext}^1_{kQ}(\tau^{-1}M, S_3) \to \mathrm{Ext}^1_{kQ}(S_2, S_3) \to \mathrm{Ext}^2_{kQ}(L, S_3) = 0.$$

Since $\mathrm{Ext}^1_{kQ}(S_2, S_3) \neq 0$, we have $\mathrm{Ext}^1_{kQ}(\tau^{-1}M, S_3) \neq 0$. On the other hand, we have by AR duality,

$$\mathrm{Ext}^1_{kQ}(\tau^{-1}M, S_3) \simeq D\,\mathrm{Hom}_{kQ}(S_3, M) = 0,$$

a contradiction.

(Step 3) To prove $\mathrm{Ext}^1_{kQ}(N,M)\neq 0$, we calculate the Euler form, see [ASS]. We have

$$\begin{aligned}\langle N,M\rangle=(\underline{\dim}N)^t\cdot(C^{-1})^t\cdot\underline{\dim}M &= (a^2b^2+2abc+c^2-1,ab^2+bc,ab+c)\begin{bmatrix}1&-a&-c\\0&1&-b\\0&0&1\end{bmatrix}\begin{bmatrix}1\\a\\0\end{bmatrix}\\ &= -1-a^2(a^2b^2-2b^2-1)-abc(2a^2-3)-c^2(a^2-1),\end{aligned}$$

which is easily shown to be negative by our assumption $a\geq 2$, $b\geq 1$ and $c\geq 0$. $\square$

Now we show the following main result in this section.

Proposition 2.8. *Let Q be a connected acyclic wild quiver with 3 vertices. Then:*
(a) *There exist M and N satisfying the conditions in Proposition 2.4.*
(b) f-tors(kQ) *is neither a join-semilattice nor a meet-semilattice.*

Proof. (a) Let a,b,c be integers such that $a\geq 2$, $b\geq 1$ and $c\geq 0$. Then Q has one of the following forms:

$$\text{(i)}:\ \begin{tikzcd} 1 \arrow[r, "(a)"] \arrow[rr, bend right, "(c)"'] & 2 \arrow[r, "(b)"] & 3 \end{tikzcd},\quad \text{(ii)}:\ \begin{tikzcd} 1 \arrow[r, "(b)"] \arrow[rr, bend right, "(a)"'] & 2 \arrow[r, "(c)"] & 3 \end{tikzcd},\quad \text{(iii)}:\ \begin{tikzcd} 1 \arrow[r, "(c)"] \arrow[rr, bend right, "(b)"'] & 2 \arrow[r, "(a)"] & 3 \end{tikzcd},$$

$$\text{(iv)}:\ \begin{tikzcd} 1 \arrow[r, "(b)"] \arrow[rr, bend right, "(c)"'] & 2 \arrow[r, "(a)"] & 3 \end{tikzcd},\quad \text{(v)}:\ \begin{tikzcd} 1 \arrow[r, "(c)"] \arrow[rr, bend right, "(a)"'] & 2 \arrow[r, "(b)"] & 3 \end{tikzcd},\quad \text{(vi)}:\ \begin{tikzcd} 1 \arrow[r, "(a)"] \arrow[rr, bend right, "(b)"'] & 2 \arrow[r, "(c)"] & 3 \end{tikzcd}.$$

First, the case (i) was shown in Lemma 2.7. Next, the case (ii) (respectively, (iii)) follows from the case (i) by using the reflection functor at the vertex 1 (respectively, 3). Finally the case (iv) (respectively, (v), (vi)) follows from the case (i) (respectively, (ii), (iii)) by using the k-dual.

(b) This follows from (a) and Proposition 2.6. $\square$

Remark 2.9. When $a,b,c\geq 1$, it is easy to check that the modules $M=S_2$ and $N:=$ $\begin{tikzcd} k \arrow[r, "(a)"] \arrow[rr, bend right, "(c)", "f=(1,\ldots,1)"'] & 0 \arrow[r, "(b)"] & k^c \end{tikzcd}$ also satisfy the conditions in Proposition 2.4 with $p>0$ and $q>0$.

2.4. **Proof of Theorem 0.3.** We need the following preparation, which is an analog of a well-known result, see [ASS, Lemma VII.2.1].

Proposition 2.10. *Let Q be a finite connected quiver. Then one of the following holds.*
(a) *Q is a Dynkin quiver.*
(b) *Q has at most two vertices.*
(c) *Q has an extended Dynkin full subquiver with at least 3 vertices.*
(d) *Q has a connected wild full subquiver with exactly 3 vertices.*

Proof. First, assume that Q has multiple arrows from i to j. If Q has exactly two vertices, then we have the case (b). If Q has at least 3 vertices, then any connected full subquiver of Q consisting of i, j and one more vertex is wild. Thus we have the case (d).

Next, assume that Q has no multiple arrows. Then it follows from [ASS, Lemma VII.2.1] that we have either the case (a) or (c). $\square$

Now we are ready to prove Theorem 0.3.

(d)$\Rightarrow$(a) If Q is a Dynkin quiver, then f-tors(kQ) forms a lattice by Proposition 2.1. If Q has exactly two vertices, then f-tors(kQ) forms a lattice by Proposition 2.2.

(a)$\Rightarrow$(b) This is clear.

(b)$\Rightarrow$(d) Assume that Q does not satisfy the condition (d). Then by Proposition 2.10, Q has either an extended Dynkin full subquiver with at least 3 vertices, or a connected wild full subquiver with exactly 3 vertices, For the former case (respectively, latter case), f-tors(kQ) is not a join-semilattice by Propositions 2.3 (respectively, 2.8) and 1.5(b).

(c)⇔(d) By Proposition 1.4, the condition (c) is equivalent to that f-tors(kQ^{op}) forms a join-semilattice. This is equivalent to that Q^{op} is either a Dynkin quiver or has at most two vertices, by using the equivalence (b)⇒(d) for the quiver Q^{op}. This is clearly equivalent to the condition (d). $\square$

2.5. **Concealed canonical algebras and tubular algebras.** Inspired by the proof that f-torsΛ is not a join-semilattice for path algebras of extended Dynkin quivers with at least 3 vertices, we have the following.

Proposition 2.11. *Let Λ be a finite dimensional k-algebra such that the set of indecomposable Λ-modules is a disjoint union $\mathbf{P} \cup \mathbf{R} \cup \mathbf{Q}$, where $\mathbf{R}$ is a family of stable standard orthogonal tubes, $\mathrm{Hom}_\Lambda(\mathbf{R}, \mathbf{P}) = 0$, $\mathrm{Hom}_\Lambda(\mathbf{Q}, \mathbf{R}) = 0$ and $\mathrm{Hom}_\Lambda(\mathbf{Q}, \mathbf{P}) = 0$. If there is a tube $\mathbf{C}$ in $\mathbf{R}$ of rank $r \geq 2$, then* f-torsΛ *is neither a join-semilattice nor a meet-semilattice.*

Proof. We only prove the assertion for join-semilattices since the other assertion follows by Proposition 1.4.

Let $S_1, \ldots, S_r$ be the indecomposable modules at the border of $\mathbf{C}$. Since $\mathbf{C}$ is standard, then $S_1, \ldots, S_r$ are τ-rigid, and hence $\mathsf{Fac}S_i$ is in f-torsΛ for $i = 1, \ldots, n$. Let $\mathcal{T} := \bigvee_{i=1}^{n} \mathsf{Fac}S_i$ in torsΛ. Then $\mathcal{T}$ is the smallest torsion class in $\mathsf{mod}\Lambda$ containing $\mathbf{C}$. Since $\mathsf{add}(\mathbf{C}, \mathbf{Q})$ is a torsion class by our assumptions, we have $\mathcal{T} \subset \mathsf{add}(\mathbf{C}, \mathbf{Q})$. Now if $\mathcal{T}$ is functorially finite, then there exists $M \in \mathcal{T}$ such that $\mathcal{T} = \mathsf{Fac}M$. Since $\mathrm{Hom}_\Lambda(\mathbf{Q}, \mathbf{C}) = 0$ holds by our assumption, the maximal direct summand N of M contained in $\mathsf{add}\mathbf{C}$ satisfies $\mathbf{C} \subset \mathsf{Fac}N$. But this is impossible since $\mathsf{add}\mathbf{C}$ is equivalent to the category of finite dimensional modules over the complete path algebra $k\widehat{Q}$ of the quiver Q of type $\widetilde{A}_{r-1}$ by our assumption, and hence there is no upper bound of Loewy length of objects. $\square$

Now we are ready to prove Theorem 0.4. It follows from Proposition 2.11 and by the properties of the concealed canonical (respectively, tubular algebras) listed in [SS, page 380] (respectively, [SS, Theorem XIX.3.20]) since there exists a tube $\mathbf{C}$ of rank $r \geq 2$ if and only if Λ has at least 3 vertices. $\square$

References

[AIR] T. Adachi, O. Iyama, I. Reiten, *τ-tilting theory*, to appear in Compos. Math., arXiv:1210.1036.

[AI] T. Aihara, O. Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. (2) 85 (2012), no. 3, 633–668.

[ASS] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory*, London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.

[AS] M. Auslander, S. O. Smalo, *Almost split sequences in subcategories*, J. Algebra 69 (1981), no. 2, 426–454.

[DIJ] L. Demonet, O. Iyama, G. Jasso, in preparation.

[HU] D. Happel, L. Unger, *On a partial order of tilting modules*, Algebr. Represent. Theory 8 (2005), no. 2, 147–156.

[IT] C. Ingalls and H. Thomas, *Noncrossing partitions and representations of quivers*, Compos. Math. 145 (2009), no. 6, 1533–1562.

[K] R. Kase, *Distributive lattices and the poset of pre-projective tilting modules*, arXiv: 1307.8247.

[M] Y. Mizuno, *Classifying τ-tilting modules over preprojective algebras of Dynkin type*, to appear in Math. Z., arXiv: 1304.0667.

[Re] N. Reading, *Cambrian lattices*, Adv. Math. 205 (2006), no. 2, 313–353.

[RS] C. Riedtmann, A. Schofield, *On a simplicial complex associated with tilting modules*, Comment. Math. Helv. 66 (1991), no. 1, 70–78.

[Ri] C. M. Ringel, *Representations of K-species and bimodules*, J. Algebra 41 (1976), no. 2, 269–302.

[SS] D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 3. Representation-infinite tilted algebras*, London Mathematical Society Student Texts, 72. Cambridge University Press, Cambridge, 2007.

[S] S. O. Smalo, *Torsion theories and tilting modules*, Bull. London Math. Soc. 16 (1984), no. 5, 518–522.

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