COLOURED QUIVER MUTATION FOR HIGHER CLUSTER CATEGORIES

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Abstract. We define mutation on coloured quivers associated to tilting objects in higher cluster categories. We show that this operation is compatible with the mutation operation on the tilting objects. This gives a combinatorial approach to tilting in higher cluster categories and especially an algorithm to determine the Gabriel quivers of tilting objects in such categories.

Introduction

A cluster category is a certain 2-Calabi-Yau orbit category of the derived category of a hereditary abelian category. Cluster categories were introduced in [BMRRT] in order to give a categorical model for the combinatorics of Fomin-Zelevinsky cluster algebras [FZ]. They are triangulated [K] and admit (cluster-)tilting objects, which model the clusters of a corresponding (acyclic) cluster algebra [CK]. Each cluster in a fixed cluster algebra comes together with a finite quiver, and in the categorical model this quiver is in fact the Gabriel quiver of the corresponding tilting object [BMRT].

A principal ingredient in the construction of a cluster algebra is quiver mutation. It controls the exchange procedure which gives a rule for producing a new cluster variable and hence a new cluster from a given cluster. Exchange is modeled by cluster categories in the acyclic case [BMR] in terms of a mutation rule for tilting objects, i.e. a rule for replacing an indecomposable direct summand in a tilting object with another indecomposable rigid object, to get a new tilting object. Quiver mutation describes the relation between the Gabriel quivers of the corresponding tilting objects.

Analogously to the definition of the cluster category, for a positive integer m, it is natural to define a certain $m+1$-Calabi-Yau orbit category of the derived category of a hereditary abelian category. This is called the *m-cluster category*. Implicitly, m-cluster categories was first studied in [K], and their (cluster-)tilting objects have been studied in [ABST, F, HJ1, HJ2, IY, KR1, KR2, T, W, Z, ZZ]. Combinatorial descriptions of m-cluster categories in Dynkin type A_n and D_n are given in [BM1, BM2].

In cluster categories the mutation rule for tilting objects is described in terms of certain triangles called *exchange triangles*. By [IY] the existence of exchange triangles generalizes to m-cluster categories. It was shown in [ZZ, W] that there are

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exactly $m+1$ non-isomorphic complements to an almost complete tilting object, and that they are determined by the $m+1$ exchange triangles defined in [IY].

The aim of this paper is to give a combinatorial description of mutation in m-cluster categories. *A priori*, one might expect to be able to do this by keeping track of the Gabriel quivers of the tilting objects. However, it is easy to see that the Gabriel quivers do not contain enough information.

We proceed to associate to a tilting object a quiver each of whose arrows has an associated colour $c \in \{0, \dots, m\}$. The arrows with colour 0 form the Gabriel quiver of the tilting object. We then define a mutation operation on coloured quivers and show that it is compatible with mutation of tilting objects. A consequence is that the effect of an arbitrary sequence of mutations on a tilting object in an m-cluster category can be calculated by a purely combinatorial procedure.

Our definition of a coloured quiver associated to a tilting object makes sense in any $m+1$-Calabi-Yau category, such as for example those studied in [IY]. We hope that our constructions may shed some light on mutation of tilting objects in this more general setting.

In section 1, we review some elementary facts about higher cluster categories. In section 2, we explain how to define the coloured quiver of a tilting object, we define coloured quiver mutation, and we state our main theorem. In sections 3 and 4, we state some further lemmas about higher cluster categories, and we prove certain properties of the coloured quivers of tilting objects. We prove our main result in sections 5 and 6. In sections 7 and 8 we point out some applications. In section 9 we interpret our construction in terms of m-cluster complexes. In section 10, we give an alternative algorithm for computing coloured quiver mutation. Section 11 discusses the example of m-cluster categories of Dynkin type A_n, using the model developed by Baur and Marsh [BM1].

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1. Higher cluster categories

Let K be an algebraically closed field, and let Γ be a finite acyclic quiver with $n = n_\Gamma$ vertices. Then the path algebra $H = K\Gamma$ is a hereditary finite dimensional basic K-algebra

Let $\operatorname{mod} H$ be the category of finite dimensional left H-modules. Let $\mathcal{D} = D^b(H)$ be the bounded derived category of H, and let $[i]$ be the i'th shift functor on $\mathcal{D}$. We let τ denote the Auslander-Reiten translate, which is an autoequivalence on $\mathcal{D}$ such that we have a bifunctorial isomorphism in $\mathcal{D}$

$$\operatorname{Hom}(A, B[1]) \simeq D \operatorname{Hom}(B, \tau A). \tag{1}$$

In other words $\nu = [1]\tau$ is a Serre functor.

Let $G = \tau^{-1}[m]$. The m-cluster category is the orbit category $\mathcal{C} = \mathcal{C}_m = \mathcal{D}/\tau^{-1}[m]$. The objects in $\mathcal{C}$ are the objects in $\mathcal{D}$, and two objects X, Y are isomorphic in $\mathcal{C}$ if and only if $X \simeq G^i Y$ in $\mathcal{D}$. The maps are given by $\operatorname{Hom}_{\mathcal{C}_m}(X, Y) = \amalg_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(X, G^i Y)$. By [K], the category $\mathcal{C}$ is triangulated and the canonical functor $\mathcal{D} \to \mathcal{C}$ is a triangle functor. We denote therefore by [1] the suspension in $\mathcal{C}$.

The m-cluster category is also Krull-Schmidt and has an AR-translate τ inherited from $\mathcal{D}$, such that the formula (1) still holds in $\mathcal{C}$. If follows that $\nu = [1]\tau$ is a Serre functor for $\mathcal{C}$ and that $\mathcal{C}$ is $m+1$-Calabi-Yau, since $\nu \simeq [m+1]$.

The indecomposable objects in $\mathcal{D}$ are of the form $M[i]$, where M is an indecomposable H-module and $i \in \mathbb{Z}$. We can choose a fundamental domain for the action of $G = \tau^{-1}[m]$ on $\mathcal{D}$, consisting of the indecomposable objects $M[i]$ with $0 \leq i \leq m-1$, together with the objects $M[m]$ with M an indecomposable projective H-module. Then each indecomposable object in $\mathcal{C}$ is isomorphic to exactly one of the indecomposables in this fundamental domain. We say that $M[d]$ has degree d, denoted $\delta(M[d]) = d$. Furthermore, for an arbitrary object $X = \amalg X_i$ in $\mathcal{C}_m$, we let $\Delta_d(X) = \amalg_j X_j[-d]$ be the H-module which is the (shifted) direct sum of all summands X_j of X with $\delta(X_j) = d$.

In the following theorem the equivalence between (i) and (ii) is shown in [ZZ, W] and the equivalence between (i) and (iii) is shown in [Z].

Theorem 1.1. *Let T be an object in $\mathcal{C}$ satisfying $\mathrm{Hom}_{\mathcal{C}}(T, T[i]) = 0$ for $i = 1, \ldots, m$. Then the following are equivalent*

(i) *If $\mathrm{Hom}_{\mathcal{C}}(T, U[i]) = 0$ for $i = 1, \ldots, m$ then U is in $\mathrm{add}\, T$.*
(ii) *If $\mathrm{Hom}_{\mathcal{C}}(U \amalg T, U[i]) = 0$ for $i = 1, \ldots, m$ then U is in $\mathrm{add}\, T$.*
(iii) *T has n indecomposable direct summands, up to isomorphism.*

Here $\mathrm{add}\, T$ denotes the additive closure of T. A (cluster-)tilting object T in an m-cluster is an object satisfying the conditions of the above Theorem. For a tilting object $T = \amalg_{i=1}^{v} T_i$, with each T_i indecomposable, and T_k an indecomposable direct summand, we call $\bar{T} = T/T_k$ an almost complete tilting object. We let $\mathrm{Irr}_{\mathcal{A}}(X, Y)$ denote the K-space of irreducible maps $X \to Y$ in a Krull-Schmidt K-category $\mathcal{A}$. The following crucial result is proved in [ZZ] and [W].

Proposition 1.2. *There are, up to isomorphism, $m+1$ complements of an almost complete tilting object.*

Let T_k be an indecomposable direct summand in an m-cluster tilting object $T = \bar{T} \amalg T_k$. The complements of $\bar{T}$ are denoted $T_k^{(c)}$ for $c = 0, 1, \ldots, m$, where $T_k = T_k^{(0)}$. By [IY], there are $m+1$ exchange triangles

$$T_k^{(c)} \xrightarrow{f_k^{(c)}} B_k^{(c)} \xrightarrow{g_k^{(c+1)}} T_k^{(c+1)} \xrightarrow{h_k^{(c+1)}} .$$

Here the $B_k^{(c)}$ are in $\mathrm{add}(T/T_k)$ and the maps $f_k^{(c)}$ (resp. $g_k^{(c)}$) are minimal left (resp. right) $\mathrm{add}(T/T_k)$-approximations, and hence not split mono or split epi. Note that by minimality, the maps $f_k^{(c)}$ and $g_k^{(c)}$ have no proper zero summands.

2. Coloured quiver mutation

We first recall the definition of quiver mutation, formulated in [FZ] in terms of skew-symmetric matrices. Let $Q = (q_{ik})$ be a quiver with vertices $1, \ldots, n$ and with no loops or oriented two-cycles, where q_{ik} denotes the number of arrows from i to k. Let j be a vertex in Q. Then, a new quiver $\mu_j(Q) = \widetilde{Q} = (\tilde{q}_{ik})$ is defined by the following data

$$\tilde{q}_{ik} = \begin{cases} q_{ki} & \text{if } j = k \text{ or } j = i \\ \max\{0, q_{ik} - q_{ki} + q_{ij}q_{jk} - q_{kj}q_{ji}\} & \text{if } i \neq j \neq k \end{cases}$$

It is easily verified that this definition is equivalent to the one of Fomin-Zelevinsky.

Now we consider coloured quivers. Let m be a positive integer. An m-coloured (multi-)quiver Q consists of vertices $1, \ldots, n$ and coloured arrows $i \xrightarrow{(c)} j$, where $c \in \{0, 1, \ldots, m\}$. Let $q_{ij}^{(c)}$ denote the number of arrows from i to j of colour (c).

We will consider coloured quivers with the following additional conditions.

(I) No loops: $q_{ii}^{(c)} = 0$ for all c.
(II) Monochromaticity: If $q_{ij}^{(c)} \neq 0$, then $q_{ij}^{(c')} = 0$ for $c \neq c'$.
(III) Skew-symmetry: $q_{ij}^{(c)} = q_{ji}^{(m-c)}$.

We will define an operation on a coloured quiver Q satisfying the above conditions. Let j be a vertex in Q and let $\mu_j(Q) = \widetilde{Q}$ be the coloured quiver defined by

$$\tilde{q}_{ik}^{(c)} = \begin{cases} q_{ik}^{(c+1)} & \text{if } j = k \\ q_{ik}^{(c-1)} & \text{if } j = i \\ \max\{0, q_{ik}^{(c)} - \sum_{t \neq c} q_{ik}^{(t)} + (q_{ij}^{(c)} - q_{ij}^{(c-1)})q_{jk}^{(0)} + q_{ij}^{(m)}(q_{jk}^{(c)} - q_{jk}^{(c+1)})\} & \text{if } i \neq j \neq k \end{cases}$$

In an m-cluster category $\mathcal{C}$, for every tilting object $T = \amalg_{i=1}^{n} T_i$, with the T_i indecomposable, we will define a corresponding m-coloured quiver Q_T, as follows.

Let T_i, T_j be two non-isomorphic indecomposable direct summands of the m-cluster tilting object T and let $r_{ij}^{(c)}$ denote the multiplicity of T_j in $B_i^{(c)}$. We define the m-coloured quiver Q_T of T to have vertices i corresponding to indecomposable direct summands T_i, and $q_{ij}^{(c)} = r_{ij}^{(c)}$. Note, in particular, that the (0)-coloured arrows are the arrows from the Gabriel quiver for the endomorphism ring of T.

By definition, Q_T satisfies condition (I). We show in Section 3 that (II) is satisfied (this also follows from [ZZ]), and in Section 4 that (III) is also satisfied.

The aim of this paper is to prove the following theorem, which is a generalization of the main result of [BMR].

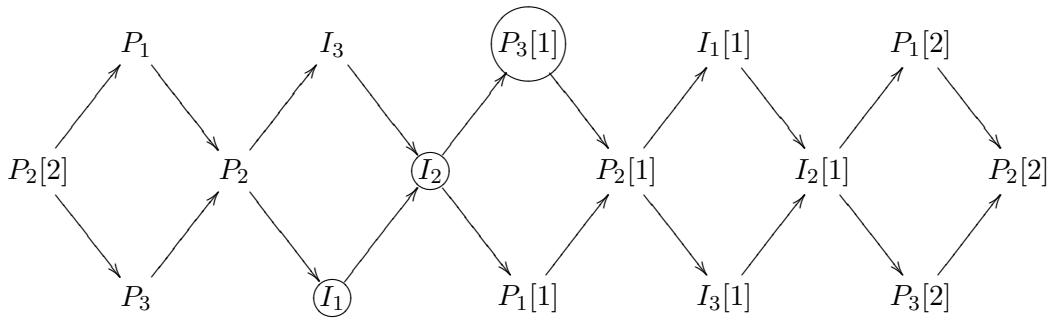
Theorem 2.1. *Let $T = \amalg_{i=1}^{n} T_i$ and $T' = T/T_j \amalg T_j^{(1)}$ be m-tilting objects, where there is an exchange triangle $T_j \to B_j^{(0)} \to T_j^{(1)} \to$. Then $Q_{T'} = \mu_j(Q_T)$.*

In the case $m = 1$ the coloured quiver of a tilting object T is given by $q_{ij}^{(0)} = \bar{q}_{ij}$ and $q_{ij}^{(1)} = \bar{q}_{ji}$ where $\bar{q}_{ij}$ denotes the number of arrows in the Gabriel quiver of T. Then coloured mutation of the coloured quiver corresponds to FZ-mutation of the Gabriel quiver.

Example: A_3, $m = 2$

Let Γ be A_3 with linear orientation, i.e. the quiver $1 \leftarrow 2 \rightarrow 3$.

The AR-quiver of the 2-cluster category of $H = K\Gamma$ is



The direct sum $T = I_1 \amalg I_2 \amalg P_3[1]$ of the encircled indecomposable objects gives a tilting object. Its coloured quiver is

$$\begin{tikzcd} I_1 \arrow[r, shift left, "(0)"] & I_2 \arrow[l, shift left, "(2)"] \arrow[r, shift left, "(0)"] & P_3[1] \arrow[l, shift left, "(2)"] \end{tikzcd}$$

Now consider the exchange triangle

$$I_2 \to P_3[1] \to I_3[1] \to$$

and the new tilting object $T' = P_1 \amalg I_3[1] \amalg P_3[1]$. The coloured quiver of T' is

$$\begin{tikzcd} I_1 \arrow[r, shift left, "(1)"] \arrow[rr, bend right, "(0)"'] & I_3[1] \arrow[l, shift left, "(1)"] \arrow[r, shift left, "(2)"] & P_3[1] \arrow[l, shift left, "(0)"] \arrow[ll, bend left, "(2)"] \end{tikzcd}$$

3. Further background on higher cluster categories

In this section we summarize some further known results about m-cluster categories. Most of these are from [Z] and [ZZ]. We include some proofs for the convenience of the reader.

Tilting objects in $\mathcal{C} = \mathcal{C}_m$ give rise to partial tilting modules in $\operatorname{mod} H$, where a *partial tilting module* M in $\operatorname{mod} H$, is a module with $\operatorname{Ext}^1_H(M, M) = 0$.

Lemma 3.1. (a) *When T is a tilting object in $\mathcal{C}_m$, then each $\Delta_d(T)$ is a partial tilting module in* $\operatorname{mod} H$.

(b) *The endomorphism ring of a partial tilting module has no oriented cycles in its ordinary quiver.*

Proof. (a) is obvious from the definition. See [HR, Cor. 4.2] for (b). □

In the following note that degrees of objects are always considered with a fixed choice of fundamental domain, and sums and differences of degrees are always computed modulo $m + 1$.

Lemma 3.2 ([Z, ZZ])**.** *Assume $m > 1$.*

(a) $\operatorname{End}(X) \simeq K$ *for any indecomposable exceptional object X.*

(b) *We have that*

$$\delta(T_i^{(c+1)}) - \delta(T_i^{(c)}) \begin{cases} = 1 & \text{if } \delta(T_i^{(c)}) = m \\ \leq 1 & \text{if } \delta(T_i^{(c)}) \notin \{m-1, m\} \\ \leq 2 & \text{if } \delta(T_i^{(c)}) = m-1 \end{cases}$$

(c) *The distribution of degrees of complements is one of the following*
- *there is exactly one complement of each degree, or*
- *there is no complement of degree m, two complements in one degree $d \neq m$, and exactly one complement in all degrees $\neq d, m$.*

(d) *If* $\mathrm{Hom}(T_i^{(c)}, T_i^{(c')}) \neq 0$, *then* $c' \in \{c, c+1, c+2\}$.

(e) *For* $t \in \{1, \ldots, m\}$ *we have*

$$\mathrm{Hom}(T_i^{(c)}, T_i^{(c')}[t]) = \begin{cases} K & \text{if } c' - c + t = 0 (\mathrm{mod}\, m+1) \\ 0 & \text{else} \end{cases}$$

Proof. (a) follows from the fact that $\mathrm{Hom}_H(X, X) = K$ for exceptional objects and the definition of maps in a m-cluster category.

(b) follows from the fact that $\mathrm{Hom}(T_i^{(c+1)}, T_i^{(c)}[1]) \neq 0$, since in the exchange triangles, the $f_i^{(c)}$ are not split mono and (c) follows from (b).

Considering the two different possible distributions of complements, we obtain from (c) that if $m \geq 3$ and $c' \geq c+3$ and $c' \neq c-1$, then $\mathrm{Hom}(T_i^{(c)}, T_i^{(c')}) = 0$. Consider the case $c' = c-1$. We can assume $m > 2$, since else the statement is void. Hence we can clearly assume that $\delta(T_i^{(c)}) = \delta(T_i^{(c-1)})$. There is an exchange triangle induced from an exact sequence in $\mathrm{mod}\, H$,

$$T_i^{(c-1)} \to B_i^{(c-1)} \to T_i^{(c)} \to T_i^{(c-1)}[1].$$

It is clear that $\mathrm{Hom}(T_i^{(c-1)}[1], T_i^{(c-1)}) = 0$, since $m > 2$. We claim that also $\mathrm{Hom}(B_i^{(c-1)}, T_i^{(c-1)}) = 0$. This holds since $B_i^{(c-1)} \amalg T_i^{(c-1)}$ is a partial tilting object in H, and so there are no cycles in the endomorphism ring, by Lemma 3.1. Hence also $\mathrm{Hom}(T_i^{(c)}, T_i^{(c-1)}) = 0$ follows, and this finishes the proof for (d).

For (e) we first apply $\mathrm{Hom}(T_i^{(c+1)}, \ )$ to the exchange triangle

$$T_i^{(c)} \to B_i^{(c)} \to T_i^{(c+1)} \to$$

and consider the corresponding long-exact sequence, to obtain that

$$\mathrm{Hom}(T_i^{(c+1)}, T_i^{(c)}[t]) = \begin{cases} K & \text{if } t = 1 \\ 0 & \text{if } t = 0 \text{ or } t \in \{2, \ldots, m\} \end{cases}.$$

Now consider $\mathrm{Hom}(T_i^{(c+u)}, T_i^{(c)}[v])$. When $0 < v \leq u \leq m$, we have that

$$\mathrm{Hom}(T_i^{(c+u)}, T_i^{(c)}[v]) \simeq \mathrm{Hom}(T_i^{(c+u+1)}, T_i^{(c)}[v+1]) \simeq$$
$$\mathrm{Hom}(T_i^{(c-1)}, T_i^{(c)}[v+m-u]) \simeq \mathrm{Hom}(T_i^{(c)}, T_i^{(c-1)}[1+u-v]).$$

When $m \geq v > u \geq 0$, we have that

$$\mathrm{Hom}(T_i^{(c+u)}, T_i^{(c)}[v]) \simeq \mathrm{Hom}(T_i^{(c+u-1)}, T_i^{(c)}[v-1]) \simeq \mathrm{Hom}(T_i^{(c)}, T_i^{(c)}[v-u]).$$

Combining these facts, (e) follows. □

Lemma 3.3. *The following statements are equivalent*

(a) $\mathrm{Hom}(T_i^{(1)}, T_j^{(1)}[1]) = 0$

(b) T_j *is not a direct summand in* $B_i^{(m)}$

(c) T_i *is not a direct summand in* $B_j^{(0)}$

Furthermore, $\mathrm{Hom}(T_i^{(c)}, T_j^{(1)}[1]) = 0$ *for* $c \neq 1$.

Proof. Note that $r_{ji}^{(0)} = r_{ij}^{(m)} = \dim \mathrm{Irr}_{\mathrm{add}\, T}(T_j, T_i)$, so (b) and (c) are equivalent. Consider the exact sequence

$$\mathrm{Hom}(T_i^{(c)}, T_j^{(0)}[1]) \to \mathrm{Hom}(T_i^{(c)}, B_j^{(0)}[1]) \to \\ \mathrm{Hom}(T_i^{(c)}, T_j^{(1)}[1]) \to \mathrm{Hom}(T_i^{(c)}, T_j^{(0)}[2]) \to$$

coming from applying $\mathrm{Hom}(T_i^{(c)},\ )$ to the exchange triangle

$$T_j^{(0)} \to B_j^{(0)} \to T_j^{(1)}.$$

The first and fourth terms are always zero. Using 3.2(e) we get that the second term (and hence the third) is non-zero if and only if $c = 1$ and T_i is a direct summand in $B_j^{(0)}$. □

Lemma 3.4. *(*[IY, ZZ]*) For* $0 \leq l \leq m$*, the composition*

$$\gamma_k^{(v,l)} = h_k^{(v)} \circ h_k^{(v-1)}[1] \circ h_k^{(v-2)}[2] \circ \cdots \circ h_k^{(v-l+1)}[l-1] \colon T_k^{(v)} \to T_k^{(v-l)}[l]$$

is non-zero and a basis for $\mathrm{Hom}(T_k^{(v)}, T_k^{(v-l)}[l])$.

Proof. For $m = 1$, see [BMRRT]. Assume $m \geq 2$. For the first claim see [IY], while the second claim then follows from Lemma 3.2(e). □

We include an independent proof of the following crucial property.

Proposition 3.5. [ZZ] $B_k^{(u)}$ *and* $B_k^{(v)}$ *has no common non-zero direct summands whenever* $u \neq v$.

Proof. When $m = 1$, this is proved in [BMR]. Assume $m > 1$. We consider two cases, $|u - v| = 1$ or $|u - v| > 1$.

Consider first the case $|u - v| = 1$. Without loss of generality we can assume $u = 0$ and $v = 1$, and that $\delta(T_k^{(0)}) = 0$. Assume that there exists a (non-zero) indecomposable T_x, which is a direct summand in $B_k^{(0)}$ and in $B_k^{(1)}$. We have that $\delta(T_k^{(1)}) \in \{0, 1\}$ by Lemma 3.2(b). Assume first $\delta(T_k^{(1)}) = 0$. Then the exchange triangle

$$T_k^{(0)} \to B_k^{(0)} \to T_k^{(1)} \to$$

is induced from the degree 0 part of the derived category, and hence from an exact sequence in $\operatorname{mod} H$. Then the endomorphism ring of the partial tilting module $T_x \amalg T_k^{(1)}$ has a cycle, which is a contradiction to Lemma 3.1. Assume now that $\delta(T_k^{(1)}) = 1$. Then $\delta(T_k^{(2)}) \in \{0, 1, 2\}$, where 0 can only occur if $m = 2$. If $\delta(T_k^{(2)}) \in \{1, 2\}$, then clearly $\delta(T_x) = 1$, and hence the partial tilting module $T_k^{(1)} \amalg T_x$ contains a cycle, which is a contradiction. Assume that $\delta(T_k^{(2)}) = 0$ (and hence $m = 2$). Then $\delta(T_x) \in \{0, 1\}$. If $\delta(T_x) = 1$, we get a contradiction as in the previous case. If $\delta(T_x) = 0$, consider the exchange triangle

$$T_k^{(2)} \to B_k^{(2)} \to T_k^{(0)} \to$$

which is induced from an exact sequence in $\operatorname{mod} H$. Hence there is a *non-zero* map $T_x \to B_k^{(2)}$ obtained by composing $T_x \to T_k^{(2)}$ with the monomorphism $T_k^{(2)} \to B_k^{(2)}$, and thus there are cycles in the endomorphism ring of the partial tilting module $T_x \amalg B_k^{(2)} \amalg T_k^{(0)}$, a contradiction. This finishes the case with $|u - v| = 1$.

Assume now that $|u - v| > 1$. Then we have $m > 2$. Since $\operatorname{Hom}(T_k^{(v)}, T_x) \neq 0$ and $\operatorname{Hom}(T_k^{(u)}, T_x) \neq 0$, we have by Lemma 3.2(c) that $|v - u| \leq 2$. So without loss of generality we can assume $v = u - 2$. Assume that $\delta(T_k^{(u)}) = 0$. Then $\delta(T_k^{(v)}) = m - 1$ using Lemma 3.2(c) and the fact that $\operatorname{Hom}(T_k^{(v)}, T_x) \neq 0$. Then also $\delta(T_k^{(v)}) \leq m$. But $\operatorname{Hom}(T_x, T_k^{(v+1)}) \neq 0$, so $\delta(T_x) \leq m$, contradicting the fact that $\operatorname{Hom}(T_k^{(u)}, T_x) \neq 0$. □

Corollary 3.6. *Q_T satisfies condition (II).*

4. Symmetry

Let $T = \widetilde{T} \amalg T_i \amalg T_j$ be a tilting object. In this section we show that the coloured quiver Q_T satisfies condition (III).

Proposition 4.1. *With the notation of the previous section, we have $r_{ji}^{(c)} = r_{ij}^{(m-c)}$.*

Proof. By Lemma 3.3 we only need to consider the case $c \notin \{0, m\}$. It is enough to show that $r_{ji}^{(c)} \leq r_{ij}^{(m-c)}$.

We first prove

Lemma 4.2. *Let $\alpha \colon T_j^{(c)} \to T_i$ be irreducible in $\operatorname{add}((T/T_j) \amalg T_j^{(c)})$. Then the composition $\alpha[-c] \circ \gamma_i^{(0,c)}[-c] \colon T_j^{(c)}[-c] \to T_i^{(m-c+1)}$ is non-zero.*

Proof. We have already assumed $c \neq 0$. Assume

$$\alpha[-c] \circ h_i^{(0)}[-c] \colon T_j^{(c)}[-c] \to T_i^{(0)}[-c] \to T_i^{(m)}[-c+1]$$

is zero. This means that $T_j^{(c)} \to T_i$ must factor through $B_i^{(m)} \xrightarrow{g_i^{(0)}} T_i$. Since T_i is by assumption a summand in $B_j^{(c)}$, we have that T_i is not a summand in $B_j^{(0)}$ by Proposition 3.5. Since $r_{ij}^{(m)} = r_{ji}^{(0)} = 0$, we have that T_j is not a direct summand in

$B_i^{(m)}$. This means that α is not irreducible in $\operatorname{add}((T/T_j) \amalg T_j^{(c)})$, a contradiction. So $\alpha[-c] \circ h_i^{(0)}[-c] \colon T_j^{(c)}[-c] \to T_i^{(m)}[-c+1]$ is non-zero.

Assume $c > 1$. If the composition $\alpha[-c] \circ h_i^{(0)}[-c] \circ h_i^{(m)}[-c+1]$ is zero, then $\alpha[-c] \circ h_i^{(0)}[-c]$ factors through

$$B_i^{(m-1)}[-c+1] \to T_i^{(m)}[-c+1].$$

We claim that $\operatorname{Hom}(T_j^{(c)}[-c], B_i^{(m-1)}[-c+1]) \simeq \operatorname{Hom}(T_j^{(c)}, B_i^{(m-1)}[1]) = 0$. This clearly holds if T_j is not a summand of $B_i^{(m-1)}$. In addition we have that $\operatorname{Hom}(T_j^{(c)}, T_j[1]) = 0$ since $c > 1$, using Lemma 3.2(e). This is a contradiction, and this argument can clearly be iterated to see that $\alpha[-c] \circ \gamma_i^{(0,c)}[-c] \colon T_j^{(c)}[-c] \to T_i^{(m-c+1)}$ is non-zero, using Lemma 3.2(e). $\square$

We now show that any irreducible map $\alpha \colon T_j^{(c)} \to T_i$ gives rise to an irreducible map $\delta \colon T_i^{(m-c)} \to T_j$.

Consider the composition

$$B_j^{(c-1)}[-c] \xrightarrow{g_j^{(c)}[-c]} T_j^{(c)}[-c] \longrightarrow T_i^{(m-c+1)}.$$

Since T_i is a summand in $B_j^{(c)}$ by assumption, it is not a summand in $B_j^{(c-1)}$. Thus, $B_j^{(c-1)}$ is in $\operatorname{add} \widetilde{T}$. Since $\operatorname{Hom}(X, T_i^{(m-c+1)}[c]) = 0$ for any X in $\operatorname{add} \tilde{T}$, the composition vanishes.

Using the exchange triangle

$$B_j^{(c-1)}[-c] \xrightarrow{g_j^{(c)}[-c]} T_j^{(c)}[-c] \xrightarrow{h_j^{(c)}[-c]} T_j^{(c-1)}[-c+1],$$

we see that $\alpha[-c] \circ \gamma_i^{(0,c)}[-c] \colon T_j^{(c)}[-c] \to T_i^{(m-c+1)}$ factors through the map $T_j^{(c)}[-c] \xrightarrow{h_j^{(c)}[-c]} T_j^{(c-1)}[-c+1]$, i.e. there is a commutative diagram

$$\begin{tikzcd}
B_j^{(c-1)}[-c] \arrow[r, "g_j^{(c)}[-c]"] & T_j^{(c)}[-c] \arrow[r, "h_j^{(c)}[-c]"] \arrow[d] & T_j^{(c-1)}[-c+1] \arrow[r] \arrow[dl, "\phi_1"] & {} \\
& T_i^{(m-c+1)} & &
\end{tikzcd}$$

Similarly, using the exchange triangle

$$B_j^{(c-2)}[-c+1] \xrightarrow{g_j^{(c-1)}[-c+1]} T_j^{(c-1)}[-c+1] \xrightarrow{h_j^{(c-1)}[-c+1]} T_j^{(c-2)}[-c+2]$$

we obtain a map $\phi_2 \colon T_j^{(c-2)}[-c+2] \to T_i^{(m-c+1)}$

Repeating this argument c times we obtain a map $\phi_c \colon T_j \to T_i^{(m-c+1)}$, such that $\gamma_j^{(c,c)}[-c] \circ \phi_c = \alpha[-c] \circ \gamma_i^{(0,c)}$.

$$\begin{tikzcd}
T_j^{(c)}[-c] \arrow[rr] \arrow[d, "h_j^{(c)}[-c]"'] & & T_i^{(m-c+1)} \\
T_j^{(c-1)}[-c+1] \arrow[urr, "\phi_1"] \arrow[d, "h_j^{(c-1)}[-c+1]"'] & & \\
T_j^{(c-2)}[-c+2] \arrow[uurr, "\phi_2"] \arrow[d, "h_j^{(c-2)}[-c+2]"'] & & \\
\vdots \arrow[d] & & \\
T_j \arrow[uuurr, "\phi_c"] & &
\end{tikzcd}$$

We claim that

Lemma 4.3. *There is a map $\beta\colon T_j \to T_i^{(m-c+1)}$, such that $\gamma_j^{(c,c)}[-c] \circ \beta = \alpha[-c] \circ \gamma_i^{(0,c)}$, and such that β is irreducible in* $\operatorname{add}((T/T_i) \amalg T_i^{(m-c+1)})$.

Proof. Let

$$T_j \xrightarrow{\begin{pmatrix}\psi' & \psi''\end{pmatrix}} (T_i^{(m-c+1)})' \amalg \widetilde{T}'$$

be a minimal left $\operatorname{add}(T_i^{(m-c+1)} \amalg \widetilde{T})$-approximation, with $\widetilde{T}'$ in $\operatorname{add} \widetilde{T}$ and $(T_i^{(m-c+1)})'$ in $\operatorname{add} T_i^{(m-c+1)}$. Let ϕ_c be as above, and factor it as

$$T_j \xrightarrow{\begin{pmatrix}\psi' & \psi''\end{pmatrix}} (T_i^{(m-c+1)})' \amalg \widetilde{T}' \xrightarrow{\begin{pmatrix}\epsilon' \\ \epsilon''\end{pmatrix}} T_i^{(m-c+1)}.$$

Since $\gamma_j^{(c,c)}$ factors through $T_j^{(1)}[-1]$, we have that $\gamma_j^{(c,c)}[-c]\psi'' = 0$, so we have

$$\gamma_j^{(c,c)}[-c](\psi'\epsilon' + \psi''\epsilon'') = \gamma_j^{(c,c)}[-c]\psi'\epsilon'.$$

Hence, let we let $\beta = \psi'\epsilon'$ and since the summands in ϵ' are isomorphisms, it is clear that β is irreducible. □

Next, assume $\{\alpha_t\}$ is a basis for the space of irreducible maps from $T_j^{(c)}$ to T_i. Then, by Lemma 4.2 the set $\{\alpha_t \circ \gamma_i^{(0,c)}\}$ is also linearly independent. For each α_t, consider the corresponding map β_t, such that $\gamma_j^{(c,c)}[-c] \circ \beta_t = \alpha_t[-c] \circ \gamma_i^{(0,c)}$, and which we by Lemma 4.3 can assume is irreducible. Assume a non-trivial linear

combination $\sum k_t\beta_t$ is zero. Then also $\sum k_t(\gamma_j^{(c,c)}[-c]\circ\beta_t) = \sum k_t\alpha_t\circ\gamma_i^{(0,c)} = 0$. But this contradicts Lemma 4.2 since $\sum k_t\alpha_t$ is irreducible. Hence it follows that $\{\beta_t\}$ is also linearly independent. Hence, in the exchange triangle $T_i^{(m-c)} \to B_i^{(m-c)} \to T_i^{(m-c+1)}$, we have that T_j appears with multiplicity at least $r_{ji}^{(c)}$ in $B_i^{(m-c)}$. So, we have that $r_{ji}^{(c)} \leq r_{ij}^{(m-c)}$, and the proof of the proposition is complete. □

5. Complements after mutation

In this section we show how mutation in the vertex j affects the complements of the almost complete tilting object T/T_i. As before, let $T = \widetilde{T} \amalg T_i \amalg T_j$ be an m-tilting object, and let $T' = T/T_j \amalg T_j^{(1)}$.

We need to consider

$$\begin{tikzcd} T_i \arrow[r, "(e)"] \arrow[rr, bend right, "(c)"'] & T_j \arrow[r, "(d)"] & T_k \end{tikzcd}$$

for all possible values of c, d, e. However, we have the following restriction on the colour of arrows.

Proposition 5.1. *Assume $q_{ij}^{(e)} > 0, q_{jk}^{(0)} > 0$ and $q_{ik}^{(c)} > 0$. Then $c \in \{e, e+1\}$.*

Proof. Consider the exchange triangle $T_i^{(e)} \to T_j \amalg X' \to T_i^{(e+1)} \to$. Note that T_j is a direct summand in the middle term $B_i^{(e)}$ by the assumption that $q_{ij}^{(e)} > 0$. Consider also the exchange triangle $T_i^{(c)} \to T_k \amalg Z \to T_i^{(c+1)} \to$. Pick an arbitrary non-zero map $h\colon T_j \to T_k$, and consider the map $\left(\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right) : T_j \amalg X' \to T_k \amalg Z$. It suffices to show that whenever $c \notin \{e, e+1\}$, then h is not irreducible in $\operatorname{add} T$. So assume that $c \notin \{e, e+1\}$. We claim that there is a commutative diagram

$$\begin{tikzcd} T_i^{(e)} \arrow[r] \arrow[d] & T_j \amalg X' \arrow[r] \arrow[d, "\left(\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & T_i^{(e+1)} \arrow[r] \arrow[d] & {} \\ T_i^{(c)} \arrow[r] & T_k \amalg Z \arrow[r] & T_i^{(c+1)} \arrow[r] & {} \end{tikzcd}$$

where the rows are the exchange triangles. The composition $T_i^{(e)} \to T_j \xrightarrow{h} T_k \to T_i^{(c+1)}$ is zero since

- if $c \neq e-1$ $\operatorname{Hom}(T_i^{(e)}, T_i^{(c+1)}) = 0$ by using $c \notin \{e, e+1\}$ and Lemma 3.2(e)
- if $c = e-1$, there is no non-zero composition $T_i^{(e)} \to T_j \to T_k \to T_i^{(c+1)} = T_i^{(e)}$

Hence the leftmost vertical map exists, and then the rightmost map exists, using that $\mathcal{C}$ is a triangulated category. Then, since $\operatorname{Hom}(T_i^{(e+1)}[-1], T_i^{(c)}) = 0$ by Lemma 3.2(e), there is a map $T_j \amalg X' \to T_i^{(c)}$, such that $T_i^{(e)} \to T_i^{(c)} = T_i^{(e)} \to T_j \amalg X' \to T_i^{(c)}$. Hence there is map $T_i^{(e+1)} \to T_k \amalg Z$ such that $T_j \amalg X' \to T_k \amalg Z = (T_j \amalg X' \to$

$T_i^{(c)} \to T_k \amalg Z) + (T_j \amalg X' \to T_i^{(e+1)} \to T_k \amalg Z)$. By restriction we get

$$h\colon T_j \to T_k = (T_j \to T_i^{(c)} \to T_k) + (T_j \to T_i^{(e+1)} \to T_k). \tag{2}$$

Under the assumption $c \notin \{e, e+1\}$ we have that $T_i^{(e+1)} \to T_k$ cannot be irreducible in $\operatorname{add}((T/T_i) \amalg T_i^{(e+1)})$. Hence $T_i^{(e+1)} \to T_k = T_i^{(e+1)} \to B_i^{(e+1)} \to T_k$, where T_k is not summand in $B_i^{(e+1)}$. Also, by Proposition 3.5 we have that T_j is not a summand in $B_i^{(e+1)}$. If $T_j \to T_i^{(c)}$ was irreducible in $\operatorname{add}((T/T_i) \amalg T_i^{(c)})$, then there would be an irreducible map $T_i^{(c-1)} \to T_j$ in $\operatorname{add}((T/T_i) \amalg T_i^{(c-1)})$, and since $c \neq e+1$, this does not hold, by Proposition 3.5. Hence, $T_j \to T_i^{(c)} = T_j \to B_i^{(c-1)} \to T_i^{(c)}$, where T_j is not a direct summand of $B_i^{(c-1)}$. Also by Proposition 3.5 we have that T_k is not a summand of $B_i^{(c-1)}$. By (2), this shows that $h\colon T_j \to T_k$ is not irreducible in $\operatorname{add} T$. □

Let $T' = (T/T_j) \amalg T_j^{(1)}$. For $i \neq j$, let $(T_i^{(u)})'$ denote the complements of T'/T_i, where there are exchange triangles

$$(T_i^{(u)})' \to (B_i^{(u)})' \to (T_i^{(u+1)})' \to$$

We first want to compare $(T_i^{(u)})'$ with $(T_i^{(u)})$.

Lemma 5.2. *Assume that $q_{ij}^{(u)} = 0$ for $0 \leq u < c$ and that $q_{ij}^{(m)} = 0$.*

(a) *For $u = 0, 1, \ldots, c-1$, the minimal left $\operatorname{add}(T/T_i)$-approximation $T_i^{(u)} \to B_i^{(u)}$ is also an $\operatorname{add}(T'/T_i)$-approximation.*
(b) *For $u = 0, 1, \ldots, c$, we have $(T_i^{(u)})' = T_i^{(u)}$.*

Proof. By assumption T_j is not a direct summand in any of the $B_i^{(u)}$. Assume there is a map $T_i^{(u)} \to T_j^{(1)}$ and consider the diagram

$$\begin{tikzcd} T_i^{(u+1)}[-1] \arrow[r] & T_i^{(u)} \arrow[r] \arrow[d] & B_i^{(u)} \arrow[r] & {} \\ & T_j^{(1)} & & \end{tikzcd}$$

Since $\operatorname{Hom}(T_i^{(u+1)}, T_j^{(1)}[1]) = 0$ by Lemma 3.3, we see that the map $T_i^{(u)} \to T_j^{(1)}$ factors through $T_i^{(u)} \to B_i^{(u)}$. Hence the minimal left $\operatorname{add}(T/T_i)$-approximation $T_i^{(u)} \to B_i^{(u)}$ is also an $\operatorname{add}(T'/T_i)$-approximation, so we have proved (a). Then (b) follows directly. □

Lemma 5.3. *Assume that $e \neq m$ and there are exchange triangles*

$$T_i^{(e)} \to (T_j)^p \amalg X \to T_i^{(e+1)} \to \tag{3}$$

and

$$T_j \to (T_k)^q \amalg Y \to T_j^{(1)} \to, \tag{4}$$

where $p = q_{ij}^{(e)} > 0$ *and* $q = q_{jk}^{(0)} \geq 0$, *i.e.* $B_i^{(e)} = (T_j)^p \amalg X$ *and* $B_j^{(0)} = (T_k)^q \amalg Y$, *where* T_k *is not isomorphic to any direct summand in* Y.

(a) *The composition* $T_i^{(e)} \to (T_j)^p \amalg X \to (T_k)^{pq} \amalg Y^p \amalg X$ *is a left* $\operatorname{add}(T'/T_i)$-*approximation.*

(b) *There is a triangle*

$$T_i^{(e)} \to (T_k)^{pq} \amalg Y^p \amalg X \to (T_i^{(e+1)})' \amalg C' \to$$

with C' *in* $\operatorname{add}(T/(T_i \amalg T_j))$ *and* $T_i^{(e)} = (T_i^{(e)})'$.

(c) *There is a triangle* $T_i^{(e+1)} \to (T_i^{(e+1)})' \amalg C' \to (T_j^{(1)})^p \to$.

Proof. Consider an arbitrary map $f\colon T_i^{(e)} \to U$ with U in $\operatorname{add}(T'/T_i)$. We have that $\operatorname{Hom}(T_i^{(e+1)}, T_j^{(1)}[1]) = 0$, by Lemma 3.3. Hence, by applying $\operatorname{Hom}(\ ,U)$ to the triangle (3) we get that f factors through $T_i^{(e)} \to (T_j)^p \amalg X$. By applying $\operatorname{Hom}(\ ,U)$ to the triangle (4), and using that $\operatorname{Hom}(T_j^{(1)}, T_j^{(1)}[1]) = 0$, we get that f factors through $T_i^{(e)} \to (T_j)^p \amalg X \to Y^p \amalg (T_k)^{pq} \amalg X$. This proves (a). For (b) and (c) we use the exchange triangles (3) and (4) and the octahedral axiom to obtain the commutative diagram of triangles

$$\begin{tikzcd}
T_i^{(e)} \arrow[r] \arrow[d, equal] & (T_j)^p \amalg X \arrow[r] \arrow[d] & T_i^{(e+1)} \arrow[r] \arrow[d] & {} \\
T_i^{(e)} \arrow[r] & (T_k)^{pq} \amalg Y^p \amalg X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & {} \\
 & (T_j^{(1)})^p \arrow[r, equal] & (T_j^{(1)})^p \arrow[r] & {}
\end{tikzcd}$$

By (a) the map $T_i^{(e)} \to (T_k)^{pq} \amalg Y^p \amalg X$ is a left $\operatorname{add}(T'/T_i)$-approximation, and by Lemma 5.2 we have that $(T_i^{(e)})' = T_i^{(e)}$. Hence $C = (T_i^{(e+1)})' \amalg C'$, where C' is in $\operatorname{add}((T_k)^{pq} \amalg Y^p \amalg X) \subset \operatorname{add}(T/(T_i \amalg T_j))$, and with no copies isomorphic to T_k in Y. □

Note that the induced $\operatorname{add}(T'/T_i)$-approximation is in general not minimal.

Lemma 5.4. *Assume* $e \neq m$ *and* $q_{ij}^{(e)} > 0$.

(a) *Then there is a triangle*

$$(T_i^{(e+1)})' \amalg C' \xrightarrow{\alpha} B_i^{(e+1)} \amalg (T_j^{(1)})^p \to T_i^{(e+2)} \to$$

where α *is a minimal left* $\operatorname{add}(T'/T_i)$-*approximation, and* C' *is as in Lemma 5.3.*

(b) *There is an induced exchange triangle*

$$(T_i^{(e+1)})' \to \frac{B_i^{(e+1)} \amalg (T_j^{(1)})^p}{\alpha(C')} \to T_i^{(e+2)} \to$$

where $\alpha(C') \simeq C'$.

(c) $(T_i^{(e+2)})' \simeq T_i^{(e+2)}$.

Proof. Consider the exchange triangle

$$T_i^{(e+2)}[-1] \to T_i^{(e+1)} \to B_i^{(e+1)} \to$$

and the triangle from Lemma 5.3 (b)

$$T_i^{(e+1)} \to (T_i^{(e+1)})' \amalg C' \to (T_j^{(1)})^p \to . \tag{5}$$

Apply the octahedral axiom, to obtain the commutative diagram of triangles

$$\begin{tikzcd}
T_i^{(e+2)}[-1] \arrow[r] \arrow[d, equal] & T_i^{(e+1)} \arrow[r] \arrow[d] & B_i^{(e+1)} \arrow[r] \arrow[d] & {} \\
T_i^{(e+2)}[-1] \arrow[r] & (T_i^{(e+1)})' \amalg C' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & {} \\
& (T_j^{(1)})^p \arrow[r, equal] & (T_j^{(1)})^p \arrow[r] & {}
\end{tikzcd}$$

Since T_j does not occur as a summand in $B_i^{(e+1)}$ by Proposition 3.5, we have that $\mathrm{Hom}(T_j^{(1)}, B_i^{(e+1)}[1]) = 0$. Hence the rightmost triangle splits, so we have a triangle

$$T_i^{(e+2)}[-1] \to (T_i^{(e+1)})' \amalg C' \to B_i^{(e+1)} \amalg (T_j^{(1)})^p \to \tag{6}$$

By Lemma 3.3 we have that $\mathrm{Hom}(T_i^{(e+2)}, T_j^{(1)}[1]) = 0$. By Lemma 3.2(e) we get that $\mathrm{Hom}(T_i^{(e+2)}, T_i[1]) = 0$, and clearly $\mathrm{Hom}(T_i^{(e+2)}, T_l[1]) = 0$, for $l \neq i$. We hence get that all maps $(T_i^{(e+1)})' \amalg C' \to U$, with U in $\operatorname{add} T'$, factor through $(T_i^{(e+1)})' \amalg C' \to B_i^{(e+1)} \amalg (T_j^{(1)})^p$. Minimality is clear from the triangle (6). This proves (a), and (b) follows from the fact that C' contains no copies of T_j, and hence splits off. (c) is a direct consequence of (b). □

Proposition 5.5. (a) *If* $q_{ij}^{(u)} = 0$ *for* $u = 0, \dots, m$, *then* $(T_i^{(v)})' \simeq T_i^{(v)}$ *for all* v.

(b) *If* $e \neq m$ *and* $q_{ij}^{(e)} > 0$, *then* $(T_i^{(v)})' \simeq T_i^{(v)}$ *for* $v \neq e+1$.

Proof. (a) is a direct consequence of 5.2. For (b) note that by Lemmas 5.2 and 5.4 we have $(T_i^{(v)})' \simeq T_i^{(v)}$ for $v = 0, \dots, e$ and $v = e+2$. For $v \geq e+2$ consider the exchange triangles

$$T_i^{(v)} \to B_i^{(v)} \to T_i^{(v+1)} \to .$$

Since $\mathrm{Hom}(T_i^{(v+1)}, T_j^{(1)}[1]) = 0$ by Lemma 3.3 and $q_{ij}^{(v)} = 0$, it is clear that the map $T_i^{(v)} \to B_i^{(v)}$ is a left $\operatorname{add} T'/T_i$-approximation. Hence (b) follows. □

6. Proof of the main result

This section contains the proof of the main result, Theorem 2.1. As before, let $T = \widetilde{T} \amalg T_i \amalg T_j$ be an m-tilting object, and let $T' = T/T_j \amalg T_j^{(1)}$.

We will compare the numbers of (c)-coloured arrows from i to k, in the coloured quivers of T and T', i.e. we will compare $q_{ik}^{(c)}$ and $\tilde{q}_{ik}^{(c)}$.

We need to consider an arbitrary T whose coloured quiver locally looks like

$$\begin{tikzcd} T_i \arrow[r, "(e)"] \arrow[rr, bend right, "(c)"'] & T_j \arrow[r, "(d)"] & T_k \end{tikzcd}$$

for any possible value of c, d, e. Our aim is to show that the formula

$$\tilde{q}_{ik}^{(u)} = \begin{cases} q_{ik}^{(u+1)} & \text{if } j = k \\ q_{ik}^{(u-1)} & \text{if } j = i \\ \max\{0, q_{ik}^{(u)} - \sum_{t \neq u} q_{ik}^{(t)} + (q_{ij}^{(u)} - q_{ij}^{u-1}) q_{jk}^{(0)} + q_{ij}^{(m)}(q_{jk}^{(u)} - q_{jk}^{(u+1)})\} & \text{if } i \neq j \neq k \end{cases}$$

holds. The case where $j = k$ is directly from the definition. The case where $j = i$ follows by condition (II) for $Q_{T'}$. For the rest of the proof we assume $j \notin \{i, k\}$. We will divide the proof into four cases, where $p \geq 0$ denotes the number of arrows from i to j, and $q = q_{jk}^{(0)}$.

I. $p = 0$
II. $p \neq 0$, $e \neq m$ and $q = 0$
III. $p \neq 0$, $e \neq m$ and $q \neq 0$.
IV. $p \neq 0$ and $e = m$

Note that in the three first cases, the formula reduces to

$$\tilde{q}_{ik}^{(u)} = \max\{0, q_{ik}^{(u)} - \sum_{t \neq u} q_{ik}^{(t)} + (q_{ij}^{(u)} - q_{ij}^{(u-1)}) q_{jk}^{(0)}\},$$

and in the first two cases it further reduces to

$$\tilde{q}_{ik}^{(u)} = q_{ik}^{(u)}.$$

CASE I. We first consider the situation where there is no coloured arrow $i \to j$, i.e. $q_{ij}^{(u)} = 0$ for all u. That is, we assume Q_T locally looks like this

$$\begin{tikzcd} T_i \arrow[rr, bend right, "(c)"'] & T_j \arrow[r, "(d)"] & T_k \end{tikzcd}$$

with c, d arbitrary. It is a direct consequence of Proposition 5.5 that $q_{ik}^{(u)} = \tilde{q}_{ik}^{(u)}$ for all u which shows that the formula holds.

CASE II. We consider the setting where we assume Q_T locally looks like this

$$\begin{tikzcd} T_i \arrow[r, "(e)"] \arrow[rr, bend right, "(c)"'] & T_j \arrow[r, "(d)"] & T_k \end{tikzcd}$$

with $e \neq m$ and $q = 0$.

We then claim that we have the following, which shows that the formula holds.

Lemma 6.1. *In the above setting $q_{ik}^{(u)} = \widetilde{q}_{ik}^{(u)}$ for all u.*

Proof. It follows directly from Proposition 5.5 that $q_{ik}^{(u)} = \widetilde{q}_{ik}^{(u)}$ for $u = 0, \dots, e-1$. We claim that $q_{ik}^{(e)} = \widetilde{q}_{ik}^{(e)}$.

By Lemma 5.3 we have the (not necessarily minimal) left $\mathrm{add}(T'/T_i)$-approximation

$$T_i^{(e)} \to (T_k)^{pq} \amalg Y^p \amalg X = Y^p \amalg X.$$

First, assume that T_k does not appear as a summand in $B_i^{(e)} = (T_j)^p \amalg X$, then the same holds for $Y^p \amalg X$, and hence for $(B_i^{(e)})'$ which is a direct summand in $Y^p \amalg X$.

Next, assume T_k appears as a summand in $B_i^{(e)}$, and hence in X. Then T_k is by Proposition 3.5 not a summand in $B_i^{(e+1)}$, and by Lemma 5.4 we have that T_k is also not a summand in C'. Therefore T_k appears with the same multiplicity in $B_i^{(e)}$ as in $(B_i^{(e)})'$, also in this case.

We now show that $q_{ik}^{(u)} = \widetilde{q}_{ik}^{(u)}$ for $u > e$.

If $q_{ik}^{(e)} \neq 0$, then $q_{ik}^{(u)} = \widetilde{q}_{ik}^{(u)} = 0$ for $u > e$ and we are finished. So assume $q_{ik}^{(e)} = 0$, i.e. T_k does not appear as a direct summand of X.

Consider the map

$$(T_i^{(e+1)})' \amalg C' \to B_i^{(e+1)} \amalg (T_j^{(1)})^p.$$

We have that $(B_i^{(e+1)})' \simeq \frac{B_i^{(e+1)} \amalg (T_j^{(1)})^p}{C'}$. By assumption, T_k is not a direct summand in $(T_k)^{pq} \amalg Y^p \amalg X = Y^p \amalg X$, and thus not in C'. Hence it follows that $q_{ik}^{(e+1)} = \widetilde{q}_{ik}^{(e+1)}$.

Since, by Proposition 5.5 we have for $u = e+2, \dots, m$, that $(T_i^{(u)})' = T_i^{(u)}$ and the left $\mathrm{add}(T/T_i)$-approximation coincide with the left $\mathrm{add}(T'/T_i)$-approximations of $T_i^{(u)}$, it now follows that $q_{ik}^{(u)} = \widetilde{q}_{ik}^{(u)}$ for all u. □

CASE III. We now consider the setting with p non-zero, $q \neq 0$ and $e \neq m$. That is, we assume Q_T locally looks like this

$$\begin{tikzcd} T_i \arrow[r, "(e)"] \arrow[rr, bend right, "(c)"'] & T_j \arrow[r, "(0)"] & T_k \end{tikzcd}$$

where $c \in \{e, e+1\}$ by Proposition 5.1, and where there are $z = q_{ik}^{(c)} \geq 0$ arrows from T_i to T_k.

Lemma 6.2. *In the above setting, we have that $Q_{T'}$ is given by*

$$\widetilde{q}_{ik}^{(e)} = \begin{cases} q_{ij}^{(e)} q_{jk}^{(0)} + q_{ik}^{(e)} & \textit{if } c = e \\ q_{ij}^{(e)} q_{jk}^{(0)} - q_{ik}^{(e+1)} & \textit{if } c = e+1 \textit{ and } q_{ik}^{(e+1)} \leq q_{ij}^{(e)} q_{jk}^{(0)} \\ 0 & \textit{otherwise} \end{cases} \tag{7}$$

$$\widetilde{q}_{ik}^{(e+1)} = \begin{cases} -q_{ij}^{(e)} q_{jk}^{(0)} + q_{ik}^{(e+1)} & \textit{if } c = e+1 \textit{ and } q_{ik}^{(e+1)} > q_{ij}^{(e)} q_{jk}^{(0)} \\ 0 & \textit{otherwise} \end{cases} \tag{8}$$

and

$$\widetilde{q}_{ik}^{(u)} = 0 \textit{ for } u \notin \{e, e+1\} \tag{9}$$

Proof. We first deal with the case where $c = e$ and $z > 0$. By assumption X in the triangle (3) has z copies of T_k, so $(T_k)^{pq} \amalg Y^p \amalg X$ has $pq + z$ copies of T_k. Hence to show (7) it is sufficient to show that C' in the triangle

$$T_i^{(e+1)} \to (T_i^{(e+1)})' \amalg C' \to (T_j^{(1)})^p \to$$

has no copies of T_k. This follows directly from the Lemma 5.4 and the fact that T_k (by the assumption that $z > 0$ and Proposition 3.5) is not a summand in $B_i^{(e+1)}$. In this case (8) and (9) follow directly from Proposition 3.5.

Consider the case with $c = e + 1$ and $0 \leq z \leq pq$. We have that X in the triangle (3) does not have T_k as a direct summand. Assume T_k appears as a direct summand of C' with multiplicity z'. We claim that $z' = z$. Assume first $z' < z$, then on one hand T_k appears with multiplicity $z - z' > 0$ in $(B_i^{(e+1)})'$. On the other hand T_k appears with multiplicity $pq - z' > 0$ in $(B_i^{(e)})'$. This contradicts Proposition 3.5. Hence $z' = z$.

Therefore $(B_i^{(e)})'$ has $pq - z$ copies of T_k and (7) and (8) hold. If $pq \neq z$, then (9) follows directly from the above and Proposition 3.5. In the case $pq = z$, we also need to show that T_k does not appear as a summand in $(B_i^{(u)})'$ for $u \neq e + 1$. Since $pq \neq 0$, we have $z \neq 0$, and the result follows from Proposition 5.5.

Now assume $c = e + 1$ and $z > pq$. Assume $C' = (T_k)^l \amalg C''$, where T_k is not a summand in C''. Now since

$$(T_i^{(e+1)})' \amalg T_k^l \amalg C'' \to (T_j^{(1)})^p \amalg T_k^z \amalg Y,$$

with T_k not a summand in Y, is a minimal left $\operatorname{add} T'$-approximation, we have that $l \leq pq < z$ and T_k appears with multiplicity $z - l > 0$ in the minimal left $\operatorname{add} T'/(T_i)$-approximation of $(T_i^{(e+1)})'$, hence T_k cannot appear as a summand in the minimal left $\operatorname{add} T'/T_i$-approximation of $(T_i^{(e)})'$. Hence $l = pq$, and we have completed the proof of (7) and (8) in this case. The case (9), i.e. $u \neq e, e+1$ follows from Proposition 3.5. □

CASE IV. We now consider the case with $q_{ij}^{(m)} \neq 0$. Assume first there are no arrows

from j to k. Then we can use the symmetry proved in Proposition 4.1 and reduce to case I. The formula is easily verified in this case.

Assume $d \neq 0$, again we can use the symmetry, this time to reduce to case III. It is straightforward to verify that the formula holds also in this case.

Assume now that $d = 0$, i.e. we need to consider the following case

$$\begin{tikzcd} T_i \arrow[r, shift left, "(m)"] \arrow[rr, bend right, "(c)"'] \arrow[rrr, bend right=60, "(m-c)"'] & T_j \arrow[l, shift left, "(0)"] \arrow[r, shift left, "(0)"] & T_k \arrow[l, shift left, "(m)"] \end{tikzcd}$$

Now by Proposition 5.1 we have that c is in $\{m, 0\}$. Assume there are $z \geq 0$ (c)-coloured arrows

The coloured quiver of T' is of the form

$$\begin{tikzcd} T_i \arrow[r, shift left, "(0)"] \arrow[rr, bend right, "(c')"'] \arrow[rrr, bend right=60, "(m-c')"'] & T_j^{(1)} \arrow[l, shift left, "(m)"] \arrow[r, shift left, "(m)"] & T_k \arrow[l, shift left, "(0)"] \end{tikzcd}$$

and applying the symmetry of Proposition 4.1 we have that if $z > 0$, then $c' \in \{0, m\}$ by Proposition 5.1. Hence for all $u \notin \{0, m\}$ we have that $\widetilde{q}_{ik}^{(u)} = q_{ik}^{(u)} = 0$. Therefore it suffices to show that $\widetilde{q}_{ik}^{(u)} = q_{ik}^{(u)}$, for $u \in \{0, m\}$. This is a direct consequence of the following.

Lemma 6.3. *Assume we are in the above setting. A map $T_i \to T_k$ or $T_k \to T_i$ is irreducible in* $\operatorname{add} T$ *if and only if it is irreducible in* $\operatorname{add} T'$.

Proof. Assume $T_i \to T_k$ is not irreducible in $\operatorname{add} T'$, and that $T_i \to T_k = T_i \to U \to T_k$ for some $U = \amalg_t U_t \in \operatorname{add} T'$, with U_t the indecomposable direct summands of U. Note that by Lemma 3.2(a), we can assume that all $T_i \to U_t$ and all $U_t \to T_k$ are non-isomorphisms. If there is some index t such that $U_t \simeq T_j^{(1)}$, the map $U_t \to T_k$ factors through some U' in $\operatorname{add}(T/(T_i \amalg T_k))$, since there are no (1)-coloured arrows $j \to i$ or $j \to k$ in the coloured quiver of T. This shows that $T_i \to T_k$ is not irreducible in $\operatorname{add} T$.

Assume $T_i \to T_k$ is not irreducible in $\operatorname{add} T$, and that $T_i \to T_k = T_i \to V \to T_k$ for some $V = \amalg_t V_t \in \operatorname{add} T$, with V_t the indecomposable direct summands of V. If there is some index t such that $V_t \simeq T_j$, the map $T_i \to V_t$ factors through $B_j^{(m)}$, which is in $\operatorname{add}(T/(T_i \amalg T_j \amalg T_k)) \subset \operatorname{add} T'$, since there are no (0)-coloured arrows $i \to j$ or $k \to j$ in the coloured quiver of T. This shows that $T_i \to T_k$ is not irreducible in $\operatorname{add} T'$.

By symmetry, the same property holds for maps $T_k \to T_i$. $\square$

Thus we have proven that the formula holds in all four cases, and this finishes the proof of Theorem 2.1.

7. m-cluster-tilted algebras

An m-cluster-tilted algebra is an algebra given as $\mathrm{End}_{\mathcal{C}}(T)$ for some tilting object T in an m-cluster category $\mathcal{C} = \mathcal{C}_m$. Obviously, the subquiver of the coloured quiver of T given by the (0)-coloured maps is the Gabriel quiver of $\mathrm{End}_{\mathcal{C}}(T)$.

An application of our main theorem is that the quivers of the m-cluster-tilted algebras can be combinatorially determined via repeated (coloured) mutation. For this one needs transitivity in the tilting graph of m-tilting objects. More precisely, we need the following, which is also pointed out in [ZZ].

Proposition 7.1. *Any m-tilting object can be reached from any other m-tilting object via iterated mutation.*

Proof. We sketch a proof for the convenience of the reader. Let T' be a tilting object in an m-cluster category $\mathcal{C}$ of the hereditary algebra $H = KQ$, and let $\mathcal{C}_1$ be the 1-cluster category of H. By [Z], there is a tilting object T of degree 0, i.e. all direct summands in T have degree 0, such that T can be reached from T' via mutation. It is sufficient to show that the canonical tilting object H can be reached from T via mutation. Since T is of degree 0, it is induced from a H-tilting module. Especially T is a tilting object in $\mathcal{C}_1$. Since T and H are tilting objects in $\mathcal{C}_1$, by [BMRRT] there are $\mathcal{C}_1$-tilting objects $T = T_0, T_1, \ldots, T_r = H$, such that T_i mutates to T_{i+1} (in $\mathcal{C}_1$) for $i = 0, \ldots, r-1$. Now each T_i is induced by a tilting module for some Q_i where all KQ_i are derived equivalent to KQ. Hence, each T_i is easily seen to be an m-cluster tilting object. Since T_{i+1} differs from T_i in only one summand the mutations in $\mathcal{C}_1$ are also mutations in $\mathcal{C}$. This concludes the proof. □

A direct consequence of the transitivity is the following.

Corollary 7.2. *For an m-cluster category $\mathcal{C} = \mathcal{C}_m$ of the acyclic quiver Q, all quivers of m-cluster-tilted algebras are given by repeated coloured mutation of Q.*

8. Combinatorial computation

In this section, we discuss concrete computation with tilting objects in an m-cluster tilting category.

An exceptional indecomposable object in $\operatorname{mod} H$ is uniquely determined by its image $[T]$ in the Grothendieck group $K_0(\operatorname{mod} H)$. There is a map from $\mathcal{D}^b(\operatorname{mod} H)$ to $K_0(\operatorname{mod} H)$ which, for $T \in \operatorname{mod} H$, takes $T[i]$ to $(-1)^i[T]$. An exceptional indecomposable in $\mathcal{D}^b(\operatorname{mod} H)$ can be uniquely specified by its class in $K_0(\operatorname{mod} H)$ together with its degree.

The map from $\mathcal{D}^b(\operatorname{mod} H)$ to $K_0(\operatorname{mod} H)$ does not descend to $\mathcal{C}$. However, if we fix our usual choice of fundamental domain in $\mathcal{D}^b(\operatorname{mod} H)$, then we can identify the indecomposable objects in it as above.

Let us define the combinatorial data corresponding to a tilting object T to be Q_T together with $([T_i], \deg T_i)$ for $1 \leq i \leq n$.

Theorem 8.1. *Given the combinatorial data for a tilting object T in $\mathcal{C}$, it is possible to determine, by a purely combinatorial procedure, the combinatorial data for the tilting object which results from an arbitrary sequence of mutations applied to T.*

Proof. Clearly, it suffices to show that, for any i, we can determine the class and degree for $T_i^{(j)}$. If we can do that then, by the coloured mutation procedure, we can determine the coloured quiver for $(T/T_i) \amalg T_i^{(j)}$, and by applying this procedure repeatedly, we can calculate the result of an arbitrary sequence of mutations.

Since we are given Q_T, we know $B_i^{(0)}$, and we can calculate $[B_i^0]$. Now we have the following lemma:

Lemma 8.2. $[T_i^{(1)}] = [B_i^{(0)}] - [T_i^{(0)}]$, *and* $\deg(T_i^{(1)}) = \deg T_i$ *or* $\deg T_i + 1$, *whichever is consistent with the sign of the class of* $[T_i^{(1)}]$, *unless this yields a non-projective indecomposable object in degree m, or an indecomposable of degree $m+1$.*

Proof. The proof is immediate from the exchange triangle $T_i \to B_i^{(0)} \to T_i^{(1)} \to$. □

Applying this lemma, and supposing that we are not in the case where its procedure fails, we can determine the class and degree $T_i^{(1)}$. By the coloured mutation procedure, we can also determine the coloured quiver for $\mu_i(T)$. We therefore have all the necessary data to apply Lemma 8.2 again. Repeatedly applying the lemma, there is some k such that we can calculate the class and degree of $T_i^{(j)}$ for $1 \leq j \leq k$, and the procedure described in the lemma fails to calculate $T_i^{(k+1)}$.

We also have the following lemma:

Lemma 8.3. $[T_i^{(m)}] = [B_i^{(m)}] - [T_i^{(0)}]$, *and* $\deg T_i^{(m)} = \deg T_i$ *or* $\deg T_i - 1$, *whichever is consistent with the sign of* $[T_i^{(m)}]$, *unless this yields an indecomposable in degree* -1.

Applying this lemma, starting again with T, we can obtain the degree and class for $T_i^{(m)}$. We can then determine the coloured quiver for $\mu_i^{-1}(T)$, and we are now in a position to apply Lemma 8.3 again. The last complement which Lemma 8.3 will successfully determine is $T_i^{(k+1)}$. It follows that we can determine the degree and class of any complement to T/T_i. □

9. The m-cluster complex

In this section, we discuss the application of our results to the study of the m-cluster complex, a simplicial complex defined in [FR] for a finite root system Φ. We shall begin by stating our results for the m-cluster complex in purely combinatorial language, and then briefly describe how they follow from the representation-theoretic perspective in the rest of the paper. For simplicity, we restrict to the case where Φ is simply laced.

Number the vertices of the Dynkin diagram for Φ from 1 to n. The m-coloured almost positive roots, $\Phi^m_{\geq -1}$, consist of m copies of the positive roots, numbered 1 to m, together with a single copy of the negative simple roots. We refer to an element of the i-th copy of Φ^+ as having colour i, and we write such an element as $\beta^{(i)}$.

Since the Dynkin diagram for Φ is a tree, it is bipartite; we fix a bipartition $\{1, \ldots, n\} = I_+ \cup I_-$.

The m-cluster complex, Δ_m, is a simplicial complex on the ground set $\Phi^m_{\geq -1}$. Its maximal faces are called m-clusters. The definition of Δ_m is combinatorial; we refer the reader to [FR]. The m-clusters each consist of n elements of $\Phi^m_{\geq -1}$ [FR, Theorem 2.9]. Every codimension 1 face of Δ_m is contained in exactly $m+1$ maximal faces [FR, Proposition 2.10]. There is a certain combinatorially-defined bijection $R_m \colon \Phi^m_{\geq -1} \to \Phi^m_{\geq -1}$, which takes faces of Δ_m to faces of Δ_m [FR, Theorem 2.4].

It will be convenient to consider *ordered m-clusters*. An ordered m-cluster is just a n-tuple from $\Phi^m_{\geq -1}$, the set of whose elements forms an m-cluster. Write Σ_m for the set of ordered m-clusters.

For each ordered m-cluster $C = (C_1, \ldots, C_n)$, we will define a coloured quiver Q_C. We will also define an operation $\mu_j : \Sigma_m \to \Sigma_m$, which takes ordered m-clusters to ordered m-clusters, changing only the j-th element.

We will define both operations inductively. The set $-\Pi$ of negative simple roots forms an m-cluster. Its associated quiver is defined by drawing, for each edge ij in the Dynkin diagram, a pair of arrows. Suppose $i \in I_+$ and $j \in I_-$. Then we draw an arrow from i to j with colour 0, and an arrow from j to i with colour m.

Suppose now that we have some ordered m-cluster C, together with its quiver Q_C. We will now proceed to define $\mu_j(C)$. Write $q^{(0)}_{jk}$ for the number of arrows in Q_C of colour 0 from j to k. Define:

$$\beta = -C_j + \sum_{k \neq j} q^{(0)}_{jk} C_k.$$

Let c be the colour of C_j. We define $\mu_j(C)$ by replacing C_j by some other element of $\Phi^m_{\geq -1}$, according to the following rules:

- If C_j is positive and β is positive, replace C_j by $\beta^{(c)}$.
- If C_j is positive and β is negative, replace C_j by $R_m(-\beta^{(c)})$.
- If C_j is negative simple $-\alpha_i$, define γ by $\gamma^{(0)} = R_m(-\alpha_i)$, and then replace C_j by $\beta + C_j - \gamma$, with colour zero.

Define the quiver for the m-cluster $\mu_j(C)$ by the coloured quiver mutation rule from Section 2. Since any m-cluster can be obtained from $-\Pi$ by a sequence of mutations, the above suffices to define $\mu_j(C)$ and Q_C for any ordered m-cluster C.

Proposition 9.1. *The operation μ_j defined above takes m-clusters to m-clusters, and the m-clusters $\mu^i_j(C)$ for $0 \leq i \leq m$ are exactly those containing all the C_i for $i \neq j$.*

The connection between the combinatorics discussed here and the representation theory in the rest of the paper is as follows. $\Phi^m_{\geq -1}$ corresponds to the indecomposable objects of (a fundamental domain for) $\mathcal{C}_m$. The cluster tilting objects in $\mathcal{C}_m$ correspond to the m-clusters. The operation R_m corresponds to [1]. For further details on the translation, the reader is referred to [T, Z]. The above proposition then follows from the approach taken in Section 8.

10. An alternative algorithm for coloured mutation

Here we give an alternative description of coloured quiver mutation at vertex j.

(1) For each pair of arrows

$$i \xrightarrow{(c)} j \xrightarrow{(0)} k$$

with $i \neq k$, the arrow from i to j of arbitrary colour c, and the arrow from j to k of colour 0, add a pair of arrows: an arrow from i to k of colour c, and one from k to i of colour $m - c$.

(2) If the graph violates property II, because for some pair of vertices i and k there are arrows from i to k which have two different colours, cancel the same number of arrows of each colour, until property II is satisfied.

(3) Add one to the colour of any arrow going into j and subtract one from the colour of any arrow going out of j.

Proposition 10.1. *The above algorithm is well-defined and correctly calculates coloured quiver mutation as previously defined.*

Proof. Fix a quiver Q and a vertex j at which the mutation is being carried out.

To prove that the algorithm is well-defined, we must show that at step 2, there are only two colours of arrows running from i to k for any pair of vertices i, k. (Otherwise there would be more than one way to carry out the cancellation procedure of step 2.)

Since in the original quiver Q, there was only one colour of arrows from i to k, in order for this problem to arise, we must have added two different colours of arrows from i to k at step 1. Two colours of arrows will only be added from i to k if, in Q, there are both (0)-coloured arrows from j to k and from j to i. In this case, by property III, there are (m)-coloured arrows from i to j and from k to j. It follows that in step 1, we will add both (0)-coloured and (m)-coloured arrows. Applying Proposition 5.1, we see that any arrows from i to k in Q are of colour 0 or m. Thus, as desired, after step 1, there are only two colours of arrows in the quiver, so step 2 is well-defined.

We now prove correctness. Let $\widetilde{Q} = \mu_j(Q)$. Write $q_{ij}^{(c)}$ for the number of c-coloured arrows from i to j in Q, and similarly $\widetilde{q}_{ij}^{(c)}$ for $\widetilde{Q}$. Write $\hat{Q}$ and $\hat{q}_{ij}^{(c)}$ for the result of applying the above algorithm.

It is clear that only the final step of the algorithm is relevant for $\hat{q}_{ik}$ where one of i or k coincides with j, and therefore that in this case $\hat{q}_{ij}^{(c)} = \widetilde{q}_{ij}^{(c)}$ as desired.

Suppose now that neither i nor k coincides with j. Suppose further that in Q there are no (0)-coloured arrows from either i or k to j, and therefore also no m-coloured arrows from k to i or j. In this case, $\widetilde{q}_{ik}^{(c)} = q_{ik}^{(c)}$. In the algorithm, no arrows will be added between i and k in step 1, and therefore no further changes will be made in step 2. Thus $\hat{q}_{ik}^{(c)} = q_{ik}^{(c)} = \widetilde{q}_{ik}^{(c)}$, as desired.

Suppose now that there are (0)-coloured arrows from j to both i and k. In this case, $\widetilde{q}_{ik}^{(c)} = q_{ik}^{(c)}$. In this case, as discussed in the proof of well-definedness, an equal

number of (0)-coloured and (m)-coloured arrows will be introduced at step 1. They will therefore be cancelled at step 2. Thus $\hat{q}_{ik}^{(c)} = q_{ik}^{(c)} = \widetilde{q}_{ik}^{(c)}$ as desired.

Suppose now that there is a (0)-coloured arrow from j to k, but not from j to i. Let the arrows from i to j, if any, be of colour c. At step 1 of the algorithm, we will add $q_{ij}^{(c)} q_{jk}^{(0)}$ arrows of colour c to Q. By Proposition 5.1, the arrows in Q from i to k are of colour c or $c+1$. One verifies that the algorithm yields the same result as coloured quiver mutation, in the three cases that the arrows from i to k in Q are of colour c, that they are of colour $c+1$ but there are fewer than $q_{ij}^{(c)} q_{jk}^{(0)}$, and that they are of colour $c+1$ and there are at least as many as $q_{ij}^{(c)} q_{jk}^{(0)}$.

The final case, that there is a (0)-coloured arrow from j to i but not from j to k, is similar to the previous one. □

11. Example: type A_n

In [BM1], a certain category $\mathcal{C}_{BM}$ is constructed, which is shown to be equivalent to the m-cluster category of Dynkin type A_n. The description of $\mathcal{C}_{BM}$ is as follows. Take an $nm+2$-gon Π, with vertices labelled clockwise from 1 to $nm+2$. Consider the set X of diagonals γ of Π with the property that γ divides Π into two polygons each having a number of sides congruent to 2 modulo m. For each $\gamma \in X$, there is an object A_γ in $\mathcal{C}_{BM}$. These objects A_γ form the indecomposables of the additive category $\mathcal{C}_{BM}$. We shall not recall the exact definition of the morphisms, other than to note that they are generated by the morphisms $p_{ijk} : A_{ij} \to A_{ik}$ which exist provided that ij and ik are both diagonals in X, and that, starting at j and moving clockwise around Π, one reaches k before i.

A collection of diagonals in X is called non-crossing if its elements intersect pairwise only on the boundary of the polygon. An inclusion-maximal such collection of diagonals divides Π into $m+2$-gons; we therefore refer to such a collection of diagonals as an $m+2$-angulation. If we remove one diagonal γ from an $m+2$-angulation Δ, then the two $m+2$-gons on either side of γ become a single $2m+2$-gon. We say that γ is a *diameter* of this $2m+2$-gon, since it connects vertices which are diametrically opposite (with respect to the $2m+2$-gon). If δ is another diameter of this $2m+2$-gon, then $(\Delta \setminus \gamma) \cup \delta$ is another maximal noncrossing collection of diagonals from X. (In particular, $\delta \in X$.)

For Δ an $m+2$-angulation, let $A_\Delta = \amalg_{\gamma \in \Delta} A_\gamma$. Then we have that A_Δ is a basic (m-cluster-)tilting object for $\mathcal{C}_{BM}$, and all basic tilting objects of $\mathcal{C}_{BM}$ arise in this way. It follows from the previous discussion that if $T = A_\Delta$ is a basic tilting object, and $\gamma \in \Delta$, then the complements to $A_{\Delta \setminus \gamma}$ will consist of the objects A_δ where δ is a diameter of the $2m+2$-gon obtained by removing γ from the $m+2$-angulation determined by Δ. In fact, we can be more precise. Define $\delta_{(i)}$ to be the diameter of the $2m+2$-gon obtained by rotating the vertices of γ by i steps counterclockwise (within the $2m+2$-gon). Then $A_\gamma^{(i)} = A_{\delta_{(i)}}$.

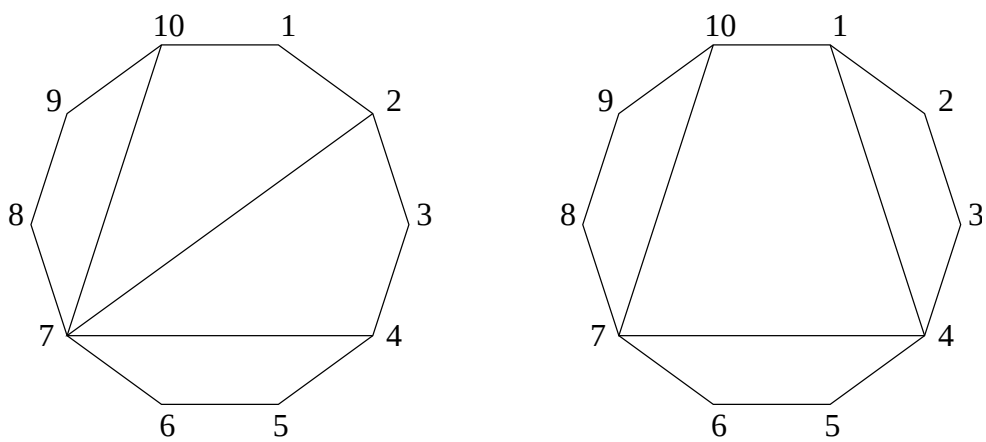
Define Q_Δ to be the coloured quiver for the tilting object A_Δ, with vertex set Δ. Using the setup of [BM1], it is straightforward to verify:

Proposition 11.1. *The coloured quiver Q_Δ of $T = A_\Delta$ has an arrow from γ to δ if and only if γ and δ both lie on some $m+2$-gon in the $m+2$-angulation defined by Δ. In this case, the colour of the arrow is the number of edges forming the segment of the boundary of the $m+2$-gon which lies between γ and δ, counterclockwise from γ and clockwise from δ.*

Given the proposition above, it is straightforward to verify directly that Q_Δ satisfies conditions (I), (II), and (III), and that mutation is indeed given by the mutation of coloured quivers.

Example: A_3**,** $m = 2$

We return to the example from Section 2. The quadrangulation of a decagon corresponding to the tilting object T is on the left. The quadrangulation corresponding to T' is on the right. Passing from the figure on the left to the figure on the right, the diagonal 27 (which corresponds to the summand I_2) has been rotated one step counterclockwise within the hexagon with vertices 1,2,3,4,7,10.



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