Combinatorial properties of double square tiles*

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Abstract

We study the combinatorial properties and the problem of generating exhaustively double square tiles, i.e. polyominoes yielding two distinct periodic tilings by translated copies such that every polyomino in the tiling is surrounded by exactly four copies. We show in particular that every prime double square tile may be obtained from the unit square by applying successively some invertible operators on double squares. As a consequence, we prove a conjecture of Provençal and Vuillon [17] stating that these polyominoes are invariant under rotation by angle π.

Keywords: Tilings, generation, polyomino, double square tile, palindromes.

1 Introduction

When considering the problem of deciding whether a given polygon tiles the plane, it is convenient to restrict ourselves to polyominoes, that is, subsets of the square lattice $\mathbb{Z}^2$ whose boundary is a non-crossing closed path (see [14] for more on tilings and [7] for related problems). Here, we consider tilings obtained by translation of a single polyomino, called *exact* in [20]. Paths are conveniently described by words on the alphabet $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, representing the elementary grid steps $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$. Beauquier and Nivat [1] characterized exact polyominoes by showing that the boundary word $b(P)$ of such a polyomino satisfies the equation $b(P) = X \cdot Y \cdot Z \cdot \widehat{X} \cdot \widehat{Y} \cdot \widehat{Z}$, where $\widehat{W}$ is the path traveled in the direction opposite to that of W (the paths W and $\widehat{W}$ are said *homologous*). From now on, this condition is referred to as the BN-factorization. In this factorization, one of the variables may be empty, in which case P is called a *square*, and *hexagon*

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otherwise. Note that a single polyomino may lead to several distinct tilings of the plane: for instance the $n \times 1$ rectangle does it in $n-1$ distinct ways as a hexagon (see Figure 1).

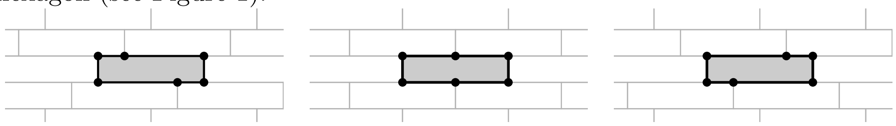

Figure 1: The three hexagonal tilings of the 4×1 rectangle.

However, it was recently established [5] that an exact polyomino tiles the plane as a square in at most two distinct ways. A polyomino having exactly two distinct square tilings is called *double square* [17] and there is a linear time algorithm to find all the square factorizations from its boundary word [11]. Double squares have a peculiar combinatorial structure and motivated developments in equations on words involving periodicities and palindromes [3] Christoffel and Fibonacci tiles were introduced in [4] as examples of infinite families of double squares (see Figure 2) but do not characterize completely the class of double square tiles (see Figure 3).

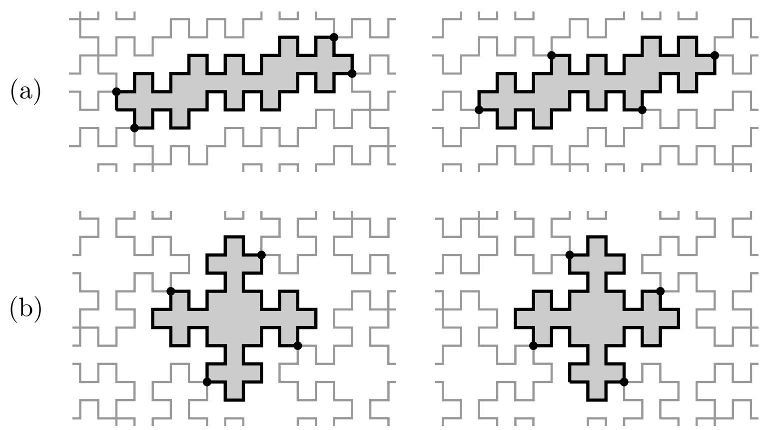

Figure 2: (a) A Christoffel tile yields two distinct nonsymmetric square tilings of the plane. (b) The Fibonacci tile of order two with its two symmetric square tilings. Note that both tiles are invariant under a rotation of angle π.

In this article, double square tiles are represented by DS-factorizations: factorization of the boundary into eight parts. We show that any DS-factorization of a double square tile can be reduced to a singular DS-factorization using only two reduction operators (Theorem 22). It appears that these operators are invertible which allows one to generate double square tiles from singular DS-factorizations. An algorithm for the generation of double square tiles is proposed, allowing thus to generate Christoffel and Fibonacci tiles under common rules. Moreover, we prove that every prime double square can be reduced to the unit square by using some reduction operators (Theorem 30). By *prime*, we

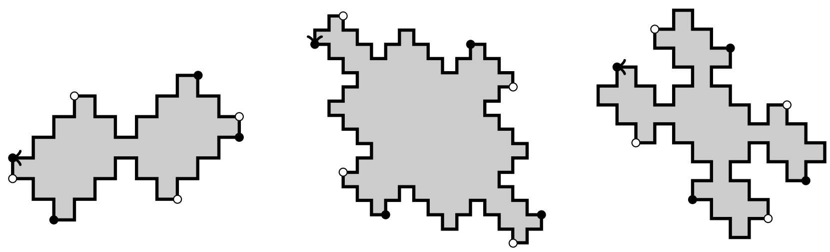

Figure 3: Some double square tiles that are neither Christoffel nor Fibonacci tiles. The two square-factorizations in each case are represented by black and white dots. The boundary of the polyomino is traveled counter-clockwise and ends with the arrow.

mean a polyomino that is not obtained from a smaller one by replacing each unit cell by another polyomino (see Figure 15, Page 25). This allows to show that the BN-factorizations of prime double squares consist of palindromes, solving positively a conjecture of Provençal and Vuillon [17]:

Theorem 1. *If $AB\widehat{A}\widehat{B}$ and $XY\widehat{X}\widehat{Y}$ are the BN-factorizations of a prime double square D, then A, B, X and Y are palindromes or equivalently D is invariant under a rotation by angle π.*

The article is divided as follows. Definitions, notations and some basic results on combinatorics on words and discrete paths are presented in Section 2. The concept of double square factorization is introduced in Section 3 and is followed by many obervations and lemmas in preparation of the next sections. Reduction of double squares is considered in Section 4, while generation operators are defined and studied in Section 5, where some algebraic properties are also established. Finally, prime double squares, homologous morphisms and the proof of Theorem 1 may be found in Section 6.

The results presented below were initially observed through computer exploration. Consequently, an implementation in Python of the different operators and algorithms involved will be made available in the open-source software Sage [16].

2 Preliminaries

The usual terminology and notation on words is from Lothaire [15]. An *alphabet* $\mathcal{A}$ is a finite set whose elements are *letters*. A finite word w is a function $w : [1, 2, \ldots, n] \to \mathcal{A}$, where w_i is the i-th letter, $1 \le i \le n$. The *length* of w, denoted by $|w|$, is the integer n. The length of the empty word denoted by ε is 0. The *free monoid* $\mathcal{A}^*$ is the set of all finite words over $\mathcal{A}$. The *reversal* of $w = w_1 w_2 \cdots w_n$ is the word $\widetilde{w} = w_n w_{n-1} \cdots w_1$. Given a nonempty word w, let $\textsc{Fst}(w) = w_1$ and $\textsc{Lst}(w) = w_n$ denote respectively the first and last letter of the word w. A word u is a *factor* of another word w if there exist $x, y \in \mathcal{A}^*$ such that $w = xuy$. If $x = \varepsilon$, then u is called *prefix* and if $y = \varepsilon$, it is called a *suffix* of w. Let u be a prefix of some word w. We denote by $u^{-1}w$ the unique word such that $uu^{-1}w = w$. Roughly speaking, $u^{-1}w$ is the word obtained from w by deleting the prefix u. The notation wu^{-1} is defined similarly for u a suffix of w. We denote by $|w|_u$ the number of occurrences of u in w. Two words u and v are *conjugate*, written $u \equiv v$ or sometimes $u \equiv_{|x|} v$, when x, y are such that $u = xy$ and $v = yx$. Conjugacy is an equivalence relation and the class of a word w is denoted $[w]$.

A *power* of a word u is a word of the form u^k for some integer $k \in \mathbb{N}$. It is convenient to set $u^0 = \varepsilon$ for each word u. When $k > 1$ is an integer we say that u^k is a *proper power* of u. A nonempty word is called *primitive* if it is not a proper power of another word. Let u be a nonempty word; then there exist a unique primitive word z and a unique integer $k \geq 1$ such that $u = z^k$. The word z is called *the primitive root* of u.

Given two alphabets A and B, a *morphism* is a function $\varphi : A^* \to B^*$ compatible with concatenation, that is, $\varphi(uv) = \varphi(u)\varphi(v)$ for any $u, v \in A^*$. It is clear that a morphism is completely defined by its action on the letters of A.

In this article, the alphabet $\mathcal{F} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ is considered as the additive group of integers modulo 4. Basic transformations on $\mathcal{F}$ are rotations $\rho^i : x \mapsto x + i$ and reflections $\sigma_i : x \mapsto i - x$, which extend uniquely to morphisms on $\mathcal{F}^*$. Another useful morphism, denoted by $\bar{\cdot}$, is the morphism defined by $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$. Given a nonempty word $w \in \mathcal{F}^*$, the *first differences word* $\Delta(w) \in \mathcal{F}^*$ of w is

$$\Delta(w) := (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}). \tag{1}$$

Given two nonempty word $w, z \in \mathcal{F}^*$, it is convenient to compare the last letter of w with the first letter of z. Hence, we define $\Delta(w, z) \in \mathcal{F}$ as the letter given by

$$\Delta(w, z) := \Delta(\textsc{Lst}(w)\textsc{Fst}(z)) = \Delta(w_n z_1) \tag{2}$$

One may verify that $\Delta(wz) = \Delta(w)\Delta(w, z)\Delta(z)$. Words in $\mathcal{F}^*$ are interpreted as paths in the square grid as usual (See Figure 4), so that we indistinctly talk of any word $w \in \mathcal{F}^*$ as the *path* w.

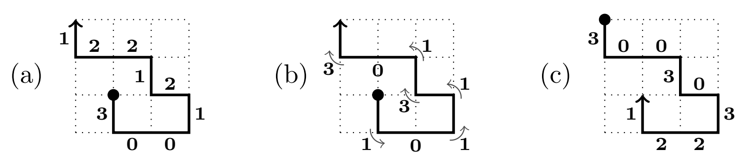

Figure 4: (a) The path w = **300121221**. (b) Its first differences word $\Delta(w)$ = **10113103**. (c) Its homologous $\widehat{w}$ = **300303221**.

Moreover, the word $\widehat{w} := \rho^2(\widetilde{w})$ is homologous to w, that is, described in direction opposite to that of w (see Figure 4). Notice that the operator $\widehat{\cdot}$ is an involutory antimorphism. A word $u \in \mathcal{F}^*$ may contain factors in $\mathcal{C} = \{\mathbf{02}, \mathbf{20}, \mathbf{13}, \mathbf{31}\}$, corresponding to cancelling steps on a path. Nevertheless, each word w can be reduced in a unique way to a word w', by sequentially applying the rewriting rules in $\{u \mapsto \varepsilon | u \in \mathcal{C}\}$. The *reduced word* w' of w is nothing but a word in $\mathcal{P} = \mathcal{F}^* \setminus \mathcal{F}^*\mathcal{C}\mathcal{F}^*$. We define the *turning number*[1] of w by

$$\mathcal{T}(w) = \frac{|\Delta(w')|_{\mathbf{1}} - |\Delta(w')|_{\mathbf{3}}}{4}, \tag{3}$$

where w' is the reduced word of w. Given two nonempty path w and z, it is practical as for the first difference word to compute the turning number of the word of length 2 consisting of the last letter of w and of the first letter of z. Thus, we define

$$\mathcal{T}(w, z) := \mathcal{T}\left(\textsc{Lst}(w)\textsc{Fst}(z)\right) \tag{4}$$

One may verify that $\mathcal{T}(wz) = \mathcal{T}(w) + \mathcal{T}(w, z) + \mathcal{T}(z)$. The turning number also satisfies $\mathcal{T}(w) = -\mathcal{T}(\widehat{w})$ and $\mathcal{T}(w, z) = -\mathcal{T}(\widehat{z}, \widehat{w})$.

A path w is *closed* if it satisfies $|w|_{\mathbf{0}} = |w|_{\mathbf{2}}$ and $|w|_{\mathbf{1}} = |w|_{\mathbf{3}}$, and it is *simple* if no proper factor of w is closed. A *boundary word* is a simple and closed path, and a *polyomino* is a subset of $\mathbb{Z}^2$ contained in some boundary word. It is convenient to represent each closed path w by its conjugacy class $[\![w]\!]$, also called *circular word*. An adjustment is necessary to the function $\mathcal{T}$, for we take into account the closing turn. The first differences also noted Δ is defined on any closed path w by setting

$$\Delta([\![w]\!]) = \begin{cases} [\![\Delta(w) \cdot \Delta(w, w)]\!] & \text{if } w \text{ is nonempty,} \\ [\![\varepsilon]\!] & \text{if } w \text{ is empty.} \end{cases} \tag{5}$$

which is also a circular word. By applying the same rewriting rules, a circular word $[\![w]\!]$ is *circularly-reduced* to a unique word $[\![w']\!]$. If w is a closed path, then the *turning number*[1] of w is

$$\mathcal{T}([\![w]\!]) = \frac{|\Delta([\![w']\!])|_{\mathbf{1}} - |\Delta([\![w']\!])|_{\mathbf{3}}}{4}. \tag{6}$$

It corresponds to its total curvature divided by 2π. Clearly, the turning number $\mathcal{T}([\![w]\!])$ of a closed path w belongs to $\mathbb{Z}$ (see [9, 10]). In particular, the Daurat-Nivat relation [12] is rephrased as follows.

Proposition 2. *The turning number of a boundary word w is $\mathcal{T}([\![w]\!]) = \pm 1$.*

Now, we may define orientation: a boundary word w is *positively oriented* (counterclockwise) if its turning number is $\mathcal{T}([\![w]\!]) = 1$. In general, if $XY\widehat{X}\widehat{Y}$ is a path (simple or not), its turning number may be computed from its square factorization by the formula

$$\mathcal{T}([\![XY\widehat{X}\widehat{Y}]\!]) = \mathcal{T}(X, Y) + \mathcal{T}(Y, \widehat{X}) + \mathcal{T}(\widehat{X}, \widehat{Y}) + \mathcal{T}(\widehat{Y}, X). \tag{7}$$

As a consequence, every square tile satifies the following condition.

Proposition 3. *Let $w \equiv XY\widehat{X}\widehat{Y}$ be the boundary word of a square, then*

$$\Delta(X, Y) = \Delta(Y, \widehat{X}) = \Delta(\widehat{X}, \widehat{Y}) = \Delta(\widehat{Y}, X) = \alpha$$

where $\alpha = \mathbf{1}$ if w is positively oriented, $\alpha = \mathbf{3}$ otherwise.

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[1] In [9, 10], the authors introduced the notion of *winding number* of w which is $4\mathcal{T}(w)$

The following result is easy to check.

Proposition 4. *Let $w \equiv XY\widehat{X}\widehat{Y}$ be an oriented boundary word of a square. Then $\textsc{Fst}(X) = \textsc{Lst}(X)$, $\textsc{Fst}(Y) = \textsc{Lst}(Y)$ and the first letter of X, $\widehat{X}$, Y, $\widehat{Y}$ are mutually distinct, that is,*

$$\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\} = \{\textsc{Fst}(X), \textsc{Fst}(\widehat{X}), \textsc{Fst}(Y), \textsc{Fst}(\widehat{Y})\}.$$

Proof. By Proposition 3, we have $\textsc{Lst}(X) + \mathbf{1} = \textsc{Fst}(Y)$. Similarly, $\textsc{Lst}(Y) + \mathbf{1} = \textsc{Fst}(\widehat{X})$, but $\textsc{Fst}(\widehat{X}) = \textsc{Lst}(X) + \mathbf{2}$. By subtracting $\mathbf{1}$ on each side, we get $\textsc{Lst}(Y) = \textsc{Lst}(X) + \mathbf{1}$. Hence, $\textsc{Lst}(Y) = \textsc{Fst}(Y)$. In the same way, we get $\textsc{Lst}(X) = \textsc{Fst}(X)$. From these equations, it follows that the set of first letters of X, Y, $\widehat{X}$ and $\widehat{Y}$ is $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$. □

3 Double square factorizations

In this section, we introduce the useful notion of double square factorization in order to describe all double squares. Its definition is motivated by the following result stating that the BN-factorizations of a double square must alternate.

Lemma 5. [11, 17] *If the boundary word of an exact polyomino satisfies $AB\widehat{A}\widehat{B} \equiv_d XY\widehat{X}\widehat{Y}$, with $0 \le d \le |A|$ and $\{A, B, \widehat{A}, \widehat{B}\} \neq \{X, Y, \widehat{X}, \widehat{Y}\}$, then the factorization must alternate, i.e., $0 < d < |A| < d + |X|$.*

Hence, we must have the situation depicted in Figure 5. Moreover, it is useful

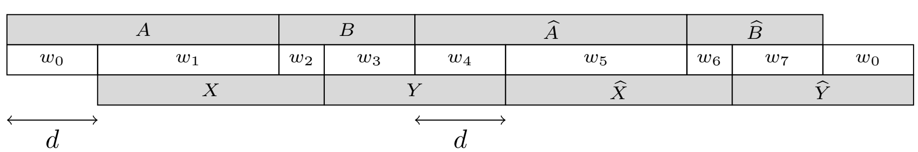


Figure 5: Finer factorization of a double square.

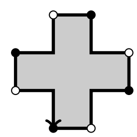
to encode double squares while keeping track of their (two) factorizations. For that purpose, we refine the BN-factorization as follows.

Definition 6. *A* double square factorization *(*DS-factorization *for short) is an* 8*-tuple $(w_i)_{i\in\mathbb{Z}_8}$, $w_i \in \mathcal{F}^*$, such that $|w_i| = |w_{i+4}|$ for $i \in \{0, 1, 2, 3\}$ and*

$$\begin{array}{rrcl rrcl} \text{(i)} & \widehat{w_0 w_1} & = & w_4 w_5; & \text{(iii)} & \widehat{w_2 w_3} & = & w_6 w_7; \\ \text{(ii)} & \widehat{w_1 w_2} & = & w_5 w_6; & \text{(iv)} & \widehat{w_3 w_4} & = & w_7 w_0. \end{array}$$

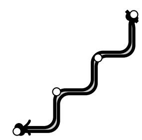
Its boundary *is the word $w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7$.*

Observe that every DS-factorization $(w_i)_{i\in\mathbb{Z}_8}$ is uniquely determined by the words w_0, w_1, w_2 and w_3. The *length* of a DS-factorization $S = (w_i)_{i\in\mathbb{Z}_8}$ is naturally defined as the length of its boundary $|S| = |w_0 w_1 \cdots w_7|$.



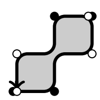
$(\mathbf{0}, \mathbf{10}, \mathbf{1}, \mathbf{21}, \mathbf{2}, \mathbf{32}, \mathbf{3})$

(a)



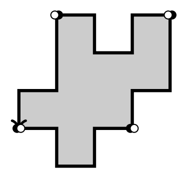
$(\varepsilon, \varepsilon, \mathbf{0101}, \mathbf{01}, \varepsilon, \varepsilon, \mathbf{3232}, \mathbf{32})$

(b)



$(\varepsilon, \mathbf{0}, \mathbf{10}, \mathbf{1}, \varepsilon, \mathbf{2}, \mathbf{32}, \mathbf{3})$

(c)



$(\varepsilon, \mathbf{03010}, \varepsilon, \mathbf{1011}, \varepsilon, \mathbf{23212}, \varepsilon, \mathbf{3323})$

(d)

Figure 6: Examples of DS-factorizations. (a) The X pentomino (following Golomb notation [13]) yields the smallest nondenegerate DS-factorization. (b) A flat DS-factorization whose corresponding circular path reduces to $[\varepsilon]$. (c) A degenerate DS-factorization: two pairs of black and white dot are the same. (d) A singular DS-factorization: the black and white dots coincide pairwise.

Some DS-factorizations play a particular role in the remainder of this article. For this purpose, it is convenient to introduce further definitions. Thus, we say that a DS-factorization S is *degenerate* if there exists i such that w_i is empty; *flat* if there exists i such that $w_i w_{i+1}$ is empty; *singular* if there exists i such that w_{i-1} and w_{i+1} are both empty. A singular DS-factorization is the *unit square* if its boundary word is conjugate to **0123** or **3210**. The choice of these adjectives are justified by the following remarks: if S is degenerate, the eight points of the two BN-factorizations partially coincide; if S is flat, then its boundary is of the form $X\widehat{X}$; if S is singular, then both square factorizations correspond to the same one. Of course, if S is singular or flat it is also degenerate. The different cases are illustrated in Figure 6.

Example 7. *Clearly, each double square yields a nonsingular DS-factorization. Indeed, consider the double square given in Figure 7: the black and white dots together with the ending arrow uniquely determine the DS-factorization*

$$(\mathbf{3}, \mathbf{03010303}, \mathbf{01030}, \mathbf{10103010}, \mathbf{1}, \mathbf{21232121}, \mathbf{23212}, \mathbf{32321232}).$$

In what follows we exhibit the properties satisfied by DS-factorizations. To fix the notation, hereafter $S = (w_i)_{i \in \mathbb{Z}_8}$ denotes a DS-factorization, and all indices are taken in $\mathbb{Z}_8$. The first result concerns periodicity. Let

$$d_i = |w_{i-1}| + |w_{i+1}|. \tag{8}$$

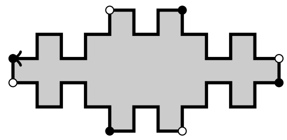


Figure 7: A double square and its DS-factorization. The black and white dots distinguish the two BN-factorizations.

Observe that if S is nonsingular, then $d_i \neq 0$ for all $i \in \mathbb{Z}_8$. Moreover, $d_i = d_{i+2}$, since $|w_i| = |w_{i+4}|$.

Lemma 8. *Let S be a DS-factorization and $i \in \mathbb{Z}_8$ such that $d_i \neq 0$. Then the following properties hold:*

(i) *There exist unique words u_i and v_i and a unique nonnegative integer n_i such that*

$$\begin{aligned} \widehat{w_{i-3}} w_{i-1} &= u_i v_i && (9) \\ w_i &= (u_i v_i)^{n_i} u_i && (10) \\ w_{i+1} \widehat{w_{i+3}} &= v_i u_i, && (11) \end{aligned}$$

where $0 \leq |u_i| < d_i$ and $0 < |v_i| \leq d_i$;

(ii) *d_i is a period of w_i;*

(iii) *$n_i = n_{i+4}$, $|u_i| = |u_{i+4}|$ and $|v_i| = |v_{i+4}|$.*

The proof of this lemma relies on the following well-known fact:

Proposition 9. [15] *Let x, z be two nonempty words and y be a word such that $xy = yz$. Then there exist unique words u, v and a unique integer $i \geq 0$ such that $x = uv$, $y = (uv)^i u$ and $z = vu$.*

Proof of Lemma 8. (i) It follows from the definition of DS-factorization that

$$\widehat{w_{i-3}} w_{i-1} w_i = \widehat{w_{i-3}} \widehat{w_{i+4}} \widehat{w_{i+3}} = \widehat{w_{i+5}} \widehat{w_{i+4}} \widehat{w_{i+3}} = w_i w_{i+1} \widehat{w_{i+3}}.$$

The two extreme members of this sequence of equalities satisfy an equation of the form $xy = yz$, with $x = \widehat{w_{i-3}} w_{i-1} \neq \varepsilon$, $y = w_i$ and $z = w_{i+1} \widehat{w_{i+3}} \neq \varepsilon$. By Proposition 9, the three equalities follow.

(ii) Since $d_i = |w_{i+1}| + |w_{i+3}| = |u_i| + |v_i|$ and $w_i = (u_i v_i)^{n_i} u_i$, we conclude that d_i is a period of w_i.

(iii) Note that n_i and $|u_i|$ are respectively the quotient and the remainder of $|w_i|$ by $d_i \neq 0$. Since $|w_i| = |w_{i+4}|$ and $d_i = d_{i+2} = d_{i+4}$, we conclude that $n_i = n_{i+4}$ and $|u_i| = |u_{i+4}|$. Finally $|v_i| = d_i - |u_i| = d_{i+4} - |u_{i+4}| = |v_{i+4}|$. $\square$

For the remainder of this article, we shall use the variables d_i, n_i, u_i and v_i to designate the numbers and words with the same label as in Lemma 8. For

i	w_i	u_i	v_i	$\lvert w_i \rvert$	d_i	n_i
0	**3**	**3**	**030103032321232**	1	16	0
1	**03010303**	**03**	**0103**	8	6	1
2	**01030**	**01030**	**10103010303**	5	16	0
3	**10103010**	**10**	**1030**	8	6	1
4	**1**	**1**	**212321210103010**	1	16	0
5	**21232121**	**21**	**2321**	8	6	1
6	**23212**	**23212**	**32321232121**	5	16	0
7	**32321232**	**32**	**3212**	8	6	1

Table 1: Values of the variables for Example 7.

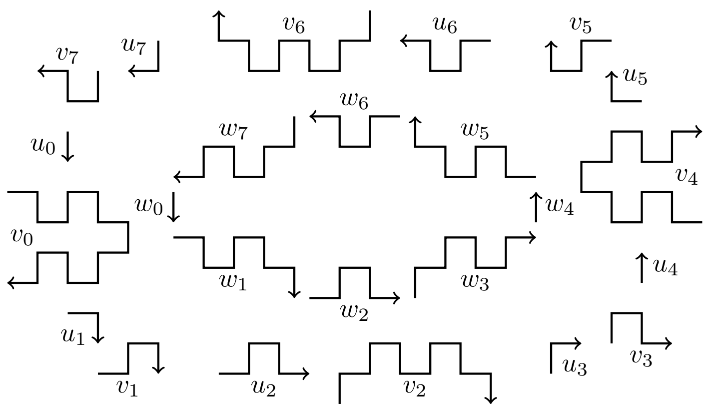


Figure 8: The paths w_i, u_i and v_i for each $i \in \mathbb{Z}_8$ for Example 7.

example, values of those variables for the double square defined in Example 7 are in the Table 1 and are illustrated in Figure 8. Moreover, a direct consequence of Lemma 8 is that the period d_i extends beyond w_i and that many commuting properties of the words u_i, v_i and w_i are verified. Those relations are used in the next sections to show that some operations on double square tiles are well-defined.

Lemma 10. *Suppose that there exists $i \in \mathbb{Z}_8$ such that $d_i \neq 0$.*

- *Then d_i is a period of $w_{i-1}w_iw_{i+1}$.*
- *The following group of equalities hold:*

$$u_iv_i \cdot w_i = w_i \cdot v_iu_i, \tag{12}$$
$$w_{i-1} \cdot u_iv_i = \widehat{u_{i+4}}\widehat{v_{i+4}} \cdot w_{i-1}, \tag{13}$$
$$v_iu_i \cdot w_{i+1} = w_{i+1} \cdot \widehat{v_{i+4}}\widehat{u_{i+4}}. \tag{14}$$

- *The following group of equalities hold:*

$$\begin{aligned} w_{i-1}u_i &= \widehat{u_{i+4}}\widehat{w_{i+3}}, && (15)\\ u_i w_{i+1} &= \widehat{w_{i+5}}\widehat{u_{i+4}}, && (16)\\ w_{i+1}\widehat{v_{i+4}} &= v_i\widehat{w_{i+5}}, && (17)\\ \widehat{v_i}w_{i+3} &= \widehat{w_{i+7}}v_{i+4}. && (18)\end{aligned}$$

Proof. Since $d_i = |u_i v_i|$ is a period of

$$(u_i v_i)^{n_i+2} = \widehat{w_{i-3}}w_{i-1} \cdot w_i \cdot w_{i+1}\widehat{w_{i+3}},$$

it is also a period of $w_{i-1}w_i w_{i+1}$.

Equation (12) is an immediate consequence of Equation (10). We prove Equations (13) and (14) by Lemma 8 applied to indices i and $i+4$:

$$\begin{aligned} w_{i-1}u_i v_i &= w_{i-1}\widehat{w_{i-3}}w_{i-1} = \widehat{u_{i+4}}\widehat{v_{i+4}}w_{i-1},\\ v_i u_i w_{i+1} &= w_{i+1}\widehat{w_{i+3}}w_{i+1} = w_{i+1}\widehat{v_{i+4}}\widehat{u_{i+4}}.\end{aligned}$$

Using Lemma 8, we obtain Equations (15) and (17) by comparing the suffixes and prefixes of the following equality

$$w_{i+1}\widehat{v_{i+4}} \cdot \widehat{u_{i+4}}\widehat{w_{i+3}} = w_{i+1}\widehat{w_{i+3}}w_{i+1}\widehat{w_{i+3}} = v_i u_i v_i u_i = v_i\widehat{w_{i-3}} \cdot w_{i-1}u_i.$$

We obtain Equations (16) and (18) similarly from

$$\widehat{w_{i-3}}\widehat{u_{i+4}} \cdot \widehat{v_{i+4}}w_{i-1} = \widehat{w_{i-3}}w_{i-1}\widehat{w_{i-3}}w_{i-1} = u_i v_i u_i v_i = u_i w_{i+1} \cdot \widehat{w_{i+3}}v_i. \quad \square$$

The numbers n_i correspond to the number of repetitions of the patterns according to the period d_i in the words w_i. Some natural constraints apply to them.

Lemma 11. *Assume that S is nonsingular and $n_i \neq 0$ for some $i \in \mathbb{Z}_8$. Then $n_{i+1} = n_{i+3} = n_{i+5} = n_{i+7} = 0$.*

Proof. We proceed by contradiction, i.e. we assume that there exists some $i \in \mathbb{Z}_8$ such that $n_i, n_{i+1} \neq 0$. Then $|w_i| \geq |w_{i-1}|+|w_{i+1}|$ and $|w_{i+1}| \geq |w_i|+|w_{i+2}|$. Therefore,

$$|w_i| \geq |w_{i-1}| + |w_{i+1}| \geq |w_{i-1}| + |w_i| + |w_{i+2}| > |w_i|, \tag{19}$$

since $|w_{i-1}| = |w_{i+3}|$ and $|w_{i+2}|$ cannot both be zero, S being nonsingular. The Inequalities (19) yield $|w_i| > |w_i|$, which is absurd. Similarly, it can be shown that $n_{i-1} = 0$ and, using the identity $n_i = n_{i+4}$, the result follows. $\square$

Before stating and proving Lemma 13, a technical observation is needed.

Proposition 12. *Let A, B, w be any words and $k > 0$ be an integer. If $A^k w = wB^k$, then $Aw = wB$.*

Proof. First we remark that $|A| = |B|$. If A and B are empty, the result is obvious. Suppose $|A| = |B| > 0$. By induction, one shows that if $A^k w = wB^k$, then $A^{kn}w = wB^{kn}$ for all positive integer n. Let n be an integer such that $|A^{kn-1}| > |w|$. Then w is a prefix of A^{kn-1} and Aw is a prefix of A^{kn}, so that Aw is a prefix of $A^{kn}w = wB^{kn}$. Hence, $Aw = wB$. $\square$

The next lemma deals with DS-factorization presenting strong periodic properties.

Lemma 13. *Assume that $d_i \neq 0$ divides $|w_i|$ and $w_{i+2} \neq \varepsilon$. Then*

(i) *u_i and u_{i+4} are empty and $n_i = n_{i+4} = |w_i|/d_i$,*

(ii) *$w_{i+1} = \widehat{w_{i+5}}$, $w_{i+3} = \widehat{w_{i+7}}$, $w_i = (w_{i+1}w_{i-1})^{n_i}$ and $w_{i+4} = (w_{i+5}w_{i+3})^{n_{i+4}}$,*

(iii) *there exist two nonempty primitive words $p, q \in \mathcal{F}^*$ and integers $k \geq 2$ and $\ell \geq 1$ such that*

$$w_{i+1}w_{i+2}w_{i+3} = p^k \quad \textit{and} \quad w_{i+6} = \widehat{p}^{\,\ell},$$

$$w_{i+5}w_{i+6}w_{i+7} = q^k \quad \textit{and} \quad w_{i+2} = \widehat{q}^{\,\ell}$$

where $|p| = |q|$ divides $g = \gcd(|w_{i+2}|, d_{i+2})$,

(iv) *p and $\widehat{q}$ are conjugate: $pw_{i+1} = w_{i+1}\widehat{q}$ and $\widehat{q}w_{i+3} = w_{i+3}p$,*

(v) *the boundary word of S is not simple.*

Proof. (i) From Lemma 8, u_i is empty because its length is equal to the remainder of the division of $|w_i|$ by d_i. Also the quotient is n_i.

(ii) From Lemma 8, we have that

$$\widehat{w_{i-3}}w_{i-1} = u_iv_i = \varepsilon \cdot v_i = v_i \cdot \varepsilon = v_iu_i = w_{i+1}\widehat{w_{i+3}}.$$

Then $w_{i+1} = \widehat{w_{i-3}} = \widehat{w_{i+5}}$ and $w_{i+3} = \widehat{w_{i-1}} = \widehat{w_{i+7}}$. Also,

$$w_i = (u_iv_i)^{n_i}u_i = (u_iv_i)^{n_i} = (\widehat{w_{i-3}}w_{i-1})^{n_i} = (w_{i+1}w_{i-1})^{n_i},$$

$$w_{i+4} = (u_{i+4}v_{i+4})^{n_{i+4}}u_{i+4} = (u_{i+4}v_{i+4})^{n_{i+4}} = (\widehat{w_{i+1}}w_{i+3})^{n_{i+4}} = (w_{i+5}w_{i+3})^{n_{i+4}}.$$

(iii) Using assertion (ii) and Definition 6, we can write

$$w_{i+1}w_{i+3}\widehat{w_{i+6}} = w_{i+1}\widehat{w_{i+7}}\widehat{w_{i+6}} = w_{i+1}w_{i+2}w_{i+3} = \widehat{w_{i+6}}\widehat{w_{i+5}}w_{i+3} = \widehat{w_{i+6}}w_{i+1}w_{i+3}.$$

Since this equation has the form $ab = ba$, with $a = w_{i+1}w_{i+3} \neq \varepsilon$ and $b = \widehat{w_{i+6}} \neq \varepsilon$, we have from Lothaire [15] that there exists $P \in \mathcal{F}^*$ such that

$$a = w_{i+1}w_{i+3} = P^{k_1} \quad \text{and} \quad b = \widehat{w_{i+6}} = P^{k_2}$$

with $|P| = \gcd(|b|, |a|) = g$. In particular, $w_{i+6} = \widehat{P}^{\,k_2}$ and $w_{i+1}w_{i+2}w_{i+3} = P^{k_1+k_2}$. Let p be the primitive root of P, i.e. the smallest word p such that $P = p^n$ with $n \in \mathbb{N}$. We have that $w_{i+1}w_{i+2}w_{i+3} = p^{n(k_1+k_2)} = p^k$ and

$w_{i+6} = \widehat{p}^{\,nk_2} = \widehat{p}^{\,\ell}$, where $k \geq 2$ and $\ell \geq 1$ are integers. The word $w_{i+5}w_{i+6}w_{i+7}$ is a conjugate of

$$w_{i+6}w_{i+7}w_{i+5} = w_{i+6}w_{i+7}\widehat{w_{i+1}} = \widehat{w_{i+3}}\widehat{w_{i+2}}\widehat{w_{i+1}} = \widehat{w_{i+3}}\widehat{w_{i+2}}\widehat{w_{i+1}} = \widehat{p}^{\,k}.$$

Then there is a primitive word $q \in \mathcal{F}^*$ conjugate of $\widehat{p}$ such that $w_{i+5}w_{i+6}w_{i+7} = q^k$. Moreover $q^k = w_{i+5}w_{i+6}w_{i+7} = \widehat{w_{i+2}}\widehat{w_{i+1}}w_{i+7}$ so that $\widehat{w_{i+2}} = q^\ell$ because $|q| = |p|$ divides $|w_{i+2}| = |w_{i+6}|$.

(iv) We have

$$\begin{aligned} p^\ell w_{i+1} &= \widehat{w_{i+6}}w_{i+1} = \widehat{w_{i+6}}\widehat{w_{i+5}} = w_{i+1}w_{i+2} = w_{i+1}\widehat{q}^{\,\ell}, \\ w_{i+3}p^\ell &= w_{i+3}\widehat{w_{i+6}} = \widehat{w_{i+7}}\widehat{w_{i+6}} = w_{i+2}w_{i+3} = \widehat{q}^{\,\ell}w_{i+3}. \end{aligned}$$

From Proposition 12, we conclude that $pw_{i+1} = w_{i+1}\widehat{q}$ and $\widehat{q}w_{i+3} = w_{i+3}p$.

(v) Without loss of generality, assume that $i = 0$. Since p and $\widehat{q}$ are conjugate, it follows that pq and qp are closed paths. Hence, it suffices to show that pq or qp is a proper factor the the boundary word of S. From (ii) and (iii), we know that the boundary word $w = w_0w_1 \cdots w_7$ of S is

$$w = w_0p^kw_4q^k = (w_1\widehat{w_3})^{n_0}p^k(\widehat{w_1}w_3)^{n_4}q^k.$$

If $n_0 = n_4 = 0$, then $w = p^kq^k$ so that pq and qp both occur in $[\![w]\!]$. Otherwise, assume that $n_0, n_4 \geq 1$. Since $|p| = |q|$ divides $d_2 = |w_1| + |w_3|$, then either $|p| \leq |w_1|$ or $|p| \leq |w_3|$. If $|p| \leq |w_1|$, then p is a prefix of w_1. Therefore w ends with q and starts with p so that qp occurs in $[\![w]\!]$. Similarly, if $|p| \leq |w_3|$, then p is a suffix of w_3. Thus w ends with pq^k so that pq occurs in $[\![w]\!]$. $\square$

Example 14. *If* $(|w_0|, |w_1|, |w_2|, |w_3|) = (5, 4, 3, 8)$, *a double square factorization might be of the following form:*

i	w_i	u_i	v_i	$\lvert w_i\rvert$	d_i	n_i
0	**32323**	**32323**	**2323010**	5	12	0
1	**2323**	**2323**	**2323**	4	8	0
2	**232**	**232**	**101012323**	3	12	0
3	**10101232**		**10101232**	8	8	1
4	**10101**	**10101**	**0101232**	5	12	0
5	**0101**	**0101**	**0101**	4	8	0
6	**010**	**010**	**323230101**	3	12	0
7	**32323010**		**32323010**	8	8	1

Here, we have that $d_3 = 8$ *divides* $|w_3| = 8$. *In this case, Lemma 13 applies and we must have that* u_3 *is empty,* $n_3 = 1$, $w_4 = \widehat{w_0}$, $w_6 = \widehat{w_2}$, $w_3 = w_4w_2$ *and* $w_7 = w_0w_6$. *Moreover, there exist two primitive words* $p = \mathbf{10}$ *and* $q = \mathbf{32}$ *and integers* $k = 6$ *and* $\ell = 2$ *such that*

$$w_4w_5w_6 = p^k = (\mathbf{10})^6 \quad \textit{and} \quad w_1 = \widehat{p}^{\,\ell} = (\widehat{\mathbf{10}})^2,$$

$$w_0w_1w_2 = q^k = (\mathbf{32})^6 \quad \textit{and} \quad w_5 = \widehat{q}^{\,\ell} = (\widehat{\mathbf{32}})^2$$

where $|p| = |q| = 2$ *divides* $g = \gcd(|w_5|, d_5) = \gcd(4, 8) = 4$*. Observe that* $|p| \neq g$*. Also,* $p = \mathbf{10}$ *and* $\widehat{q} = \mathbf{01}$ *are conjugates:*

$$pw_4 = \mathbf{10} \cdot \mathbf{10101} = \mathbf{10101} \cdot \mathbf{01} = w_4\widehat{q},$$

$$\widehat{q}w_6 = \mathbf{01} \cdot \mathbf{010} = \mathbf{010} \cdot \mathbf{10} = w_6 p.$$

Finally, we verify that $pq = \mathbf{1032}$ *and* $qp = \mathbf{3210}$ *are closed proper factors of the boundary word.*

The turning number of a DS-factorization $S = (w_i)_{i\in\mathbb{Z}_8}$ is naturally defined from the circular word it defines: $\mathcal{T}(S) = \mathcal{T}([\![w_0w_1w_2w_3w_4w_5w_6w_7]\!])$. Whenever S is nonsingular, its turning number can be computed from both its square factorizations using Equation (7):

$$\mathcal{T}(S) = \sum_{i\in\{0,2,4,6\}} \mathcal{T}(w_{i-2}w_{i-1}, w_iw_{i+1}) = \sum_{i\in\{1,3,5,7\}} \mathcal{T}(w_{i-2}w_{i-1}, w_iw_{i+1}) \quad (20)$$

These formulas are used in Lemma 20. Proposition 4 translates directly as follows for DS-factorizations.

Lemma 15. *If* S *is nonflat and* $\mathcal{T}(S) = \pm 1$ *then* $\textsc{Fst}(w_iw_{i+1}) = \textsc{Lst}(w_iw_{i+1})$ *for all* i*.*

Proof. It is a direct consequence of Proposition 4. □

Under some conditions, we may guarantee that some DS-factorizations do not yield double squares. More precisely:

Lemma 16. *Assume that* S *is nondegenerate. If there exists* $i \in \mathbb{Z}_8$ *such that* $d_i = d_{i+1}$*, then* $\mathcal{T}(S) \notin \{-1, 1\}$*.*

Proof. Let $d = |w_i| + |w_{i+2}| = |w_{i+1}| + |w_{i+3}|$. We first show that there exists $j \in \mathbb{Z}_8$ such that $|w_{j-1}w_j| \geq d$ and $|w_jw_{j+1}| \geq d$. Arguing by contradiction, assume that the contrary holds. Then for all $j \in \mathbb{Z}_8$ either $|w_{j-1}w_j| < d$ or $|w_jw_{j+1}| < d$. By the pigeonhole principle, there must exist $k \in \mathbb{Z}_8$ with $|w_k| + |w_{k+1}| < d$ and $|w_{k+2}| + |w_{k+3}| < d$. Thus,

$$2d = |w_k| + |w_{k+1}| + |w_{k+2}| + |w_{k+3}| < 2d,$$

which is absurd. Now, we know from Lemma 10 that the words $x = w_{j-2}w_{j-1}w_j$, $y = w_{j-1}w_jw_{j+1}$ and $z = w_jw_{j+1}w_{j+2}$ all have period d. Moreover, x has a suffix of length at least d that is a prefix of y, and y has a suffix of length at least d that is a prefix of z, so that the period d propagates on the whole word $w_{j-2}w_{j-1}w_jw_{j+1}w_{j+2}$. First, since $|w_{j-2}w_{j-1}w_jw_{j+1}| = 2d$, we have $\textsc{Fst}(w_{j-2}) = \textsc{Fst}(w_{j+2})$. On the other hand, $w_{j+2}w_{j+3} = \widehat{w_{j-1}}\widehat{w_{j-2}}$ implies $\textsc{Fst}(w_{j+2}) = \overline{\textsc{Lst}(w_{j-1})}$. To conclude, we proceed again by contradiction. Assume that $\mathcal{T}(S) \in \{-1, 1\}$. Then Lemma 15 applies. In particular, $\textsc{Lst}(w_{j-1}) = \textsc{Fst}(w_{j-2})$. Gathering these three equalities, we obtain

$$\textsc{Fst}(w_{j-2}) = \textsc{Fst}(w_{j+2}) = \overline{\textsc{Lst}(w_{j-1})} = \overline{\textsc{Fst}(w_{j-2})},$$

which is impossible. Hence, $\mathcal{T}(S) \notin \{-1, 1\}$. □

It is worth mentioning that Lemma 16 is false if one of the w_i is empty. For instance, the double square $S = (\mathbf{3}, \varepsilon, \mathbf{0}, \mathbf{10}, \mathbf{1}, \varepsilon, \mathbf{2}, \mathbf{32})$ is such that $d_1 = d_2$ but its turning number is 1.

Lemma 17. *Let S be a DS-factorization. If S is flat, then $\mathcal{T}(S) = 0$.*

Proof. There exists $i \in \mathbb{Z}_8$ such that $w_i w_{i+1} = \varepsilon$. Then

$$\mathcal{T}(S) = \mathcal{T}(\llbracket w_{i+2} w_{i+3} \widehat{w_{i+2} w_{i+3}} \rrbracket) = \mathcal{T}(\llbracket \varepsilon \rrbracket) = 0. \qquad \square$$

4 Reduction of double squares

The goal of this section is to show that any DS-factorization can be reduced to a singular DS-factorization (Theorem 22) using two simple operators. First, we show that those operators are well-defined (Proposition 18) and give conditions under which they reduce the size of a DS-factorization (Lemma 19) and preserve the turning number (Lemma 20). We observe that these conditions form a partition so that one or the other operator can reduce any non singular DS-factorization to a smaller one (Proposition 21). This leads to Algorithm 1 which reduces any DS-factorization of a double square tile to a singular DS-factorization. It is worth mentioning both operators are invertible under mild conditions described in Section 5 and preserve remarkable topological properties studied in Section 6. Below, we describe them and illustrate their action on double squares.

Let $S = (w_i)_{i \in \mathbb{Z}_8}$ be a nonsingular DS-factorization. Recall that Lemma 8 applies, so that the words u_i, v_i and the integers n_i as discussed in the previous section may be uniquely determined from S. Hence, if $|w_0| \geq d_0$, we define

$$\textsc{trim}(S) = (w_0(v_0u_0)^{-1}, w_1, w_2, w_3, w_4(v_4u_4)^{-1}, w_5, w_6, w_7).$$

Moreover, for any nonsingular DS-factorization S, let

$$\textsc{swap}(S) = (\widehat{w_4}, (v_1u_1)^{n_1}v_1, \widehat{w_6}, (v_3u_3)^{n_3}v_3, \widehat{w_0}, (v_5u_5)^{n_5}v_5, \widehat{w_2}, (v_7u_7)^{n_7}v_7).$$

The basic operators $\textsc{trim}$, and $\textsc{swap}$ are generalized to act on any w_i by using a shift operator. Let $\textsc{shift}$ be the operator defined by

$$\textsc{shift}(S) = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_0).$$

It is obvious that $\textsc{shift}(S)$ is a DS-factorization. Therefore for each $i \in \mathbb{Z}_8$ and every $\Theta \in \{\textsc{swap}, \textsc{trim}\}$, we define the operator $\Theta_i(S)$ as

$$\Theta_i(S) = \textsc{shift}^{-i} \circ \Theta \circ \textsc{shift}^i(S).$$

The reason for shifting back is simply to keep fixed the positions of other factors. In particular, $\Theta_0(S) = \Theta(S)$. The effects of these operators are illustrated in Figures 9 and 10. Their inverse are detailed in the next section. We first prove that $\textsc{trim}_i$ and $\textsc{swap}_i$ yield DS-factorizations.

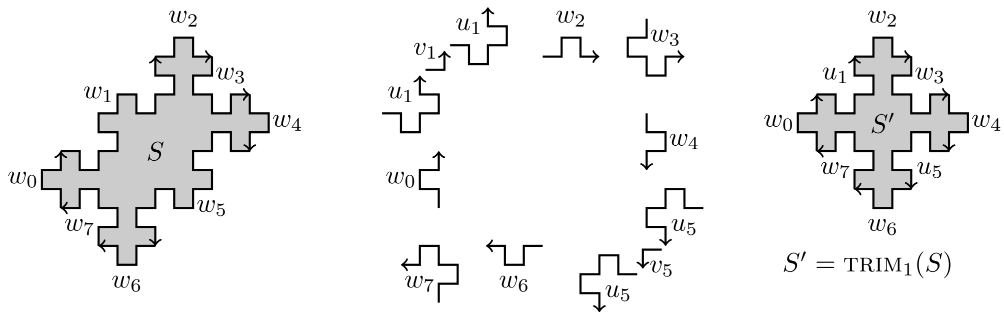


Figure 9: $S' = \text{TRIM}_1(S)$ is obtained from S by removing one period to w_1 and w_5.

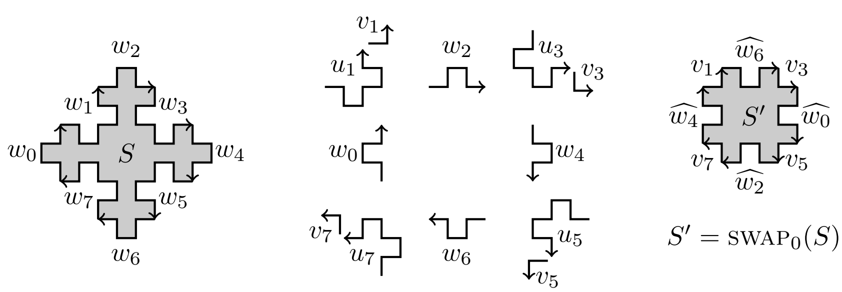


Figure 10: For $S' = \text{SWAP}_0(S)$, w_0 and w_4 as well as w_2 and w_6 are interverted and reversed. Moreover $w_1 = u_1$, $w_3 = u_3$, $w_5 = u_5$ and $w_7 = u_7$ are respectively replaced by v_1, v_3, v_5 and v_7.

Proposition 18. *Let $S = (w_i)_{i\in\mathbb{Z}_8}$ be a nonsingular DS-factorization. The following properties hold.*

(i) $\text{SWAP}_i(S)$ *is a nonsingular DS-factorization.*

(ii) *If $|w_i| \geq d_i$, then $\text{TRIM}_i(S)$ is a DS-factorization.*

Proof. Notice that S nonsingular implies that Lemma 8 applies for any index in $\mathbb{Z}_8$. Without loss of generality, we prove it for $i = 0$.
(i) Let $(w'_i)_{i\in\mathbb{Z}_8} = \text{SWAP}_0(S)$. It suffices to verify the equations in Definition 6. First we show that $w'_0w'_1 = \widehat{w'_4w'_5}$. We know from Equation (13) that $\widehat{w_4}v_1u_1 = \widehat{v_5}\widehat{u_5}\widehat{w_4}$ and from Equation (18) that $\widehat{w_4}v_1 = \widehat{v_5}w_0$. Hence

$$w'_0w'_1 = \widehat{w_4}(v_1u_1)^{n_1}v_1 = (\widehat{v_5}\widehat{u_5})^{n_1}\widehat{w_4}v_1 = (\widehat{v_5}\widehat{u_5})^{n_1}\widehat{v_5}w_0 = \widehat{w'_4w'_5}.$$

Now we show that $w'_1w'_2 = \widehat{w'_5w'_6}$. We know from Equation (14) that $u_1v_1\widehat{w_6} = \widehat{w_6}\widehat{u_5}\widehat{v_5}$ and from Equation (17) that $v_1\widehat{w_6} = w_2\widehat{v_5}$. Hence,

$$w'_1w'_2 = v_1(u_1v_1)^{n_1}\widehat{w_6} = v_1\widehat{w_6}(\widehat{u_5}\widehat{v_5})^{n_1} = w_2\widehat{v_5}(\widehat{u_5}\widehat{v_5})^{n_1} = \widehat{w'_5w'_6}.$$

The proof for $w_2'w_3' = \widehat{w_6'w_7'}$ and $w_3'w_4' = \widehat{w_7'w_0'}$ are done as above. Then $\text{SWAP}_0(S)$ satisfy the hypothesis to be a DS-factorization. Finally, we observe that d_1 is preserved by SWAP_0 and

$$d_0' = (|w_1| - |u_1|) + |v_1| + (|w_3| - |u_3|) + |v_3| \geq 0 + |v_1| + 0 + |v_3| > 0.$$

Thus $\text{SWAP}_0(S)$ is nonsingular.
(ii) Remark that $n_0 = n_4 \geq 1$ since $|w_0| \geq d_0$. Let $w_0' = (u_0v_0)^{n_0-1}u_0$ and $w_4' = (u_4v_4)^{n_4-1}u_4$. We want to show that

$$\text{TRIM}(S) = (w_0', w_1, w_2, w_3, w_4', w_5, w_6, w_7)$$

is a DS-factorization. We know from Equation (13) that $w_7u_0v_0 = \widehat{u_4}\widehat{v_4}w_7$. Then we can write

$$\widehat{u_4}\widehat{v_4}w_7w_0' = w_7u_0v_0w_0' = w_7w_0 = \widehat{w_4}\widehat{w_3} = \widehat{u_4}\widehat{v_4}\widehat{w_4'}\widehat{w_3}$$

and $w_7w_0' = \widehat{w_3w_4'}$. The proof that $w_0'w_1 = \widehat{w_4'w_5}$ is about the same, using the equalities $v_0u_0w_1 = w_1\widehat{v_4}\widehat{u_4}$ (Equation (17)). Hence, $\text{TRIM}(S)$ is a DS-factorization. $\square$

As we know from Lemma 10, there exists a local periodicity in the neighborhood of w_i. The effect of the operator TRIM may be interpreted as removing one instance of the repetead pattern, as illustrated in Figure 9. The operator SWAP is defined from the relations between the w_i's and the periods $|u_jv_j|$ (Lemma 10). An example is in Figure 10. Both TRIM and SWAP are invertible (see Section 5 for the definition of the inverses).

It is not difficult to verify that TRIM produces smaller DS-factorizations whenever it is applicable. However, the operator SWAP might increase the size of the tile depending on the lengths of the words u_i and v_i. These facts are detailed in the following lemma.

Lemma 19. *Let $S = (w_i)_{i\in\mathbb{Z}_8}$ be a nonsingular DS-factorization such that $\mathcal{T}(S) = \pm 1$. The following properties hold.*

(i) *If $|w_i| \geq d_i$ for some i, then $|\text{TRIM}_i(S)| < |S|$.*

(ii) *If $0 < |w_i| < d_i$ for all i, then there exists i such that $|\text{SWAP}_i(S)| < |S|$.*

(iii) *If there exists i such that $|w_i| = 0$ and $0 < |w_j| < d_j$ for all $j \notin \{i, i+4\}$, then $|\text{SWAP}_i(S)| < |S|$.*

Proof. (i) Follows directly from the definition of TRIM_i.
(ii) We have that $w_i = u_i$. Suppose by contradiction that $\text{SWAP}_i(S)$ does not reduce S for all i. Then $|u_{i+1}| + |u_{i+3}| \leq |v_{i+1}| + |v_{i+3}|$ for all i. Using Equation (10), we have $|v_i| = |w_{i+3}| + |w_{i+1}| - |u_i|$, this implies $|u_{i+1}| + |u_{i+3}| \leq |u_{i+2}| + |u_{i+4}|$ for all $i \in \mathbb{Z}_8$. Then we deduce that $|u_0| + |u_2| = |u_1| + |u_3|$. But Lemma 16

implies $\mathcal{T}(S) \neq \pm 1$ which is a contradiction.
(iii) Since $|w_i| = 0$, we have $d_{i+1} = |w_{i+2}|$. Also, the hypothesis implies that

$$\begin{array}{lcl} w_{i+1} = u_{i+1} & & |v_{i+1}| = d_{i+1} - |u_{i+1}| \\ w_{i+2} = u_{i+2} & \text{and} & |v_{i+2}| = d_{i+2} - |u_{i+2}| \\ w_{i+3} = u_{i+3} & & |v_{i+3}| = d_{i+3} - |u_{i+3}| \end{array}.$$

If $|\text{SWAP}_i(S)| \geq |S|$, then

$$|u_{i+1}| + |u_{i+3}| \leq |v_{i+1}| + |v_{i+3}| = (d_{i+1} - |u_{i+1}|) + (d_{i+3} - |u_{i+3}|).$$

This implies that $2|u_{i+1}| + 2|u_{i+3}| \leq d_{i+1} + d_{i+3} = 2d_{i+1}$ which leads to $d_{i+2} = |w_{i+1}| + |w_{i+3}| = |u_{i+1}| + |u_{i+3}| \leq d_{i+1} = |w_{i+2}|$, a contradiction. □

A remarkable property of the reduction operators is that they preserve the absolute value of the turning number. For example, the reader may check that

$$\text{SWAP}_1(\mathbf{3}, \varepsilon, \mathbf{0}, \mathbf{10}, \mathbf{1}, \varepsilon, \mathbf{2}, \mathbf{32}) = (\mathbf{2}, \varepsilon, \mathbf{1}, \mathbf{01}, \mathbf{0}, \varepsilon, \mathbf{3}, \mathbf{23})$$

but $\mathcal{T}([\![\mathbf{30101232}]\!]) = 1$ and $\mathcal{T}([\![\mathbf{21010323}]\!]) = -1$.

Lemma 20. *Let $S = (w_i)_{i\in\mathbb{Z}_8}$ be a DS-factorization.*

(i) *We have $\mathcal{T}(\text{TRIM}_i(S)) = \mathcal{T}(S)$ for all i.*

(ii) *If S is nondegenerate, then $\mathcal{T}(\text{SWAP}_i(S)) = \mathcal{T}(S)$ for all i.*

(iii) *If S is degenerate for only one $i \in \mathbb{Z}_8$, then $\mathcal{T}(\text{SWAP}_i(S)) = -\mathcal{T}(S)$.*

Proof. (i) We prove it for TRIM_0, the other cases being similar. Let $S' = (w'_i) = \text{TRIM}_0(S)$ and first assume that S' is flat. Then

$$S' = ((u_0v_0)^{n_0-1}u_0, w_1, w_2, w_3, (u_4v_4)^{n_4-1}u_4, w_5, w_6, w_7).$$

S' being flat implies $\mathcal{T}(S') = 0$, by Lemma 17. If S is flat, the result follows. Otherwise, there exists $j \in \{-1, 0\}$ such that $w'_jw'_{j+1} = \varepsilon$, since TRIM modifies only positions 0 and 4. Without loss of generality, assume that $j = 0$. Then $w_1 = \varepsilon$ and $|w_0| = d_0 = |w_1| + |w_3| = |w_3|$. Also, Lemma 13 applies at position 1 since any positive integer divides $|w_1| = 0$. Hence, $w_1w_2w_3w_4 = \varepsilon \cdot p^k = p^k$ and $w_5w_6w_7w_0 = \varepsilon \cdot q^k = q^k$ for some primitive words p, q and an integer $k \geq 2$, where $pw_2 = w_2\widehat{q}$ and $|p| = |q|$ divides $\gcd(|w_3|, |w_0| + |w_2|)$. But $|w_0| = |w_3|$ implies that $|p|$ divides $|w_2|$, so that $p = \widehat{q}$ and then $\mathcal{T}(S) = \mathcal{T}([\![p^k\widehat{p}^{\,k}]\!]) = \mathcal{T}([\![\varepsilon]\!]) = 0$. This solves the case where S' is flat since $\mathcal{T}(S) = \mathcal{T}(S') = 0$.

Now, assume that S' is not flat. Then $\text{FST}(w_iw_{i+1}) = \text{FST}(w'_iw'_{i+1})$ and $\text{LST}(w_iw_{i+1}) = \text{LST}(w'_iw'_{i+1})$ for all $i \in \mathbb{Z}_8$, so that

$$\begin{aligned} \mathcal{T}(\text{TRIM}_0(S)) &= \sum_{i\in\{0,2,4,6\}} \mathcal{T}(w'_{i-2}w'_{i-1}, w'_iw'_{i+1}) \\ &= \sum_{i\in\{0,2,4,6\}} \mathcal{T}(w_{i-2}w_{i-1}, w_iw_{i+1}) \\ &= \mathcal{T}(S). \end{aligned}$$

(ii) We do the proof only for $\textsc{swap}_0$. If all the w_i are nonempty, then

$$\begin{aligned}\mathcal{T}(\textsc{swap}_0(S)) &= \sum_{i\in\{0,2,4,6\}} \mathcal{T}(w'_{i-1}, w'_i) = \sum_{i\in\{0,2,4,6\}} \mathcal{T}(v_{i-1}, \widehat{w_{i+4}}) \\ &= \sum_{i\in\{0,2,4,6\}} \mathcal{T}(w_{i-2}, w_{i-1}) = \sum_{i\in\{1,3,5,7\}} \mathcal{T}(w_{i-1}, w_i) = \mathcal{T}(S)\end{aligned}$$

(iii) Without loss of generality, we suppose $i = 0$. Then w_0 and w_4 are both empty. Therefore,

$$\begin{aligned}&\mathcal{T}(\textsc{swap}_0(S)) = \sum_{i\in\{0,2,4,6\}} \mathcal{T}(w'_{i-2}w'_{i-1}, w'_i w'_{i+1}) \\ = \;& \mathcal{T}(w'_6w'_7, w'_0w'_1) + \mathcal{T}(w'_0w'_1, w'_2w'_3) + \mathcal{T}(w'_2w'_3, w'_4w'_5) + \mathcal{T}(w'_4w'_5, w'_6w'_7) \\ = \;& \mathcal{T}(w'_7, w'_1) + \mathcal{T}(w'_1, w'_2) + \mathcal{T}(w'_3, w'_5) + \mathcal{T}(w'_5, w'_6)\end{aligned}$$

But, using Equations (12), (13), (14), the fact that w_0 and w_4 are empty and Lemma 13, we get

$$\mathcal{T}(w'_7, w'_1) = \mathcal{T}(v_7, v_1) = \mathcal{T}(w_6, w_2) = \mathcal{T}(\widehat{p}, \widehat{q}) = -\mathcal{T}(q, p) = -\mathcal{T}(w_7, w_1),$$

$$\mathcal{T}(w'_1, w'_2) = \mathcal{T}(v_1, \widehat{w_6}) = \mathcal{T}(\widehat{w_6}, p) = \mathcal{T}(\widehat{w_6}, w_1) = \mathcal{T}(\widehat{w_6}, \widehat{w_5}) = -\mathcal{T}(w_5, w_6),$$

$$\mathcal{T}(w'_3, w'_5) = \mathcal{T}(v_3, v_5) = \mathcal{T}(w_2, w_6) = \mathcal{T}(\widehat{q}, \widehat{p}) = -\mathcal{T}(p, q) = -\mathcal{T}(w_3, w_5),$$

$$\mathcal{T}(w'_5, w'_6) = \mathcal{T}(v_5, \widehat{w_2}) = \mathcal{T}(\widehat{w_2}, q) = \mathcal{T}(\widehat{w_2}, w_5) = \mathcal{T}(\widehat{w_2}, \widehat{w_1}) = -\mathcal{T}(w_1, w_2).$$

Hence, we conclude that

$$\mathcal{T}(\textsc{swap}_0(S)) = -\mathcal{T}(w_7, w_1) - \mathcal{T}(w_5, w_6) - \mathcal{T}(w_3, w_5) - \mathcal{T}(w_1, w_2) = -\mathcal{T}(S) \quad \square$$

Before proving Theorem 22, let us define what we mean by reduction of a DS-factorization. Let S and S' be two DS-factorizations such that $S' = \Theta_n \circ \cdots \circ \Theta_2 \circ \Theta_1(S)$ where $\Theta_j \in \{\textsc{trim}_i, \textsc{swap}_i\}$ are operators on DS-factorizations. Let $S_k = \Theta_k \circ \Theta_{k-1} \circ \cdots \circ \Theta_1(S)$, so that $S_0 = S$ and $S_n = S'$. Then we say that S *reduces to* S' if $|S_k| < |S_{k-1}|$ for all $k \in [1..n]$.

Proposition 21. *Let S be a DS-factorization such that $\mathcal{T}(S) = \pm 1$. Then, one of the conditions below holds:*

(i) *S is singular,*

(ii) *$\textsc{trim}_i$ reduces S for some $i \in \mathbb{Z}_8$ and $\mathcal{T}(\textsc{trim}_i(S)) = \pm 1$,*

(iii) *$\textsc{swap}_i$ reduces S for some $i \in \mathbb{Z}_8$ and $\mathcal{T}(\textsc{swap}_i(S)) = \pm 1$.*

Proof. If there is $i \in \{0, 1, 2, 3\}$ such that $|w_i| \geq d_i$, then S reduces to $\textsc{trim}_i(S)$ by Lemma 19 (i). Moreover, by Lemma 20 (i), $\mathcal{T}(\textsc{trim}_i(S)) = \mathcal{T}(S) = \pm 1$.

Also, if $0 < |w_i| < d_i$ for all i, then S reduces to $\textsc{swap}_i(S)$ for some i by Lemma 19 (ii). Moreover, by Lemma 20 (ii), $\mathcal{T}(\textsc{swap}_i(S)) = \mathcal{T}(S) = \pm 1$.

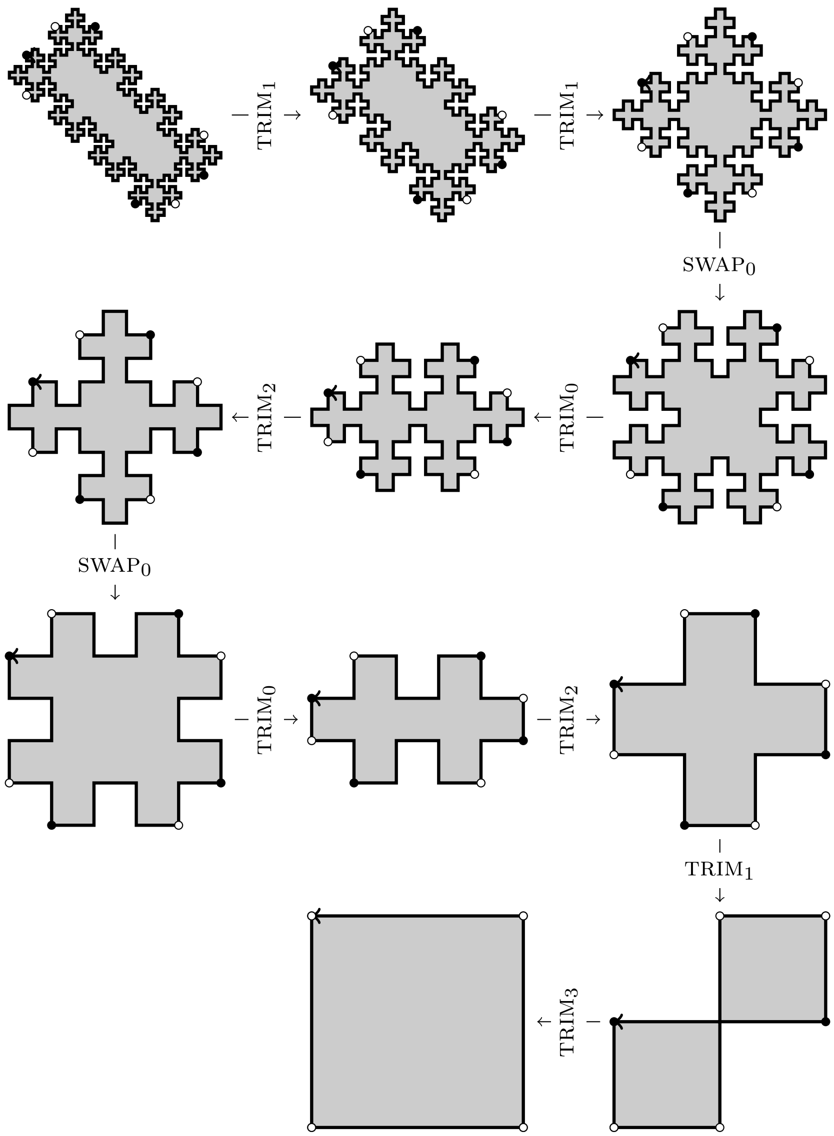


Figure 11: Reduction of a double square tile to a singular DS-factorization which in this case is the unit square.

Otherwise, assume that there exists i such that $|w_i| = 0$ and $0 < |w_j| < d_j$ for all $j \notin \{i, i+4\}$. Then S reduces to $\text{SWAP}_i(S)$ by Lemma 19 (iii). Moreover, by Lemma 20 (iii), $\mathcal{T}(\text{SWAP}_i(S)) = -\mathcal{T}(S) = \pm 1$.

Next, assume that there exists i such that $|w_i| = 0$ and $|w_{i+2}| = 0$. Then S

is singular.

It remains to consider the case where there exists i such that $|w_i| = 0$ and $|w_{i+1}| = 0$ or $|w_{i+3}| = 0$, then S is flat so that $\mathcal{T}(S) = 0$ (Lemma 17) which is a contradiction. $\square$

Algorithm 1 Reduction of a double square tile

function REDUCE(S)
  Input: a DS-factorization $S = (w_i)_{i \in \mathbb{Z}_8}$ s.t. $\mathcal{T}(S) = \pm 1$
  Output: an ordered list L of operators.
  $L \leftarrow [\ ]$ (empty list)
  while S is not singular **do**
    if there is i such that $|w_i| \geq d_i$ **then**
      $S \leftarrow \text{TRIM}_i(S)$, $L \leftarrow L + [\text{TRIM}_i]$
    else if $0 < |w_i| < d_i$ for all i **then**
      Let $i \in \{0, 1\}$ such that $|\text{SWAP}_i(S)| < |S|$
      $S \leftarrow \text{SWAP}_i(S)$, $L \leftarrow L + [\text{SWAP}_i]$
    else if there is i s.t. $|w_i| = 0$ and $0 < |w_j| < d_j$ for all $j \notin \{i, i+4\}$ **then**
      $S \leftarrow \text{SWAP}_i(S)$, $L \leftarrow L + [\text{SWAP}_i]$
    else ▷ S is singular or flat
      Error: impossible case.
    end if
  end while
  return L ▷ S corresponds to a singular DS-factorization
end function

Using the previous results, we are now ready to show that every double square is reducible.

Theorem 22. *Every DS-factorization of double square reduces to a singular DS-factorization.*

Proof. Let S be the DS-factorization of a double square. From Proposition 2, the turning number of S is ± 1. Hence, from Proposition 21, if S is nonsingular, either S may be reduced by TRIM or SWAP to some DS-factorization S', which both preserve the turning number ± 1. If S' is not singular, then this procedure can be done recursively. Since the length of the DS-factorization gets strictly smaller at each reduction (Lemma 19), Fermat's infinite descent principle applies. It follows that the number of iterations is finite and S reduces to a singular DS-factorization. $\square$

Algorithm 1 contains the pseudocode for the reduction and Figure 11 illustrates the execution of the reduction on a double square tile. The correctness of Algorithm 1 follows directly from Proposition 21 and Theorem 22.

Remark that in the reduction algorithm, the last operator must be TRIM since SWAP(S) is never singular when S is not. This observation is useful for proving Theorem 30 in Section 6 giving conditions under which a DS-factorization reduces to the unit square.

5 Generation of double square tiles

The previous section ended with Theorem 22 stating that every DS-factorization of a double square reduces to a singular DS-factorization. Therefore, it raises the question whether this leads to an algorithm that generates all double squares by inverting the reduction operators. It appears that SWAP is its own inverse under some conditions and that TRIM can be inverted easily. In this section, we introduce and study more deeply the inverses of the reduction operators. Moreover, we give some relations between these operators and we provide an algorithm that generates all double squares up to a given perimeter length.

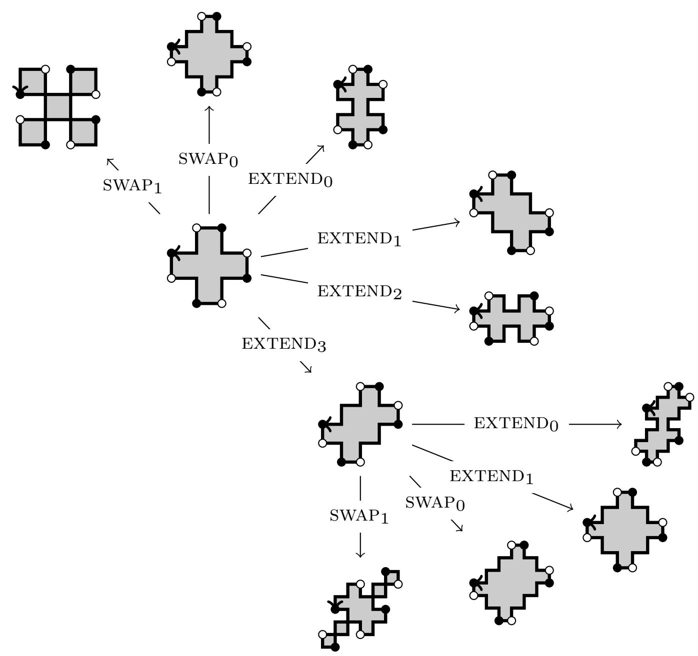


Figure 12: Subtree of the space of DS-factorizations generated when starting from the X pentomino.

Let $S = (w_i)_{i\in\mathbb{Z}_8}$ be a DS-factorization such that $d_0 \neq 0$. We define

$$\text{EXTEND}(S) = (w_0(v_0u_0), w_1, w_2, w_3, w_4(v_4u_4), w_5, w_6, w_7).$$

For any $i \in \mathbb{Z}_8$ such that $d_i \neq 0$, the operator EXTEND$_i$ is naturally defined by

$$\textsc{extend}_i = \textsc{shift}^{-i} \circ \textsc{extend} \circ \textsc{shift}^i.$$

Proposition 23. *Let $S = (w_i)_{i\in\mathbb{Z}_8}$ be a DS-factorization such that $d_i \neq 0$. Then $\textsc{extend}_i(S)$ is a DS-factorization.*

Proof. Without loss of generality, we only consider the case $i = 0$. By Lemma 8, we know that $w_0 = (u_0v_0)^{n_0}$ and $w_4 = (u_4v_4)^{n_4}$ for unique words u_0, v_0, u_4, v_4 and unique nonnegative integers n_0 and n_4. Let $w_0' = (u_0v_0)^{n_0+1}u_0$ and $w_4' = (u_4v_4)^{n_4+1}u_4$. We show that

$$\textsc{extend}(S) = (w_0', w_1, w_2, w_3, w_4', w_5, w_6, w_7)$$

is a DS-factorization. First we prove that $\widehat{w_3w_4'} = w_7w_0'$. Indeed, from Equation (13), we have $w_7u_0v_0 = \widehat{u_4}\widehat{v_4}w_7$ and we can write $\widehat{w_3w_4'}$ as

$$\widehat{u_4}\widehat{v_4}\widehat{w_4} \cdot \widehat{w_3} = \widehat{u_4}\widehat{v_4}w_7w_0 = w_7 \cdot u_0v_0w_0 = w_7w_0'$$

so that $\widehat{w_3w_4'} = w_7w_0'$. The proof that $\widehat{w_4'w_5} = w_0'w_1$ is about the same, using the equalities $v_0u_0w_1 = w_1\widehat{v_4}\widehat{u_4}$ (Equation (17)) and $w_0w_1 = \widehat{w_5}\widehat{w_4}$. $\square$

Under mild conditions, all operators are invertible, as shown by the next proposition.

Proposition 24. *Let $S = (w_i)_{i\in\mathbb{Z}_8}$ be a DS-factorization.*

(i) *$\textsc{shift}^8(S) = S$;*

(ii) *If $d_i \neq 0$, then $\textsc{trim}_i \circ \textsc{extend}_i(S) = S$;*

(iii) *If $0 < d_i \leq |w_i|$, then $\textsc{extend}_i \circ \textsc{trim}_i(S) = S$;*

(iv) *If S is nonsingular and u_{i+1}, u_{i+3}, u_{i+5} and u_{i+7} are nonempty, then $\textsc{swap}_i^2(S) = S$.*

Proof. (i) Trivial.
(ii) This follows from the fact that all factors but w_i and w_{i+4} are not modified by $\textsc{trim}_i$ nor $\textsc{extend}_i$ and from the equalities $w_i = w_i(v_iu_i)(v_iu_i)^{-1}$ and $w_{i+4} = w_{i+4}(v_{i+4}u_{i+4})(v_{i+4}u_{i+4})^{-1}$.
(iii) Same idea as (ii) but since $|w_i| > d_i$, the equalities $w_i = w_i(v_iu_i)^{-1}(v_iu_i)$ $w_{i+4} = w_{i+4}(v_{i+4}u_{i+4})^{-1}(v_{i+4}u_{i+4})$ hold as well.
(iv) Without loss of generality, assume that $i = 0$. Notice that by definition, $v_j \neq \varepsilon$, for all $j \in \mathbb{Z}_8$ but it is possible to have $u_j = \varepsilon$. Clearly, $\textsc{swap}$ is involutory for the factors at even positions $j \in \mathbb{Z}_8$, since the map $w_j \mapsto \widehat{w_{j+4}}$ is involutory. For odd positions, if $u_j \neq \varepsilon$, then the map $(u_jv_j)^{n_j}u_j \mapsto (v_ju_j)^{n_j}v_j$ is clearly involutory while, if $u_j = \varepsilon$, then $v_j^{n_j} = (u_jv_j)^{n_j}u_j \mapsto (v_ju_j)^{n_j}v_j = v_j^{n_j+1} \mapsto (u_jv_j)^{n_j+1} \neq v_j^{n_j}$. Hence, the assumptions $u_j \neq \varepsilon$ for odd $j \in \mathbb{Z}_8$ is necessary. $\square$

Algorithm 2 Generation of double squares from a DS-factorization

function GENERATE(S_0, n)
  Input: S_0 any DS-factorization s.t. $\mathcal{T}(S_0) = \pm 1$, n a positive integer
  Output: all double squares that reduce to S_0 of perimeter at most n
  $T \leftarrow \emptyset$
  $Q \leftarrow \{S_0\}$
  while $Q \neq \emptyset$ **do**
    $S \leftarrow$ POP(Q)
    $T \leftarrow T \cup \{S\}$
    $U \leftarrow \{\text{EXTEND}_i(S) : i = 0, 1, 2, 3\} \cup \{\text{SWAP}_i(S) : i = 0, 1\}$
    $Q \leftarrow Q \cup \{C \in U \setminus T : |S| < |C| \leq n\}$
  end while
  return $\{S \in T : S$ describes a polyomino $\}$
end function

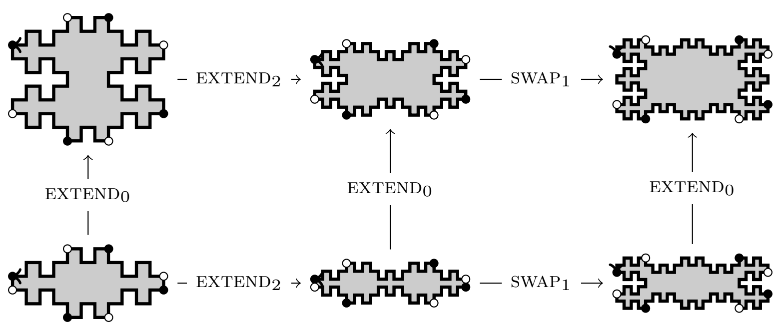


Figure 13: Two distinct ways of generating the same double square tile. The diagram commutes in virtue of Proposition 25(iii) and (iv).

In order to improve the efficiency of Algorithm 2, it is worth mentioning that the operators EXTEND and SWAP satisfy commuting properties (see Figure 13).

Proposition 25. *Let* $\Theta \in \{\text{EXTEND}, \text{SWAP}\}$ *and* $i \in \mathbb{Z}_8$.

(i) $\Theta_i = \Theta_{i+4}$;

(ii) *If S is nonsingular, then* $\text{SWAP}_i(S) = \text{SWAP}_{i+2}(S)$;

(iii) *If* $d_i, d_{i+2} \neq 0$, *then* $\text{EXTEND}_{i+2} \circ \text{EXTEND}_i(S) = \text{EXTEND}_i \circ \text{EXTEND}_{i+2}(S)$;

(iv) *If S is nonsingular, then* $\text{EXTEND}_{i+1} \circ \text{SWAP}_i = \text{SWAP}_i \circ \text{EXTEND}_{i+1}$;

Proof. We only show (iv), the other proofs being similar. Without loss of generality, we may assume $i = 0$. On one hand, we have

$$\begin{aligned}
& (\text{SWAP}_0 \circ \text{EXTEND}_1)(S) \\
= \quad & \text{SWAP}_0(\text{EXTEND}_1(S)) \\
= \quad & \text{SWAP}_0(w_0, w_1(v_1u_1), w_2, w_3, w_4, w_5(v_5u_5), w_6, w_7) \\
= \quad & (\widehat{w_4}, (v_1u_1)^{n_1+1}v_1, \widehat{w_6}, (v_3u_3)^{n_3}v_3, \widehat{w_0}, (v_5u_5)^{n_5+1}v_5, \widehat{w_2}, (v_7u_7)^{n_7}v_7).
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
& (\text{EXTEND}_1 \circ \text{SWAP}_0)(S) \\
= \quad & \text{EXTEND}_1(\text{SWAP}_0(S)) \\
= \quad & \text{EXTEND}_1(\widehat{w_4}, (v_1u_1)^{n_1}v_1, \widehat{w_6}, (v_3u_3)^{n_3}v_3, \widehat{w_0}, (v_5u_5)^{n_5}v_5, \widehat{w_2}, (v_7u_7)^{n_7}v_7) \\
= \quad & (\widehat{w_4}, (v_1u_1)^{n_1+1}v_1, \widehat{w_6}, (v_3u_3)^{n_3}v_3, \widehat{w_0}, (v_5u_5)^{n_5+1}v_5, \widehat{w_2}, (v_7u_7)^{n_7}v_7). \quad \square
\end{aligned}$$

Based on the preceding results, Algorithm 2 allows to generate all double squares of perimeter at most n reducing to a given double square. As mentioned above, it can be enhanced in virtue of Proposition 25. More precisely, it is possible to avoid exploring all paths involving commuting operators by choosing precedence on the operators. For instance, we could avoid using the operator EXTEND_1 if the last applied operator is either EXTEND_3 or SWAP_0, i.e. EXTEND_1 would precede EXTEND_3 and SWAP_0. Figure 12 illustrates a partial trace of Algorithm 2 when starting with the X pentomino.

It is not clear whether Algorithm 2 is efficient for generation purposes, since it yields infinitely many nonsimple DS-factorization. It is worth mentioning that Line 13 can be achieved in linear time by using an algorithm of Brlek, Koskas and Provençal [8]. Moreover, a discussion in the next section shows that all prime double squares may be generated by setting S_0 to the unit square. Finally, the reader may notice that the enumeration of double squares according to increasing perimeter length can be accomplished by making Q a priority heap.

6 Prime double squares and palindromes

This section is devoted to the proof that double square tiles have a palindromic structure. For this purpose, it is convenient to define homologous morphisms: A morphism φ is called *homologous* if

$$\varphi(\widehat{w}) = \widehat{\varphi(w)}, \quad \text{for any word } w \in \mathcal{F}^*. \tag{21}$$

Clearly, if $AB\widehat{A}\widehat{B}$ is the boundary of a square tile Q, then the morphism

$$\varphi_{A,B} : \mathbf{0} \mapsto A, \mathbf{1} \mapsto B, \mathbf{2} \mapsto \widehat{A}, \mathbf{3} \mapsto \widehat{B}$$

is homologous.

A polyomino P is called *prime* if for any boundary word w of P, any boundary word u and any homologous morphism $\varphi_{A,B}$, the equality $w = \varphi_{A,B}(u)$

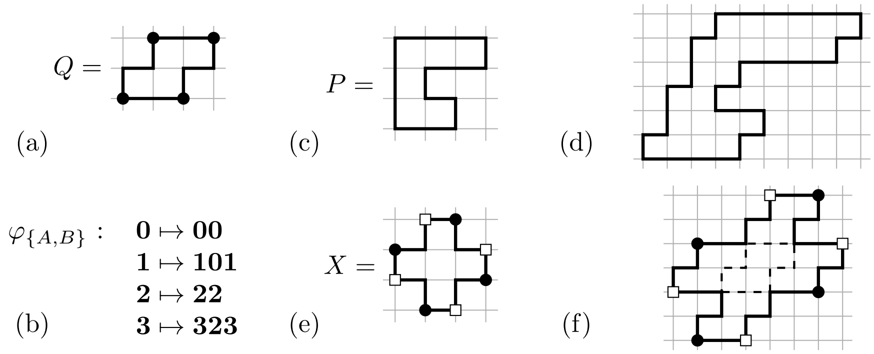


Figure 14: (a) A square tile Q with boundary word $AB\widehat{A}\widehat{B} = \mathbf{00} \cdot \mathbf{101} \cdot \mathbf{22} \cdot \mathbf{323}$. (b) The homologous morphism $\varphi_{A,B}$. (c) A polyomino P with boundary word $u = \mathbf{00121001222333}$. (d) The composed tile having boundary word $\varphi_{A,B}(u)$. (e) The prime double square X pentomino having boundary word $v = \mathbf{010121232303}$. (f) The composed tile with boundary word $\varphi_{A,B}(v)$, which is also a double square tile.

implies that either $AB\widehat{A}\widehat{B}$ or u is a boundary word of the unit square. Otherwise, P is called *composed* (see Figure 14). More intuitively, a polyomino is composed if it may be square-tiled by a smaller nontrivial polyomino.

Note that if D is composed, Theorem 1 does not apply as illustrated by Figure 15 which considers $\varphi_{A',B'} : \mathbf{0} \mapsto \mathbf{0100}, \mathbf{1} \mapsto \mathbf{11}, \mathbf{2} \mapsto \mathbf{2232}, \mathbf{3} \mapsto \mathbf{33}$.

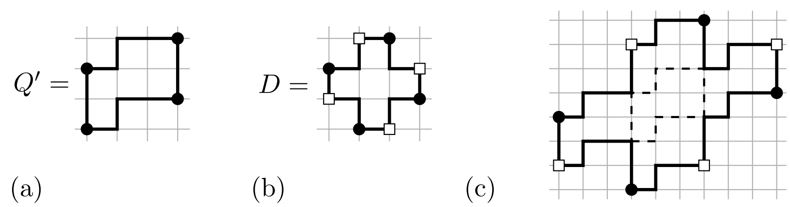


Figure 15: (a) A square tile Q' with boundary word $AB\widehat{A}\widehat{B} = \mathbf{0100} \cdot \mathbf{11} \cdot \mathbf{2232} \cdot \mathbf{33}$. (b) The prime double square pentomino X with boundary word $v = \mathbf{010121232303}$. (c) The composed tile having boundary word $\varphi_{A,B}(v)$, which is not invariant under ρ^2, where $\varphi_{A,B}$ is defined by $A = \mathbf{0100}$ and $B = \mathbf{11}$.

In the sequel, we suppose that φ is homologous. Moreover, if $S = (w_i)_{i \in \mathbb{Z}_8}$ is a DS-factorization then we define

$$\varphi(S) = (\varphi(w_0), \varphi(w_1), \varphi(w_2), \varphi(w_3), \varphi(w_4), \varphi(w_5), \varphi(w_6), \varphi(w_7)).$$

The first statement we prove is that homologous morphisms preserve DS-factorizations.

Proposition 26. *Let $S = (w_i)_{i \in \mathbb{Z}_8}$ be a DS-factorization and φ be an homologous morphism. Then, $\varphi(S)$ is a DS-factorization.*

Proof. We verify the first condition, the other three being similar:

$$\varphi(\widehat{w_0)\varphi(}w_1) = \varphi(\widehat{w_0 w_1}) = \varphi(\widehat{w_0 w_1}) = \varphi(w_4 w_5) = \varphi(w_4)\varphi(w_5). \qquad \square$$

Let $w_i' = \varphi(w_i)$ and $d_i' = |w_{i+1}'| + |w_{i+3}'|$. From Lemma 8, if $d_i' \neq 0$, then there exist u_i', v_i' and n_i' such that

$$\begin{array}{rcl} \widehat{w_{i-3}'} w_{i-1}' & = & u_i' v_i' \\ w_i' & = & (u_i' v_i')^{n_i'} u_i' \\ w_{i+1}' \widehat{w_{i+3}'} & = & v_i' u_i', \end{array}$$

where $0 \leq |u_i'| < d_i'$, for all i. The next technical lemma show that homologous morphisms also preserve the words u_i and v_i as well as the numbers n_i.

Lemma 27. *For any $i \in \mathbb{Z}_8$ such that $d_i \neq 0$, $\varphi(u_i) = u_i'$, $\varphi(v_i) = v_i'$ and $n_i' = n_i$.*

Proof. Without loss of generality, we only show the case $i = 0$. First we observe that

$$u_0' v_0' = \widehat{w_5'} w_7' = \widehat{\varphi(w_5)} \varphi(w_7) = \varphi(\widehat{w_5})\varphi(w_7) = \varphi(\widehat{w_5} w_7) = \varphi(u_0 v_0).$$

We want to show that $|u_0'| = |\varphi(u_0)|$. We also have

$$(u_0' v_0')^{n_0'} u_0' = w_0' = \varphi(w_0) = \varphi((u_0 v_0)^{n_0} u_0) = \varphi(u_0 v_0)^{n_0} \varphi(u_0) = (u_0' v_0')^{n_0} \varphi(u_0).$$

Then we have

$$d_0' \cdot n_0' + |u_0'| = d_0' \cdot n_0 + |\varphi(u_0)|$$

with $0 \leq |u_0'| < d_0'$ and $0 \leq |\varphi(u_0)| < |\varphi(u_0 v_0)| = |u_0' v_0'| = d_0'$. The unicity of the quotient and remainder of the division of $|w_0'|$ by d_0' yields $n_0' = n_0$ and $|u_0'| = |\varphi(u_0)|$. We conclude that $\varphi(u_0) = u_0'$ and $\varphi(v_0) = v_0'$. $\square$

Another useful fact is that homologous morphisms and the generation operators commute (see Figure 16).

Proposition 28. *Let $i \in \mathbb{Z}_8$. Then*

(i) *φ and EXTEND_i commute;*

(ii) *φ and SWAP_i commute.*

Proof. We prove the result for $i = 0$, the other cases being symmetric.

(i) Let $S = (w_i)_{i \in \mathbb{Z}_8}$. Then by Lemma 27

$$\begin{array}{rcl} \varphi(\text{EXTEND}_0(S)) & = & \varphi(w_0 v_0 u_0, w_1, w_2, w_3, w_4 v_4 u_4, w_5, w_6, w_7) \\ & = & (w_0' \varphi(v_0 u_0), w_1', w_2', w_3', w_4' \varphi(v_4 u_4), w_5', w_6', w_7') \\ & = & (w_0' v_0' u_0', w_1', w_2', w_3', w_4' v_4' u_4', w_5', w_6', w_7') \\ & = & \text{EXTEND}_0(w_0', w_1', w_2', w_3', w_4', w_5', w_6', w_7') \\ & = & \text{EXTEND}_0(\varphi(S)). \end{array}$$

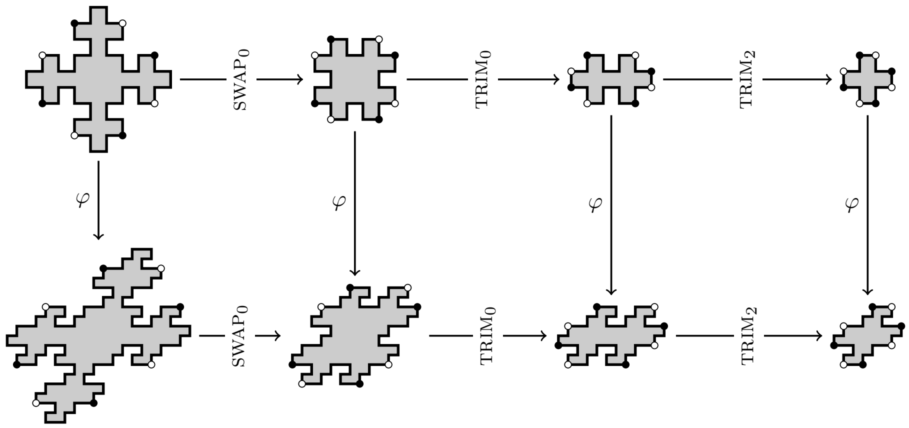


Figure 16: Generation operators and homologous morphisms commute.

(ii) Let $S = (w_i)_{i\in\mathbb{Z}_8}$. Then by Lemma 27

$$\begin{aligned}
\varphi(\text{SWAP}_0(S)) &= \varphi(\widehat{w_4}, (v_1u_1)^{n_1}v_1, \cdots) \\
&= (\varphi(\widehat{w_4}), \varphi((v_1u_1)^{n_1}v_1), \cdots) \\
&= (\widehat{\varphi(w_4)}, (\varphi(v_1)\varphi(u_1))^{n_1}\varphi(v_1), \cdots) \\
&= (\widehat{w'_4}, (v'_1u'_1)^{n'_1}v'_1, \cdots) \\
&= \text{SWAP}_0(w'_0, w'_1, w'_2, w'_3, w'_4, w'_5, w'_6, w'_7) \\
&= \text{SWAP}_0(\varphi(S)). \quad \square
\end{aligned}$$

As a consequence, we conclude that composed tiles are preserved by generation operators while prime tiles are preserved by reduction operators.

Proposition 29. *Let $i \in \mathbb{Z}_8$. Then*

(i) EXTEND$_i$ *and* SWAP$_i$ *preserve composed double square tiles.*

(ii) TRIM$_i$ *and* SWAP$_i$ *preserve prime double square tiles.*

Proof. (i) Let S be DS-factorization. If S is composed, there exists a homologous morphism φ and another DS-factorization T such that $S = \varphi(T)$ where T is not the unit square. From Proposition 28, we have

$$\text{SWAP}_i(S) = \text{SWAP}_i(\varphi(T)) = \varphi(\text{SWAP}_i(T))$$

and

$$\text{EXTEND}_i(S) = \text{EXTEND}_i(\varphi(T)) = \varphi(\text{EXTEND}_i(T))$$

so that SWAP$_i(S)$ and EXTEND$_i(S)$ are composed double square tile.

(ii) This statement is the contrapositive of (i). $\square$

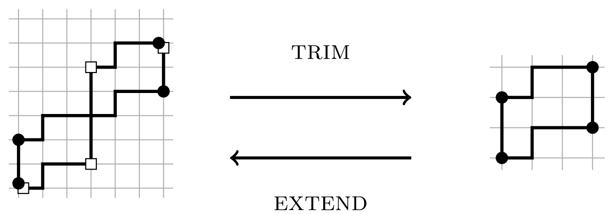


Figure 17: The degenerate double square to the left is a composed tile whereas the singular double square tile to the right is prime. This illustrates that TRIM_i does not preserve composed tiles and EXTEND_i does not preserve prime tiles.

Note that TRIM_i do not preserve composed tiles and EXTEND_i do not preserve prime tiles as illustrated in Figure 17.

Theorem 30. *Let D be a double square tile. If D is prime, then D reduces to the unit square.*

Proof. Let D be a double square tile and S be its nonsingular DS-factorization. From Theorem 22, S reduces to a singular DS-factorization T of the form:

$$T = (\varepsilon, w_1, \varepsilon, w_3, \varepsilon, \widehat{w_1}, \varepsilon, \widehat{w_3})$$

where each of the intermediate DS-factorization is prime since D is prime (Proposition 29). Also, the last operator used in the reduction algorithm must be TRIM_i where $i \in \{0, 2, 4, 6\}$, since $\text{SWAP}_i(S)$ cannot be singular by Proposition 18. Therefore, the penultimate DS-factorization in the reduction algorithm is of the form

$$\text{EXTEND}_0(T) = (w_1\widehat{w_3}, w_1, \varepsilon, w_3, \widehat{w_1}w_3, \widehat{w_1}, \varepsilon, \widehat{w_3}).$$

But, this last double square is prime only if w_1 and w_3 are letters, so that T is the unit square. $\square$

We conclude this section by proving Theorem 1. Before doing so, we need two last technical lemmas.

Lemma 31. *The following conditions are equivalent:*

(i) w_iw_{i+1} *is a palindrome for all* i*;*

(ii) $w_i = \overline{w_{i+4}}$ *for all* i*;*

(iii) $u_i = \overline{u_{i+4}}$ *and* $v_i = \overline{v_{i+4}}$ *for all* i*.*

Proof. We first show that (i) and (ii) are equivalent. Since $w_iw_{i+1} = \widehat{w_{i+5}}\widehat{w_{i+4}}$, we have that w_iw_{i+1} is a palindrome if and only if $\widetilde{w_{i+1}}\widetilde{w_i} = w_iw_{i+1} = \widehat{w_{i+5}}\widehat{w_{i+4}}$. But $|w_i| = |w_{i+4}|$ and the result follows.

Next, we prove that (ii) and (iii) are equivalent. Since $|u_i| = |u_{i+4}|$ and $|v_i| = |v_{i+4}|$ for all $i \in \mathbb{Z}_8$, we deduce that $w_i = \overline{w_{i+4}}$ for all $i \in \mathbb{Z}_8$ if and only if

$$u_i v_i = \widehat{w_{i-3}} w_{i-1} = \widetilde{w_{i+1}}\,\overline{w_{i+3}} = \overline{\widehat{w_{i+1}} w_{i+3}} = \overline{u_{i+4} v_{i+4}}. \qquad \square$$

The next lemma states that palindromicity is preserved by the reduction and generation operators.

Lemma 32. *Let S be a DS-factorization such that $w_i w_{i+1}$ is a palindrome for all $i \in \mathbb{Z}_8$ and $S' = \Theta(S)$ be DS-factorization, where $\Theta \in \{\textsc{shift}, \textsc{extend}, \textsc{trim}, \textsc{swap}\}$. Then $w'_i w'_{i+1}$ is a palindrome as well for all $i \in \mathbb{Z}_8$.*

Proof. Since the operator SHIFT preserved exactly the factors $w_i w_{i+1}$, the result is trivial in this case. Consider now $\textsc{extend}_0$. We have to show that $w'_0 w'_1 = w_0 \cdot w_1 v_1 u_1$ and $w'_1 w'_2 = w_1 u_1 v_1 \cdot w_2$ are palindromes. From Equations (12), (15) and (18), we can write $w_0 w_1 v_1 u_1 = \widehat{u_5}\widehat{v_5} w_0 w_1$. Since $\widehat{u_5}\widehat{v_5} = \widetilde{u_1}\widetilde{v_1}$ (Lemma 31 (iii)), this proves that $w_0 w_1 v_1 u_1$ is a palindrome. Similarly, we prove that $w_1 v_1 u_1 w_2 = w_1 w_2 \widehat{v_5}\widehat{u_5} = u_1 v_1 w_1 w_2$ is a palindrome. To show that $\textsc{trim}_0$ preserve the palindromes, the steps to do are the same as for $\textsc{extend}_0$, removing a period instead of adding one. Consider now the operator $\textsc{swap}_0$. From Lemma 31 and Equations (13) and (18), we obtain

$$\begin{aligned} w'_0 \cdot w'_1 &= w_0 v_0 u_0 w_1 = w_0 w_1 \widehat{v_4}\widehat{u_4} = \widetilde{w_1}\widetilde{w_1}\widehat{v_4}\widehat{u_4} = \widetilde{w_1}\widetilde{w_1}\widetilde{v_0}\widetilde{u_0} \\ &= u_0 \widetilde{v_0 w_0} w_1 = \widetilde{w'_0 w'_1} \end{aligned}$$

The argument is the same to prove that the other factors $w'_i w'_{i+1}$ also are palindromes. $\square$

As a consequence, we solve the conjecture of Provençal and Vuillon [17]:

Proof of Theorem 1. If D is a prime double square tile, it reduces to the unit square by Theorem 30 which is made of palindromes but generation operators preserve palindromes $w_i w_{i+1}$ by Lemma 32. $\square$

7 Concluding remarks and open problems

Although we have described an algorithm that exhaustively generate double squares, there is room for efficiency improvements. First, we have not yet been able to find a double square tile having turning number ± 1 such that Line 11 of Algorithm 1 is used. Hence, we are tempted to conjecture that it could be removed. Also, it would be interesting to study and perhaps improve the complexity of Algorithms 1 and 2.

Moreover, we observed that, for instance, all DS-factorizations starting from the DS-factorization $(\textsc{extend}_0 \circ \textsc{extend}_1 \circ \textsc{extend}_1)(\varepsilon, \mathbf{2}, \varepsilon, \mathbf{3}, \varepsilon, \mathbf{0}, \varepsilon, \mathbf{1})$ are not simple (see Figure 18). It would be interesting to verify if it is indeed the case, i.e. if the paths generated by Algorithm 2 starting from particular DS-factorizations are all nonsimple. This would significantly reduce the size of the explored space. Lemma 13 (v) is probably a good starting point for further investigations in that direction. In the same spirit, it seems that every double square tile might be generated uniquely up to the commutative properties of the TRIM_i and SWAP_i operators stated in Proposition 25.

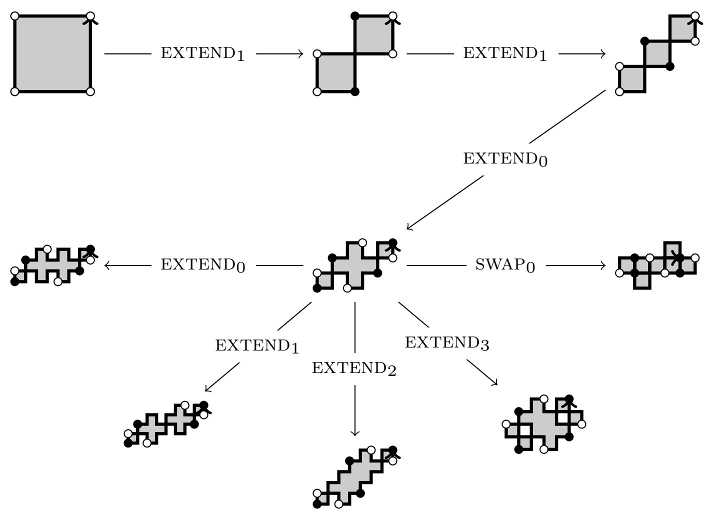


Figure 18: Subtree of DS-factorizations yielding only nonsimple paths.

Empirical observations suggests that when applying Algorithm 1 to a double square, it reduces to a composed X pentomino in a way such that each intermediate DS-factorization describes a simple closed path.

Conjecture 33. *Let S be the nonsingular DS-factorization of a double square tile and X be the X pentomino. There exist an homologous morphism φ and a reduction from S to $\varphi(X)$, such that each intermediate DS-factorization in the reduction describes a polyomino.*

Theorem 30 states that prime double squares reduce to the unit square. Can one prove the converse? This would lead to a primality test for double square tiles. We believe the converse is true and propose the following conjecture.

Conjecture 34. *Let D be a double square tile. If D reduces to the unit square, then D is prime.*

Also, it seems that the boundary of a prime double square is made only of left and right turns. More formally, we make the following conjecture.

Conjecture 35. *Let $w \in \mathcal{F}^*$ be the boundary word of a prime double square tile. Then for all letter $\alpha \in \mathcal{F}$, $\alpha\alpha$ is not a factor of $[\![w]\!]$. Equivalently, $\Delta([\![w]\!]) \in \{\mathbf{1}, \mathbf{3}\}^*$.*

A natural question comes when working with homologous morphisms : *Does homologous morphisms preserve primitive words?* It turns out to be false for the morphism $\varphi_{A,B}$ with $A = \mathbf{0}$, $B = \mathbf{101}$. Indeed, **01** is primitive but $\varphi_{A,B}(\mathbf{01}) =$ **0101** is not. But we believe it is true if $AB\widehat{A}\widehat{B}$ is the boundary of a polyomino. This question can be written in terms of codes (see chapter on circular codes in [2]). A submonoid M of the free monoid $\mathcal{A}^*$ is *very pure* if for all $u, v \in \mathcal{A}^*$, $uv, vu \in M$ implies that $u, v \in M$. Indeed, very pure code preserve primitive words.

Conjecture 36. *Let $A, B \in \mathcal{F}^*$ be two primitive words. If $w = AB\widehat{A}\widehat{B}$ is the boundary word of a square tile, i.e. none of the proper factors of conjugates of w is closed then $\{A, B, \widehat{A}, \widehat{B}\}^*$ is a very pure submonoid.*

For example, **010121232303** is the boundary of a square tile containing no closed proper factors and $M = \{\mathbf{010}, \mathbf{121}, \mathbf{232}, \mathbf{303}\}^*$ is very pure. However, the reciprocal is not true as observed by Hugo Tremblay [19] who provided a counter example: the submonoid $M = \{\mathbf{0010}, \mathbf{1101}, \mathbf{2322}, \mathbf{3233}\}^*$ is very pure but $\mathbf{0010} \cdot \mathbf{1101} \cdot \mathbf{2322} \cdot \mathbf{3233}$ is a closed path containing the closed proper factor **0123**.

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