ON CHARACTER VARIETIES, SETS OF DISCRETE CHARACTERS, AND NON-ZERO DEGREE MAPS

MICHEL BOILEAU AND STEVEN BOYER

ABSTRACT. A *knot manifold* is a compact, connected, irreducible, orientable 3-manifold whose boundary is an incompressible torus. We first investigate virtual epimorphisms between the fundamental groups of small knot manifolds and prove minimality results for small knot manifolds with respect to non-zero degree maps. These results are applied later in the paper where we fix a small knot manifold M and investigate various sets of characters of representations $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ whose images are discrete. We show that the topology of these sets is intimately related to the algebraic structure of the $PSL_2(\mathbb{C})$-character variety of M as well as dominations of manifolds by M and its Dehn fillings. We apply our results to the following question of Shicheng Wang: *Are non-zero degree maps between infinitely many distinct Dehn fillings of two hyperbolic knot manifolds M and N induced by a non-zero degree map $M \to N$?* We show that the answer is yes generically. Using this we show that if a small $\mathcal{H}$-minimal hyperbolic knot manifold admits non-homeomorphic $\mathcal{H}$-minimal Dehn fillings, it admits infinitely many such fillings. We also construct the first infinite families of small, closed, connected, orientable manifolds which are minimal in the sense that they do not admit non-zero degree maps, other than homotopy equivalences, to any aspherical manifold.

November 3, 2011

1. INTRODUCTION

1.1. **General introduction.** Character variety methods have proven an essential tool for the investigation of problems in low-dimensional topology and have been instrumental in the resolution of many well-known problems. In particular, they have been used to study homomorphisms between the fundamental groups of 3-manifolds. In this paper we focus on their application to the study of homomorphisms induced by non-zero degree maps. We assume throughout that our manifolds are compact, connected, orientable, and 3-dimensional. A *knot manifold* is a compact, connected, irreducible, orientable 3-manifold whose boundary is an incompressible torus. We shall restrict our attention, for the most part, to *small* knot manifolds, that is, those which contain no closed essential surfaces. A small knot manifold is atoroidal and so by geometrisation it is either hyperbolic or admits a Seifert fibred structure with base orbifold of the form $D^2(p, q)$ for some integers $p, q \geq 2$.

We call a homomorphism $\varphi : \Gamma_1 \to \Gamma_2$ between two groups a *virtual epimorphism* if its image is of finite index in Γ_2. For instance, a non-zero degree map between manifolds induces a virtual

epimorphism on the level of fundamental groups. In §3 we investigate virtual epimorphisms between the fundamental groups of small knot manifolds and proves minimality results for small knot manifolds with respect to non-zero degree maps. This work will be applied to illustrate the results of the later sections of the paper. There we fix a small knot manifold M and investigate various sets of characters of representations $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ whose images are discrete. It turns out that the topology of these sets is intimately related to the algebraic structure of the $PSL_2(\mathbb{C})$-character variety of M as well as dominations of manifolds by M and its Dehn fillings. In particular, we apply our results to study families of non-zero degree maps $f_n : M(\alpha_n) \to V_n$ where $M(\alpha_n)$ is the α_n-Dehn filling M and V_n is either a hyperbolic manifold or $\widetilde{SL_2}$ manifold. Using this, the existence of infinite families of small, closed, connected, orientable manifolds which do not admit non-zero degree maps, other than homotopy equivalences, to any hyperbolic manifold, or even manifolds which are either reducible or aspherical is determined.

In the remainder of the introduction we give a more detailed description of our results and the organization of the paper. Here is some notation and terminology we shall use.

Throughout, Γ will denote a finitely generated group. We call a homomorphism $\rho : \Gamma \to PSL_2(\mathbb{C})$ discrete, non-elementary, torsion free, abelian, etc. if its image has this property. If $\chi_\rho \in X_{PSL_2}(M)$ is the character of ρ we will call it discrete, non-elementary, torsion free, abelian, etc. if each representation $\rho' : \pi_1(M) \to PSL_2(\mathbb{C})$ with $\chi_{\rho'} = \chi_\rho$ has this property. For instance we can unambiguously refer to a character as being either irreducible, non-elementary, or torsion free.

We will denote the projective space of $H_1(M; \mathbb{R})$ by $\mathbb{P}(H_1(\partial M; \mathbb{R}))$ and the projective class of a non-zero element $\beta \in H_1(\partial M; \mathbb{R})$ by $[\beta]$.

A *slope* on the boundary of a knot manifold M is a ∂M-isotopy class of essential simple closed curves. Slopes correspond bijectively with $\pm$ pairs of primitive elements of $H_1(\partial M)$ in the obvious way. The *longitudinal slope* on ∂M is the unique slope λ_M having the property that it represents a torsion element of $H_1(M)$. When M is the exterior of a knot K in a closed 3-manifold W, there is a unique slope μ_K on ∂M, called the *meridional slope*, which is homologically trivial in a tubular neighbouhood of the knot. If W is a $\mathbb{Z}$-homology 3-sphere, then μ_K and λ_M are dual in the sense that the homology classes they carry form a basis for $H_1(\partial M)$.

Each slope α on ∂M determines an element of $\pi_1(M)$ well-defined up to conjugation and taking inverse. We will sometimes use this connection to evaluate a representation on a slope, but only in a context where the statement being made is independent of the choice of element of $\pi_1(M)$. For instance we may say that $\rho(\alpha) \in PSL_2(\mathbb{C})$ is parabolic, or loxodromic, or trivial.

A representation $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ is *peripherally nontrivial* if $\rho(\pi_1(\partial M))$ does not equal $\{\pm I\}$. When M is small, there are only finitely many characters of representations which are not peripherally non-trivial. Indeed, there are only finitely many characters χ_ρ for which $\rho(\pi_1(\partial M))$ is trivial or a parabolic subgroup of $PSL_2(\mathbb{C})$ (cf. Corollary 2.8). Thus, apart from finitely many exceptions, a discrete, torsion-free character is the character of a representation ρ for which there is a unique slope α on ∂M such that $\rho(\alpha) = \pm I$. In this case we call α the *slope of ρ*.

A *hyperbolic manifold* is one whose interior admits a complete, finite volume, hyperbolic structure. A closed manifold which admits an $\widetilde{SL_2}$ structure is called an $\widetilde{SL_2}$ manifold. Similarly we will refer to closed *Nil* manifolds, $\mathbb{E}^3$ manifolds, etc. Two families of manifolds we will focus on are the family $\mathcal{H}$ of hyperbolic 3-manifolds and the family $\mathcal{M}$ of compact, connected, orientable 3-manifolds.

We say that M *dominates* N, written $M \geq N$, if there is a continuous, proper map from M to N of non-zero degree. Moreover, if N is not homeomorphic to M we say that M *strictly dominates* N. The relation $\geq$ is a partial order when restricted to manifolds in $\mathcal{M}$ which are aspherical but are neither torus (semi) bundles or Seifert manifolds with zero Euler number [Wan1], [Wan2]. This partial order is far from well-understood, even when restricted to hyperbolic 3-manifolds.

A knot manifold is *minimal* if the only knot manifold it dominates is itself. (Note that each knot manifold dominates $S^1 \times D^2$, but also that the latter is not a knot manifold.) For example, a punctured torus bundle is minimal if and only if its monodromy is not a proper power (see [BWa, Prop. 2.6]). A closed, connected, orientable 3-manifold is *minimal* if the only manifold it dominates is one with finite fundamental group. (It is easy to see that every closed, connected, orientable 3-manifold admits a degree 1 map to the 3-sphere and hence it dominates any manifold with finite fundamental group.) A manifold V is $\mathcal{H}$-*minimal* if the only manifold in $\mathcal{H}$ it dominates is itself. Note that we do not require that $V \in \mathcal{H}$.

1.2. **Main results.** Our first result yields apriori bounds on the length of sequences $\pi_1(M) \xrightarrow{\varphi_1} \pi_1(N_1) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} \pi_1(N_n)$ of non-injective virtual epimorphisms between small knot manifolds in terms of a certain invariant of the $PSL_2(\mathbb{C})$-character variety of M. See Theorem 3.8. Precise calculations of this invariant can be made for various families of knot manifolds. In the case where M is the exterior of a two-bridge knot we obtain the following explicit bounds. (See Theorem 3.11 and Corollaries 3.12 and 3.15.)

Theorem 1.1. *Let $M_{p/q}$ denote the exterior of the $\left(\frac{p}{q}\right)$ two-bridge knot and consider a sequence of virtual epimorphisms none of which is injective:*

$$\pi_1(M_{p/q}) \xrightarrow{\varphi_1} \pi_1(N_1) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} \pi_1(N_n)$$

(1) If N_i is small for each i, then $n < \frac{p-1}{2}$.
(2) If each φ_i is induced by a non-zero degree map, then $n + 1$ is bounded above by the number of distinct divisors of p.
(3) If each φ_i is induced by a degree one map, then $n + 1$ is bounded above by the number of distinct prime divisors of p.

This result immediately yields an infinite family of minimal two-bridge knot exteriors.

Corollary 1.2.
(1) If p is an odd prime and $M_{p/q}$ is hyperbolic then $M_{p/q}$ is minimal.
(2) If p is a prime power, any degree one map $M_{p/q} \to N$, N a knot manifold, is homotopic to a homeomorphism.

We note that two-bridge knot exteriors are not minimal in general. For instance, T. Ohtsuki, R. Riley, and M. Sakuma have given a systematic construction of degree one maps between such knot manifolds [ORS].

The fundamental group of a small knot manifold M admits many discrete, non-elementary representations with values in $PSL_2(\mathbb{C})$. For instance when M is hyperbolic, its holonomy representation is discrete and non-elementary, as are the holonomy representations of the hyperbolic Dehn fillings of M. Similarly, when M is Seifert fibred but not a twisted I-bundle over the Klein bottle, a holonomy representation of its base orbifold is discrete and non-elementary, as are those of the base orbifolds of the generic Dehn fillings of M. One of the problems we investigate in the paper is to what extent these are the only discrete non-elementary representations of $\pi_1(M)$.

Set

$$D(M) = \{\chi_\rho \in X_{PSL_2}(M) : \rho \text{ is discrete and non-elementary}\}.$$

Classic work of Jørgensen and Marden shows that $D(M)$ is closed in $X_{PSL_2}(M)$ (see §4.1). Their results combine with the work of Culler and Shalen on ideal points of curves of $PSL_2(\mathbb{C})$-characters to show that if $D(X_0) = D(M) \cap X_0$ is not compact for some component X_0 of $X_{PSL_2}(M)$, there is a connected essential surface S in M such that the restriction of each character in X_0 to $\pi_1(S)$ is elementary. This is a key point in the proof by Morgan and Shalen that if the set of discrete faithful characters of the fundamental group of a compact 3-manifold is not compact, then the group splits non-trivially along a virtually abelian subgroup [MS2]. Similarly we use these ideas to construct various infinite families of small hyperbolic knot exteriors M for which $D(M)$ is compact (see §4.2 and §4.3).

To each representation $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ is associated a volume $vol(\rho) \in \mathbb{R}$ defined by taking any pseudo-developing map from the universal cover $\widetilde{M}$ into $\mathbb{H}^3$ and integrating the pull-back of the hyperbolic volume form on a fundamental domain of M. (See [Dun], [Fra] for more details.) This value depends only on the character of ρ so it makes sense to talk of the volume of a character. Moreover, the associated volume function $vol : X_{PSL_2}(M) \to \mathbb{R}$ is continuous (indeed analytic). We will see below (Corollary 4.3) that discrete, non-elementary, torsion free, non-zero volume representations of the fundamental group of a knot manifold M correspond to non-zero degree maps of M or its Dehn fillings to a hyperbolic manifold.

A *principal component* X_0 of the $PSL_2(\mathbb{C})$-character variety of a finitely generated group Γ is a component which contains the character of a discrete, faithful, irreducible representation of $\Gamma/Z(\Gamma)$, where $Z(\Gamma)$ denotes the centre of Γ. Our next result is a combination of Theorem 4.14 and Lemma 4.5.

Theorem 1.3. *Let M be a small hyperbolic knot manifold, X_0 a component of $X_{PSL_2}(M)$, and suppose that $\{\chi_n\} \subset X_0$ is a sequence of distinct characters of non-zero volume representations $\rho_n : \pi_1(M) \to PSL_2(\mathbb{C})$ with image a torsion-free, cocompact, discrete group Γ_n. For $n \gg 0$, let α_n be the slope of ρ_n. Then up to taking a subsequence, one of the following two possibilities arises:*
(a) the slopes α_n converge projectively to the projective class of a boundary slope of M; or
(b) $\lim \chi_n$ exists and is the character of a discrete, non-elementary, torsion free, non-zero

volume representation ρ_0 such that:

  (i) $\rho_0|\pi_1(\partial M)$ is 1-1 and $\rho_0(\pi_1(M))$ is a finite index subgroup of the fundamental group of a 1-cusped hyperbolic manifold V.

  (ii) there are slopes β_n on ∂V such that for each n the fundamental group of the Dehn filled manifold $V(\beta_n)$ is isomorphic to Γ_n and the character χ_n is induced by the composition $\pi_1(M) \to \rho_0(\pi_1(M)) \hookrightarrow \pi_1(V) \to \pi_1(V(\beta_n)) \cong \Gamma_n$.

  (iii) $X_0 = \rho_0^*(Y_0)$ for a principal component Y_0 of $X_{PSL_2}(V)$

In certain circumstances we can guarantee that conclusion (b) of the theorem holds. For instance, this is the case when M is hyperbolic and the characters χ_n lie on a principal component X_0 of the $PSL_2(\mathbb{C})$-character variety of $\pi_1(M)$ (Corollary 4.8). More generally, it holds if we suppose that the characters χ_n lie on a curve component X_0 of $X_{PSL_2}(M)$ such that one of the following two conditions is satisfied:

  (a) for each ideal point x_0 of X_0 there are a component S_0 of an essential surface associated to x_0 and a character $\chi \in X_0$ such that $\chi|\pi_1(S_0)$ is non-elementary; or

  (b) the Culler-Shalen seminorm of X_0 is a norm and each ideal point of X_0 has an associated essential surface S_0 with $|\partial S_0| \leq 2$.

See Corollary 4.7 for the justification of case (a) and Corollary 4.12 for that of case (b). The following is a consequence of Theorem 1.3 (see Corollary 4.15.)

Corollary 1.4. *Let M be a small hyperbolic knot manifold. Then all but finitely many of the discrete, non-zero volume characters on a principal curve X_0 of the $PSL_2(\mathbb{C})$-character variety of $\pi_1(M)$ are induced by the complete hyperbolic structure on the interior of M or by Dehn fillings of manifolds finitely covered by M.*

If $f : M \to N$ is a domination between hyperbolic knot manifolds, then to each slope α on ∂M we can associate a slope $\beta = \beta(\alpha)$ on ∂N such that f induces a domination $M(\alpha) \to N(\beta)$. This domination is strict as long as f is strict. We say that the dominations $M(\alpha) \to N(\beta)$ are *induced* by f.

Here is a version of Theorem 1.3 for non-zero degree maps (see §4.4).

Theorem 1.5. *Let M be a small hyperbolic knot manifold and $\{\alpha_n\}_{n\geq 1}$ a sequence of distinct slopes on ∂M which do not subconverge projectively to the projective class of a boundary slope. If there are dominations $f_n : M(\alpha_n) \geq V_n$ where $\{V_n\}$ is a sequence of mutually distinct hyperbolic manifolds, then there exist a compact hyperbolic manifold V_0 with a domination $f : M \geq V_0$, a subsequence $\{j\}$ of $\{n\}$, and slopes β_j on ∂V_0 such that for each j:*

  (i) $V_0(\beta_j) \cong V_j$; and

  (ii) the following diagram is homotopy commutative:

$$
\begin{array}{ccc}
M & \xrightarrow{f} & V_0 \\
\downarrow & & \downarrow \\
M(\alpha_j) & \xrightarrow{f_j} & V_j \cong V_0(\beta_j)
\end{array}
$$

Thus infinitely many of the dominations $f_n : M(\alpha_n) \geq V_n$ are induced by f. If we assume further that the dominations $f_n : M(\alpha_n) \geq V_n$ are strict, then $f : M \geq V_0$ is strict as well.

One of our principal motivations for investigating families of discrete characters was to address the following question posed by Shicheng Wang: *If there are non-zero degree maps between infinitely many distinct Dehn fillings of two knot manifolds M and N, are they induced by a non-zero degree map $M \to N$?* The methods of this paper show that the answer is yes generically. For instance, for small manifolds we have the following corollary.

Corollary 1.6. *Let M and N be two small hyperbolic knot manifolds and suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of slopes on ∂M and ∂N for which there are dominations $M(\alpha_n) \geq N(\beta_n)$. If the $\{\alpha_n\}$ do not subconverge projectively to the projective class of a boundary slope of M, then there is a subsequence $\{j\}$ of $\{n\}$ and a domination $M \geq N$ which induces the dominations $M(\alpha_j) \geq N(\beta_j)$ for all j.* □

The results above can be applied to the study of minimal manifolds. For instance the only known example of closed hyperbolic $\mathcal{H}$-minimal 3-manifold is $\frac{1}{2}$ surgery on the figure eight knot [ReW]. The following consequences of Theorem 1.5 show that closed $\mathcal{H}$-minimal manifolds are actually quite plentiful. The following is part (1) of Theorem 5.1.

Corollary 1.7. *Let M be a small, hyperbolic $\mathcal{H}$-minimal knot manifold and suppose that there is a slope α_0 on ∂M such that $M(\alpha_0)$ is $\mathcal{H}$-minimal. If $M(\alpha_0)$ is hyperbolic, assume that the core of the α_0 filling solid torus is not null-homotopic in $M(\alpha_0)$. If $U \subset \mathbb{P}(H_1(\partial M; \mathbb{R}))$ is the union of disjoint closed arc neighbourhoods of the finite set of boundary slopes of M, then $\mathbb{P}(H_1(\partial M; \mathbb{R})) \setminus U$ contains only finitely many projective classes of slopes α such that $M(\alpha)$ is not $\mathcal{H}$-minimal. In particular, M admits infinitely many $\mathcal{H}$-minimal Dehn fillings.*

Remark 1.8.
(1) The condition that the core of the α_0 filling solid torus is not null-homotopic in $M(\alpha_0)$ when $M(\alpha_0)$ is hyperbolic is necessary. Take, for instance, M to be the exterior of a null-homotopic knot in an $\mathcal{H}$-minimal hyperbolic 3-manifold and let α_0 be the meridian slope of this knot. Then each Dehn filling of M dominates $M(\alpha_0)$, so at most finitely many are $\mathcal{H}$-minimal.
(2) The existence of a slope α_0 satisfying the conditions of the corollary guarantees that for any sequence of distinct slopes $\{\alpha_n\}$ on ∂M which does not subconverge projectively to a boundary slope, and dominations $M(\alpha_n) \geq V_n$, where V_n each is hyperbolic, there are infinitely many distinct V_n (cf. the proof of Theorem 1.5). Hence the hypotheses of Theorem 1.5 are satisfied.

If a knot manifold M admits a filling $M(\alpha)$ in which the core of the surgery torus is null-homotopic, then each slope β on ∂M is homotopically trivial $M(\alpha)$. Hence there is a degree one map $M(\beta) \to M(\alpha)$. It follows that if an $\mathcal{H}$-minimal hyperbolic knot manifold admits two non-homeomorphic $\mathcal{H}$-minimal Dehn fillings, then either one of them is non-hyperbolic or both are hyperbolic and the cores of their respective filling solid tori are not null-homotopic. Thus,

Corollary 1.9. *Let M be a small, hyperbolic $\mathcal{H}$-minimal knot manifold which admits two non-homeomorphic $\mathcal{H}$-minimal Dehn fillings. Then it admits infinitely many $\mathcal{H}$-minimal Dehn fillings.* □

Corollary 1.7 applies to many hyperbolic knot manifolds. For instance, it applies to punctured torus bundles whose monodromies are pseudo-Anosov and not proper powers, or the exterior of the $(-2, 3, n)$ pretzel $(n \neq 1, 3, 5)$.

The meridional slope of a knot in the 3-sphere whose exterior is small is never a boundary slope (see Theorem 2.0.3 of [CGLS]). Thus Corollary 1.7 implies:

Corollary 1.10. *Let M be the exterior of a small hyperbolic knot in S^3 and let $\mu, \lambda \in H_1(\partial M)$ represent the meridional and longuitudinal slope respectively. If M is $\mathcal{H}$-minimal, then for all but finitely many $n \in \mathbb{Z}$, the Dehn filled manifold $M(n\mu + \lambda)$ is $\mathcal{H}$-minimal.* $\square$

For certain two-bridge knot exteriors we can say more (see §5).

Corollary 1.11. *Let M be the exterior of a hyperbolic $\frac{p}{q}$ two-bridge knot with p prime. Then all but finitely many Dehn fillings of M yield $\mathcal{H}$-minimal manifolds.*

In order to construct families of closed minimal manifolds it is necessary to prove a version of Theorem 1.3 for discrete representations to $PSL_2(\mathbb{R})$. Set

$$D(M; \mathbb{R}) = \{\chi_\rho \in D(M) : \rho \text{ has image in } PSL_2(\mathbb{R})\}$$

and $D(X_0; \mathbb{R}) = D(M; \mathbb{R}) \cap X_0$ where X_0 is a subvariety of $X_{PSL_2}(M)$. An example of the sort of result we obtain is the next proposition (see Corollary 6.16).

Proposition 1.12. *Let M be a small hyperbolic knot manifold with $H_1(M) \cong \mathbb{Z}$, $\{\alpha_n\}$ a sequence of distinct slopes on ∂M, and $\{\chi_n\} \subset D(M; \mathbb{R})$ a sequence of characters of representations ρ_n such that $\rho_n(\alpha_n) = \pm I$ for all n. If there are infinitely many distinct characters χ_n and the sequence $\{\chi_n\}$ subconverges to a character χ_{ρ_0} such that $\rho_0(\lambda_M) \neq \pm I$, then M strictly dominates a Seifert manifold with incompressible boundary.*

This last result can be used to construct infinite families of closed minimal manifolds. For instance, we have the following theorem (see Theorem 7.2).

Theorem 1.13. *Suppose that M is a small $\mathcal{H}$-minimal hyperbolic knot manifold which has the following properties:*

(a) There is a slope α_0 on ∂M such that $M(\alpha_0)$ is $\mathcal{H}$-minimal. Suppose as well that the core of the α_0 filling solid torus is not null-homotopic in $M(\alpha_0)$ when $M(\alpha_0)$ is hyperbolic.
(b) For each norm curve $X_0 \subset X_{PSL_2}(M)$ and for each essential surface S associated to an ideal point of X_0 there is a character $\chi_\rho \in X_0$ which restricts to a strictly irreducible character on $\pi_1(S)$.
(c) There is no surjective homomorphism from $\pi_1(M)$ onto a Euclidean triangle group.
(d) There is no epimorphism $\rho : \pi_1(M) \to \Delta(p, q, r) \subset PSL_2(\mathbb{R})$ such that the elements of $\rho(\pi_1(\partial M))$ are either elliptic or trivial.

Then all but finitely many Dehn fillings $M(\alpha)$ yield a minimal manifold.

As a consequence of the preceding result and the non-existence of certain epimomorphisms from the fundamental group of a hyperbolic twist knot exterior to a Euclidean or hyperbolic triangle group (Appendix B) we show (Corollary 7.3):

Corollary 1.14. *If M is the exterior of a hyperbolic twist knot, then all but finitely many Dehn fillings of M yield a minimal manifold.*

Our final results show that quite general hypotheses on a minimal knot exterior imply that it admits infinitely many minimal Dehn fillings. See Theorems 7.5 and 7.7 and Examples 7.6 and 7.9. For instance,

Corollary 1.15. *Let M be the exterior of a hyperbolic $\frac{p}{q}$ two-bridge knot with p prime, or of a $(-2, 3, n)$ pretzel with $n \not\equiv 0 \pmod 3$. Then there are infinitely many slopes α on ∂M such that $M(\alpha)$ is minimal.*

1.3. **Acknowledgements.** The authors wish to thank Shicheng Wang for many stimulating conversations and for bringing to their attention his question about Dehn fillings and non-zero degree maps. They also wish to thank the referee for many valuable suggestions.

2. Varieties of $PSL_2(\mathbb{C})$-characters

In this section we gather some of the basic properties of $PSL_2(\mathbb{C})$-character varieties and Culler-Shalen theory that will be needed in the later sections.

2.1. **Generalities.** In what follows we shall refer to the elements of $PSL_2(\mathbb{C})$ as matrices. Denote by $\mathcal{D}$ the abelian subgroup of $PSL_2(\mathbb{C})$ consisting of diagonal matrices and by $\mathcal{N}$ the subgroup consisting of those matrices which are either diagonal or have diagonal coefficients 0. Note that $\mathcal{D}$ has index 2 in $\mathcal{N}$ and any element in $\mathcal{N} \setminus \mathcal{D}$ has order 2.

The action of $SL_2(\mathbb{C})$ on $\mathbb{C}^2$ descends to one of $PSL_2(\mathbb{C})$ on $\mathbb{C}P^1$. We call a representation ρ with values in $PSL_2(\mathbb{C})$ *irreducible* if the associated action on $\mathbb{C}P^1$ is fixed point free, otherwise we call it *reducible*. We call it *strictly irreducible* if the action has no invariant subset in $\mathbb{C}P^1$ with fewer than three points. Note that

- ρ is reducible if and only if it is conjugate to a representation whose image consists of upper-triangular matrices.
- ρ is conjugate to a representation with image in $\mathcal{D}$ if and only if the action on $\mathbb{C}P^1$ has at least two fixed points. It is conjugate into $\mathcal{N}$ if and only if it leaves a two point subset of $\mathbb{C}P^1$ invariant.
- ρ is is strictly irreducible if and only if it is irreducible but not conjugate into $\mathcal{N}$.
- if ρ is irreducible and $A \in PSL_2(\mathbb{C})$ satisfies $A\rho A^{-1} = \rho$, then either $A = \pm I$ or up to conjugation, $A = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and ρ conjugates into $\mathcal{N}$. Thus if ρ is strictly irreducible, then $A = \pm I$.

The action of $PSL_2(\mathbb{C})$ on $\mathbb{C}P^1 = S_\infty^2$ extends over $\mathbb{H}^3$ yielding an identification $PSL_2(\mathbb{C}) = \text{Isom}_+(\mathbb{H}^3)$. A representation is called *elementary* if the associated action on $\overline{\mathbb{H}}^3$ has a finite orbit. Equivalently, the representation is reducible or conjugates to one with image in either $SO(3) = PSU(2)$ or $\mathcal{N}$.

Let Γ be a finitely generated group. The set $R_{PSL_2}(\Gamma)$ of representations of Γ with values in $PSL_2(\mathbb{C})$ admits the structure of a $\mathbb{C}$-affine algebraic set [LM] called the $PSL_2(\mathbb{C})$-*representation variety* of Γ. The action of $PSL_2(\mathbb{C})$ on $R_{PSL_2}(\Gamma)$ determines an algebro-geometric quotient $X_{PSL_2}(\Gamma)$ whose coordinate ring is $\mathbb{C}[R_{PSL_2}(\Gamma)]^{PSL_2(\mathbb{C})}$ and a regular map $t : R_{PSL_2}(\Gamma) \to X_{PSL_2}(\Gamma)$ [LM]. This quotient is called the $PSL_2(\mathbb{C})$-*character variety* of Γ. For $\rho \in R_{PSL_2}(\Gamma)$, we denote $t(\rho)$ by χ_ρ and refer to it as the *character* of ρ. If $\chi_{\rho_1} = \chi_{\rho_2}$ and ρ_1 is irreducible, then ρ_1 and ρ_2 are conjugate representations. We can therefore call a character χ_ρ reducible, irreducible, or strictly irreducible if ρ has that property. Each reducible character is the character of a diagonal representation. That is, one with image in $\mathcal{D}$. The property of an irreducible representation being conjugate into $SO(3)$ is also determined by its character (see Proposition III.1.1 of [MS1] for instance) and so if $\chi_{\rho_1} = \chi_{\rho_2}$ and ρ_1 is elementary, then so is ρ_2. In this case we call the character elementary.

When Γ is the fundamental group of a path-connected space Y, we write $R_{PSL_2}(Y)$ for $R_{PSL_2}(\pi_1(Y))$, $X_{PSL_2}(Y)$ for $X_{PSL_2}(\pi_1(Y))$, and refer to them respectively as the $PSL_2(\mathbb{C})$-representation variety of Y and $PSL_2(\mathbb{C})$-character variety of Y.

Each $\gamma \in \Gamma$ determines an element f_γ of the coordinate ring $\mathbb{C}[X_{PSL_2}(\Gamma)]$ according to the formula

$$f_\gamma(\chi_\rho) = \text{trace}(\rho(\gamma))^2 - 4.$$

A homomorphism $\varphi : \Gamma_1 \to \Gamma_2$ determines morphisms $\varphi^* : R_{PSL_2}(\Gamma_2) \to R_{PSL_2}(\Gamma_1)$, $\rho \mapsto \rho \circ \varphi$ and $\varphi^* : X_{PSL_2}(\Gamma_2) \to X_{PSL_2}(\Gamma_1)$, $\chi_\rho \mapsto \chi_{\rho \circ \varphi}$. For $\gamma \in \Gamma_1$ and $\chi_\rho \in X_{PSL_2}(\Gamma_2)$ we have

$$f_\gamma(\varphi^*(\chi_\rho)) = f_{\varphi(\gamma)}(\chi_\rho). \tag{2.1.1}$$

We end this section with a useful observation

Lemma 2.1. *If the image of* $\varphi : \Gamma_1 \to \Gamma_2$ *is of finite index* n *in* Γ_2, *then* $\varphi^* : X_{PSL_2}(\Gamma_2) \to X_{PSL_2}(\Gamma_1)$ *is a closed map with respect to the Zariski topology.*

Proof. Let X_0 be a Zariski closed subset of $X_{PSL_2}(\Gamma_2)$ and let $Y_0 = \overline{\varphi^*(X_0)}$. If $\bar{X}_0, \bar{Y}_0$ are projective closures of X_0, Y_0, then φ^* determines a surjective projective morphism $\bar{\varphi}^* : \bar{X}_0 \to \bar{Y}_0$. Let $y_0 \in Y_0$ and choose $x_0 \in \bar{X}_0$ such that $\bar{\varphi}^*(x_0) = y_0$, and a projective curve $C \subseteq \bar{X}_0$ which contains x_0. Set $C_0 = C \cap C_0$ and note that if $x_0 \notin C_0$, there is some $\gamma \in \pi_1(M)$ such that $f_\gamma(x_0) = \infty$ (cf. Theorem 2.1.1 of [CS]). For $A, B \in SL_2(\mathbb{C})$ we have $\text{trace}(AB) + \text{trace}(A^{-1}B) = \text{trace}(A)\text{trace}(B)$, and this identity can be used inductively to show that f_{γ^n} is a degree $|n|$ polynomial in f_γ. In particular, $f_{\gamma^n}(x_0) = \infty$. On the other hand there is some $\delta \in \Gamma_1$ such that $\varphi(\delta) = \gamma^n$. Then $f_\delta(y_0) = f_\delta(\bar{\varphi}^*(x_0)) = f_{\varphi(\delta)}(x_0) = f_{\gamma^n}(x_0) = \infty$. But this contradicts the fact that $y_0 \in Y_0$. Thus $x_0 \in C_0 \subseteq X_0$ and so φ^* is onto Y_0. $\square$

2.2. **Subvarieties of** $X_{PSL_2}(\Gamma)$. The set of reducible representations $R_{PSL_2}^{red}(\Gamma) \subseteq R_{PSL_2}(\Gamma)$ (Γ a finitely generated group) is a closed algebraic subset (cf. the proof of Corollary 1.4.5 of [CS]). The sets $R_{SO(3)}(\Gamma), R_{\mathcal{D}}(\Gamma)$, and $R_{\mathcal{N}}(\Gamma)$ of representations of Γ with values in $SO(3), \mathcal{D}$, and $\mathcal{N}$ are also closed algebraic subsets of $R_{PSL_2}(\Gamma)$. A similar statement holds for their

images $X_{SO(3)}(\Gamma), X^{red}_{PSL_2}(\Gamma)$, and $X_{\mathcal{N}}(\Gamma)$ in $X_{PSL_2}(\Gamma)$. In particular, the set $X_{Elem}(\Gamma) = X_{SO(3)}(\Gamma) \cup X_{\mathcal{N}}(\Gamma)$ of characters is Zariski closed in $X_{PSL_2}(\Gamma)$.

A subvariety X_0 of $X_{PSL_2}(\Gamma)$ is called *non-trivial* if it contains the character of an irreducible representation. It is called *strictly non-trivial* if it contains the character of a strictly irreducible representation. The property of being (strictly) irreducible is open so that the generic character of a (strictly) non-trivial subvariety of $X_{PSL_2}(\Gamma)$ is (strictly) irreducible. Let $X^{irr}_+(\Gamma)$ denote the union of the positive dimensional non-trivial components of $X_{PSL_2}(\Gamma)$ and $X^{str}_+(\Gamma)$ the union of its positive dimensional strictly non-trivial components.

For each non-trivial subvariety X_0 of $X_{PSL_2}(\Gamma)$ there is a subvariety R_{X_0} of $R_{PSL_2}(\Gamma)$ uniquely determined by the condition that it is conjugation invariant and $t(R_{X_0}) = X_0$ (cf. Lemma 4.1 of [BZ1]). We define the *kernel* of X_0 to be the normal subgroup of Γ given by

$$\mathrm{Ker}(X_0) = \bigcap_{\rho \in R_{X_0}} \mathrm{kernel}(\rho).$$

For instance $\mathrm{Ker}(X_0) = \{1\}$ if R_{X_0} contains an injective representation.

Lemma 2.2. *Let X_0 be a non-trivial subvariety of $X_{PSL_2}(\Gamma)$.*
(1) There is a subset V of R_{X_0} which is a countable union of proper, closed, conjugation invariant algebraic subsets of R_{X_0} such that for $\rho \in R_{X_0} \setminus V$, $\mathrm{kernel}(\rho) = \mathrm{Ker}(X_0)$.
(2) If $\varphi : \Gamma_1 \to \Gamma_2$ is a homomorphism and X_0 is a subvariety of $X_{PSL_2}(\Gamma_2)$ such that $Y_0 = \overline{\varphi^*(X_0)}$ is non-trivial, then $\mathrm{Ker}(Y_0) = \varphi^{-1}(\mathrm{Ker}(X_0))$. In particular, $\mathrm{kernel}(\varphi) \subseteq \mathrm{Ker}(Y_0)$.

Proof. (1) For each $\gamma \in \pi_1(M)$ set $V_\gamma = \{\rho \in R_{X_0} \mid \rho(\gamma) = \pm I\}$. Then V_γ is a closed, conjugation invariant algebraic subset of R_{X_0}. It is clear that $\gamma \in \mathrm{Ker}(X_0)$ if and only if $V_\gamma = R_{X_0}$. Set $V = \bigcup_{\gamma \notin \mathrm{Ker}(X_0)} V_\gamma$ and observe that $\rho \in R_{X_0} \setminus V$ if and only if $\rho(\gamma) \neq \pm I$ for each $\gamma \notin \mathrm{Ker}(X_0)$. In particular, $\mathrm{kernel}(\rho) = \mathrm{Ker}(X_0)$ for such ρ. This proves (1).

(2) Now $\varphi^*(X_0) = t(\varphi^*(R_{X_0})) \subseteq t(\overline{\varphi^*(R_{X_0})}) \subseteq \overline{t(\varphi^*(R_{X_0}))} = \overline{\varphi^*(X_0)} = Y_0$ and since $\overline{\varphi^*(R_{X_0})}$ is closed and conjugation invariant in $R_{PSL_2}(\Gamma_1)$, Theorem 3.3.5(iv) of [Ne] implies that $t(\overline{\varphi^*(R_{X_0})})$ is Zariski closed in $X_{PSL_2}(\Gamma_1)$. It follows that $R_{Y_0} = \overline{\varphi^*(R_{X_0})}$. Hence noting that $\varphi^*(\rho)(\gamma) = \rho(\varphi(\gamma)) = \pm I$ whenever $\gamma \in \varphi^{-1}(\mathrm{Ker}(X_0))$ and $\rho \in R_{X_0}$, it follows that $\rho'(\gamma) = \pm I$ for all $\rho' \in R_{Y_0}$. In other words, $\gamma \in \mathrm{Ker}(Y_0)$. Conversely if $\gamma \in \mathrm{Ker}(Y_0)$ and $\rho \in R_{X_0}$, then $\rho(\varphi(\gamma)) = \varphi^*(\rho)(\gamma) = \pm I$. Thus $\gamma \in \varphi^{-1}(\mathrm{Ker}(X_0))$. This proves (2). $\qquad\square$

We call a component X_0 of $X^{irr}_+(\Gamma)$ *principal* if it contains the character of a discrete, faithful, irreducible representation of $\Gamma/Z(\Gamma)$ where $Z(\Gamma)$ denotes the centre of Γ. It is clear that $\mathrm{Ker}(X_0) \subseteq Z(\Gamma)$.

Lemma 2.3.
(1) If X_0 is a principal component of $X^{irr}_+(\Gamma)$, then $\mathrm{Ker}(X_0) = Z(\Gamma)$.
(2) If $\varphi : \Gamma_1 \to \Gamma_2$ is a homomorphism and X_0 is a subvariety of $X_{PSL_2}(\Gamma_2)$ such that $\overline{\varphi^*(X_0)}$ is principal, then $\mathrm{kernel}(\varphi) \subseteq Z(\Gamma_1)$.

Proof. (1) It suffices to show that $Z(\Gamma) \subseteq \mathrm{Ker}(X_0)$. To that end we note that if $\rho \in R_{PSL_2}(\Gamma)$ is irreducible, then every element in $\rho(Z(\Gamma))$ has order 1 or 2. In particular for $\gamma \in Z(\Gamma)$ and

$\rho \in R_{X_0}$ we have $f_\gamma(\chi_\rho) \in \{0, -4\}$. Hence $f_\gamma|X_0$ is constant and since it vanishes at a discrete faithful character, it is identically zero. In particular as $\rho(\gamma)^2 = \pm I$ we have $\rho(\gamma) = \pm I$. We must show that $\rho(\gamma) = \pm I$ for all $\rho \in R_{X_0}$. To that end consider the closed algebraic subset $R_{X_0}^Z = \{\rho \in R_{X_0} : Z(\Gamma) \subseteq \mathrm{kernel}(\rho)\}$ of R_{X_0}. Now $R_{X_0}^{red} = R_{X_0} \cap R^{red}(\Gamma)$ is a proper closed algebraic subset of R_{X_0} and we have just seen that $R_{X_0} \backslash R_{X_0}^{red} \subseteq R_{X_0}^Z$. Thus $R_{X_0} = \overline{R_{X_0} \backslash R_{X_0}^{red}} \subseteq R_{X_0}^Z \subseteq R_{X_0}$. Hence $R_{X_0} = R_{X_0}^Z$ and so $Z(\Gamma) \subset \mathrm{Ker}(X_0)$.

Part (2) follows from part (1) and part (2) of the previous lemma. □

2.3. **Restriction.** If $\varphi : \Gamma_1 \to \Gamma$ is surjective, it is easy to see that $\varphi^* : X_{PSL_2}(\Gamma) \to X_{PSL_2}(\Gamma_1)$ is injective. The goal of this section is to show that a similar conclusion is true for virtual epimorphisms $\varphi : \Gamma_1 \to \Gamma$ as long as we restrict φ^* to $X_+^{str}(\Gamma)$. This turns out to be a mild generalisation of a result of Long and Reid [LR, Corollary 3.2], who proved an analogous result for the characters of *strongly irreducible* representations. These are the irreducible representations with values in $PSL_2(\mathbb{C})$ whose images contain non-abelian free groups. Note that such representations are strictly irreducible.

Let $D_n, T_{12}, O_{24}, I_{60}$ denote, respectively, the dihedral group of order $2n$, the tetrahedral group of order 12, the octahedral group of order 24, and the icosahedral group of order 60. Each of these is isomorphic to a subgroup of $PSL_2(\mathbb{C})$.

Lemma 2.4. *Let Γ be a group and $\rho \in R_{PSL_2}(\Gamma)$. If ρ is strictly irreducible representation but not strongly irreducible, then its image is isomorphic to either T_{12}, O_{24} or I_{60}.*

Proof. The inclusion of a finite subgroup of $PSL_2(\mathbb{C})$ into $PSL_2(\mathbb{C})$ is strictly irreducible if and only if the group is isomorphic to T_{12}, O_{24} or I_{60}. Thus it suffices to show that ρ has finite image.

The Tits alternative implies that the image G of ρ is virtually solvable [Ti]. Hence there is a finite index normal solvable subgroup G_0 of G. If G_0 is finite we are done. Otherwise choose a non-trivial normal abelian subgroup A of G_0. Then up to conjugation, A is contained in either $\mathcal{D}$ or $\mathcal{P}$, the group of upper-triangular parabolic matrices, or is the subgroup $\mathcal{K} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ of $\mathcal{N}$. Since A is normal in G_0 it follows that up to conjugation we can assume that G_0 is contained in either $\mathcal{N}$ or the group $\mathcal{U}$ of upper-triangular matrices. The only case which requires comment is when $A = \mathcal{K}$. In this case it's easy to see that the kernel of the conjugation-induced homomorphism $G_0 \to \mathrm{Aut}(\mathcal{K})$ is finite, so G_0 is finite, contrary to our assumptions.

If G_0 is contained in $\mathcal{N}$, then G has an infinite finite index normal subgroup contained in $\mathcal{D}$. But then G is contained in $\mathcal{N}$, contrary to the fact that ρ is strictly irreducible. If G_0 is contained in $\mathcal{U}$ but not conjugate into $\mathcal{D}$, it fixes a unique point of $\mathbb{C}P^1$. But this is impossible since then G necessarily leaves this point invariant and therefore ρ would be reducible. □

Taking Lemma 2.4 into consideration, our next result is an immediate consequence of [LR, Corollary 3.2].

Proposition 2.5. *Let Γ_1 be a finitely generated group and $\varphi : \Gamma_1 \to \Gamma$ be a virtual epimorphism. Then $\varphi^*|X_+^{str}(\Gamma)$ is a Zariski closed map with image contained in $X_+^{str}(\Gamma_1)$. Further, $\varphi^*|X_+^{str}(\Gamma)$ is generically one-to-one.*

Proof. It suffices to prove the result when Γ_1 is a finite index subgroup of Γ and φ is the inclusion. In this case $\varphi^* : X_{PSL_2}(\Gamma) \to X_{PSL_2}(\Gamma_1)$ is a Zariski closed map by Lemma 2.1. Hence the same holds for $\varphi^*|X_+^{str}(\Gamma)$.

Since any two finite subgroups of $PSL_2(\mathbb{C})$ which are isomorphic are conjugate in $PSL_2(\mathbb{C})$ and Γ is finitely generated, there are only finitely many characters of representations $\rho \in R_{PSL_2}(\Gamma)$ with image isomorphic to T_{12}, O_{24} or I_{60}. Thus Lemma 2.4 shows that $\{\chi_\rho \in X_{PSL_2}(\Gamma) : \chi_\rho$ is strictly irreducible but not strongly irreducible$\}$ is finite. Corollary 3.2 of [LR] implies that φ^* one-to-one on the set of strongly irreducible characters. Moreover the image of a strongly irreducible character under φ^* is strongly irreducible. Hence $\varphi^*(X_+^{str}(\Gamma)) \subseteq X_{PSL_2}^{str}(\Gamma_1)$. This completes the proof. $\square$

2.4. **Culler-Shalen theory.** In this section, M will denote compact, connected, orientable, irreducible 3-manifold whose boundary is a torus. We collect various results on the $PSL_2(\mathbb{C})$-character varity of M which will be used throughout the paper.

Any complex affine curve C admits an affine desingularisation $C^\nu \xrightarrow{\nu} C$ where ν is surjective and regular. Moreover, the smooth projective model $\widetilde{C}$ of C is obtained by adding a finite number of ideal points to C^ν. Thus $\widetilde{C} = C^\nu \cup \mathcal{I}(C)$ where $\mathcal{I}(C)$ is the set of ideal points of C. There are natural identifications between the function fields of C, C^ν, and $\widetilde{C}$. Thus to each $f \in \mathbb{C}(C)$ we have corresponding $f^\nu \in \mathbb{C}(C^\nu) = \mathbb{C}(C)$ and $\tilde{f} \in \mathbb{C}(\widetilde{C}) = \mathbb{C}(C)$ where $f^\nu = f \circ \nu = \tilde{f}|C^\nu$.

Recall that each $\gamma \in \pi_1(M)$ determines an element f_γ of the coordinate ring $\mathbb{C}[X_{PSL_2}(M)]$ satisfying

$$f_\gamma(\chi_\rho) = \text{trace}(\rho(\gamma))^2 - 4$$

where $\rho \in R_{PSL_2}(M)$. Each $\alpha \in H_1(\partial M) = \pi_1(\partial M)$ defines an element of $\pi_1(M)$ well-defined up to conjugation and therefore determines an element $f_\alpha \in \mathbb{C}[X_{PSL_2}(M)]$. Similarly each slope α on ∂M determines an element of $\pi_1(M)$ well-defined up to conjugation and taking inverse, and so defines $f_\alpha \in \mathbb{C}[X_{PSL_2}(M)]$.

To each curve X_0 in $X_{PSL_2}(M)$ we associate the function

$$d_{X_0} : \pi_1(M) \to \mathbb{Z}, \ d_{X_0}(\gamma) = \text{degree}(f_\gamma : X_0 \to \mathbb{C}).$$

Standard trace identities imply that for $n \in \mathbb{Z}$,

$$d_{X_0}(\gamma^n) = |n| d_{X_0}(\gamma).$$

More generally, it was shown in [CGLS] that d_{X_0} has nice properties when restricted to abelian subgroups of $\pi_1(M)$. For instance, when restricted to $\pi_1(\partial M)$ it gives rise to a *Culler-Shalen seminorm*

$$\| \cdot \|_{X_0} : H_1(\partial M; \mathbb{R}) \to [0, \infty)$$

where for each $\alpha \in H_1(\partial M) = \pi_1(\partial M)$ we have $\|\alpha\|_{X_0} = d_{X_0}(\alpha)$.

We say that a curve $X_0 \subset X_{PSL_2}(M)$ is a *norm curve* if $\|\cdot\|_{X_0}$ is a norm. If $\|\cdot\|_{X_0} \neq 0$, though it is not a norm, there is a primitive element $\beta \in H_1(\partial M)$ well-defined up to sign such that $\|\beta\|_{X_0} = 0$. In this case we say that X_0 is a *β-curve*.

For $x \in \widetilde{X}_0$ and $\gamma \in \pi_1(M)$, we denote by $Z_x(\tilde{f}_\gamma), \Pi_x(\tilde{f}_\gamma)$ the multiplicity of x as a zero, respectively pole, of $\tilde{f}_\gamma$. From the definition of $\|\cdot\|_{X_0}$ we see that for each $\alpha \in H_1(\partial M)$ we have

$$\|\alpha\|_{X_0} = \sum_{x \in \widetilde{X}_0} Z_x(\tilde{f}_\alpha) = \sum_{x \in \mathcal{I}(X_0)} \Pi_x(\tilde{f}_\alpha). \tag{2.4.1}$$

When M is hyperbolic and X_0 a principal component of $X_{PSL_2}(M)$, $\|\cdot\|_{X_0}$ is a norm [CGLS].

Consider a curve X_0 in $X_{PSL_2}(M)$. We say that a sequence of characters $\chi_n \in X_0$ *converges to an ideal point* $x_0 \in \widetilde{X}_0$ if there are a sequence $\{x_n\}$ in $X_0^\nu \subset \widetilde{X}_0$ and an ideal point $x_0 \in \mathcal{I}(X_0)$ such that $\nu(x_n) = \chi_n$ for all n and $\lim_n x_n = x_0$.

For a path-connected space X, a representation $\rho \in R_{PSL_2}(X)$, a path-connected subspace Q of a space X with inclusion map $i : Q \to X$, set

$$\rho^Q := \rho \circ i_\# : \pi_1(Q) \to PSL_2(\mathbb{C}).$$

Since ρ^Q is determined up to conjugation, there is a well-defined

$$\chi_\rho^Q = \chi_{\rho^Q}.$$

Proposition 2.6. ([CS]) *Suppose that X_0 is a curve in $X_{PSL_2}(M)$ and $\rho_n \in R_{X_0} \subset R_{PSL_2}(M)$ is a sequence of representations whose characters χ_n converge to an ideal point x_0 of $\widetilde{X}_0$. Then there is an essential surface $S \subset M$ whose complementary components $A_1, A_2, \ldots, A_n$ satisfy the following properties:*
(a) For each i, the characters $\chi_n^{A_i}$ converge to a character $\chi_0^{A_i}$. Thus if S_j is a component of S, then $\chi_0^{S_j} := \lim_n \chi_n^{S_j} \in X_{PSL_2}(S_j)$ exists. Further, $\chi_0^{S_j}$ is reducible.
(b) For each i, there are conjugates $\sigma_n^{A_i}$ of $\rho_n^{A_i}$ which converge to a representation $\sigma_0^{A_i} \in R_{PSL_2}(A_i)$ for which $\chi_{\sigma_0^{A_i}} = \chi_0^{A_i}$. $\qquad\square$

A representation $\sigma_0^{A_i} \in R_{PSL_2}(A_i)$ obtained as a limit of some conjugates of $\rho_n^{A_i}$ is said to be a *limiting representation associated to the sequence $\{\rho_n\}$*.

Any essential surface $S \subset M$ as described in Proposition 2.6 is said to be *associated* to the ideal point x_0.

Proposition 2.7. ([CS], [CGLS], [CCGLS]) *Let x_0 be an ideal point of a curve X_0 in $X_{PSL_2}(M)$. There is at least one primitive class $\alpha \in H_1(\partial M)$ such that $\tilde{f}_\alpha(x_0) \in \mathbb{C}$. Further,*
(1) if there is exactly one such class (up to sign), then it is a boundary class and any surface S associated to x_0 has non-empty boundary of slope α. Further, $\tilde{f}_\alpha(x_0) = (\lambda - \lambda^{-1})^2$ where λ is a root of unity.
(2) if there are rationally independent classes $\alpha, \beta \in H_1(\partial M)$ such that $\tilde{f}_\alpha(x_0), \tilde{f}_\beta(x_0) \in \mathbb{C}$, then $\tilde{f}_\gamma(x_0) \in \mathbb{C}$ for each $\gamma \in H_1(\partial M)$ and the surface S can be chosen to be closed. $\qquad\square$

Corollary 2.8. *Suppose that M is a small knot manifold.*
(1) If X_0 is a non-trivial component of $X_{PSL_2}(M)$ and x_0 an ideal point of X_0, there is a

primitive class $\alpha \in H_1(\partial M)$ such that $\Pi_{x_0}(\tilde{f}_\alpha) > 0$. Thus $\| \cdot \|_{X_0}$ is either a norm curve or a β-curve for some primitive $\beta \in H_1(\partial M)$.

(2) If $\alpha \in H_1(\partial M)$ is a slope such that $X_{PSL_2}(M(\alpha))$ is infinite, then α is a boundary slope.

Proof. The first statement follows immediately from the previous proposition. For the second, assume that $X_{PSL_2}(M(\alpha))$ is infinite and choose a curve $X_0 \subset X_{PSL_2}(M(\alpha)) \subset X_{PSL_2}(M)$. Since M is small, any essential surface in M associated to an ideal point x_0 has boundary. Moreover, since $X_0 \subset X_{PSL_2}(M(\alpha))$, $\tilde{f}_\alpha(x_0) = 0$. Part (1) of the corollary shows that α is the unique slope with this property and therefore part (1) of the previous proposition shows that it is a boundary slope. $\square$

Corollary 2.9. *Let X_0 be a curve in $X_{PSL_2}(M)$ containing the characters of two discrete representations ρ_1, ρ_2 such that $\rho_j(\pi_1(\partial M))$ contains a non-trivial loxodromic element of $PSL_2(\mathbb{C})$. If there are rationally independent classes $\alpha_1, \alpha_2 \in H_1(\partial M)$ such that $\rho_j(\alpha_j) = \pm I$ for $j = 1, 2$, then $\| \cdot \|_{X_0}$ is a norm.*

Proof. Our hypotheses imply that $\rho_j(\pi_1(\partial M)) \cong \mathbb{Z} \oplus \mathbb{Z}/c_j$ for some $c_j \geq 1$ and that any element of infinite order in this group is loxodromic. In particular this is the case for any element of $H_1(\partial M)$ which is rationally independent of α_j and therefore $f_{\alpha_1}(\chi_{\rho_2}) = |\text{trace}(\rho_2(\alpha_1))|^2 - 4 \neq 0$. Since $f_{\alpha_1}(\chi_{\rho_1}) = 0$, $f_{\alpha_1}|X_0$ is not constant. If there is a primitive class $\beta \in H_1(\partial M)$ such that $f_\beta|X_0$ is constant, then for some j, α_j and β are rationally independent and so $\rho_j(\beta)$ is loxodromic. It follows that $f_\beta \equiv f_\beta(\chi_j) = (\lambda - \lambda^{-1})^2$ where λ is not a root of unity. In particular $\tilde{f}_\beta$ takes on this value at each ideal point of X_0. Proposition 2.7 now shows that $\tilde{f}_{\alpha_1}(x_0) \in \mathbb{C}$ for each ideal point x_0 of X_0. But this impossible as it would imply that $f_{\alpha_1}|X_0$ is constant. Thus $\| \cdot \|_{X_0}$ is a norm. $\square$

3. DOMINATIONS BETWEEN SMALL KNOT MANIFOLDS

We assume that M is a small knot manifold in this section.

3.1. **Two infinite families of minimal knot manifolds.** The character varieties of small knot manifolds have dimension 1 [CCGLS] and since they are either hyperbolic or Seifert fibred, they contain at least one principal component. Moreover, such a component is contained in $X_+^{str}(M)$ unless M is a twisted I-bundle over the Klein bottle.

Proposition 3.1. *Suppose that M and N are small knot manifolds and $\varphi : \pi_1(M) \to \pi_1(N)$ is a virtual epimorphism.*

(1) φ^* induces a birational isomorphism between $X_+^{str}(N)$ and a union of algebraic components of $X_+^{str}(M)$. In particular if Y_0 is a component of $X_+^{str}(N)$, then $X_0 = \varphi^*(Y_0)$ is a component of $X_+^{str}(M)$ and for each $\gamma \in \pi_1(M)$ we have

$$d_{X_0}(\gamma) = d_{Y_0}(\varphi(\gamma)).$$

(2) If there is a principal component X_0 of $X_{PSL_2}(M)$ contained in $\varphi^*(X_+^{str}(N))$, then φ is injective.

Proof. Part (1) follows from Proposition 2.5 given that $X^{str}_+(M)$ has dimension 1. We consider part (2) then.

Suppose that $\varphi^*(Y_0) = X_0$ for some component Y_0 of $X^{str}_+(N)$ and principal component X_0 of $X_{PSL_2}(M)$. Lemma 2.2 (2) shows that kernel$(\varphi) \subseteq \text{Ker}(X_0) \subseteq Z(\pi_1(M))$. Thus if kernel$(\varphi) \neq \{1\}$, M is Seifert fibred, and as $\pi_1(M)$ and $\pi_1(N)$ are torsion free, kernel$(\varphi) = Z(\pi_1(M)) \cong \mathbb{Z}$. But this is impossible as it would imply that $\pi_1(N)$ contains a subgroup isomorphic to $\pi_1(M)/Z(\pi_1(M))$, which is the free product of two finite cyclic groups. Thus kernel$(\varphi) = \{1\}$. $\qquad\square$

Lemma 3.2. *Let M be the exterior of a small knot K in the 3-sphere. If K admits no free symmetries and $X_{PSL_2}(M)$ has a unique non-trivial component, then M is minimal.*

Proof. If $f : (M, \partial M) \to (N, \partial N)$ has non-zero degree then N is small and therefore $X_{PSL_2}(N)$ contains a principal component, say Y_0. There is a finite degree cover $p : \widetilde{N} \to N$ for which f factors $f = p \circ \tilde{f}$ and $\tilde{f}_\# : \pi_1(M) \to \pi_1(\widetilde{N})$ is surjective. Note that $\widetilde{N}$ has connected boundary. Homological considerations show that $\widetilde{N}$ isn't a twisted I-bundle over the Klein bottle and therefore the same is true for N. It follows that $Y_0 \subseteq X^{str}_{PSL_2}(N)$. Then $f^*(Y_0)$ is a non-trivial component of $X^{str}_{PSL_2}(M)$ and so by hypothesis, must be principal. Proposition 3.1(2) implies that $f_\# : \pi_1(M) \to \pi_1(N)$ is injective and therefore $\tilde{f}$ is homotopic to a homeomorphism [Wal1]. It follows that f is homotopic to a covering map. But M covers no orientable manifold but itself as otherwise the cover would be regular [GW] and so the knot would admit a free symmetry. Hence f is homotopic to a homeomorphism, so M is minimal. $\qquad\square$

Example 3.3. The $PSL_2(\mathbb{C})$-character variety of the exterior M of a non-trivial twist knot or a $(-2, 3, n)$ pretzel knot, $n \not\equiv 0 \pmod 3$, has a unique non-trivial component X_0 ([Bu], [Mat]). By [GLM] and [BolZ] (see also [Ha]), M has a free symmetry only in the case that M is the trefoil knot exterior, which is known to be minimal. Thus Lemma 3.2 implies that M is minimal in all cases. It is interesting to note that when $n \equiv 0 \pmod 3$, the character variety of the exterior of the $(-2, 3, n)$ pretzel knot has precisely two non-trivial components, one principal and the other corresponding to a strict domination of the trefoil knot exterior. (This follows from the analysis in the section "r-curves" of [Mat].) Thus M is $\mathcal{H}$-minimal in this case.

3.2. **Rigidity in $\pi_1(M)$ and bounds on sequences of dominations.** Call $\gamma \in \pi_1(M)$ *rigid* if $f_\gamma|X_0$ is constant for some principal curve X_0 of $X_{PSL_2}(M)$. Equivalently, $d_{X_0}(\gamma) = 0$ (cf. §2.4). (This condition is independent of the choice of principal curve.) For instance, if a positive power of $\gamma \in \pi_1(M)$ is central, then γ is rigid. If γ is not rigid, we call it *non-rigid*. Finally we call $\gamma \in \pi_1(M)$ *totally non-rigid* if $f_\gamma|X_0$ is non-constant for all curves $X_0 \subseteq X^{irr}_+(M)$.

Lemma 3.4. *Let M be a small knot manifold.*
(1) *If M is hyperbolic, every non-trivial element of $\pi_1(\partial M)$ is non-rigid.*
(2) *If M is Seifert, an element of $\pi_1(M)$ is rigid if and only if some non-zero power of it is central.*
(3) *If $\alpha \in H_1(\partial M) = \pi_1(\partial M) \subset \pi_1(M)$ is a slope which is not a boundary slope, then α is totally non-rigid.*

Proof. Part (1) is proved in Proposition 1.1.1 of [CGLS].

(2) Suppose that M is Seifert. Since it is small, its base orbifold is of the form $\mathcal{B} = D^2(p, q)$ for some integers $p, q \geq 2$. If no positive power of $\gamma \in \pi_1(M)$ is central, then γ projects to an element $\bar{\gamma} \in \pi_1(\mathcal{B}) \cong \mathbb{Z}/p * \mathbb{Z}/q$ of reduced length at least 2 with respect to any generators x of $\mathbb{Z}/p$ and y of $\mathbb{Z}/q$. It follows as in the proof of Theorem 1 of [BMS] that $f_{\bar{\gamma}}$ is non-constant on each non-trivial curve of $X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$. Thus γ is non-rigid.

(3) Let X_0 be a non-trivial curve in $X_{PSL_2}(M)$ and suppose that $f_\alpha|X_0$ is constant. Then for any ideal point x of X_0, $f_\alpha(x) \in \mathbb{C}$. But this impossible as otherwise Proposition 2.7 implies that either M is large or α is a boundary slope. Thus $f_\alpha|X_0$ is not constant. $\qquad\square$

We define the *strict degree* of an element γ of the fundamental group of a small knot manifold M to be the sum

$$d_M(\gamma) = \sum_{\substack{components\ X_0\ of \\ X_+^{str}(M)}} d_{X_0}(\gamma).$$

Note that $d_M(\gamma) > 0$ if γ is non-rigid as long as M is not a twisted I-bundle over the Klein bottle. The following lemma is of use in this case.

Lemma 3.5. *Let M, N be small knot manifolds and $\varphi : \pi_1(M) \to \pi_1(N)$ a virtual epimorphism.*
(1) If N is a twisted I-bundle over the Klein bottle and X_0 the principal curve in $X_+^{irr}(N)$, then $\varphi^*(X_0)$ is a non-trivial curve in $X_{PSL_2}(M)$.
(2) If M is a twisted I-bundle over the Klein bottle then so is N. Further, φ is injective.

Proof. (1) Let $\widetilde{N} \xrightarrow{g} N$ be the cover corresponding to $\varphi(\pi_1(M))$. A finite cover of N is either homeomorphic to N or $S^1 \times S^1 \times I$ and so as M has first Betti number 1, $\widetilde{N}$ is also a twisted I-bundle over the Klein bottle. The reader will then verify that $Y_0 = (g_\#)^*(X_0)$ is a principal curve for $\widetilde{N}_i$. But if $\tilde{\varphi} : \pi_1(M) \to \pi_1(\widetilde{N})$ is the surjection induced by φ, $\tilde{\varphi}^*(Y_0) = \varphi^*(X_0)$ is a non-trivial curve in $X_{PSL_2}(M)$.

(2) If M is a twisted I-bundle over the Klein bottle, then $\pi_1(N)$ has a finite index abelian subgroup and therefore is also a twisted I-bundle over the Klein bottle. As in the proof of (1), if $\widetilde{N} \to N$ is the cover corresponding to $\varphi(\pi_1(M))$, then $\widetilde{N}$ is also a twisted I-bundle over the Klein bottle. But the fundamental group of such a manifold is Hopfian so the induced epimorphism $\pi_1(M) \to \pi_1(\widetilde{N})$ is an isomorphism. Thus φ is injective. $\qquad\square$

Proposition 3.6. *Let $\varphi : \pi_1(M) \to \pi_1(N)$ be a virtual epimorphism.*
(1) $d_N(\varphi(\gamma)) \leq d_M(\gamma)$ for all $\gamma \in \pi_1(M)$.
(2) If $\gamma \in \pi_1(M)$ is not rigid and $d_N(\varphi(\gamma)) = d_M(\gamma)$, then φ is injective.

Proof. The first assertion is a consequence of part (1) of Proposition 3.1. To prove the second, note that Proposition 3.1(2) shows that we can suppose there is a principal component X_0 of $X_{PSL_2}(M)$ which is not contained in the Zariski closure of the image of φ^*. Lemma 3.5(2) shows that we can also suppose that M is not a twisted I-bundle over the Klein bottle. Thus as γ is not rigid, $d_{X_0}(\gamma) > 0$ and therefore $d_M(\gamma) \geq d_N(\varphi(\gamma)) + d_{X_0}(\gamma) > d_N(\varphi(\gamma)) = d_M(\gamma)$, which is impossible. $\qquad\square$

Remark 3.7. Note that under the hypotheses of part (2) of the proposition, work of Waldhausen [Wal1] implies that φ is induced by a covering map $M \to N$ as long as it preserves the peripheral subgroups of $\pi_1(M)$ and $\pi_1(N)$. This is automatically satisfied if N is hyperbolic.

Our next result gives an a priori bound on the length of certain sequences of homomorphisms between the fundamental groups of small knot manifolds.

Theorem 3.8. *Let M be a small knot manifold and consider a sequence of homomorphisms*

$$\pi_1(M) \xrightarrow{\varphi_1} \pi_1(N_1) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} \pi_1(N_n)$$

none of which is injective. If N_i is small and φ_i is a virtual epimorphism for each i, then $n \leq d_M(\gamma)$ for each totally non-rigid element $\gamma \in \pi_1(M)$. Moreover, if $n = d_M(\gamma)$ for some such γ, then N_n is a twisted I-bundle over the Klein bottle.

Proof. Set $\psi_i = \varphi_i \circ \cdots \circ \varphi_1$ and let $\gamma \in \pi_1(M)$ be totally non-rigid. If $\psi_i(\gamma)$ is rigid for some $1 \leq i \leq n$ and $X_0 \subset X_+^{irr}(N_i)$ is a principal curve, then $f_{\psi_i(\gamma)}|X_0$ is constant, or equivalently, $f_\gamma|\psi_i^*(X_0)$ is constant (Identity (2.1.1)). Since γ is totally non-rigid, $\psi_i^*(X_0)$ cannot be a non-trivial curve and therefore $X_0 \not\subset X_+^{str}(N_i)$ (Proposition 2.5). Thus N_i is a twisted I-bundle over the Klein bottle. But Lemma 3.5(1) shows that this case does not arise under our assumptions. It follows that $\psi_i(\gamma)$ is non-rigid for $1 \leq i \leq n$. Moreover, Lemma 3.5(2) shows that if N_i is a twisted I-bundle over the Klein bottle for some i, then $i = n$. Proposition 3.6 now implies that

$$d_M(\gamma) > d_{N_1}(\psi_1(\gamma)) > \cdots > d_{N_i}(\psi_i(\gamma)) \cdots > d_{N_n}(\psi_n(\gamma)) \geq 0$$

with $d_{N_n}(\psi_n(\gamma)) = 0$ if and only if N_n is a twisted I-bundle over the Klein bottle. This completes the proof. $\qquad\square$

3.3. **Dominations by two-bridge knot exteriors.** Consider relatively prime integers p, q where $p \geq 1$ is odd and let $k_{p/q}$ denote the two-bridge knot corresponding to the rational number p/q. Thus the 2-fold cover of S^3 branched over $k_{p/q}$ is the lens space $L(p, q)$. It is a theorem of Schubert that $k_{p/q}$ is equivalent to $k_{p'/q'}$ if and only if $L(p, q)$ is homeomorphic to $L(p', q')$. The exterior $M_{p/q}$ of $k_{p/q}$ is known to be small [HT]. Moreover it is hyperbolic unless $q \equiv \pm 1 \pmod{p}$, in which case it is a $(p, 2)$ torus knot.

The proof of the following unpublished result of Tanguay is contained in Appendix A.

Proposition 3.9. ([Tan]) *Let M be the exterior of the two-bridge knot of type p/q. If $\mu \in \pi_1(M)$ is a meridional class, then $d_M(\mu) = \frac{p-1}{2}$.*

As a consequence we deduce:

Corollary 3.10. *Consider a sequence of homomorphisms*

$$\pi_1(M_{p/q}) \xrightarrow{\varphi_1} \pi_1(N_1) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} \pi_1(N_n)$$

none of which is injective. If N_i is small and φ_i is a virtual epimorphism for each i, then $n < \frac{p-1}{2}$.

Proof. The meridional slope μ of a two-bridge knot is not a boundary slope [HT] so Lemma 3.4(3) shows that it is totally non-rigid in $\pi_1(M_{p/q})$. Theorem 3.8 then yields the inequality $n \leq d_{M_{p/q}}(\mu) = \frac{p-1}{2}$ with equality only if N_n is a twisted I-bundle over the Klein bottle. We saw in the proof of Lemma 3.5(1) that if $\widetilde{N}_n \to N_n$ is the cover corresponding to the image of $\varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1$, then $\widetilde{N}_n$ is also a twisted I-bundle over the Klein bottle. But this is impossible since $H_1(M_{p/q})$ is cyclic while $H_1(\widetilde{N}_n)$ is not. Thus $n < \frac{p-1}{2}$. $\qquad\square$

This result can be significantly strengthened if the homomorphisms are induced by non-zero degree maps. This is the goal of the remainder of this section.

Theorem 3.11. *Let N be a knot manifold and $\varphi : \pi_1(M_{p/q}) \to \pi_1(N)$ a homomorphism such that the image $\varphi(\mu)$ of a meridian μ is peripheral.*
(1) If φ is an epimorphism, then N is homeomorphic to the exterior $M_{p'/q'}$ of a 2-bridge knot in S^3. Moreover either $M_{p'/q'} = M_{p/q}$ or $p = kp'$ with $k > 1$.
(2) If $\varphi(\pi_1(M_{p/q}))$ is of finite index d in $\pi_1(N)$, then either $d = 1$ and the conclusions of the first part of this theorem hold, or N is Seifert fibred and φ factors through an epimorphism $\tilde{\varphi} : \pi_1(M_{p/q}) \to \pi_1(M_{p'/1})$ (for some p' dividing p) to which the conclusions of part (1) apply. Further, $\gcd(2p', d) = 1$.

Proof. (1) Since μ normally generates $\pi_1(M_{p/q})$, $\varphi(\mu)$ does the same for $\pi_1(N)$. In particular $\varphi(\mu) \neq 1$ so if μ' is the slope on ∂N corresponding to the projective class of $\varphi_*(\mu) \in H_1(\partial N)$, then the manifold $W = N(\mu')$ obtained by Dehn filling ∂N along the slope μ' is a homotopy 3-sphere.

Let k' be the core of the surgery in $W = N(\mu')$ and let $\widehat{W}_2(k')$ be the 2-fold cover of W branched over k'. There is an induced surjective homomorphism $\mathbb{Z}/p \cong \pi_1(L(p,q)) \to \pi_1(\widehat{W}_2(k'))$ and so the latter group is finite cyclic $\mathbb{Z}/p'$ with p' dividing p. Since $\pi_1(M_{p/q})$ is generated by two elements, the same holds for $\pi_1(N)$, hence k' is a 2-generator knot in the homotopy sphere W. It follows as in [Wed] that k' is prime and thus N cannot contain an essential annulus with slope μ'. Thus the 2-fold branched covering $\widehat{W}_2(k')$ of k' is irreducible and by geometrisation of orbifolds of cyclic type ([BoP], [CHK]) $\widehat{W}_2(k')$ it is itself a lens space and the covering involution conjugates to an orthogonal involution. Therefore $W = N(\varphi(\mu)) \cong S^3$ and k' is a two-bridge knot. In other words, $(N(\varphi(\mu)), k') \cong (S^3, k_{p'/q'})$ for some integers $p' \geq 1, q'$ with p' dividing p and q' coprime with p'. Property P for two-bridge knots [Tak] implies that $\varphi(\mu) = \mu'$ is a meridian of $k_{p'/q'}$.

According to Proposition 3.6 and Proposition 3.9, either φ is an isomorphism or $\frac{p-1}{2} = d_{M_{p/q}}(\mu) > d_{M_{p'/q'}}(\mu') = \frac{p'-1}{2}$. In the first case $k_{p/q} = k_{p'/q'}$ while in the second $p = kp'$ with $k > 1$. This is the conclusion of (1).

(2) Let $\widetilde{N} \to N$ be the cover corresponding to the image of φ and define $\tilde{\varphi} : \pi_1(M_{p/q}) \to \pi_1(\widetilde{N})$ and $\psi : \pi_1(\widetilde{N}) \to \pi_1(N)$ in the obvious way. Part (1) implies that $\widetilde{N}$ is homeomorphic to the exterior $M_{p'/q'}$ of a 2-bridge knot in S^3. Hence by [GW], the cover $\widetilde{N} \to N$ is regular and cyclic. If N is hyperbolic, $\widetilde{N} = N$ since hyperbolic 2-bridge knot exteriors admit no free symmetries by [GLM] (see also [Ha]), and therefore we are in case (1). Otherwise N is Seifert and so $\widetilde{N}$ is the exterior $M_{p'}$ of $k_{p'} = k_{p'/1}$, which is the $(p', 2)$ torus knot. Thus the Seifert structure on $\widetilde{N}$

has base orbifold $D^2(2, p')$. Let $D^2(a, b)$ be the base orbifold of N. There is an induced cover $D^2(2, p') \to D^2(a, b)$ of degree $d' \geq 1$ say. Since p' is odd an elementary calculation based on Euler characteristics shows that $d' = 1$. Thus the cover is a degree d unwinding of a regular fibre of N and so if F is the fibre and $h : F \to F$ the monodromy of the realization of N as a surface bundle over the circle, then h^d is the monodromy of the realization of $\widetilde{N}$ as a surface bundle. The induced homeomorphism on the level of orbifolds is $F/h \to F/h^d$ and so d must be coprime with the order of h. But since $F/h \cong D^2(2, p')$, this order is a multiple of $2p'$.   □

The following two results are immediate consequences of the previous theorem:

Corollary 3.12. *Consider a sequence of non-zero degree maps*

$$M_{p_0/q_0} = N_0 \xrightarrow{f_1} N_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} N_n$$

between knot manifolds, none of which is homotopic to a homeomorphism. If N_{n-1} is hyperbolic, there are coprime pairs p_j, q_j ($1 \leq j \leq n$) such that $N_j = M_{p_j/q_j}$ ($1 \leq j \leq n-1$), N_n is finitely covered by some M_{p_n/q_n}, and $p_{j-1} = k_j p_j$ for some integer $k_j > 1$ ($1 \leq j \leq n$). Hence, $n+1$ is bounded above by the number of distinct multiplicative factors of p.   □

Corollary 3.13. *If p is an odd prime and $M_{p/q}$ is hyperbolic then $M_{p/q}$ is minimal.*   □

If we consider domination via degree-one maps (i.e. 1-domination), we obtain stronger results:

Corollary 3.14. *Let N be a knot manifold and $f : M_{p/q} \to N$ a degree-one map. Then either f is homotopic to a homeomorphism or N is a two-bridge knot exterior $M_{p'/q'}$ where $p = kp', k > 1$, and $\gcd(k, p') = 1$.*

Proof. Since a degree-one map induces an epimorphism on the level of fundamemtal groups, case (1) of the Theorem 3.11 shows that if f is not homotopic to a homeomorphism, then $N = M_{p'/q'}$ where $p = kp', k > 1$. Moreover, f induces a degree-one map $L(p, q) \to L(p', q')$ between the 2-fold branched covers. By Corollary 6 of [RoW], there is an integer c such that $q' \equiv (\frac{p}{p'})c^2 q \pmod{p'}$. In particular this implies that $\gcd(\frac{p}{p'}, p') = 1$.   □

Corollary 3.15. *Consider a sequence of degree-one maps*

$$M_{p/q} \xrightarrow{f_1} N_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} N_n$$

between knot manifolds, none of which is homotopic to a homeomorphism. Then $n+1$ is bounded above by the number of distinct prime factors of p.   □

Corollary 3.16. *If p is a prime power, the two-bridge knot exterior $M_{p/q}$ does not stictly 1-dominate any knot manifold.*   □

4. Sets of discrete $PSL_2(\mathbb{C})$-characters

We investigate sequences of $PSL_2(\mathbb{C})$ characters of representations of the fundamental groups of small knot manifolds whose images are discrete. This leads us in particular to proofs of Theorems 1.3 and 1.5. Our analysis relies fundamentally on the convergence theory of Kleinian groups and hyperbolic 3-manifolds.

4.1. **Convergence of Kleinian groups and hyperbolic 3-manifolds.** A metric space is *proper* if all of its closed and bounded subsets are compact. A sequence of proper pointed metric spaces (X_n, x_n) is said to converge *geometrically* to a metric space (X_0, x_0) if for every $r > 0$, the sequence of compact metric balls $\{B_{X_n}(x_n; r)\}$ converges in the Gromov bilipschitz topology to $B_{X_0}(x_0; r)$. (See chapter 3 of [Gro] and also chapter E [BeP], chapter 7 [MT].)

We recall the thick/thin decomposition of a complete, finite volume hyperbolic 3-manifold V (chapter D of [BeP]): given a positive constant $0 < \mu \leq \mu_0$, where μ_0 is the Margulis constant, V decomposes as $V_{[\mu,\infty)} \cup V_{(0,\mu]}$ such that:

$$V_{[\mu,\infty)} = \{x \in V : \text{inj}(x) \geq \mu\} \text{ is the } \mu\text{-thick part of } V$$

$$V_{(0,\mu]} = \{x \in V : \text{inj}(x) \leq \mu\} \text{ is the } \mu\text{-thin part of } V.$$

For $\mu \leq \mu_0$, each component of μ-thin part of V is either empty, or a geodesic neighborhood of a closed geodesic (a Margulis tube, homeomorphic to $S^1 \times D^2$) or a cusp with torus cross sections (homeomorphic to $T^2 \times [0, \infty)$).

Let $\{V_n\}$ be a sequence of pointed, closed, connected, orientable, hyperbolic 3-manifolds whose volumes are bounded above. There is a sequence of base points $x_n \in (V_n)_{[\mu_0,\infty)}$ such that some subsequence $\{(V_j, x_j)\}$ converges to a pointed, complete, finite volume, hyperbolic 3-manifold (V, x). In particular this implies that given $\varepsilon > 0$ and $0 < \mu \leq \mu_0$, for $j \geq n_0(\varepsilon, \mu)$ the μ-thick parts of V_j and V are $(1 + \varepsilon)$-bilipschitz homeomorphic. Moreover

$$vol(V) = \lim_j vol(V_j),$$

(see chapter E of [BeP], theorem 7.9 of [MT]). Further, if V is closed, then $V = V_j$ for $j \gg 0$ and if V is not closed, V_j is obtained by Dehn filling V for $j \gg 0$ (see chapter 5 of [Thu], chapter E of [BeP]). By a Dehn filling of a complete, non-compact, finite volume hyperbolic 3-manifold V we mean a Dehn filling of some compact core V_0 of V.

In order to simplify the presentation, base points for fundamental groups and pointed metric spaces will often be supressed from the notation. In particular we will say that a sequence $\{V_n\}$ of hyperbolic manifolds converges geometrically to a hyperbolic manifold V if it does so under a suitable choice of base points.

We come now to the algebraic counterpart of this notion of geometric convergence. A good source on this topic is the paper [JM] of Jørgensen and Marden. The torsion-free case is dealt with in chapter 7 of [MT].

The *envelope* of a sequence $\{\Gamma_n\}$ of subgroups of $PSL_2(\mathbb{C})$ is defined as

$$Env(\{\Gamma_n\}) := \{\gamma = \lim_n \gamma_n : \gamma_n \in \Gamma_n \text{ for all } n\} \subset PSL_2(\mathbb{C}).$$

Clearly, $Env(\{\Gamma_n\})$ is a subgroup of $PSL_2(\mathbb{C})$.

A *Kleinian* group is a discrete subgroup of $PSL_2(\mathbb{C})$. A *Fuchsian* group is a discrete subgroup of $PSL_2(\mathbb{R})$.

A sequence $\{\Gamma_n\}$ of subgroups of $PSL_2(\mathbb{C})$ is said to *converge geometrically* to a subgroup Γ_0 of $PSL_2(\mathbb{C})$ if $\Gamma_0 = Env(\{\Gamma_j\})$ for every subsequence $\{j\}$ of $\{n\}$. The sequence $\{\Gamma_n\}$ is

said to *converge algebraically* to Γ_0 if there is a finitely generated group π and representations $\rho_n \in R_{PSL_2}(\pi)$ $(n \geq 0)$ such that $\Gamma_n = \rho_n(\pi)$ and $\lim_n \rho_n = \rho_0$. Note that if $\{\Gamma_n\}$ converges algebraically to Γ_0 and geometrically to Γ, then $\Gamma_0 \subseteq \Gamma \subseteq Env(\{\Gamma_n\})$.

We record the following result for later use. Proofs in the torsion-free case can be found in Theorems 7.6, 7.7, 7.12, 7.13, and 7.14 of [MT]. The general case can be dealt with using the results of [JM].

Proposition 4.1. *Suppose that π is a finitely generated group and $\rho_n : \pi \to PSL_2(\mathbb{C})$ is a sequence which converges to $\rho_0 \in R_{PSL_2}(\pi)$. For $n \geq 0$ set $\Gamma_n = \rho_n(\pi)$ and suppose that for $n \geq 1$, Γ_n is a non-elementary Kleinian group. Then*
(1) Γ_0 is a non-elementary Kleinian group.
(2) for $n \gg 0$ there is a homomorphism $\theta_n : \Gamma_0 \to \Gamma_n$ such that $\rho_n = \theta_n \circ \rho_0$. Further, $\lim \theta_n = 1_{\Gamma_0}$.
(3) there are a non-elementary Kleinian group Γ containing Γ_0 and a subsequence $\{j\}$ of $\{n\}$ such that $\{\Gamma_j\}$ converges geometrically to Γ. Moreover, the homomorphisms θ_j of part (2) extend to homomorphisms $\Gamma \to \Gamma_j$, which we continue to denote θ_j, in such a way that $\lim_j \theta_j = 1_\Gamma$.
(4) the quotient spaces $\mathbb{H}^3/\Gamma_j$ converge geometrically to $\mathbb{H}^3/\Gamma$ $\qquad\qquad\square$

In the remainder of the paper we investigate sets of discrete characters and apply our results to study sequences of non-zero degree maps between closed manifolds. Let M be a knot manifold and X_0 a subvariety of $X_{PSL_2}(M)$. Set

$$D(X_0) = \{\chi_\rho \in X_0 : \rho \text{ is discrete and non-elementary}\}$$

$$D^*(X_0) = \{\chi_\rho \in D(X_0) : \rho \text{ is torsion free}\}$$

$$D_0^*(X_0) = \{\chi_\rho \in D^*(X_0) : \rho \text{ has non-zero volume}\}.$$

Note that the image of any $\rho \in R_{PSL_2}(M)$ whose character is contained in $D_0^*(X_0)$ is the fundamental group of a complete hyperbolic 3-manifold.

The following lemma shows that sets of non-zero volume discrete characters arise naturally in the study of dominations by hyperbolic knot manifolds.

Lemma 4.2. *Let $\chi_\rho \in D_0^*(X_0)$, $\Gamma = \rho(\pi_1(M))$, and $V = \mathbb{H}^3/\Gamma$.*
(1) If $\rho|\pi_1(\partial M)$ is injective, then a compact core V_0 of V is a hyperbolic knot manifold and there is a proper non-zero degree map $f : M \to V_0$ such that $f_\# : \pi_1(M) \to \pi_1(V_0) = \Gamma$ is conjugate to ρ.
(2) If $\rho|\pi_1(\partial M)$ is not injective, V is closed and there is a slope α on ∂M and a non-zero degree map $f : M(\alpha) \to V$ such that the composition $\pi_1(M) \to \pi_1(M(\alpha)) \xrightarrow{f_\#} \pi_1(V) = \Gamma$ is conjugate to ρ.
(3) If $v_0 > 0$ is the minimal volume for complete, connected, orientable, hyperbolic 3-manifolds, then $|vol(\chi_\rho)| \geq v_0$.

Proof. If $\rho|\pi_1(\partial M)$ is injective, there is a compact core V_0 of V and a torus T in ∂V_0 such that $\rho(\pi_1(\partial M)) \subset \pi_1(T)$. Thus there is a proper map $f : (M, \partial M) \to (V_0, T)$ realizing ρ. By the definition of the volume of a representation ([Dun]), $|degree(f)|vol(V) = |vol(\rho)| \neq 0$.

In particular $|\text{degree}(f)| \neq 0$, which implies that $\partial V_0 = T$ and $|vol(\rho)| \geq vol(V) \geq v_0$. On the other hand, if $\text{kernel}(\rho|\pi_1(\partial M)) \neq \{\pm I\}$, ρ factors $\pi_1(M) \to \pi_1(M(\alpha)) \xrightarrow{\bar{\rho}} \pi_1(V) = \Gamma$ for some slope α since the image of ρ is torsion free. There is a map $f : M(\alpha) \to V$ associated to the homomorphism $\bar{\rho}$ and Lemma 2.5.4 of [Dun] implies that $vol(\bar{\rho}) = vol(\rho)$. Then $|\text{degree}(f)|vol(V) = |vol(\bar{\rho})| = |vol(\rho)| \neq 0$ and again we see that $|\text{degree}(f)| \neq 0$ so that V must be closed and $|vol(\rho)| \geq vol(V) \geq v_0$. This completes the proof. $\qquad\square$

Corollary 4.3. *Discrete, non-elementary, torsion free, non-zero volume representations of the fundamental group of a knot exterior M correspond to non-zero degree maps of M, or its Dehn fillings, to a hyperbolic manifold.*

Proof. Let V be a compact, connected, orientable, hyperbolic manifold with holonomy representation $\rho_V : \pi_1(V) \to PSL_2(\mathbb{C})$. If V is a knot manifold, $f : M \to V$ a non-zero degree map, and $\rho = \rho_V \circ f_\#$, then $vol(\rho) = \text{degree}(f)vol(V) \neq 0$. Similarly, if V is closed, $M(\alpha)$ is a Dehn filling of M, $f : M(\alpha) \to V$ a non-zero degree map, and ρ the composition $\pi_1(M) \to \pi_1(M(\alpha)) \xrightarrow{f_\#} \pi_1(V) \xrightarrow{\rho_V} PSL_2(\mathbb{C})$, then $vol(\rho) = vol(\rho_V \circ f_\#) = \text{degree}(f)vol(V) \neq 0$. Each of these non-zero volume representations is discrete and torsion free. The converse is a consequence of the previous lemma. $\qquad\square$

Here is a simple application of the results of this section.

Proposition 4.4. $D(X_0), D^*(X_0),$ *and* $D_0^*(X_0)$ *are closed in* X_0.

Proof. Suppose that $\lim_n \chi_n = \chi_0 \in X_0$ where $\chi_n \in D(X_0)$ for all n. Proposition 1.4.4 of [CS] (or Corollary 2.1 of [CL]) shows that there are a subsequence $\{j\}$ of $\{n\}$ and a convergent sequence of representations $\{\rho_j\} \subset R_{X_0}$ such that $\chi_j = \chi_{\rho_j}$. Set $\rho_0 = \lim_j \rho_j$ and note that $\chi_0 = \chi_{\rho_0}$. Proposition 4.1 implies that $\chi_{\rho_0} \in D(X_0)$. Moreover, if we assume that each $\chi_n \in D^*(X_0)$, then part (2) of Proposition 4.1 implies that $\chi_{\rho_0} \in D^*(X_0)$. Thus $D(X_0)$ and $D^*(X_0)$ are closed in X_0. In particular, if $\chi_n \in D_0^*(X_0)$ for all n, then $\chi_0 \in D^*(X_0)$. From the previous lemma and the continuity of the volume function we have $|vol(\chi_0)| = \lim_n |vol(\chi_n)| \geq v_0 > 0$. Thus $\chi_0 \in D_0^*(X_0)$, which completes the proof. $\qquad\square$

4.2. **Unbounded sequences of discrete** $PSL_2(\mathbb{C})$**-characters.** In this section M will be a small knot manifold and X_0 a non-trivial component of $X_{PSL_2}(M)$. We are interested in the asymptotic behaviour of the sets $D(X_0)$ and $D^*(X_0)$. Consider a sequence $\{\chi_n\} \subset D(X_0)$ which converges to an ideal point x_0 of X_0. Fix $\rho_n \in R_{X_0}$ such that $\chi_n = \chi_{\rho_n}$ and let α_0 be the ∂-slope associated to x_0.

Lemma 4.5. *Let M be a small knot manifold, X_0 a curve component of $X_{PSL_2}(M)$, and $\{\chi_n\} \subset D(X_0)$ a sequence which converges to an ideal point x_0 of X_0. Fix $\rho_n \in R_{X_0}$ such that $\chi_n = \chi_{\rho_n}$ and let α_0 be the ∂-slope associated to x_0.*
(1) *For $n \gg 0$, $\text{kernel}(\rho_n|\pi_1(\partial M)) \cong \mathbb{Z}$ and $\rho_n(\pi_1(\partial M)) \cong \mathbb{Z} \oplus \mathbb{Z}/c_n$ where the $\mathbb{Z}$ factor is generated by a loxodromic element and $c_n \geq 1$.*
(2) *Let $\alpha_n \in H_1(\partial M)$ be the element, unique up to sign, which generates the kernel of $\rho_n|\pi_1(\partial M)$ $(n \gg 0)$. Then $\lim_n[\alpha_n] = [\alpha_0]$.*
(3) *If $[\alpha_n] \neq [\alpha_m]$ for some $m, n \gg 0$, then X_0 is a norm curve.*

Proof. (1) Since M is small, there is a slope β on ∂M such that $\tilde{f}_\beta$ has a pole at x_0. Thus $\rho_n(\beta)$ is loxodromic for large n and so for such n, $\rho_n|\pi_1(\partial M))$ contains no parabolics. On the other hand, a discrete subgroup of $PSL_2(\mathbb{C})$ isomorphic to $\mathbb{Z}^2$ contains parabolic matrices. Thus kernel$(\rho_n|\pi_1(\partial M)) \neq \{\pm I\}$, which implies (1).

(2) Since M is small, Proposition 2.7 shows that there is a unique slope $\alpha_0 \in H_1(\partial M)$ such that $\tilde{f}_{\alpha_0}(x_0) \in \mathbb{C}$. Further, α_0 is a boundary slope, any surface S associated to x_0 has non-empty boundary of slope α_0, and if $\alpha_0^* \in H_1(\partial M)$ is a slope dual to α_0 (i.e. $\alpha_0 \cdot \alpha_0^* = 1$), then $\lim_n f_{\alpha_0^*}(\chi_n) = \infty$. We must show that $\lim_n[\alpha_n] = [\alpha_0]$.

To that end set $\alpha_n = p_n\alpha_0 + q_n\alpha_0^*$. By construction, $(p_n, q_n) \neq (0,0)$ and for n large, $\rho_n(\alpha_0^*)$ is loxodromic. By choice of α_n we have $(\rho_n(\alpha_0))^{p_n} = (\rho_n(\alpha_0^*))^{-q_n}$ and therefore the minimal translation lengths $\ell(\rho_n(\alpha_0))$ and $\ell((\rho_n(\alpha_0^*))$ of $\rho_n(\alpha_0)$ and $\rho_n(\alpha_0^*)$ satisfy:

$$|p_n|\ell(\rho_n(\alpha_0)) = |q_n|\ell((\rho_n(\alpha_0^*)) > 0.$$

If $\pm A \in PSL_2(\mathbb{C})$, then $\ell(\pm A) = |\log(|\frac{\text{trace}(A)}{2} + \sqrt{(\frac{\text{trace}(A)}{2})^2 - 4}|)|$ and so our hypotheses imply that $\lim_n \ell((\rho_n(\alpha_0^*)) = \infty$ while $\lim_n \ell(\rho_n(\alpha_0))$ is bounded. Thus $\lim_n \frac{q_n}{p_n} = 0$ or equivalently, α_n converge projectively to $[\alpha_0]$.

(3) follows from (1) and Corollary 2.9. $\qquad\square$

Proposition 4.6. *Let M be a small knot manifold, X_0 a non-trivial component of $X_{PSL_2}(M)$, and $\{\chi_n\} \subset D(X_0)$ a sequence which converges to an ideal point x_0 of X_0. If S_0 is a component of an essential surface associated to x_0 and $i_\# : \pi_1(S_0) \to \pi_1(M)$ is the inclusion induced homomorphism, then either*
(a) $\overline{i_\#^*(X_0)} \subset X_\mathcal{N}(S_0)$, or
(b) $i_\#^*(X_0) = \{\chi_\rho\}$ where $\rho(\pi_1(S_0))$ is either the tetrahedral group, the octahedral group, or the icosahedral group.

Proof. Fix $\rho_n \in R_{X_0}$ such that $\chi_n = \chi_{\rho_n}$ and let S_0 be a component of an essential surface S in M associated to x_0. Since $\chi_n|\pi_1(S_0)$ converges to a character $\chi_\sigma \in X_{PSL_2}(S_0)$ (Proposition 2.6), we can replace the ρ_n by conjugate representations so that after passing to a subsequence $\{j\}$, we have $\lim \rho_j|\pi_1(S_0) = \sigma$ where $\chi = \chi_\sigma$ (see Proposition 1.4.4 of [CS] or Corollary 2.1 of [CL]). We also know that σ is reducible (Proposition 2.6) and so by taking $j \gg 0$, $\rho_j^{S_0}$ is discrete and elementary (Proposition 4.1). A discrete elementary subgroup of $PSL_2(\mathbb{C})$ is either reducible, conjugates into $\mathcal{N}$, or is isomorphic to a *polyhedral group* (i.e. the tetrahedral group, the octahedral group, or the icosahedral group). Thus $i_\#^*(\chi_j)$ is contained in $X_\mathcal{N}(S_0)$ or ρ_j has polyhedral image. This proves the lemma when $i_\#^*(X_0)$ is a single character. Suppose, on the other hand, that $\overline{i_\#^*(X_0)}$ is a curve $Y_0 \subset X_{PSL_2}(S_0)$. Then $i_\#^* : X_0 \to Y_0$ is finite-to-one and since there only finitely many characters of representations in $R_{PSL_2}(S_0)$ with image a polyhedral group, $Y_0 \cap X_\mathcal{N}(S_0)$ is infinite. But $X_\mathcal{N}(S_0)$ is Zariski closed in $X_{PSL_2}(S_0)$, and so it contains Y_0. $\qquad\square$

Corollary 4.7. *Let M be a small knot manifold, X_0 a non-trivial component of $X_{PSL_2}(M)$. Suppose that for each ideal point x_0 of X_0 there are a component S_0 of an essential surface associated to x_0 and a character $\chi \in X_0$ such that $\chi|\pi_1(S_0)$ is non-elementary. Then $D(X_0), D^*(X_0)$, and $D_0^*(X_0)$ are compact.*

Proof. Proposition 4.6 shows that $D(X_0)$ does not accumulate to an ideal point of X_0. The result then follows from Proposition 4.4. $\qquad\square$

Corollary 4.8. *Let M, N be small hyperbolic knot manifolds and suppose that $\varphi : \pi_1(M) \to \pi_1(N)$ is a virtual epimorphism. Fix a principal component $Y_0 \subset X_{PSL_2}(N)$ and set $X_0 = \varphi^*(Y_0)$. Then $D(X_0), D^*(X_0)$, and $D_0^*(X_0)$ are compact. In particular this is true for a principal component of $X_{PSL_2}(M)$.*

Proof. First suppose that $M = N$ and φ is the identity. By Corollary 4.7 it suffices to show that for each connected essential surface S_0 in M, there is a character $\chi_\rho \in X_0$ such that $\rho(\pi_1(S_0))$ is non-elementary. Fix such a surface and note that $\pi_1(S_0)$ is a non-abelian free group since M is small and hyperbolic. Moreover, since X_0 is principal, it contains the character of a discrete faithful representation ρ_0 of $\pi_1(M)$. Thus $\rho_0(\pi_1(S_0))$ is a discrete and free of rank at least 2 and as such is non-elementary. Thus $D(X_0), D^*(X_0)$, and $D_0^*(X_0)$ are compact.

Now consider the general case and let $\chi_\rho \in D(X_0)$. By Lemma 2.1 there is a $\chi_{\rho'} \in Y_0$ such that $\varphi^*(\chi_{\rho'}) = \chi_\rho$. Since χ_ρ is irreducible we can suppose that $\rho = \rho' \circ \varphi$. Then ρ' is non-elementary and since the image of φ has finite index in $\pi_1(N)$, it is also discrete. In other words, $\chi_{\rho'} \in D(Y_0)$ and so $\chi_\rho = \varphi^*(\chi_{\rho'}) \in \varphi^*(D(Y_0))$. Hence $D(X_0)$ is contained in the compact subset $\varphi^*(D(Y_0))$ of X_0. $\qquad\square$

Example 4.9. Corollary 4.8 implies that the set of discrete, non-elementary characters in the $PSL_2(\mathbb{C})$ character variety of the exterior of either a hyperbolic twist knot or the $(-2, 3, n)$ pretzel knot, $n \not\equiv 0 \pmod 3$ is compact (cf. Example 3.3).

Remark 4.10. The corollary is false if we assume that N is Seifert fibred but not a twisted I-bundle over the Klein bottle. Indeed, suppose that N has base orbifold $D(p, q)$ where $p, q \geq 2, (p, q) \neq (2, 2)$. Each pair $\pm I \neq A_0, B_0 \in \mathcal{D}$ such that $A_0^p = B_0^q = \pm I$ determines a curve $Y_0 \subset X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ consisting of the characters of homomorphisms sending a generator of $\mathbb{Z}/p$ to A_0 and one of $\mathbb{Z}/q$ to a conjugate B of B_0 (see Example 3.2, [BZ1]). Further, if a sequence $\{B_n\}$ of such conjugates is chosen so that $\lim_n |\mathrm{trace}(A_0 B_n)| = \infty$, the associated characters tend to the unique ideal point of Y_0 (Example 3.2, [BZ1]). On the other hand, if $A_0, B_0 \in PSL_2(\mathbb{R})$ are chosen to have extreme negative trace (page 293, [Kn]), they generate a discrete group isomorphic to $\mathbb{Z}/p * \mathbb{Z}/q \cong \pi_1(D(p, q))$ as long as $|\mathrm{trace}(A_0 B)| \geq 2$ (Theorem 2.3 [Kn]). In particular, they determine a principal component $Y_0 \subset X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q) = X_{PSL_2}(\pi_1(D(p, q)) \subset X_{PSL_2}(N)$ for which $D(Y_0)$ is non-compact. By hypothesis, $Y_0 \subset X_+^{str}(N)$, so $X_0 := \varphi^*(Y_0) \subset X_+^{str}(M)$ (Proposition 2.5) and by construction, $D(X_0)$ is non-compact.

Proposition 4.11. *Let M be a small knot manifold, X_0 a norm curve component of $X_{PSL_2}(M)$, and $\{\chi_n\} \subset D^*(X_0)$ a sequence which converges to an ideal point x_0 of X_0. If S is an essential*

surface associated to x_0 and S_0 a component of S, then S_0 is separating and there is a complementary component A of S_0 such that $\rho(\pi_1(A)))$ is abelian for each $\rho \in R_{X_0}$. There is also a subsequence $\{j\}$ of $\{n\}$ such that $\rho_j(\pi_1(A)))$ is cyclic for all j.

Proof. Choose $\rho_n \in R_{X_0}$ such that $\chi_n = \chi_{\rho_n}$. By Lemma 4.5 we may suppose that $\rho_n(\pi_1(\partial M))$ is loxodromic and since ρ_n is torsion free, the lemma implies that there is a unique slope α_n on ∂M satisfying $\rho_n(\alpha_n) = \pm I$. We may suppose that the α_n are distinct, since X_0 is a norm curve, and that none of them are boundary slopes [Hat].

Since M is small, S_0 has non-empty boundary of slope α_0, say. Then by construction, $\rho_n(\alpha_0) \neq \pm I$ is loxodromic for $n \gg 0$. According to Proposition 4.6, $\rho_n(\pi_1(S_0))$ is elementary and since it is discrete, torsion free, and contains a loxodromic ($n \gg 0$), it is a cyclic subgroup of $PSL_2(\mathbb{C})$. In this case we can apply the bending construction to χ_n along $\pi_1(S_0)$ (Appendix C). We claim that for $n \gg 0$, the bending of χ_n along $\pi_1(S_0)$ is trivial. For such n, $\chi_n \in X_0$ is contained in a unique component of $X_{PSL_2}(M)$, and so if the claim is false X_0 is obtained by bending χ_n along $\pi_1(S_0)$. In particular $\rho_m(\alpha_0)$ is independent of m, at least up to conjugation. But then $f_{\alpha_0}|X_0$ is constant and so X_0 cannot be a norm curve, contrary to our hypotheses.

Suppose that S_0 is non-separating and write $M = A/\{S_0^+ = S_0^-\}$ where A is the complementary component of S_0 in M and $S_0^+ \sqcup S_0^- \subseteq \partial A$ are parallel copies of S_0. Note that $\pi_1(M)$ is generated by $\pi_1(A)$ and γ, a homotopy class represented by a loop which intersects S_0 once transversely. Fix $n \gg 0$ and note that since ρ_n cannot be bent non-trivially along $\pi_1(S_0)$, either $\rho_n(\pi_1(M)) \subset \mathcal{N}$ or ρ_n is reducible (Lemma C.3). As neither of these possibilities is satisfied in our situation, S_0 must be separating. Hence if $M = A \cup_{S_0} B$ where A and B are the complementary components of S_0 in M, the fact that for large n the bending of χ_n along $\pi_1(S_0)$ is trivial, at least one of ρ_n^A, ρ_n^B has cyclic image. Further, this image is trivial if the image of $\rho_n^{S_0}$ is trivial (Lemma C.2). By passing to a subsequence and possibly exchanging A and B, we can assume that for $n \gg 0$, ρ_n^A has cyclic image and $\rho_n(\pi_1(A)) = \{\pm I\}$ if $\rho_n(\pi_1(S_0)) = \{\pm I\}$.

Let $\mathcal{O}(\rho_n)$ denote the $PSL_2(\mathbb{C})$ orbit of ρ_n. Since $\cup_{m \geq n}\mathcal{O}(\rho_n)$ is Zariski dense in R_{X_0}, $n \geq 1$, the previous paragraph shows that $\rho(\pi_1(S_0))$ is abelian for each $\rho \in R_{X_0}$. $\qquad\square$

Corollary 4.12. *Let M be a small knot manifold and X_0 a norm curve of $X_{PSL_2}(M)$. Then $D^*(X_0)$ is a compact subset of X_0 as long as the following condition holds: Any ideal point of a norm curve in $X_{PSL_2}(M)$ has an associated essential surface with a component S_0 having no more than two boundary components.*

Proof. By Proposition 4.4, it suffices to show that $D^*(X_0)$ is contained in a compact subset of X_0. Suppose then that $\{\chi_n\} \subset D^*(X_0)$ is a sequence which converges to an ideal point x_0 of X_0 and choose $\rho_n \in R_{X_0}$ whose character is χ_n. Fix a component S_0 of an essential surface S associated to x_0 with $|\partial S_0| \leq 2$. Proposition 4.11 implies that that there is a complementary region A of S_0 such that $\rho(\pi_1(A)))$ is abelian for each $\rho \in R_{X_0}$. Since X_0 is a norm curve, $f_{\alpha_0}|X_0$ is non-constant and so there is a Zariski dense subset in R_{X_0} of representations $\rho \in R_{X_0}$ such that $\rho(\alpha_0)$ is loxodromic. Fix such a representation ρ_0 and conjugate it so that $\rho_0(\alpha_0)$ is diagonal. Then both $\rho_0(\pi_1(A))$ and $\rho_0(\pi_1(\partial M))$ are contained in $\mathcal{D}$.

Let B be the other complementary component of S_0. A maximal compression of ∂B in B must yield a family of 2-spheres as M is small. Thus B is a handlebody and therefore $\pi_1(\partial B) \to \pi_1(B)$ is surjective. Consider a class $\sigma \in \pi_1(M)$ represented by a product of a path in S_0 followed by one in $\partial M \cap B$. By hypothesis $|\partial S_0| = 2$ and so σ is the product of an element of $\pi_1(A)$ and one in $\pi_1(\partial M)$. It follows that $\rho_0(\sigma) \in \mathcal{D}$. Since $\pi_1(B)$ is generated by such classes and $\pi_1(S_0)$, we see that $\rho_n(\pi_1(B)) \subset \mathcal{D}$. But then the image of ρ_0 is abelian, which is impossible as R_{X_0} contains a Zariski dense subset of such representations. Thus $D^*(X_0)$ is contained in a compact subset of X_0. $\qquad\square$

Ohtsuki [Oht] has shown that two-bridge knot exteriors satisfy the condition of the previous corollary.

Corollary 4.13. *Let X_0 be a norm curve in the character variety of a two-bridge knot exterior. Then $D^*(X_0)$ and $D_0^*(X_0)$ are compact subsets of X_0.* $\qquad\square$

4.3. Convergent sequences of discrete, co-compact $PSL_2(\mathbb{C})$-characters with non-zero volume.

Let M be a small hyperbolic knot manifold and X_0 a non-trivial component of $X_{PSL_2}(M)$. Consider a sequence $\{\chi_{\rho_n}\} \subset D_0^*(X_0)$ of distinct characters which converge to some $\chi_{\rho_0} \in X_0$. Fix $\rho_n \in R_{X_0}$ whose character is χ_n, set $\Gamma_n = \rho_n(\pi_1(M))$, and let $V_n = \mathbb{H}^3/\Gamma_n$.

Since M is small, $\|\cdot\|_{X_0} \neq 0$ and so there are only finitely many n such that $\rho_n(\pi_1(\partial M))$ is either $\{\pm I\}$ or contains a paraboic element of $PSL_2(\mathbb{C})$. (Otherwise $\rho_n(\pi_1(\partial M))$ would be contained in a paraboic subgroup for infinitely many n and therefore $f_\gamma|X_0 \equiv 0$ for every peripheral γ). We suppose then that $\rho_n(\pi_1(\partial M))$ contains a loxodromic for each n. Since Γ_n is torsion free, this implies that $\rho_n(\pi_1(\partial M)) \cong \mathbb{Z}$ and so there is a unique slope α_n on ∂M which generates $\mathrm{kernel}(\rho_n|\pi_1(\partial M))$. It follows from Lemma 4.2 that V_n is a closed hyperbolic 3-manifold. If $\bar{\rho}_n \in R_{PSL_2}(M(\alpha_n))$ is the homomorphism induced by ρ_n, the proof of part (3) of this lemma shows that

$$vol(V_n) \leq |vol(\bar{\rho}_n)| \leq |vol(\rho_n)| \leq vol(M(\alpha_n)) < vol(M).$$

Theorem 4.14. *Assume that the sequence $\{\chi_n\} \subset D_0^*(X_0)$, as above, converges to a character χ_{ρ_0} and that the slopes α_n associated to ρ_n are distinct. Then there are*
(a) a subsequence $\{j\}$ of $\{n\}$ such that $\{V_j\}$ converges geometrically to a 1-cusped hyperbolic 3-manifold V whose fundamental group containes $\rho_0(\pi_1(M))$ as a finite index subgroup.
(b) a proper non-zero degree map $f_0 : M \to V_0$ such that V_0 is a compact core of V and if $k_0 : V_0 \to V$ is the inclusion, then $\rho_0 = (k_0)_\# \circ (f_0)_\#$.
(c) slopes β_j on ∂V_0 and identifications $V_j \cong V_0(\beta_j)$, such that $(f_0|\partial M)_*(\alpha_j)$ is a multiple of $\beta_j \in H_1(\partial V)$ and if $k_j : V_0 \to V_0(\beta_j)$ is the inclusion, then χ_j is the character of the composition $(k_j)_\# \circ (f_0)_\#$.
(d) non-zero degree maps $f_j : M(\alpha_n) \to V_0(\beta_j)$ such that the following diagrams are commutative up to homotopy:

$$
\begin{array}{ccc}
M & \xrightarrow{f_0} & V_0 \\
\downarrow & & \downarrow \\
M(\alpha_j) & \xrightarrow{f_j} & V_j \cong V_0(\beta_j)
\end{array}
$$

Moreover, $X_0 = (f_0)^*_\#(Y_0)$ where Y_0 is a principal curve for V_0. In particular, X_0 is a norm curve.

Proof. After replacing the ρ_n by conjugate representations and passing to a subsequence, we may suppose that $\lim \rho_n = \rho_0$ (see Corollary 2.1 of [CL] for example). Then $\{\Gamma_n\}$ converges algebraically to $\Gamma_0 = \rho_0(\pi_1(M))$. By Proposition 4.1, Γ_0 is a non-elementary Kleinian group and there are homomorphisms $\theta_n : \Gamma_0 \to \Gamma_n$ such that $\rho_n = \theta_n \circ \rho_0$ for $n \gg 0$. By passing to a subsequence we may suppose that this is true for $n \geq 1$.

Since $\text{kernel}(\rho_n|\pi_1(\partial M)) = \langle \alpha_n \rangle$, $\rho_n = \theta_n \circ \rho_0$, and the slopes α_n are distinct, it follows that $\rho_0|\pi_1(\partial M)$ is injective. Proposition 4.1 implies that after passing to a subsequence we can suppose that

    (i) $\{\Gamma_n\}$ converges geometrically to a non-elementary Kleinian group Γ containing Γ_0 and θ_n extends to homomorphisms $\Gamma \to \Gamma_n$ which we still denote by θ_n.

    (ii) $\lim V_n = V := \mathbb{H}^3/\Gamma$ in the sense of Gromov bilipschitz topology.

As we noted above, $vol(V_n) < vol(M)$ and therefore $vol(V) = \lim vol(V_n) \leq vol(M)$. It follows that V is a complete, connected, orientable, finite volume hyperbolic 3-manifold. Further, V has at least one cusp since Γ contains $\rho_0(\pi_1(\partial M)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus V_n is obtained from V by hyperbolic Dehn filling for large n (cf. §4.1).

Let $k_0 : V_0 \to V$ be the inclusion of a compact core of V. Since M and V_0 are $K(\pi, 1)$-spaces, there is a map $f_0 : M \to V_0$ such that $\rho_0 = (k_0)_\# \circ (f_0)_\#$. (We have fixed an identification of $\pi_1(V)$ with Γ here.) Since V_0 is atoroidal, there is a torus $T \subseteq \partial V_0$ such that $\rho_0(\pi_1(\partial M)) \subseteq \pi_1(T)$, at least up to conjugation. Homotope f_0 so that $f_0(\partial M) \subseteq T$. Then $(f_0)\# : \pi_1(\partial M) \to \pi_1(T)$ and is injective (by construction), which shows that $\text{degree}(f_0| : \partial M \to T) \neq 0$. On the other hand, if $[\partial M] \in H_2(M)$ and $[T] \in H_2(V_0)$ are fundamental classes for ∂M and T, then $0 = [\partial M]$ so that $0 = (f_0)_*([\partial M]) = \text{degree}(f_0|)[T]$ and therefore $\partial V_0 = T$.

Recall that V_n is obtained by hyperbolic Dehn filling on V. There is some slope β_n on T such that $V_n = V_0(\beta_n)$. If $k_n : V_0 \to V_n$ denotes the inclusion, then $\theta_n \circ (k_0)_\# = (k_n)_\#$ (cf. Theorem 7.17 of [MT]) and so $\text{kernel}(\theta_n) = (k_0)_\#(\langle\langle \beta_n \rangle\rangle_{\pi_1(V_0)})$ where $\langle\langle \beta_n \rangle\rangle_{\pi_1(V_0)}$ is the normal closure in $\pi_1(V_0)$ of the element corresponding to the slope β_n. Thurston's hyperbolic Dehn filing theorem (see chapter 5 of [Thu] or the appendix to [BoP]) implies that $\pi_1(\partial V_0) \cap \text{kernel}(\theta_n) = \langle \beta_n \rangle \cong \mathbb{Z}$ for large n. By passing to a subsequence we can arrange for it to hold for all n.

Since $\rho_n = \theta_n \circ \rho_0 = \theta_n \circ (k_0)_\# \circ (f_0)_\# = (k_n)_\# \circ (f_0)_\#$, we have $1 = \rho_n(\alpha_n) = (k_n)_\#((f_0)_\#(\alpha_n))$ and therefore $(f_0)_\#(\alpha_n) \in \pi_1(\partial V_0) \cap \text{kernel}(\theta_n) = \langle \beta_n \rangle$. Thus f_0 induces a map $f_n : M(\alpha_n) \to V(\beta_n)$ with $\text{degree}(f_n) = \text{degree}(f_0)$. If $i_n : M \to M(\alpha_n)$ denotes the inclusion, we have $(f_n)_\# \circ (i_n)_\# = \theta_n \circ (k_0)_\# \circ (f_0)\# = (k_n)_\# \circ (f_0)\# = \rho_n$. Hence for large n the following diagrams are commutative up to homotopy

$$
\begin{array}{ccc}
M & \xrightarrow{f_0} & V_0 \\
\downarrow & & \downarrow \\
M(\alpha_j) & \xrightarrow{f_j} & V_j \cong V_0(\beta_j)
\end{array}
$$

To complete the proof, we must show that $X_0 = (f_0)^*_\#(Y_0)$ where Y_0 is a principal curve for V_0. To that end we note that Thurston's hyperbolic Dehn filling theorem proves that if Y_0 is the principal component of $X_{PSL_2}(V_0)$ which contains the character χ'_0 of $\pi_1(V_0) = \Gamma$, then $\chi'_0 = \lim_n \chi'_n$ where $\chi'_n \in Y_0$ is the character of our identification $\pi_1(V_0(\beta_n)) = \Gamma_n$. By construction $(f_0)^*_\#(\chi'_n) = \chi_n$ so that $X_0 \cap (f_0)^*_\#(Y_0)$ is infinite. Lemma 2.1 then shows that $X_0 = (f_0)^*_\#(Y_0)$. $\qquad\square$

Corollary 4.15. *Let M be a small hyperbolic knot manifold and X_0 a principal component of $X_{PSL_2}(M)$. Then all but finitely many of the elements of $D^*_0(X_0)$ are induced by the complete hyperbolic structure on the interior of M or by Dehn fillings of manifolds finitely covered by M.*

Proof. We know from Corollary 4.8 that $D^*_0(X_0)$ is contained in a compact subset of X_0, Thus if the result is false, we could find a convergent sequence $\{\chi_n\} \subset D^*_0(X_0)$ of distinct characters no one of which is induced by a holonomy character of M or one of the Dehn fillings of an oriented manifold it finitely covers. As above we can assume that χ_n is the character of a representation ρ_n which is peripherally non-trivial. Let α_n be its slope and note that since $\|\cdot\|$ is a norm curve, the function $n \mapsto \alpha_n$ is finite-to one. Thus we can take a subsequence $\{j\}$ of $\{n\}$ for which the α_j are distinct and apply Theorem 4.14 to see that there are a non-zero degree map $f_0 : M \to N$ where N is hyperbolic, a principal component Y_0 of $X_{PSL_2}(N)$ such that $X_0 = (f_0)^*_\#(Y_0)$, and, for infinitely many j, χ_j is the image under $(f_0)^*_\#$ of the holonomy character of some Dehn filling of N. Lemma 2.3(2) shows that $(f_0)_\#$ is injective and so we can take $f_0 : M \to N$ to be a covering map ([Wal1]). But then infinitely many χ_j are induced from a Dehn filling of an orientable manifold covered by M contrary to our hypotheses. This completes the proof. $\qquad\square$

Corollary 4.16. *Let M be a small knot manifold and suppose that there is a norm curve X_0 in $X_{PSL_2}(M)$ for which $D^*_0(X_0)$ has an accumulation point in X_0. Then*
(1) there are a hyperbolic manifold N, a non-zero degree map $f_0 : M \to N$, and a principal component Z_0 of $X_{PSL_2}(N)$ such that $X_0 = (f_0)^*_\#(Z_0)$;
(2) all but finitely many characters in $D^*_0(X_0)$ are the images under $(f_0)^*_\#$ of the holonomy character of N or one of the Dehn fillings of an oriented manifold finitely covered by N;
(3) $D^*_0(X_0)$ is compact and has a unique accumulation point corresponding to the holonomy character of N under $(f_0)^*_\#$.

Proof. The hypotheses can be used with Theorem 4.14 to see that there are a hyperbolic manifold N_0, a non-zero degree map $f_0 : M \to N_0$, and a principal component Y_0 of $X_{PSL_2}(N_0)$ such that $X_0 = f^*_\#(Y_0)$. Let $N \to N_0$ be the cover corresponding to the image of $(f_0)_\#$, $\tilde{f}_0 : M \to N$ a lift of f_0, and Z_0 the principal curve in $X_{PSL_2}(N)$ obtained by restriction from Y_0. Clearly $X_0 = (\tilde{f}_0)^*_\#(Z_0)$ and final claim of the corollary is a consequence of the previous result applied to Z_0. The corollary follow from these observations. $\qquad\square$

The last two results of this section follow from Proposition 4.11 and Corollaries 4.12, 4.13, and 4.16.

Corollary 4.17. *Let M be a small knot manifold and $X_0 \subset X_{PSL_2}(M)$ a norm curve. Then $D^*_0(X_0)$ is compact in X_0 with at most one accumulation point if for each ideal point x_0 of X_0*

there is a component S_0 of an essential surface associated to x_0 such that at least one of the following two conditions holds:

(i) $\chi|\pi_1(S_0)$ is non-elementary for some $\chi \in X_0$; or

(ii) $|\partial S_0| \leq 2$. □

Proposition 4.18. *Let X_0 be a norm curve in the character variety of a two-bridge knot exterior M. Then $D_0^*(X_0)$ is either finite or is a compact subset of X_0 with a unique accumulation point. In the latter case there are a two-bridge knot exterior N, a non-zero degree map $f : M \to N$, and a principal component Y_0 of $X_{PSL_2}(N)$ such that $X_0 = f_\#^*(Y_0)$.*

Proof. The proposition follows from the results cited above and Theorem 3.11 once we note that M must be hyperbolic if $X_{PSL_2}(M)$ is to contain a norm curve. □

4.4. **Domination and hyperbolic Dehn filling.** Let M be a small knot manifold and $\{\alpha_n\}_{n \geq 1}$ a sequence of distinct slopes on ∂M such that for each n we have a map $f_n : M(\alpha_n) \to V_n$ of degree $d_n \geq 1$ where V_n is a family of mutually distinct hyperbolic manifolds. We suppose as well that $\{\alpha_n\}$ does not subconverge projectively to a boundary slope.

Proof of Theorem 1.5. Let $p_n : \widetilde{V}_n \to V_n$ be the finite cover corresponding to $(f_n)_\#(\pi_1(M))$. We can suppose that p_n is a local isometry. Fix a lift $\tilde{f}_n : M \to \widetilde{V}_n$ of f_n of degree $\tilde{d}_n \geq 1$ say. If $v_0 > 0$ is the minimal volume for closed, connected, orientable, hyperbolic 3-manifolds, then for each n we have $vol(M) > vol(M(\alpha_n)) \geq \tilde{d}_n vol(\widetilde{V}_n) = d_n vol(V_n) \geq d_n v_0 \geq \tilde{d}_n v_0$. Thus the d_n and $\tilde{d}_n$ are bounded so we can assume, after passing to a subsequence, that they are constant, say $\text{degree}(f_n) = d \geq 1$, $\text{degree}(\tilde{f}_n) = \tilde{d}$. The degree of each p_n is $d/\tilde{d}$.

We identify $\pi_1(V_n)$ with a subgroup Γ_n of $PSL_2(\mathbb{C})$ and set $(p_n)_\#(\pi_1(\widetilde{V}_n)) = \widetilde{\Gamma}_n \subseteq \Gamma_n$. Let $i_n : M \to M(\alpha_n)$ be the inclusion and define $\rho_n \in R_{PSL_2}(M)$ to be the composition $\pi_1(M) \xrightarrow{(i_n)_\#} \pi_1(M(\alpha_n)) \xrightarrow{(f_n)_\#} \widetilde{\Gamma}_n \subseteq \Gamma_n \subset PSL_2(\mathbb{C})$. The character of ρ_n will be denoted by χ_n. These objects combine in the following commutative diagram.

$$
\begin{array}{ccc}
& & \pi_1(\widetilde{V}_n) \\
& \overset{(\tilde{f}_n)_\#}{\nearrow} & \Big\downarrow {\scriptstyle (p_n)_\#} \\
\pi_1(M(\alpha_n)) & \xrightarrow{\ (f_n)_\#\ } & \Gamma_n = \pi_1(V_n) \\
{\scriptstyle (i_n)_\#}\Big\uparrow & & \Big\downarrow \\
\pi_1(M) & \xrightarrow{\ \ \rho_n\ \ } & PSL_2(\mathbb{C})
\end{array}
$$

Since the V_n are distinct, the characters χ_n are distinct. After passing to a subsequence we can suppose that they are contained in a non-trivial curve $X_0 \subset X_{PSL_2}(M)$. Proposition 2.9 shows that X_0 is a norm curve. Finally, noting that $vol(\chi_n) = d \, vol(V_n) \neq 0$ we see that $\chi_n \in D_0^*(X_0)$. Since the slopes $\{\alpha_n\}_{n \geq 1}$ do not projectivly subconverge to a ∂-slope, Lemma 4.5 shows that

there is a subsequence of characters $\{\chi_k\}$ which converge to a character $\chi_{\rho_0} \in D_0^*(X_0)$, and so the conditions of Theorem 4.14 are satisfied.

By Theorem 4.14, the sequence $\widetilde{V}_k$ converges geometrically to a 1-cusped hyperbolic 3-manifold $\widetilde{V}$ for which there are:

(a) a proper non-zero degree map $\tilde{f}_0 : M \to \widetilde{V}_0$ such that $\widetilde{V}_0$ is a compact core of $\widetilde{V}$ and if $j_0 : \widetilde{V}_0 \to \widetilde{V}$ is the inclusion map, then $\rho_0 = (j_0)_\# \circ (\tilde{f})_\#$;

(b) slopes $\tilde{\beta}_k$ on $\partial\widetilde{V}_0$ and identifications $\widetilde{V}_k = \widetilde{V}_0(\tilde{\beta}_k)$, such that $(\tilde{f}_0|\partial M)_*(\alpha_k)$ is a multiple of $\tilde{\beta}_k \in H_1(\partial\widetilde{V}_0)$ and if $\tilde{j}_k : \widetilde{V}_0 \to \widetilde{V}_0(\tilde{\beta}_k)$ is the inclusion, then χ_k is induced by the composition $(p_k)_\# \circ (\tilde{j}_k)_\# \circ (\tilde{f}_0)_\#$;

(c) non-zero degree maps $\tilde{f}'_k : M(\alpha_k) \to \widetilde{V}(\tilde{\beta}_k)$ such that the following diagrams are commutative up to homotopy:

$$
\begin{array}{ccc}
M & \xrightarrow{\ \tilde{f}_0\ } & \widetilde{V}_0 \\
\downarrow & & \downarrow \\
M(\alpha_k) & \xrightarrow{\ \tilde{f}'_k\ } & \widetilde{V}_k \cong \widetilde{V}_0(\tilde{\beta}_k)
\end{array}
\tag{4.2.1}
$$

Since $(p_k)_\# \circ (\tilde{f}_k)_\# \circ (i_k)_\# = \rho_k = (p_k)_\# \circ (\tilde{j}_k)_\# \circ (\tilde{f}_0)_\# = (p_k)_\# \circ (\tilde{f}'_k)_\# \circ (i_k)_\#$, it follows that $f_k = p_k \circ \tilde{f}_k$ is homotopic to $f'_k = p_k \circ \tilde{f}'_k$. In particular $\mathrm{degree}(\tilde{f}_0) = \mathrm{degree}(\tilde{f}'_k) = \mathrm{degree}(\tilde{f}_k) = \tilde{d}$ and $\mathrm{degree}(f'_k) = \mathrm{degree}(f_k) = d$.

Now $\lim_k vol(V_k) = \lim_k (\frac{\tilde{d}}{d})vol(\widetilde{V}_k) = (\frac{\tilde{d}}{d})vol(\widetilde{V})$, and so after passing to a subsequence we may assume that $\{V_k\}$ converges geometrically to a complete hyperbolic 3-manifold V with finite volume $vol(V) = (\frac{\tilde{d}}{d})vol(\widetilde{V})$. For $k \gg 0$, $vol(\widetilde{V}) > vol(\widetilde{V}_k)$ and therefore $vol(V) > vol(V_k)$. Thus V has at least one cusp. On the other hand, p_k is a local isometry so for $\tilde{x} \in \widetilde{V}_k$ we have $\mathrm{inj}(\tilde{x}) \leq (\frac{d}{\tilde{d}})\mathrm{inj}(p_k(\tilde{x}))$. Thus if μ_0 is the Margulis constant and $\mu \leq \frac{\tilde{d}\mu_0}{d}$, we have $p_k^{-1}((V_k)_{(0,\mu]}) \subseteq (\widetilde{V}_k)_{(0,\frac{d\mu}{\tilde{d}}]}$ $(k \gg 0)$. Since there is a sequence $\mu_k \to 0$ such that $(\widetilde{V}_k)_{(0,\frac{d\mu_k}{\tilde{d}}]}$ is a Margulis tube about a geodesic $\tilde{\gamma}_k$, $(V_k)_{(0,\mu_k]}$ is a Margulis tube about a geodesic γ_k, because a geodesic is unique in its homotopy class. Thus V has only one cusp. We note, moreover, that $p_k^{-1}(\gamma_k) = \tilde{\gamma}_k$ and therefore $(\widetilde{V}_k)_{(0,\frac{d\mu}{\tilde{d}}]} \setminus p_k^{-1}((V_k)_{(0,\mu]}) \cong \partial(\widetilde{V}_k)_{(0,\frac{d\mu}{\tilde{d}}]} \times I$. Thus for large k we can identify $(V_k)_{[\mu,\infty)}$ with a compact core V_0 of V and $p_k^{-1}((V_k)_{[\mu,\infty)})$ with a compact core $\widetilde{V}_0$ of $\widetilde{V}$. In this way p_k induces a covering map $p_k^0 : \widetilde{V}_0 \to V_0$ of degree $d/\tilde{d}$. Since V_0 and $\widetilde{V}_0$ admit complete finite volume hyperbolic structures on their interiors, after pre-composition by an isotopy of $\widetilde{V}_0$, we can take p_n^0 to be a local isometry on the interior of $\widetilde{V}_0$. Now V_0 has only finitely many (pointed) covers of degree $d/\tilde{d}$ up to equivalence and the isometry group of $\mathrm{int}(\widetilde{V}_0)$ is finite, therefore we can restrict to a subsequence and suppose that for all n, m, we have $p_n^0 = p_m^0 = p$, say.

The geometric convergence of V_k to V implies that for large k there are slopes β_k on ∂V_0 such that $V_k = V_0(\beta_k)$. From the previous paragraph we see that any component of $p^{-1}(\beta_k)$ is isotopic to $\tilde{\beta}_k$ on $\partial\widetilde{V}_0$. Therefore the following diagrams are commutative up to homotopy:

$$\begin{array}{ccc} \widetilde{V}_0 & \xrightarrow{p} & V_0 \\ \downarrow & & \downarrow \\ \widetilde{V}_k \cong \widetilde{V}_0(\widetilde{\beta}_k) & \xrightarrow{p_k} & V_k \cong V_0(\beta_k) \end{array} \qquad (4.2.2)$$

Since $f_k = p_k \circ \tilde{f}_k$ is homotopic to $f'_k = p_k \circ \tilde{f}'_k$, by putting together diagrams (4.2.1) and (4.2.2) one deduces that the proper map $f = p \circ \tilde{f}_0 : M \to V_0$ of degree $d \geq 1$ makes the following diagrams commute up to homotopy:

$$\begin{array}{ccc} M & \xrightarrow{f} & V_0 \\ \downarrow & & \downarrow \\ M(\alpha_k) & \xrightarrow{f_k} & V_k \cong V_0(\beta_k) \end{array}$$

If we assume further that the dominations $f_k : M(\alpha_k) \to V_k \cong V(\beta_k)$ are strict, then the domination $f_0 : M \to V$ must also be strict. Otherwise V_0 is homeomorphic to M and $d = \mathrm{degree}(f_0) = 1$ (since $vol(M) = vol(V_0)$). Then $(f_0)_\# : \pi_1(M) \to \pi_1(V_0)$ is surjective and therefore an isomorphism since $\pi_1(M)$ is Hopfian. By Mostow rigidity theorem we can suppose that f_0 is a homeomorphism. But then the induced maps $f'_k : M(\alpha_k) \to V(\beta_k)$ are homeomorphisms homotopic to f_k, in contradiction with our assumption that f_k is a strict domination. $\qquad \square$

5. $\mathcal{H}$-MINIMAL DEHN FILLING

The goal of this section is to construct collections of infinitely many $\mathcal{H}$-minimal closed hyperbolic 3-manifolds by proving Corollaries 1.7 and 1.11. The proofs of these results rely on the following theorem:

Theorem 5.1. *Let M be an $\mathcal{H}$-minimal, small hyperbolic knot manifold and suppose that there is a slope α_0 on ∂M such that $M(\alpha_0)$ is $\mathcal{H}$-minimal. If $M(\alpha_0)$ is hyperbolic, assume that the core of the α_0 filling solid torus is not null-homotopic in $M(\alpha_0)$.*

(1) If $U \subset \mathbb{P}(H_1(\partial M; \mathbb{R}))$ is the union of disjoint closed arc neighbourhoods of the finite set of boundary slopes of M, then $\mathbb{P}(H_1(\partial M; \mathbb{R})) \setminus U$ contains only finitely many projective classes of slopes α such that $M(\alpha)$ is not $\mathcal{H}$-minimal. In particular, M admits infinitely many $\mathcal{H}$-minimal Dehn fillings.

(2) If $D_0^*(X_0)$ is a compact subset of X_0 for each norm curve in $X_{PSL_2}(M)$, then $M(\alpha)$ is not $\mathcal{H}$-minimal for almost all slopes α on ∂M. In particular, this conclusion holds if for each ideal point x_0 of a norm curve X_0, there is a component S_0 of an essential surface associated to x_0 such that at least one of the following two conditions holds:

(i) $\chi|\pi_1(S_0)$ is non-elementary for some $\chi \in X_0$; or
(ii) $|\partial S_0| \leq 2$.

Proof. (1) Suppose that there are infinitely many projective classes of slopes α in $\mathbb{P}(H_1(\partial M; \mathbb{R})) \setminus U$ such that $M(\alpha)$ is not $\mathcal{H}$-minimal. Then there are an infinite sequence of distinct slopes α_n on ∂M which does not subconverge to a ∂-slope and strict finite dominations $f_n : M(\alpha_n) \to V_n$,

where V_n are closed hyperbolic 3-manifolds. We claim that up to taking a subsequence, we can suppose that the χ_n are pairwise distinct. To prove this, first note that almost all ρ_n are peripherally non-trivial. Suppose that $\rho_n(\pi_1(\partial M)) \subset \{\pm I\}$. Then the composition $M \to M(\alpha_n) \xrightarrow{f_n} V_n$ extends to a map non-zero degree $f_n^0 : M(\alpha_0) \to V_n$. By $\mathcal{H}$-minimality, $M(\alpha_0)$ is hyperbolic and f_n^0 is homotopic to a homeomorphism. Now ρ_n induces an epimorphism $\pi_1(M)/\langle\langle\pi_1(\partial M)\rangle\rangle \to \pi_1(V_n) \cong \pi_1(M(\alpha_0))$. Since the natural homomorphism $\varphi : \pi_1(M(\alpha_0)) \to \pi_1(M)/\langle\langle\pi_1(\partial M)\rangle\rangle$ is surjective and $\pi_1(M(\alpha_0))$ is hopfian, then φ is an isomorphism. But this is true if and only if the core of the α_0 filling solid torus is null-homotopic in $M(\alpha_0)$. As we have explicitly excluded this possibility, ρ_n is peripherally non-trivial. Therefore as $\alpha_n \neq \alpha_m$ for $n \neq m$, the characters χ_n are distinct. As in the proof of Theorem 1.5 we can apply Theorem 4.14 to produce a strict domination $M > V_0$ for some hyperbolic knot manifold V_0, which contradicts our hypotheses.

(2) The first assertion follows from the argument in the proof of part (1) while the second follows from Corollary 4.17. $\qquad\square$

Proofs of Corollaries 1.7 and 1.11. Corollary 1.7 is the first assertion of Theorem 5.1 while Corollary 1.11 follows from second assertion and Corollary 4.13. $\qquad\square$

6. Sets of discrete $PSL_2(\mathbb{R})$-characters

6.1. **Discrete $PSL_2(\mathbb{R})$-representations of the fundamental groups of small knot manifolds.** In this section we specialize our study to sets of discrete $PSL_2(\mathbb{R})$-characters and apply our conclusions to obtain results on $\widetilde{SL_2}$-minimality. This will lead us, for instance, to a proof of Corollary 1.14 and thus the construction of infinitely many closed minimal manifolds.

Let M be a small knot manifold and set

$$D(M;\mathbb{R}) = \{\chi_\rho \in X_{PSL_2}(M) : \rho \text{ is a discrete, non-elementary } PSL_2(\mathbb{R}) \text{ representation}\}.$$

If X_0 is a component of $X_{PSL_2}(M)$ let

$$D(X_0;\mathbb{R}) = \{\chi_\rho \in X_0 : \rho \text{ is a discrete, non-elementary } PSL_2(\mathbb{R}) \text{ representation}\}.$$

Thus $D(X_0;\mathbb{R}) = D(X_0) \cap X_{PSL_2(\mathbb{R})}(M)$ and so is closed in X_0 (cf. Proposition 4.4).

Fix $\rho \in R_{PSL_2}(M)$ such that $\chi_\rho \in D(M;\mathbb{R})$, set $\Delta = \rho(\pi_1(M))$, and let $\mathcal{B} = \mathbb{H}^2/\Delta$. The underlying surface $|\mathcal{B}|$ of $\mathcal{B}$ is orientable and therefore has only cone singularities.

Lemma 6.1.
(1) Δ is either a hyperbolic triangle group or a free product of two finite cyclic groups.
(2) $\rho(\pi_1(\partial M)) \cong \mathbb{Z}/c$ for some $c \geq 0$ and if $c > 0$, Δ is a hyperbolic triangle group.
(3) Suppose that $\chi_{\rho_n} \in D(X_0;\mathbb{R}), (n \geq 1)$ are distinct and $\rho_n(\pi_1(\partial M)) \cong \mathbb{Z}/c_n$ with $c_n \geq 1$. Then $\lim_n c_n = \infty$.

Proof. (1) If $|\mathcal{B}|$ is non-compact, then $\Delta = \pi_1(\mathcal{B}) \cong \pi_1(|\mathcal{B}|) * \mathbb{Z}/a_1 * \ldots * \mathbb{Z}/a_k$ where $a_1, a_2, \ldots, a_k \geq 2$ are the orders of the cone points. On the other hand, we can identify $X_{PSL_2}(\Delta)$ with a closed algebraic subset of $X_{PSL_2}(M)$. Since the latter has complex dimension 1, either $\pi_1(|\mathcal{B}|) \cong \{1\}$ and $k \leq 2$ or $\pi_1(|\mathcal{B}|) \cong \mathbb{Z}$ and $k = 0$. The latter is impossible since

it implies that $\Delta \cong \mathbb{Z}$. Thus Δ is a free product of two finite cyclic groups. In this case, if $\alpha \in \mathrm{kernel}(\rho|\pi_1(\partial M))$, $X_{PSL_2}(M(\alpha))$ has dimension 1 and since M is small, α is a boundary slope.

Next suppose that $|\mathcal{B}|$ is closed. The relation which associates a holonomy representation to a hyperbolic structure determines an embedding of the Teichmüller space $\mathcal{T}(\mathcal{B})$ of $\mathcal{B}$ in $X_{PSL_2(\mathbb{R})}(\pi_1(\mathcal{B})) \subset X_{PSL_2}(\pi_1(\mathcal{B})) \subset X_{PSL_2}(M)$. Thus $\mathcal{T}(\mathcal{B})$ has real dimension at most 1. But this dimension is given by $-3\chi(|\mathcal{B}|) + 2k$ where k is the number of cone points in $\mathcal{B}$ (Corollary 13.3.7 [Thu]). Since $|\mathcal{B}|$ is orientable, the only possibility is for it to be of the form $S^2(a, b, c)$ so that Δ is a hyperbolic triangle group.

(2) The first assertion of (2) follows from the elementary observation that an abelian subgroup of $PSL_2(\mathbb{R})$ is cyclic. For the second, suppose that $c > 0$ and note that there are infinitely many slopes in $\mathrm{kernel}(\rho|\pi_1(\partial M))$. Fix one such slope α and suppose that Δ is a free product of two finite cyclic groups. There is a principal curve $Y_0 \subset X_{PSL_2}(\Delta) \subset X_{PSL_2}(M(\alpha))$ and so $M(\alpha)$ admits a closed essential surface. Since M is small, α is a boundary slope, and as there are only finitely many such slopes [Hat], we obtain a contradiction. Thus Δ is a hyperbolic triangle group.

(3) Otherwise there is a subsequence $\{j\}$ and $c \geq 1$ such that $c_n = c$ for all j. Then for any peripheral class γ there are only finitely many possibilities for $f_\gamma(\chi_j)$. Since the χ_j are distinct this implies that each f_γ is constant, which contradicts the smallness of M. $\qquad\square$

Lemma 6.2. *If the image of $\rho \in R_{PSL_2}(M)$ is a discrete hyperbolic triangle group, then χ_ρ is an isolated point of $D(M; \mathbb{R})$.*

Proof. Suppose that there is a sequence $\{\chi_{\rho_n}\}$ in $D(M; \mathbb{R}) \setminus \{\chi_\rho\}$ which converges to χ_ρ. By passing to a subsequence and replacing the ρ_n by conjugate representations we may suppose that $\lim_n \rho_n = \rho$ (Lemma 2.1 [CL]) and find homomorphisms $\theta_n : \rho(\pi_1(M)) \to PSL_2(\mathbb{C})$ such that $\rho_n = \theta_n \circ \rho$ (Proposition 4.1). We claim that the $PSL_2(\mathbb{C})$ character varieties of triangle groups are finite. Assuming this for the moment, by again passing to a subsequence we may find $A_n \in PSL_2(\mathbb{C})$ such that $\theta_n = A_n \theta_1 A_n^{-1}$. Then $\rho_n = \theta_n \circ \rho = A_n(\theta_1 \circ \rho)A_n^{-1}$. Hence $\chi_\rho = \lim_n \chi_{\rho_n} = \lim_n \chi_{\rho_1} = \chi_{\rho_1} \in D(M; \mathbb{R}) \setminus \{\chi_\rho\}$, which is impossible. Thus χ_ρ is an isolated point of $D(M; \mathbb{R})$.

To see that the character variety of the (p, q, r)-triangle group $\Delta(p, q, r) = \langle x, y : x^p = y^q = (xy)^r = 1 \rangle$ is finite, note that there is a natural embedding $X_{PSL_2}(\Delta(p, q, r)) \subset X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$. Indeed, $X_{PSL_2}(\Delta(p, q, r))$ is contained in the set of points where the regular function $f : X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q) \to \mathbb{C}, \chi_\rho \mapsto \mathrm{trace}(\rho(xy))^2$ takes on the value $4\cos^2(\frac{\pi j}{r})$ for some integer j. Now $X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ consists of a finite union of curves and isolated points (Example 3.2 [BZ1]) and it is simple to see from the parameterizations given in that example that the restriction of f to any of the curves is non-constant. Thus it takes on the value $4\cos^2(\frac{\pi j}{r})$ at only finitely many points and therefore $X_{PSL_2}(\Delta(p, q, r))$ is finite. $\qquad\square$

Lemma 6.3. *Suppose that the image Δ of $\rho \in R_{PSL_2}(M)$ is isomorphic to $\mathbb{Z}/p * \mathbb{Z}/q$. Then χ_ρ is an accumulation point of $D(X_0; \mathbb{R})$ where $X_0 = \rho^*(Y_0)$ for some principal component Y_0 of $X_{PSL_2}(\Delta)$. Further, $D(X_0; \mathbb{R})$ is non-compact in X_0 and there is a compact subset $K \subset X_0$*

such that

(a) $int(K)$ *contains all characters in* $D(X_0; \mathbb{R})$ *of representations whose images are hyperbolic triangle groups; and*

(b) $\overline{(X_0 \setminus K)} \cap D(X_0; \mathbb{R})$ *contains all characters in* $D(X_0; \mathbb{R})$ *of representations whose images are* $\mathbb{Z}/p * \mathbb{Z}/q$.

Proof. The inclusion $\Delta \to PSL_2(\mathbb{C})$ is contained in a unique curve $Y_0 \subset X_{PSL_2}(\Delta)$ (cf. Example 3.2 [BZ1]). Set $X_0 = \rho^*(Y_0)$. The remaining assertions of the lemma are a consequence of the discussion in Remark 4.10 and Theorem 2.3 of [Kn]. $\qquad\square$

Lemma 6.4. *Let* $\{\chi_n\} \subset D(X_0; \mathbb{R})$ *be a sequence of distinct characters of representations* ρ_n *with image a free product of two finite cyclic groups. Then there are an epimorphism* $\pi_1(M) \to \mathbb{Z}/p * \mathbb{Z}/q$ $(2 \leq p, q)$ *and a principal curve* $Y_0 \subset X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ *which maps bijectively to* X_0 *under the inclusion* $X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q) \subset X_{PSL_2}(M)$. *In particular,* X_0 *is an* α_0-*curve for some slope* α_0 *on* ∂M *and* $\rho_n(\pi_1(M)) \cong \mathbb{Z}/p * \mathbb{Z}/q$ *for all* n.

Proof. Choose $n \gg 0$ such that χ_n is a simple point of $X_{PSL_2}(M)$. By hypothesis, the image Δ of ρ_n is isomorphic to $\mathbb{Z}/p * \mathbb{Z}/q$ for some $2 \leq p, q$. There is a principal curve $Y_0 \subset X_{PSL_2}(\Delta)$ containing the inclusion $\Delta \to PSL_2(\mathbb{C})$ and since χ_n is a simple point, its image in $X_{PSL_2}(M)$ is X_0. Lemma 6.1(2) implies that X_0 is an α_0-curve for some slope α_0. Finally, for each n, there is an epimorphism $\mathbb{Z}/p * \mathbb{Z}/q \cong \Delta \to \rho_n(\pi_1(M)) \cong \mathbb{Z}/r * \mathbb{Z}/s$ for some $r, s \geq 2$. It follows from Example 3.2 of [BZ1] that the induced homomorphisms $\mathbb{Z}/p, \mathbb{Z}/q \to \mathbb{Z}/r * \mathbb{Z}/s$ are injective. Further, since these images conjugate into one of $\mathbb{Z}/r, \mathbb{Z}/s$ and generate $\mathbb{Z}/r * \mathbb{Z}/s$, we must have $\mathbb{Z}/r * \mathbb{Z}/s \cong \mathbb{Z}/p * \mathbb{Z}/q$. $\qquad\square$

6.2. **Unbounded sequences of discrete $PSL_2(\mathbb{R})$-characters.** Let M be a small knot manifold and X_0 a component of $X_{PSL_2}(M)$. Consider a sequence $\{\chi_n\}$ in $D(X_0; \mathbb{R})$ which converges to an ideal point x_0 of X_0. The following lemma is a consequence of Lemmas 4.5(1) and 6.1.

Lemma 6.5. *Let* M *be a small knot manifold,* X_0 *a curve component of* $X_{PSL_2}(M)$, *and* $\{\chi_n\} \subset D(X_0; \mathbb{R})$ *a sequence which converges to an ideal point* x_0 *of* X_0. *Fix* $\rho_n \in R_{X_0}$ *such that* $\chi_n = \chi_{\rho_n}$ *and let* α_0 *be the* ∂-*slope associated to* x_0. *For* $n \gg 0$, *kernel*$(\rho_n|\pi_1(\partial M)) \cong \mathbb{Z}$ *and* $\rho_n(\pi_1(\partial M)) \cong \mathbb{Z}$ *where the* $\mathbb{Z}$ *factor is generated by a loxodromic.* $\qquad\square$

There is no subgroup of $PSL_2(\mathbb{R})$ isomorphic to the tetrahedral, octahedral, or icosahedral group. Thus the next result follows directly from Proposition 4.6.

Proposition 6.6. *Let* M *be a small knot manifold,* X_0 *a component of* $X_{PSL_2}(M)$, *and* $\{\chi_n\} \subset D(X_0; \mathbb{R})$ *a sequence which converges to an ideal point* x_0 *of* X. *If* S_0 *is a component of an essential surface associated to* x_0, *then for* $n \gg 0$, *the image of* X_0 *in* $X_{PSL_2}(S_0)$ *is contained in* $X_{\mathcal{N}}(S_0)$. $\qquad\square$

6.3. **Convergent sequences of discrete $PSL_2(\mathbb{R})$-characters.** Let M be a small knot manifold and X_0 a non-trivial component of $X_{PSL_2}(M)$. We are interested in the accumulation points of $D(X_0; \mathbb{R})$ in X_0.

Proposition 6.7. *Let M be a small knot manifold, X_0 a non-trivial component of $X_{PSL_2}(M)$, and $\{\chi_n\} \subset D(X_0; \mathbb{R})$ a sequence of distinct characters which converges to some $\chi_{\rho_0} \in X_0$. Then*

(1) $\rho_0(\pi_1(M)) = \Delta_0$ is discrete, non-elementary, and isomorphic to $\mathbb{Z}/p * \mathbb{Z}/q$ for some integers $2 \le p, q$;

(2) there is a principal component $Y_0 \subset X_{PSL_2}(\Delta_0)$ such that $X_0 = \rho_0^*(Y_0)$;

(3) there is a unique slope α_0 on ∂M such that $\rho_0(\alpha_0) = \pm I$ and X_0 is an α_0-curve;

(4) if $\rho_n(\pi_1(\partial M))$ is finite for infinitely many n, $\rho_0(\pi_1(\partial M)) \cong \mathbb{Z}$ is generated by a parabolic.

Proof. Fix $\rho_n \in R_{X_0}$ whose character is χ_n. After replacing the ρ_n by conjugate representations (over $PSL_2(\mathbb{R})$) and passing to a subsequence, we may suppose that $\lim \rho_n = \rho_0$. Let Δ_n the image of ρ_n ($n \ge 0$) and $\mathcal{B}_n = \mathbb{H}^2/\Delta_n$ ($n \ge 1$). Lemma 6.1 shows that for $n \ge 1$, Δ_n is either a free product of two finite cyclic groups or a hyperbolic triangle group. Thus the topological orbifold type of $\mathcal{B}_n$ is either $\mathbb{R}^2(p, q)$ or $S^2(p, q, r)$. Since $\{\Delta_n\}$ converges algebraically to Δ_0, Proposition 4.1 implies that Δ_0 is a non-elementary Kleinian group and after passing to a subsequence we may suppose that $\{\Delta_n\}$ converges geometrically to a Fuchsian group Δ containing Δ_0. Further, there are homomorphisms $\theta_n : \Delta \to \Delta_n$ such that $\rho_n = \theta_n \circ \rho_0$ ($n \ge 1$) and $\lim_n \theta_n$ is the inclusion $\Delta \to PSL_2(\mathbb{C})$.

Assume first that the image of ρ_n is a free product of finite cyclic groups for infinitely many n. By Lemma 6.4 there are an integer $n \gg 0$, integers $p, q \ge 2$, and a principal component Z_0 of $X_{PSL_2}(\Delta_n)$ such that $\Delta_n \cong \mathbb{Z}/p * \mathbb{Z}/q$ and $X_0 = \rho_n^*(Z_0) \subset \rho_n^*(X_{PSL_2}(\Delta_n))$. Hence as ρ_0 is irreducible, we have $\rho_0 = \psi \circ \rho_n$ for some $\psi \in R_{PSL_2}(\Delta_n)$. It follows that we have surjective homomorphisms $\Delta_n \xrightarrow{\psi} \Delta_0$ and $\Delta_0 \xrightarrow{\theta_n} \Delta_n$. Since Δ_0 and Δ_n are Hopfian, θ_n is an isomorphism. Thus $Y_0 = \theta_n^*(Z_0)$ is a principal component of $X_{PSL_2}(\Delta_0)$ and $X_0 = \rho_n^*(Z_0) = \rho_0^*(Y_0)$. Lemma 6.1 shows that the remaining conclusions (3) and (4) of the proposition hold.

Next assume that Δ_n is isomorphic to the (a_n, b_n, c_n) triangle group $\Delta(a_n, b_n, c_n)$ where $2 \le a_n \le b_n \le c_n$. Then $\mathcal{B}_n = S^2(a_n, b_n, c_n)$. We know that $\rho_0(\pi_1(\partial M)) \cong \mathbb{Z}/d$ for some $d \ge 0$. If $d > 0$, then $\rho_n(\pi_1(\partial M)) = \theta_n(\rho_0(\pi_1(\partial M)))$ is a quotient of the finite group $\mathbb{Z}/d$ for all n, which contradicts Lemma 6.1. Thus $\rho_0(\pi_1(\partial M)) \cong \mathbb{Z}$. Let α_0 be the unique slope such that $\rho_0(\alpha_0) = \pm I$.

Claim 6.8. *The sequence $\{c_n\}$ tends to infinity and after passing to a subsequence we can find integers $2 \le p \le q$ such that $a_n = p, b_n = q$ for all n. Further, $\Delta \cong \mathbb{Z}/p * \mathbb{Z}/q$ and Δ_0 has index at most 2 in Δ. If it has index 2, then $\Delta \cong \mathbb{Z}/2 * \mathbb{Z}/q, \Delta_0 \cong \mathbb{Z}/q * \mathbb{Z}/q$, and c_n is odd.*

Proof. If $\{c_n\}$ is a bounded sequence, then so are $\{a_n\}, \{b_n\}$ and so after passing to a subsequence we can suppose that they are constants a, b, c. We know that $\mathbb{H}^2/\Delta = \lim_n \mathbb{H}^2/\Delta_n = \lim_n \mathcal{B}_n = S^2(a, b, c)$. Thus $\Delta \cong \Delta(a, b, c)$ and as this group is Hopfian, it follows that $\theta_n : \Delta \to \Delta_n$ is an isomorphism for all n. Since the groups Δ_n are conjugate in $PSL_2(\mathbb{R})$ and the outer automorphism group of $\Delta(a, b, c)$ is finite, it follows that there are only finitely many conjugacy classes among the representations $\rho_n = \theta_n \circ \rho_0$, which contradicts our assumptions. Thus after passing to a subsequence we may suppose that $\lim_n c_n = \infty$.

If $\{a_n\}$ is not bounded, then up to passing to a subsequence we may suppose that $\lim_n a_n = \infty$. It follows that $\lim_n b_n = \infty$ and therefore $\mathbb{H}^2/\Delta = \lim_n \mathcal{B}_n$ is a thrice-punctured sphere. But then Δ is a free group on two generators, and therefore the non-abelian group Δ_0 is free. Thus the dimension of $X_{PSL_2}(\Delta_0)$ is at least 3. But this is impossible as $\rho_0^* : X_{PSL_2}(\Delta_0) \to X_{PSL_2}(M)$ is injective. Thus $\{a_n\}$ is bounded so that after passing to a subsequence we may suppose that $a_n = p \geq 2$ for all n.

A similar argument shows that if $\{b_n\}$ is unbounded, then Δ_0 is a non-abelian subgroup of $\Delta \cong \mathbb{Z}/p * \mathbb{Z}$. Then Δ_0 is a free product of at least two cyclic groups, each of which is either free or has order dividing p. If there are either three such factors or two with one of them free, a contradiction is obtained as in the previous paragraph. On the other hand if $\Delta_0 \cong \mathbb{Z}/r * \mathbb{Z}/s$ where r and s divide p, then $\Delta(p, b_n, c_n) = \theta_n(\Delta_0)$ is generated by two elements of order dividing p. Knapp [Kn] studied when two elliptics can generate a triangle group and determined necessary and sufficient conditions on their orders and the coefficients of the triangle group for this to occur. It follows from Theorem 2.3 of [Kn] (and its proof) that if $\Delta(p, b_n, c_n)$ is generated by elements of bounded order, then $\{b_n\}$ is a bounded sequence, contrary to our assumptions. Thus by passing to a subsequence we may suppose that $b_n = q \geq p$ for all n.

The work above shows that $\mathbb{H}^2/\Delta = \lim_n S^2(p, q, c_n) = \mathbb{R}^2(p, q)$ so that $\Delta \cong \mathbb{Z}/p * \mathbb{Z}/q$. Hence $\Delta_0 \subset \Delta$ is a free product of cyclic groups. and the smallness of M implies that it must be of the form $\mathbb{Z}/r * \mathbb{Z}/s$ where each of r, s divides at least one of p, q. It follows that $\Delta(p, q, c_n)$ is generated by two elements whose orders divide r, s respectively. Given our constraints on c_n and r, s, Theorem 2.3 of [Kn] shows that the conclusion of the claim holds. $\square$(Claim 6.8)

There is a principal component Y_0 of $X_{PSL_2}(\Delta_0)$ which contains the character of the inclusion $\Delta_0 \to PSL_2(\mathbb{C})$. Since $\lim_n \theta_n$ is this inclusion and the algebraic components of $X_{PSL_2}(\Delta_0)$ are topological components (see Example 3.2 of [BZ1]), if $n \gg 0$, $\chi_{\theta_n} \in Y_0$. On the other hand, χ_n is a simple point of $X_{PSL_2}(M)$ for $n \gg 0$. Since $\chi_{\rho_0} = \lim_n \chi_n = \lim_n \rho_0^*(\chi_{\theta_n})$ it follows that $X_0 = \rho_0^*(Y_0)$. This proves (1) and (2) while (3) is a consequence of the (1), (2), and Lemma 6.1. Finally, to prove (4), note that if $\alpha_1 \neq \alpha_0$ is a slope, then $|\text{trace}(\rho_0)(\alpha_1)| = \lim_n |\text{trace}(\rho_n)(\alpha_1)| \leq 2$. On the other hand if $\rho_0(\alpha_1)$ is elliptic, then $\rho_n(\alpha_1)$ is elliptic of the same order for $n \gg 0$. This contradicts Lemma 6.1. Thus $\rho_0(\alpha_1)$ is parabolic. $\square$

Corollary 6.9. *Suppose that M is a small knot manifold.*
(1) If X_0 is a norm curve component of $X_{PSL_2}(M)$, then the intersection of $D(X_0; \mathbb{R})$ with any compact subset of X_0 is finite.
(2) If $\pi_1(M)$ does not surject onto a free product of non-trivial cyclic groups, then the intersection of $D(M; \mathbb{R})$ with any compact subset of $X_{PSL_2}(M)$ is finite. $\square$

Corollary 6.10.
(1) If M is a small knot manifold and X_0 is a non-trivial component of $X_{PSL_2}(M)$ such that for each connected, essential surface S_0 in M there is a character $\chi \in X_0$ such that $\chi|\pi_1(S_0)$ is strictly irreducible, then $D(X_0; \mathbb{R})$ is finite.
(2) Let M and N be small hyperbolic knot manifolds and suppose that $\varphi : \pi_1(M) \to \pi_1(N)$ is a virtual epimorphism. If Y_0 is a principal component of $X_{PSL_2}(N)$ and $X_0 = \varphi^*(Y_0)$, then $D(X_0; \mathbb{R})$ is finite.

Proof. (1) By Proposition 6.6 we deduce that $D(X_0; \mathbb{R})$ is compact. If it has an accumulation point then Proposition 6.7 implies that there is a surjection $\rho : \pi_1(M) \to \mathbb{Z}/p * \mathbb{Z}/q$ and a principal component Y_0 of $X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ such that $X_0 = \rho^*(Y_0)$. But then Lemma 6.3 shows that $D(X_0; \mathbb{R})$ is not compact. Thus $D(X_0; \mathbb{R})$ is finite.

(2) Corollary 4.8 implies that $D(X_0; \mathbb{R})$ is compact. If it has an accumulation point then Proposition 6.7 implies that for each irreducible $\chi_\rho \in X_0$, $\rho(\pi_1(M))$ is generated by two torsion elements (cf. Remark 4.10). But this is clearly not the case for the image by ρ^* of the discrete faithful character of $\pi_1(N)$. Thus $D(X_0; \mathbb{R})$ is finite. $\qquad\square$

Example 6.11. The character variety of a knot manifold M whose fundamental group admits a discrete epimorphism onto a free product of finite cyclic groups contains an α_0-curve for some slope α_0. Hence Example 3.3 gives many examples for which this does not occur. In particular Corollaries 4.8 and 6.10 show that if M is the exterior of a hyperbolic twist knot or a $(-2, 3, n)$ pretzel knot with $n \not\equiv 0 \pmod 3$, then $D(M; \mathbb{R})$ is finite. Gonzàlez-Acuña and Ramirez [GR1], [GR2] have studied the problem of when the fundamental group of the exterior M of a knot in the 3-sphere admits an epimorphism onto a free product $\mathbb{Z}/p * \mathbb{Z}/q$ for some integers $p, q \geq 2$. It is simple to see that in this case p, q are relatively prime. Hartley and Murasugi showed [HM] that the epimorphism factors through a homomorphism $\pi_1(M) \to \pi_1(M_{p,q})$ whose image is normal with cokernel finite cyclic. This implies that the Alexander polynomial of M is divisible by that of $M_{p,q}$. These conclusions hold more generally for manifolds M with $H_1(M) \cong \mathbb{Z}$ (cf. the proof of Theorem 6.12). Gonzàlez-Acuña and Ramirez [GR1] have given an algorithm which determines which two-bridge knot exteriors have fundamental groups which admit such a representation. This work easily shows that the fundamental group of the exterior of the $\frac{p}{q}$ two-bridge knot, p prime, admits no such representation.

6.4. **Discrete $PSL_2(\mathbb{R})$-representations and domination.**

Theorem 6.12. *Let M be a knot manifold with $H_1(M) \cong \mathbb{Z}$ and suppose that there is a homomorphism $\rho_0 \in R_{PSL_2}(M)$ with discrete, non-elementary image $\Delta_0 \cong \mathbb{Z}/p * \mathbb{Z}/q$. Suppose further that $\rho_0(\lambda_M)$ is parabolic for any longitudinal class $\lambda_M \in \pi_1(\partial M)$. Then there are a Seifert fibred manifold N whose interior has base orbifold $\mathbb{H}^2/\Delta_0 \cong \mathbb{R}^2(p, q)$ and a domination $f : (M, \partial M) \to (N, \partial N)$ such that the composition $\pi_1(M) \xrightarrow{f_\#} \pi_1(N) \to \Delta_0$ is conjugate to ρ_0.*

Proof. Consider the central extension

$$1 \to K \to \mathrm{Isom}_0(\widetilde{SL_2}) \xrightarrow{\psi} PSL_2(\mathbb{R}) \to 1$$

where $\mathrm{Isom}_0(\widetilde{SL_2})$ is the component of the identity in $\mathrm{Isom}(\widetilde{SL_2})$ and $K \cong \mathbb{R}$ (cf. pp. 464-465 of [Sc]). It is simple to see that for each torsion element $x \in PSL_2(\mathbb{R})$, there is a unique torsion element $\widetilde{A} \in \psi^{-1}(A) \subset \mathrm{Isom}_0(\widetilde{SL_2})$. Thus ρ_0 lifts to a representation $\tilde{\rho}_0 : \pi_1(M) \to \mathrm{Isom}_0(\widetilde{SL_2})$ whose image is isomorphic to Δ_0. Fix a non-zero homomorphism $\phi : \pi_1(M) \to K$ and note that

$$\tilde{\rho} : \pi_1(M) \to \mathrm{Isom}_0(\widetilde{SL_2}), \gamma \mapsto \phi(\gamma)\tilde{\rho}_0(\gamma)$$

is another homomorphism which lifts ρ_0. Set $\widetilde{\Delta}_\phi = \tilde{\rho}(\pi_1(M))$.

Claim 6.13. $\widetilde{\Delta}_\phi$ *is discrete, torsion free, and is the fundamental group of a Seifert manifold* N *with base orbifold* $D^2(p,q)$.

Proof. Since Δ_0 is discrete in $PSL_2(\mathbb{R})$, $\widetilde{\Delta}_\phi$ is discrete in $\text{Isom}_0(\widetilde{SL_2})$ if and only if it intersects the central subgroup K of $\text{Isom}_0(\widetilde{SL_2})$ in a discrete subgroup. This intersection is precisely $\tilde{\rho}(\text{kernel}(\rho_0)) = \phi(\text{kernel}(\rho_0)) \subset \phi(\pi_1(M)) \subset K$. The latter group is isomorphic to $\mathbb{Z}$ by construction, and so is discrete. Thus $\widetilde{\Delta}_\phi$ is discrete.

Suppose that $\gamma \in \pi_1(M)$ and $\tilde{\rho}(\gamma)^n = 1$ for some positive n. Then up to conjugation, $\rho_0(\gamma)^n = \pm I$ is also torsion and therefore $\tilde{\rho}_0(\gamma)^n = 1$ as well. But then $1 = \tilde{\rho}(\gamma)^n = \phi(\gamma)^n \tilde{\rho}_0(\gamma)^n = \phi(\gamma)^n$. Since K is torsion free we conclude that $\gamma \in \text{kernel}(\phi)$, and since $H_1(M) \cong \mathbb{Z}$ and $\phi \neq 0$, $\text{kernel}(\phi) = [\pi_1(M), \pi_1(M)]$. Hence $\gamma \in [\pi_1(M), \pi_1(M)]$ and therefore the image of $\rho_0(\gamma)$ in $H_1(\Delta_0)$ is zero. But $\Delta_0 \cong \mathbb{Z}/p * \mathbb{Z}/q$ so that $\Delta_0 \to H_1(\Delta_0)$ is injective on torsion elements. Thus $\rho_0(\gamma) = 1$ and therefore $\tilde{\rho}(\gamma) = \phi(\gamma)\tilde{\rho}_0(\gamma) = 1$. This proves that $\widetilde{\Delta}_\phi$ is torsion free.

The conclusions of the two previous paragraphs imply that $\widetilde{\Delta}_\phi$ acts freely and properly discontinuously on $\widetilde{SL_2}$. Let $W = \widetilde{SL_2}/\widetilde{\Delta}_\phi$ be the quotient manifold. Now $\widetilde{\Delta}_\phi \cap K \neq \{0\}$ as otherwise $\psi|\widetilde{\Delta}_\phi \to \Delta_0$ would be an isomorphism, which contradicts the result of the last paragraph. Thus $\widetilde{\Delta}_\phi \cap K \cong \mathbb{Z}$ and so $K/(\widetilde{\Delta}_\phi \cap K) \cong S^1$. On the other hand, $\mathbb{H}^2/\Delta_0 \cong \mathbb{R}^2(p,q)$. Thus there is an orbifold bundle $S^1 \to W \to \mathbb{R}^2(p,q)$ so that W admits a compactification N with boundary a torus. Further, N admits a Seifert fibering with base orbifold $D^2(p,q)$. This completes the proof of the claim. $\qquad\square$(Claim 6.13)

To complete the proof of the proposition we must show that there is a domination $M \geq N$. To that end, fix a map $f : M \to N$ which realizes $\tilde{\rho} : \pi_1(M) \to \widetilde{\Delta}_\phi = \pi_1(N)$. We must show that $f_\#|\pi_1(\partial M)$ is injective and has image contained in a peripheral subgroup of $\pi_1(N)$.

By hypothesis $\rho_0(\lambda_M)$ is parabolic. In particular it has infinite order and is distinct from a primitive element $\alpha_0 \in \text{kernel}(\rho_0|\pi_1(\partial M))$ (cf. Proposition 6.7(3)). It follows that $1 \neq \phi(\alpha_0) \in K$. It is easy to see that the restriction of $\tilde{\rho}$ to $\langle \lambda_M, \alpha_0 \rangle \cong \mathbb{Z}^2$ is injective and since $\widetilde{\Delta}_\phi$ is torsion free, the same holds for its restriction to $\pi_1(\partial M)$.

Finally, to show that $\tilde{\rho}(\pi_1(\partial M))$ is peripheral, it suffices to see that $\rho_0(\pi_1(\partial M))$ is peripheral in $\Delta_0 = \pi_1(D^2(p,q))$. But this is clear since it is a parabolic subgroup of Δ_0. This completes the proof. $\qquad\square$

Remark 6.14. The condition that $H_1(M) \cong \mathbb{Z}$ was used to guarantee that $\widetilde{\Delta}$ is torsion free. Without this condition we can still construct a proper non-zero degree map from M to a 3-dimensional Seifert orbifold, but the underlying space of the orbifold might be $S^1 \times D^2$.

Corollary 6.15. *Let* M *be a small hyperbolic knot manifold with* $H_1(M) \cong \mathbb{Z}$, X_0 *a nontrivial component of* $X_{PSL_2}(M)$, *and* $\{\chi_{\rho_n}\} \subset D(X_0; \mathbb{R})$ *a sequence of distinct characters which converges to* $\chi_{\rho_0} \in X_0$. *Suppose further that for each* n, $\rho_n(\pi_1(\partial M))$ *is finite. Then* ρ_0 *has discrete, non-elementary image isomorphic to a free product of two finite cyclic groups and* $\rho_0(\pi_1(\partial M))$ *is parabolic. If* $\rho_n(\lambda_M) \neq \pm I$ *for infinitely many* n, *there is a strict domination* $M \geq N$ *for some Seifert manifold* N *with incompressible boundary.*

Proof. Suppose that $\lim_n \chi_{\rho_n} = \chi_{\rho_0}$ and set $\Delta_0 = \rho_0(\pi_1(M))$. By Proposition 6.7, $\Delta_0 \subset PSL_2(\mathbb{R})$ is discrete and isomorphic to $\mathbb{Z}/p * \mathbb{Z}/q$ for some integers $2 \leq p, q$. Our hypotheses imply that $\rho_0(\lambda_M)$ is parabolic (cf. [JM, Lemma 3.6] and Proposition 6.7(4)). Thus Theorem 6.12 implies the desired conclusion. $\square$

Corollary 6.16. *Let M be a small hyperbolic knot manifold, $\{\alpha_n\}$ a sequence of distinct slopes on ∂M, and $\{\chi_n\} \subset D(M; \mathbb{R})$ a sequence of characters of representations ρ_n such that $\rho_n(\alpha_n) = \pm I$ for all n. If there are infinitely many distinct χ_n and the sequence $\{\chi_n\}$ subconverges to a character χ_{ρ_0}, then*
(1) the image of ρ_0 is isomorphic to a discrete, non-elementary free product of two finite cyclic groups;
(2) $\rho_n(\pi_1(\partial M))$ is finite for infinitely many n and $\rho_0(\pi_1(\partial M))$ is parabolic;
(3) if $H_1(M) \cong \mathbb{Z}$ and $\rho_0(\lambda_M) \neq \pm I$, M strictly dominates a Seifert manifold with incompressible boundary.

Proof. After passing to a subsequence we can assume that the χ_n are distinct. Part (3) of Proposition 6.7 shows that there is a unique slope α_0 on ∂M such that $\rho_0(\alpha_0) = \pm I$. Then for $n \gg 0$ we have $\rho_n(\alpha_0) = \pm I$ ([JM, Lemma 3.6]). Since the α_n are distinct, this implies that for large n, $\rho_n(\pi_1(\partial M))$ is a finite cyclic group of order dividing $\Delta(\alpha_0, \alpha_n)$. Corollary 6.15 then yields a strict domination $f : M \to N$ where N is a Seifert manifold with incompressible boundary. $\square$

7. Minimal Dehn fillings

In the section we use the results of the paper to construct various infinite families of minimal closed 3-manifolds.

Lemma 7.1.
(1) If M is a small knot manifold, there are only finitely many slopes α on ∂M such that $M(\alpha)$ is either reducible or Haken.
(2) A closed, connected, orientable manifold with infinite fundamental group is either reducible, Haken, or admits a geometric structure modeled on $Nil, \mathbb{H}^3$, or $\widetilde{SL_2}$.

Proof. (1) If $M(\alpha)$ contains an essential surface S and we isotope S so as to minimize $|S \cap \partial M|$, then $S_0 := S \cap M$ is an essential surface in M. Since M is small, $\partial S_0 \neq \emptyset$ and has slope α. Thus α is a boundary slope. By [Hat], there are at most finitely many such α. Thus (1) holds.

(2) By the geometrization theorem of Perelman we see that a closed, connected, orientable manifold W which is irreducible though not Haken admits a geometric structure. If the structure is *Sol*, W is Haken since it is irreducible and contains an essential torus ([Sc]). If it is $\mathbb{S}^2 \times \mathbb{R}, \mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$, then W amits a Seifert fibre structure with zero Euler number and therefore it is either reducible or Haken ([Sc]). If it is $\mathbb{S}^3$, $\pi_1(W)$ is finite. This proves (2). $\square$

Theorem 7.2. *Suppose that M is a small $\mathcal{H}$-minimal hyperbolic knot manifold which has the following properties:*

(a) *There is a slope α_0 on ∂M such that $M(\alpha_0)$ is $\mathcal{H}$-minimal. Suppose as well that the core of the α_0 filling solid torus is not null-homotopic in $M(\alpha_0)$ when $M(\alpha_0)$ is hyperbolic.*
(b) *For each norm curve $X_0 \subset X_{PSL_2}(M)$ and for each essential surface S associated to an ideal point of X_0 there is a character $\chi_\rho \in X_0$ which restricts to a strictly irreducible character on $\pi_1(S)$.*
(c) *There is no surjective homomorphism from $\pi_1(M)$ onto a Euclidean triangle group.*
(d) *There is no epimorphism $\rho : \pi_1(M) \to \Delta(p, q, r) \subset PSL_2(\mathbb{R})$ such that $\rho(\pi_1(\partial M))$ is elliptic or trivial.*

Then all but finitely many Dehn fillings $M(\alpha)$ yield a minimal manifold.

Proof. By Theorem 5.1(2) and Lemma 7.1, we need only show that $M(\alpha)$ is Nil-minimal and $\widetilde{SL_2}$-minimal for all but finitely many slopes α on ∂M.

Suppose that there is a slope α and a domination f from $M(\alpha)$ to a closed Nil-manifold V with base orbifold $\mathcal{B}_n$. By passing to a cover of V we may suppose that $f_\#$ is surjective. We can also suppose that α is not a boundary slope so that $M(\alpha)$ is not Haken. Since $\mathcal{B}$ is Euclidean, the only possibility is that $\mathcal{B} \cong S^2(a, b, c)$ for some Euclidean triple (a, b, c). But then we would have an epimorphism $\pi_1(M) \to \pi_1(M(\alpha)) \to \pi_1(V) \to \pi_1(S^2(a, b, c)) \cong \Delta(a, b, c)$, which contradicts (c). Thus $M(\alpha)$ is Nil-minimal for all but finitely many α.

Suppose next that there are a sequence of distinct slopes α_n and dominations f_n from $M(\alpha_n)$ to a closed $\widetilde{SL_2}$-manifold V_n with base orbifold $\mathcal{B}_n$. By passing to a cover of V_n, we may suppose that $(f_n)_\#$ is surjective for all n. Let ρ_n be the composition $\pi_1(M) \to \pi_1(M(\alpha_n)) \overset{(f_n)_\#}{\longrightarrow} \pi_1(V_n) \to \pi_1(\mathcal{B}_n) \subset PSL_2(\mathbb{R}) \subset PSL_2(\mathbb{C})$. By passing to a subsequence we may suppose that $\chi_n \in X_0$ for some non-trivial curve X_0. Hypothesis (d) implies that $\text{kernel}(\rho_n|\pi_1(\partial M)) = \langle \alpha_n \rangle$ so that there are infinitely many distinct χ_n and $\rho_n(\pi_1(\partial M))$ is infinite. It follows that $\rho_n(\pi_1(\partial M))$ contains loxodromics. Thus Corollary 2.9 implies that X_0 is a norm curve. But then hypothesis (b) and Corollary 6.10(1) imply that there are only finitely many χ_n, contrary to the construction. Thus there is no sequence $\{\alpha_n\}$ as above and so $M(\alpha)$ is $\widetilde{SL_2}$-minimal for all but finitely many α. $\qquad\square$

Corollary 7.3. *If M is the exterior of a hyperbolic twist knot, then all but finitely many Dehn fillings $M(\alpha)$ yield a minimal manifold.* $\qquad\square$

Proof. Hypothesis (a) of Theorem 7.2 clearly holds for M, and since the only non-trivial curve in $X_{PSL_2}(M)$ is a principal curve ([Bu]), hypothesis (b) holds as well (cf. the proof of Corollary 4.8). Finally, hypotheses (c) and (d) are true for M by Proposition B.1 and Proposition B.2. $\quad\square$

Condition (d) of Theorem 7.2 is difficult to verify in general. Nevertheless, the following results show that we can still construct infinite families of minimal Dehn fillings in quite general situations. First we need to prove an elementary lemma.

Lemma 7.4. *Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be projectively distinct primitive elements of $\mathbb{Z}^2$ and suppose that $L_1, L_2, \ldots, L_m$ are subgroups of $\mathbb{Z}^2$ none of which contains α_0. For each $i = 1, 2, \ldots, n$, let U_i be an arc neighbourhood of $[\alpha_i] \in \mathbb{P}(\mathbb{R}^2)$ and suppose that $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$. Then there are infinitely many primitive $\alpha \in \mathbb{Z}^2$ such that $\alpha \notin L_1 \cup \ldots \cup L_m$ and $[\alpha] \notin U_1 \cup \ldots \cup U_n$.*

Proof. Since each L_j is contained in a rank 2 subgroup of $\mathbb{Z}^2$ in the complement of α_0, we can assume, without loss of generality, that each L_j has rank 2. Let $\beta_0 \in \mathbb{Z}^2$ be dual to α_0 and fix coprime integers a, b such that $\delta_0 := a\alpha_0 + b\beta_0 \neq \alpha_i$, $0 \leq i \leq n$. Set $L_0 = \{\alpha_0 + n\delta_0 : n \in \mathbb{Z}\}$ and note that from the definition of $U = U_1 \cup \ldots \cup U_n$ and choice of δ_0, there is some $k_0 > 0$ such that if $|k| \geq k_0$, $[\alpha_0 + k\delta_0] \notin U$. Define $d \geq 1$ to be the index of $L_1 \cap L_2 \cap \ldots \cap L_m$ in $\mathbb{Z}^2$ and note that for each $k \in \mathbb{Z}$, the class $\alpha_k = \alpha_0 + dkb\delta_0 \notin (L_1 \cup L_2 \cup \ldots \cup L_m)$. The proof is completed by observing that $\alpha_k = (1 + abkd)\alpha_0 + b^2 kd\beta_0$ is primitive and $[\alpha_k] \notin U$ for $|k| \geq k_0$. $\qquad\square$

Theorem 7.5. *Let M be an $\mathcal{H}$-minimal, small, hyperbolic knot manifold and suppose that $H_1(M) \cong \mathbb{Z} \oplus T$ where*

(a) $H_1(\partial M) \to H_1(M)/T \cong \mathbb{Z}$ is surjective, and
(b) $\mathbb{Z}/a \oplus \mathbb{Z}/b$ is not a quotient of T for $(a, b) = (2, 3), (2, 4), (3, 3)$.

Suppose as well that

(c) there is no discrete, non-elementary representation $\rho \in R_{PSL_2(\mathbb{R})}(M)$ such that $\rho(\pi_1(M))$ is isomorphic to a free product of two non-trivial cyclic groups and $\rho(\pi_1(\partial M))$ is parabolic;
(d) there is a slope α_0 on ∂M such that $\pi_1(M(\alpha_0))$ admits no homomorphism onto a non-elementary Kleinian group or a Euclidean triangle group.

If $U \subset \mathbb{P}(H_1(\partial M; \mathbb{R}))$ is the union of disjoint closed arc neighbourhoods of the finite set of boundary slopes of M, then there are infinitely many slopes α such that $[\alpha] \in \mathbb{P}(H_1(\partial M; \mathbb{R})) \setminus U$ and $M(\alpha)$ is minimal.

Proof. By Lemma 7.1 and Theorem 5.1(1), it suffices to show that there are infinitely many slopes α such that $[\alpha] \in \mathbb{P}(H_1(\partial M; \mathbb{R})) \setminus U$ and $M(\alpha)$ is both Nil-minimal and $\widetilde{SL_2}$-minimal. As we argued in the proof of Theorem 7.2, if α is not a boundary slope and there is a domination $M(\alpha) \to V$ where V is a Nil or $\widetilde{SL_2}$ manifold, there is an epimorphism $\rho : \pi_1(M) \to \Delta(a, b, c)$ which can suppose lies in $D(M; \mathbb{R})$ if (a, b, c) is a hyperbolic triple.

Suppose first of all that $\rho : \pi_1(M) \to \Delta(a, b, c)$ is surjective and (a, b, c) is a Euclidean triple with $a \leq b \leq c$. There is an epimorphism $\mathbb{Z} \oplus T = H_1(M) \to H_1(\Delta(a, b, c)) \cong \mathbb{Z}/a \oplus \mathbb{Z}/b$ where (a, b) is one of the pairs $(2, 3), (2, 4), (3, 3)$. Hence our hypotheses imply that there is some $\gamma \in \pi_1(\partial M)$ which is sent to a non-zero element of $H_1(\Delta(a, b, c))$ under the composition $\pi_1(M) \xrightarrow{\rho} \Delta(a, b, c)) \to H_1(\Delta(a, b, c))$. It is a simple exercise to then show that $\rho(\gamma)$ has non-trivial finite order in $\Delta(a, b, c)$. (For instance, use the fact that $\Delta(a, b, c)$ can be considered a subgroup of the upper-triangular matrices in $PSL_2(\mathbb{C})$.) Since an abelian subgroup of an infinite triangle group is cyclic, $\rho(\pi_1(\partial M)) \cong \mathbb{Z}/d$ where $d \in \{2, 3, 4, 6\}$. Thus there are only finitely many possibilities for kernel$(\rho|\pi_1(\partial M))$, say $L_1, \ldots, L_k$. By hypothesis none of them contain α_0. Further, if $\alpha \notin L_1 \cup \ldots \cup L_k$, then $\pi_1(M(\alpha))$ admits no homomorphism onto a Euclidean triangle group.

Next set $D_U(M; \mathbb{R}) := \{\chi_\rho \in D(M; \mathbb{R}) : \rho(\alpha) = \pm I$ for some slope α such that $[\alpha] \notin U\}$ and suppose it is infinite. If α is a slope such that $[\alpha] \notin U$, then α is not a boundary slope and so there are only finitely many $\chi_\rho \in D_U(M; \mathbb{R})$ such that $\rho(\alpha) = \pm I$ (Corollary 2.8). Hence we

can find a sequence of distinct slopes α_n, a sequence of distinct characters $\chi_{\rho_n} \in D_U(M; \mathbb{R})$, and a component X_0 of $X_{PSL_2}(M)$ such that $[\alpha_n] \notin U$, $\rho_n(\alpha_n) = \pm I$, and $\chi_{\rho_n} \in X_0$. Lemma 4.5 shows that $\{\chi_{\rho_n}\}$ does not accumulate to an ideal point of X_0. Thus we can suppose that it converges to some $\chi_{\rho_0} \in X_0$. Proposition 6.7 implies that $\rho_0(\pi_1(M))$ is a free product of two finite cyclic groups and $\rho_0(\pi_1(\partial M))$ is parabolic. But this contradicts our hypotheses. Thus $D_U(M; \mathbb{R})$ is finite, say $D_U(M; \mathbb{R}) = \{\chi_{\rho_1}, \chi_{\rho_2}, \ldots, \chi_{\rho_l}\}$. Set $L_j' = \text{kernel}(\rho_j | \pi_1(\partial M))$ ($1 \leq j \leq l$). Then $\alpha_0 \notin (L_1' \cup L_2' \cup \ldots \cup L_m')$ and if $\alpha \notin (L_1' \cup L_2' \cup \ldots \cup L_m')$ is a slope such that $[\alpha] \notin U$, $\pi_1(M(\alpha))$ admits no homomorphism onto a hyperbolic triangle group. The proof is completed by applying Lemma 7.4 to the subgroups $L_1, \ldots, L_k, L_1', \ldots, L_l'$. $\qquad\square$

Example 7.6. The theorem applies to the exterior of many knots in lens spaces. For instance, it follows from work of Indurskis [In] that if M_m is the manifold obtained by m-Dehn filling on one component of the right-hand Whitehead link, then $X_{PSL_2}(M_m)$ has exactly one non-trivial component and is therefore minimal. For $|m| > 4$, M_m is hyperbolic. If μ is the slope on ∂M_m corresponding to a meridian of the Whitehead link, $M_m(\mu) \cong L(m, 1)$. Let $T_1(M_m)$ denote the torsion subgroup of $H_1(M_m)$. Since $H_1(M_m) \cong \mathbb{Z} \oplus \mathbb{Z}/m$ and $H_1(\partial M_m) \to H_1(M_m)/T_1(M_m)$ is onto, the hypotheses of Theorem 7.5 are satisfied as long as $m \not\equiv 0 \pmod 6$. For such m, $M_m(\alpha)$ is minimal for infinitely many slopes α.

Theorem 7.7. *Suppose that M is a minimal small hyperbolic knot manifold such that $H_1(M) \cong \mathbb{Z}$ and that*

(a) there is no homomorphism $\rho : \pi_1(M(\lambda_M)) \to PSL_2(\mathbb{R})$ such that $\rho(\pi_1(M(\lambda_M)))$ is a free product of two non-trivial cyclic groups and $\rho(\pi_1(\partial M))$ parabolic.
(b) there is a slope α_0 on ∂M such that $\pi_1(M(\alpha_0))$ admits no homomorphism onto a non-elementary Kleinian group or a Euclidean triangle group.

Then there are infinitely many slopes α on ∂M such that $M(\alpha)$ is minimal.

Proof. The proof is similar to that of Theorem 7.5. As before it suffices to show that there are subgroups $L_1, L_2, \ldots, L_m$ of $H_1(\partial M)$, none of which contain α_0, such that if $\alpha \notin L_1 \cup \ldots \cup L_m$ is a slope, though not a boundary slope, then $\pi_1(M(\alpha))$ admits no surjective homomorphism onto an infinite triangle group. Since $H_1(M) \cong \mathbb{Z}$, the homological conditions (a) and (b) from the statement of Theorem 7.5 hold and so there are subgroups $L_1, \ldots, L_k$ of $H_1(\partial M)$, none of which contain α_0, such that if $\alpha \notin L_1 \cup \ldots \cup L_k$, then $\pi_1(M(\alpha))$ admits no homomorphism onto a Euclidean triangle group.

To derive a similar conclusion for hyperbolic triangle groups, the proof of Theorem 7.5 shows that it suffices to fix a disjoint union $U \subset \mathbb{P}(H_1(\partial M; \mathbb{R}))$ of closed arc neighbourhoods of the finite set of boundary slopes of M and prove that $D_U(M; \mathbb{R}) := \{\chi_\rho \in D(M; \mathbb{R}) : \rho(\alpha) = \pm I$ for some slope α such that $[\alpha] \notin U\}$ is finite. Suppose otherwise and note that as in the proof of Theorem 7.5, we can find a representation $\rho_0 : \pi_1(M) \to PSL_2(\mathbb{R})$ with discrete image isomorphic to a free product of non-trivial cyclic groups such that $\rho_0(\pi_1(\partial M))$ is parabolic. Hypothesis (a) implies that $\rho_0(\lambda_M) \neq \pm I$ and so Theorem 6.12 implies that M strictly dominates some Seifert manifold N with incompressible boundary. This contradicts the minimality of M. Thus $D_U(M; \mathbb{R})$ is finite and the proof proceeds as in that of Theorem 7.5. $\qquad\square$

Corollary 7.8. *Let M be a minimal, small, hyperbolic 3-manifold which is the exterior of a knot K in the 3-sphere. If there is no homomorphism $\rho : \pi_1(M(\lambda_M)) \to PSL_2(\mathbb{R})$ such that $\rho(\pi_1(M(\lambda_M)))$ is a free product of two non-trivial cyclic groups and $\rho(\pi_1(\partial M))$ is parabolic, then there are infinitely many slopes α on ∂M such that $M(\alpha)$ is minimal.* $\square$

Example 7.9. If the Alexander polynomial of a knot $K \subset S^3$ with exterior M is not divisible by the Alexander polynomial of a non-trivial torus knot, there is no homomorphism of $\pi_1(M)$ onto the free product of two non-trivial finite cyclic groups (cf. Remark 6.11). Thus if its exterior is minimal, small, and hyperbolic, there are infinitely many slopes α on ∂M such that $M(\alpha)$ is minimal. This provides many examples. For example, let K be the $(-2, 3, n)$ pretzel where $n \not\equiv 0 \pmod 3$. We noted in Example 3.3 that there is a unique non-trivial component of $X_{PSL_2}(M)$, necessarily principal, and used this to deduce that M is minimal. It also implies that there is no homomorphism $\rho : \pi_1(M(\lambda_M)) \to PSL_2(\mathbb{R})$ such that $\rho(\pi_1(M(\lambda_M)))$ is a free product of two non-trivial cyclic groups since such a representation would yield an extra non-trivial component of $X_{PSL_2}(M)$. Thus Corollary 7.8 may be applied to see that there are infinitely many slopes α on ∂M such that $M(\alpha)$ is minimal. As a final example, Riley has shown that if K is a two-bridge knot and $\rho \in R_{PSL_2}(M)$ is irreducible with $\rho(\pi_1(\partial M))$ parabolic, then $\rho(\lambda_M) \neq \pm I$ (Lemma 1 [Ri]). In particular, if M is the exterior of a $\frac{p}{q}$ two-bridge knot, it is minimal (Corollary 3.13), small, and hyperbolic if p is prime and $q \not\equiv \pm 1 \pmod p$. Thus the corollary implies that there are infinitely many slopes α on ∂M such that $M(\alpha)$ is minimal.

APPENDIX A. ON THE SMOOTHNESS OF DIHEDRAL CHARACTERS

One goal of this appendix is to prove that if μ is a meridional class of the p/q two-bridge knot, then $d_{M_{p/q}}(\mu) = \frac{p-1}{2}$. In order to do this, we determine a useful criterion for the smoothness of dihedral characters.

A.1. A cohomological calculation.
Let Γ be a finitely generated group, V is a complex vector space, and $\theta : \Gamma \to GL(V)$ a homomorphism. We use $b_1(\Gamma; \theta)$ to denote the complex dimension of $H^1(\Gamma; V_\theta)$. For instance if $\rho \in R_{PSL_2}(\Gamma)$, the induced action of Γ on $sl_2(\mathbb{C})$ given by the composition $\Gamma \xrightarrow{\rho} PSL_2(\mathbb{C}) \xrightarrow{Ad} Aut(sl_2(\mathbb{C}))$ gives rise to the cohomology group $H^1(\Gamma; sl_2(\mathbb{C})_{Ad\rho})$ whose dimension is $b_1(\Gamma; Ad\rho)$.

Identify the dihedral group of $2n$ elements D_n with the subgroup of $\mathcal{N}$ generated by the matrices $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\pm \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ where $\zeta = \exp(\frac{2\pi i}{2n})$. Any subgroup of $PSL_2(\mathbb{C})$ abstractly isomorphic to D_n is conjugate in $PSL_2(\mathbb{C})$ to D_n.

For each divisor $d \geq 1$ of n we have surjections $\theta_{n,d} : D_n \to D_d$ given by

$$\theta_{n,d}(\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \theta_{n,d}(\pm \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}) = \pm \begin{pmatrix} \zeta^{\frac{n}{d}} & 0 \\ 0 & \zeta^{-\frac{n}{d}} \end{pmatrix}.$$

Lemma A.1. *Let $\rho : \Gamma \to PSL_2(\mathbb{C})$ be a representation whose image is D_n, $n > 1$. For each divisor $d \geq 1$ of n let ρ_d be the composition of ρ with $\theta_{n,d}$ and set $\Gamma_{2d} = ker(\rho_d)$. Then*

$$b_1(\Gamma; Ad\rho) = b_1(\Gamma_2) - b_1(\Gamma) + \frac{1}{\phi(n)} \sum_{d|n} \mu(\frac{n}{d}) b_1(\Gamma_{2d})$$

where ϕ is Euler's ϕ-function and μ is the Möbius function.

Proof. Consider the real basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $sl_2(\mathbb{C})$. Let $\Theta = Ad| : \mathcal{N} \to Aut(sl_2(\mathbb{C}))$. The reader will verify that the span $\langle e_1 \rangle \cong \mathbb{C}$ of e_1 is invariant under $\Theta(\mathcal{N})$ as is $\langle e_2, e_3 \rangle \cong \mathbb{C}^2$. Thus

$$sl_2(\mathbb{C})_\Theta = \mathbb{C}_{\Theta_1} \oplus \mathbb{C}^2_{\Theta_2} \tag{A.1.1}$$

where $\Theta_1 : \mathcal{N} \to GL_1(\mathbb{C})$ is given by

$$\Theta_1(A) = \begin{cases} 1_\mathbb{C} & \text{if } A \in \mathcal{D} \\ -1_\mathbb{C} & \text{if } A \in \mathcal{N} \setminus \mathcal{D} \end{cases}$$

and, in terms of the ordered basis $\{e_2, e_3\}$, $\Theta_2 : \mathcal{N} \to GL_2(\mathbb{C})$ is given by

$$\Theta_2(\pm \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}) = \begin{pmatrix} u^2 & 0 \\ 0 & u^{-2,} \end{pmatrix}, \Theta_2(\pm \begin{pmatrix} 0 & v \\ -v^{-1} & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -v^2 \\ -v^{-2} & 0 \end{pmatrix}.$$

Without loss of generality we may suppose that the image of ρ lies in $\mathcal{N}$. Then (A.1.1) yields the decomposition $sl_2(\mathbb{C})_{Ad\rho} = \mathbb{C}_{\theta_1} \oplus \mathbb{C}^2_{\theta_2}$ where $\theta_j = \Theta_j \circ \rho$. Hence

$$b_1(\Gamma; Ad\rho) = b_1(\Gamma; \theta_1) + b_1(\Gamma; \theta_2).$$

The proof of the lemma now follows from Claims (1) and (2) below.

Claim 1. $b_1(\Gamma; \theta_1) = b_1(\Gamma_2) - b_1(\Gamma)$.

Proof. The Γ-module $\mathbb{C}[\Gamma/\Gamma_2] = \mathbb{C}[\mathbb{Z}/2]_\Gamma$ splits into two 1-dimensional modules

$$\mathbb{C}[\mathbb{Z}/2]_\Gamma = \mathbb{C}_1 \oplus \mathbb{C}_{\theta_1}$$

where $\mathbb{C}_1$ is the trivial Γ-module. Then $H^1(\Gamma_2; \mathbb{C}) \cong H^1(\Gamma; \mathbb{C}[\Gamma/\Gamma_2]) = H^1(\Gamma; \mathbb{C}[\mathbb{Z}/2]_\Gamma) = H^1(\Gamma; \mathbb{C}_1) \oplus H^1(\Gamma; \mathbb{C}_{\theta_1})$. Thus $b_1(\Gamma_2) = b_1(\Gamma) + b_1(\Gamma; \theta_1)$. $\square$(of Claim 1)

Claim 2. $b_1(\Gamma; \theta_2) = \frac{1}{\phi(n)} \sum_{d|n} \mu(\frac{n}{d}) b_1(\Gamma_{2d})$ *where ϕ is Euler's ϕ-function and μ is the Möbius function.*

Proof. Fix a divisor $d \geq 1$ of n and observe that the Γ-module $\mathbb{C}[\Gamma/\Gamma_{2d}] \cong \mathbb{C}[D_d]$ splits as a sum

$$\mathbb{C}[\Gamma/\Gamma_{2d}] = \mathbb{C}_1 \oplus \mathbb{C}_{\theta_1} \oplus \bigoplus_{r=1}^{d} \mathbb{C}^2_{\delta_r} \tag{A.1.2}$$

where $\delta_r = \Theta \circ \delta_r^0 \circ \rho_d : \Gamma \to GL_2(\mathbb{C})$ with $\delta_r^0 : D_d \to D_d$ given by

$$\delta_r^0(\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \delta_r^0(\pm \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}) = \pm \begin{pmatrix} \zeta^{\frac{rn}{d}} & 0 \\ 0 & \zeta^{-\frac{rn}{d}} \end{pmatrix}.$$

(see §5.3 of [Se] for example). If r divides d, then $\delta_r^0 \circ \rho_d = \rho_{\frac{d}{r}}$. Moreover, if r_1 and r_2 have the same order in $\mathbb{Z}/d$, there is a Γ-module isomorphism between $\mathbb{C}_{\delta_{r_1}}^2$ and $\mathbb{C}_{\delta_{r_2}}^2$. (For under this condition there is an automorphism ψ of the group $\theta_{r_1}(\Gamma)$ such that $\delta_{r_2} = \psi \circ \delta_{r_1}$.) Combining these observations with (A.1.2) and Claim 1 shows that

$$b_1(\Gamma_{2d}) - b_1(\Gamma_2) = \sum_{e|d} \phi(e) b_1(\Gamma; \Theta \circ \rho_e).$$

This formula holds for each d which divides n, so the Möbius inversion formula (see §16.4 of [HW] for example) yields

$$b_1(\Gamma; \theta_2) = b_1(\Gamma; \Theta \circ \rho_n) = \frac{1}{n} \sum_{d|n} \mu(\frac{n}{d})(b_1(\Gamma_{2d}) - b_1(\Gamma_2)) = \frac{1}{\phi(n)} \sum_{d|n} \mu(\frac{n}{d}) b_1(\Gamma_{2d}),$$

as $n > 1$. $\square$(of Claim 2)

This completes the proof of Lemma A.1. $\square$

A.2. **A criterion for the smoothness of dihedral characters of knot groups.** For a knot K in a $\mathbb{Z}$-homology 3-sphere W we use $\widehat{W}_2(K) \to W$ to denote the 2-fold cover of W branched over K. It is well-known that any irreducible representation of the fundamental group of the exterior of K with values in $\mathcal{N}$ has image D_n for some odd n. Moreover, Klassen observed that if Δ_K is the Alexander polynomial of K, there are exactly $\frac{|\Delta_K(-1)|-1}{2}$ characters of such representations (compare Theorem 10 of [Kl]).

A *simple point* of a complex affine algebraic set V is a point of V which is contained in a unique algebraic component of V and is a smooth point of that component.

Lemma A.2. *Let M be the exterior of a knot K in a $\mathbb{Z}$-homology 3-sphere W. Suppose that $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ has image D_n where $n > 1$. Then the associated $2n$-fold cover $\widetilde{M}_\rho \to M$ extends to a branched cover $p : \widehat{W}_\rho(K) \to W$, branched over K. Moreover*
(1) p factors through an n-fold cyclic (unbranched) cover $\widehat{W}_\rho(K) \to \widehat{W}_2(K)$ and the 2-fold branched cyclic cover $\widehat{W}_2(K) \to W$; and
(2) if $b_1(\widehat{W}_\rho(K)) = 0$, then $H^1(M; Ad\rho) \cong \mathbb{C}$ and χ_ρ is a simple point of $X_{PSL_2}(M)$.

Proof. (1) Fix meridional and longitudinal classes μ and λ in $\pi_1(\partial M) \subset \pi_1(M)$. Denote by $\widetilde{M}_2 \to M$ the 2-fold cover of M. Since W is a $\mathbb{Z}$-homology 3-sphere we have $b_1(M) = b_1(\widetilde{M}_2) = 1$.

The subgroup $\rho^{-1}(\mathcal{D})$ has index 2 in $\pi_1(M)$ and so equals $\pi_1(\widetilde{M}_2)$. Hence $\rho|\pi_1(\widetilde{M}_2)$ has image $D_n \cap \mathcal{D} \cong \mathbb{Z}/n$. Since $\pi_1(M)$ is generated by μ and $\pi_1(\widetilde{M}_2)$ we see that $\rho(\mu) \in \mathcal{N} \setminus \mathcal{D}$ and therefore has order 2. Further, since λ is a double commutator, $\pi_1(\partial \widetilde{M}_2) \subset \text{kernel}(\rho)$. In particular if $\widetilde{M}_\rho \to \widetilde{M}_2$ is the regular cover associated to $\rho|\pi_1(\widetilde{M}_2)$, then $|\partial \widetilde{M}_\rho| = n$.

Since $\rho(\mu^2) = \pm I$, $\rho|\pi_1(\widetilde{M}_2)$ factors through $\pi_1(\widehat{W}_2(K))$ and defines an n-fold cyclic cover $\widehat{W}_\rho(K) \to \widehat{W}_2(K)$ which composes with $\widehat{W}_2(K) \to W$ to produce the desired cover of W branched over K. It is clear that $\widehat{W}_\rho(K) \to \widehat{W}_2(K)$ is obtained from $\widetilde{M}_\rho \to \widetilde{M}_2$ by equivariant Dehn filling.

(2) Suppose now that $b_1(\widehat{W}_\rho(K)) = 0$. For each $d \geq 1$ which divides n, let $\rho_d = \theta_{n,d} \circ \rho$ and $\widetilde{M}_{2d} \to M$ the associated cover. The second paragraph of the proof of (1) shows that $|\partial \widetilde{M}_{2d}| = d$ and since each boundary component is a torus, $b_1(\widetilde{M}_{2d}) \geq d$. On the other hand, the third paragraph shows that there is a Dehn filling of $\widetilde{M}_{2d}$ which yields $\widehat{W}_{\rho_d}(K)$. Now by construction, $\widehat{W}_{\rho_d}(K)$ is covered by $\widehat{W}_\rho(K)$ and therefore $b_1(\widehat{W}_{\rho_d}(K)) \leq b_1(\widehat{W}_\rho(K)) = 0$. Thus $d \leq b_1(\widetilde{M}_{2d}) \leq b_1(\widehat{W}_{\rho_d}(K)) + |\partial \widetilde{M}_{2d}| \leq d$. Plugging $b_1(\widetilde{M}_{2d}) = d$ into the conclusion of Lemma A.1 shows that

$$b_1(M; Ad\rho) = b_1(\widetilde{M}_2) - b_1(M) + \frac{1}{\phi(n)} \sum_{d|n} \mu(\frac{n}{d})d = \frac{1}{\phi(n)} \sum_{d|n} \mu(\frac{n}{d})d.$$

It is well-known that for $n > 1$ we have $\frac{1}{\phi(n)} \sum_{d|n} \mu(\frac{n}{d})d = 1$ (see Identity 16.3.1 of [HW] for example) and thus, $b_1(M; Ad\rho) = 1$. Theorem 3 of [BZ2] now shows that χ_ρ is a simple point of $X_{PSL_2}(M)$. This completes the proof. □

Corollary A.3. *If K is a two-bridge knot with exterior M and $\rho \in R_{PSL_2}(M)$ has image D_n where $n > 1$, then $H^1(M; Ad\rho) \cong \mathbb{C}$ and χ_ρ is a simple point of $X_{PSL_2}(M)$.*

Proof. Since the 2-fold branched cyclic cover of $W = S^3$ over K is a lens space, Proposition A.2(1) implies that $b_1(\widehat{W}_\rho(K)) = 0$. The desired conclusion now follows from conclusion (2) of Proposition A.2. □

A.3. **Proof of Proposition 3.9.** Let $p \geq 1, q$ be relatively prime integers where p is odd. We observed in §A.2 that given a knot $K \subset S^3$ with exterior M, the image of any homomorphism $\rho : \pi_1(M) \to \mathcal{N}$ with non-abelian image is D_n for some odd $n \geq 3$. Moreover, the number of characters of such representations is exactly $\frac{|\Delta_K(-1)|-1}{2} = \frac{|H_1(\hat{S}^3(K))|-1}{2} < \infty$. For $K = k_{p/q}$ we have $|\Delta_K(-1)| = |H_1(L(p,q))| = p$. This discussion yields our next lemma.

Lemma A.4. *Every non-trivial curve in the $PSL_2(\mathbb{C})$ character variety of the exterior of a knot in S^3 is strictly non-trivial.* □

Consider a non-trivial curve $X_0 \subset X_{PSL_2}(M_{p/q})$. It is shown in [HT] that the meridional slope μ of $k_{p/q}$ is not a boundary slope. Since $M_{p/q}$ is small, Propositions 2.7 shows that for each ideal point x of $\widetilde{X}_0$, $\Pi_x(\tilde{f}_\mu) > 0$. Thus $\Pi_x(\tilde{f}_{\mu^2}) = \Pi_x(\tilde{f}_\mu(\tilde{f}_\mu + 4)) > 0$ as well. Then $Z_x(\tilde{f}_\mu) = Z_x(\tilde{f}_{\mu^2}) = 0$ and so by Identity (2.4.1) we have

$$0 < d_M(\mu) = d_M(\mu^2) - d_M(\mu) = \sum_{\substack{non-trivial \\ X_0}} \sum_{x \in X_0^\nu} (Z_x(\tilde{f}_{\mu^2}) - Z_x(\tilde{f}_\mu)). \tag{A.3.1}$$

It follows from Proposition 1.1.3 of [CGLS] that $Z_x(\tilde{f}_\mu) \leq Z_x(\tilde{f}_{\mu^2})$ for each $x \in X_0^\nu$. Moreover, Proposition 1.5.4 of that paper shows that if $Z_x(\tilde{f}_\mu) < Z_x(\tilde{f}_{\mu^2})$ for some $x \in X_0^\nu$ and $\nu(x) = \chi_\rho$, then $\rho(\mu^2) = \pm I$. In particular, the restriction of ρ to the fundamental group of the 2-fold cover of $M_{p/q}$ factors through the fundamental group of $L(p,q)$, the 2-fold cover of S^3 branched over $k_{p/q}$. Thus the image of ρ is finite and as we can suppose that it is not cyclic (Proposition 1.5.5 of [CGLS]), it must be a non-abelian dihedral group.

Now $\rho(\mu)$ is neither parabolic nor $\pm I$, so that $Z_x(\tilde{f}_\mu(x)) = 0$. We know that χ_ρ is a simple point of $X_{PSL_2}(M_{p/q})$ by Corollary A.3 and we claim (see Lemma A.5 below) that $Z_x(\tilde{f}_{\mu^2}) = 1$. Note that these two facts and Identity (A.3.1) show that $d_M(\mu)$ equals the number of irreducible, dihedral characters which lie on some non-trivial curve in $X_{PSL_2}(M_{p/q})$. But by a result of Thurston (see [CS, Proposition 3.2.1]), every such character lies on such a curve, and since there are $\frac{p-1}{2}$ irreducible, dihedral characters of $\pi_1(M_{p/q})$, we have $d_M(\mu) = \frac{p-1}{2}$, which is what we set out to prove. $\qquad\square$

Lemma A.5. *Let χ_ρ be an irreducible, dihedral character of $\pi_1(M_{p/q})$, X_0 the unique curve in $X_{PSL_2}(M_{p/q})$ which contains it, and x the unique point of X_0^ν such that $\nu(x) = \chi_\rho$. Then $Z_x(\tilde{f}_{\mu^2}) = 1$.*

Proof. The proof that $Z_x(\tilde{f}_{\mu^2}) = 1$ is essentially identical to the proof of Theorem 2.1 (2) of [BB], though with with some slight modifications as 2μ is not a primitive class in $H_1(\partial M)$. These modifications are simple and we describe them next.

Let $M_{p/q}(2\mu)$ be the space obtained by attaching a solid torus to $M_{p/q}$ by a covering map which maps $S^1 \times \{1\}$ homeomorphically to $\lambda_{M_{p/q}}$ and is a 2-fold cover of $\{*\} \times \partial D^2$ to μ. Note that there is a unique 2-fold cover of $M_{p/q}(2\mu)$ and its total space is $L(p,q)$, the 2-fold cover of S^3 branched over $k_{p/q}$. Note as well that ρ factors through $\pi_1(M_{p/q}(2\mu))$. Lemma A.1 applied to this situation shows that $H^1(M_{p/q}(2\mu)) = 0$. The proof of Lemma 1.8 of [BB] shows that if $u \in Z^1(\pi_1(M_{p/q}); Ad\rho)$ represents a non-zero class in $H^1(\pi_1(M_{p/q}); Ad\rho)$ then $u(\mu^2) \neq 0$. The calculation $Z_x(\tilde{f}_{\mu^2}) = 1$ now follows in a similar fashion to the calculation in the proof of Theorem 2.1 (2) of [BB]. $\qquad\square$

APPENDIX B. PERIPHERAL VALUES OF HOMOMORPHISMS OF TWIST KNOT GROUPS

In this appendix we show that the trefoil knot is the only twist knot whose group admits a homomorphism onto an infinite triangle group such that the image of the peripheral subgroup is finite.

After Hoste and Shanahan, we identify the n-twist knot K_n with the knot $J(2, 2n)$ of [HS]. When $n = -1, 0, 1$, K_n is the figure 8 knot, the trivial knot, and the trefoil knot respectively.

The fundamental group of the exterior M_n of K_n admits a presentation

$$\pi_1(M_n) = \langle a, b : a(ab^{-1}a^{-1}b)^n = (ab^{-1}a^{-1}b)^n b \rangle$$

where a and b are meridional classes (cf. Proposition 1 of [HS]). Set $w = ab^{-1}a^{-1}b$ so that the relation becomes $aw^n = w^n b$.

Proposition B.1. *If there is a surjective homomorphism $\rho : \pi_1(M_n) \to \Delta(p, q, r)$ where (p, q, r) is a Euclidean triple, then $n = 1$ and hence K_n is the trefoil knot. Further, $(p, q, r) = (2, 3, 6)$ up to permutation.*

Proof. First we observe that any two elements of $\Delta(p, q, r)$ which are of infinite order commute since they correspond to translations under the natural embedding $\Delta(p, q, r) \to \text{Isom}_+(\mathbb{E}^2)$.

Thus $\rho(a)$ and $\rho(b) = \rho(w)^{-n}\rho(a)\rho(w)^n$ must be elliptic. Since $H_1(\Delta(p,q,r))$ is necessarily cyclic, we have $(p,q,r) = (2,3,6)$ up to permutation. Further, $\rho(a)$ generates $H_1(\Delta(2,3,6)) \cong \mathbb{Z}/6$ so that $\rho(a)$ and $\rho(b)$ have order 6. Then up to replacing ρ by a conjugate representation we may suppose that

$$\rho(a) = xy \in \langle x, y : x^2 = y^3 = (xy)^6 = 1 \rangle = \Delta(2,3,6).$$

We claim that up to conjugating ρ by a power of xy, we can suppose that $\rho(b) = yx$. To see this, fix a tessellation $\mathcal{T}$ of $\mathbb{E}^2$ by triangles with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$, and identify $\Delta(2,3,6) \subset \mathrm{Isom}(\mathbb{E}^2)$ with the set of orientation preserving symmetries of $\mathcal{T}$. the elements xy, yx are conjugate elements of order 6 in $\Delta(2,3,6)$ and form a generating set. Moreover, the tessellation $\mathcal{T}$ can be described as follows. Let A be the fixed point of xy, $B \neq A$ that of yx, and let C be the midpoint of $[A,B]$. Denote by L the line through C which is orthogonal to $[A,B]$ and let $T(A,D,E)$ be the triangle with vertices A,D,E where $D,E \in L$ are equidistant to C and the angles at A,D,E are $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6}$ respectively. The triangle $T(A,C,D)$ is a face of $\mathcal{T}$ and so the tessellation is its orbit under the action of $\Delta(2,3,6)$. Moreover, $T(A,D,E)$ is the union of the two adjacent faces $T(A,C,D)$ and $T(A,C,E)$ and so is a fundamental domain for $\Delta(2,3,6)$. Since $\mathbb{E}^2$ admits self-similarities of arbitrary scale factor, it is clear that any two elements of order 6 in $\mathrm{Isom}(\mathbb{E}^2)$ with distinct fixed points generate a subgroup isomorphic to $\Delta(2,3,6)$ with invariant tessellation and fundamental domain constructed as above. In particular this is the case for $\rho(a), \rho(b)$. Let $\mathcal{T}'$ and $T(A,C',E')$ be the associated tessellation and fundamental domain. Since $\rho(a), \rho(b)$ generate $\Delta(2,3,6)$, $\mathcal{T} = \mathcal{T}'$ and so $T(A,C,E)$ can be obtained from $T(A,C',E')$ by a rotation about A of angle $\frac{2\pi j}{3}$ for some integer j. This rotation is given by $(xy)^{\epsilon j}, \epsilon \in \{\pm 1\}$, so if we replace ρ by $(xy)^{-\epsilon j}\rho(xy)^{\epsilon j}$, the new fixed point of $\rho(b)$ is B. Since $\rho(a)$ and $\rho(b)$ are conjugate, it follows that $\rho(b) = yx$.

With these calculations in hand, we see that $v := [a, b^{-1}] = x(yx)^3$ is a product of two elements of order 2 with distinct fixed points. Thus v is a translation which leaves $\mathcal{T}$ invariant. Since $\rho(b) = v^{-n}\rho(a)v^n$, B, the fixed point of $\rho(b)$, equals $v^{-n}(A)$. But examination of $\mathcal{T}$ shows the only way this is possible is for $n = 1$. $\qquad\square$

Proposition B.2. *If there is a surjective homomorphism $\rho : \pi_1(M_n) \to \Delta(p,q,r) \subset PSL_2(\mathbb{R})$ such that (p,q,r) is a hyperbolic triple and $\rho(a)$ is elliptic, then $n = 1$, so that K_n is the trefoil knot. Further, $(p,q,r) = (2,3,r)$, $r \geq 7$ up to permutation.*

Proof. Suppose that there is a surjective homomorphism $\rho : \pi_1(M_n) \to \Delta(p,q,r) \subset PSL_2(\mathbb{R})$ such that (p,q,r) is a hyperbolic triple and $\rho(a)$ is elliptic. Clearly $n \neq 0$ and $\Delta(p,q,r)$ is generated by two conjugate elliptics. Theorem 2.3 of [Kn] implies that up to permuting p,q,r, one of the following two scenarios arises.

(a): $(p,q,r) = (2,q,r)$ where $\rho(a)$ has order q and $r \geq 3$ is odd. Further, there is an integer s relatively prime to q such that in the standard presentation $\Delta(2,q,r) = \langle x, y, z : x^2, y^q, z^r \rangle$ we have $\rho(a) = y^s, \rho(b) = xy^s x^{-1}$.

(b): $(p,q,r) = (2,3,r)$ where where $\rho(a)$ has odd order $r \geq 7$. Further, there is an integer s relatively prime to r such that in the standard presentation $\Delta(2,3,r) = \langle x, y, z : x^2, y^3, z^r \rangle$ we have $\rho(a) = z^s, \rho(b) = yxy^{-1}z^s yxy^{-1}$.

Set $v = \rho(w)$ so that

$$v = [\rho(a), \rho(b^{-1})] = \begin{cases} (y^s x y^{-s})((xy^{-s})x(xy^{-s})^{-1}) & \text{in scenario (a)} \\ ((z^s y)x(z^s y)^{-1})((yxy^{-1}z^{-s}y)x(yxy^{-1}z^{-s}y)^{-1}) & \text{in scenario (b).} \end{cases}$$

In either case $v = uu'$ where u, u' are of order 2. Denote by R, R' the fixed points of u, u' and observe that if $R = R'$, then $u = u'$ and therefore $v = uu' = 1$. Then $\rho(b) = \rho(w^{-n}aw^n) = v^{-n}\rho(a)v^n = \rho(a)$, which is impossible. Thus $R \neq R'$ and it is easy to see that if γ denotes the geodesic in $\mathbb{H}^2$ which contains both R and R', then v is a hyperbolic element of $PSL_2(\mathbb{R})$ with invariant geodesic γ and translation length $2d_{\mathbb{H}^2}(R, R')$. The proof of the proposition is similar in the two possible scenarios. We analyse them separately.

Assume first that we are in scenario (a). There is a fundamental domain for $\Delta(2, q, r)$ in $\mathbb{H}^2$ which is a geodesic triangle $T = T(A, B, C)$ having vertices $A = \text{Fix}(y), B = \text{Fix}(xyx^{-1}), C = \text{Fix}(z)$ and the midpoint P of $[A, B]$ is $\text{Fix}(x)$ (so $x(A) = B$). The angles of T at A, B, C are $\frac{\pi}{q}, \frac{\pi}{q}, \frac{2\pi}{r}$ respectively. The hyperbolicity of v implies that for $l \neq 0$, $v^l(C), v^l(P) \notin T$ and so if $T \cap v^l(T) \neq \emptyset$, then up to replacing l by its negative we have $T \cap v^l(T) = \{A\}$ and $v^l(B) = A$.

Since $xy^s x = \rho(b) = \rho(w^{-n}aw^n) = v^{-n}y^s v^n$, there is an integer m such that $v^n = y^m x$. Then $v^n(B) = y^m x(B) = A$ so that $v^n(T) \cap T = \{A\}$. It now follows from the previous paragraph that if for some $l \neq 0$ we have $T \cap v^l(T) \neq \emptyset$, then up to replacing l by its negative we have $v^l(B) = A = v^n(B)$. Thus $l = \pm n$ and so for $d \neq e$,

$$v^{dn}(T) \cap v^{en}(T) = \begin{cases} v^{dn}(B) & e = d - 1 \\ v^{(d+1)n}(B) & e = d + 1 \\ \emptyset & e \neq d \pm 1 \end{cases}$$

and the reader will verify that $\Gamma_0 = \cup_d v^{dn}(T)$ is an infinite chain of geodesic triangles which is closed, properly embedded, and separating in $\mathbb{H}^2$. It follows that $n = \pm 1$ as otherwise $v(\Gamma_0) \cap \Gamma_0 = \emptyset$ and so the side of Γ_0 in $\mathbb{H}^2$ containing $v(\Gamma_0)$ is invariant under v. Thus $v^l(\Gamma_0) \cap \Gamma_0 = \emptyset$ for all $l > 0$, contrary to the fact that $v^{|n|}(\Gamma_0) = \Gamma_0$.

If $n = -1$, then $y^m x = v^{-1}$ so that $xy^m x^{-1} = (y^{-s}xy^s)x(y^s xy^{-s})$. In particular, $(y^{-s}xy^s)x(y^s xy^{-s})$ fixes $x(A) = B$. But $(y^{-s}xy^s)x(y^s xy^{-s})$ is a product of three conjugates of x with fixed points $P, y^s(P), y^{-s}(P)$ and the reader will verify that this is impossible because such a configuration of order 2 elliptics cannot fix B. (Alternately, we refer the reader to the proof of Theorem 11.5.2 of [Beardon] where the fixed points of a product of three order 2 elliptics are analysed. The analysis implies that if $(y^{-s}xy^s)x(y^s xy^{-s})$ has a fixed point, then this fixed point and B lie on opposite sides of the geodesic through $y^s(P)$ and $y^{-s}(P)$.)

Finally, if $n = 1$, K_n is the trefoil knot and we have $y^m x = v = y^s xy^{-s}xy^{-s}xy^s x$ so that $y^{m-3s} = (xy^{-s})^3$. If $(xy^{-s})^3 \neq 1$ then xy^{-s} fixes A, which is impossible. Thus $y^{m-3s} = (xy^{-s})^3 = 1$. We will show that $r = 3$ to complete this part of the proof. Let D be the fixed point of xy^{-s}. Since $xy^{-s}(A) = B$, D lies on the perpendicular L to $[A, B]$ through P. Now $D \neq P$ as otherwise $y^s = 1$. On the other hand, C and $x(C)$ are the closest points to P of the given tessellation of $\mathbb{H}^2$ which lie on L. Further, since $z(B) = (xz^{-1})(B) = A$ we see that $\frac{2\pi}{3}$, the absolute value of the angle of rotation of xy^{-s} is bounded above by that of z at C or $xz^{-1}x$

at $x(C)$, which is $\frac{2\pi}{r}$, with equality if and only if $D \in \{C, x(C)\}$. Thus $\frac{2\pi}{3} \le \frac{2\pi}{r}$ which implies that $r = 3$ and we are done.

Now suppose that we are in scenario (b) and consider the geodesic triangle T_0 in $\mathbb{H}^2$ with vertices A, B, C such that $A = \text{Fix}(x), B = \text{Fix}(y), C = \text{Fix}(z)$. The angles of T_0 at A, B, C are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{r}$ respectively where $r \ge 7$ is odd. Recall that

$$v = [(z^s y)x(z^s y)^{-1}][(yxy^{-1}z^{-s}y)x(yxy^{-1}z^{-s}y)^{-1}]$$

is a product of two conjugates of x and observe that that form of $\rho(b)$ given in scenario (b) implies that

$$v^n = z^m yxy^{-1}$$

for some integer m.

Consider the geodesic triangle $T = T(C, D, E)$ containing T_0 with vertices $C, D = (yxy^{-1})(C) = \text{Fix}((yxy^{-1})z(yxy^{-1})), E = y(C) = \text{Fix}(yzy^{-1})$ of angles $\frac{\pi}{r}, \frac{\pi}{r}, \frac{4\pi}{r}$ respectively.

Claim B.3. *For any integer $l \ne 0$, $v^l(T) \cap T$ is either empty or one of the vertices C, D.*

Proof. (of Claim B.3) First we show that $v^l(\text{int}(T)) \cap \text{int}(T) = \emptyset$. Decompose T as $T_1 \cup T_2 \cup T_3$ where $T_1 = T(B, C, E), T_2 = T(B, E, B')$ where $B' = (yxy^{-1})(B)$, and $T_3 = T(B', D, E)$ and note that each of T_1, T_2, T_3 is a fundamental domain for the action of $\Delta(2, 3, r)$ on $\mathbb{H}^2$. Since T_1 is sent to the geodesic triangle T_3 by the elliptic element $(yxy^{-1})(xzx^{-1})(yxy^{-1})^{-1}$, $v^l(\text{int}(T_1)) \cap \text{int}(T_3) = \emptyset$. Similarly $v^l(\text{int}(T_1)) \cap \text{int}(T_2) = \emptyset$ so that $v^l(\text{int}(T_1)) \cap \text{int}(T) = \emptyset$. In the same way we see that $v^l(\text{int}(T_2)) \cap \text{int}(T) = v^l(\text{int}(T_3)) \cap \text{int}(T) = \emptyset$, which is what we needed to prove.

Second we claim that $v^l(E) \notin T$. If this is false we have $v^l(E) \in \{C, D\}$ (i.e. the only valency $2r$ vertices in T are C, D, E and $v^l(E) \ne E$ as it is hyperbolic), say $v^l(E) = C$. Then the axis of v is perpendicular to the perpendicular bisector L of $[C, E]$. On the other hand, $v^n(D) = z^m yxy^{-1}(D) = C$, so the axis of v is also perpendicular to the perpendicular bisector L' of either $[C, D]$. But this is impossible since $L \cap L' = \{y(E)\} \ne \emptyset$. Hence $v^l(E) \ne C$ and a similar argument shows it does not equal D.

Third we observe that $v^l(T) \cap T$ contains no edge of the tessellation. By the first paragraph, such an edge would have to lie in ∂T and by the second it could not contain E. Since v^l preserves the combinatorial type of the vertices it is now easy to use the method of the first paragraph to obtain a contradiction.

These observations imply that if $v^l(T) \cap T$ is non-empty, then it is a vertex of T. This proves Claim B.3 $\qquad\square$

Since $v^n(D) = C$, the claim implies that if $v^l(T) \cap T \ne \emptyset$, then after possibly replacing l by its negative we have $v^l(D) = C$. Thus $l = \pm n$ and the intersection is C if $l = n$ and D otherwise. Arguing as in scenario (a) we have $|n| = 1$. If $n = -1$, we have $z^m yxy^{-1} = v^{-1}$ and therefore

$$(yxy^{-1})z^m(yxy^{-1}) = [(z^{-s}y)x(z^{-s}y)^{-1}](yxy^{-1})[(z^s y)x(z^s y)^{-1}].$$

Hence D is fixed by the product $[(z^{-s}y)x(z^{-s}y)^{-1}](yxy^{-1})[(z^sy)x(z^sy)^{-1}]$ of three conjugates of x with fixed points $y(P), (z^sy)(P), (z^{-s}y)(P)$. As in the analysis of the case $n = -1$ in scenario (a), it can be verified that this cannot occur. Thus $n = 1$ and K_n is the trefoil knot. $\qquad\square$

Appendix C. Bending

We collect some basic information on the bending construction here. This material is well-known to experts but we haven't found it written down in the literature.

Let Γ be a finitely generated group which splits over a subgroup Γ_0 and $\rho : \Gamma \to PSL_2(\mathbb{C})$ a homomorphism such that $\rho(\Gamma_0)$ is abelian but not isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Under this condition we can perform a deformation operation on χ_ρ known as bending. The details of the construction depend on whether the splitting is a free product with amalgamation or an HNN extension and are dealt with in §C.1 and §C.2 respectively.

Recall the subgroups $\mathcal{D}, \mathcal{N}$ of $PSL_2(\mathbb{C})$ defined in §2.1 and set

$$\mathcal{P}_+ = \{\pm \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C}\} \subset \mathcal{T}_+ = \{\pm \begin{pmatrix} z & w \\ 0 & z^{-1} \end{pmatrix} \mid z, w \in \mathbb{C}, z \neq 0\}$$

Under the natural action of $PSL_2(\mathbb{C})$ on $\mathbb{C}P^1$, the fixed point sets of $\mathcal{T}_+$ and $\mathcal{P}_+$ coincide and consist of a single line L_+. That of $\mathcal{D}$ consists of two lines $\{L_+, L_-\}$.

The centraliser of a subset E of $PSL_2(\mathbb{C})$ wil be denoted by $Z_{PSL_2}(E)$ and the component of the identity of $Z_{PSL_2}(E)$ will be denoted $Z^0_{PSL_2}(E)$. For $\pm I \neq A \in PSL_2(\mathbb{C})$ we have

$$Z^0_{PSL_2}(A) \text{ is conjugate to } \begin{cases} \mathcal{D} & \text{if } A \text{ is diagonalisable} \\ \mathcal{P}_+ & \text{if } A \text{ is parabolic} \end{cases}$$

Thus when $E \neq \{\pm I\}$, $Z^0_{PSL_2}(E)$ is abelian and reducible. For a group π and representation $\rho \in R_{PSL_2}(\pi)$, we use $Z_{PSL_2}(\rho), Z^0_{PSL_2}(\rho)$ to denote, respectively, the centraliser and the component of the identity of the centraliser of $\rho(\pi)$.

C.1. $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$. Fix $\rho : \Gamma \to PSL_2(\mathbb{C})$ a homomorphism such that $\rho(\Gamma_0)$ is abelian but not isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Denote by ρ_j the restriction of ρ to Γ_j. For each $S \in Z_{SL_2}(\rho_0)$ define $\rho_S : \Gamma \to PSL_2(\mathbb{C})$ to be the homomorphism determined by the push-out diagram

We say that the character χ_{ρ_S} is obtained by *bending* χ_ρ by S. The *bending function* is given by

$$\beta_\rho : Z^0_{PSL_2}(\rho_0) \to X_{PSL_2}(\Gamma), \ S \mapsto \chi_{\rho_S}.$$

We say that ρ can be *bent non-trivially* if β_ρ is non-constant. Our next result determines necessary and sufficient conditions for this to occur.

Lemma C.1. *Suppose that $\rho \in R_{PSL_2}(\Gamma$ is such that $\rho(\Gamma_0)$ is abelian but not isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. The bending function $\beta_\rho : Z^0_{PSL_2}(\rho_0) \to X_{PSL_2}(\Gamma)$ is constant if and only if one of the following two situations arises:*
(a) $\rho_0(\Gamma_0) = \{\pm I\}$ and either $\rho_1(\Gamma_1) = \{\pm I\}$ or $\rho_2(\Gamma_2) = \{\pm I\}$.
(b) $\rho_0(\Gamma_0) \neq \{\pm I\}$ and either $\rho_1(\Gamma_1)$ is abelian and reducible, or $\rho_2(\Gamma_2)$ is abelian and reducible, or ρ is reducible.

Proof. Suppose that the correspondence $S \mapsto \chi_{\rho_S}$ is constant. We leave the justification of the following claim to the reader.

Claim C.2. *Let $A, B, C \in SL_2(\mathbb{C})$. Then $\text{trace}(ASBS^{-1}) = \text{trace}(AB)$ for all $S \in Z_{SL_2}(C)$ if and only if one of the following two situations arise:*
(a) $C = \pm I$ and either $A = \pm I$ or $B = \pm I$.
(b) $C \neq \pm I$ and either $[A, C] = I$ or $[B, C] = I$ or A, B and C have a common eigenvector. □

Set $G_j = \text{image}(\rho_j)$ for $j = 0, 1, 2$. The claim shows that if $G_0 = \{\pm I\}$, then either $G_1 = \{\pm I\}$ or $G_2 = \{\pm I\}$, and we are done. Assume then that $G_0 \neq \{\pm I\}$ and that both G_1 and G_2 are either irreducible or non-abelian subgroups of $PSL_2(\mathbb{C})$. It follows that neither $G_1 \subset Z^0_{PSL_2}(G_0)$ nor $G_2 \subset Z^0_{PSL_2}(G_0)$. We will show that ρ is reducible.

Fix $C \in G_0 \setminus \{\pm I\}$ and observe that our hypotheses show

$$Z^0_{PSL_2}(C) = Z^0_{PSL_2}(G_0) = \begin{cases} \mathcal{D} & \text{if } G_0 \subset \mathcal{D} \\ \mathcal{P}_+ & \text{if } G_0 \subset \mathcal{P}_+ \end{cases}$$

If there is some $A_0 \in G_1 \setminus Z^0_{PSL_2}(G_0)$, then Claim C.2 implies that for each $B \in G_2$, either $B \in Z^0_{PSL_2}(G_0) \subset \mathcal{T}_+$ or A_0, B and C have a common fixed point in $\mathbb{C}P^1$. It follows that each element of G_2 fixes at least one of L_+ and L_- and it is simple to deduce from this that G_2 fixes one of these lines. A similar argument shows that G_1 fixes one of them as well. If G_1 and G_2 have a common fixed point, then ρ is reducible, so suppose that they do not. One of them, say G_1, fixes L_+ and not L_-, while G_2 fixes L_- and not L_+. Since $G_0 \subset G_1 \cap G_2$, it fixes both L_+ and L_- and therefore we must have $G_0 \subset \mathcal{D}$. By choice, $A_0 \in G_1 \setminus \mathcal{D}$ and so its fixed point set is L_+. On the other hand we have assumed that there is some $B_0 \in G_2 \setminus Z^0_{PSL_2}(G_0) = G_2 \setminus \mathcal{D}$. Its fixed point set is L_-. But this is impossible as Claim C.2 implies that A_0, B_0 and C have a common fixed point in $\mathbb{C}P^1$. Thus G_1 and G_2 do have a common fixed point, and so ρ is reducible.

Conversely if either $G_1 \subset Z^0_{PSL_2}(G_0)$, or $G_2 \subset Z^0_{PSL_2}(G_0)$, or $G_0 \neq \{\pm I\}$ and ρ is reducible, then Claim C.2 implies that the correspondence $S \mapsto \chi_{\rho_S}$, where $S \in Z^0_{PSL_2}(G_0)$, is constant. This completes the proof of the lemma. □

C.2. $\Gamma = (\Gamma_1)_{\Gamma_0}$. In this case there is an injective homomorphism $\varphi : \Gamma_0 \to \Gamma$ such that

$$\Gamma = \langle \Gamma_1, \mu : \mu \gamma \mu^{-1} = \varphi(\gamma), \gamma \in \Gamma_0 \rangle.$$

Set $\Gamma_0' = \varphi(\Gamma_0)$ and for $\rho \in R_{PSL_2}(\Gamma)$ we take ρ_1, ρ_0, ρ_0' to be its restriction to $\Gamma_1, \Gamma_0, \Gamma_0'$ respectively. The correspondence $\rho \in R_{PSL_2}(\Gamma) \mapsto (\rho_1, \rho(\mu)) \in R_{PSL_2}(\Gamma_1) \times PSL_2(\mathbb{C})$ determines an identification

$$R_{PSL_2}(\Gamma) = \{(\rho_1, A) \in R_{PSL_2}(\Gamma_1) \times PSL_2(\mathbb{C}) : A\rho_0(\gamma)A^{-1} = \rho_0'(\varphi(\gamma)) \text{ for all } \gamma \in \Gamma_0\}.$$

Note that $(\rho_1, A), (\rho_1, B) \in R_{PSL_2}(\Gamma)$ if and only if $B = AS$ for some $S \in Z_{PSL_2}(\rho(\Gamma_0))$. In particular, if $\rho(\Gamma_0)$ is abelian but not $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, we have a bending function

$$\beta_{(\rho_1, A)} : Z^0_{PSL_2}(\rho(\Gamma_0)) \to X_{PSL_2}(\Gamma_0), S \mapsto \chi_{(\rho, AS)}.$$

Lemma C.3. *Suppose that $(\rho_1, A) = \rho \in R_{PSL_2}(\Gamma)$ is a representation with $\rho(\Gamma_0)$ is abelian but not $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. The bending function $\beta_{(\rho_1, A)}$ is constant if and only if $\rho(\Gamma_0) \neq \{\pm I\}$ and, after a possible conjugation, one of the following two situations arises:*
(a) $\rho(\Gamma_1) \subset \mathcal{D}$ and $A = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
(b) $\rho(\Gamma_1) \subset \mathcal{T}_+$ and $A \in \mathcal{T}_+$.
In particular, ρ_1 is reducible and ρ is either reducible or conjugate into $\mathcal{N}$.

Proof. Let G_1, G_0, G_0' denote the images of ρ_1, ρ_0, ρ_0' respectively. After possibly replacing $\rho = (\rho_1, A)$ by a conjugate representation, we may suppose that either $G_0 = \{\pm I\}$, or $\{\pm I\} \neq G_0 \subset \mathcal{D}$, or $\{\pm I\} \neq G_0 \subset \mathcal{P}$. We consider these three cases separately.

Case 1. $G_0 = \{\pm I\}$.

Then $Z^0_{PSL_2}(\rho_0) = PSL_2(\mathbb{C})$ and so in general, $\beta_{(\rho_1, A)}(\mu) = \pm \text{trace}(A) \neq \pm \text{trace}(AS) = \beta_{(\rho_1, AS)}(\mu)$ for $S \in Z^0_{PSL_2}(\Gamma_0)$, $\beta_{(\rho_1, A)}$ is not constant.

Case 2. $\{\pm I\} \neq G_0 \subset \mathcal{D}$.

Then $Z^0_{PSL_2}(\rho_0) = \mathcal{D}$. If $\beta_{(\rho_1, A)}$ is constant, then $\pm \text{trace}(\rho(\gamma)A) = \beta_{(\rho_1, A)}(\gamma\mu) = \beta_{(\rho, AS)}(\gamma\mu) = \pm \text{trace}(\rho(\gamma)AS)$ for each $\gamma \in \Gamma_1$ and $S \in \mathcal{D}$. It follows that $\rho(\gamma)A \in \mathcal{N} \setminus \mathcal{D}$ for each $\gamma \in \Gamma_1$. Hence $A \in \mathcal{N} \setminus \mathcal{D}$ and $\rho(\Gamma_1) \subset \mathcal{D}$. After a further conjugation we may suppose that $A = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Conversely suppose that $A = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\rho(\Gamma_1) \subset \mathcal{D}$. Consider a word $w = \Pi_j \mu^{a_j} x_j$ where $a_j \in \mathbb{Z}$ and $x_j \in \Gamma_1$. Set $D_j = \rho(x_j) \in \mathcal{D}$. Then for any $S \in \mathcal{D}$ we have

$$(\rho, AS) : w \mapsto \Pi_j (AS)^{a_j} D_j = (AS)^{a_1 + a_2 + \dots + a_n} \Pi_j D_j^{(-1)^{a_{j+1} + a_{j+2} + \dots + a_n}}.$$

The trace of the right-hand side of this identity is independent of S, so that $\beta_{(\rho_1, A)}$ is constant.

Case 3. $\{\pm I\} \neq G_0 \subset \mathcal{P}$.

Then $Z_{PSL_2}^0(\rho_0) = \mathcal{P}$. If $\beta_{(\rho_1, A)}$ is constant, then $\mathrm{trace}(\rho(\gamma)A) = \beta_{(\rho_1, A)}(\gamma\mu) = \beta_{(\rho_1, AS)}(\gamma\mu) = \mathrm{trace}(\rho(\gamma)AS)$ for each $\gamma \in \Gamma_1$ and $S \in \mathcal{P}$. It follows that $\rho(\gamma)A \in \mathcal{T}_+$ for each $\gamma \in \Gamma_1$. Hence $A \in \mathcal{T}_+$ and $\rho(\Gamma_1) \subset \mathcal{T}_+$.

Conversely suppose that $A \in \mathcal{T}_+$ and $\rho(\Gamma_1) \subset \mathcal{T}_+$. Consider a word $w = \Pi_j \mu^{a_j} x_j$ where $a_j \in \mathbb{Z}$ and $x_j \in \Gamma_1$. Set $U_j = \rho(x_j) \in \mathcal{T}_+$. Then for any $S \in \mathcal{P}$ we have

$$(\rho, AS) : w \mapsto \Pi_j (AS)^{a_j} U_j.$$

The trace of the right-hand side of this identity is independent of S, so that $\beta_{(\rho_1, A)}$ is constant.

This completes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

References

[BMS]   G. Baumslag, J. Morgan and P. Shalen, *Generalized triangle groups*, Math. Proc. Camb. Phil. Soc. **102** (1987), 25–31.

[BB]   L. Ben Abdelghani and S. Boyer, *A calculation of the Culler-Shalen seminorms associated to small Seifert Dehn fillings*, Proc. London Math. Soc. **83** (2001), 235–256.

[BeP]   R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry*, Universitext, Springer, Berlin, 1992.

[BoP]   M. Boileau and J. Porti, *Geometrization of 3-orbifolds of cyclic type*, Astérisque Monograph, **272**, 2001.

[BWa]   M. Boileau and S.C. Wang, *Non-zero degree maps and surface bundles over S^1*, J. Diff. Geom. **43** (1996), 789–806.

[BolZ]   M. Boileau and B. Zimmermann, *Symmetries of nonelliptic Montesinos links*. Math. Ann. **277** (1987), no. 3, 563–584.

[BZ1]   S. Boyern and X. Zhang, *On Culler-Shalen seminorms and Dehn filling*, Ann. Math. 148 (1998), 737–801.

[BZ2]   ———, *On simple points of the character varieties of 3-manifolds*, Knots in Hellas '98 (Delphi), Ser. Knots Everything **24**, World Sci. Publishing, River Edge, NJ, 2000, 27–35.

[Bu]   G. Burde, *$SU(2)$-representation spaces for two-bridge knot groups*, Math. Ann. **288** (1990), 103–119.

[CCGLS]   D. Cooper, M. Culler, H. Gillet, D. D. Long and P. Shalen, *Plane curves associated to character varieties of 3-manifolds*, Invent. Math. **118** (1994), 47–84.

[CHK]   D. Cooper, C. Hodgson and S. Kerchkoff *Three-dimensional Orbifolds and Cone- Manifolds*, Math. Soc. of Japan Mem. **5**, Math. Soc. of Japan, Tokyo, 2000.

[CL]   D. Cooper and D. Long, *Roots of unity and the character variety of a knot complement*, J. Aus. Math. Soc. **55** (1993), 90–94.

[CGLS]   M. Culler, C. Gordon, J. Luecke and P. Shalen, *Dehn surgery on knots*, Ann. of Math., **125** (1987), 237–300.

[CS]   M. Culler and P. B. Shalen. *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. **117** (1983), 109–146.

[Dun]   N. Dunfield, *Cyclic surgery, degrees of maps of characters curves and volume rigidity for hyperbolic manifolds*, Invent. Math. **136** (1999), 623–657.

[Fra]   S. Francaviglia, *Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds*, Int. Math. Res. Not. **9** (2004), 425–459.

[GLM]   C.McA. Gordon, R. Litherland and K. Murasugi, *Signatures of covering links*, Can. J. Math. **33**, 1981, 331-394.

[GR1]   F. Gonzàlez-Acuña and A. Ramirez *Two-bridge knots with property Q*, Quart. J. Math. **52** (2001), 447–454.

[GR2]   ———, *Epimorphisms of knot groups onto free products*, Topology **42** (2003), 1205–1227.

[GW]   F. Gonzàlez-Acuña and W. Whitten *Imbeddings of three-manifold groups*, Mem. Amer. Math. Soc. **99**, 1992.

[Gro]   M. Gromov, *Metric structure for riemannian and non-riemannian spaces*, Progress in Math. **110**, Birkhäuser 1999.

[HW]   G. Hardy and G. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Oxford University Press, 1975.

[Ha]   R. Hartley, *Knots with free period*, Canad. J. Math. **33** (1981), 91–102

[HM]   R. Hartley and K. Murasugi. *Homology invariants*, Canad. J. Math. **30** (1978), 655–670.

[Hat]   A. Hatcher, *On the boundary curves of incompressible surfaces*, Pac. J. Math. **99** (1982), 373–377.

[HT]   A. Hatcher and W. Thurston. *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. **79** (1985), 225–246.

[HS]   J. Hoste and P.D. Shanahan, *A formula for the A-polynomial of twist knots*, J. Knot Theory and Ramifications **13** (2004), 193–209.

[In]   G. Indurskis, *Remplissages d'une composante du bord de l'extérieur de l'entrelacs de Whitehead*, UQAM doctoral dissertation, 2005.

[JM]   T. Jørgensen and J. Marsden, *Algebraic and geometric convergence of Kleinian groups*, Math. Scand. **66** (1990), 47–72.

[Kl]   E. Klassen, *Representations of knots groups in $SU(2)$*, Trans. Amer. Math. Soc. **326** (1991), 795–828.

[Kn]   A.W. Knapp, *Doubly generated Fuchsian groups*, Mich. Math. J. **15** (1968), 289–304.

[LM]   A. Lubotzky and A.R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58**, 1985.

[LR]   D. Long and A. Reid, *Commensurability and the character variety*, Math. Research Letters **6** (1999), 581–591.

[MT]   K. Matsuzaki and M. Taniguchi, *Hyperbolic manifolds and Kleinian groups*, Oxford Math. Monographs, Oxford, 1998.

[Mat]   T. Mattman, *The Culler-Shalen seminorms of the $(-2, 3, n)$ pretzel knot*, J. Knot Theory and Ramifications **11** (2002), 1251–1289.

[MS1]   J. Morgan and P. Shalen, *Valuations, trees, and degenerations of hyperbolic structures, I*, Ann. Math. **120** (1984), 401–476.

[MS2]   ———, *Valuations, trees, and degenerations of hyperbolic structures, III*, Ann. Math. **127** (1988), 467–519.

[Ne]   P. E. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Institute Lecture Notes, Narosa Publishing House, New Delhi, 1978.

[Oht]   T. Ohtsuki, *Ideal points and incompressible surfaces in two-bridge knot complements*, J. Math. Soc. Japan **46** (1994), 51–87.

[ORS]   T. Ohtsuki, R. Riley and M Sakuma, *Epimorphisms between 2-bridge link groups*, in The Zieshang Gedenkschrift, Geom. Topol. Monogr. **14** (2008), 417450.

[ReW]   A.W. Reid and S. Wang, *Non-Haken 3-manifolds are not large with respect to mappings of non-zero degree*, Comm. Anal. Geom, **7** (1999), 105–132.

[Ri]   Y.R. Riley, *Knots with parabolic property P*, Quart. J. Math. **35** (1974), 273–283.

[RoW]   Y. Rong and S.C. Wang, *The preimage of submanifolds*, Math. Proc. Camb. Phil. Soc. **112** (1992), 271–279.

[Sc]   P. Scott, *The geometries of 3-manifolds*, Bull. Lond. Math. Soc. **15** (1983), 401–487.

[Se]   J.-P. Serre, *Représentations linéaires des groupes finis*, Hermann, Paris, 3rd ed, 1978.

[Tak]   M. Takahashi, *Two-bridge knots have property P*, Mem. Amer. Math. Soc. **29**, 1981.

[Tan]   D. Tanguay, *Chirurgie finie et noeuds rationnels*, Ph. D. thesis, UQAM, 1995.

[Thu]   W. Thurston, *Topology and Geometry of 3-manifolds*, Princeton Lecture Notes, 1978.

[Ti]   J. Tits, *Free subgroups in linear groups*, J. of Alg. **20** (1970), 250–270.

[Wan1]   S.C. Wang, *The existence of maps of nonzero degree between aspherical 3-manifolds*, Math. Z. **208** (1991), 147–160.

[Wan2]   ———, *Non-zero degree one maps between 3-manifolds*, Proceedings of ICM 2002, Beijing, 457–468.

[Wal1]   F. Waldhausen, *Gruppen mit Zentrum un 3-dimensionale Mannigfaltigkeiten*, Topology **6** (1967), 505–517.

[Wed]   R. Weidmann, *On the rank of amalgamated products and product knot groups*, Math. Ann. **312** (1998), 761–771.

INSTITUT DE MATHÉTIQUES DE TOULOUSE, UMR 5219 ET INSTITUT UNIVERSITAIRE DE FRANCE, UNIVERSITÉ PAUL SABATIER 31062 TOULOUSE CEDEX 9, FRANCE.

E-mail address: `boileau@math.univ-toulouse.fr`

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DU QUÉBEC À MONTRÉAL, 201 AVENUE DU PRÉSIDENT-KENNEDY, MONTRÉAL, QC H2X 3Y7.

E-mail address: `boyer.steven@uqam.ca`

URL: `http://www.cirget.uqam.ca/boyer/boyer.html`