Frequency and Severity Modelling with Multifractal Processes: an Application to US tornadoes

Donatien Hainaut† Jean-Philippe Boucher‡

† *Rennes Business School - CREST, France.*
Email: donatien.hainaut@esc-rennes.fr

‡ *Département de mathématiques, UQAM. Québec, Canada.*
Email: boucher.jean-philippe@uqam.ca

This paper proposes a statistical model for claims related to climatic events that exhibit huge volatility both in frequency and intensity, such these caused by tornadoes hitting the US. To duplicate this volatility and the seasonality, we introduce a new claim arrival process modeled by a Poisson process of intensity equal to the product of a periodic function with a multifractal process. The amplitudes of claims are modeled in a similar way, with gamma random variables. We show that this method allows simulation of the peaks of damage. The two dimension multifractal model is also investigated. The work concludes with an analysis of the impact of the model on spreads of weather bonds related to claims caused by tornadoes.

Key Words: Multifractal process, Claims process, Poisson, Gamma, Dependence, CAT bonds.

1. Introduction

As mentioned in the work of Barrieu and Scaillet (2008), weather is not only an environmental issue but also a key economic factor. W. Daley in 1998, the former US commerce secretary stated that at least $1 trillion of the world economy is weather sensitive. There are mainly two solutions to hedge against economic losses caused by weather risk. The first one is to contract an insurance policy but it is not always a well suited solution as if could be for climatic events such storm or drought, or for events that exhibits a huge volatility in the frequency of occurences, such tornadoes. The second way to hedge weather risk is to purchase financial contracts depending on weather conditions. This type of contracts are most of the time tailor made transactions, traded on the OTC (other the counter market) market. Some basic weather derivatives (mainly designed for the US) are however also traded on the Chicago Mercantile Exchange (CME). The Weather Risk Management Association (WRMA) conducts every year a survey of the weather derivatives market. The value of trades in the year to March 2011 totalled $11.8 billion, nearly 20% up on the previous year, though far below the peak reached before the financial crisis took the steam out of the business. In 2005-06 the value of contracts had hit $45 billion.

The first weather contract was concluded in 1997 between Enron and Kock Industries and was based upon temperature indices. In parallel to the development of futures and options, whose the price is mainly related to the evolution of indices, weather and catastrophe (cat) bonds have appeared on the market. These bonds deliver coupons that are directly related to the occurences of climatic events. The weather or cat-bonds are interesting tools of investment for investors looking for diversifaction, given that they have a very small correlation with traditional financial markets. The interested reader may refer to the work of Schmock (1999) for a detailed analysis of the WINCAT bond, a cat bond linked to damages caused by hail and storm to motor vehicules insured with Winterthur in Switzerland. A survey of products and their applications is available in Barrieu and Dischel (2002).

Physical models for the analysis and forecasting of claims related to recurrent meteorological events have a limited tractability for financial applications such as the pricing of climatic products, given their complexity. For this reason, the existing literature on the pricing of weather derivatives mainly develop statistical models. For a survey, we recommend the PhD dissertation of Lopez Cabrera (2010). In Vaugirard (2003) or in Lee and Yub (2007), claims caused by weather catastrophes are modelled as a jump diffusion process. In Alaton et al. (2002) or Campbell and Diebold (2005), the index of temperatures is modelled by a Brownian motion with a seasonal drift**.** Other climatic indexes are modeled by an Ornstein-Uhlenbeck process such as in Dornier and Queruel (2000) and Benth and Benth (2007) and (2009). In Hainaut (2010), we have used a similar approach to model the arrival process of seasonal claims.

The first purpose of this paper is to propose a new statistical model for the claims arrival and cost processes duplicating the seasonality of meteorologic events and the huge volatility exhibited by the frequency and amplitude of claims. The second goal is to illustrate how the proposed model can be used to price weather derivatives such cat bonds. The novelty of our approach is that it considers that the parameters defining the claims arrival and cost processes are stochastic multifractal processes. The literature about these models in statistics is rather sparse, even if multifractals are used since the early sixties in geophysics. The interested reader may refer to the survey of Lovejoy and Schertzer (2007) for an overview. Recent applications of fractals to meteorology may be found in Sachs et al. (2002) and Tchiguirinskaia and al. (2006). The model that we propose is based on Markov-Switching Multifractals processes that have been studied by Calvet and Fisher (2008). These are similar to on-off processes used to model data transmission (e.g. see Resnick and Samorodnitsky (2003)). Some applications of on-off processes in weather prediction have been studied by Mu and Zheng (2005). This type of process is well adapted to duplicate memory effects that are often exhibited by empirical observations of weather indexes (see e.g. Brody et al. (2002) for an attempt to model these effecs with a fractional brownian). To illustrate the utility of this model, we show that it is particularly efficient to model the volatility claims frequency and damage caused by tornadoes in the US. We show next that the fitted model can eventually be used to design a cat bond linked to these climatic events. The word "fractal' emerged on the scientific scene with the work of Mandelbrot (1982) in the 1960s and 1970s. Subsequently, multifractal processes became popular means of modelling financial times series. We refer the interested reader to the numerous publications of Mandelbrot, e.g. (1997) and (2001), for applications of these processes to finance. In actuarial sciences, apart from the work of Major and Lantsman (2001) that proposes methods to fit and simulate multifractal models in the context of two-dimensional fields, there are very few applications. Our work aims to show that this type of models is nonetheless well adapted to introduce volatility in the traditional claims model.

In the first part of this study, the claims arrivals and costs are assumed to be independent. We model those processes by a Poisson and a gamma distribution, whose parameters depend on a multifractal process and on a periodic function respectively. Those models are next calibrated to data related to tornadoes that hit the US. In the second part of this work, we propose a multivariate analysis of the claims process. We attempt to apply the two-dimensional framework developed by Calvet and Fisher (2008), to eliminate the assumption of independence between costs and frequencies. To conclude, we explore the influence of model choice on the pricing of catastrophe bonds.

2. The Claims Arrival Process

2.1 The model

The frequency of many natural phenomena such as tornadoes or hurricanes exhibit seasonality combined with a huge volatility. Figure 1, presents the monthly numbers of tornadoes that hit the US between 1990 and 2008 (data retrieved on Sheldus[i]). It clearly shows that most of tornadoes are observed during the second term compared with the remainder of the year. Modeling the number of claims by a Poisson process with a constant intensity, as usually done for claim arrivals process, is consequently insufficient to capture this trend.

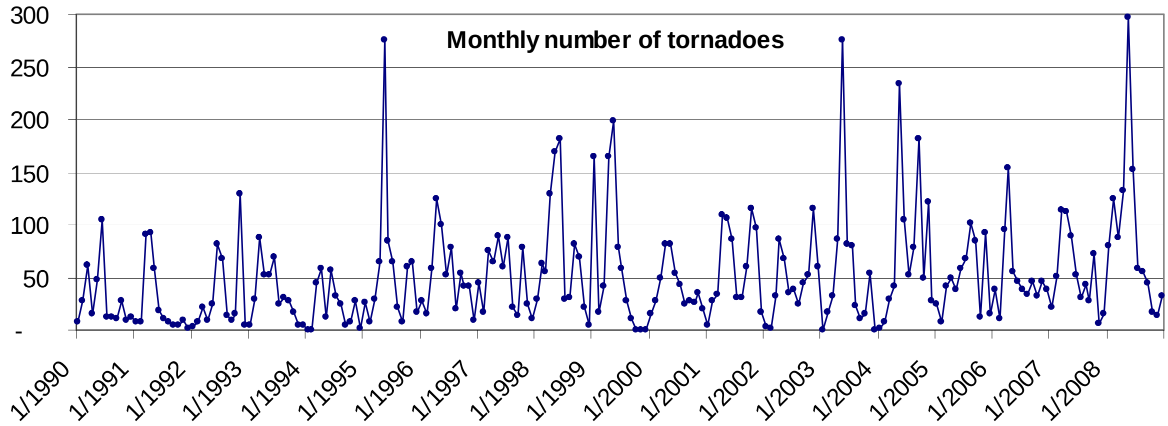

Figure 1: number of tornadoes per month from 1/1990 to 12/2008.

The calculation of the smoothed average number of claims per month, on data from 1990 to 2008, confirms a peak of intensity as illustrated in table 1. Replacing the constant intensity of the Poisson process by a constant piecewise intensity partly improves the modeling of the claims arrival process. However, we will see in the next section that the volatility of this process is still significantly lower than the overdispersion exhibited by real data.

One way to model overdispersion is to insert a stochastic component in the intensity. In a previous work, see Hainaut (2010), the intensity has been modeled as the sum of one seasonal function and of one Browian mean reverting process. We have fitted this model to the arrival process of tornadoes but the results were not satisfying (the high volatility of the Brownian motion leads to a negative intensity with a significant probability). This is why we have chosen in this work to multiply the intensity by a simple multifractal process (a binomial cascade). This category of processes have been successfully applied in econometry or in data transmissions to model stochastic volatility of time series. In finance, Calvet and Fisher (2008, chapter 3) have shown that in many cases, multifractal volatility models outperform the Garch model. In the remainder of this section we adapt this theoretical framework to model the claims arrival process. More precisely, the monthly number of tornadoes is modelled by a Poisson random variable whose intensity is driven by a multifractal process having some persistence properties. As illustrated in numerical results, this approach is efficient to generate peaks of tornadoes occurences.

The number of claims observed on period t, is noted as N_t in the remainder of this work. This process is defined on a filtration $\mathcal{F}_t$, in a probability space Ω coupled with a probability measure, noted as P. The intensity of N_t is a stochastic process, noted as λ_t, and defined on a filtration $\mathcal{H}_t$. We note Δt the length of the period, during which the intensity is constant. Conditionally on $\mathcal{H}_t \vee \mathcal{F}_0$, the process N_t is a Poisson process for which the probability of observing k jumps is given by the formula:

$$P(N_t = n | \mathcal{H}_t \vee \mathcal{F}_0) = \frac{\left(\sum_{i=0}^{t} \lambda_i \Delta t\right)^n}{n!} e^{-\sum_{i=0}^{t} \lambda_i \Delta t} . \tag{1}$$

Interested readers may refer to Bremaud (1981, chapter 2) and Bielecki & Rutkowski (2004, chapter 6) for details on this kind of processes, called doubly stochastic. The intensity of our Poisson process is modeled as the product of a constant piecewise function $\lambda(t)$, with a multifractal process F_t^N that will be defined later. The function $\lambda(t)$ will be constructed to replicate the seasonality observed in the claims arrival process while the process F_t^N introduces volatility in the claims frequency:

$$\lambda_t = \lambda(t) F_t^N . \tag{2}$$

The function $\lambda(t)$ is piecewise constant and periodic. In the remainder of this work, we work on a monthly basis. $\lambda(t)$ is in this case equal to:

$$\lambda(t) = \lambda_i \quad i \equiv t \bmod 12 \tag{3}$$

We set λ_i to the smoothed average of claims observed during the i^{th} month of the year, between 1990 and 2008 (see table 1).

Month	λ_i
January	20
February	25
March	53
April	102
May	118
June	103
July	60
August	42
September	33
October	25
November	21
December	21

Table 1: $\lambda(t)$ average number of claims per month

The process F_t^N, adding volatility in the intensity λ_t, is modeled by a multifractal process, as in the framework proposed by Calvet and Fisher(2008). This process is defined on the filtration H_t. We assume that there exists m climatic factors affecting the frequency of tornado occurences. Those climatic factors are unobservable and are modeled by a Markov state vector, M_t^N, of m components:

$$M_t^N = \left(M_{1,t}^N, M_{2,t}^N \dots M_{m,t}^N\right) \in \mathrm{R}_+^m .$$

The process F_t^N is the product of those climatic factors:

$$\lambda_t = \lambda_i F_t^N = \lambda_i \prod_{k=1}^{m} M_{k,t}^N \quad i \equiv t \bmod 12.$$

M_t^N is built in a recursive manner. Let us assume that the vector M_t^N has been built until period t. For each $k = \{1, \dots, m\}$, the next period multiplier $M_{k,t}^N$ is drawn from a fixed distribution M with probability γ_k, and is otherwise equal to its previous value $M_{k,t}^N = M_{k,t-1}^N$. The distribution of M is positive and is such that $E(M_{k,t}^N) = 1$. This last constraint ensures that on average, the intensity λ_t is equal to $\lambda(t)$. Calvet and Fisher recommend the following distribution for M:

$$M = \begin{cases} m_0 & p_0 = \frac{1}{2} \\ 2 - m_0 & 1 - p_0 = \frac{1}{2} \end{cases} \tag{4}$$

which is fully determined by the parameters $m_0 \in [0,1]$. A Markov process $M_{i,t}^N$, that equals $2 - m_0$, increases the intensity. Conversely, if $M_{i,t}^N = m_0$, the intensity of the claims arrival process is reduced. The probabilities $\gamma_{k=1\ldots m}$ depend on two parameters $\gamma_1 \in (0,1)$ and $b \in (1,\infty)$ as follows:

$$\gamma_k \equiv 1-(1-\gamma_1)^{b^{k-1}} \quad k=1,\ldots,m \tag{5}$$

This rule of construction guarantees that $\gamma_1 \le \ldots \le \gamma_m < 1$. If we note $\mathbf{1}_{k,t}^N$ the indicator function equals 1 if there is a new draw from the distribution M at time t, for the k^{th} components, we have

$$P(\mathbf{1}_{k,t}^N = 1) = \gamma_k \quad k=1\ldots m$$

This means that the last climatic factor $M_{m,t}^N$ changes value more frequently than the first climatic factor $M_{1,t}^N$. The figure 2 illustrates this. It presents simulated trajectories of three climatic factors, involved in the evolution of the claims arrival process (the calibration of these factors is detailed in the next section). Clearly, the third component oscillates more frequently than the first one.

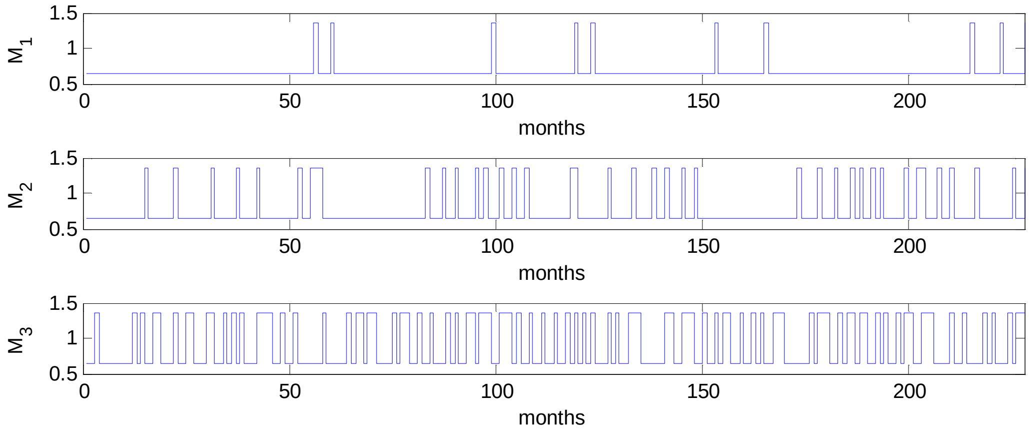

Figure 2: frequencies of fractal components.

The main advantage of this model is its ability to capture low-frequency regime shifts and long volatility cycles of the claims arrival process. Furthermore, it allows a parsimonious representation (only three parameters) of a high dimensional state space. If we consider that there are $m = 8$ hidden climatic factors, the intensity at time t can have $2^8 = 256$ values.

2.2 Calibration

As mentioned in the previous paragraph, the function F_t^N can take $d = 2^m$ values. the Markov state vector M_t^N then takes finitely many values $m^1,\ldots m^d \in \mathbb{R}_+^m$. The transition matrix $A = (a_{i,j})_{1\le i,j\le d}$ is fully determined by the $\gamma_{k=1\ldots m}$. It has the following components:

$$\begin{aligned} a_{i,j} &= P(M^N_{t+1} = m^j | M^N_t = m^i) \\ &= \prod_{k=1}^{m}\left(\gamma_k \frac{1}{2} + (1-\gamma_k) \mathrm{I}_{\{m^j(k) = m^i(k)\}}\right). \end{aligned} \tag{6}$$

For a given combination m^j, the number of tornadoes occurring on the time interval $[t, t+\Delta t]$ is distributed as a Poisson random variable with intensity:

$$\lambda_t^j = \lambda_i F_t^N = \lambda_i \prod_{k=1}^{m} m^j(k) \quad i \equiv t \bmod 12 \quad j = 1 \ldots d., \tag{7}$$

where $m^j(k)$ is the k^{th} elements of the vector m^j. In this case, the probability of observing n_t claims, given λ_t^j, during this period is:

$$p(t, j, n_t) = P(N_{t+h} - N_t = n_t | \lambda_t^j) \quad = \quad \frac{\left(\lambda_t^j \Delta t\right)^n}{n!} e^{-\lambda_t^j \Delta t}\ . \tag{8}$$

At time t, the vector of these probabilities for each combination of climatic factors is noted as $p(t, n_t) = \left(p(t, j, n_t)\right)_{j=1\ldots d}$. The climatic factors $\left(M^N_{k,t}\right)_{k=1\ldots m}$ are not directly observable, but the filtering technique developed by Hamilton (1989) and inspired by Kalman's filter (1960) allows us to retrieve the probabilities of being in a state given all the previous observations. Let briefly summarize this filter. Let us note as $n_{i=0,\ldots,t}$ the number of tornadoes observed in previous periods. Let us define the probabilities of presence in a certain state j as:

$$\Pi_t^j \quad = \quad P\left(M_t^N = m^j | n_1, \ldots, n_t\right)$$

Hamilton has proved that the vector $\Pi_t = \left(\Pi_t^j\right)_{j=1\ldots m}$ can be calculated as a function of the probabilities of presence during the previous period:

$$\Pi_t^j = \frac{p(t, n_t) * \left(\Pi_{t-1} A\right)}{\left\langle p(t, n_t) * \left(\Pi_{t-1} A\right), \mathbf{1}\right\rangle}, \tag{9}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathrm{R}^d$ and $x * y$ is the Hadamard product $(x_1 y_1, \ldots, x_d y_d)$. To start the recursion, we assume that the Markov processes have reached their stable distribution. Π_0 are then set to the ergodic distribution, which is the eigenvector of the matrix A, coupled to the eigenvalue equal to 1. If we observed the claims process on T months, the loglikelihood is:

$$\ln L(n_1 \ldots n_T | m_0, \gamma_1, b) = \sum_{t=0}^{T} \ln \left\langle p(t, n_t), \left(\Pi_{t-1} A\right)\right\rangle. \tag{10}$$

The most likely parameters are obtained by numerical maximization of ((10)).

2.3 Empirical Illustration

The calibration of the claim arrival process has been performed on monthly data from 1990 to 2008, comprising 576 observations. We have first fitted a basic Poisson process, with a constant intensity, by loglikelihood maximization. We get on average 51 tornadoes per month, for a loglikelihood of -5246. Next, the loglikelihood of a Poisson process having a time dependent intensity given in table 1 has been computed. The loglikelihood is improved. Nonetheless, a comparison of simulated N_t with real number of claims indicates that the volatility of this model is significantly lower than the real one.

Table 2 presents the calibrated parameters of multifractal models, counting five to nine hidden climatic factors. Calculations have been performed in SAS. The highest likelihood is obtained with nine components (512 states). Note that we observe a certain stability of parameter values between models.

m	Parameter	Estimates	Std. Err.	Loglik.
9	γ_1	0.1247	0.0875	-1110.849
	b	1.7750	0.5521	
	m_0	0.7405	0.0043	
8	γ_1	0.2218	0.1170	-1133.168
	b	1.5524	0.3532	
	m_0	0.7477	0.0059	
7	γ_1	0.2620	0.1558	-1159.252
	b	1.5696	0.4701	
	m_0	0.7448	0.0055	
6	γ_1	0.2300	0.1372	-1150.353
	b	1.7850	0.6402	
	m_0	0.7056	0.0049	
5	γ_1	0.2460	0.1323	-1194.588
	b	1.7283	0.5362	
	m_0	0.6412	0.0055	
4	γ_1	0.1295	0.0617	-1311.508
	b	4.1602	2.8746	
	m_0	0.5784	0.0082	

Table 2 : Parameters of N_t

In figure 3, we have plotted 2 simulated trajectories of the claims arrival process, with nine fractals, versus the observed number of tornadoes from January 2001 to December 2008. This graph reveals that our model is able to generate peaks of activities, similar to real ones.

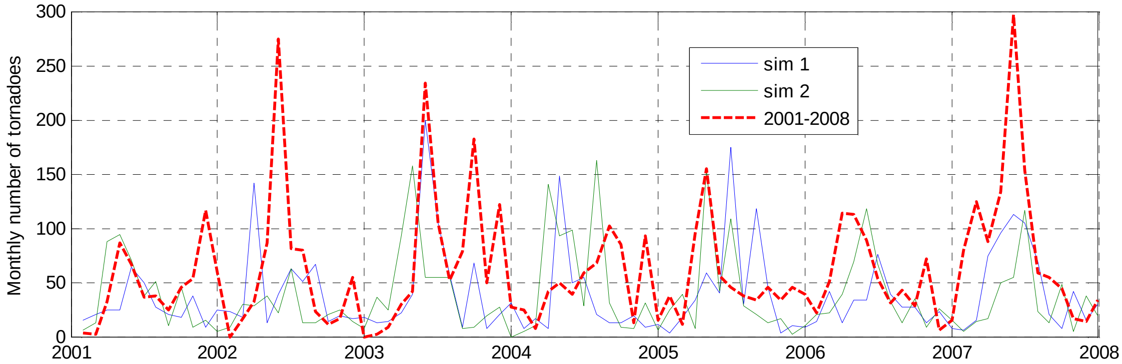

Figure 3: Example of simulated numbers of tornadoes.

In table 3, we compare the first two moments of the observed monthly number of tornadoes (from 1990 to 2008), with the moments of a sample of 1000 simulations. These figures tend to confirm that the model duplicates the seasonality and volatility of the claims arrival process reasonably well.

Month	Historical mean	Historical std	Simulated mean	Simulated std
January	26	39	20	17
February	25	29	25	20
March	57	27	54	43
April	86	44	101	81
May	120	92	121	95
June	77	39	100	80
July	48	23	59	50
August	33	23	42	36
September	40	43	32	26
October	38	30	24	20
November	48	43	22	18
December	15	15	20	15

Table 3: average number of claims and standard deviations per month

3. The Size of Claims

3.1 The model

Figure 4 presents the mean monthly cost of damage caused by one tornado. The amplitude of claims varies considerably between months. An analysis of the smoothed average deflated cost of one tornado per month, contained in table 4 (second column), computed on a period from 1990 to 2008, shows that damage costs seem higher in April and May. From June to October, the average costs exhibit small oscillations.

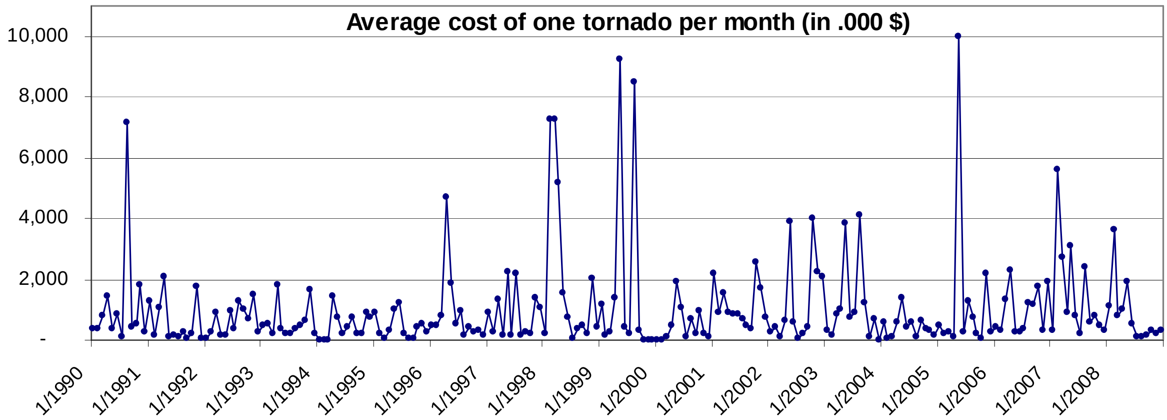

Figure 4: Observed mean monthly cost caused by one tornado.

In the remainder of this work, we denote by C_t, the (deflated) cost of damage caused by one tornado, during the period $[t, t+\Delta t]$. The choice of the probability distribution for claim costs should ideally take into account a certain degree of seasonality. Given the observations, the damage are more expensive in April and May. Furthermore, costs exhibit huge volatility. To capture these trends, the cost process is modeled by a gamma random variable, whose mean parameter is the product of a time dependent function and a multifractal process. Note that we have tested different laws such as exponential or Pareto to model the costs of tornadoes but none of them are satisfactory.

C_t as N_t are defined on the filtration F_t. The mean cost of C_t at time t is noted as τ_t and is a stochastic process defined on a filtration E_t. τ_t will be defined later. Conditionally to $\mathsf{E}_t \vee \mathsf{F}_0$, the density of costs caused by one tornado C_t occurring in the period $[t, t+\Delta t]$, is gamma distributed:

$$f(c|\tau_t) = \frac{1}{\Gamma(\nu)}\left(\frac{\nu c}{\tau_t}\right)^{\nu} \exp(-\frac{c\,\nu}{\tau_t})\frac{1}{c} I_{]0,\infty[}$$

where $\nu \in \mathsf{R}^{+}$. The mean cost of damage is defined as the product of a piecewise constant periodic function $\tau(t)$ with a multifractal process F_t^C :

$$\tau_t = \tau(t) F_t^C . \tag{11}$$

As we work by steps Δt of one month, $\tau(t)$ is in this case equal to:

$$\tau(t) = \tau_i \quad i \equiv t \bmod 12 \tag{12}$$

where the chosen τ_i are provided in the third column of table 4. The values of $\tau_{i=1\ldots12}$ differs slightly from the smoothed average cost, presented in the second column of the same table. In particular, we have removed the oscillations observed from July to October.

Month	Observed average costs (\$)	Model costs τ_i (\$)
January	673 123	673 123
February	673 123	673 123
March	1 059 905	1 059 905
April	1 351 749	1 351 749
May	1 241 208	1 241 208
June	840 978	840 978
July	676 593	676 593
August	645 338	645 338
September	645 338	645 338
October	645 338	645 338
November	645 338	645 338
December	645 338	645 338

Table 4: $\tau(t)$ average claim cost per month in \$.

The amplitude of cost is assumed to be independent from the claims arrival process. In this setting, the cost of claims is also influenced by m^C unobservable factors, independent from those driving the claims arrival process. These factors are modeled by a Markov state vector, M_t^C

$$M_t^C = \left(M_{1,t}^C, M_{2,t}^C \ldots, M_{m,t}^C\right) \in \mathrm{R}_+^n.$$

is built in a similar way to the state vector M_t affecting the number of tornadoes, and is fully parametrized by three parameters (m_0^C, b^C, γ_1^C). The multipliers $M_{k,t}^C$ are drawn from a fixed distribution M^C, with probability γ_k^C , as defined by equations (4) and (5). Otherwise, the multiplier is equal to its previous value $M_{k,t}^C = M_{k,t-1}^C$. The distribution of M^C is also such that $E(M_{k,t}^C) = 1$. The process F_t^C is the product of these factors:

$$\tau_t = \tau(t) F_t^C = \tau_i \prod_{k=1}^{m^C} M_{k,t}^C \quad i \equiv t \bmod 12.$$

This process is defined on the filtration E_t. The calibration is done as for claims arrival process by the Hamilton filter (1989). M_t^C can take $d = 2^{m^C}$ values $m^{C, j=1..d}$. The matrix of transition probabilities between these states is noted as A^C. The vector of probabilities of presence is noted as Π_t^C and is computed by equation ((9)). If $c_1, \ldots, c_T$ are the costs observed on the past T periods, the loglikelihood is given by:

$$\ln L(c_1 \ldots c_T | m_0^C, \gamma_1^C, b^C, \nu) = \sum_{t=0}^{T} \ln \left\langle f^C(c_t), \left(\Pi_{t-1}^C A^C\right) \right\rangle$$

where $f^C(c_t)$ is the vector of density $\left(f^C(c_t | \tau_j)\right)_{j=1\ldots d}$. The parameters m_0^C, γ_1^C, b^C and ν are obtained numerically by maximization of this loglikelihood.

3.2 Empirical Illustration

As for the claim arrival process, the cost process is fitted on monthly data from 1990 to 2008, that is 576 observations. We have first fitted a basic gamma distribution to claim costs. In this model, a claim caused by one tornado costs on average $1.0314 milliion and has a volatility equal to $1.108 million. The loglikelihood is -3264. The next table presents the loglikelihoods and parameters of multifractal models with five to nine components. Increasing the number of volatility components does not reveal a significant improvement.

m	Parameter	Estimates	Std. Err.	Loglik.
9	γ_1	0,1283	0,2132	-3223,746
	b	4,0087	5,5231	
	m_0	0,7181	0,0326	
	ν	2,4828	0,8762	
8	γ_1	0,1491	0,2017	-3223,805
	b	4,9044	19,0991	
	m_0	0,7028	0,0326	
	ν	2,4802	0,8332	
7	γ_1	0,1646	0,1617	-3223,846
	b	7,3779	13,9051	
	m_0	0,6816	0,0368	
	ν	2,5341	0,8875	
6	γ_1	0,1822	0,1545	-3223,898
	b	8,5550	13,4898	
	m_0	0,6602	0,0345	
	ν	2,4658	0,7193	
5	γ_1	0,2094	0,1705	-3224,011
	b	9,4178	17,9029	
	m_0	0,6260	0,0439	
	ν	2,5280	0,9224	
4	γ_1	0,2335	0,1721	-3224,115
	b	9,9730	20,6153	
	m_0	0,5852	0,0383	
	ν	2,4140	0,5996	

Table 5: Parameters for the claims cost.

In figure 5, we have plotted 2 simulated trajectories of the cost process, versus the mean monthly cost of damage caused by one tornado, from January 2001 to December 2008. This graph reveals that our model is able to generate peaks of activities, similar to real ones.

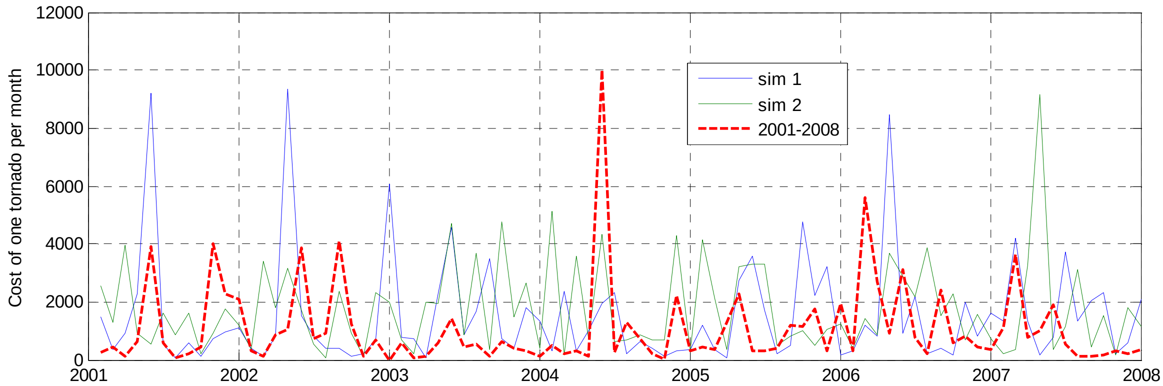

Figure 5: Example of simulated average costs (.000$).

4. Multivariate analysis

4.1 Two-dimensional Multifractal Process

Instead of modelling frequency and severity independently, as done in the previous two sections, it could be interesting to analyze a modelling strategy for simultaneous fitting. We test a bivariate model in which the multifractal processes influencing claims costs and numbers are dependent.

In this subsection we consider a Poisson-Gamma model whose means depend on dependent multifractal processes. As previously, the number of claims N_t, observed in the period $[t,t+\Delta t]$ is assumed to be Poisson, and its intensity is modeled as the product of a constant piecewise function $\lambda(t)$, and of a multifractal process F_t^N :

$$\begin{aligned} \lambda_t &= \lambda(t) F_t^N \\ &= \lambda_i \prod_{k=1}^{m} M_{k,t}^N \quad i \equiv t \bmod 12. \end{aligned} \tag{13}$$

where λ_i are those presented in table 1. The amplitude of claims, C_t, caused by one tornado observed in the period $[t,t+\Delta t]$ is modelled as in section 3 by a Gamma random variable. Its shape parameter is noted as ν and its mean is the product of a piecewise constant function and of a multifractal process F_t^C , having as much components, m, as F_t^N:

$$\begin{aligned} \tau_t &= \tau(t) F_t^C \\ &= \tau_i \prod_{k=1}^{m} M_{k,t}^C \quad i \equiv t \bmod 12. \end{aligned}$$

where τ_i are those presented in table 4. $M_{k,t}^N$ and $M_{k,t}^C$ are characterized by the same triplet (m_0, γ_1, b).

In this paragraph, the assumption of independence between frequencies and costs is dropped. We use a two-dimensional model allowing dependence between claims and costs with multifractal mean. We suppose the same multifractal structure as in sections 2 and 3. However, in the bidimensional multifractal process, not only do the frequency and severity models share some parameters, but also another parameter models the unconditional correlation between the arrival of $M_{k,t}^{C}$ and $M_{k,t}^{N}$.

As previously, the probability of having a new draw from the distribution M, for the component $M_{k,t}^{N}$, is noted as γ_k:

$$P(\mathbf{1}_{k,t}^{N}=1)=\gamma_k \quad k=1\ldots m$$

Following Fisher and Calvet (2008, chapter 4), we assume that there exists $\theta\in[0,1]$ such that the condition

$$\begin{cases} P(\mathbf{1}_{k,t}^{C}=1|\mathbf{1}_{k,t}^{N}=1)=(1-\theta)\gamma_k+\theta \\ P(\mathbf{1}_{k,t}^{C}=0|\mathbf{1}_{k,t}^{N}=1)=(1-\theta)(1-\gamma_k) \end{cases} \quad k=1\ldots m$$

is satisfied. If $\theta=0$ or if $\theta=1$, $M_{k,t}^{N}$ and $M_{k,t}^{C}$ are respectively independent or dependent. Furthermore, it is assumed that the arrivals vector is symmetrically distributed $\left(\mathbf{1}_{k,t}^{C},\mathbf{1}_{k,t}^{N}\right)\overset{d}{=}\left(\mathbf{1}_{k,t}^{N},\mathbf{1}_{k,t}^{C}\right)$. Its distribution is then defined as follows:

$$\begin{cases} p_{11}^{k}=P(\mathbf{1}_{k,t}^{C}=1,\mathbf{1}_{k,t}^{N}=1) & =\gamma_k\left((1-\theta)\gamma_k+\theta\right) \\ p_{01}^{k}=P(\mathbf{1}_{k,t}^{C}=0,\mathbf{1}_{k,t}^{N}=1) & =\gamma_k(1-\theta)(1-\gamma_k) \\ p_{10}^{k}=P(\mathbf{1}_{k,t}^{C}=1,\mathbf{1}_{k,t}^{N}=0) & =\gamma_k(1-\theta)(1-\gamma_k) \\ p_{00}^{k}=P(\mathbf{1}_{k,t}^{C}=0,\mathbf{1}_{k,t}^{N}=0) & =(1-\gamma_k)\left(1-\gamma_k(1-\theta)\right) \end{cases} \tag{14}$$

The third equality is a direct consequence of the symmetry of $\left(\mathbf{1}_{k,t}^{C},\mathbf{1}_{k,t}^{N}\right)$. The expression of p_{00}^{k} is obtained from the relation $p_{10}^{k}+p_{00}^{k}=1-\gamma_k$. The marginal distributions of $\mathbf{1}_{k,t}^{C}$ is identical to the marginal distribution of $\mathbf{1}_{k,t}^{N}$:

$$P(\mathbf{1}_{k,t}^{C}=1)=\gamma_k \quad k=1\ldots m$$

Furthermore, from ((14)), we can infer the conditionial probabibilities when $\mathbf{1}_{k,t}^{N}=0$:

$$\begin{cases} P(\mathbf{1}_{k,t}^{C}=1|\mathbf{1}_{k,t}^{N}=0)=(1-\theta)\gamma_k \\ P(\mathbf{1}_{k,t}^{C}=0|\mathbf{1}_{k,t}^{N}=0)=1-(1-\theta)\gamma_k \end{cases} \quad k=1\ldots m$$

Let us denote $M_t=(M_{1,t}^{N}\ldots M_{m,t}^{N},M_{1,t}^{C}\ldots M_{m,t}^{C})$ the m vector of volatility components. M_t can take $d=4^m$ possible values, $m^1,\ldots,m^d\in\mathbb{R}_{+}^{2m}$. The probabilities of switching from one state to another one are given by the transition matrix $A=(a_{i,j})_{1\le i,j\le d}$ where

$$
\begin{aligned}
a_{i,j} &= P(M_{t+1} = m^j | M_{t+1} = m^i) \\
&= \prod_{k=1}^{m} \Bigg[\left(((1-\theta)\gamma_k + \theta)\frac{1}{2} + ((1-\theta)(1-\gamma_k))\mathrm{I}_{\{m^j(m+k)=m^i(m+k)\}} \right) \gamma_k \frac{1}{2} + \\
&\quad \left((1-\theta)\gamma_k \frac{1}{2} + (1-(1-\theta)\gamma_k)\mathrm{I}_{\{m^j(m+k)=m^i(m+k)\}} \right) (1-\gamma_k)\mathrm{I}_{\{m^j(k)=m^i(k)\}} \Bigg]
\end{aligned}
$$

Let us note as $o_{i=0,\ldots,t} = (n_i, c_i)_{i=0,\ldots,t}$ the observed number and costs of tornadoes on the past periods. The probabilities of presence in a certain state j are denoted as in the previous section $\Pi_t^j = P\left(M_t = m^j | o_1, \ldots, o_t\right)$. The vector $\Pi_t = \left(\Pi_t^j\right)_{j=1\ldots m}$ If we observed the arrivals and costs processes on T months, the loglikelihood is:

$$\ln L(o_1 \ldots o_T | m_0, \gamma_1, b, \theta, \nu) = \sum_{t=0}^{T} \ln\left\langle p(t, o_t), (\Pi_{t-1} A)\right\rangle \tag{15}$$

where $p(t, o_t)$ is the vector of probability density functions of the claim arrivals and costs processes. Again, The parameters are obtained numerically by maximization of this loglikelihood.

7.2 Empirical Illustration

The next table presents the parameters fitting the bidimension process to frequencies and amplitudes of claims caused by tornadoes. With an equivalent number of fractal components and fewer parameters, the 2D model has a log-likelihood slightly lower than the sum of log-likelihoods of standalone arrivals and claims models. Note that the dependence parameter θ is close to zero for two fractal components and increases with m. In our opinion, this reveals a higher degree of dependence between high frequency fractal components than between low frequency components.

m	Parameter	Estimates	Std. Err.	Loglik.
6	γ_1	0,1934	0,0699	-4364,056
	b	2,2967	0,4677	
	m_0	0,7038	0,0048	
	θ	0,4215	0,7360	
	ν	1.9573	0,2700	
5	γ_1	0,1870	0,0650	-4409,264
	b	2,3501	0,4241	
	m_0	0,6394	0,0052	
	θ	0,5512	0,3485	
	ν	2,1318	0,3201	
4	γ_1	0,1654	0,0596	-4529,002
	b	3,5859	0,8235	

	m_0	0,5796	0,0084	
	θ	0,0692	0,5654	
	ν	2,2928	0,3774	
3	γ_1	0,2827	0,0791	-4671,987
	b	3,6959	0,9883	
	m_0	0,5890	0,0066	
	θ	0,0000	0,5317	
	ν	1,9070	0,2399	
2	γ_1	0.4416	0.0892	-4925.479
	b	3.8246	2.1702	
	m_0	0.5150	0.0080	
	θ	0.0000	0.4331	
	ν	1.5721	0.1802	

Table 6: Calibration of a 2D multifractal process.

It would be interesting to analyze a bimensional multifractal model with $Nbk \geq 7$, but it needs a transition matrix of more than 4^7 rows. Calvet and Fisher propose to use a numerical procedure for the inference, via a particle filter. This area of research should be investigated.

9. Pricing of Catastrophe bonds

9.1 Spread calculations

A reinsurer can securitize a portfolio of reinsurance treaties so as to tranfer the risk to other potential investors looking for diversification. The reinsurance treaties are transferred to a special purpose vehicle (SPV), and in exchange for collateral, investors receive a periodic floating payment, linked to the amount of claims covered by treaties. The success of the issuance of such weather derivatives depends on the pricing and the specification of the model chosen to replicate the costs caused by the natural catastrophes covered by reinsurance.

The aims of this section are twofold. First, we underline the impact of working with multifractal models on the pricing of catastrophe bonds linked to claims caused by hurricanes hitting the US. Our results are compared with those obtained with a basic Poisson Gamma model. The second objective is to exhibit the influence of seasonality on pricing.

In this section, we assume that the risk faced by an insurance bondholder is inherent to his exposure to accumulated insured property losses. This process of accumulated losses, which is denoted by X_t in the sequel of this work, depends both upon the frequency of claims N_t and on the magnitude of claims. The process of aggregated losses is defined by the following expression:

$$X_{t_n} = \sum_{t=t_1}^{t_n} N_t C_t$$

The insurance bond priced in this section pays a periodic coupon equal to a constant percentage of the nominal reduced by the amount of aggregated losses, exceeding a certain trigger level. At maturity, what is left of the nominal is repaid. To compensate for this eventual loss of nominal, the coupon rate always exceeds the risk free rate. If few claims occur, the bondholder is rewarded at a higher rate than the one obtained by investing in risk-free assets with the same maturities. Conversely, in the case of catastrophic losses, the nominal of the bond can fall to zero and the payment of coupons can be interrupted. To understand how the spread of this bond is priced, we need to introduce some additional mathematical notations.

Let us note as BN the initial nominal of the bond. The level above which the excess of aggregated losses is deduced from the nominal, is noted as K_1 and called ``attachment point''. If the total insured losses reach the amount of $K_2 = K_1 + BN$, before maturity, the bond stops delivering coupons and the nominal is depleted. The bond, issued at time t_0, pays n coupons, at regular intervals of time, Δt, ranging from t_1 to t_n. The coupon rate is the sum of the constant risk free rate of maturity t_n , and of a spread, that are respectively noted as r and sp. The coupons paid at times $t_{i=1\ldots n}$ are noted as $cp(t_i)$ and defined as follows:

$$cp(t_i) = (r+sp)\Delta t \underbrace{\left[(K_2 - K_1) I_{X_{t_i} \in [0, K_1]} + \left(K_2 - X_{t_i}\right) I_{X_{t_i} \in (K_1, K_2]} \right]}_{BN_{t_i}} . \tag{16}$$

The term between brackets is the (stochastic) nominal of bond at time t_i and is written BN_{t_i} in the sequel of our developments. Note that BN_{t_0} is worth BN. Based upon the principle of absence of arbitrage, the spread of the insurance bond is chosen such that the expectations of future discounted spreads and of future discounted cutbacks of nominal are equal. The expectations of future discounted spreads and reductions of nominal are respectively termed the ``spreads leg'' and the ``claims leg'' (this terminology is inspired by that of credit derivatives). They are defined by the following expressions:

$$\begin{aligned} Spreadsleg(t_0) &= sp\Delta t \sum_{t_i=t_1}^{t_n} e^{-r(t_i - t_0)} \mathrm{E}\left(BN_{t_i} \mid \mathcal{F}_{t_0}\right), \\ Claimsleg(t_0) &= \sum_{t_i=t_1}^{t_n} e^{-r(t_i - t_0)} \mathrm{E}\left(BN_{t_{i-1}} - BN_{t_i} \mid \mathcal{F}_{t_0}\right). \end{aligned} \tag{17)(18}$$

By making equal equations ((17)) and ((18)), we infer the following fair spread rate that will added to the risk free rate, at the issuance of the insurance bond:

$$sp = \frac{\sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathrm{E}\left(BN_{t_{i-1}} - BN_{t_i} | \mathcal{F}_{t_0}\right)}{\Delta t \sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathrm{E}\left(BN_{t_i} | \mathcal{F}_{t_0}\right)} \tag{19}$$

Despite the apparent simplicity of this last expression, the expected future nominals are not calculable by a closed form equation and we have to rely on numerical methods to appraise them. Among the numerical tools available, we have chosen a Monte Carlo method.

9.2 Numerical applications

In this section, we have attempted to price insurance bonds linked to the aggregated costs caused by US tornadoes. The exposure of the insurer issuing the insurance bonds is assumed to be $1/1000$ of the total claims cost. As in Vaugirard (2003), we have computed by Monte Carlo simulations the spreads of insurance bonds of maturities ranging from one to five years, and paying quarterly, biannual and annual coupons. The risk free rate is constant and set to 3%. The nominal, NB , is 15 million and is reduced if the aggregated losses breach the trigger of 5 million.

Two approaches are compared. In the first one, the claims and arrivals are modelled by independent multifractal processes, with nine fractal components. Parameters used to simulate claims scenarios are those presented in tables 2 and 5. In the second approach, the claims and arrivals processes are modeled by a two-dimensional multifractal process, with six components. Parameters used in this simulation are those of table 6.

Tables 7 and 8 presents the spreads obtained with 10000 simulations. The spreads obtained with the 2D model are clearly higher than those obtained under the assumption of independence between claims and costs processes. An analysis of simulations points to positive dependence between the number of losses and the amplitude of damage caused by tornadoes. We also observe that for long-term bonds, the spreads are very high. This is directly related to the fact that the attachement point is breached in most of scenarios after one year. To confirm this intuition, we have plotted in figure 6, the average evolution of the nominals, for the two considered models. On average, the nominal is reduced of 3 and 5 million after five years, depending on the model chosen. This graph reveals the influence of seasonlity on the trajectory of the nominal. From July to March, the nominal decreases more slowly than during the spring. Note that the volatility of the nominal is high. The 5% percentile of the nominal distribution after five years is null. We have also priced the insurance bonds with a simple Poisson Gamma process. This method produces quasi null spreads!

	Quarterly	Semi-annual	Annual
1 y	1.03%	2.06%	4.11%
2 y	2.72%	5.45%	10.89%
3 y	5.42%	10.89%	21.77%

4 y	8.54%	17.20%	34.40%
5 y	11.35%	22.89%	45.84%

Table 7. Spreads in %, independent fractal models.

	Quarterly	Semi-annual	Annual
1 y	0.96%	1.92%	3.84%
2 y	5.94%	11.94%	23.89%
3 y	11.25%	22.69%	45.47%
4 y	14.93%	30.19%	60.55%
5 y	17.57%	35.55%	71.36%

Table 8. Spreads in %, bivariate fractal models.

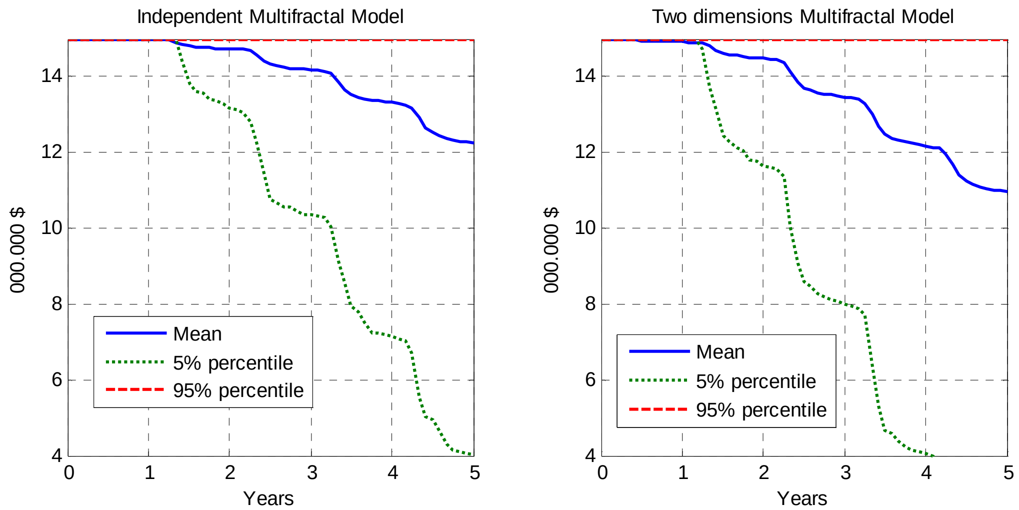

Figure 6: Evolution of nominal.

Conclusion

This paper proposes an new statistical model based on fractals able to duplicate the seasonality of meteorologic events and the huge volatility exhibited by the frequency and amplitude of claims. The innovation of our approach is that it considers a traditional Poisson-gamma model for claims aggregated costs, in which parameters are Markov switching multifractals. To justify the utility of this model, it is fitted to the claims process, caused by tornadoes, that hit the US in the last decades. We observe a significant improvement of loglikelihoods with our model, compared to traditional Poisson-Gamma models.

In the first part of this study, the claims arrivals and costs are assumed to be independent. In the second part of this work, we perform a multivariate analysis of the claim process. We attempt to apply the 2-dimensional framework developed by Calvet and Fisher

(2008), to drop the assumption of independence between costs and frequencies. This last model fits the tornado process with fewer parameters much better than previous methods do. Apparently, dependence appears when the number of fractals increases.

In the last part of this work, we investigated the impact of adopting a multifractal model on the pricing of generic catastrophe bonds, linked to US tornadoes. Our results reveals that the two-dimensional multifractal model leads to the highest spreads. This is due to the positive dependence between the number and amplitude of claims. We believe that using a multifractal model can lead to a better pricing of a wide category of insurance bonds, but this point needs further investigations.

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