

EXPLICIT MARTINGALE REPRESENTATIONS FOR BROWNIAN FUNCTIONALS AND APPLICATIONS TO OPTION HEDGING

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ABSTRACT. Using Clark-Ocone formula, explicit martingale representations for path-dependent Brownian functionals are computed. As direct consequences, explicit martingale representations of the extrema of geometric Brownian motion and explicit hedging portfolios of path-dependent options are obtained.

1. INTRODUCTION

The representation of functionals of Brownian motion by stochastic integrals, also known as martingale representation, has been widely studied over the years. The first proof of what is now known as Itô's representation theorem was implicitly provided by Itô (1951) himself. This theorem states that any square-integrable Brownian functional is equal to a stochastic integral with respect to Brownian motion. Many years later, Dellacherie (1974) gave a simple new proof of Itô's theorem using Hilbert space techniques. Many other articles were written afterward on this problem and its applications but one of the pioneer work on explicit descriptions of the integrand is certainly the one by Clark (1970). Those of Haussmann (1979), Ocone (1984), Ocone and Karatzas (1991) and Karatzas et al. (1991) were also particularly significant. A nice survey article on the problem of martingale representation was written by Davis (2005).

Even though this problem is closely related to important issues in applications, for example finding hedging portfolios in finance, much of the work on the subject did not seem to consider explicitness of the representation as the ultimate goal, at least as it is intended in this

Date: November 24, 2006.

2000 Mathematics Subject Classification. 60H05, 60J65, 60G44, 60H07, 90A09.

Key words and phrases. Martingale representation; stochastic integral representation; Brownian functionals; Clark-Ocone formula; Black-Scholes model; hedging; path-dependent options.

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work. In many papers using Malliavin calculus or some kind of differential calculus for stochastic processes, the results are quite general but unsatisfactory from the explicitness point of view: the integrands in the stochastic integral representations always involve predictable projections or conditional expectations and some kind of gradients.

Recently, Shiryaev and Yor (2004) proposed a method based on Itô's formula to find explicit martingale representations for Brownian functionals. They mention in their introduction that the search for explicit representations is an *uneasy business*. Even though they consider Clark-Ocone formula as a general way to find stochastic integral representations, they raise the question if it is possible to handle it efficiently even in simple cases.

In the present paper, we show that Clark-Ocone formula is easier to handle than one might think in the first place. Using this tool from Malliavin calculus, explicit martingale representations for path-dependent Brownian functionals, i.e. random variables involving Brownian motion and its running extrema, are computed. No conditional expectations nor gradients appear in the closed-form representations obtained.

The method of Shiryaev and Yor (2004) yields in particular the explicit martingale representation of the running maximum of Brownian motion. In the following, this representation will be obtained once more as an easy consequence of our main result. Moreover, the explicit martingale representations of the maximum and the minimum of geometric Brownian motion will be computed. Using these representations in finance, hedging portfolios will be obtained for strongly path-dependent options such as lookback and spread lookback options, i.e. options on some measurement of the volatility.

The rest of the paper is organized as follows. In Section 2, the problem of martingale representation is presented and, in Section 3, the martingale representations of the maximum and the minimum of Brownian motion are recalled. Martingale representations for more general Brownian functionals are given in Section 4 and those for the extrema of geometric Brownian motion are given in Section 5. Finally, in Section 6, our main result is applied to the maximum of Brownian motion and, in Section 7, explicit hedging portfolios of exotic options are computed.

2. MARTINGALE REPRESENTATION

Let $B = (B_t)_{t \in [0, T]}$ be a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$, where $(\mathcal{F}_t)_{t \in [0, T]}$ is the augmented

Brownian filtration which satisfies *les conditions habituelles*. If F is a square-integrable random variable, Itô's representation theorem tells us that there exists a unique adapted process $(\varphi_t)_{t \in [0, T]}$ in $L^2([0, T] \times \Omega)$ such that

$$(1) \quad F = \mathbb{E}[F] + \int_0^T \varphi_t dB_t.$$

In other words, there exists a unique martingale representation or, more precisely, the integrand φ in the representation exists and is unique in $L^2([0, T] \times \Omega)$. The expression martingale representation comes from the fact that Itô's representation theorem is essentially equivalent to the representation of Brownian martingales (see Karatzas and Shreve (1991)). Unfortunately, the problem of finding explicit representations is still unsolved.

2.1. Clark-Ocone representation formula. When F is Malliavin differentiable, the process φ appearing in Itô's representation theorem, i.e. in Equation (1), is given by

$$\varphi_t = \mathbb{E}[D_t F | \mathcal{F}_t]$$

where $t \mapsto D_t F$ is the Malliavin derivative of F . This is Clark-Ocone representation formula.

More precisely, let $W(h) = \int_0^T h(s) dB_s$ be defined for $h \in L^2([0, T])$. For a smooth Brownian functional F , i.e. a random variable of the form

$$F = f(W(h_1), \dots, W(h_n))$$

where f is a smooth bounded function with bounded derivatives of all orders, the Malliavin derivative is defined by

$$D_t F = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n)) h_i(t)$$

where ∂_i stands for the i^{th} partial derivative. Note that $D_t(\int_0^T h(s) dB_s) = h(t)$ and in particular $D_s(B_t) = \mathbb{I}_{\{s \leq t\}}$.

The operator D being closable, it can be extended to obtain the Malliavin derivative

$$D: \mathbb{D}^{1,2} \rightarrow L^2([0, T] \times \Omega)$$

where the domain $\mathbb{D}^{1,2}$ is the closure of the set of smooth functionals under the seminorm

$$\|F\|_{1,2} = \left\{ \mathbb{E}[F^2] + \mathbb{E} \left[\|DF\|_{L^2([0, T])}^2 \right] \right\}^{1/2}.$$

Random variables in $\mathbb{D}^{1,2}$ are said to be Malliavin differentiable. An interesting fact is that $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$. This means that Clark-Ocone representation formula is not restricted to a small subset of Brownian functionals.

Fortunately, the Malliavin derivative satisfies some chain rules. First of all, if $g: \mathbb{R}^m \rightarrow \mathbb{R}$ is a continuously differentiable function with bounded partial derivatives and if $F = (F^1, \dots, F^m) \in (\mathbb{D}^{1,2})^m$, then $g(F) \in \mathbb{D}^{1,2}$ and $D(g(F)) = \sum_{i=1}^m \partial_i g(F) DF^i$. If $g: \mathbb{R}^m \rightarrow \mathbb{R}$ is instead a Lipschitz function, then $g(F) \in \mathbb{D}^{1,2}$ still holds. If in addition the law of F is absolutely continuous with respect to Lebesgue measure on \mathbb{R}^m , then $D(g(F)) = \sum_{i=1}^m \partial_i g(F) DF^i$. These last results will be useful in the sequel.

For more on Malliavin calculus, a concise presentation is available in the notes of Øksendal (1996) and a more detailed and general one in the book of Nualart (1995).

2.2. Hedging portfolios. As mentioned in the introduction, stochastic integral representations appear naturally in mathematical finance. Since the work of Harrison and Pliska (1983), it is known that the completeness of a market model and the computation of hedging portfolios, relies on these representations. One can illustrate this connection by considering the classical Black-Scholes model. Under the probability measure \mathbb{P} , the price dynamics of the risky and the risk-free assets follow respectively

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t, & S_0 = s; \\ dA_t = r A_t dt, & A_0 = 1, \end{cases}$$

where r is the interest rate, μ is the drift and σ is the volatility. Let \mathbb{Q} be the unique equivalent martingale measure of this complete market model and let $B^\mathbb{Q}$ be the corresponding \mathbb{Q} -Brownian motion. Note that under the risk neutral measure \mathbb{Q} ,

$$dS_t = r S_t dt + \sigma S_t dB_t^\mathbb{Q}, \quad S_0 = s,$$

so that for any $t \geq 0$,

$$S_t = s e^{(r-\sigma^2/2)t + \sigma B_t^\mathbb{Q}}.$$

Let G be the payoff of an option on S and (η_t, ξ_t) the self-financing trading strategy replicating this option, i.e. a process over the time interval $[0, T]$ such that

$$dV_t = \eta_t dA_t + \xi_t dS_t \quad \text{and} \quad V_T = G$$

where $V_t = \eta_t A_t + \xi_t S_t$, where ξ_t is the number of shares of the risky asset, $\xi_t S_t$ being the amount invested in it, while η_t is the number of

shares of the risk-free asset, so $\eta_t A_t$ is the amount invested without risk. Then, the price of the option at time t is given by V_t . It is clear that η_t is a linear combination of ξ_t and V_t . When the price is known, the problem of finding the hedging portfolio is the same as finding ξ_t .

It is easily deduced that

$$(2) \quad \xi_t = e^{-r(T-t)} (\sigma S_t)^{-1} \varphi_t,$$

where φ_t is the integrand in the martingale representation of $\mathbb{E}^{\mathbb{Q}} [G \mid \mathcal{F}_t]$, i.e.,

$$e^{r(T-t)} V_t = \mathbb{E}^{\mathbb{Q}} [G \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}} [G] + \int_0^t \varphi_s dB_s^{\mathbb{Q}}.$$

We will use Equation (2) extensively in the section on financial applications.

For example, let $G = (S_T - K)^+$, where K is a constant. This is the payoff of a call option. Since S_T is a Malliavin differentiable random variable and since $f(x) = (x - K)^+$ is a Lipschitz function, one obtains that $D_t G = \sigma S_T \mathbb{I}_{\{S_T > K\}}$. Then

$$\varphi_t = \mathbb{E}^{\mathbb{Q}} [\sigma S_T \mathbb{I}_{\{S_T > K\}} \mid \mathcal{F}_t] = g(t, S_t),$$

where

$$\begin{aligned} g(t, a) &= \sigma a e^{(r-\sigma^2/2)(T-t)} \mathbb{E} \left[e^{\sigma\sqrt{T-t}Z} \mathbb{I}_{\left\{ Z > \frac{\log(K/a) - (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right\}} \right] \\ &= \sigma a e^{r(T-t)} \Phi \left(\frac{\log(a/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right), \end{aligned}$$

with $Z \sim N(0, 1)$. Therefore

$$\xi_t = \Phi \left(\frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right),$$

recovering the well-known formula of the Black-Scholes hedging portfolio for the call option.

Note that even if the payoff of the option involves the non-smooth function $f(x) = (x - K)^+$, the Malliavin calculus approach is applicable. As mentioned in the preceding subsection, f only needs to be a continuously differentiable function with bounded derivative, or a Lipschitz function if it is applied to a random variable with an absolutely continuous law with respect to Lebesgue measure. This was the case for S_T .

3. MAXIMUM AND MINIMUM OF BROWNIAN MOTION

Let $B^\theta = (B_t^\theta)_{t \in [0, T]}$ be a Brownian motion with drift θ , i.e. $B_t^\theta = B_t + \theta t$, where $\theta \in \mathbb{R}$. Its running maximum and its running minimum are respectively defined by

$$M_t^\theta = \sup_{0 \leq s \leq t} B_s^\theta \quad \text{and} \quad m_t^\theta = \inf_{0 \leq s \leq t} B_s^\theta.$$

When $\theta = 0$, M_t and m_t will be used instead. The range process of B_t^θ is then defined by $R_t^\theta = M_t^\theta - m_t^\theta$ and $R \equiv R^\theta$ if $\theta = 0$.

A result of Nualart and Vives (1988) leads to the following lemma; see also Section 2.1.4 in Nualart (1995).

Lemma 3.1. *The random variables M_T^θ and m_T^θ are elements of $\mathbb{D}^{1,2}$ and their Malliavin derivatives are given by*

$$D_t(M_T^\theta) = \mathbb{I}_{[0, \tau_M^\theta]}(t) \quad \text{and} \quad D_t(m_T^\theta) = \mathbb{I}_{[0, \tau_m^\theta]}(t)$$

for $t \in [0, T]$, where $\tau_M^\theta = \inf\{0 \leq t \leq T \mid M_t^\theta = M_T^\theta\}$ and $\tau_m^\theta = \inf\{0 \leq t \leq T \mid m_t^\theta = m_T^\theta\}$ are the almost surely unique random points where B^θ attains respectively its maximum and its minimum.

This lemma will be of great use in the sequel.

3.1. The case $\theta = 0$. If $\theta = 0$, the martingale representation of the maximum of Brownian motion is

$$(3) \quad M_T = \sqrt{\frac{2T}{\pi}} + \int_0^T 2 \left[1 - \Phi \left(\frac{M_t - B_t}{\sqrt{T-t}} \right) \right] dB_t$$

where $\Phi(x) = \mathbb{P}\{N(0, 1) \leq x\}$ and $\mathbb{E}[M_T] = \sqrt{\frac{2T}{\pi}}$ because

$$M_T \stackrel{d}{=} |B_T| \stackrel{d}{=} \sqrt{T} |N(0, 1)|.$$

This representation can be found in the book of Rogers and Williams (1987). Their proof uses Clark's formula (see Clark (1970)), which is essentially a Clark-Ocone formula on the canonical space of Brownian motion. As mention in the introduction, it can also be computed using the completely different method of Shiryaev and Yor (2004). Obviously, the martingale representation of the minimum of Brownian motion is a direct consequence:

$$m_T = \sqrt{\frac{2T}{\pi}} - \int_0^T 2 \left[1 - \Phi \left(\frac{B_t - m_t}{\sqrt{T-t}} \right) \right] dB_t.$$

3.2. The general case. If one extends this by adding a drift to Brownian motion, the results are similar. In the papers of Graversen et al. (2001) and Shiryaev and Yor (2004), the integrand in the martingale representation of M_T^θ is computed. Indeed, the stationary and independent increments of B^θ yield

$$\mathbb{E} [M_T^\theta | \mathcal{F}_t] = M_t^\theta + \int_{M_t^\theta - B_t^\theta}^{\infty} \mathbb{P} \{M_{T-t}^\theta > z\} dz.$$

Thus, $t \mapsto M_t^\theta + \int_{M_t^\theta - B_t^\theta}^{\infty} \mathbb{P}\{M_{T-t}^\theta > z\} dz$ is a martingale and a function of (B_t^θ, M_t^θ) . An application of Itô's formula to this martingale and coefficients analysis yield the martingale representation of M_T^θ . The integrand in this integral representation is given by

$$(4) \quad 1 - \Phi \left(\frac{M_t^\theta - B_t^\theta - \theta(T-t)}{\sqrt{T-t}} \right) + e^{2\theta(M_t^\theta - B_t^\theta)} \left[1 - \Phi \left(\frac{M_t^\theta - B_t^\theta + \theta(T-t)}{\sqrt{T-t}} \right) \right].$$

The integrand in the representation of m_T^θ , the minimum of Brownian motion with drift θ , is then easily deduced and given by

$$(5) \quad - \left[1 - \Phi \left(\frac{B_t^\theta - m_t^\theta - \theta(T-t)}{\sqrt{T-t}} \right) \right] - e^{2\theta(B_t^\theta - m_t^\theta)} \left[1 - \Phi \left(\frac{B_t^\theta - m_t^\theta + \theta(T-t)}{\sqrt{T-t}} \right) \right].$$

Consequently, the integrand in the martingale representation of R^θ , i.e. the range process of B^θ , is given by the difference of Equation (4) and Equation (5).

It is worth mentioning that all these stochastic integral representations can be easily derived with the main result of this paper, i.e. Theorem 4.1.

4. PATH-DEPENDENT BROWNIAN FUNCTIONALS

For a function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ with gradient $\nabla F = (\partial_x F, \partial_y F, \partial_z F)$, define $\text{Div}_{x,y}(F) = \partial_x F + \partial_y F$, $\text{Div}_{x,z}(F) = \partial_x F + \partial_z F$, and so on. Then, $\text{Div}(F)$ is the divergence of F , i.e. $\text{Div}(F) = \partial_x F + \partial_y F + \partial_z F$.

Before stating and proving our main result, let's mention that the joint law of (B_t, m_t, M_t) is absolutely continuous with respect to Lebesgue measure. The joint probability density function will be denoted by $g_{B,m,M}(x, y, z; t)$.

Theorem 4.1. *If $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuously differentiable function with bounded partial derivatives or a Lipschitz function, then the Brownian functional $X = F(B_T^\theta, m_T^\theta, M_T^\theta)$ admits the following martingale representation:*

$$X = \mathbb{E}[X] + \int_0^T f(B_t^\theta, m_t^\theta, M_t^\theta; t) dB_t,$$

where

$$\begin{aligned} f(a, b, c; t) &= e^{-\frac{1}{2}\theta^2\tau} \\ &\times \mathbb{E} \left[\text{Div} F(B_\tau + a, m_\tau + a, M_\tau + a) e^{\theta B_\tau} \mathbb{I}_{\{m_\tau \leq b-a, c-a \leq M_\tau\}} \right. \\ &\quad + \text{Div}_{x,y} F(B_\tau + a, m_\tau + a, c) e^{\theta B_\tau} \mathbb{I}_{\{m_\tau \leq b-a, M_\tau \leq c-a\}} \\ &\quad + \text{Div}_{x,z} F(B_\tau + a, b, M_\tau + a) e^{\theta B_\tau} \mathbb{I}_{\{b-a \leq m_\tau, c-a \leq M_\tau\}} \\ &\quad \left. + \partial_x F(B_\tau + a, b, c) e^{\theta B_\tau} \mathbb{I}_{\{b-a \leq m_\tau, M_\tau \leq c-a\}} \right] \end{aligned}$$

for $b < a < c$, $b < 0$, $c > 0$, and $\tau = T - t$.

Proof. If F is a continuously differentiable function with bounded partial derivatives or if F is a Lipschitz function, then, using one of Malliavin calculus chain rules given in Subsection 2.1, the Brownian functional X is an element of the space $\mathbb{D}^{1,2}$ and its Malliavin derivative is given by

$$D_t X = \nabla F(B_T^\theta, m_T^\theta, M_T^\theta) \cdot (D_t(B_T^\theta), D_t(m_T^\theta), D_t(M_T^\theta)).$$

This is true in both cases since the law of $(B_T^\theta, m_T^\theta, M_T^\theta)$ is absolutely continuous with respect to Lebesgue measure.

Define an equivalent probability measure \mathbb{Q} on \mathcal{F}_T by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$, where $Z_t = \exp\{-\theta B_t - \frac{1}{2}\theta^2 t\}$ for $t \in [0, T]$. Notice that $\frac{d\mathbb{P}}{d\mathbb{Q}} = (Z_T)^{-1}$. Since $D_t X$ is \mathcal{F}_T -measurable for each $t \in [0, T]$, using the abstract Bayes rule (see Lemma 5.3 in Karatzas and Shreve (1991)), one obtains

$$\begin{aligned} \mathbb{E}[D_t X \mid \mathcal{F}_t] &= Z_t \mathbb{E}^{\mathbb{Q}}[(Z_T)^{-1} D_t X \mid \mathcal{F}_t] \\ &= e^{\frac{1}{2}\theta^2(T-t)} \mathbb{E}^{\mathbb{Q}}[e^{\theta(B_T - B_t)} D_t X \mid \mathcal{F}_t] \\ &= e^{-\frac{1}{2}\theta^2(T-t)} \mathbb{E}^{\mathbb{Q}}[e^{\theta(B_T^\theta - B_t^\theta)} D_t X \mid \mathcal{F}_t]. \end{aligned}$$

By Girsanov's theorem, B^θ is a standard Brownian motion under \mathbb{Q} with respect to the filtration generated by B .

Using Lemma 3.1 and the fact that for any random variable Z and partition $(A_i)_i$ of Ω the equality $Z = \sum_i Z \mathbb{I}_{A_i}$ holds, one gets

$$\begin{aligned} (6) \quad D_t X &= \text{Div}(F) \mathbb{I}_{[0, \tau_m^\theta]}(t) \mathbb{I}_{[0, \tau_M^\theta]}(t) + \text{Div}_{x,y}(F) \mathbb{I}_{[0, \tau_m^\theta]}(t) \mathbb{I}_{(\tau_M^\theta, \infty)}(t) \\ &\quad + \text{Div}_{x,z}(F) \mathbb{I}_{(\tau_m^\theta, \infty)}(t) \mathbb{I}_{[0, \tau_M^\theta]}(t) + \partial_x(F) \mathbb{I}_{(\tau_m^\theta, \infty)}(t) \mathbb{I}_{(\tau_M^\theta, \infty)}(t). \end{aligned}$$

Next, note that $t < \tau_M^\theta$ if and only if $M_t^\theta < M_T^\theta$ and $t < \tau_m^\theta$ if and only if $m_t^\theta > m_T^\theta$. It can be shown that the random variables τ_M^θ and τ_m^θ have absolutely continuous laws. In fact, under \mathbb{Q} , τ_M^θ/T and τ_m^θ/T both have a Beta(1/2, 1/2) distribution. As a result,

$$\begin{aligned}\mathbb{I}_{[0, \tau_M^\theta)}(t) &= \mathbb{I}_{[0, \tau_M^\theta)}(t) = \mathbb{I}_{\{M_t^\theta < M_T^\theta\}}(t) \\ \mathbb{I}_{[0, \tau_m^\theta)}(t) &= \mathbb{I}_{[0, \tau_m^\theta)}(t) = \mathbb{I}_{\{m_t^\theta > m_T^\theta\}}(t)\end{aligned}$$

almost surely. Hence, the expression of $D_t X$ previously obtained in Equation (6) becomes

$$\begin{aligned}D_t X &= \text{Div}(F) \mathbb{I}_{\{m_t^\theta > m_T^\theta, M_t^\theta < M_T^\theta\}}(t) + \text{Div}_{x,y}(F) \mathbb{I}_{\{m_t^\theta > m_T^\theta, M_t^\theta = M_T^\theta\}}(t) \\ &\quad + \text{Div}_{x,z}(F) \mathbb{I}_{\{m_t^\theta = m_T^\theta, M_t^\theta < M_T^\theta\}}(t) + \partial_x(F) \mathbb{I}_{\{m_t^\theta = m_T^\theta, M_t^\theta = M_T^\theta\}}(t).\end{aligned}$$

Then, using the Markov property of $(B^\theta, m^\theta, M^\theta)$ under \mathbb{Q} , we get that $\mathbb{E}[D_t X \mid \mathcal{F}_t] e^{\frac{1}{2}\theta^2\tau}$ is equal to

$$\begin{aligned}\mathbb{E}^\mathbb{Q} &\left[\text{Div} F(B_\tau^\theta + a, m_\tau^\theta + a, M_\tau^\theta + a) e^{\theta B_\tau^\theta} \mathbb{I}_{\{m_\tau^\theta \leq b-a, c-a \leq M_\tau^\theta\}} \right] \\ &+ \mathbb{E}^\mathbb{Q} \left[\text{Div}_{x,y} F(B_\tau^\theta + a, m_\tau^\theta + a, c) e^{\theta B_\tau^\theta} \mathbb{I}_{\{m_\tau^\theta \leq b-a, M_\tau^\theta \leq c-a\}} \right] \\ &+ \mathbb{E}^\mathbb{Q} \left[\text{Div}_{x,z} F(B_\tau^\theta + a, b, M_\tau^\theta + a) e^{\theta B_\tau^\theta} \mathbb{I}_{\{b-a \leq m_\tau^\theta, c-a \leq M_\tau^\theta\}} \right] \\ &+ \mathbb{E}^\mathbb{Q} \left[\partial_x F(B_\tau^\theta + a, b, c) e^{\theta B_\tau^\theta} \mathbb{I}_{\{b-a \leq m_\tau^\theta, M_\tau^\theta \leq c-a\}} \right]\end{aligned}$$

where $\tau = T - t$, $a = B_t^\theta$, $b = m_t^\theta$ and $c = M_t^\theta$. Since the law of $(B^\theta, m^\theta, M^\theta)$ under \mathbb{Q} and the law of (B, m, M) under \mathbb{P} are equal, $\mathbb{E}[D_t X \mid \mathcal{F}_t] e^{\frac{1}{2}\theta^2\tau}$ is then equal to

$$\begin{aligned}\mathbb{E} &\left[\text{Div} F(B_\tau + a, m_\tau + a, M_\tau + a) e^{\theta B_\tau} \mathbb{I}_{\{m_\tau \leq b-a, c-a \leq M_\tau\}} \right] \\ &+ \mathbb{E} \left[\text{Div}_{x,y} F(B_\tau + a, m_\tau + a, c) e^{\theta B_\tau} \mathbb{I}_{\{m_\tau \leq b-a, M_\tau \leq c-a\}} \right] \\ &+ \mathbb{E} \left[\text{Div}_{x,z} F(B_\tau + a, b, M_\tau + a) e^{\theta B_\tau} \mathbb{I}_{\{b-a \leq m_\tau, c-a \leq M_\tau\}} \right] \\ &+ \mathbb{E} \left[\partial_x F(B_\tau + a, b, c) e^{\theta B_\tau} \mathbb{I}_{\{b-a \leq m_\tau, M_\tau \leq c-a\}} \right].\end{aligned}$$

The statement follows from Clark-Ocone formula. \square

Using Theorem 4.1, the results of Section 3, i.e. the martingale representations of the extrema of Brownian motion, are easily derived. For

example, one obtains the martingale representation of M_T by considering the function $F(x, y, z) = z$. Indeed,

$$\begin{aligned}
f(a, b, c; t) &= \mathbb{E} \left[\mathbb{I}_{\{m_{T-t} \leq b-a, c-a \leq M_{T-t}\}} + \mathbb{I}_{\{b-a \leq m_{T-t}, c-a \leq M_{T-t}\}} \right] \\
&= \mathbb{P} \{c - a \leq M_{T-t}\} \\
&= \int_{c-a}^{\infty} \sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{z^2}{2(T-t)}} dz \\
&= 2 \int_{\frac{c-a}{\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= 2 \left[1 - \Phi \left(\frac{c-a}{\sqrt{T-t}} \right) \right]
\end{aligned}$$

since the density function of M_t is given by $z \mapsto \sqrt{\frac{2}{\pi t}} e^{-\frac{z^2}{2t}} \mathbb{I}_{\{z \geq 0\}}$.

Remark 4.1. *As mentioned before, the expectation appearing in the integrand of the martingale representation of Theorem 4.1 is a simple expectation, i.e. it is not a conditional expectation, and the integrand does not involve any gradient. This expectation can also be written in the following form:*

$$\begin{aligned}
(7) \quad & \int_{c-a}^{\infty} \int_{-\infty}^{b-a} \int_{-\infty}^{\infty} \text{Div} F(x+a, y+a, z+a) g(dx, dy, dz; \tau) \\
& + \int_0^{c-a} \int_{-\infty}^{b-a} \int_{-\infty}^{\infty} \text{Div}_{x,y} F(x+a, y+a, c) g(dx, dy, dz; \tau) \\
& + \int_{c-a}^{\infty} \int_{b-a}^0 \int_{-\infty}^{\infty} \text{Div}_{x,z} F(x+a, b, z+a) g(dx, dy, dz; \tau) \\
& \quad + \int_0^{c-a} \int_{b-a}^0 \int_{-\infty}^{\infty} \partial_x F(x+a, b, c) g(dx, dy, dz; \tau)
\end{aligned}$$

where $g(dx, dy, dz; s) = e^{\theta x + \frac{1}{2}\theta^2 s} g_{B,m,M}(x, y, z; s) dx dy dz$.

Here,

$$\int_A \int_B \int_C G(x, y, z) g(dx, dy, dz; s)$$

means

$$\int_{z \in A} \int_{y \in B} \int_{x \in C} G(x, y, z) g(dx, dy, dz; s).$$

In order to apply Theorem 4.1, one needs the joint distribution of the random vector (B_t, m_t, M_t) . This is recalled next.

4.1. The joint probability density function. The expression of the joint law of (B_t, m_t, M_t) was obtained by Feller (1951). Let $y, z > 0$ and $-y \leq a < b \leq z$. If $\phi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$, then it is known that

$$\begin{aligned} & \mathbb{P} \{B_t \in (a, b), -y \leq m_t, M_t \leq z\} \\ &= \int_a^b \left(\sum_{k \in \mathbb{Z}} \phi_t(2k(y+z) + x) - \sum_{k \in \mathbb{Z}} \phi_t(2k(y+z) + 2z - x) \right) dx. \end{aligned}$$

Hence,

$$\begin{aligned} g_{B,m,M}(x, y, z; t) &= 4 \sum_{k \in \mathbb{Z}} [k^2 \phi_t''(2k(y+z) + x) \\ &\quad - n(n-1) \phi_t''(2k(y+z) + 2z - x)] \end{aligned}$$

where $\phi_t''(x) = \frac{(x^2-1)}{t^2} \phi_t(x)$.

Rearranging terms, one obtains that $g_{B,m,M}(x, y, z; t)$ is also given by

$$\begin{aligned} & 4 \sum_{k \geq 1} k^2 \phi_t''(2k(y+z) + x) + 4 \sum_{k \geq 1} k^2 \phi_t''(2k(y+z) - x) \\ & \quad - 4 \sum_{k \geq 2} k(k-1) \phi_t''(2k(y+z) + 2z - x) \\ & \quad - 4 \sum_{k \geq 1} k(k+1) \phi_t''(2k(y+z) - 2z + x). \end{aligned}$$

Integrating with respect to z , one obtains the joint PDF of (B_t, m_t) :

$$g_{B,m}(x, y; t) = \frac{2(x-2y)}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}(x-2y)^2} \mathbb{I}_{\{y \leq x\}} \mathbb{I}_{\{y \leq 0\}}.$$

The same work can be done to compute the joint PDF of (B_t, M_t) . Its expression is given in the proof of Proposition 5.1.

5. MAXIMUM AND MINIMUM OF GEOMETRIC BROWNIAN MOTION

In this section, Theorem 4.1 is applied to produce explicit martingale representations for the maximum and the minimum of geometric Brownian motion. These particular Brownian functionals are important in finance and fortunately the upcoming representations are plainly explicit.

For a stochastic process $(X_t)_{t \in [0, T]}$, let its running extrema be denoted respectively by

$$M_t^X = \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad m_t^X = \inf_{0 \leq s \leq t} X_s.$$

The range process of X is then given by $R_t^X = M_t^X - m_t^X$.

Proposition 5.1. *If X is a geometric Brownian motion, i.e. $X_t = e^{\mu t + \sigma B_t}$ for $\mu \in \mathbb{R}$ and $\sigma > 0$, its maximum on $[0, T]$ admits the following martingale representation:*

$$M_T^X = \mathbb{E}[M_T^X] + \int_0^T g(B_t^\theta, M_t^\theta) dB_t.$$

Here, $\theta = \frac{\mu}{\sigma}$ and $g(a, b)$ is given by

$$(8) \quad \frac{\sigma e^{\sigma a}}{\mu + \frac{\sigma^2}{2}} \left\{ (\mu + \sigma^2) e^{(\mu + \frac{\sigma^2}{2})(T-t)} \right. \\ \times \left[1 - \Phi \left(\frac{\sigma(b-a) - (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \right] \\ \left. + \mu \left(e^{\sigma(b-a)} \right)^{\frac{2}{\sigma^2}(\mu + \frac{\sigma^2}{2})} \left[1 - \Phi \left(\frac{\sigma(b-a) + \mu(T-t)}{\sigma\sqrt{T-t}} \right) \right] \right\},$$

for $a < b$ and $b > 0$.

Proof. Applying Theorem 4.1 with $F(x, y, z) = e^{\sigma z}$ and using the density function of (B_s, M_s) , i.e.

$$g_{B,M}(x, y; s) = \frac{2(2y-x)}{\sqrt{2\pi s^3}} e^{-\frac{1}{2s}(2y-x)^2} \mathbb{I}_{\{y \geq x\}} \mathbb{I}_{\{y \geq 0\}},$$

the integrand in the representation of M_T^X is given by

$$(9) \quad 2\sigma e^{\sigma a - \frac{1}{2}\theta^2\tau} \int_{b-a}^{\infty} \int_{-\infty}^y e^{\sigma y + \theta x} \frac{(2y-x)}{\sqrt{2\pi\tau^3}} e^{-\frac{1}{2\tau}(2y-x)^2} dx dy$$

where $a = B_t^\theta$, $b = M_t^\theta$ and $\tau = T - t$. Hopefully, in this case, the integrand can be greatly simplified and so the rest of the proof involves only elementary calculations.

For $a \leq b$, let I denote only the integral in Equation (9). If $z = 2y - x$, then

$$I = e^{\frac{1}{2}\theta^2\tau} \int_{b-a}^{\infty} e^{(\sigma+2\theta)y} \int_y^{\infty} \frac{z}{\sqrt{2\pi\tau^3}} e^{-\frac{1}{2\tau}(z+\theta\tau)^2} dz dy \\ = e^{\frac{1}{2}\theta^2\tau} \int_{b-a}^{\infty} e^{(\sigma+2\theta)y} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2\tau}(y+\theta\tau)^2} dy \\ \quad - \theta e^{\frac{\theta^2\tau}{2}} \int_{b-a}^{\infty} e^{(\sigma+2\theta)y} \left(1 - \Phi \left(\frac{y+\theta\tau}{\sqrt{\tau}} \right) \right) dy \\ = e^{\frac{1}{2}\theta^2\tau} I_1 - \theta e^{\frac{1}{2}\theta^2\tau} I_2,$$

where the integrals I_1 and I_2 are obviously defined. If $z = \frac{y+\theta\tau}{\sqrt{\tau}}$ and $\beta = \sigma + 2\theta$, then

$$I_1 = e^{-\beta\theta\tau} e^{\frac{1}{2}\beta^2\tau} \left[1 - \Phi \left(\frac{b-a - (\theta + \sigma)\tau}{\sqrt{\tau}} \right) \right],$$

and

$$I_2 = -\frac{e^{\beta(b-a)}}{\beta} \left(1 - \Phi \left(\frac{b-a + \theta\tau}{\sqrt{\tau}} \right) \right) + \frac{1}{\beta} I_1.$$

Finally,

$$I = e^{\frac{1}{2}\theta^2\tau} \left(1 - \frac{\theta}{\beta} \right) I_1 + \frac{\theta}{\beta} e^{\frac{1}{2}\theta^2\tau} e^{\beta(b-a)} \left(1 - \Phi \left(\frac{b-a + \theta\tau}{\sqrt{\tau}} \right) \right).$$

The statement follows. \square

The martingale representation of the minimum of geometric Brownian motion is not a completely direct consequence of the last corollary since the exponential function is not linear. However, the proof is almost identical to the proof of Proposition 5.1.

Corollary 5.1. *If X is a geometric Brownian motion, i.e. $X_t = e^{\mu t + \sigma B_t}$ for $\mu \in \mathbb{R}$ and $\sigma > 0$, its minimum m_T^X admits the following martingale representation:*

$$m_T^X = \mathbb{E} [m_T^X] + \int_0^T h(B_t^\theta, m_t^\theta) dB_t.$$

Here, $\theta = \frac{\mu}{\sigma}$ and $h(a, b)$ is given by

$$(10) \quad \frac{\sigma e^{\sigma a}}{\mu + \frac{\sigma^2}{2}} \left\{ (\mu + \sigma^2) e^{(\mu + \frac{\sigma^2}{2})(T-t)} \right. \\ \times \left[1 - \Phi \left(\frac{\sigma(a-b) + (\mu - \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \right] \\ \left. + \mu (e^{\sigma(b-a)})^{\frac{2}{\sigma^2}(\mu + \frac{\sigma^2}{2})} \left[1 - \Phi \left(\frac{\sigma(a-b) - \mu(T-t)}{\sigma\sqrt{T-t}} \right) \right] \right\},$$

for $a > b$ and $b < 0$.

In Proposition 5.1, the expression of g is not simplified further because its actual form will be useful to get Black-Scholes like formulas in the upcoming financial applications. Moreover, it gives this interesting

other expression of $g(B_t^\theta, M_t^\theta)$ in terms of (X_t, M_t^X) :

$$(11) \quad \frac{\sigma X_t}{\mu + \frac{\sigma^2}{2}} \left\{ (\mu + \sigma^2) e^{(\mu + \frac{\sigma^2}{2})(T-t)} \right. \\ \times \left[1 - \Phi \left(\frac{\ln(M_t^X/X_t) - (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \right] \\ \left. + \mu \left(\frac{M_t^X}{X_t} \right)^{\frac{2}{\sigma^2}(\mu + \frac{\sigma^2}{2})} \left[1 - \Phi \left(\frac{\ln(M_t^X/X_t) + \mu(T-t)}{\sigma\sqrt{T-t}} \right) \right] \right\}.$$

Of course, a similar expression for $h(B_t^\theta, m_t^\theta)$ in terms of (X_t, m_t^X) is available. The representation of R_T^X , the range process of geometric Brownian motion $X_t = \exp\{\mu t + \sigma B_t\}$ at time T , is now obvious.

Corollary 5.2. *The random variable R_T^X admits a martingale representation with the following integrand:*

$$g(B_t^\theta, M_t^\theta) - h(B_t^\theta, m_t^\theta) \equiv \frac{\sigma X_t}{\mu + \frac{\sigma^2}{2}} \left\{ (\mu + \sigma^2) e^{(\mu + \frac{\sigma^2}{2})(T-t)} \right. \\ \times \left[\Phi \left(\frac{\ln(X_t/m_t^X) + (\mu - \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \right. \\ \left. - \Phi \left(\frac{\ln(M_t^X/X_t) - (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \right] \\ \left. + \mu \left(\frac{M_t^X}{X_t} \right)^{\frac{2}{\sigma^2}(\mu + \frac{\sigma^2}{2})} \left[1 - \Phi \left(\frac{\ln(M_t^X/X_t) + \mu(T-t)}{\sigma\sqrt{T-t}} \right) \right] \right. \\ \left. - \mu \left(\frac{m_t^X}{X_t} \right)^{\frac{2}{\sigma^2}(\mu + \frac{\sigma^2}{2})} \left[1 - \Phi \left(\frac{\ln(X_t/m_t^X) - \mu(T-t)}{\sigma\sqrt{T-t}} \right) \right] \right\}$$

the difference of the integrands in Equation (8) and Equation (10), i.e. the integrands in the representations of M_T^X and m_T^X respectively.

6. APPLICATIONS: HEDGING FOR PATH-DEPENDENT OPTIONS

As mentioned earlier, martingale representations results are important in mathematical finance for option hedging. With the previous explicit representations, one can compute explicit hedging portfolios for some strongly path-dependent options. For example, options involving the maximum and/or the minimum of the risky asset can be replicated explicitly. To get the complete hedging portfolio of such options, i.e. (η_t, ξ_t) (see the introduction), recall that one also needs the

price of the option. The prices of the options in consideration can be found in the literature.

Recall from the introduction the classical Black-Scholes risk-neutral market model:

$$\begin{cases} dS_t = rS_t dt + \sigma S_t dB_t, & S_0 = 1; \\ dA_t = rA_t dt, & A_0 = 1, \end{cases}$$

where \mathbb{P} and B stand respectively for the risk-neutral probability measure and the corresponding \mathbb{P} -Brownian motion. In this case, $S_t = e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t}$ and then all the notation introduced earlier is adapted, i.e.

$$\begin{aligned} \mu &= r - \frac{1}{2}\sigma^2 \\ \theta &= \frac{\mu}{\sigma} = \frac{r - \frac{1}{2}\sigma^2}{\sigma}. \end{aligned}$$

From Equation (2), the amount to invest in the risky asset to replicate an option with payoff G is

$$(12) \quad \xi_t = e^{-r(T-t)} (\sigma S_t)^{-1} \varphi_t.$$

6.1. Standard lookback options. Let's compute the explicit hedging portfolio of a standard lookback put option. The payoff of a standard lookback put option is given by $G = [M_T^S - S_T]^+ = M_T^S - S_T$.

Corollary 6.1. *The number of shares to invest in the risky asset to replicate a standard lookback put option is*

$$\xi_t = \left(1 - \frac{\sigma^2}{2r}\right) \left\{ [1 - \Phi(d_1(t))] + e^{-r(T-t)} \left(\frac{M_t^S}{S_t}\right)^{\frac{2r}{\sigma^2}} [1 - \Phi(d_2(t))] \right\},$$

for $t \in [0, T[$, where

$$\begin{aligned} d_1(t) &= \frac{\sigma M_t^\theta - \sigma B_t^\theta - (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \\ d_2(t) &= \frac{\sigma M_t^\theta - \sigma B_t^\theta + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

Proof. Apply Equation (12) and Proposition 5.1 with the representation in Equation (11). \square

The preceding portfolio was computed by Bermin (2000) in a slightly different manner.

The payoff and the hedging portfolio of a standard lookback call option are similar to those of the standard lookback put option. The

prices of these two options are in the article of Conze and Viswanathan (1991) and in the book of Musiela and Rutkowski (1997).

6.2. Options on the volatility. The range of the risky asset is a particular measure of the volatility. Payoffs involving the range are therefore very sensitive to the volatility of the market. First, consider a contract who gives its owner $G = M_T^S - m_T^S$ at maturity, i.e. a payoff equivalent to buying the maximum at the price of the minimum.

Corollary 6.2. *The number of shares to invest in the risky asset to replicate a contingent claim with payoff $M_T^S - m_T^S$ is*

$$(13) \quad \xi_t = e^{-r(T-t)} \left(1 - \frac{\sigma^2}{2r}\right) \left(\frac{M_t^S}{S_t}\right)^{\frac{2r}{\sigma^2}} [1 - \Phi(d_2(t))] \\ + \left(1 + \frac{\sigma^2}{2r}\right) [\Phi(d_3(t)) - \Phi(d_1(t))] \\ + e^{-r(T-t)} \left(1 - \frac{\sigma^2}{2r}\right) \left(\frac{m_t^S}{S_t}\right)^{\frac{2r}{\sigma^2}} [1 - \Phi(d_4(t))] ,$$

for $t \in [0, T[$, where

$$d_3(t) = \frac{\sigma B_t^\theta - \sigma m_t^\theta + (r - \frac{3\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \\ d_4(t) = \frac{\sigma B_t^\theta - \sigma m_t^\theta - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

Proof. Apply Equation (12) with the representation given by Corollary 5.2. \square

The price of this option is easily derived from those of the standard lookback put and call options. Since

$$M_T^S - m_T^S = (M_T^S - S_T) - (m_T^S - S_T)$$

and since the pricing operator is linear, the price of an option with payoff $M_T^S - m_T^S$ is the difference of the prices of the standard lookback options just considered.

One can generalize the previous payoff by considering a spread lookback call option, i.e. an option with payoff

$$[(M_T^S - m_T^S) - K]^+$$

where $K \geq 0$ is the strike price. The amount to invest in the risky asset will depend if the option is *in-the-money* or *out-of-the-money*. Notice that $t \mapsto M_t^S - m_t^S$ is an increasing function.

Corollary 6.3. *If $\Psi(y, z; s) = \int_{-\infty}^{\infty} e^{\theta x} g_{B, m, M}(x, y, z; s) dx$, $\tau = T - t$ and $A = \{(y, z) \mid K \leq e^{\sigma z} - e^{\sigma y}\}$, the number of shares to invest in the risky asset to replicate a spread lookback call option ξ_t is equal to*

$$\begin{aligned} & \int_{M_t^\theta - B_t^\theta}^{\infty} \int_{-\infty}^{m_t^\theta - B_t^\theta} (e^{\sigma z} - e^{\sigma y}) \mathbb{I}_A(y + B_t^\theta, z + B_t^\theta) \Psi(y, z; \tau, \theta) dy dz \\ & - \mathbb{I}_{\{M_t^S \geq K\}} \int_0^{M_t^\theta - B_t^\theta} \int_{-\infty}^{m_t^\theta - B_t^\theta} e^{\sigma y} \mathbb{I}_A(y + B_t^\theta, M_t^\theta) \Psi(y, z; \tau, \theta) dy dz \\ & + \mathbb{I}_{\{m_t^S \leq 1 - K\}} \int_{M_t^\theta - B_t^\theta}^{\infty} \int_{m_t^\theta - B_t^\theta}^0 e^{\sigma z} \mathbb{I}_A(m_t^\theta, z + B_t^\theta) \Psi(y, z; \tau, \theta) dy dz \end{aligned}$$

times $\exp\{-r\tau - \frac{1}{2\sigma^2}(r - \frac{\sigma^2}{2})^2\tau\}$ when $R_t^S = M_t^S - m_t^S < K$, i.e. when the option is out-of-the-money, and ξ_t is as in Equation (13) as soon as $R_t^S = M_t^S - m_t^S \geq K$, i.e. as soon as the option is in-the-money.

Proof. Define $F(B_T^\theta, m_T^\theta, M_T^\theta) = [(M_T^S - m_T^S) - K]^+$ where F is the Lipschitz function $F(x, y, z) = (e^{\sigma z} - e^{\sigma y}) \mathbb{I}_{\{e^{\sigma z} - e^{\sigma y} \geq K\}}$. Clearly, $\partial_x F \equiv 0$, $\partial_y F = -\sigma e^{\sigma y} \mathbb{I}_{\{e^{\sigma z} - e^{\sigma y} \geq K\}}$ and $\partial_z F = \sigma e^{\sigma z} \mathbb{I}_{\{e^{\sigma z} - e^{\sigma y} \geq K\}}$. Using Theorem 4.1 and Equation (12), one gets that ξ_t is equal to

$$e^{-r\tau} (S_t)^{-1} e^{\sigma B_t^\theta - \frac{1}{2}\theta^2\tau}$$

times

$$\begin{aligned} & \int_{M_t^\theta - B_t^\theta}^{\infty} \int_{-\infty}^{m_t^\theta - B_t^\theta} \int_{-\infty}^{\infty} (e^{\sigma z} - e^{\sigma y}) e^{\theta x} \mathbb{I}_A(y + B_t^\theta, z + B_t^\theta) \\ & \quad g(x, y, z; \tau) dx dy dz \\ & - \int_0^{M_t^\theta - B_t^\theta} \int_{-\infty}^{m_t^\theta - B_t^\theta} \int_{-\infty}^{\infty} e^{\sigma y + \theta x} \mathbb{I}_A(y + B_t^\theta, M_t^\theta) g(x, y, z; \tau) dx dy dz \\ & + \int_{M_t^\theta - B_t^\theta}^{\infty} \int_{m_t^\theta - B_t^\theta}^0 \int_{-\infty}^{\infty} e^{\sigma z + \theta x} \mathbb{I}_A(m_t^\theta, z + B_t^\theta) g(x, y, z; \tau) dx dy dz, \end{aligned}$$

since $A = \{(y, z) \mid K \leq e^{\sigma z} - e^{\sigma y}\}$ and where $g = g_{B, m, M}$. This completes the proof. \square

It is possible to simplify the function Ψ . The details are given in Appendix A.

Of course, the payoff and the hedging portfolio of a spread lookback put option are similar and the computations of the latter follow the same steps. Numerical prices of these options can be found in He et al. (1998).

In Corollary 6.3, if $K = 0$ then $\mathbb{I}_A \equiv 1$ and the payoff becomes $M_T^S - m_T^S$. Consequently, the hedging portfolio in Corollary 6.2 is recovered, as one would expect.

7. ACKNOWLEDGMENTS

The authors wish to thank the Associate Editor and the referees for a careful reading of the paper and helpful comments. Partial funding in support of this work was provided by a doctoral scholarship of the Institut de Finance Mathématique de Montréal, as well as by the Natural Sciences and Engineering Research Council of Canada and by the Fonds québécois de la recherche sur la nature et les technologies.

APPENDIX A. SOME INTEGRAL MANIPULATIONS

In the way toward computing

$$\Psi(y, z; s) = \int_{\mathbb{R}} e^{\theta x} g_{B, m, M}(x, y, z; s) dx$$

where $g(\cdot; s)$ is the joint PDF of (B_s, m_s, M_s) , one has to compute integrals of the form:

$$\int_y^z e^{\theta x} \phi''\left(\frac{x+a}{\sqrt{s}}\right) dx,$$

for some constant a and where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Integrating by parts twice yields the following:

$$\begin{aligned} \int_y^z e^{\theta x} \phi''\left(\frac{x+a}{\sqrt{s}}\right) dx &= \sqrt{s} \left[e^{\theta z} \phi'\left(\frac{z+a}{\sqrt{s}}\right) - e^{\theta y} \phi'\left(\frac{y+a}{\sqrt{s}}\right) \right] \\ &\quad - s\theta \left[e^{\theta z} \phi\left(\frac{z+a}{\sqrt{s}}\right) - e^{\theta y} \phi\left(\frac{y+a}{\sqrt{s}}\right) \right] \\ &\quad + \sqrt{s} \theta^2 e^{s\frac{\theta^2}{2} - a\theta} \left[\Phi\left(\frac{z+a-s\theta}{\sqrt{s}}\right) \right. \\ &\quad \quad \left. - \Phi\left(\frac{y+a-s\theta}{\sqrt{s}}\right) \right] \\ &= \sqrt{s} e^{s\frac{\theta^2}{2}} e^{-a\theta} \left\{ \phi'(\bar{z}) - \phi'(\bar{y}) \right. \\ &\quad \quad \left. + \theta^2 \Phi'(\bar{z}) - \theta^2 \Phi'(\bar{y}) \right\}, \end{aligned}$$

where $\bar{w} = \frac{w+a-s\theta}{\sqrt{s}}$, for $w = y, z$. To simplify, define

$$H(y, z, a; s) = e^{-a\theta} \left\{ \phi'(\bar{z}) - \phi'(\bar{y}) + \theta^2 \Phi'(\bar{z}) - \theta^2 \Phi'(\bar{y}) \right\}.$$

Then,

$$\int_y^z e^{\theta x} \phi''\left(\frac{x+a}{\sqrt{s}}\right) dx = \sqrt{s} e^{s\frac{\theta^2}{2}} H(y, z, a; s).$$

Consequently, $\Psi(y, z; s)$ is given by

$$\begin{aligned} & \frac{4}{s} e^{s\frac{\theta^2}{2}} \sum_{n=1}^{\infty} n^2 H(y, z, 2ny - 2nz; s) \\ & - \frac{4}{s} e^{s\frac{\theta^2}{2}} \sum_{n=2}^{\infty} n(n-1) H(y, z, -2ny + 2(n-1)z; s) \\ & + \frac{4}{s} e^{s\frac{\theta^2}{2}} \sum_{n=1}^{\infty} n^2 H(y, z, -2ny + 2nz; s) \\ & - \frac{4}{s} e^{s\frac{\theta^2}{2}} \sum_{n=1}^{\infty} n(n+1) H(y, z, 2ny - 2(n+1)z; s). \end{aligned}$$

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