On a matrix representation for polynomially recursive sequences

Christophe Reutenauer

Laboratoire de combinatoire et d'informatique mathématique,
Université du Québec à Montréal
Case postale 8888, succursale Centre-ville
Montréal (Québec) H3C 3P8, Canada

`reutenauer.christophe@uqam.ca`

Abstract

In this article we derive several consequences of a matricial characterization of P-recursive sequences. This characterization leads to canonical representations of these sequences. We show their uniqueness for a given sequence, up to similarity. We study their properties: operations, closed forms, d'Alembertian sequences, field extensions, positivity, extension of the sequence to $\mathbb{Z}$, difference Galois group.

1 Introduction

Polynomial recursivity (Stanley [S1]), equivalently holonomy (Zeilberger [Z]), is a basic notion in the theory of integer sequences. It is known that P-recursive sequences coincide with D-finite series. These sequences are interesting in particular since many sequences appearing in mathematics are P-recursive, and that they contain rational and algebraic series; moreover, they may be effectively computed and manipulated, see the book by Petkovšek, Wilf and Zeilberger [PWZ].

Following ideas in algebraic automata theory, we define *representations* of P-recursive sequences. Such a matrix representation may be seen as an analogue of the representation $r(1)r(2)\cdots r(n)$ of a hypergeometric sequence, by replacing the rational function $r(t)$ by a matrix of such functions. The basic characterization is Th.2.1. This result motivates the notion of representation: the form we give is close to the Fliess representation (λ, μ, γ) of noncommutative rational series [F], following the work of Schützenberger (see [BR] and the references therein).

These representations behave nicely with respect to the usual operations with P-recursive sequences, notably the multiplivation. Minimal representations are studied: for

a fixed P-recursive *germ* (that is, a sequence viewed at infinity), they are all similar, in the sense of pseudo-linear algebra. It is shown how general representations of a fixed germ are related to minimal ones. From this it follows that closed forms and d'Alembertian germs (introduced by Abramov and Petkovšek [AP]) are nicely characterized by minimal representations: they have respectively diagonal and triangular form.

We study several questions of arithmetic nature: extension of the ground field, positivity, extension of sequences to $\mathbb{Z}$ (in a way different from Stanley in [S1]), Galois group in the sense of difference fields. We conclude by some comments on effectiveness and integer arithmetic.

The author thanks Marko Petkovšek for useful e-mails in 1998. Also François Bergeron, Sébastien Labbé and Claudine Mitschi for useful discussions and mails. Also the anonymous referee, who gave a long list of stylistic, grammatical and mathematical suggestions, which helped to improve the article.

2   Matrix representation

Let K be a field of characteristic 0. Recall that a sequence $f(n), n \in \mathbb{N}$, is said to be *polynomially recursive* (or *P-recursive*) if there exist $d \in \mathbb{N}^*$, $n_0 \in \mathbb{N}$ and d rational functions $r_1(t), \ldots, r_d(t) \in K(t)$ such that the $r_i(n)$ are defined for $n \geqslant n_0$ and that one has the recursion

$$\forall n \geqslant n_0, f(n+d) = r_1(n)f(n+d-1) + \cdots + r_d(n)f(n). \tag{1}$$

We call *length* of the recursion the integer d (whatever is the value of n_0).

The basic theorem, on which the whole paper rests, is the following. This result is implicitly contained in Proposition 4.1 of the book of van der Put and Singer [PS]. The "if" part is Theorem 1 of [LRZ]. We formulate it in a more direct manner. Although this result can be derived from these two articles, we give here a direct and simple proof [1].

Before stating the result, recall that a *hypergeometric* sequence is a sequence f such that for some $r(t) \in K(t)$, one has $f(n+1) = f(n)r(n+1)$ if n is large enough. Then for some n_0, $f(n) = f(n_0)r(n_0+1)r(n_0+2)\cdots r(n)$ for any $n \geqslant n_0$. Thus, in view of the theorem below, P-recursive sequences appear as a matrix version of hypergeometric sequences.

Theorem 2.1. *A sequence $f(n)$ is P-recursive if and only if there exist $d \in \mathbb{N}^*$, a matrix $M(t) \in K(t)^{d \times d}$, a row matrix $\lambda \in K^{1 \times d}$ and a column matrix $\gamma(t) \in K(t)^{d \times 1}$ such that $M(n)$ is defined for each integer $n \geqslant n_0 + 1$, that $\gamma(n)$ is defined for $n \geqslant n_0$ and that*

$$\forall n \geqslant n_0, f(n) = \lambda M(n_0+1)M(n_0+2)\cdots M(n)\gamma(n). \tag{2}$$

We give two examples. The number D_n of derangements (permutations without fixed points) in the symmetric group S_n satisfies $D_0 = 1, D_1 = 0$ and $D_{n+2} = (n+1)D_{n+1} +$

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[1]This was originally obtained independently by the author in 1997.

$(n+1)D_n$. Hence, we see by the proof below that

$$D_n = (1,0) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \cdots \begin{pmatrix} 0 & n \\ 1 & n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The number I_n of involutions in S_n satisfies $I_0 = I_1 = 1$ and $I_{n+2} = I_{n+1} + (n+1)I_n$. Thus

$$I_n = (1,1) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & n \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In these two examples, we have $n_0 = 0$, with the notations of the theorem.

Proof. Suppose that $f(n)$ satisfies Eq.(1). Define $M(t) \in K(t)^{d \times d}$ to be the companion matrix

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & r_d(t-1) \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & r_2(t-1) \\ 0 & \cdots & & 1 & r_1(t-1) \end{pmatrix}$$

Note that $M(n)$ is defined for $n \geqslant n_0+1$. Let also $R(n)$ be the row matrix $(f(n),\ldots,f(n+d-1)) \in K^{d \times 1}$. Then

$$R(n+1) = (f(n+1),\ldots,f(n+d))$$

$$= (f(n),\ldots,f(n+d-1)) \begin{pmatrix} 0 & \cdots & \cdots & 0 & r_d(n) \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & r_2(n) \\ 0 & \cdots & & 1 & r_1(n) \end{pmatrix} = R(n)M(n+1)$$

for any $n \geqslant n_0$. We deduce that for any such n,

$$R(n) = R(n_0)M(n_0+1)M(n_0+2) \cdots M(n).$$

It suffices now to take $\lambda = R(n_0)$ and $\gamma(t) = (1,0,\ldots,0)^T$.

Conversely, suppose that Eq.(2) holds. Consider the column vectors $\gamma(t), M(t+1)\gamma(t+1), M(t+1)M(t+2)\gamma(t+2),\ldots$, in the d-dimensional vector space $K(t)^{d \times 1}$ over $K(t)$. There exists $l \leqslant d$ and $r_1(t),\ldots,r_l(t) \in K(t)$ such that

$$M(t+1) \cdots M(t+l)\gamma(t+l) = r_1(t)M(t+1) \cdots M(t+l-1)\gamma(t+l-1)$$

$$+ \cdots + r_{l-1}(t)M(t+1)\gamma(t+1) + r_l(t)\gamma(t).$$

Let $n_1 \geqslant n_0$ such that the $r_i(n)$ are defined for $n \geqslant n_1$. By letting $t = n$ and multiplying at the left by $\lambda M(n_0+1)M(n_0+2) \cdots M(n)$ and using Eq.(2), we obtain that for any $n \geqslant n_1$

$$f(n+l) = \lambda M(n_0+1)M(n_0+2) \cdots M(n+l)\gamma(n+l)$$

$$= \lambda M(n_0 + 1)M(n_0 + 2) \cdots M(n)M(n+1) \cdots M(n+l)\gamma(n+l)$$
$$= r_1(n)\lambda M(n_0 + 1)M(n_0 + 2) \cdots M(n)M(n+1) \cdots M(n+l-1)\gamma(n+l-1)$$
$$+ \cdots + r_{l-1}(n)\lambda M(n_0 + 1)M(n_0 + 2) \cdots M(n)M(n+1)\gamma(n+1)$$
$$+ r_l(n)\lambda(n)M(n_0 + 1)M(n_0 + 2) \cdots M(n)\gamma(n)$$
$$= r_1(n)f(n+l-1) + \cdots + r_{l-1}(n)f(n+1) + r_l(n)f(n).$$

This proves that f is P-recursive. □

The right-hand side of Eq.(2) may be interpreted as a sum of labels of paths. We give this automata-like interpretation in the case $n_0 = 0$ for simplicity. Consider indeed the directed graph with vertices $1, \ldots, d$ and edge $i \to j$ labelled $M(n)_{ij}$ if the latter is nonzero. To each path $\pi = i_0 i_1 \cdots i_n$ of length n in this graph, we assign the label

$$l(\pi) = M(1)_{i_0 i_1} M(2)_{i_1 i_2} \cdots M(n)_{i_{n-1} i_n}.$$

Then

$$f(n) = \sum_{ij} \lambda_i \left(\sum l(\pi) \right) \gamma_j(n), \tag{3}$$

where the second sum is over all paths π from i to j of length n.

We illustrate this by in Figure 1: the incoming (resp. outgoing) arrows indicate the row vector λ (resp. column vector $\gamma(n)$); they are labelled with the corresponding element of the vector, with the following conventions: no label indicates that the label is 1; no arrow indicates that the label is 0. Thus, the number of involutions in S_n is equal to the sum of the labels of the paths of length n which end at 1; for example, for $n = 3$, we have the paths $1 \to 2 \to 2 \to 1$ with label $1 \times 1 \times 1$, $2 \to 2 \to 2 \to 1$ with label $1 \times 1 \times 1$ and $2 \to 1 \to 2 \to 1$ with label $1 \times 2 \times 1$, so that the number of involutions is 4, as it must be.
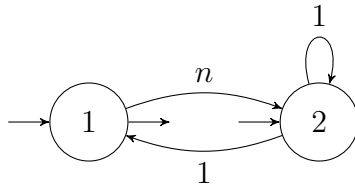

Figure 1: Involutions

3 Germs

Following [S1], [S2] Section 6.4, we define an equivalence on the set of sequences over K by $f \sim g$ if for some n_0, one has: $\forall n \geqslant n_0, f(n) = g(n)$. The equivalence class of f is denoted by $[f]$ and called a *germ*. The set $\mathcal{G}$ of germs over K is a vector space over $K(t)$.

The result of the action of $r(t) \in K(t)$ on $\phi \in \mathcal{G}$ is $\gamma \in \mathcal{G}$ as follows: if the sequence f is a representative of the germ ϕ, then the sequence $g(n)$ defined by $g(n) = r(n)f(n)$, n large enough, is a representative of γ; this is well-defined (*loc. cit*).

Let $\phi = [f]$. The *shift mapping E* is defined on germs by $E(\phi) = \gamma$, where $\gamma = [g]$, with g the sequence $g(n) = f(n+1)$. It is a K-linear automorphism of the K-vector space $\mathcal{G}$. It is not $K(t)$-linear, but satisfies the equation

$$E(r\phi) = \theta(r)E(\phi), \tag{4}$$

where θ is the K-automorphism of $K(t)$ which sends t onto $t + 1$.

A sequence is P-recursive if and only if all the sequences in its equivalence class are P-recursive, [S1], Th.1.4. In this case, we say that the class is a *P-recursive germ*.

4   Representations

Let ϕ be a P-recursive germ. We call *representation* of ϕ a quadruple $(\lambda, M(t), \gamma(t), n_0)$, satisfying the conditions stated in Th.2.1, and such that the sequence f defined by Eq.(2) is a representative of ϕ; its *dimension* is d. We also say that $(\lambda, M(t), \gamma(t), n_0)$ *represents* ϕ and f.

For $k \in \mathbb{N}$, the *k-translate* of representation $(\lambda, M(t), \gamma(t), n_0)$ is the representation $(\lambda', M(t), \gamma(t), n_0')$, with $\lambda' = \lambda M(n_0 + 1)M(n_0 + 2) \cdots M(n_0 + k)$ and $n_0' = n_0 + k$. It represents the same germ: indeed, for any $n \geqslant n_0'$,

$$\lambda' M(n_0' + 1)M(n_0' + 2) \cdots M(n)\gamma(n) =$$

$$\lambda M(n_0 + 1)M(n_0 + 2) \cdots M(n_0 + k)M(n_0 + k + 1)M(n_0 + k + 2) \cdots M(n)\gamma(n)$$

$$= \lambda M(n_0 + 1)M(n_0 + 2) \cdots M(n)\gamma(n).$$

A *translate* of a representation is a k-translate for some natural number k. Note that translating a representation $(\lambda, M(t), \gamma(t), n_0)$ changes neither $M(t)$ nor $\gamma(t)$.

The next result is then evident.

Lemma 4.1. *A P-recursive germ has representations $(\lambda, M(t), \gamma(t), n_0)$, with arbitrarily large n_0.*

For later use, we state the following lemmas.

Lemma 4.2. *If $(\lambda, M(t), \gamma(t), n_0)$ represents ϕ, then for any natural integer i, $E^i(\phi)$ is represented by $(\lambda, M(t), M(t + 1) \cdots M(t + i)\gamma(t + i), n_0)$.*

Proof. Indeed, we have for any $n \geqslant n_0$, $f(n+i) = \lambda M(n_0+1)M(n_0+2) \cdots M(n+i)\gamma(n+i) = \lambda M(n_0 + 1)M(n_0 + 2) \cdots M(n)(M(n + 1) \cdots M(n + i)\gamma(n + i))$. $\qquad \square$

Lemma 4.3. *Let ϕ, ϕ_i, $i = 1, \ldots, d$ be germs and $M(t) \in K(t)^{d \times d}$, $\gamma(t) \in K(t)^{d \times 1}$ such that $E(\phi_j) = \sum_{1 \leqslant i \leqslant d} M(t + 1)_{ij}\phi_i$ for $j = 1, \ldots, d$, and a column vector $\gamma(t)$ over R such that $\phi = \sum_{1 \leqslant i \leqslant d} \gamma(t)_i\phi_i$. Then there exist $\lambda \in K^{1 \times d}$ and $n_0 \in \mathbb{N}$ such that $(\lambda, M(t), \gamma(t), n_0)$ is a representation of ϕ.*

Proof. Let f, f_i be representatives of ϕ, ϕ_i and let n_0 be such that

$$f_j(n+1) = \sum_{1 \leqslant i \leqslant d} M(n+1)_{ij} f_i(n)$$

and $f(n) = \sum_{1 \leqslant i \leqslant d} \gamma(n)_i f_i(n)$ for any $n \geqslant n_0$. Let $F(n) = (f_1(n), \ldots, f_d(n))$ and $\lambda = (f_1(n_0), \ldots, f_d(n_0))$. Then $F(n+1) = F(n)M(n+1)$ and therefore $F(n) = F(n_0)M(n_0 + 1)M(n_0+2)\ldots M(n)$. Since $f(n) = F(n)\gamma(n)$, the lemma follows. $\qquad\square$

5   Product and other operations

We consider here several operations which preserve P-recursiveness. The results are of course not new. However, matrix representations are particularly well-adapted for the proofs.

Recall that the *product* of two germs $[f(n)]$ and $[g(n)]$ is the germ $[f(n)g(n)]$. This is well-defined.

In order to understand the generality of the next result (since there is an implicit assumption on the constant n_0), recall Lemma 4.1.

Theorem 5.1. *Consider the representations* $(\lambda_i, M_i(t), \gamma_i(t), n_0)$ *of two germs* ϕ_i, $i = 1, 2$
.

(i) $\phi_1 + \phi_2$ *has the representation*

$$\left((\lambda_1, \lambda_2), \begin{pmatrix} M_1(t) & 0 \\ 0 & M_2(t) \end{pmatrix}, \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} \right).$$

(ii) $\phi_1\phi_2$ *has the representation* $(\lambda_1 \otimes \lambda_2, M_1(t) \otimes M_2(t), \gamma_1(t) \otimes \gamma_2(t), n_0)$.

Consider now a single germ $\phi = [f]$ *with representation* $(\lambda, M(t), \gamma(t), n_0)$. *Let* $r(t) \in K(t)$. *Let* i, p *be natural integers.*

(iii) $r(t)\phi$ *has the representation* $(\lambda, M(t), r(t)\gamma(t), n_0)$, *provided that* $r(t)$ *is defined for* $n \geqslant n_0$.

(iv) The germ $[f(i + pn)_{n \in \mathbb{N}}]$ *has the representation* $(\lambda', M'(t), \gamma'(t), n'_0)$, *where the natural integers* k, n'_0 *have been chosen in such a way that* $n_0 + k = i + pn'_0$, *with* $\lambda' = \lambda M(n_0 + 1) \cdots M(n_0 + k)$, $M'(t) = M(i + p(t-1) + 1)M(i + p(t-1) + 2) \cdots M(i + pt)$ *and* $\gamma'(t) = \gamma(i + pt)$.

Proof. (i) and (iii) are straighforward. (ii) is a consequence of the multiplicativity of tensoring matrices: $(A_1 \otimes A_2)(B_1 \otimes B_2) = (A_1 B_1) \otimes (A_2 B_2)$. We verify (iv): let $n \geqslant n'_0$; then

$$\lambda' M'(n'_0 + 1) \cdots M'(n)\gamma'(n) =$$

$$\lambda M(n_0 + 1) \cdots M(n_0 + k)M(i + p(n'_0 + 1 - 1) + 1) \cdots M(i + p(n'_0 + 1)) \cdots$$

$$\cdots M(i + p(n-1) + 1) \cdots M(i + pn)\gamma(i + pn) = f(i + pn).$$

$$\square$$

Corollary 5.2. *Let p be a natural integer and f be a sequence. Then f is P-recursive if and only if for any $i = 0, \ldots, p-1$, the sequence $f(i + pn)_{n \in \mathbb{N}}$ is P-recursive.*

Proof. The "only if" part follows from Th.5.1 (iv). Let us prove the converse. Fix p and $i, 0 \leqslant i \leqslant p-1$. Since P-recursiveness is preserved by addition, it is enough to show that if a sequence $g(n)$ is P-recursive, then the following sequence $f(n)$ is also P-recursive: $f(i + pn) = g(n)$, and $f(n) = 0$ if n is not congruent to i modulo p.

Suppose that g satisfies

$$\forall n \geqslant n_0, \; g(n + d) = r_1(n)g(n + d - 1) + \cdots + r_d(n)g(n). \tag{5}$$

Then f satisfies the equation $f(n + dp) = r_1((n - i)/p)f(n + (d - 1)p) + \cdots + r_d((n - i)/p)f(n)$ for n large enough. Indeed, if n is not congruent to i modulo p, then so are $n + dp, \, n + (d - 1)p, \ldots, n$, hence the corresponding values of f vanish and the equation is satisfied. Now, in the other case, replace in this equation n by $i + pn$, obtaining: $f(i + pn + dp) = r_1(n)f(i + pn + (d - 1)p) + \cdots + r_d(n)f(i + pn)$, that is $g(n + d) = r_1(n)g(n + d - 1) + \cdots + r_d(n)g(n)$, which is Eq.(5). $\qquad \square$

We shall also need the next result.

Lemma 5.3. *Let ϕ_i have the representation $(\lambda, M(t), \gamma_i(t), n_0)$, for $i = 1, \ldots, l$. If $\sum_i r_i(t)\gamma_i(t) = 0$, then $\sum_i r_i(t)\phi_i = 0$.*

Proof. Without harm, we may assume that the $r_i(t)$ are polynomials. In this case, $r_i(n)$ is defined for $n \geqslant n_0$ and it follows from Th.5.1 (iii) that $r_i(t)\phi_i$ has the representation $(\lambda, M(t), r_i(t)\gamma_i(t), n_0)$. It is easily deduced that $\sum_i r_i(t)\phi_i$ has the representation $(\lambda, M(t), \sum_i r_i(t)\gamma_i(t), n_0)$. Hence $\sum_i r_i(t)\phi_i = 0$. $\qquad \square$

6 Pseudo-linear algebra

In this section, we follow [J], see also [BP], with the addition of some apparently well-known results on diagonalizability and triangularizability, with their proofs, for which we could not find references. We restrict the presentation of his work to the aspects that we need here [2]. Note that pseudo-linear algebra has already been used previously in the study of P-recursive sequences; see [BP].

Let V, W be vector spaces over $F = K(t)$. An additive homomorphism E from V into W is called *pseudo-linear* if for any r in $K(t)$ and v in V, one has $E(rv) = \theta(r)E(v)$, where θ is as in Section 3. Thus the mapping E in Section 3 is pseudo-linear, by Eq.(4).

Given bases (v_j) and (w_i) of V, W, the matrix of E is as usually the matrix $M(E) = (m_{ij})$ with $E(v_j) = \sum_i m_{ij}w_i$. If the column vector $\gamma \in F^{\dim V \times 1}$ represents some vector $v \in V$ in the basis (v_j), then $E(v)$ is represented in the basis (w_i) by the column vector

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[2] In full generality, one considers a field F with an automorphism θ and a θ-derivation; K is then the *subfield of constants*, that is the subset of elements fixed by θ and which are in the kernel of the derivation; here we take $F = K(t)$ with $\theta(t) = t + 1$ and the zero derivation.

$M(E)\gamma^\theta$. If u is a *linear* mapping, left composable with E, then $u \circ E$ is a pseudo-linear mapping and $M(u \circ E) = M(u)M(E)$; if u is right composable with E, then $E \circ u$ is pseudo-linear and $M(E \circ u) = M(E)M(u)^\theta$. In particular, if v is a $K(t)$-linear isomorphism, and if $v^{-1} \circ E \circ v$ is defined, then

$$M(v^{-1} \circ E \circ v) = M(v)^{-1}M(E)M(v)^\theta. \tag{6}$$

Viewed otherwise, if E is a pseudo-linear endomorphism of V and has the matrices M_e and M_v in the bases (e_i) and (v_i) respectively, then these matrices are related by $M_v = P^{-1}M_e P^\theta$, where P is the matrix whose j-th column represents the coefficients of (v_j) expressed in the basis (e_i) (see [J] p. 486). We say that two matrices M, M' are *pseudo-similar* if for some square invertible matrix P one has $M' = P^{-1}MP^\theta$.

Let E be a pseudo-linear endomorphism of V. A subspace W of V is called *stable* under E if $E(W) \subset W$. In that case E induces a pseudo-linear endomorphism of W and of V/W.

As in the classical case, if $E(v) = rv$ for some $v \in V \setminus \{0\}$ and $r \in F$, we call v an *eigenvector* of E, and r the corresponding *eigenvalue*. Note that in this case, for any $a \in F^*$, av is also an eigenvector; however, its eigenvalue is not r in general, but $r\frac{a^\theta}{a}$; indeed, $E(av) = a^\theta E(v) = a^\theta rv = r\frac{a^\theta}{a}(av)$. One therefore cannot define eigenspaces as in the classical case. Observe that however the set of eigenvectors with a given eigenvalue, together with the zero vector, is a K-subspace.

We say that the pseudo-linear endomorphism E of V is *diagonalizable* if there is a basis of V consisting of eigenvectors. Note that if V has a spanning set of eigenvectors of E, then it is diagonalizable. Equivalently, E is diagonalizable if and only if in some basis, its matrix is a diagonal matrix. Correspondingly, we say that a square matrix M over F is *pseudo-diagonalizable* if it is the matrix of some diagonalizable pseudo-linear endomorphism; in other words, there exists a square invertible matrix P such that $P^{-1}MP^\theta$ is a diagonal matrix.

If W is a subspace of V, stable under E, a diagonalizable pseudo-linear endomorphism of V, then the endomorphisms induced by E on W and V/W are diagonalizable. This is easy to see for V/W, since the basis of eigenvectors of E gives a spanning set of eigenvectors in V/W. For W, we do as follows: consider the ring of skew polynomials $R = F[x]$, where the product is subject to the rule $xa = a^\theta x$ for any $a \in F$. Then V is a left R-module, extending its structure of vector space over F, by the rule $xv = E(v)$. Now, stable subspaces in the previous sense coincide with R-submodules. If E is assumed to be diagonalizable, then V is a semisimple R-module, since for any eigenvector v, the line Fv is stable, hence a simple submodule. Hence V is a sum of simple submodules, hence semisimple. It is well-known that each simple submodule must be one of these. Now each submodule of a semisimple module is also semisimple, which implies that W is a sum of simple submodules, and so the restriction of E to W is diagonalizable.

We say that the endomorphism E of V is *triangularizable* if there exists a maximal chain of subspaces of V, each of them being moreover stable; in other words, there exists a basis in which the matrix of E is an upper (or equivalently lower) triangular matrix. Correspondingly, we say that a square matrix M over F is *pseudo-triangularizable* if it is

the matrix of some triangularizable pseudo-linear endomorphism; in other words, there exists a square invertible matrix P such that $P^{-1}MP^\theta$ is a triangular matrix.

If W is a subspace of V, stable under E, triangularizable pseudo-linear endomorphism of V, then the endomorphisms induced by E on W and V/W are triangularizable. Indeed, the chain of subspaces above induces a similar chain in W and V/W.

7   Minimal representations

A representation of the germ ϕ is called *minimal* if it has the minimal dimension among all representations of ϕ. We call this dimension the *rank* of ϕ. The lemma below follows from the proof of Th.2.1.

Lemma 7.1. *Let f be a sequence over K. The rank of $[f]$ is equal to the smallest length of a recursion for f.*

The following lemma is a consequence of the previous one and of [S1], Th. 1.6 and its proof.

Lemma 7.2. *The rank of $[f]$ is equal to the dimension over $K(t)$ of the vector subspace spanned by the germs $E^i[f]$, $i \in \mathbb{N}$. More precisely, a basis of this subspace is formed by these elements for $i = 0, \ldots, d-1$, where d is the rank.*

Similarly, we have, using the proof of Th.2.1,

Lemma 7.3. *Let ϕ be a P-recursive germ. The following conditions are equivalent, for elements $r_1(t), \ldots, r_d(t)$ of $K(t)$:*
(i) the shortest recursion satisfied by a representative f of ϕ is Eq.(1);
(ii) ϕ has a representation $(\lambda, M(t), \gamma(t), n_0)$ with $M(t+1) \cdots M(t+d)\gamma(t+d) = r_1(t)M(t+1) \cdots M(t+d-1)\gamma(t+d-1) + \cdots + r_{d-1}(t)M(t+1)\gamma(t+1) + r_d(t)\gamma(t)$ and d is the smallest possible.

We can give an intrinsic characterization of minimal representations.

Proposition 7.4. *Representation $(\lambda, M(t), \gamma(t), n_0)$ of dimension d is minimal if and only if the following two conditions hold:*
(i) the d column vectors $\gamma(t)$, $M(t+1)\gamma(t+1), \ldots, M(t+1) \cdots M(t+d-1)\gamma(t+d-1)$ form a basis of the $K(t)$-vector space $K(t)^{d\times 1}$;
(ii) let e_i be the canonical basis of $K(t)^{d\times 1}$. Then the d germs represented by the representations $(\lambda, M(t), e_i, n_0)$ are linearly independent over $K(t)$.

Note that condition (i) means equivalently that the vectors $M(t+1)M(t+2) \ldots M(t+i)$, $i \in \mathbb{N}$ span the $K(t)$-vector space $K(t)^{d\times 1}$; this follows from the calculations in the second part of the proof of Theorem 2.1. Moreover, condition (ii) means equivalently that for no nonzero vector $v(t) \in K[t]^{d\times 1}$, the germ represented by $(\lambda, M(t), v(t), n_0)$ is the zero germ.

Proof. Suppose that the given representation is minimal. Then d is the rank of the germ ϕ represented by it. Condition (i) follows from the proof of Th. 2.1: indeed, if these vectors do not form a basis, they are not linearly independent; then that proof shows that the germ satisfies a linear recursion of length smaller than d, contradicting Lemma 7.1.

Suppose that condition (ii) does not hold. Then these d germs ϕ_i are linearly dependent and the subspace they span is therefore of dimension less than d over $K(t)$. The germs $E^j(\phi)$ are represented by $(\lambda, M(t), M(t+1)\cdots M(t+j)\gamma(t+j), n_0)$ (Lemma 4.2). We have, for some elements $r_i(t)$ of $K(t)$: $M(t+1)\cdots M(t+j)\gamma(t+j) = \sum_{i=1}^d r_i(t)e_i$; by Lemma 5.3, we deduce that $E^j(\phi) = \sum_{i=1}^d r_i(t)\phi_i$; thus $E^j(\phi)$ belongs to a subspace of dimension $< d$, contradicting Lemma 7.2.

Conversely, suppose that these two conditions hold. By condition (i), the e_i are linear combinations over $K(t)$ of the vectors in (i). Now, by Lemma 5.3, the d germs of (ii) are linear combinations of the d germs represented by the representations $(\lambda, M(t), M(t+1)\cdots M(t+i-1)\gamma(t+i-1), n_0)$, $i = 0, \ldots, d-1$. Thus, by (ii), the d latter germs are linearly independent over $K(t)$. But these germs are $E^i(\phi)$, which shows by Lemma 7.2 that d is $\leqslant$ the rank of ϕ. Hence this rank is d and the representation is therefore minimal. $\qquad\square$

We give now two examples which show that the conditions (i) and (ii) in the proposition are independent. Consider the representation

$$\left((1,0), \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0\right).$$

Since $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, condition (i) is satisfied. Now the representation

$$\left((1,0), \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0\right)$$

represents the zero sequence, so that condition (ii) is not satisfied.

For the converse counter-example, we choose the transpose representation

$$\left((1,0), \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0\right).$$

Since $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, condition (i) is not satisfied; moreover, this representation represents the constant sequence (1). Now the representation

$$\left((1,0), \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0\right)$$

represents the sequence $(2^n - 1)$, since $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$. Since the germs represented by the sequences (1) and $(2^n - 1)$ are linearly independent over $K(t)$ (by Petkovsek's algorithm), (ii) is satisfied.

8 Similarity

The aim of this section is to show how the different minimal representations of ϕ are related to each other.

Two representations $(\lambda, M(t), \gamma(t), n_0)$ and $(\lambda', M'(t), \gamma'(t), n_0')$ are *strongly similar* if they have the same dimension d, if $n_0 = n_0'$ and if there exists a matrix $P(t) \in GL_d(K(t))$ such that $\lambda' = \lambda P(n_0)$, $M'(t) = P(t-1)^{-1} M(t) P(t)$, $\gamma'(t) = P(t)^{-1} \gamma(t)$ and that $P(n), P(n)^{-1}$ are defined for any $n \geqslant n_0$. Then they represent the same germ: indeed, for any $n \geqslant n_0$, one has

$$\lambda' M'(n_0 + 1) M'(n_0 + 2) \cdots M'(n) \gamma'(n)$$

$$= \lambda P(n_0) P(n_0)^{-1} M(n_0 + 1) P(n_0 + 1) P(n_0 + 1)^{-1} M(n_0 + 2) P(n_0 + 2) \cdots$$

$$\cdots P(n-1)^{-1} M(n) P(n) P(n)^{-1} \gamma(n) = \lambda M(n_0 + 1) M(n_0 + 2) \cdots M(n) \gamma(n).$$

Two representations are called *similar* if they have translates which are strongly similar. They represent therefore the same germ. The following result is the converse, under the hypothesis of minimality.

Theorem 8.1. *Any two minimal representations of the same P-recursive germ are similar.*

In particular, we obtain

Corollary 8.2. *If $(\lambda, M(t), \gamma(t), n_0)$ and $(\lambda', M'(t), \gamma'(t), n_0')$ are two minimal representations of the same germ, then the matrices $M(t)$ and $M'(t)$ are pseudo-similar.*

We begin by some lemmas.

Lemma 8.3. *Let $(\lambda, M(t), \gamma(t), n_0)$ be a minimal representation of dimension d, and let $(\lambda', M(t), \gamma(t), n_0')$, where $n_0' = n_0 + k$, be its k-translate. Then for all sufficiently large k the column vectors $\gamma(n_0')$, $M(n_0' + 1)\gamma(n_0' + 1),\ldots,\ M(n_0' + 1)M(n_0' + 2)\ldots M(n_0' + d - 1)\gamma(n_0' + d - 1)$ are K-linearly independent.*

Proof. We know by Proposition 7.4 that the d column vectors $\gamma(t)$, $M(t + 1)\gamma(t + 1),\ldots,M(t+1)\cdots M(t+d-1)\gamma(t+d-1)$ form a basis of the $K(t)$-vector space $K(t)^{d \times 1}$. Hence the $d \times d$ determinant over $K(t)$ formed by these columns is nonzero. Thus, if k is large enough, this determinant is nonzero when we put $t = n_0 + k$. Let $n_0' = n_0 + k$. Then the k-translate of $(\lambda, M(t), \gamma(t), n_0)$ is $(\lambda', M(t), \gamma(t), n_0')$ with $n_0' = n_0 + k$. Thus the lemma follows. $\square$

Lemma 8.4. *Let $(\lambda, M(t), \gamma(t), n_0)$ and $(\lambda', M'(t), \gamma'(t), n_0')$ be two minimal representations of dimension d of the same germ. If there exists a matrix $P(t) \in GL_d(K(t))$ such that $M'(t) = P(t-1)^{-1} M(t) P(t)$ and $\gamma'(t) = P(t)^{-1} \gamma(t)$, then these representations are similar.*

Proof. By taking translates of these representations, we may assume that $n_0 = n'_0$, that the sequences that they represent coincide for $n \geqslant n_0$, that $P(n), P(n)^{-1}$ are defined for $n \geqslant n_0$, and that the column vectors $\gamma(n_0)$, $M(n_0 + 1)\gamma(n_0 + 1),\ldots$, $M(n_0 + 1)M(n_0 + 2)\ldots M(n_0 + d - 1)\gamma(n_0 + d - 1)$ are K-linearly independent (Lemma 8.3).

Consider the representation $(\lambda_1, M'(t), \gamma'(t), n_0)$ with $\lambda_1 = \lambda P(n_0)$. It is strongly similar to $(\lambda, M(t), \gamma(t), n_0)$. Moreover, since strongly similar representations represent the same sequence for $n \geqslant n_0$, we have $\lambda_1 M'(n_0 + 1)M'(n_0 + 2)\cdots M'(n_0 + i)\gamma'(n_0 + i) = \lambda P(n_0)P(n_0)^{-1}M(n_0+1)P(n_0+1)\cdots P(n_0+i-1)^{-1}M(n_0+i)P(n_0+i)P(n_0+i)^{-1}\gamma(n_0+i) = \lambda M(n_0 + 1)\cdots M(n_0 + i)\gamma(n_0 + i) = \lambda' M'(n_0 + 1)M'(n_0 + 2)\cdots M'(n_0 + i)\gamma'(n_0 + i)$ for any $i \in \mathbb{N}$. Thus, by the linear independence above, we have $\lambda_1 = \lambda'$. This proves the lemma. $\square$

We now prove the theorem above.

Proof. Let $(\lambda, M(t), \gamma(t), n_0)$ be a minimal representation of a germ ϕ. Then

$$(\lambda, M(t), M(t + 1) \cdots M(t + i)\gamma(t + i), n_0)$$

is a representation of the germ $E^i(\phi)$ for any $i \in \mathbb{N}$ (Lemma 4.2). By Lemma 7.2 and Proposition 7.4, we obtain therefore a $K(t)$-linear isomorphism u from the vector subspace of $\mathcal{G}$ spanned by the $E^i(\phi)$ into the vector space $K(t)^{d\times1}$, which sends $E^i(\phi)$ onto $M(t + 1)\cdots M(t + i)\gamma(t + i)$, for $i = 0,\ldots, d - 1$. Let G be the mapping sending any $\gamma'(t) \in K(t)^{d\times1}$ onto $M(t + 1)\gamma'(t + 1)$. The mapping G is pseudo-linear and its matrix in the canonical basis is $M(t + 1)$. One has $u \circ E = G \circ u$, by Lemma 7.3.

Taking another minimal representation $(\lambda', M'(t), \gamma'(t), n'_0)$ and doing the same thing, we have $u' \circ E = G' \circ u'$. Therefore we obtain by composition a $K(t)$-linear automorphism $v = u \circ u'^{-1}$ of $K(t)^{d\times1}$ such that $G' = v^{-1} \circ G \circ v$ and $v^{-1}(\gamma) = \gamma'$. Therefore, the matrix of v being $P(t)$, we have by Eq.(6), $M'(t + 1) = P(t)^{-1}M(t + 1)P(t + 1)$, hence $M'(t) = P(t - 1)^{-1}M(t)P(t)$, and $P(t)^{-1}\gamma(t) = \gamma'(t)$. This implies the the theorem, by Lemma 8.4. $\square$

Similarity of minimal representations has the following two consequences.

Corollary 8.5. *If $(\lambda, M(t), \gamma(t), n_0)$ is a minimal representation of a P-recursive germ, then $M(t)$ is an invertible matrix.*

Proof. This follows from the proof of Theorem 8.1, noting that E is bijective; in the proof, G is therefore bijective, and its matrix is invertible; one concludes using Corollary 8.2. $\square$

Let $(\lambda, M(t), \gamma(t), n_0)$ be a representation of ϕ. By taking a suitable k-translate of this representation, we may by Corollary 8.5 assume that the matrices $M(n)$ are invertible over K for $n \geqslant n_0$. Let V denote the set of germs represented by $(v, M(t), \gamma(t), n_0)$, $v \in K^{1\times d}$; V is clearly a K-subspace of $\mathcal{G}$.

Corollary 8.6. *Suppose that $(\lambda, M(t), \gamma(t), n_0)$ is a minimal representation of ϕ with $M(n)$ invertible for $n \geqslant n_0$. Then the subspace V defined above depends only on ϕ.*

Proof. If two representations $(\lambda, M(t), \gamma(t), n_0)$ and $(\lambda', M'(t), \gamma'(t), n_0')$ are strongly similar, then the associated subspaces are equal, because $n_0 = n_0'$ and

$$M'(n_0 + 1)M'(n_0 + 2)\cdots M'(n)\gamma'(n)$$

$$= P(n_0)^{-1}M(n_0 + 1)P(n_0 + 1)P(n_0 + 1)^{-1}M(n_0 + 2)P(n_0 + 2)\cdots$$

$$\cdots P(n-1)^{-1}M(n)P(n)P(n)^{-1}\gamma(n) = P(n_0)^{-1}M(n_0 + 1)M(n_0 + 2)\cdots M(n)\gamma(n)$$

and because $P(n_0)$ is an invertible matrix over K.

Moreover, if $(\lambda', M'(t), \gamma'(t), n_0')$ is a k-translate of $(\lambda, M(t), \gamma(t), n_0)$, then we have $M' = M, \gamma' = \gamma$ and $n_0' = n_0 + k$. Thus for $n \geqslant n_0 + k$

$$M(n_0 + 1)M(n_0 + 2)\cdots M(n)\gamma(n) =$$

$$M(n_0 + 1)M(n_0 + 2)\cdots M(n_0 + k)M(n_0 + k + 1)M(n_0 + k + 2)\cdots M(n)\gamma(n)$$

$$= M(n_0 + 1)M(n_0 + 2)\cdots M(n_0 + k)M(n_0' + 1)M(n_0' + 2)\cdots M(n)\gamma'(n).$$

Since the matrix $M(n_0 + 1)M(n_0 + 2)\cdots M(n_0 + k)$ is invertible over K, the associated subspaces are equal.

Thus the corollary follows from Th. 8.1. $\qquad \square$

9 Minimal and nonminimal representations

We show how general representations of ϕ are related to minimal representations: they have a block decomposition, with 3 diagonal blocks, such that the middle block is a minimal decomposition.

Theorem 9.1. *Let $(\lambda, M(t), \gamma(t), n_0)$ be a representation of dimension d of a P-recursive germ ϕ. Then it is similar to a representation of the form $(\bar{\lambda}, \bar{M}(t), \bar{\gamma}(t), \bar{n}_0)$ with the compatible block decompositions $\bar{\lambda} = (0, \lambda', \times)$,*

$$\bar{M} = \begin{pmatrix} M_1 & \times & \times \\ 0 & M' & \times \\ 0 & 0 & M_2 \end{pmatrix}$$

and

$$\bar{\gamma} = \begin{pmatrix} \times \\ \gamma' \\ 0 \end{pmatrix},$$

where $(\lambda', M'(t), \gamma'(t), \bar{n}_0)$ is a minimal representation of ϕ.

Proof. 1. Consider the $K(t)$-vector space $J = K(t)^{d \times 1}$ and its pseudo-linear automorphism G, having matrix $M(t+1)$ in the canonical basis. It sends each vector $v(t)$ onto $M(t+1)v(t+1)$.

2. Consider the subset H of J of vectors $v(t)$ such that for any large enough $n \geqslant n_0$, $v(n)$ is defined and satisfies

$$\lambda M(n_0 + 1)M(n_0 + 2)\ldots M(n)v(n) = 0. \tag{7}$$

The subset H is a $K(t)$-subspace of J, as is seen by multiplying the latter equation by $r(n)$, for any $r(t) \in K(t)$. It is stable under G: indeed, if $v(t) \in H$, let $w(t) = G(v(t)) = M(t+1)v(t+1)$. Then we have Eq.(7), hence also for any $n \geqslant n_0$, $w(n)$ is defined for n large enough and $\lambda M(n_0+1)M(n_0+2)\ldots M(n)w(n) = \lambda M(n_0+1)M(n_0+2)\ldots M(n)M(n+1)v(n+1) = 0$. Thus $w(t) \in H$.

3. Now consider the $K(t)$-subspace I of J spanned by the vectors $M(t+1)M(t+2)\ldots M(t+i)\gamma(t+i)$ for $i \in \mathbb{N}$. It is also stable under G. Note that $\gamma(t) \in I$.

4. Note that if we translate the representation $(\lambda, M(t), \gamma(t), n_0)$, then the subspaces H, I are unchanged. Indeed, $M(t)$ and $\gamma(t)$ are unchanged, by Section 4, and this implies that I remains unchanged. For I, it follows from Eq.(7).

5. The subspace $H \cap I$ is also stable under G. Let $v_1(t), \ldots, v_d(t)$ be a basis of J such that $v_1, \ldots, v_{dim(H \cap I)}$ is a basis of $H \cap I$ and $v_1, \ldots, v_{dim(I)}$ is a basis of I. Consider the $K(t)$-linear automorphism u of J that sends e_j onto $v_j(t)$ ((e_j) is the canonical basis). Let $P(t)$ denote its matrix in the canonical basis; in other words, the j-th column of $P(t)$ is the vector $v_j(t)$. Choose $\bar{n}_0 \geqslant n_0$ large enough so that $P(n), P(n)^{-1}$ and the $v_i(n)$ are defined for $n \geqslant \bar{n}_0$, and that the vectors $v_i(\bar{n}_0)$, $i = 1, \ldots, dim(H \cap I)$, are linearly independent over K; moreover, we may assume that Eq.(7) holds for $n \geqslant \bar{n}_0$.

6. Now, we translate the representation $(\lambda, M(t), \gamma(t), n_0)$ by $k = \bar{n}_0 - n_0$: so we are reduced to the case $n_0 = \bar{n}_0$.

7. The matrix of the pseudo-linear mapping $v = u^{-1} \circ G \circ u$ in the canonical basis is by Eq.(6) equal to $P(t)^{-1}M(t+1)P(t+1)$. Let $\bar{M}(t) = P(t-1)^{-1}M(t)P(t)$ and $\bar{\lambda} = \lambda P(\bar{n}_0)$. The first $dim(H \cap I)$ coordinates of $\bar{\lambda}$ vanish, since by Eq.(7), one has $\lambda v_i(\bar{n}_0) = 0$ for $i = 1, \ldots, dim(H \cap I)$ (recall that $n_0 = \bar{n}_0$). Since $H \cap I$ and I are stable under G, the matrix of v, hence also the matrix $\bar{M}(t)$, has the block form indicated in the statement. The last $d - dim(I)$ coordinates of the vector $u^{-1}(\gamma(t))$ vanish since $\gamma(t)$ is in I; it is represented by the column vector $\bar{\gamma}(t) = P(t)^{-1}\gamma(t)$. Thus $\bar{\lambda}$ and $\bar{\gamma}$ have the block form of the statement.

8. It follows that $(\lambda', M'(t), \gamma'(t), \bar{n}_0)$ is a representation of ϕ, of dimension $e = dim(I) - dim(H \cap I) = dim(I/(H \cap I))$. We may identify $K(t)^{e \times 1}$ and $I/(H \cap I)$. We deduce from the construction and from Proposition 7.4 that it is minimal.

9. Indeed, if condition (ii) were not satisfied, there would exist a nonvanishing $v(t) \in K(t)^{e \times 1}$ such that $(\lambda', M'(t), v(t), \bar{n}_0)$ is the zero germ; thus we could find a nonvanishing element in H which is a linear combination of the vectors $v_i(t)$, $i = dim(H \cap I) + 1, \ldots, dim(I)$, contradicting the construction of the $v_i(t)$.

10. Moreover, condition (i) is satisfied, since the space I is spanned by the elements $M(t+1)M(t+2)\ldots M(t+i)\gamma(t+i)$ for $i \in \mathbb{N}$; hence the space $I/(H \cap I)$ is spanned by the vectors $M'(t+1)M'(t+2)\ldots M'(t+i)\gamma'(t+1)$ for $i \in \mathbb{N}$; so that these vectors, for $i = 0, \ldots, e-1$ form a basis of $K(t)^{e \times 1}$ and (i) holds (see the remark after Prop. 7.4). $\square$

10   Closed forms

We call a germ *hypergeometric* if it is of the form $[f]$ where f is a hypergeometric sequence. Clearly, hypergeometric germs are the P-recursive germs of rank 1; this follows since a $K(t)$-multiple of a hypergeometric germ is hypergeometric.

Following [P1] p. 259, or [PWZ] Definition 8.1.1, we say that a germ ϕ is a *closed form* if it is a sum of hypergeometric germs.

Theorem 10.1. *The following conditions are equivalent for a P-recursive germ:*
(i) it is a closed form;
(ii) it has a representation $(\lambda, M(t), \gamma(t), n_0)$ such that $M(t)$ is a diagonal matrix;
(iii) it has a minimal representation $(\lambda, M(t), \gamma(t), n_0)$ such that $M(t)$ is a diagonal matrix.

Proof. (i) implies (ii): suppose that germ ϕ is a sum of hypergeometric germs. Then clearly, ϕ has a representation $(\lambda, M(t), \gamma(t), n_0)$, not necessarily minimal, where $M(t)$ is a diagonal matrix.

(ii) implies (iii): By Section 6, we know that diagonalizable pseudo-linear endomorphisms induce, on stable subspaces and quotients, pseudo-linear diagonalizable endomorphisms. Thus, by Th.9.1, ϕ has a minimal representation $(\lambda', M'(t), \gamma'(t), n_0')$ such that $M'(t)$ is pseudo-diagonalizable.

Then we can find an invertible square matrix $Q(t)$ over $K(t)$ such that

$$Q(t)^{-1} M'(t) Q(t+1)$$

is diagonal. Writing $P(t) = Q(t+1)$, we see that $P(t)$ is square and invertible and that $M(t) = P(t-1)^{-1} M'(t) P(t)$ is diagonal. Let $\gamma(t) = P(t)^{-1} \gamma'$. Then by Lemma 8.4, our representation is similar to the representation $(\lambda, M(t), \gamma(t), n_0)$.

(iii) implies (i): Clearly, ϕ is a linear combination over $K(t)$ of hypergeometric germs. Since a $K(t)$-multiple of a hypergeometric germ is hypergeometric, we see that actually ϕ is a sum of hypergeometric germs. $\square$

The proof also shows the following two corollaries.

Corollary 10.2. *If a germ is a closed form, then it is a sum of exactly d hypergeometric germs, where d is its rank, and not less.*

According to [P1] Proposition 5.4, this sum is even unique.

Corollary 10.3. *A germ is a closed form if and only if for each minimal representation $(\lambda, M(t), \gamma(t), n_0)$, $M(t)$ is pseudo-diagonalizable.*

11 D'Alembertian germs

D'Alembertian germs have been introduced in [AP]. We follow Definition 1 of [AB]: a germ ϕ is *d'Alembertian* if there exist rational functions $r_1(t), \ldots, r_d(t)$ such that

$$(E - r_1(t)) \circ \cdots \circ (E - r_d(t))(\phi) = 0. \tag{8}$$

Theorem 11.1. *The following conditions are equivalent for a P-recursive germ:*
   (i) it is a d'Alembertian germ;
   (ii) it has a representation $(\lambda, M(t), \gamma(t), n_0)$ with $M(t)$ triangular.
   (iii) it has a minimal representation $(\lambda, M(t), \gamma(t), n_0)$ with $M(t)$ triangular.

Proof. (i) implies (ii): let ϕ be a d'Alembertian germ. Then it satisfies Eq. (8). We prove (ii) by induction on d. If $d = 1$, then ϕ is hypergeometric and it has a representation of dimension 1. Suppose that $d > 1$ and let $\phi' = (E - r_d(t))(\phi)$. Then ϕ' is d'Alembertian with $d - 1$. We conclude by induction that ϕ' has a representation $(\lambda', M'(t), \gamma'(t), n_0')$ of dimension d', where $M'(t)$ is an upper triangular matrix. For $n \geqslant n_0'$, consider the row vector $F(n) = \lambda' M'(n_0' + 1) M'(n_0' + 2) \cdots M'(n)$. Then $F(n + 1) = F(n) M'(n + 1)$ and, by definition of representations, the sequence f' defined by $f'(n) = F(n) \gamma'(n)$, $n \geqslant n_0$, is a representative of the germ ϕ'. Let f be a representative of the germ ϕ. Since $\phi' = (E - r_d(t))(\phi)$, we have for n large enough, $f(n + 1) = f'(n) + r_d(n) f(n) = F(n) \gamma'(n) + r_d(n) f(n)$. Let $n_0 \geqslant n_0'$ such that this equality holds for $n \geqslant n_0$. Then we have the matrix equality

$$(F(n + 1), f(n + 1)) = (F(n), f(n)) \begin{pmatrix} M'(n + 1) & \gamma'(n) \\ 0 & r_d(n) \end{pmatrix}.$$

Let $M(n + 1)$ denote the latter matrix, $\lambda = (F(n_0), f(n_0))$ and $\gamma = (0, \ldots, 0, 1)^T$. Then $(\lambda, M(t), \gamma(t), n_0)$ is a representation of ϕ. Moreover, $M(n)$ is clearly upper triangular.

(ii) implies (iii): similar to the proof of the similar implication in Theorem 10.1.

(iii) implies (i): we show by induction on d that if ϕ has a triangular representation of dimension d, then Eq.(8) holds. Let

$$((\lambda, a), \begin{pmatrix} M(t) & \delta(t) \\ 0 & b(t) \end{pmatrix}, \begin{pmatrix} \gamma(t) \\ c(t) \end{pmatrix}, n_0)$$

be a representation of $f(n)$ in the class of ϕ, of dimension $d + 1$, with a block decomposition such that $a, b(t), c(t)$ are one-by-one matrices. By translating the representation if necessary, we may assume that $c(n)$ is nonzero for $n \geqslant n_0$; indeed, if $c(t) = 0$, then we may directly apply induction. For $n \geqslant n_0$, let $A(n) = M(n_0 + 1) M(n_0 + 2) \cdots M(n)$ and $B(n) = b(n_0 + 1) b(n_0 + 2) \cdots b(n)$.

We see that $f(n)$ is equal to

$$\lambda A(n) \gamma(n) + \lambda \sum_{i=n_0+1}^{n} M(n_0 + 1) \cdots M(i - 1) \delta(i) b(i + 1) \cdots b(n) c(n) + a B(n) c(n).$$

Applying the operator $E - \frac{c(t+1)}{c(t)}b(t+1)$ to ϕ, we obtain a germ ϕ' which is represented by the sequence $g(n) = f(n+1) - \frac{c(n+1)}{c(n)}b(n+1)f(n)$. We have

$$g(n) = \lambda A(n+1)\gamma(n+1) - \lambda A(n)\gamma(n)\frac{c(n+1)}{c(n)}b(n+1)$$

$$+\lambda \sum_{i=n_0+1}^{n+1} M(n_0+1)\cdots M(i-1)\delta(i)b(i+1)\cdots b(n+1)c(n+1)$$

$$-\lambda \sum_{i=n_0+1}^{n} M(n_0+1)\cdots M(i-1)\delta(i)b(i+1)\cdots b(n)c(n)\frac{c(n+1)}{c(n)}b(n+1)$$

$$+aB(n+1)c(n+1) - aB(n)c(n)\frac{c(n+1)}{c(n)}b(n+1).$$

After simplifications, we have $g(n) = \lambda A(n)\gamma'(n)$, where

$$\gamma'(n) = M(n+1)\gamma(n+1) - \gamma(n)\frac{c(n+1)}{c(n)}b(n+1) + \delta(n+1)c(n+1).$$

Thus the germ $\phi' = [g]$ has the representation $(\lambda, M(t), \gamma'(t), n_0)$, which shows by induction that we have $(E - r_1(t)) \circ \cdots \circ (E - r_d(t))(\phi') = 0$ for some $r_i(t)/inK(t)$. Thus we obtain $(E - r_1(t)) \circ \cdots \circ (E - r_d(t)) \circ (E - \frac{c(t+1)}{c(t)}b(t+1))(\phi) = 0$ and ϕ is d'Alembertian. $\qquad\square$

For $r, r' \in K(t)$, define an equivalence relation: $r \sim r'$ if $r = r's^\theta/s$ for some nonzero $s \in K(t)$.

Corollary 11.2. *If a germ ϕ is d'Alembertian, then its rank is equal to the smallest d such that an equation of the form (8) holds. Moreover, if one has also $(E - r'_1(t)) \circ \cdots \circ (E - r'_d(t))(\phi) = 0$, then for some permutation $\sigma \in S_d$, one has $r_{\sigma(i)} \sim r'_i$ for $i = 1, \ldots, d$.*

Proof. This follows from the proof of the previous theorem, knowing that the diagonal elements of two pseudo-similar triangular matrices are related as in the statement. This latter fact follows from the Jordan-Hölder theorem. $\qquad\square$

We obtain also the following two corollaries.

Corollary 11.3. *A germ is d'Alembertian if and only if for each minimal representation $(\lambda, M(t), \gamma(t), n_0)$, $M(t)$ is pseudo-triangularizable.*

Corollary 11.4. *The set of d'Alembertian germs is a $K(t)$-subalgebra of $\mathcal{G}$.*

This follows since the tensor product of triangular matrices is triangular. This result was already known (see [PWZ] page 157).

The germ $[I_n]$, where I_n is the number of involutions in S_n, is not d'Alembertian. Indeed, the associated recursion $f(n+2) = f(n+1) + (n+1)f(n)$ has no nonzero hypergeometric solution, as follows from Petkovšek's algorithm *Hyper*, see [P1] Example 4.4, or [PWZ] Example 8.4.3. Now, if a minimal representation of a germ ϕ is triangular, then by modifying λ, one obtains a representation of a nonzero hypergeometric germ, which satisfies the shortest linear recursion of ϕ; it follows then from Cor. 8.6 that $[I_n]$ has no minimal triangular representation.

12   Field extension

The next result was known to Petkovšek (personal communication) and its proof is completely similar to the proof of [P1] Theorem 5.2, who gives the similar result for hypergeometric germs.

Proposition 12.1. *Let $K \subset L$ be a field extension. If a germ $\phi = [f]$ is P-recursive over L and if $f(n)$ is in K for n large enough, then ϕ is P-recursive over K. Moreover, its rank over K and L are equal.*

Proof. We use the following well-known result: if an infinite system of linear equations in finitely many variables over K has a nonzero solution in L, then it has a nonzero solution in K.

By hypothesis, we know that there exist polynomials $p_0(t), \ldots, p_d(t)$ over L, with $p_0(t)$ nonzero, such that for $n \geqslant n_0$, $p_0(n)f(n+d) + p_1(n)f(n+d-1) + \cdots + p_d(n)f(n) = 0$ and that $f(n) \in K$. We may assume that $f(n) \in K$ for $n \geqslant n_0$. Consider this as an infinite system of linear equations in the coefficients of the p_i; this system has coefficients in K (since $f(n)$ and n^i are in K), and a nonzero solution in L (since $p_i(t) \in L[t]$ and $p_0(t) \neq 0$). Hence it has a solution in K. Thus, ϕ is P-recursive over K. If we choose d minimal, then d is the rank over L; thus the rank over K is not larger; hence the ranks are equal, since a representation over K is also a representation over L. □

Observe that, unlike the rank, the property for a germ to be a closed form, or d'Alembertian, depends on the ground field. This is already seen by the simple example of the Fibonacci sequence: over any field containing the golden number, it is a closed form, by the Binet formula; however, over the rationals, it is not a closed form, and not even d'Alembertian. Indeed, by Petkovšek's algorithm *Hyper* (see [P1] Example 5.1), the Fibonacci recurrence has no hypergeometric solution over $\mathbb{Q}$.

The *rank* of a sequence satisfying a *linear recursion with constant coefficients* is classically defined as the length of the shortest such recursion; equivalently of the rank of its *Hankel matrix*, see [F], [BR], chapter 6. Such a sequence is clearly P-recursive. It is however not true that both ranks coincide. Indeed, for such a sequence f, $f(n)$ may be expressed by an *exponential polynomial*; the number of exponentials appearing in its is equal to the number of nonzero roots of the companion polynomial of the shortest recursion with constant coefficients (loc cit.). This number is equal to the rank of $f(n)$ as P-recursive sequence, as it follows from Corollary 5.1 in [P1] or Theorem 8.7.1 in [PWZ] (see also Cor. 10.2). For example, $(n2^n)$ has rank 1 as P-recursive sequence, but rank 2 in the other sense.

13   Positivity

Let R denote a subsemiring of $K(t)$. We say that f or ϕ is *P-recursive on R* if it has a representation $(\lambda, M(t), \gamma(t), n_0)$ whose matrices are over R.

Proposition 13.1. *If ϕ is P-recursive germ on $\mathbb{Z}[t]$, then it is the difference of two germs which are P-recursive on $\mathbb{N}[t]$.*

This result is similar to a result of Schützenberger on noncommutative rational series over $\mathbb{Z}$ and $\mathbb{N}$ (see [BR] Exercise 7.2.3). For P-recursive sequences, a similar result, due to Kotek and Makowsky [KM], is known; I do not know how the result above relates to theirs.

Proof. It will be useful to adopt the following notation: if $r(t)$ is a polynomial over $\mathbb{Z}$, choose two polynomials $r_+(t)$ and $r_-(t)$ over $\mathbb{N}$ such that $r = r_+ - r_-$.

Instead of representations, we use the graph model given at the end of Section 2. We assume that $n_0 = 0$ for simplicity. Starting with a representation over $\mathbb{Z}[t]$ and the corresponding graph, we double the number of vertices: a vertex i gives two vertices i_+ and i_-. For each arrow $i \to j$ in the original graph, we know that its label is $r(t) \in \mathbb{Z}[t]$; this arrow gives four arrows in the new graph: $i_s \to j_u$, with $s, u \in \{+, -\}$, labelled $r_+(t)$ if s, u are equal, and labelled $r_-(t)$ otherwise. This new graph corresponds to a matrix $N(t)$.

Now define two row vectors λ_+ and λ_- indexed by the i_s: $(\lambda_s)_{i_u} = (\lambda_i)_+$ if $s = u$, and $= (\lambda_i)_-$ otherwise. Likewise, we define two row vectors $\gamma_+(t)$ and $\gamma_-(t)$ indexed by the i_s: $(\gamma_s(t))_{i_u} = (\gamma_i)(t)_+$ if $s = u$, and $= (\gamma_i)(t)_-$ otherwise.

We then obtain two representations $(\lambda_s, N(t), \gamma_s(t), 0)$, $s \in \{+, -\}$, which represent two P-recursive germs over $\mathbb{N}[t]$, whose difference is ϕ. Indeed, this follows from Eq.(3), which may be rewritten as

$$\sum_{ij} ((\lambda_i)_+ - (\lambda_i)_-)(\sum_{\pi} ((s_1)_+(1) - (s_1)_-(1)) \ldots ((s_n)_+(n) - (s_n)_-(n)))$$

$$\times ((\gamma_j)_+(t) - (\gamma_j)_-(t)),$$

where the second sum is over all paths $\pi = i_0 i_1 \ldots i_n$ of length n from $i = i_0$ to $j = i_n$, where $s_j(t) = M(t)_{i_{j-1} i_j}$. The expansion of this formula implies the result. $\qquad\square$

14 Extension of a sequence to $\mathbb{Z}$

In Th.3.4 of [S1], Stanley considers a recursion of the form Eq.(1) where the $r_i(n)$ are defined for each $n \in \mathbb{Z}$ (the formulation is different but equivalent). He then considers extensions of $f(n)$ to all $n \in \mathbb{Z}$ (and he obtains for P-recursive sequences a generalization of Popoviciu's theorem on rational series, see Th.3.4 in [S1]). Our aim here is to replace Stanley's condition by a matricial condition, which will allow to compute the values $f(n)$ for all integers n.

We say that a representation $(\lambda, M(t), \gamma(t), n_0)$ is *strict* if $M(n)$ and $\gamma(n)$ are defined for all n in $\mathbb{Z}$, and if moreover $M(n)$ is invertible for any $n \in \mathbb{Z}$ satisfying $n \leqslant n_0$ (equivalently for this latter condition, $det(M(n))$ is nonzero for each $n \leqslant n_0$).

Let R be the subring of $K(t)$ of fractions which are defined for any $n \in \mathbb{Z}$. This ring is principal, since it is a localization of the principal ring $K[t]$. Note that the condition for

$M(t)$ and $\gamma(t)$ to be defined for any integer n is equivalent to the fact that their entries are in R.

In order to have the next result, we need to extend slightly the notion of representation: we allow here the integer n_0 to be negative.

Proposition 14.1. *The following conditions are equivalent, for a P-recursive germ ϕ.*
  (i) ϕ has a strict representation;
  (ii) ϕ has a strict minimal representation;
  (iii) the R-submodule of $\mathcal{G}$ spanned by the shifts $E^i(\phi)$, $i \in \mathbb{N}$ of ϕ is a finitely generated R-module.

Proof. (i) implies (iii): let $(\lambda, M(t), \gamma(t), n_0)$ be a strict representation of ϕ. Let ϕ_i have the representation $(\lambda, M(t), e_i, n_0)$, where e_i is the i-th canonical column vector. Let $\mathcal{M}$ be the R-submodule of $\mathcal{G}$ spanned by the ϕ_i; $\mathcal{M}$ is a finitely generated R-module. Since $E(\phi_i)$ is represented by $(\lambda, M(t), M(t+1)e_i, n_0)$, we see by Lemma 5.3 that $\mathcal{M}$ is closed under E, because by hypothesis the entries of $M(t+1)$ are in R. Moreover, the germ ϕ is in $\mathcal{M}$, because the entries of $\gamma(t)$ are in R. It follows, since R is principal, that the R-submodule of $\mathcal{G}$ spanned by the shifts $E^i(\phi)$, $i \in \mathbb{N}$ of ϕ is a finitely generated R-module.

(iii) implies (ii): note that the field of fractions of R is $K(t)$. Since the R-submodule $\mathcal{M}_0$ of $\mathcal{G}$ spanned by the shifts $E^i(\phi)$, $i \in \mathbb{N}$ is a finitely generated R-module, and since it is contained in the $K(t)$-vector space $\mathcal{G}$, it is a free R-module, of rank equal to the dimension d of the $K(t)$-vector space spanned by the shifts $E^i(\phi)$, $i \in \mathbb{N}$; thus d is the rank of ϕ, by Lemma 7.2. Let $\phi_1, \ldots, \phi_d$ be a basis of the R-module $\mathcal{M}_0$. Then we can find a matrix $M(t)$ over R such that $E(\phi_j) = \sum_{1 \leqslant i \leqslant d} M(t+1)_{ij}\phi_i$ and and a column vector $\gamma(t)$ over R such that $\phi = \sum_{1 \leqslant i \leqslant d} \gamma(t)_i \phi_i$. Hence, by Lemma 4.3, we may find a representation of ϕ with matrix $M(t)$; $M(t)$ is invertible by Corollary 8.5, and there exists $n_0 \in \mathbb{Z}$ such that $det(M(n)) \neq 0$ for any $n \leqslant n_0$. It follows that $(\lambda, M(t), \gamma(t), n_0)$ is a strict representation of ϕ since the entries of $M(t)$ and $\gamma(t)$ are in R. It is minimal since its dimension is the rank of ϕ.

(ii) implies (i) is clear. $\qquad\square$

A natural way to extend a P-recursive sequence $f(n)$, satisfying Eq.(1), to all of $\mathbb{Z}$ is to assume that the $r_i(n)$ are defined for all n in $\mathbb{Z}$ (Stanley's condition considered above) and moreover to assume that $r_d(n)$ is nonzero for all $n \in \mathbb{Z}$. Note that in this case f has a (minimal) strict representation $(\lambda, M(t), \gamma(t), 0)$; indeed, it suffices to follow the first part of the proof of Theorem 2.1, noting that $\gamma(t)$ is constant, that $M(t)$ is the companion matrix of the recursion and that $\pm r_d(t)$ is its determinant.

Let f have the minimal strict representation $(\lambda, M(t), \gamma(t), n_0)$ of dimension d. Then we extend f to all of $\mathbb{Z}$ by the formula

$$f(n_0 - n) = \lambda M(n_0)^{-1} M(n_0 - 1)^{-1} \cdots M(n_0 - n + 1)^{-1} \gamma(n_0 - n), \qquad (9)$$

for any $n \geqslant 0$. Note that this is well defined by the "strict" hypothesis.

Proposition 14.2. *Let f be as above and suppose that for n large enough, $f(n)$ satisfies the recursion*

$$p_0(n)f(n+e) + p_1(n)f(n+e-1) + \cdots + p_e(n)f(n) = 0, \tag{10}$$

where the $p_i(t)$ are in $K[t]$. Then this recursion is true for any $n \in \mathbb{Z}$.

Proof. Suppose that $(\lambda, M(t), \gamma(t), n_0)$ is a minimal representation of $f(n)$, which is therefore strict. Let $v(t) = p_0(t)M(t+1)\cdots M(t+e)\gamma(t+e) + p_1(t)M(t+1)\cdots M(t+e-1)\gamma(t+e-1) + \cdots + p_e(t)\gamma(t)$. Then by hypothesis $(\lambda, M(t), v(t), n_0)$ represents 0. Thus by Proposition 7.4 (ii), $v(t)$ vanishes.

We thus have

$$p_0(t)M(t+1)\cdots M(t+e)\gamma(t+e) + p_1(t)M(t+1)\cdots$$
$$\times M(t+e-1)\gamma(t+e-1) + \cdots + p_{e-1}(t)M(t+1)\gamma(t+1) + p_e(t)\gamma(t) = 0. \tag{11}$$

In Eq.(11), put $t = n$ with $n \geqslant n_0$. Thus

$$p_0(n)M(n+1)\cdots M(n+e)\gamma(n+e) + p_1(n)M(n+1)\cdots M(n+e-1)$$
$$\times \gamma(n+e-1) + \cdots + p_{e-1}(n)M(n+1)\gamma(n+1) + p_e(n)\gamma(n) = 0.$$

Multiply on the left by $\lambda M(n_0+1)\cdots M(n)$. We obtain that $p_0(n)f(n+e) + p_1(n)f(n+e-1) + \cdots + p_e(n)f(n) = 0$.

Now, in Eq.(11), put $t = n_0 - i$ with $1 \leqslant i \leqslant e$. We obtain

$$p_0(n_0-i)M(n_0-i+1)\cdots M(n_0-i+e)\gamma(n_0-i+e)$$
$$+p_1(n_0-i)M(n_0-i+1)\cdots M(n_0-i+e-1)\gamma(n_0-i+e-1)$$
$$+\cdots+p_{e-i}(n_0-i)M(n_0-i+1)\cdots M(n_0)\gamma(n_0)+\cdots$$
$$+p_{e-1}(n_0-i)M(n_0-i+1)\gamma(n_0-i+1)+p_e(n_0-i)\gamma(n_0-i) = 0.$$

Multiply by $\lambda M(n_0)^{-1}\cdots M(n_0-i+1)^{-1}$ on the left, obtaining

$$p_0(n_0-i)\lambda M(n_0+1)\cdots M(n_0-i+e)\gamma(n_0-i+e)$$
$$+p_1(n_0-i)\lambda M(n_0+1)\cdots M(n_0-i+e-1)\gamma(n_0-i+e-1)$$
$$+\cdots+p_{e-i}(n_0-i)\lambda\gamma(n_0)+\cdots$$
$$+p_{e-1}(n_0-i)\lambda M(n_0)^{-1}\cdots M(n_0-i+2)^{-1}\gamma(n_0-i+1)$$
$$+p_e(n_0-i)\lambda M(n_0)^{-1}\cdots M(n_0-i+1)^{-1}\gamma(n_0-i) = 0.$$

Thus, taking in account Eq.(9) we obtain $p_0(n_0-i)f(n_0-i+e) + p_1(n_0-i)f(n_0-i+e-1) + \cdots + p_{e-i}(n_0-i)f(n_0) + \cdots + p_{e-1}(n_0-i)f(n_0-i+1) + p_e(n_0-i)f(n_0-i) = 0$.

Now put $t = n_0 - n$ in Eq.(11) with $n > e$. We obtain

$$p_0(n_0-n)M(n_0-n+1)\cdots M(n_0-n+e)\gamma(n_0-n+e)$$

$$+p_1(n_0-n)M(n_0-n+1)\cdots M(n_0-n+e-1)\gamma(n_0-n+e-1)$$

$$+\cdots+p_{e-1}(n_0-n)M(n_0-n+1)\gamma(n_0-n+1)+p_e(n_0-n)\gamma(n_0-n) = 0.$$

Multiply at the left by $\lambda M(n_0)^{-1} \cdots M(n_0 - n + 1)^{-1}$. We obtain

$$p_0(n_0 - n)\lambda M(n_0)^{-1} \cdots M(n_0 - n + e + 1)^{-1}\gamma(n_0 - n + e)$$

$$+ p_1(n_0 - n)\lambda M(n_0)^{-1} \cdots M(n_0 - n + e)^{-1}\gamma(n_0 - n + e - 1) + \cdots$$

$$+ p_{e-1}(n_0 - n) M(n_0)^{-1} \cdots M(n_0 - n + 2)^{-1}\gamma(n_0 - n + 1)$$

$$+ p_e(n_0 - n) M(n_0)^{-1} \cdots M(n_0 - n + 1)^{-1}\gamma(n_0 - n) = 0.$$

Thus we obtain $p_0(n_0 - n)f(n_0 - n + e) + p_1(n_0 - n)f(n_0 - n + e - 1) + \cdots + p_{e-1}(n_0 - n)f(n_0 - n + 1) + p_e(n_0 - n)f(n_0 - n)$.

Thus Eq.(10) holds for any $n \in \mathbb{Z}$. $\qquad\square$

Corollary 14.3. *The extension to $\mathbb{Z}$ given by Eq.(9) is independent of the chosen strict minimal representation.*

As mentioned before, Stanley considers in [S1] recursion like Eq.(10), with the assumption that p_0 has no integer roots. We show by an example that a sequence may have a strict representation without satisfying any such recursion. Take indeed the representation $(\lambda, M(t) = \begin{pmatrix} 1 & 1 \\ t & t+1 \end{pmatrix}, \gamma(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0)$ with any nonvanishing λ. Then one has

$$M(t+1)M(t+2)\gamma(t) = \frac{t^2 + 5t + 5}{t+1}M(t+1)\gamma(t+1) - \frac{t+2}{t+1}\gamma(t).$$

This implies that the associated sequence satisfies the recursion $f(n+2) = \frac{n^2 + 5n + 5}{n+1}f(n+1) - \frac{n+2}{n+1}f(n)$. It may be shown by using Petkovšek's algorithm *Hyper* that no hypergeometric sequence satisfies this recursion. Hence, f is not hypergeometric, the above recursion is its shortest one and does not satisfy Stanley's condition since it is not defined at $n = -1$.

15   Galois group

The field $K(t)$ is a *difference field*, that is, a field with an automorphism, which here is defined by $\theta(t) = t + 1$. Likewise, the ring $\mathcal{G}$ is a *difference ring*, because E is a ring automorphism. Since $\mathcal{G}$ contains $K(t)$ (the embedding is $r(t) \mapsto [f]$, with $f(n) = r(n)$, n large), and since E restricts to θ, $\mathcal{G}$ is a difference ring extension of $K(t)$. The field of constants (that is, the fixpoints of the automorphism), in both cases, is K. We assume in this section that K is algebraically closed.

Let ϕ be a P-recursive germ. Let $(\lambda, M(t), \gamma(t), n_0)$ be a minimal representation of ϕ. We may by Corollary 8.5 assume that $M(n) \in GL_d(K)$ for $n \geqslant n_0$. Define for $n \geqslant n_0$, $Z(n) = M(n_0 + 1)M(n_0 + 2) \cdots M(n)$. This defines a $d \times d$ matrix Z over $\mathcal{G}$, which by construction is in $GL_d(\mathcal{G})$; indeed, its determinant is the invertible hypergeometric germ associated to the sequence $d(n_0 + 1)d(n_0 + 2) \cdots d(n)$, where $d(t)$ is the determinant of $M(t)$. Moreover, $E(Z) = ZM(t+1)$, by definition of the embedding of $K(t)$ into $\mathcal{G}$.

We follow now [PS] and [HS]. Define the subring R of $\mathcal{G}$ generated by $K(t)$, by the d^2 entries of Z and by the inverse of the determinant of Z. By [PS] Proposition 4.1 and [HS] Proposition 3.1, R is the *Picard-Vessiot ring associated to the difference equation*

$$EY = YM(t+1). \tag{12}$$

It follows from Corollary 8.6 that R does not depend on the chosen minimal representation. This ring is closed under E (by Eq.(12) and since the inverse of a hypergeometric germ is hypergeometric), and is therefore a difference subring of $\mathcal{G}$.

The *Galois group* of Eq.(12) is by definition the group of ring automorphisms of R that fix $K(t)$ pointwise and that commute with E. We call this group the *Galois group* of the P-recursive germ ϕ.

We use the following result of van der Put and Singer (see Proposition 1.21 in [PS] or Theorem 2.1 and 2.2 in [HS]): let G be an algebraic subgroup of GL_d, defined over K; recall that $G(K(t))$ denotes the set of matrices in $GL_d(K(t))$ which satisfy the polynomial equations over K defining the algebraic group G. Then: (a) if $M(t+1)$ is a point in the extension $G(K(t))$, then the Galois group of Eq.(12) is a subgroup of G; (b) if G is the Galois group of the equation, then there exists a matrix $P(t) \in GL_d(K(t))$ such that $P(t)^{-1}M(t+1)P(t+1)$ is in G.

Theorem 15.1. *Let ϕ be a P-recursive germ and $G(\phi)$ its Galois group. Then ϕ is d'Alembertian if and only if $G(\phi)$ is triangularizable.*

Proof. If ϕ is d'Alembertian, then we know by Th.11.1 that ϕ has a minimal representation with $M(t)$ upper triangular. It follows from (a) above that G is upper triangular. Conversely, if G is upper triangular, then it follows from (b) above that $M(t)$ is pseudo-similar to a triangular matrix; it follows that ϕ has a minimal representation with a triangular matrix and we deduce from Th.11.1 that ϕ is d'Alembertian. $\square$

A germ is hypergeometric if and only if its Galois group is contained in $K^{*\,3}$; likewise, a germ is a closed form if and only if its Galois group is contained in a torus (that is, product of several K^*). This follows from [PS], Chapter 2, were the reader may find all the details of the action of the Galois group as a group of automorphisms of the ring R (see also [HS]).

In [HS], Hendriks and Singer define a class $\mathcal{L}$ of P-recursive germs, called *Liouvillian*. It is by definition the smallest subring of $\mathcal{G}$ such that

1. $K(t) \subset \mathcal{L}$;

2. $\mathcal{L}$ is closed under E and E^{-1};

3. it contains all the hypergeometric germs;

---

[3]An example: the Picard-Vessiot extension of $K(t)$ corresponding to the hypergeometric sequence $f(n) = n!$ is isomorphic with the polynomial ring $K(t)[f, f^{-1}]$, and the action of $a \in K^*$ is induced by the mapping $f \mapsto af$.

4. if ϕ is in L and $E(\psi) - \psi = \phi$, then $\psi \in \mathcal{L}$;

5. the interlacing of elements in $\mathcal{L}$ is in $\mathcal{L}$.

Here, the *interlacing* of m sequences $f_0, \ldots, f_{m-1}$ is the sequence g defined by $g(n) = f_r(q)$ if $n = qm + r$ (Euclidean division of m by n). Thus the interlacing of germs is well-defined. Note that "interlacing" is called "merge" in [BR].

The main result of [HS] is that a germ is Liouvillian if and only if its Galois group is solvable. Now, the celebrated Lie-Kolchin theorem asserts that a *connected* algebraic group is solvable if and only if it is triangularizable. However, the Galois group of difference equations are in general not connected; if G^0 is the connected component of 1 in the algebraic group G, then it is a normal subgroup of finite index with cyclic quotient, see [PS] Proposition 1.20 or [HS] Theorem 2.1.

Corollary 15.2. *A germ ϕ is Liouvillian if and only if it is a interlacing of d'Alembertian germs.*

Proof. It follows from [HS] that $\mathcal{L}$ contains the d'Alembertian germs; hence also their interlacing. Conversely, the proof of Theorem 3.4 in [HS] shows that if the Galois group of a germ is solvable, then it is a interlacing of germs belonging to the class defined by items 1 to 4 above. This class is exactly the class of d'Alembertian germs by Section 8.6 in [PWZ]. Thus the corollary follows. $\square$
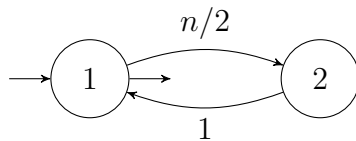

Figure 2: A Liouvillian germ which is not d'Alembertian

In Figure 2 is shown the representation (in graph form, with $n_0 = 0$) of a Liouvillian germ which is not d'Alembertian. Its Galois group is the set of matrices

$$\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in K^* \} \cup \{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, a, b \in K^* \},$$

which is a non connected solvable algebraic group. The germ is the class of f, with $f(2n) = \frac{1.3...(2n-1)}{2^n}$ and $f(2n+1) = 0$. The recursion for f is $f(n+2) = \frac{n+1}{2} f(n)$ and $f(0) = 1, f(1) = 0$.

16 Effectiveness

A major result which is not proved here is the effectiveness of computing a minimal representation of a given P-recursive germ. This is equivalent to computing the shortest

P-recursion Eq.(1). Petkovšek has shown that, if $K = \mathbb{Q}$, a shortest recursion is effectively computable (personal communication [P2]); his algorithm uses the resolution of algebraic equations over the field $\bar{\mathbb{Q}}$ of algebraic numbers and the fact that the shortest recursions over both fields coincide, for a germ over $\mathbb{Q}$, see Section 12.

The difficulty in computing a minimal representation lies apparently in condition (ii) of Prop.7.4, since it is easy to find a representation satisfying condition (i), by following for example the proof of Th.2.1.

A more general algorithmic question is the following: given several P-recursive sequences, decide if they are linearly dependent over $K(t)$. An algorithm for this problem will imply an algorithm for the computation of the minimal representation.

17 Arithmetic

A very natural question is the following: when has a given P-recursive germ coefficients in $\mathbb{Z}$? (that is, is it the class $[f]$ of a sequence over $\mathbb{Z}$?). This question, even for the subclass of hypergeometric germs, seems very difficult.

A particular case is the study of ratios of products of factorials, a question studied by Landau [L], who showed that their integrality is equivalent to the positivity of a certain step function; the latter problem is related to the NymanBeurling formulation of the Riemann hypothesis, see [B, Bo] for details and references. Picon [Pi2] has given new insight into Landau's problem, and also sufficient conditions for the integrality of hypergeometric sequences [Pi1, Pi3].

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