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Inventing (in) Early Geometry, or How Creativity Inheres in the Doing of Mathematics

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Inventing (in) Geometry, or How Creativity Inheres in the Doing of Mathematics

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Abstract

Inventing is fundamental to mathematical activity, should one be a professional mathematician or a primary school student. Research on mathematical creativity generally is organized along three axes according to its focus on the final product, the overall process, or the individual person. Through these conceptualizations, however, research rarely considers how mathematical actions themselves are fundamentally creative. In this article, we conceptualize mathematical actions as inherently creative of the activity within which professional mathematicians and primary school students experience (some) mathematics for a first time. To make our case, we develop the microanalysis of an exemplary episode of third-grade geometry (age 8–9 years) in which two children and an adult work with a tangram set. Our analysis characterizes inventing (in) geometry as a serendipitous, open-ended experience of working with traces in the receiving and the offering of something novel. In concluding, we propose considering that inventing in early geometry is also inventing geometry itself: an inventing-in-the-act which also result in being invented as a (professional or school) geometer.

Keywords: Creativity, invention, student, geometry, phenomenology

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Inventando (en) Geometría Inicial, o Cómo la Creatividad está Inherente en el Hacer Matemáticas

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(*Recibido: 17 Junio 2014; Aceptado: 5 Febrero 2015; Publicado: 24 Febrero 2015)*

Resumen

Inventar es clave para la actividad matemática, tanto si uno es matemático/a profesional, como estudiante de primaria. La investigación sobre creatividad matemática generalmente se organiza alrededor de 3 ejes según el producto final, el proceso general, o la persona. A través de estas conceptualizaciones la investigación raramente considera cómo las acciones matemáticas en sí mismas son creativas. En este artículo las acciones matemáticas se conceptualizan como inherentemente creativas en la actividad que tanto los/as matemáticos/as profesionales, como los estudiantes de primaria, experimentan por primera vez. Para desarrollar nuestra investigación llevamos a cabo microanálisis de un episodio experimental en 3º de geometría (8-9 años de edad) donde dos niños y un adulto trabajan con un juego de tangram. Nuestro análisis caracteriza *inventing* en geometría como una experiencia fortuita y abierta a partir de trabajar con trazos como algo novedoso. Proponemos considerar que inventar en geometría inicial es también inventar la geometría misma: un proceso de inventar-en-la-situación que da lugar a inventarse como geómetra (profesional o estudiante).

Palabras clave: Creatividad, invención, estudiantes, geometría, fenomenología

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Inventing has long been a concern in mathematics, with Hadamar as one of the most well-known figure in this history. Research in mathematics education is also often concerned with questions of invention and creativity, and some scholars go as far as to say that even in school, the essence of mathematics is creative thinking (Dreyfus & Eisenberg, 1996). Inventing (in) mathematics is also at the heart of the realistic mathematics education movement, in which "context problems can function as anchoring points for the *reinvention* of mathematics by the students themselves" (Gravemeijer & Doorman, 1999, p. 111, our emphasis). Recently, three lines of discourse about mathematical creativity have been identified according to a focus on (a) the final product, (b) the overall process, or (c) the individual person (Liljedahl & Allan, 2013). In the first case, "creativity is assessed on the basis of the external and observable products" (p. 1233), and it is the originality of the "thing" itself that constitutes the heart of the matter. The second discourse pays greater attention to the conditions within which such things come about, and is concerned with "phases" such as inspiration, incubation, and realization through which the process of inventing is carried on—sometimes including the social dimension that is at work in the approbation of problems and solutions within the community (e.g., Csikszentmihalyi, 1996). The third focus is on the traits and habits of those individuals considered to be particularly inventive. Many test of mathematical creativity have been developed over the years along those axis (e.g., Evans, 1964; Balka, 1974; Haylock, 1984; Mann, 2006).

One area, however, has received little attention in mathematics education: the actual work in the instant of *inventing* and especially in terms of ordinary, everyday creativity (Mann, 2006; Runco, 2004). How can "ordinary" mathematical action be creative in essence? Getting closer to how mathematical invention concretely/observably takes place, research inquiring into the heuristics of mathematical activity give texture to what creative mathematical work might actually look like. For example, five principles in relation to creativity in mathematical work have been identified (e.g., Sriraman & Dahl, 2009): (a) the experienced of sudden inspiration, (b) the following of aesthetic appeal, (c) the offering of ideas, (d) the widening of considerations, and (e) the welcoming of ambiguity and uncertainty. It is quite possible to imagine such moments as part of the work of, for example, students dwelling on a geometry problem. The

concatenation of professional mathematicians and school children might first seem hasty, but we follow in this Hadamar's (1945) suggestion to consider that "[b]etween the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of a similar nature" (p. 104). Thus, for the conceptualization of a more fleshed and blooded account of inventing in mathematics generally, we might also investigate what happens when mathematicians are thinking. Putting forward the double ascension of from abstract to concrete and from concrete to abstract as essential to those moments when mathematics is created (Roth & Hwang, 2006), Zhang (2013) moves us toward this. Another lead for the conceptualization of everyday mathematical actions as inventive is found in the observation that "the environment is itself a source for ideas" (Runco, 1993, p. 5). A recent study close to our concerns conceptualizes mathematical inventiveness based on Châtelet's insights on the physical nature of mathematical activity (Sinclair, de Freitas, & Ferrara, 2013). In their study, the authors identify creative acts at the classroom level as the "[introduction] of the new in an unpredictable way that transgresses current habits of behavior and exceed existent meanings" (p. 252). Through this take on creativity in mathematics education, the authors however perpetuate a tendency to situate creative moments in relatively objective "special instants" in which creativity is thought to take place. That is, these authors still situate inventing as characteristic of unordinary (series of) actions. Conversely, in this study we introduce the notion of *inventing-in-the-act*, that is, in the most mundane moment of doing something mathematical.

Well aware of the numerous ways in which creativity can be defined—over 100 were identified in one study (Treffinger, Young, Selby, & Shepardson, 2002)—our intent is not to define creativity in mathematics, but rather to expand the range of phenomena in which mathematical invention is considered. Following Merleau-Ponty's (and others, as we see below) insights, we do so from a phenomenological perspective, starting from the micro-analysis of an episode in which two children and an adult work with a tangram set as part of the mathematics curriculum requirement to have children replicate composite 2-D shapes. As the analysis unfolds, we thus approach inventing (in) geometry from the ground up, progressively conceptualizing *inventing-in-the-act* as the serendipitous,

open and ongoing experience of working with traces. In section 2, we draw on this first hand experience of inventing to bring forth its theorisation as we can read it across various literature including Derrida, Châtelet, Kadinsky, and others.

Inventing-in-the-Act: Case from an Early Geometry Lesson

In his book about the invention of the double helix model of the DNA structure, James Watson recalls his sudden realization of pieces of a puzzle fitting together while moving about physical shapes of four bases that were known to be part of the DNA molecule:

> [I] began shifting the bases in and out of various other pairing possibilities. Suddenly I became aware that an adenine-thymine pair held together by two hydrogen bonds was identical in shape to a guanine-cytosine pair held together by at least two hydrogen bonds. All the hydrogen bonds seemed to form naturally; no fudging was required to make the two types of base pairs identical in shape. (Watson, 2012, pp. 207–208)

In this statement, Watson emphasizes the *becoming aware* of something that resulted from his actions without being intentionally constructed as such. Watson's *invention* thus consisted in recognizing something significant and unforeseen in what he had done before. That is, his realization of the new *followed* its production. This episode is helpful in rethinking how we approach inventing in mathematical activity, stressing the importance of fully considering how, for example, it is commonly realized by students in and as the act of *doing* geometry, when they find mathematics in the shapes they produce. To illustrate such *inventing-in-the-act* to phenomenological account for inventing (in) early geometry, we describe a fragment of a lesson in a second grade classroom in which pupils were prompted to produce a series of composite figures (a square, an oblong, etc.) from a tangram set (Fig. 1). In the fragment we selected Nelly and Kelly, who manipulate triangles of various size, and subsequently identify and attend to different configurations that had resulted from their actions. The two girls were videotaped by an early childhood educator (Mary), who also takes an active part in the girls' work.

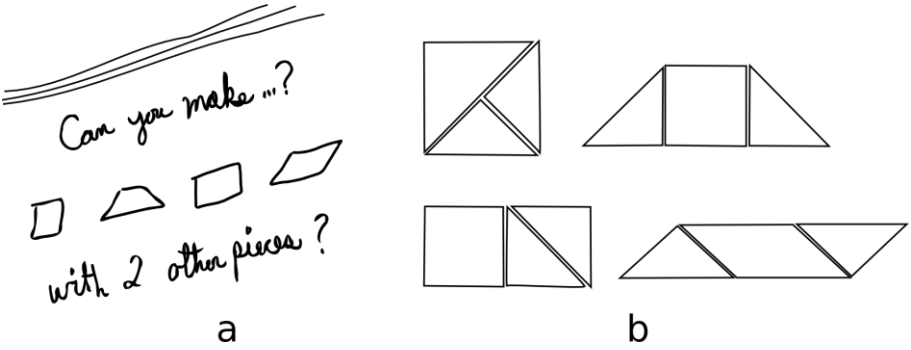

Figure 1. The task and the expected solutions.

The lesson fragment beings with Nelly stating that she "made a very very interesting shape" made of loosely arranged pieces (fig. 2). After saying this, she keeps moving the pieces as if she wanted to adjust them more neatly. Mary then jumps in, apparently trying to help her, and then steps back with a "no," as if she realized that the pieces could not be tidily fitted. At the same time, Nelly also changes her orientation, pushing away some of the shapes and apparently starting up on something new. She begins with adjusting a non-rectangular parallelogram piece and a medium triangle. She then adds a small triangle (called in the lesson "the missing piece") [fig. 3a], and observing that "they fit together," continues her bricolage ("And then") with the addition of a third triangle. In a high-pitched voice, Mary comments: "A boat!" while Nelly adds a fifth piece which completes the configuration as a convex hexagon, stepping back from her creation, while Mary keeps commenting: "Ohhh! Interesting!" We then notice another change in Nelly's work when she points to the vertices following a statement about counting "the sides." As Nelly reaches "6," the configuration is called "hexagon," and Kelly adds: "a true hexagon." Mary seems to confirm this result by stating "Another one!" before encouraging the girls to move on and "see if you can make the rectangle one."

Nelly: I made a very very interesting shape! [fig 2]
Nelly goes back to the pieces and gently moves them.
Mary: Or maybe here, you know… and… no…
Mary reaches to the pieces; also moving them gently, and then steps back.
Nelly then pushes away some of the triangles and starts adjusting the parallelogram and the medium triangle, and then adds a small one [fig. 3a]:
Nelly: Here I'm using the missing piece [*fig. 3*], they fit together! And then [*adding an other small triangle*]
Mary: A boat! [*Nelly keeps going, adds an other triangle, fig. 4, and then briefly pauses*] Ohh! Interesting!
Nelly: Lets count the sides. 1,2,3,4,5,6 [*pointing to the vertices*]. An oc…
Kelly and Mary: Hexagon
Nelly: Hexagon.
Kelly: A true hexagon [*flipping through the sheets where they drew the figures they previously found*]
Mary: Another one! Now see if you can make the rectangle one.
Nelly: I'm gonna put this put keep it safe so it doesn't wreck [*placing 2 large triangles on opposite sides of the hexagon, fig. 5*]
Sally: That's a ribbon.
Mary: Oh it's beautiful, yeah.
Nelly turns the two triangles in a symmetrical (but opposite!) movement and then quickly flips one over [fig 6].
Nelly: What did I made, an airplane or something?
Sally: Hum, yeah. Trace it now.
Nelly: I can't. Someone has to hold these so I can oh [*sliding the central, hexagonal part towards her, and then disturbing the puzzle*]. I cant… It's ok, I think I can remember it [*placing triangles on her sheet*]. I forgot my memory now [*taking apart what she was doing, and then re-starting from the first 2, 3 shapes she began with*, fig, 7].

Figure 2. Nelly's very very interesting shape.

Figure 3. Nelly adding the missing piece.

Figure 4. Adding another triangle.

Figure 5. Keeping it safe.

Figure 6. An airplane?

Figure 7. Experiencing "memory loss"

In the last part of the fragment, we see Nelly preserve the hexagonal configuration ("I'm gonna put this put keep it safe so it doesn't wreck") by enclosing it with 2 other large triangles, the result of which Kelly declares to be "a ribbon." But there is only a very short pause before Nelly moves the last two pieces, first rotating them in a symmetrical fashion hence producing a clearly unsymmetrical figure from an apparently symmetrical one, and then flips the left-hand piece over (figure 6). Commenting on the result, she utters in a questioning fashion "What did I made, an airplane or something?" to wich Sally responds "Hum, yeah. Trace it now." Nelly answers saying that she cannot do it, while trying to slide her configuration on a sheet of paper. She ends up completely disturbing the organisation, and apparently try to put it back together, but does not achieve it.

Provided that we accept to approach this episode in terms of inventing, and generally consider that inventing can be thought about from a phenomenological perspective, as *inventing-in-the-act*, what can be said about such phenomena? In the following subsections, we discuss three aspect of *inventing-in-the-act* as it takes place in those children's doing of mathematics. Following this analysis, we turn to the literature to situate and add consistency to those first-level observations.

Inventing as Serendipitous

In this episode, Nelly appears to be both playing with the shapes and moving towards recognizable, yet undefined and certainly unstated configurations from which an identification could be made—much like Watson might have appeared to an observer. Taking the perspective of *inventing-in-the-act* we might say that the shapes Nelly produces are in a sense true inventions: they come out of her exploration with the pieces and do not resemble any of the "models" presented at the beginning of the lesson. While possibly trying to make "something" that might or might not have been specific (e.g. a square) with the tangram pieces, she suddenly stops in the face of a configuration that reveals itself to her as an "interesting shape," which eventually becomes a "hexagon" that turns into a ribbon and airplane before its inadvertent disintegration. *Inventing-in-the-act* thus appears as a (series of) serendipitous events, as moments marked up in the flow of mathematical exploration (adjusting pieces together,

reorienting them, etc.) out of which something not precisely intended emerges.

The fortuitous nature of the find appears particularly important in at least two ways. If Nelly were to merely construct, as from a plan, the configuration we see her creating in this episode, it would make little sense to call it inventing. *Inventing-in-the-act* thus refers to the opened-ended and indeterminate feature of mathematical doings, and the idea that we do not fully intend our creations, while nevertheless acting in a way that makes their realization possible. *Inventing-in-the-act* means trying out possibilities, offering oneself something to work with, and then seizing it for further exploration. In a different way, it thus also applies to circumstances in which one moves toward a more precise product, while finding oneself having to "discover" a way to do so. An example of this can be found earlier in the lesson, when Nelly explicitly tries to make the fourth figure presented on the board as stated in the lesson goals. She is then clearly in the situation of having to *invent* a method to (re)produce the configuration (rather than inventing a figure). Trying out possibilities and even pushing them to the limit, we see her at some point partially superposing two shapes. As a result, the expected figure briefly appears (its outline at least), but Nelly quickly rejects the procedure (and its outcome), and continues her search. The way by which the configuration is to be produced has yet to be found, stumbled across and, once it shows up, recognized as something valuable. This last example thus also takes us to the second aspect by means of which inventing is not only in part unintentional, but also propitious. Had Nelly taken the superposition technique as an acceptable part of realizing shapes, it would have counted (in her experience, that is from a phenomenological perspective) as inventing a (new) way, a mathematical procedure, to arrive at the configuration. *Inventing-in-the-act* thus also requires this sort of recognition without which emerging forms can hardly be regarded as being invented. Nelly gives us examples of this positive disposition toward what one comes across when she highlights the apparition of the "interesting shape" and the "hexagon" configuration, and which in fact also orient us toward their identification *qua* invention.

It seems important, however, not to reduce the recognition of an invention to some sort of conscious reflective action. Recognition can also be found, for example, in *repetition*: when someone does something first,

and then keeps doing it without necessarily stepping back thinking "I'm going to do it this way from now on." In fact, *re*cognition (again) also is a *first* cognition, namely of the repetition of something that is cognized for a second time. Thus, it is precisely the absence of such repetition that allows us to comment on Nelly's "not inventing" superposition as an acceptable move. Moreover so, what we refer to as reflective consciousness also has an unintended, serendipitous aspect to it. We mostly find ourselves realizing that something works or does not, and only rarely objectively (observably) engage in reflexive thinking. In this case, there is for instance no evidence that Nelly reflexively invented the superposition as a "non-acceptable" technique. Had she (or Kelly, or Mary!) commented on the move, this would be a different story. It is of course possible that Nelly did tell it to herself, but the only recognition *we* have access too is the non-reappearance of the move.

Inventing as Open Ended

Another important aspect of *inventing-in-the-act* is its open-endedness. The transcript shows many instances of how the configuration created by Nelly are never simply present, but always in the course of becoming, fundamentally *in the making* (Roth, 2014a). The "interesting shape" for example is only virtually present: Nelly and Mary move to something else without actually fitting the pieces together. That is, even if (from a phenomenal perspective) a configuration seemed to exist for them both, it was yet to be attained, something to work on, to work with. Inventing then is not, as we might think, limited to the find. The "interesting shape" is one for which no name or pattern are easily identified, hence its deictic rather that descriptive label. Something is given, but what that something holds is still unknown. Such open-endedness of *inventing-in-the-act* can also be observed following the emergence of the "hexagon" shape in the middle of the episode. The apparition of the configuration is evocative of something that is at the very same time present and absent: something *still* to be brought forth. The hexagon that might already reveal itself to Mary (or the reader) *as a hexagon* does not necessarily have that specific quality (yet) for Nelly. She found something, and now explores that thing that she made. When Nelly pauses (accompanying Mary's "Oh, interesting") and announces "Lets count the sides," she engages in a mathematical

observation (the counting of sides/vertices) of what was found: from this the status of the find *as* a hexagon is produced. That is, we have to think of *inventing-in-the-act* as unfinished and ongoing. Nelly moves from one invention to another, which can also be the inventing the invention as something else, as something more. Following what seems like the recognition of a convex figure (what we see in fig. 3 is *also* a hexagon), something called a hexagon is made present, again re-invented through Kelly's intervention: it is not only a hexagon, but a "true hexagon" (perhaps alluding to the nearly regular aspect of the configuration).

At the same time, the dis-closure of *inventing-in-the-act* also signifies fragility and temporality. When Nelly notes the emergence of an "interesting shape" and actively attends to it, she does so despite the non-existence of a configuration in which the pieces would "fit" in the way we generally expect (tangram) puzzles to be made. Nelly thus invents a shape that cannot be fully actualized, an ephemeral invention that lasts exactly as long as it could be worked with as such. But from the moment one receives the impression that what seems to be there cannot be actually realized, the invention disappears. Open-endedness is thus also in regard to a potential disappearance, while *inventing-in-the-act* might only fleetingly assemble something as an invention. More so, whereas for Mary that invention soon has been lived through ("Now see if you can make the rectangle one"), Nelly and Kelly seem to experience it otherwise. Another powerful example of this happens at the end of the episode, when the hexagonal configuration is lost. As if there where still something about the arrangement that had to be encountered (i.e., its existence *as* that specific disposition of tangram pieces), Nelly gives the impression to be stifled by its disorganization. The presence of a convex hexagon appears to be very fragile, almost vibrant, still very much alive. Hence Nelly's stated concern with safekeeping the configuration ("I'm gonna put this put keep it safe so it doesn't wreck"), which suggests that it is still somehow elusive to her. The discovery needs to be preserved because it *can* disappear like the word that remains at the tip of the tongue in its absence/presence, meaning that *inventing-in-the-act* does not guaranty the permanence of what is invented.

Inventing as Tracing

A third element we bring into light for this first conceptualization of *inventing-in-the-act* concerns traces. While "tracing" can mean finding or discovering by investigation (e.g. the police traced the missing van), the word also connotes all the marks, sings, prints, and residues that so remarkably play in our episode. Arrangements of tangram pieces, in their material dispositions on the table, are traces with which we observe inventing taking place. In other segments, we also see, e.g., the children drawing their find on paper, which is a form a traces closer to what we most spontaneously think of as marks of mathematical work. Observing how the girls actually transact with those traces highlights their dual role with respect to inventing: tracing is both the origin and the conclusion, the constituent (as in raw material, the medium) and the product of *inventing-in-the-act*.

The "interesting shape" Nelly initially finds—just as the convex hexagonal configuration that follows her moving around the tangram pieces— will result in something eventually identified as relevant. From configuration to configuration, even starting with one or two pieces placed on the table, inventing creates traces with which the girls move forward, adding to or removing from arrangements, repositioning and so on. And in a certain way, those traces are also the resting place in which inventing is reified. The hexagonal configuration is the material proposition of a hexagon made of tangram pieces they discovered so much so that its loss made the find unrecoverable. But with this disappearance, it would not be fair to say that everything has gone. Traces are also in the memory of inventing, including the verbal and gestural (amongst others) traces left behind. Although Nelly did not, this time, remember the configuration, it is quite reasonable to think that she would have been able to recall that "a hexagon" was made, and describe the invention in certain ways. This naming of a figure is thus another way in which *inventing-in-the-act* is observable here. Exploring the hexagon by tracing out its contour through counting with the vertices and them calling it a "hexagon," naming the figure is an action that we might describe as half cognition, half recognition: it is both a thinking of the new as some thing created and a recalling of some thing(s). Both parts of the action are essential to inventing the figure as a hexagon. The case of the "interesting shape" is quite

convincing in that matter: something that is not even "fully" present in the material disposition of the pieces stands out when Nelly calls it "a very very interesting shape." This shows that as tracing, *inventing-in-the act* hinges on a difference between the inventing as ongoing and those traces it proceeds from and arrive at. The difference here appears as a surplus, the traces of inventing bringing in the action *more* than what the action brought to the traces. Nelly created a configuration, which turned out to be hexagonal, just as something like a "true" hexagon was spotted by Kelly in this 'almost' regular configuration, and just as an interesting shape emerged from the verbal trace of naming it even though that shape was and remained materially absent. In a essential way, the physical fitting of the shapes coming together in a tangram puzzle is always very approximate, and no clear line can be drawn between the shapes (mal-) adjustment in Nelly's "very very interesting shape" and the formation of vertices in the hexagon configuration (especially the top one, on fig. 4), as illustrated in the famous puzzle invented by Lewis Carol, where a tinny unfit is sufficient to give the illusion that 64 = 65 (fig. 8).

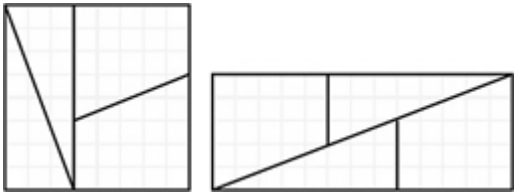

Figure 8. Lewis Carol's puzzle suggests that 8x8 = 5x13

From a phenomenological perspective, naming a "shape" from spread-out tangram pieces, or sides and corner from more or less adjusted plastics blocs requires discovering, unveiling those concepts in material arrangements. We might thus want to take this as an extreme case for *inventing-in-the-act*, one that takes into account as instances of inventing all forms of (mathematical) interpretations, all cases of working with traces to bring about something that springs from traces but at the same time constitutes something new to the ongoing experience.

Inventing (in) Early Geometry: Reinventing Invention from a Theoretical Standpoint

> Prégnance: les psychologues oublient que cela veut dire pouvoir d'éclatement, productivité (praegnans futuri), fécondité (Merleau-Ponty, 1964, p. 258)

In the preceding section, we show how inventing-in-the-act manifests itself in the traces of children's activity. From a phenomenological perspective, we see little if any difference between those experiences and Watson's recollection of his sudden awareness that DNA bases could be assembled to make identically shaped pairs. That is, a fine-grained study of mathematical acts blurs the everyday distinction between a small, personal invention and largely recognizable creations. Such observation might even take us to question Hadamar's (1945) suggestion of a "difference of degree" between the student's and the professional mathematician's work of inventing. But more importantly, our analysis exhibits the inherently creative nature of doing mathematics, observable in most "ordinary" episode of young children doing mathematics (including composing shapes from tangram pieces), when inventing-in-the-act reveals itself as the serendipitous, open and ongoing experience of working with traces. In this section, we turn to a literature rarely considered in mathematics education research concerned with creativity to show how these observations can be theorized as inventing (in) early geometry, thus reinventing invention from a theoretical standpoint.

In everyday language, inventing means either creating something that did not exist before or the making up of something. For example, it is often heard that the Greek invented geometry, but this statement is also an invention especially since we know that many of those "Greeks" (e.g. Aristotle, Herodotus, Democritus) themselves believed that geometry was invented by the Egyptians (e.g., Anglin & Lambek, 1995). Taken as devising, inventing also often evokes discovering the solution to a more or less well defined problem—e.g., the invention of dynamic geometry software somehow "solves" the problem of animating geometrical constructions in the way Nicolet or McLaren did in their films. Invention may also refer to the identification of something that was already there

although not "as such": in the garbage can model of invention, the solution precedes the problem to which it becomes the invention (Roth, 1995).

In the preceding description, we note a dialectic of finding | being-found dialectic at the heart of inventing-in-the-act. This aspect is necessary to account for the passive, unconscious, unintentional dimension in mathematical creation (Hadamar, 1945; Pointcaré, 1952; Sriraman & Dahl, 2009). The non-teleological approach to inventing emphasizes the dual nature of mathematical doing: intended and unintended. For example, Pointcaré (1952) insisted on the importance of voluntary effort—which is "absolutely fruitless and whence nothing good seems to have come [thereof]" (p. 27)—as a condition for fruitful unconscious work. In this way, blind exploration, random trials, and sterile lines of action sometimes do produce events that lead to an unexpected breakthrough. The inherent excess in any intentional action is an immediate source of potential discovery through which inventing is finding what was there to be found. As the Watson episode shows, inventing means being affected by a "something" that he somehow managed to put together unintentionally. If inventing is finding something for the first time, to invent is first of all the experience of finding, it is finding oneself in the presence of something (Derrida, 2007). Rorty (1989) writes about how poets inventing new language cannot say what they are doing until they have done it, and yet this identification of the language once it has arrived as new is also fundamental to its existence as such. Otherwise it would only be an evolution or an expansion of language, not an invention *per se*.

In the lesson fragment, Nelly comes upon something that in part has resulted from her movements. To come [Lat. *venīre*] upon [Lat. *in*-] is to invent (in German: er*finden*, from *er*-, prefix marking success + *finden*, to find), to find something (unexpected). The etymological roots of the verb *invent* go far back to the Proto-Indo-European *g^wā-, g^wem*- to go, to come, to come to the world, to be born. That is, in the movement that we call inventing, something (inherently new) comes to be born that had not existed before and that did not foresee its own coming. One interesting take on this is that of epistemic actions (e.g. Baltag, 1999): actions performed to change the states of affairs (as opposed to specifically goal-oriented actions). What is found through largely unintended movements transcends any intent to create something as the term is commonly understood: inventing as the production of new mathematical possibilities is like a bifurcation point

where new system states emerges that could not be anticipated from the previous states, a continuity and discontinuity simultaneously (Roth, 2014b). Similarly, Merleau-Ponty's (1964) mentions that "when a 'good' form appears it either modifies its environment by its radiation or obtains from my body movements until …" (Merleau-Ponty, 1964, p. 259). This gives us a new understanding of the observations that the environment can be a source of ideas (Runco, 1993), now reconceptualised phenol-menologically.

In the episode, we observe how a surplus in the traces of inventing keeps calling, imposing itself as something "unsolved." Something is there to work with, traces, and in this work of inventing we can appreciate the "dance of agency," in the pieces to pieces craft, involving all material "bodies" (de Freitas & Sinclair, 2013) contributing to actual mathematical doings. This is also consistent with an enactivist account of creating/inventing the world that we live in as an emerging feature of our socio-material encounters (e.g. Maturana, 2009), where the generative power of making something new cannot be limited to situations in which a clear line can be drawn between some before and after. As an in-between involving presence and absence, the old and the new, *inventing-in-the-act* requires the active | passive aspect of being disposed to(ward), of responding. Conceptually, the opening created by this superabundance is, in its very strangeness, a form of otherness. Phenomenologically, it thus represents the intrusion of the virtual found to be essential to mathematical invention (Châtelet, 1993). And the surplus coming with inventing then has to be viewed as the creation of opportunities: *inventing-in-the-act* is the opening of the space of the possible.

There is a fundamental contradiction inherent to inventing: the arrival of something impossible which becomes a source for new possibilities. In a recent study on mathematical inventiveness drawing on Sinclair, de Freitas and Ferrra (2013) discuss this at the classroom level, showing how inventing demands bringing forth what was not present before and "without given content in that its meaning cannot be exhausted by existent meanings" (p. 242). In his deconstruction of the concept, Derrida also arrived at such an observation: "[invention] would be in conformity with its concept, with the dominant feature of the word and concept 'invention,' only insofar as, paradoxically, invention invents nothing, when in invention the other does not come, and when nothing comes to the other or from the

other" (Derrida, 2007, p. 44). Invention of the impossible is the *only* possible invention, which then "has to declare itself to be the invention of that which did not appear to be possible" (p. 44). The configuration Nelly found is not only an undetermined actualization on "the difference between creative acts of actualizing versus logical inferences that realize the possible" (Sinclair et al., 2013, p. 250): It is also the self-revelation of something already there, suddenly dis-covered as the result of indeterminate epistemic actions that seek rather than already know. As long as a configuration—Watson's base pairs, Nelly's hexagon—up to its appearance, remains unforeseen, nothing is invented; and a soon as it appears, it is too late already since this appearance automatically completes the declaration: the configuration is heard, seen, received, and responded to, re-cognized, and thus offered up as an object of consciousness.

As an unfinished and inherently open experience, *inventing-in-the act* thus have the features of an event*-in-the-making (Roth, 2013), which is precisely concerned with unpredictability, excess over intention, and temporal-provisional signifying in place of fixed meanings. Theoretically, this lead us to conceive of mathematical doings as the "making of" mathematics, within which geometrical *inventing-in-the-act* is both inventing *in* geometry and inventing geometry itself. Eventing is making present, and present again, by means of re-presentations while geometry, in Derrida's (1962) analysis of Husserls' work, is born again in and through the movements and the *un-intended* changes that these have brought about in the world. Nelly's hexagon is a re-invention, a *finding again* of what the tangram shapes afford as potentialities, and these events (of inventing (in) geometry) are, in their phenomenological structure, similar to that of the historical first inventions emerging on a ground of proto-geometrical experiences that come from being in the world (Husserl, 1976; Roth, 2011). That is, if mathematical invention is characterized by an accidental character in the sense that it is strictly unforeseeable from a historical perspective (Hadamar, 1945; Pointcaré, 1952), the term "historical" commonly heard in relation to long time spans (notable at societal-historical levels) also applies to the moment-to-moment temporality of human acts. If tracing, for example, is taken as moment in the course of inventing, a separation in time between the production of a trace and its (re)interpretation (or conversely: the re-interpretation in tracing of something previously done, or said, or thought) introduces a generative

difference (Derrida, 1993). Naming a configuration turns it into a thing about which more is being said as long as inventing "it" is taking place. And once all is said, as Châtelet (2010) humorously explains, nothing else remains to be done than going to bed. But, Châtelet argues (based on his study of great mathematician drawings), when a diagram is experienced in its materiality, mathematical possibilities are exceeded by virtuality which, as a result of inventing, degrades into new actuality and possibilities. There is always and at any moment an interval, a distance that is created with/in such mathematical doing, a gap where a new zone of virtuality emerges: "this is where a full dialectic of virtuality develops" (Châtelet, 1993, p. 18). Moreover, *Inventing-in-the-act* here actually helps us understand Derrida's (1962) mesmerizing analysis of Husserl's *Origin of Geometry* when he rhetorically asks: "Must we not say that geometry then an infinity of birth, of act of being born, in which each time is announce, while hiding itself, another birth? Should we not say that the geometry is en route to its origin rather than proceeding from it?" (p. 131, our translation). *Inventing-in-the-act* thus phenomenologically resonates with Freudenthal's insight regarding the reinvention of mathematics by the students (e.g. Gravemeijer & Doorman, 1999) but without situating it in the often mentioned impasse of having to answer how children can be ask to build alone and from scratch what mathematicians took centuries to come up with. Mathematical doing *are* inherently creative of one's mathematics including instances of working with others' traces (carrying on mathematics' ongoing history). Geometry is endlessly born again, re-invented in actual geometrical doings that do not establish it once and for all (Roth, 2011), but essentially calls for further *inventing-in-the-act* to bring it to live. Traces of others' mathematical work can set inventing into motion, gets one to "do" mathematics, thereby keeping mathematics alive. When someone is active with traces, like Nelly working with a tangram puzzle–an invention Chinese were already inventing with in the first century–every movement changes the body, not only creating new (more or less ephemeral) traces, but changing its practices so that "the individual, who develops its capabilities in producing also expends these, consumes these in the act of production" (Marx/Engels, 1983, p. 25). Thinking back though this quote, we realize that, obviously, mathematics always not only existed but also evolved through mathematicians' *inventing-in-the-act*, every invention being the result of some concrete and lived mathematical doings.

Meaning is what is produced when a sound complex is attached to trace in the world, when a wild otherness, suddenly discovered, is tamed in a name, pinned down, captured, comprehended, contained. There is always, however, the possibility to once again discover more in those traces, taking them up and get the inventing going afresh. This act of finishing, taking up, and then finishing again an object is fundamentally similar in regard to the work of an Andrew Wiles' proof of Fermat's Last theorem (Fermat negligently noting in a margin that he already *had* a proof thereof) and that of Mary naming her find, or Kelly offering the concept/category of "true" figures with the configuration as an example. The experience of offering and receiving something that as known but also partially unknown makes inventing (in) mathematics equally available to and performed by all who actually engage in mathematical activity. And even though Nelly's or Kelly's work is not, for someone who encountered such productions already, as "original" as the first formulation of the Lindemann–Weierstrass theorem (which, however, essentially solved *again* a problem—the quadrature of the circle—so many time "solved" in previous centuries), the girls still had to re-discover, to re-invent these in the way geometry always is rediscovered and reinvented, as for the first time. Reinventing inventing (in) geometry phenomenologically, geometry is on the way to its origin because geometry is coming to us, coming to Being in the very acts through which it comes to live. Doing geometry is over and over again inventing it (Husserl, 1976), while being already within geometry, and thus inventing "in" the field of actions it opens for us. As for the social dimension of inventing regarding how something comes to count as a creation (Csikszentmihalyi, 1996), it can also be re-conceptualized phenomenologically in those actions, deemed essential to *inventing-in-the-act*, through which traces are made relevant for oneself or another. Just the way we saw it in Nelly's calling of an "interesting shape", Kelly's taking up of the "hexagon" confirming the find, or the absence of a response moving forward her offer to consider the configuration as a "true" hexagon (a case of inventing that did not seems resonate for the others). Thus, we find here evidence in support of the conceptualization of *inventing-in-the-act* in the very sense that Merleau-Ponty (1964) and Husserl before him (see e.g. Husserl, 1976) suggestions that those learning to do geometry today are reproducing the same kind of creative act that originally produced geometry during the times of the early Greek.

Finally, it is worth mentioning how the serendipitous aspect of *inventing-in-the-act* we observed comes very close to artistic creation in the way great artists—including Kandinsky (1913) and Klee (1953)—report it. Providing accounts similar to what we discuss in this article, Kandinsky (1913) for example explains: "All forms that I ever used arrived 'on their own,' they placed themselves complete in front of my eyes and all that remained was for me to copy them, or they formed themselves already during the work, often taking me by surprise" (p. XVI). That is, even in cases of aesthetic productions, each invention is related to but not merely derived from preceding ones, and there is interplay between the accumulation of minor transformations to the sudden mutation of what was there into something new (Dufrenne, 1953). It is always possible, *after the fact*, to find a sort of prefiguration for the new, and it is precisely the temporality which makes room for inventing. We observe this not only in external accounts on invention, but also in the phenomenological moment of inventing: "In short, [the artist] knows the work only when he has created it. Until then, he knows only that something new wills itself within him and that, because it is new, he must first create it" (p. 34). Mathematical work as *inventing-in-the-act* is artistic in the sense that it is an *explosion* of what is and was is merely possible, breaking from the technicalities of logic and proof to step into the creative realm of bringing entities to life (Châtelet, 2010). Merleau-Ponty (1964) introduces for this kind of phenomenon the concept of *pregnance*, the productive power of explosion at the interface of the visible and the invisible: "My body *obeys* the pregnance [of the visible], responds to it. . . . When the 'right' form appears, either it's shine modifies the surroundings, or obtains from my body a movement" (p. 259, our translation). That is Merleau-Ponty (1964) provided one response to the question of how can "ordinary" mathematical action be creative in essence by insisting on the fact that the effect of actions always exceeds the intentions, which allows individuals to encounter unexpected and surprising results. This encounter with the unexpected changes the field within which objects appear, and, therefore, the objects themselves; or they change the movements of the inquirer, leading to the emergence of new ways of doing. Simply doing (some) mathematics, one could then come to see mathematical objects in a new light, turning them into something new, and opening for new ways to act with or upon them. With our episode, we saw how inventing-in-the-act

takes us to consider that Nelly does not merely constructs something predetermine, but rather to observe movement until something appears to offer itself to her, at which point the orientation in the movements changes. A shift has occurred in which a new perceptual arose from the ground. Such a gestalt switch through which the invisible becomes visible can occur because "every '[ful-]filled' intuition surrounds itself by a horizon of potential appearances, every effective presence by a horizon of non-presence or of virtual presence" (Henry, 2000, p. 53). A similar take on inventing in relation with traces is found in the theoretical work of the painter Paul Klee (1953), who shows how a line can come to circumscribe itself with the present in its absence of an imaginary line indicated by secondary lines and so on. In that sense, *inventing-in-the-act* requires thinking the trace as different from itself, which also can be found in Kandinsky's (1913) discussion of the visible and invisible life in painting, or the hearable and the silent dynamic force in music.

Coda

The purpose of this article is to show that inventing (in) mathematics education has to do with mathematical ideas borne with/in mathematical doings. If someone were to make something already conceived in mind, it would not be new at all from a phenomenological perspective, and, therefore, would not count as inventing. Novelty is not an objective/realist quality, but the result of the very experience of the unexpected *qua* unexpected. This means being able to take into consideration approximate or momentary creations as integral parts of inventing in mathematics education, realizing that it does can take place at any moment in the most mundane, everyday mathematical action. As a case of *inventing-in-the-act*, we observed school children coming up with configuration of tangram piece that where not intended from the outset, but encountered while fitting pieces together (recall Nelly saying: "I'm using the missing piece, they fit together!"). Conceptually, we found that *inventing-in-the-act* as the serendipitous, open and ongoing experience of working with traces involves the recognition that some/thing is now present, from an indeterminate being-affected to the presence as presence, and to overturn it, move it beyond what it might have be found/made for, to see it as something other than what was. As such, we assert that there is then no phenomenological

distinction possible between students or "professional mathematician" inventing (in) geometry (or mathematics more generally). This also possible because mathematics is never fully invented.

On a larger scale, this study is part of our effort to further develop a "dynamic" approach to mathematics and mathematical activity, one in which movement, flow, and transformation are at the center and origin of our conceptualizations. Approaching inventing (in) geometry in terms of *inventing-in-the-act* contributes to that endeavour, which in turn opens up for new ways of thinking in mathematics education. Our work with data involving a student doing mathematics is crucial to that undertaking in at least two ways. First, it illustrates the concrete, substantial, observable presence of what otherwise might have seem "sheer philosophical considerations" on mathematical doing. A case is made that, in our fragment, two seven -years -old students are engaged in the moment-to-moment of inventing (in) geometry as part of a quite ordinary episode of mathematics teaching and learning. Second, it shows how, and for that very reason, observing students'' mathematical activity qua *inventing-in-the-act* is a mater of orientation; it is in the way we look, in how we observe. No special setup is required to situate mathematics students a creative "inventors." Attending to their mathematical activity in such a way, however, gives raise to number of new questions upon which mathematics educators are invited to dwell: How to engage with a (mathematics student) inventor in the moment of inventing? How to prepare oneself to do so? How would the distinction of various forms of inventing be interesting?

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