PERMUTAHEDRA AND GENERALIZED ASSOCIAHEDRA

CHRISTOPHE HOHLWEG, CARSTEN LANGE, AND HUGH THOMAS

Abstract. Given a finite Coxeter system (W, S) and a Coxeter element c, or equivalently an orientation of the Coxeter graph of W, we construct a simple polytope whose outer normal fan is N. Reading's Cambrian fan $\mathcal{F}_c$, settling a conjecture of Reading that this is possible. We call this polytope the c-generalized associahedron. Our approach generalizes Loday's realization of the associahedron (a type A c-generalized associahedron whose outer normal fan is not the cluster fan but a coarsening of the Coxeter fan arising from the Tamari lattice) to any finite Coxeter group. A crucial role in the construction is played by the c-singleton cones, the cones in the c-Cambrian fan which consist of a single maximal cone from the Coxeter fan.

Moreover, if W is a Weyl group and the vertices of the permutahedron are chosen in a lattice associated to W, then we show that our realizations have integer coordinates in this lattice.

Contents

Date: April 16, 2008.
Hohlweg was partially supported by Canadian Research Chair and the Fields Institute. Hohlweg is partly supported by a startup grant from PAFARC. Thomas is partly supported by a Discovery Grant from NSERC.

Introduction

Let (W, S) be a finite Coxeter system acting by reflections on an $\mathbb{R}$-Euclidean space. Let $\boldsymbol{a}$ be a point in the complement of the hyperplanes corresponding to the reflections in W. The convex hull of the W-orbit of $\boldsymbol{a}$ is a simple convex polytope: the well-known *permutahedron* $\mathsf{Perm}^{\boldsymbol{a}}(W)$. The normal fan of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is the *Coxeter fan* $\mathcal{F}$.

Cluster fans were introduced by S. Fomin and A. Zelevinsky in their work on cluster algebras [7]. To each Weyl group there corresponds a cluster fan which encodes important algebraic information including the exchange graph of the corresponding cluster algebra. One very natural question is to find realizations of these fans as normal fans of simple polytopes. This was first answered by F. Chapoton, S. Fomin, and A. Zelevinsky in [5]: for each Weyl group W, they construct a simple convex polytope whose normal fan is the cluster fan. Such a polytope is called a *generalized associahedron of type* W.

In [20], N. Reading introduced *Cambrian lattices* which generalize the Tamari lattice to any finite Coxeter group W and Coxeter element $c \in W$. The c-Cambrian lattice is defined as a quotient of the weak order on W, and therefore has an associated fan, the c-*Cambrian fan* $\mathcal{F}_c$, which is a coarsening of the Coxeter fan $\mathcal{F}$. N. Reading conjectured that any c-Cambrian fan is the normal fan of a simple convex polytope [20, Conjecture 1.1].

We will call a polytope with the c-Cambrian fan as normal fan a c-*generalized associahedron* $\mathsf{Asso}_c(W)$. In [23], N. Reading and D. Speyer showed that $\mathcal{F}_c$ is a simplicial fan, so c-generalized associahedra are simple polytopes. In the case where W is a Weyl group, they also showed that c-Cambrian fans are all combinatorially isomorphic to the cluster fan. Moreover, for *bipartite* c, they show that $\mathcal{F}_c$ is linearly isomorphic to the cluster fan. In other words, $\mathsf{Asso}_c(W)$ has the same combinatorial type as a generalized associahedron of type W, and for bipartite c, $\mathsf{Asso}_c(W)$ is linearly isomorphic to a generalized associahedron of type W.

In this article, we construct c-generalized associahedra for all c, settling Reading's conjecture.

Let us review the history of some realizations. For symmetric groups, that is for Coxeter groups of type A, c-generalized associahedra are combinatorially isomorphic to the *classical associahedron* $\mathsf{Asso}(S_n)$, whose combinatorial structure was first described by J. Stasheff in 1963 [29]. The combinatorial structure in question is the face lattice of a simple convex polytope whose 1-skeleton is isomorphic to the undirected Hasse diagram of the Tamari lattice on the set Y_n of planar binary trees with $(n+1)$-leaves (see for instance [14]) and therefore a fundamental example of a secondary polytope as described in [8]. Numerous realizations of the associahedron have been given, see [5, 15] and the references therein. An elegant and simple realization of $\mathsf{Asso}(S_n)$ by picking some of the inequalities for the permutahedron $\mathsf{Perm}(S_n)$ is due to S. Shnider & S. Sternberg [26] (for a corrected version consider J. Stasheff & S. Shnider [30, Appendix B]). This realization implies a surjective map from the vertices of the permutahedron to the vertices of the associahedron, which turns out to be the well-known surjection from S_n to Y_n that maps the weak order on S_n to the Tamari lattice as described in [3, Sec. 9]. Recently, J.-L. Loday presented an algorithm to compute the vertex coordinates of this realization, [15]: label the vertices of the associahedron by planar binary trees with $n+2$ leaves and apply a simple algorithm on trees to obtain integer coordinates in $\mathbb{R}^n$. This polytopal realization of the associahedron from the permutahedron is referred to as *Loday's realization*. An important aspect of this construction is the fact that the Tamari lattice is a quotient lattice, as well as a sublattice, of the weak order on S_n.

For Coxeter groups of types A and B, the first two authors gave recently a Loday-type realization of any c-generalized associahedron [9], and they showed that Loday's realization of the associahedron is a c-generalized associahedron for a particular c. For hyperoctahedral groups, that is Coxeter groups of type B, the generalized associahedron is called a *cyclohedron*. It was first described by R. Bott and C. Taubes in 1994 [4] in connection with knot theory, and rediscovered independently by R. Simion [27]. See also [5, 16, 18, 24, 27]; none of these realizations is similar to Loday's (type A) realization.

The realization of $\mathsf{Asso}_c(W)$ for any finite Coxeter group W and any Coxeter element c given in this article generalizes to any finite Coxeter group the approach initiated by Shnider & Sternberg in type A, and extended by the two first authors to types A and B.

Our construction of the c-generalized associahedron is very straightforward. Start from $\mathsf{Perm}^{\boldsymbol{a}}(W)$ and its H-representation by non-redundant half spaces and fix a particular reduced expression of the longest element w_0 of W determined by c. The prefixes of w_0 up to commutation of commuting simple reflections will be called *c-singletons*. A half space $\mathscr{H}$ of the H-representation of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is *c-admissible* if its boundary contains a vertex of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ that corresponds to a c-singleton. The intersection of all c-admissible half spaces is a c-generalized associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ and its normal fan is the c-Cambrian fan $\mathcal{F}_c$ (Corollary 3.5). The selection of c-admissible and the removal of not c-admissible half spaces for the permutahedron of type A and distinct choices for the Coxeter element is illustrated in Figure 1 and Figure 2.

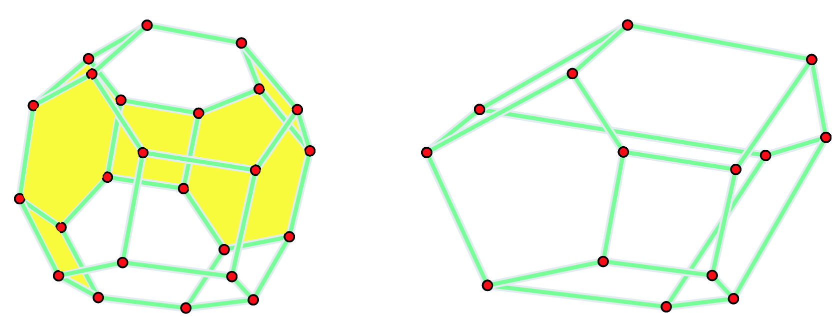

FIGURE 1. Obtaining the associahedron from the permutahedron for the Coxeter group S_4 and Coxeter element $c = s_1s_2s_3$. The left picture shows the permutahedron with the facets contained in the boundary of c-admissible half spaces translucent and the facets contained in the boundary of non c-admissible half spaces shaded. The picture to the right shows the associahedron obtained from the permutahedron after removal of all non c-admissible half spaces.

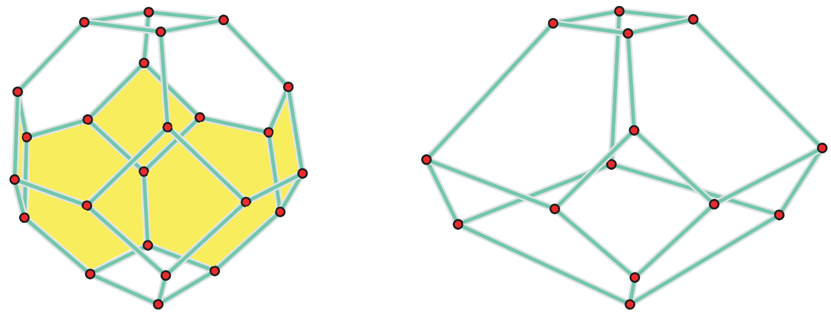

FIGURE 2. Obtaining the associahedron from the permutahedron for the Coxeter group S_4 and Coxeter element $c = s_2 s_1 s_3$. The left picture shows the permutahedron with the facets contained in the boundary of c-admissible half spaces translucent and the facets contained in the boundary of non c-admissible half spaces shaded. The picture to the right shows the associahedron obtained from the permutahedron after removal of all non c-admissible half spaces.

Moreover, if W is a Weyl group and the vertices of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$ are chosen in a suitable lattice associated to W, then we show that $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ has integer coordinates in this lattice (Theorem 3.15).

Another interesting aspect of this construction is that we are able to recover the c-cluster complex: relating cluster fans to quiver theory, R. Marsh, M. Reineke and A. Zelevinsky introduce in [17] what N. Reading and D. Speyer call the *c-cluster fan* in [23], and its associated simplicial complex *the c-cluster complex*. A c-cluster fan is a generalization of the cluster fan to any finite Coxeter group W and Coxeter element $c \in W$ (c bipartite is then the traditional case); its applications in quiver representations are most natural for W of types A, D and E.

By replacing the natural labeling of the maximal faces of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ by a labeling that uses almost positive roots only, we obtain the c-cluster complex. This replacement is determined by an easy combinatorial rule as stated in Theorem 2.6. This suggests that these constructions will play an important role in the study of c-cluster complexes and related structures.

This article is organized as follows. In §1, we recall some facts about finite Coxeter groups, Coxeter sortable elements, and Cambrian lattices. Additionally, the important notion of a c-singleton is defined and fundamental properties are proven. In §2, we recall some facts about fans, in particular of Coxeter and Cambrian fans, and give a precise combinatorial description of the rays of Cambrian fans. In §3, we state and prove our main result (Theorem 3.4). Finally, in §4 we study some specific examples of finite reflection groups. We work out the dihedral case explicitly to show that the vertex barycentres of the permutahedra and associahedra coincide and we explain how the realizations given in [9] for type A and B are a particular instances of the construction described in this article.

In a sequel [1], we describe the isometry classes of these realizations.

The construction presented in this article has been implemented as the set of functions `CAMBRIAN` to be used with the library `CHEVIE` for `GAP` [25, 6] and can be found on the first author's web page, [10].

1. Coxeter-singletons and Cambrian lattices

Let (W, S) be a finite Coxeter system. We denote by e the identity of W and by $\ell : W \to \mathbb{N}$ the length function on W. Let $n = |S|$ be the rank of W. Denote by w_0 the unique element of maximal length in W.

The *(right) weak order* $\leq$ on W can be defined by $u \leq v$ if and only if there is a $v' \in W$ such that $v = uv'$ and $\ell(v) = \ell(u) + \ell(v')$. The *descent set* $D(w)$ of $w \in W$ is $\{s \in S \,|\, \ell(ws) < \ell(w)\}$. A *cover* of $w \in W$ is an element ws such that $s \notin D(w)$.

The subgroup W_I generated by $I \subseteq S$ is a *(standard) parabolic subgroup* of W and the set of minimal length (left) coset representatives of W/W_I is given by

$$W^I = \{x \in W \,|\, \ell(xs) > \ell(x),\ \forall s \in I\} = \{x \in W \,|\, D(x) \subseteq S \setminus I\}.$$

Moreover, each $w \in W$ has a unique decomposition $w = w^I w_I$ where $w^I \in W^I$ and $w_I \in W_I$. Moreover, $\ell(w) = \ell(w^I) + \ell(w_I)$, see [11, §5.12]. The pair (w^I, w_I) is often called the *parabolic components* of w along I. For $s \in S$ we follow N. Reading's notation and set $\langle s \rangle := S \setminus \{s\}$.

Let c be a Coxeter element of W, that is, the product of the simple reflections of W taken in some order, and fix a reduced expression for c.

1.1. c-sortable elements.

For $I \subset S$, we denote by $c_{(I)}$ the subword of c obtained by considering only simple reflections in I. Obviously, $c_{(I)}$ is a Coxeter element of W_I. For instance, take $W = S_5$ and $S = \{s_i \,|\, 1 \leq i \leq 4\}$ where s_i denotes the simple transposition $(i, i+1)$. If $c = s_1 s_3 s_4 s_2$ and $I = \{s_2, s_3\}$ then $c_{(I)} = s_3 s_2$. Consider the possible ways to write $w \in W$ as a reduced subword of the infinite word $c^\infty = cccccc\ldots$. In [21, §2], N. Reading defines the *c-sorting word* of $w \in W$ as the reduced subword of c^∞ for w which is lexicographically first as a sequence of positions. The c-sorting word of w can be written as $c_{(K_1)} c_{(K_2)} \ldots c_{(K_p)}$ where p is minimal for the property:

$$w = c_{(K_1)} c_{(K_2)} \ldots c_{(K_p)} \quad \text{and} \quad \ell(w) = \sum_{i=1}^{p} |K_i|.$$

The sequence $c_{(K_1)}, \ldots, c_{(K_p)}$ associated to the c-sorting word for w is called *the c-factorization* of w. The c-factorization of w is independent of the chosen reduced word for c but depends on the Coxeter element c. In general the c-factorization does not yield a nested sequence $K_1, \ldots, K_p$ of subsets of S. An element $w \in W$ is called *c-sortable* if $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_p$. It is clear that for any chosen Coxeter element c, the identity e is c-sortable, and Reading proves in [21] that the longest element $w_0 \in W$ is c-sortable as well. The c-factorization of w_0 is of particular importance for us and is denoted by $\mathbf{w_0}$. To illustrate these notions, consider $W = S_4$ with generators $S = \{s_1, s_2, s_3\}$. The weak order of S_4 with elements represented by their c-factorization is shown in Figure 3 for $c = s_1 s_2 s_3$ and Figure 4 for $c = s_2 s_1 s_3$ (the delimiter '|' indicates the end of K_i and the beginning of K_{i+1}). Moreover, the background colour carries additional information: The background of w is white if and only if w is c-sortable.

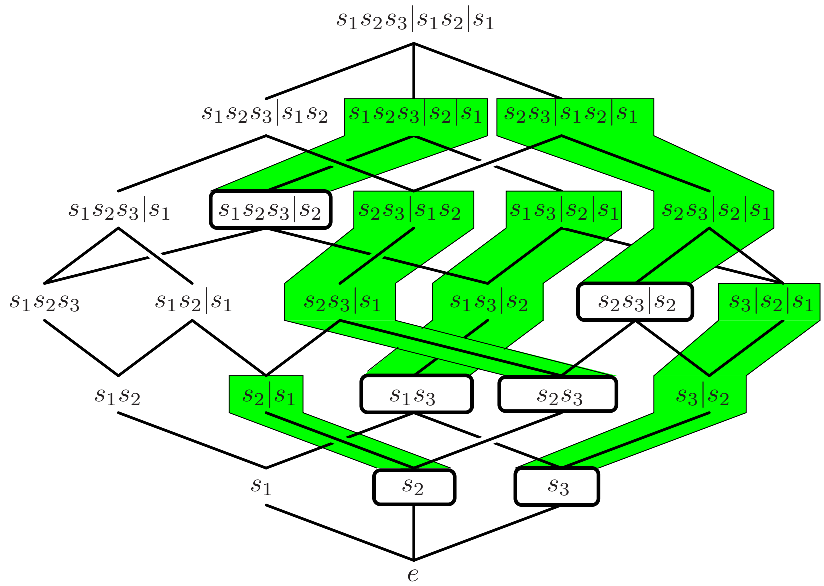

FIGURE 3. $c = s_1s_2s_3$

1.2. **c-Cambrian lattice.** N. Reading shows that the c-sortable elements constitute a sublattice of the weak order of W which is called the *c-Cambrian lattice*, [20, 22]. A Cambrian lattice is also a lattice quotient of the weak order on W. In particular, there is a downward projection $\pi_\downarrow^c$ from W to the c-sortable elements of W which maps w to the maximal c-sortable element below w. Hence, w is c-sortable if and only if $\pi_\downarrow^c(w) = w$, [22, Proposition 3.2]. It is easy to recover $\pi_\downarrow^c$ in Figure 3 and 4. A c-sortable element w (white background) is projected to itself; an element w which is not c-sortable (coloured background) is projected to the (maximal) boxed c-sortable element below the coloured component containing w. For instance in Figure 4, we consider $c = s_2s_1s_3$ and have $\pi_\downarrow^c(s_2s_3s_2) = s_2s_3s_2$ and $\pi_\downarrow^c(s_3s_2s_1) = \pi_\downarrow^c(s_3s_2) = \pi_\downarrow^c(s_3) = s_3$.

We say that $w \in W$ is *c-antisortable* if ww_0 is c^{-1}-sortable. We have therefore a projection $\pi_c^\uparrow$ from W to the set of c-antisortable elements of W which takes w to the minimal c-antisortable element above w. For example in Figure 4 we have $\pi_c^\uparrow(s_1s_3) = s_1s_3s_2s_1s_3$. The maps $\pi_\downarrow^c$ and $\pi_c^\uparrow$ have the same fibres, that is,

$$\left(\pi_\downarrow^c\right)^{-1}\pi_\downarrow^c(w) = \left(\pi_c^\uparrow\right)^{-1}\pi_c^\uparrow(w).$$

These fibers are intervals in the weak order as shown by N. Reading, [22, Theorem 1.1] and the fibre that contains w is $[\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$.

1.3. **c-singletons.** We introduce a particularly important subclass of c-sortable elements: an element $w \in W$ is a *c-singleton* if $\left(\pi_\downarrow^c\right)^{-1}(w)$ is a singleton. It is easy to read off c-singletons in Figures 3 and 4: An element is a c-singleton if and only if its background colour is white and it is not boxed, that is, $s_2s_1s_3$ in Figure 4 is a c-singleton while neither $s_1s_3s_2$ nor $s_2s_3s_2$ are c-singletons.

We now prove some useful properties of c-singletons.

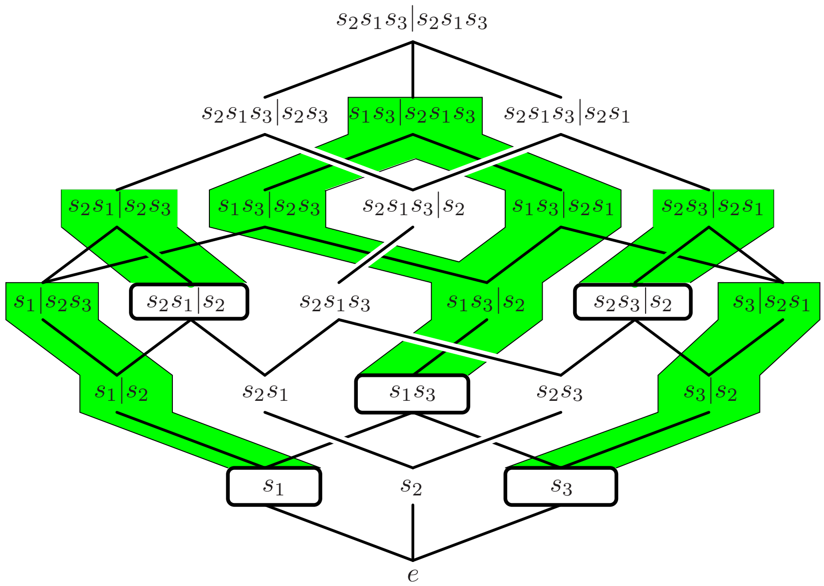

FIGURE 4. $c = s_2s_1s_3$.

Proposition 1.1. *Let $w \in W$. The following propositions are equivalent.*

(i) w is a c-singleton;

(ii) w is c-sortable and ws is c-sortable for all $s \notin D(w)$;

(iii) w is c-sortable and c-antisortable.

Proof. '(i) is equivalent to (iii)' and '(i) is equivalent to (ii)' follow from the fact that the fibre containing w is $[\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$ and that the map $\pi_\downarrow^c$ is order preserving. $\square$

It follows that w_0 and e are c-singletons.

The word property says that any pair of reduced expressions for $w \in W$ can be linked by a sequence of braid relation transformations. In particular, the set

$$S(w) := \{s_i \in S \mid w = s_1 \dots s_{\ell(w)} \text{ is reduced}\} = \bigcap_{\substack{I \subset S \\ w \in W_I}} I$$

is independent of the chosen reduced expression for w. It is clear that $w \in W_{S(w)}$ and that $S(w) = K_1$ if w is c-sortable with c-factorization $c_{(K_1)}c_{(K_2)} \dots c_{(K_p)}$.

Two reduced expressions for $w \in W$ are equivalent *up to commutations* if they are linked by a sequence of braid relation of order 2, that is, by commutations. Let $\mathbf{u}, \mathbf{w}$ be reduced expressions for $u, w \in W$. Then $\mathbf{u}$ is a *prefix of* $\mathbf{w}$ *up to commutations* if $\mathbf{u}$ is the prefix of a reduced expression $\mathbf{w'}$ and $\mathbf{w'}$ is equivalent to $\mathbf{w}$ up to commutations. We now state the main result of this section. Its proof is deferred until after Proposition 1.7.

Theorem 1.2. *Let w be in W. Then w is a c-singleton if and only if w is a prefix of $\mathbf{w_0}$ up to commutations.*

Remark 1.3. For computational purposes, it would be interesting to find a nice combinatorial description of $\mathbf{w_0}$.

Example 1.4. Let $W = S_4$ with set of generators $S = \{s_i \mid 1 \le i \le 3\}$ and Coxeter element $c = s_2 s_1 s_3$. The c-singletons of W are

$$\begin{array}{lll} e, & s_2s_3, & s_2s_1s_3s_2s_1, \\ s_2, & s_2s_1s_3, & s_2s_1s_3s_2s_3, \text{ and} \\ s_2s_1, & s_2s_1s_3s_2, & w_0 = s_2s_1s_3s_2s_1s_3. \end{array}$$

We see here that s_2s_3 is not a prefix of $s_2s_1s_3s_2s_1s_3$, but it does appear as a prefix after commutation of the commuting simple reflections s_1 and s_3.

Proposition 1.5. *The c-singletons constitute a distributive sublattice of the (right) weak order on W.*

Examples of these distributive lattices for $W = S_4$ are given in Figure 5.

Proof. Let L be the set of subsets $P \subset \{1, \dots, \ell(w_0)\}$ with the property that the reflections at positions $i \in P$ of $\mathbf{w_0}$ can be moved by commutations to form a prefix $\mathbf{w}_P$ of $\mathbf{w_0}$. This prefix $\mathbf{w}_P$ represents $w_P \in W$. Note that $\ell(w_P) = |P|$ because $\mathbf{w}_P$ is a prefix up to commutation of a reduced word for $\mathbf{w_0}$. The set L is partially ordered by inclusion and forms a distributive lattice with $P_1 \vee P_2 = P_1 \cup P_2$ and $P_1 \wedge P_2 = P_1 \cap P_2$ according to [28, Exercise 3.48]. (In particular, $P_1 \cup P_2,\ P_1 \cap P_2 \in L$ if $P_1, P_2 \in L$).

We claim that $P \longmapsto w_P$ is an injective lattice homomorphism.

First we check injectivity. Suppose $w_P = w_Q$ for $P \neq Q$. Since $\mathbf{w}_P$ and $\mathbf{w}_Q$ are reduced expressions, we have $|P| = |Q| = r$. Let $P = \{p_1, \dots, p_r\}$ and $Q = \{q_1, \dots, q_r\}$ with $p_i < p_{i+1}$ and $q_i < q_{i+1}$. Without loss of generality, let the smallest element in $(P \cup Q) \setminus (P \cap Q)$ be p_i. Let $s \in S$ be the reflection at position p_i of $\mathbf{w_0}$, then s also appears at some q_j with $q_j > p_i$. When moving the reflections in Q to the front of $\mathbf{w_0}$, the s that started at q_j must pass the s at p_i, but this implies that the expression for $\mathbf{w_0}$ would not be reduced at this step, which is contrary to our assumption. Thus the map is injective.

We show that $P \longmapsto w_P$ respects the lattice structures of L and W. Let $P, Q \in L$ and $R = P \cap Q$. Since $R \in L$, $\mathbf{w}_R$ is a prefix of $\mathbf{w_0}$ up to commutation. In

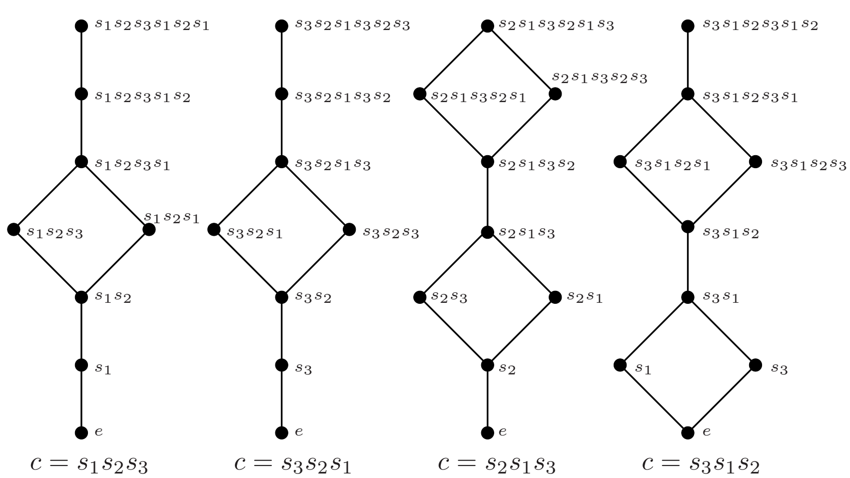

FIGURE 5. There are four Coxeter elements in S_4. Each yields a distributive lattice of c-singletons.

particular, it is also a prefix of $\mathbf{w}_P$ and $\mathbf{w}_Q$ up to commutation. Hence $w_R \leq w_P$ and $w_R \leq w_Q$. We obtain $\mathbf{w}_{P\setminus R}$ and $\mathbf{w}_{Q\setminus R}$ from $\mathbf{w}_P$ and $\mathbf{w}_Q$ by deletion of all reflections that correspond to an element of R and conclude $w_P = w_R w_{P\setminus R}$ and $w_Q = w_R w_{Q\setminus R}$. We have $S(\mathbf{w}_{P\setminus R}) \cap S(\mathbf{w}_{Q\setminus R}) = \varnothing$ since $\mathbf{w_0}$ is reduced. The proof is by contradiction and is similar to the proof of injectivity. Therefore, no element of W is above $\mathbf{w}_R$ and below $\mathbf{w}_P$ and $\mathbf{w}_Q$. We have shown $w_R = w_P \wedge w_Q$ with respect to the weak order of W. A similar argument proves $w_T = w_P \vee w_Q$ with respect to the weak order of W where $T = P \cup Q$: $S(\mathbf{w}_{T\setminus P}) \cap S(\mathbf{w}_{T\setminus Q}) = \varnothing$ implies that no $w \in W$ below w_R and above w_P and w_Q exists. $\square$

The following lemma characterizes the elements that cover a c-singleton.

Lemma 1.6. *Let $c_{(K_1)} \dots c_{(K_p)}$ be the c-factorization of the c-singleton w and $s \notin D(w)$. The c-factorization of the cover ws of w is either $c_{(K_1)} \dots c_{(K_p)} c_{(s)}$ or $c_{(K_1)} \dots c_{(K_i \cup \{s\})} \dots c_{(K_p)}$.*

If $ws = c_{(K_1)} \dots c_{(K_i \cup \{s\})} \dots c_{(K_p)}$ then i is uniquely determined and s commutes with every $r \in K_{i+1} \cup L$ where L satisfies $c_{(K_i \cup \{s\})} = c_{(K_i \setminus L)}\, s\, c_{(L)}$.

Proof. If $s \in K_p$ then $c_{(K_1)} \dots c_{(K_p)} c_{(s)}$ is obviously the c-factorization for ws. So we assume $s \notin K_p$. As w is a c-singleton, ws is c-sortable with c-factorization $c_{(L_1)} \dots c_{(L_q)}$ where $L_1 \supseteq \dots \supseteq L_q$. As $s \in D(ws)$, there is a unique $1 \leq i \leq q$ and $r \in L_i$ such that $w = (ws)s = c_{(L_1)} \dots c_{(L_i \setminus \{r\})} \dots c_{(L_q)}$ by the exchange condition.

Case 1: Suppose $i = 1$, i.e. $i = 1$ is the unique index such that

$$w = (ws)s = c_{(L_1 \setminus \{r\})} c_{(L_2)} \dots c_{(L_q)}. \tag{1}$$

Case 1.1: $r \notin K_1$. Then $r \notin S(w)$ and $s = r$ because $r \in S(ws) = S(w) \cup \{s\}$. Since any two reduced expressions of ws are linked by braid relations according to Tits' Theorem, [2, Theorem 3.3.1] and since $s \notin S(w)$, we conclude that we have to move s from the rightmost position to the left by commutation. In other words, s commutes with $K_2 \cup L$.

Case 1.2: $r \in K_1 = S(w)$. As $c_{(L_1 \setminus \{r\})} c_{(L_2)} \dots c_{(L_q)}$ is reduced and $L_2 \supseteq \dots \supseteq L_q$ is nested, we have $r \in L_2$. Hence

$$K_1 = S(w) = (L_1 \setminus \{r\}) \cup L_2 = L_1 \cup L_2 = L_1.$$

Thus $c_{(L_2)} \dots c_{(L_q)}$ and $c_{(K_2)} \dots c_{(K_p)} s$ are reduced expressions for some $\widehat{w} \in W$ and $s \in D(\widehat{w})$. The exchange condition implies the existence of a unique index $2 \leq j \leq q$ and $t \in L_j$ such that

$$\widehat{w}s = c_{(L_2)} \dots c_{(L_j \setminus \{t\})} \dots c_{(L_q)}.$$

In other words

$$w = c_{(L_1)} \hat{w} = c_{(L_1)} c_{(L_2)} \dots c_{(L_j \setminus \{t\})} \dots c_{(L_q)}$$

is reduced. But this contradicts the uniqueness of $i = 1$ in Equation (1). So $r \notin K_1$ and we finished the first case.

Case 2: Suppose $i > 1$, then $K_1 = S(w) = L_1$. Set $\nu := \min(p, i-1)$ and iterate the argument for $c_{(L_1)}^{-1} w$, $c_{(L_2)}^{-1} c_{(L_1)}^{-1} w, \dots$ to conclude $L_j = K_j$ for $1 \leq j \leq \nu$. If $\nu = p$ then $i = q = p + 1$ and $L_i \setminus \{r\} = \varnothing$. So $L_i = \{s\} \subseteq L_{i-1} = K_p$ which contradicts $s \notin K_p$. Thus $\nu = i - 1$ for some $i \leq p$ and $L_j = K_j$ for $1 \leq j \leq i - 1$. We may assume $i = 1$ and are done by Case 1. $\square$

Proposition 1.7. *Let w be a c-singleton and $\mathbf{w}$ its c-sorting word. Any prefix of $\mathbf{w}$ up to commutations is a c-singleton.*

Proof. Let $c_{(K_1)} \dots c_{(K_p)}$ denote the c-factorization of w. It is sufficient to show that the prefix w' up to commutations of length $\ell(w) - 1$ is a c-singleton. There is $1 \le i \le p$ and $r \in K_p$ such that $w' = c_{(K_1)} \dots c_{(K_i \setminus \{r\})} \dots c_{(K_p)}$ is the c-factorization of w'. It remains to show that $w's$ is c-sortable for $s \notin D(w')$.

Case 1: Suppose $s \in D(w)$. Recall the definition of the *Bruhat order $\le_{\mathcal{B}}$ on W*: $u \le_{\mathcal{B}} v$ in W if an expression for u can be obtained as a subword of a reduced expression of v, see [2, Chapter 2]. The lifting property of the Bruhat order implies $w's \le_{\mathcal{B}} w$. Moreover $\ell(w's) = \ell(w') + 1 = \ell(w)$. Thus $w = w's$ and $s = r$. In particular $w's = w$ is c-sortable.

Case 2: Suppose $s \notin D(w)$, in particular $s \neq r$. So ws is c-sortable and by Lemma 1.6 there are two cases to distinguish.
Case 2.1: If $c_{(K_1)} \dots c_{(K_p)} c_{(s)}$ is the c-factorization of ws then $s \in K_p$ and $c_{(K_1)} \dots c_{(K_i \setminus \{r\})} \dots c_{(K_p)} c_{(s)}$ is the c-factorization of $w's$. In particular, the sequence $K_1 \supseteq \dots \supseteq K_i \setminus \{r\} \supseteq \dots \supseteq K_p \supseteq \{s\}$ is nested and $w's$ is c-sortable.
Case 2.2: If $c_{(K_1)} \dots c_{(K_j \cup \{s\})} \dots c_{(K_p)}$ is the c-factorization of ws then either s and r commute or not.

If s and r do not commute then $j = i$ and r appears before s in the chosen reduced expression of c, since s commutes with every simple reflections to the right of the rightmost copy of s in the c-factorization of ws by Lemma 1.6. Then $w's = c_{(K_1)} \dots c_{(K_i \setminus \{r\} \cup \{s\})} \dots c_{(K_p)}$ is c-sortable.

If s and r commute, suppose first $j \le i$. Then

$$w's = wrs = wsr = c_{(K_1)} \dots c_{(K_j \cup \{s\})} \dots c_{(K_i \setminus \{r\})} \dots c_{(K_p)}$$

is the c-factorization of $w's$. As $K_1 \supseteq \dots \supseteq K_j \cup \{s\} \supseteq \dots \supseteq K_i \setminus \{r\} \supseteq \dots \supseteq K_p$ is nested, $w's$ is c-sortable. The case $j > i$ is proved similarly.

We conclude that $w's$ is c-sortable for any $s \notin D(w')$, so w' is a c-singleton. □

Proof of Theorem 1.2. We know by Proposition 1.1 that w is a c-singleton if and only if w is c-sortable and ww_0 is c^{-1}-sortable.

Suppose w is a c-singleton. Let s be the rightmost simple reflection appearing in the c^{-1}-factorization for ww_0, so $ww_0 = us$ for some c^{-1}-sortable element u.

We have $su^{-1}w = w_0$ and hence $u^{-1}w = sw_0$. Since $S = w_0 S w_0$, $t := w_0 s w_0$ is a simple reflection. Now $u^{-1}wt = w_0$ implies $\ell(wt) > \ell(w)$ and we conclude that wt is c-sortable by Proposition 1.1. But wt is also c-antisortable since $wtw_0 = u$ is c^{-1}-sortable. Hence, wt is a c-singleton that covers w in the weak order.

Repeating this process, we show that every c-singleton is on an unrefinable chain of c-singletons leading up to w_0. By downwards induction, every element of that chain is a prefix of w_0 up to commutations. This is clearly true for w_0. As we went up each step, though, we added a simple reflection which commuted with every reflection to its right (or was added at the rightmost end), by Lemma 1.6. Thus, when we want to remove the element we added at the last step, we can rewrite w_0 using commutations only such that this simple reflection is on the right. □

2. Coxeter fans, Permutahedra, and Cambrian fans

In this section, we describe the geometry of Coxeter fans and c-Cambrian fans. We first recall some fact about the geometric representation of W and use the notation of [11] on Coxeter groups and root systems. Let W act by reflections on an $\mathbb{R}$-Euclidean space $(V, \langle \cdot, \cdot \rangle)$.

Let Φ be a root system corresponding to W with simple roots $\Delta = \{\alpha_s \mid s \in S\}$, positive roots $\Phi^+ = \Phi \cap \mathbb{R}_{>0}[\Delta]$ and negative roots $\Phi^- = -\Phi^+$. Without loss of generality, we assume that the action of W is essential relative to V, that is, Δ is a basis of V. The set Φ^+ parametrizes the set of reflections in W: to each reflection $t \in W$ there corresponds a unique positive root $\alpha_t \in \Phi^+$ such that t maps α_t to $-\alpha_t$ and fixes the hyperplane $H_t = \{v \in V \mid \langle v, \alpha_t \rangle = 0\}$.

The *Coxeter arrangement* $\mathcal{A}$ for W is the collection of all reflecting hyperplanes for W. The complement $V \setminus (\bigcup \mathcal{A})$ of $\mathcal{A}$ consists of open cones. Their closures are called *chambers*. The chambers are in canonical bijective correspondence with the elements of W. The *fundamental chamber* $D := \bigcap_{s \in S} \{v \in V \mid \langle v, \alpha_s \rangle \geq 0\}$ corresponds to the identity $e \in W$ and the chamber $w(D)$ corresponds to $w \in W$.

A subset U of V is *below* a hyperplane $H \in \mathcal{A}$ if every point in U is on H or on the same side of H as D. The subset U is *strictly below* $H \in \mathcal{A}$ if U is below H and $U \cap H = \varnothing$. Similarly, U is *above* or *strictly above* a hyperplane $H \in \mathcal{A}$. The inversions of $w \in W$ are the reflections that correspond to the hyperplanes H which $w(D)$ is above.

For a simple reflection $s \in S$, we have $\ell(sw) < \ell(w)$ if and only if $s \leq w$ in the weak order if and only if $w(D)$ is above H_s. To decide whether $w(D)$ is above or below H_s is therefore a weak order comparison. These notions will be handy in §3.

A *fan* $\mathcal{G}$ is a family of nonempty closed polyhedral (convex) cones in V such that
(i) every face of a cone in $\mathcal{G}$ is in $\mathcal{G}$, and
(ii) the intersection of any two cones in $\mathcal{G}$ is a face of both.
A fan $\mathcal{G}$ is *complete* if the union of all its cones is V, *essential* (or *pointed*) if the intersection of all non-empty cones of $\mathcal{G}$ is the origin, and *simplicial* if every cone is simplicial, that is, spanned by linearly independent vectors. A 1-dimensional cone is called a *ray* and a ray is *extremal* if it is a face of some cone. The set of k-dimensional cones of $\mathcal{G}$ is denoted by $\mathcal{G}^{(k)}$ and two cones in $\mathcal{G}^{(k)}$ are *adjacent* if they have a common face in $\mathcal{G}^{(k-1)}$. A fan $\mathcal{G}$ coarsens a fan $\mathcal{G}'$ if every cone of $\mathcal{G}$ is the union of cones of $\mathcal{G}'$ and $\bigcup_{C \in \mathcal{G}} C = \bigcup_{C \in \mathcal{G}'} C$. We refer to [31, Lecture 7] for more details and examples.

The chambers and all their faces of a Coxeter arrangement $\mathcal{A}$ define the *Coxeter fan* $\mathcal{F}$. The Coxeter fan $\mathcal{F}$ is known to be complete, essential, and simplicial, [11, Sections 1.12–1.15]. The fundamental chamber $D \in \mathcal{F}$ is a (maximal) cone spanned by the (extremal) rays $\{\rho_s \mid s \in S\}$, where ρ_s is the intersection of D with the subspace orthogonal to the hyperplane spanned by $\{\alpha_t \mid t \in \langle s \rangle\}$.

Recall that the rays of $\mathcal{F}$ decompose into n orbits under the action of W and each orbit contains exactly one ρ_s, $s \in S$. Thus, any ray $\rho \in \mathcal{F}^{(1)}$ is $w(\rho_s)$ for some $w \in W$ where $s \in S$ is uniquely determined by ρ but w is not unique. In fact, $w(\rho_s) = g(\rho_s)$ if and only if $w \in gW_{\langle s \rangle}$.

2.1. **Permutahedra.** We illustrate Coxeter fans by means of permutahedra, that is, polytopes that have a Coxeter fan as normal fan.

Take a point $\boldsymbol{a}$ of the complement $V \setminus (\bigcup \mathcal{A})$ of the Coxeter arrangement $\mathcal{A}$, and consider its W-orbit. The convex hull of this W-orbit is a W-*permutahedron* denoted by $\mathsf{Perm}^{\boldsymbol{a}}(W)$. There is a bijection between the rays of $\mathcal{F}$ and the facets of $\mathsf{Perm}^{\boldsymbol{a}}(W)$: there is a halfspace associated to each ray $\rho \in \mathcal{F}$ such that its supporting hyperplane is perpendicular to ρ and such that the permutahedron is the intersection of these halfspaces. Let us be more precise.

Let $\Delta^* := \{v_s \in V \mid s \in S\}$ be the fundamental weights of Δ, that is, Δ^* is the dual basis of Δ for the scalar inner product. The fundamental chamber D is spanned by the fundamental weights, that is, $D = \mathbb{R}_{\geq 0}[\Delta^*]$. Hence, the rays of $\mathcal{A}$ are easily expressed in terms of Δ^*: We have $\rho_s = \mathbb{R}_{\geq 0}[v_s]$ and therefore $w(\rho_s) = \mathbb{R}_{\geq 0}[w(v_s)]$ for any $w \in W$ and $s \in S$.

Without loss of generality, we choose $\boldsymbol{a} = \sum_{s\in S} a_s v_s$ in the interior of D, that is $a_s > 0$ for $s \in S$, and define $M(w) := w(\boldsymbol{a})$. All points $M(w)$ are distinct and the convex hull of $\{M(w) \mid w \in W\}$ yields a realization of the W-permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$. It is not difficult to describe this polytope as an intersection of half-spaces.

For each $\rho = w(\rho_s) \in \mathcal{F}^{(1)}$, we define the closed half space

$$\mathscr{H}_\rho^{\boldsymbol{a}} := \{v \in V \mid \langle v, w(v_s)\rangle \leq \langle \boldsymbol{a}, v_s\rangle\}.$$

This definition does not depend on the choice of $w \in W$ such that $\rho = w(\rho_s)$, but only of the coset $W/W_{\langle s\rangle}$. The open half space $\mathscr{H}_\rho^{\boldsymbol{a},+}$ and the hyperplane $H_\rho^{\boldsymbol{a}}$ are defined by strict inequality and equality respectively. Now, the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is given by

$$\mathsf{Perm}^{\boldsymbol{a}}(W) = \bigcap_{\rho\in\mathcal{F}^{(1)}} \mathscr{H}_\rho^{\boldsymbol{a}}.$$

As for the rays of the Coxeter fan, we have $H_\rho^{\boldsymbol{a}} = H_{(w,s)}^{\boldsymbol{a}} = H_{(w',s)}^{\boldsymbol{a}}$ if and only if $w \in w'W_{\langle s\rangle}$ and $\rho = w(\rho_s) = w'(\rho_s)$. Moreover, $M(w) \in H_{(w',s)}^{\boldsymbol{a}}$ if and only if $H_{(w',s)}^{\boldsymbol{a}} = H_{(w,s)}^{\boldsymbol{a}}$. A simple description of the vertex $M(w)$ of the permutahedron follows:

$$M(w) = \bigcap_{s\in S} H_{(w,s)}^{\boldsymbol{a}}.$$

Example 2.1 (Realization of $\mathsf{Perm}(S_3)$)**.** We consider the Coxeter group $W = S_3$ of type A_2 acting on $\mathbb{R}^2$. The reflections s_1 and s_2 generate W and the simple roots that correspond to s_1 and s_2 are α_1 and α_2. They are normal to the reflection hyperplanes H_{s_1} and H_{s_2}. The fundamental weight vectors that correspond to the simple roots are the vectors v_1 and v_2 and determine the ray $L = \{\mu(a_1v_1 + a_2v_2) \mid \mu > 0\}$, $a_1, a_2 > 0$, which contains $M(e) = \boldsymbol{a} \in L$. We obtain the permutahedron $\mathsf{Perm}(S_3)$ as convex hull of the W-orbit of $M(e)$. Alternatively, the permutahedron is described as intersection of the half spaces $\mathscr{H}_{(x,s)}^{\boldsymbol{a}}$ with bounding hyperplanes $H_{(x,s)}^{\boldsymbol{a}}$ for $x \in W$ and $s \in S$. All objects are indicated in Figure 6.

2.2. **Cambrian fans.** For any lattice congruence Θ of the weak order on W, N. Reading constructs a complete fan $\mathcal{F}_\Theta$ that coarsens the Coxeter fan $\mathcal{F}$, [19]. A maximal cone $C_\vartheta \in \mathcal{F}_\Theta$ corresponds to a congruence class ϑ of Θ and C_ϑ is the union of the chambers of $\mathcal{A}$ that correspond to the elements of ϑ. In [19, Section 5] N. Reading proves that these unions are indeed convex cones and that the collection $\mathcal{F}_\Theta$ of these cones and their faces is a complete fan.

The *c-Cambrian fan* $\mathcal{F}_c$ of W is obtained by this construction if we consider the lattice congruence with congruence classes $[\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$ for $w \in W$ and chosen Coxeter element c. The n-dimensional cone that corresponds to the c-sortable element w is denoted by $C(w)$. It is the union of the maximal cones of $\mathcal{F}$ that

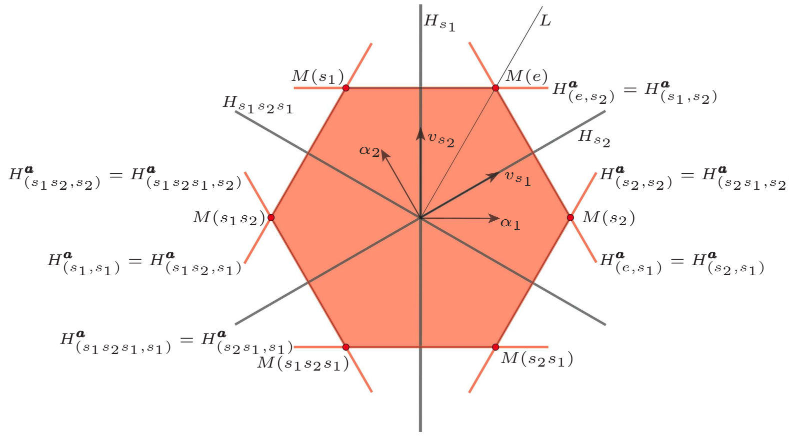

FIGURE 6. The permutahedron $\mathsf{Perm}(S_3)$ obtained as convex hull of the S_3-orbit of $M(e) \in L$ or as intersection of the half spaces $\mathscr{H}^{\boldsymbol{a}}_{(x,s)}$.

correspond to the elements of $(\pi^c_\downarrow)^{-1}\pi^c_\downarrow(w) = [\pi^c_\downarrow(w), \pi^\uparrow_c(w)]$. In particular, $C(w)$ is a maximal cone of $\mathcal{F}_c$ and of $\mathcal{F}$ if and only if w is a c-singleton.

In [23], N. Reading and D. Speyer define a bijection between the set of rays of $\mathcal{F}_c$ and the set of *almost positive roots* $\Phi_{\geq -1} := \Phi^+ \cup (-\Delta)$. To describe this labeling of the rays, we first define a set of almost positive roots for any c-sortable w. For $s \in S(w)$, let $1 \leq j_s \leq \ell(w)$ be the unique integer such that s_{j_s} is the rightmost occurrence of s in the c-sorting word $s_1 \dots s_{\ell(w)}$ of w and define

$$\mathsf{Lr}_s(w) := \begin{cases} s_1 \dots s_{j_s - 1}(\alpha_s) & \text{if } s \in S(w) \\ -\alpha_s & \text{if } s \notin S(w), \end{cases} \quad \text{and} \quad \mathrm{cl}_c(w) := \bigcup_{s \in S} \mathsf{Lr}_s(w).$$

Example 2.2. To illustrate these maps, we consider the Coxeter group $W = S_3$ with generators $S = \{s_1, s_2\}$ as shown in Figure 6. Choose $c = s_1 s_2$ as Coxeter element. It is easy to check that $w \in W \setminus \{s_2 s_1\}$ is c-sortable and that $w \in W \setminus \{s_2, s_2 s_1\}$ is a c-singleton. From the above definition follows

$$\begin{aligned} &\mathsf{Lr}_{s_1}(e) = \mathsf{Lr}_{s_1}(s_2) = -\alpha_1, && \mathsf{Lr}_{s_2}(e) = \mathsf{Lr}_{s_2}(s_1) = -\alpha_2, \\ &\mathsf{Lr}_{s_1}(s_1) = \mathsf{Lr}_{s_1}(s_1 s_2) = \alpha_1, && \mathsf{Lr}_{s_2}(s_1 s_2 s_1) = \mathsf{Lr}_{s_2}(s_1 s_2) = \alpha_1 + \alpha_2, \\ &\mathsf{Lr}_{s_1}(s_1 s_2 s_1) = \alpha_2, && \mathsf{Lr}_{s_2}(s_2) = \alpha_2, \end{aligned}$$

and therefore

$$\begin{aligned} &\mathrm{cl}_c(e) = \{-\alpha_1, -\alpha_2\}, && \mathrm{cl}_c(s_1 s_2) = \{\alpha_1, \alpha_1 + \alpha_2\}, \\ &\mathrm{cl}_c(s_1) = \{\alpha_1, -\alpha_2\}, && \mathrm{cl}_c(s_1 s_2 s_1) = \{\alpha_2, \alpha_1 + \alpha_2\}, \\ &\mathrm{cl}_c(s_2) = \{-\alpha_1, \alpha_2\}. \end{aligned}$$

N. Reading and D. Speyer use the cluster map cl_c to prove that c-Cambrian fans and cluster fans have the same combinatorics: the maximal cone $C(w)$ of the c-Cambrian fan represented by the c-sortable element w is mapped to the set $\mathrm{cl}_c(w)$ of almost positive roots. The cardinality of $\mathrm{cl}_c(w)$ matches the number of extremal rays of $C(w)$ and cl_c induces a bijection f_c between the rays of $\mathcal{F}_c$ and the almost positive roots by extending cl_c to intersection of cones:

$\mathrm{cl}_c(C_1 \cap C_2) := \mathrm{cl}_c(C_1) \cap \mathrm{cl}_c(C_2)$. To put it slightly differently, N. Reading and D. Speyer showed the following Theorem.

Theorem 2.3 (Reading-Speyer [23, Theorem 7.1])**.** *There is a bijective labeling $f_c : \mathcal{F}_c^{(1)} \to \Phi_{\geq -1}$ of the rays of the c-Cambrian fan $\mathcal{F}_c$ by almost positive roots such that the extremal rays of $C(w)$ are labeled by $\mathrm{cl}_c(w)$.*

We now aim for an explicit description of f_c that relates nicely to c-singletons, but first need the following two lemmas.

Lemma 2.4. *For $\beta \in \Phi_{\geq -1}$, there exists a c-sortable element w and a simple reflection s such that $\mathsf{Lr}_s(w) = \beta$.*

Proof. The identity e is a c-singleton and $\mathrm{cl}_c(e) = \Phi_{\geq -1} \setminus \Phi^+$, so we are done if β is a negative simple root. Suppose that $\beta \in \Phi^+$ and consider the longest element w_0 with c-sorting word $\mathbf{w_0} = s_{j_1} s_{j_2} \dots s_{j_N}$. Since $w_0(D)$ is above all reflecting hyperplanes,

$$\Phi^+ = \{s_{j_1} s_{j_2} \dots s_{j_{p-1}}(\alpha_{s_{j_p}}) \,|\, 1 \leq p \leq N\}$$

and $\beta = s_{j_1} \dots s_{j_{i-1}}(\alpha_{s_{j_i}})$ for some $1 \leq i \leq N$. Since $w = s_{j_1} \dots s_{j_i}$ is a prefix of $\mathbf{w_0}$, it is a c-singleton and $\mathsf{Lr}_{s_{j_i}}(w) = \beta$. □

Lemma 2.5. *Let $\rho \in \mathcal{F}_c^{(1)}$. There is a c-singleton w such that ρ is an extremal ray of $C(w)$.*

Proof. Pick $\rho \in \mathcal{F}_c^{(1)}$. According to Theorem 2.3, $f_c(\rho) = \beta$ for some almost positive root β. By Lemma 2.4, there is a c-singleton w and a simple reflection $s \in S$ such that $\mathsf{Lr}_s(w) = \beta$. But this implies that ρ is an extremal ray of $C(w)$. □

If w is a c-singleton, then $C(w) \in \mathcal{F}_c^{(n)}$ is the maximal cone $w(D)$ which is spanned by the rays $\{w(\rho_s) \,|\, s \in S\}$. The main result of this section is

Theorem 2.6. *Let $\rho \in \mathcal{F}_c^{(1)}$. There is a unique simple reflection $s \in S$ and there is a c-singleton w such that $\rho = w(\rho_s)$ and $f_c(w(\rho_s)) = \mathsf{Lr}_s(w)$.*

Proof. The uniqueness of $s \in S$ follows from the fact that any ray of the Coxeter fan is of the form $w(\rho_s)$ where $s \in S$ is uniquely determined (but not w!).

The first claim follows directly from Lemma 2.5. We proceed by induction on the length of w. If $\ell(w) = 0$ then $w = e$. In particular, e is a c-singleton and $s = es$ is c-sortable for any $s \in S$. Fix some $s \in S$. Since $\mathrm{cl}_c(e) = -\Delta = \{-\alpha_t \,|\, t \in S\}$ and $\mathrm{cl}_c(s) = \{-\alpha_t \,|\, t \in \langle s \rangle\} \cup \{\alpha_s\}$, we conclude $f_c(e(\rho_s)) = -\alpha_s$ as $s(D) \subset C(s)$ and the rays of $\mathcal{F}$ in $s(D) \cap D$ are $\{\rho_t \,|\, t \in \langle s \rangle\}$.

Suppose that $\ell(w) > 0$ and let $t \in S$ be the last simple reflection of the c-sorting word of w. By Proposition 1.7, wt is a c-singleton with $\ell(wt) < \ell(w)$. By induction, $f_c(wt(\rho_s)) = \mathsf{Lr}_s(wt)$ for some $s \in S$. If $s \neq t$ then $t \in W_{\langle s \rangle}$ and we conclude $wt(\rho_s) = w(\rho_s)$ and $\mathsf{Lr}_s(wt) = \mathsf{Lr}_s(w)$. Hence, we suppose $s = t$. We have $C(w) \cap C(ws) = w(D) \cap ws(D)$, the extremal rays of this cone are $\{w(\rho_t) \,|\, t \in \langle s \rangle\}$, and their image under f_c is

$$\mathrm{cl}_c(w) \cap \mathrm{cl}_c(ws) = \{\mathsf{Lr}_t(w) \,|\, t \in \langle s \rangle\} = \mathrm{cl}_c(w) \setminus \{\mathsf{Lr}_s(w)\}.$$

So $f_c(w(\rho_s)) = \mathsf{Lr}_s(w)$. □

3. Realizing generalized associahedra

3.1. **A general result.** A fan has to satisfy some obvious conditions in order to have the combinatorics of the normal fan of a polytope, in particular, the fan has to be pointed and complete. This condition is far from sufficient and in general it is quite hard to decide whether a given fan is polytopal or not. Coarsenings or refinements of a (polytopal or non-polytopal) fan may or may not be polytopal, examples are easily derived from the non-polytopal fan of [31, Example 7.5] which is a pointed, complete and simplicial fan in $\mathbb{R}^3$. We aim for something less ambitious and prove a criterion that implies that a given realization of a fan is the normal fan of a polytope. Our notation is inspired by Section 2.

Consider a pointed, complete, and simplicial fan $\mathcal{G} \subseteq \mathbb{R}^n$ with d-dimensional cones $\mathcal{G}^{(d)}$. To $\rho \in \mathcal{G}^{(1)}$ we associate a vector v_ρ such that $\rho = \mathbb{R}_{\geq 0}[v_\rho]$. Suppose that we are given a collection of positive real numbers λ_ρ, one for each $\rho \in \mathcal{G}^{(1)}$. We then define a hyperplane

$$H_\rho = \{x \in \mathbb{R}^n \ \mid \langle x, v_\rho \rangle = \lambda_\rho\}$$

and a half space

$$\mathscr{H}_\rho := \{x \in \mathbb{R}^n \ \mid \langle x, v_\rho \rangle \leq \lambda_\rho\}.$$

We write $\mathscr{H}_\rho^+$ if the inequality is strict. Since $\mathcal{G}$ is simplicial, we have for every maximal cone $C \in \mathcal{G}^{(n)}$ a point $x(C)$ defined by $\{x(C)\} := \bigcap_{\rho \in C^{(1)}} H_\rho$. Then

$$P := \operatorname{ConvexHull}\left\{ x(C) \ \middle|\ C \in \mathcal{G}^{(n)} \right\} \quad \text{and} \quad \widetilde{P} := \bigcap_{\rho \in \mathcal{G}^{(1)}} \mathscr{H}_\rho$$

are well-defined polytopes of dimension at most n.

For instance the case of a W-permutahedron constructed from the Coxeter fan $\mathcal{F}$ as explained in §2.1 fits nicely in this context: $x(w(D))$ is by definition $M(w)$ and the half spaces $\mathscr{H}_\rho$ are precisely the half spaces $\mathscr{H}_\rho^{\mathbf{a}}$, for $\rho \in \mathcal{F}^{(1)}$. In this case the two polytopes P and $\widetilde{P}$ coincide.

Let $C \in \mathcal{G}^{(n)}$ and $f \in C^{(n-1)}$ a $(n-1)$-dimensional face of C. An *outer normal of C relative to f* is a vector v normal to f, that is normal to the hyperplane spanned by f, and such that $C \subseteq \{x \in \mathbb{R}^n \ \mid \langle x, v \rangle \leq 0\}$.

Let $C_i, C_j \in \mathcal{G}^{(n)}$ be two adjacent maximal cones in $\mathcal{G}$, that is, $C_i \cap C_j \in \mathcal{G}^{(n-1)}$. A vector u is said to be *pointing to C_i from C_j* if there is an outer normal v of C_j relative to $C_i \cap C_j$ such that $\langle u, v \rangle > 0$. In particular, observe that:

(i) Any outer normal of C_j relative to $C_i \cap C_j$ is pointing to C_i from C_j;
(ii) If $x_i \in C_i$ and $x_j \in C_j$ are points in the interior of these cones, then the vector $x_i - x_j$ is pointing to C_i from C_j;
(iii) Any vector not contained in the span of $C_i \cap C_j$ is either pointing to C_i from C_j or pointing to C_j from C_i.

Notice that the vector $x(C_i) - x(C_j)$ is a normal vector to $C_i \cap C_j$, but not necessarily pointing to C_i from C_j, since $x(C_i)$ is not necessarily a point in C_i.

Theorem 3.1. *Use the notation as above and suppose that $x(C_i) - x(C_j)$ points to C_i from C_j whenever $C_i, C_j \in \mathcal{G}^{(n)}$ with $C_i \cap C_j \in \mathcal{G}^{(n-1)}$. Then $P = \widetilde{P}$ has (outer) normal fan $\mathcal{N}(P) = \mathcal{G}$ and is of dimension n.*

Remark 3.2. The hypothesis of Theorem 3.1 is satisfied in (at least) three cases.

First, the case of W-permutahedra constructed from the Coxeter fan $\mathcal{F}$. Indeed, the points $M(w)$ are strictly in the cone $w(D)$. So $M(w) - M(w')$ points to $w(D)$ from $w'(D)$ whenever $w(D)$ and $w'(D)$ are adjacent cones.

Second, the case of the cube constructed from the fan $\mathcal{G}$ which is the skew coordinate hyperplane arrangement obtained from the hyperplanes which bound the fundamental chamber in the Coxeter arrangement. (Note that this fan corresponds to the usual construction of the Boolean lattice as a quotient of weak order, via the descent map, see [13]).

Third, as we shall show, the case of Cambrian fans described on the two next sections.

Remark 3.3. A similar theorem, which includes the assumption that $\mathcal{G}$ should be a coarsening of a Coxeter fan, appears as [12, Theorem A.3]. That theorem would suffice for our purposes, but we prefer to give the following independent proof of the more general theorem.

Proof of Theorem 3.1. Since P and $\widetilde{P}$ are convex polytopes, it is enough to show that the vertex set $V(P)$ of P is equal to the vertex set $V(\widetilde{P})$ of $\widetilde{P}$.

Let us prove first that $P \subseteq \widetilde{P}$. It suffices to prove $\langle x(C), v_\rho \rangle < \lambda_\rho$ for $C \in \mathcal{G}^{(n)}$ and $\rho \in \mathcal{G}^{(1)} \setminus C^{(1)}$.

Let $C \in \mathcal{G}^{(n)}$ and $\rho \in \mathcal{G}^{(1)} \setminus C^{(1)}$. We will show that there is a finite sequence $C_0 := C, \ldots, C_k = C'$ of maximal cones of $\mathcal{G}$ such that $\rho \subseteq C'$, $C_i \cap C_{i+1} \in \mathcal{G}^{(n-1)}$ and v_ρ is pointing to C_{i+1} from C_i, for $0 \leq i < k$.

For x in C, we write $x + \rho$ for the half line $\{x + \lambda v_\rho \ \mid \lambda \geq 0\}$ parallel to ρ and starting at x. Write C_ρ for the union of all maximal cones of $\mathcal{G}$ that contain ρ. Since $\mathcal{G}$ is a pointed complete fan, C_ρ contains n-dimensional balls of arbitrary diameter centered at points of ρ. In particular, C_ρ contains such a ball of diameter d, where d is the distance between the lines containing ρ and $x+\rho$. So $(x+\rho) \cap C_\rho \neq \varnothing$ for any point $x \in C$. Hence there is a maximal cone C' of $\mathcal{G}$ such that ρ is an extremal ray of C' and $(x + \rho) \cap C' \neq \varnothing$ for any point $x \in C$. For any $x \in C$, the line segment between C and C' on $x + \rho$ determines a sequence of cones $C_0 = C,\ C_1, \ldots,\ C_p = C'$ of $\mathcal{G}$ of arbitrary dimension, those are the cones that $x + \rho$ meets between C and C' in the natural order on the C_i induced by the order of points of $x+\rho$ given by the parametrization of this half line. From this point, we are looking for a sequence $C = C_0,\ C_{0,1},\ C_1,\ C_{1,2}, \ldots, C_{k-1,k},\ C_k = C'$ such that C_i is a maximal cone and $C_{i,i+1} = C_i \cap C_{i+1}$ is a cone of codimension 1. Since the number of cones in $\mathcal{G}$ is finite, the number of cones met by all possible half lines $x+\rho$ for $x \in C$ is finite. Since C is a full dimensional cone, we may move x in C and then may assume that $x + \rho$ does not intersect any cone of $\mathcal{G}$ of codimension larger than 1. In other words, there is a finite sequence $C_0 = C,\ C_1, \ldots,\ C_k = C'$ of maximal cones of $\mathcal{G}$ such that $C_{i,i+1} = C_i \cap C_{i+1} \in \mathcal{G}^{(n-1)}$ and $(x + \rho) \cap C_i \neq \varnothing$. Pick y_i in the interior of C_i and in $x + \rho$. So $y_{i+1} - y_i$ points to C_{i+1} from C_i. Since the cones $C_0, \ldots, C_k$ have the same order as the points on $x + \rho$, the distance from x to y_i is strictly smaller than the distance from x to y_{i+1}. This means the vector $y_{i+1} - y_i = \kappa v_\rho$ with $\kappa > 0$. Hence v_ρ points to C_{i+1} from C_i.

Now, we consider the piecewise linear path from $x(C_0)$ to $x(C_k)$ that traverses from $x(C_i)$ to $x(C_{i+1})$. Since $x(C_{i+1}) - x(C_i)$ points to C_{i+1} from C_i, the vector $x(C_{i+1}) - x(C_i)$ is an outer normal to C_i relative to $C_i \cap C_{i+1}$, and since v_ρ points to C_{i+1} from C_i for $0 \leq i < k$, we conclude that $\langle x(C_{i+1}) - x(C_i), v_\rho \rangle > 0$.

Hence

$$\langle x(C_0), v_\rho\rangle < \ldots < \langle x(C_k), v_\rho\rangle = \lambda_\rho.$$

This proves $P \subseteq \widetilde{P}$.

For any vertex $x(C)$ of P we know that $x(C)$ is a vertex of $\widetilde{P}$ since $x(C)$ is defined as the intersection of hyperplanes that bound half spaces defining $\widetilde{P}$ and $P \subseteq \widetilde{P}$. Moreover, for any maximal cone $C \in \mathcal{G}$, the point $x(C)$ must be on the boundary of P for that reason. But now it follows that the points $x(C)$ are in convex position because $\mathcal{G}$ is simplicial and $\bigcap_{\rho\in C^{(1)}} H_\rho$ is a singleton. Hence

$$\left\{x(C) \;\middle|\; C \in \mathcal{G}^{(n)}\right\} = V(P) \subseteq V(\widetilde{P}).$$

The rays $\rho \in C^{(1)}$ span the outer normal cone of $x(C) \in P$ for every maximal cone $C \in \mathcal{G}$. The hyperplanes H_ρ, $\rho \in C^{(1)}$, bound the half spaces of $x(C) \in \widetilde{P}$, so the normal cones agree. But the union of these cones equals the complete fan $\mathcal{G}$, so $\widetilde{P}$ does not have any additional vertices.

The claim that $\dim(P) = n$ follows from the fact that $\lambda_\rho > 0$ for all $\rho \in \mathcal{G}^{(1)}$: a neighborhood of 0 is contained in P. $\square$

3.2. **Realizations of generalized associahedra.** We apply Theorem 3.1 to show how c-Cambrian fans $\mathcal{F}_c$ and associahedra $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ relate. The associahedron is described as intersection of certain half spaces of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$ determined by the rays of $\mathcal{F}_c$ and the common vertices of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ and $\mathsf{Perm}^{\boldsymbol{a}}(W)$ are characterized in terms of c-singletons. The proof of Theorem 3.4 is deferred to Section 3.3.

Theorem 3.4. *Let c be a Coxeter element of W and choose a point $\boldsymbol{a}$ in the interior of the fundamental chamber D, to fix a realization of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$.*

(i) The polyhedron

$$\mathsf{Asso}_c^{\boldsymbol{a}}(W) = \bigcap_{\rho\in\mathcal{F}_c^{(1)}} \mathscr{H}_\rho^{\boldsymbol{a}}$$

is a simple polytope of dimension n with c-Cambrian fan $\mathcal{F}_c$ as normal fan.

(ii) The vertex sets $V(\mathsf{Asso}_c^{\boldsymbol{a}}(W))$ and $V(\mathsf{Perm}^{\boldsymbol{a}}(W))$ satisfy

$$V(\mathsf{Asso}_c^{\boldsymbol{a}}(W)) \cap V(\mathsf{Perm}^{\boldsymbol{a}}(W)) = \{M(w) \mid w \text{ is a } c\text{-singleton}\}.$$

The first statement implies that the facet-supporting half spaces of the associahedron form a subset of the facet-supporting half spaces of the permutahedron. We mentioned this in the introduction in the context of c-admissible half spaces. A facet-supporting half space $\mathscr{H}_\rho^{\boldsymbol{a}}$ of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is *c-admissible* if $M(w) \in H_\rho^{\boldsymbol{a}}$ for some c-singleton w. We rephrase the first statement as follows:

Corollary 3.5. *The associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ is the intersection of all c-admissible halfspaces of $\mathsf{Perm}^{\boldsymbol{a}}(W)$.*

We illustrate these results with a basic example.

Example 3.6. The first statement of Theorem 3.4 claims that the intersection of a subset of the half spaces $\mathscr{H}_\rho^{\boldsymbol{a}}$ of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ yields a generalized associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ if we restrict to half spaces such that ρ is a ray of the c-Cambrian fan $\mathcal{F}_c$. Figures 7 and 8 illustrate this for $W = S_4$ generated by $S = \{s_1, s_2, s_3\}$. We use the following conventions: The point $\boldsymbol{a}$ used to fix a realization of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ is labeled A. A facet of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ is labeled by the ray $\rho_j \in \mathcal{F}_c$

that is perpendicular to that facet. Recall that each ray ρ can be written as $w(\rho_{s_i})$ for some (non-unique) c-singleton w and some (unique) simple reflection s_i by Lemma 2.5. If the Coxeter element $c = s_1 s_2 s_3$ is chosen (Figure 7), then we can express the ray $\rho_i \in \mathcal{F}_c$ that corresponds to the (c-admissible) half space $\mathscr{H}^{\boldsymbol{a}}_{\rho_i}$ as follows:

$$
\begin{aligned}
&\rho_1 = e(\rho_{s_1}),\\
&\rho_2 = e(\rho_{s_3}) = s_1(\rho_{s_3}) = s_1 s_2(\rho_{s_3}) = s_1 s_2 s_1(\rho_{s_3}),\\
&\rho_3 = e(\rho_{s_2}) = s_1(\rho_{s_2}),\\
&\rho_4 = s_1 s_2(\rho_{s_2}) = s_1 s_2 s_3(\rho_{s_2}) = s_1 s_2 s_1(\rho_{s_2}) = s_1 s_2 s_3 s_1(\rho_{s_2}),\\
&\rho_5 = s_1(\rho_{s_1}) = s_1 s_2(\rho_{s_1}) = s_1 s_2 s_3(\rho_{s_1}),\\
&\rho_6 = s_1 s_2 s_3 s_1 s_2 s_1(\rho_{s_1}),\\
&\rho_7 = s_1 s_2 s_3 s_1 s_2(\rho_{s_2}) = s_1 s_2 s_3 s_1 s_2 s_1(\rho_{s_2}),\\
&\rho_8 = s_1 s_2 s_3(\rho_{s_3}) = s_1 s_2 s_3 s_1(\rho_{s_3}) = s_1 s_2 s_3 s_1 s_2(\rho_{s_3}) = s_1 s_2 s_3 s_1 s_2 s_1(\rho_{s_3}),\ \textit{and}\\
&\rho_9 = s_1 s_2 s_1(\rho_{s_1}) = s_1 s_2 s_3 s_1(\rho_{s_1}) = s_1 s_2 s_3 s_1 s_2(\rho_{s_1}).
\end{aligned}
$$

The claim of the second statement of Theorem 3.4 is that the common vertices of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ and $\mathsf{Asso}^{\boldsymbol{a}}_c(W)$ are the points $M(w) = w(M(e))$ for w a c-singleton. It is straightforward to verify this claim directly if $c = s_1 s_2 s_3$ in Figure 7: The common vertices of $\mathsf{Asso}^{\boldsymbol{a}}_c(W)$ and $\mathsf{Perm}^{\boldsymbol{a}}(W)$ are labeled A through H and we have

$$
\begin{array}{lll}
A = M(e), & B = M(s_1), & C = M(s_1 s_2),\\
D = M(s_1 s_2 s_1), & E = M(s_1 s_2 s_3), & F = M(s_1 s_2 s_3 s_1),\\
G = M(s_1 s_2 s_3 s_1 s_2), & H = M(s_1 s_2 s_3 s_1 s_2 s_1). &
\end{array}
$$

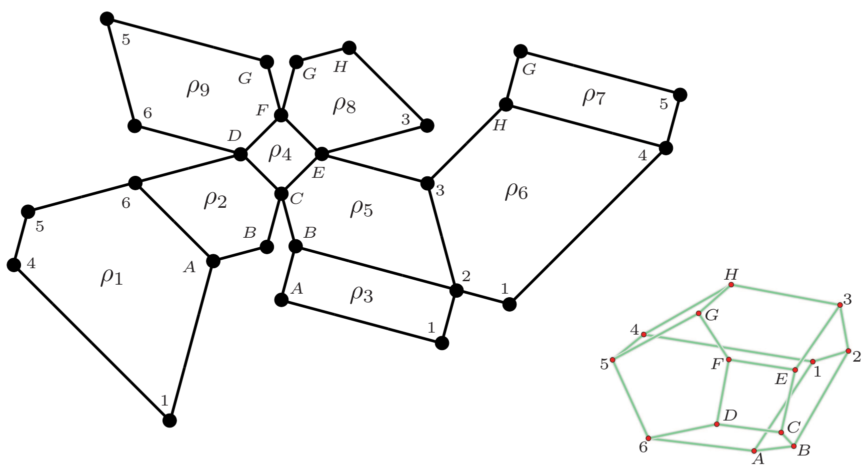

FIGURE 7. An unfolding of the associahedron $\mathsf{Asso}^{\boldsymbol{a}}_c(S_4)$ with $c = s_1 s_2 s_3$. The 2-faces are labelled by $\rho_i \in \mathcal{F}^1_c$ for the facet-defining hyperplane $H^{\boldsymbol{a}}_{\rho_i}$.

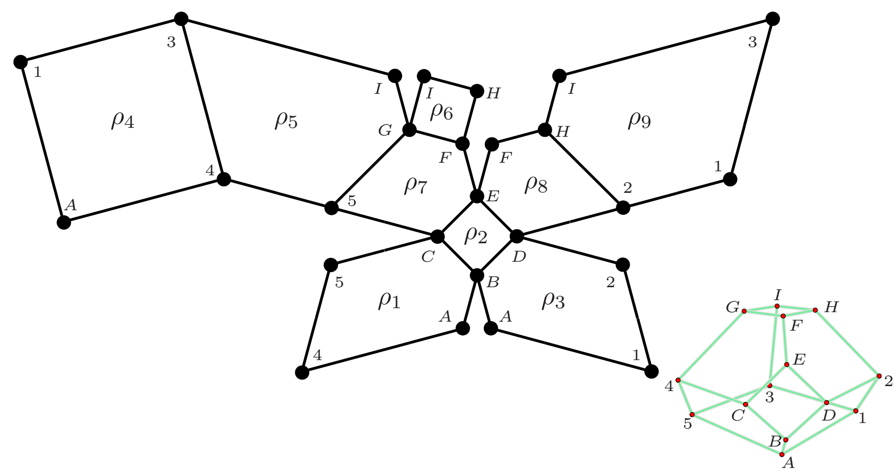

FIGURE 8. An unfolding of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_4)$ with $c = s_2s_1s_3$. The 2-faces are labeled by $\rho = (w, s) \in \mathcal{F}_c^1$ for the facet-defining hyperplane $H_\rho^{\boldsymbol{a}}$.

If the Coxeter element is $c = s_2s_1s_3$ (Figure 8) then we have the following list of expressions for $\rho_i \in \mathcal{F}_c$ (we do not list all possible expressions for ρ_i):

$$\begin{array}{lll}
\rho_1 = e(\rho_{s_3}), & \rho_2 = s_2(\rho_{s_2}), & \rho_3 = e(\rho_{s_1}), \\
\rho_4 = e(\rho_{s_2}), & \rho_5 = s_2s_1s_3s_2s_3(\rho_{s_3}) & \rho_6 = s_2s_1s_3s_2s_1s_3(\rho_{s_2}), \\
\rho_7 = s_2s_1(\rho_{s_1}), & \rho_8 = s_2s_3(\rho_{s_3}), & \rho_9 = s_2s_1s_3s_2s_1(\rho_{s_1}).
\end{array}$$

The common vertices of the permutahedron and associahedron are labeled A through I and we have

$$\begin{array}{lll}
A = M(e), & B = M(s_2), & C = M(s_2s_1), \\
D = M(s_2s_3), & E = M(s_2s_1s_3), & F = M(s_2s_1s_3s_2), \\
G = M(s_2s_1s_3s_2s_3), & H = M(s_2s_1s_3s_2s_1), & I = M(s_2s_1s_3s_2s_1s_3).
\end{array}$$

3.3. **Proof of Theorem 3.4.** Let $c \in W$ be a Coxeter element and let $\boldsymbol{a}$ be in the interior of D. The proof of Theorem 3.4 is based on Theorem 3.1. We use the same notations as in §3.1 applied to the c-Cambrian fan $\mathcal{F}_c$ and denote by $x(C)$ the intersection point of the hyperplanes $H_\rho^{\boldsymbol{a}}$ for ρ the extremal rays of a maximal cone C of $\mathcal{F}_c$.

Let w and w' be distinct c-sortable elements such that the associated maximal cones $C := C(w)$ and $C' := C(w')$ of the Cambrian fan $\mathcal{F}_c$ intersect in a cone of codimension 1. So either w is a cover of w' or w' is a cover of w in the lattice of c-sortable elements. Without loss of generality, we may assume that w is a cover of w'. To meet the requirement of Theorem 3.1, we have to prove that the vector $x(C) - x(C')$ points to C from C'.

We use the following notations. Set $\mathcal{R} := \{w(v_s) \mid w \in W, s \in S\}$ so that the set of rays of the Coxeter fan is $\mathbb{R}_{\geq 0}\mathcal{R} := \{\lambda v \mid \lambda \geq 0, v \in \mathcal{R}\}$. For $v \in \mathcal{R}$, we write $\nu_v := \langle \boldsymbol{a}, v_s \rangle = \langle M(w), v \rangle > 0$ which depends only on s.

The intersection $C \cap C'$ is contained in a hyperplane H_t for some reflection $t \in W$ since $\mathcal{F}_c$ is a coarsening of the Coxeter fan $\mathcal{F}$. We now show which of the two roots associated to H_t is an outer normal to C' relative to $C' \cap C$.

Lemma 3.7. *Let $w, w' \in W$ be c-sortable elements such that w is a cover of w' in the lattice of c-sortable elements. $C(w) \cap C(w')$ is an $n-1$-dimensional cone of $\mathcal{F}_c$, and suppose it lies in H_t for some reflection t. Let the roots associated to t be $\pm\beta$. If β is an outer normal to C' relative to $C' \cap C$, then β is a negative root.*

Proof. Let $\widetilde{w} \in (\pi_\downarrow^c)^{-1}(w')$ such that w is a cover of $\widetilde{w}$ in the right weak order. Then $w(D) \cap \widetilde{w}(D) \subset H_t$ is an $(n-1)$-dimensional cone of the Coxeter fan $\mathcal{F}$. Since β is an outer normal for $C(w')$ relative to $C(w) \cap C(w')$, it is an outer normal for $\widetilde{w}(D)$ with respect to $\widetilde{w}(D) \cap w(D)$.

Case 1: Suppose that $w' = \widetilde{w} = e$. Then $w = s$ for some $s \in S$. The hyperplane dividing $w(D)$ from D is H_s, perpendicular to α_s. D lies on the side of H_s having positive inner product with α_s. Thus the outer normal $\beta = -\alpha_s$ is a negative root.

Case 2: Suppose $\widetilde{w} \neq e$. Then $\widetilde{w}^{-1}w = s \in S$ since w covers $\widetilde{w}$ in right weak order. By the previous case, $\widetilde{w}(-\alpha_s)$ is an outer normal. Since $\ell(w) = \ell(\widetilde{w}s) > \ell(\widetilde{w})$, we have that $\beta = \widetilde{w}(-\alpha_s)$ is a negative root, as desired. □

We have that $C \cap \mathbb{R}_{\geq 0}\mathcal{R} = \{\rho_{u_1}, \ldots, \rho_{u_p}\}$ and since $\mathcal{F}_c$ is simplicial, we may assume that the extremal rays of C are the first $n := |S|$ rays. Similarly, we may assume $\{\rho_{u_1'}, \rho_{u_2}, \ldots, \rho_{u_n}\}$ are the extremal rays of C'. Hence, H_t is spanned by $\{u_2, \ldots, u_n\}$. As we have $x(C) := \bigcap_{i=1}^n H^{\boldsymbol{a}}_{\rho_{u_i}}$ and $x(C) - x(C') \in \bigcap_{i=2}^n H^{\boldsymbol{a}}_{\rho_{u_i}}$, we conclude $x(C) - x(C') = \mu\beta$ for some $\mu \in \mathbb{R}$ and β a negative root. Thus, $x(C) - x(C')$ is pointing to C from C' if and only if $x(C) - x(C') = \mu\beta$ with $\mu > 0$.

Lemma 3.8. *Let $w, w' \in W$ be c-sortable elements such that w covers w' in the lattice of c-sortable elements and $C(w) \cap C(w') \subset H_t$ is an $(n-1)$-dimensional cone of $\mathcal{F}_c$ for a reflection t. Let the extremal rays of $C := C(w)$ and $C' := C(w')$ be generated by $\{u_1, \ldots, u_n\}$ and $\{u_1', u_2, \ldots, u_n\}$ and suppose $u_1 + u_1' = \sum_{i=2}^n b_i u_i \in H_t$ with $b_i \geq 0$. Then the following statements are equivalent:*

(i) $x(C) - x(C')$ is pointing to C from C';
(ii) $x(C) - x(C') = \mu\beta$ with $\beta \in \Phi^-$ and $\mu > 0$;
(iii) $\langle x(C) - x(C'), u_1 \rangle > 0$;
(iv) $\nu_{u_1} + \nu_{u_1'} > \sum_{i=2}^n b_i \nu_i$.

Proof. The first equivalence follows from Lemma 3.7 and the preceding discussion.

As $u_1 \in C$ and C is spanned by the vectors in $C \cap C'$ together with u_1, we have $\langle \beta, u_1 \rangle > 0$ if and only if β is an outer normal of C' relative to $C \cap C'$. This shows the second equivalence.

The last equivalence follows from

$$\langle x(C) - x(C'), u_1 \rangle = \nu_{u_1} - \langle x(C'), -u_1' + \sum_{i=2}^n b_i u_i \rangle = \nu_{u_1} + \nu_{u_1'} - \sum_{i=2}^n b_i \nu_{u_i}.$$

□

We apply Theorem 3.1 to conclude:

> If one of the equivalences in Lemma 3.8 is achieved for all pairs of adjacent cones in $\mathcal{F}_c$, then $\mathcal{F}_c$ is the outer normal fan of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ which proves Theorem 3.4.

This will be done in Lemma 3.13 below.

We first consider the special case that e is covered by $s \in S$ and there is a reduced expression for c that starts with s, that is s is *initial* in c.

Lemma 3.9. *Let $s \in S$ be initial in c. Then $C(e) \cap C(s) \subseteq H_s$, $u_1' = v_s$, $u_1 = s(v_s)$, and*

$$u_1' + u_1 = \sum_{r \neq s} b_r u_r \in H_s$$

with $b_r = -2\frac{\langle \alpha_s, \alpha_r \rangle}{\langle \alpha_s, \alpha_s \rangle} \geq 0$. Moreover, $\nu_{u_1'} + \nu_{u_1} > \sum_{r \neq s} b_r \nu_{u_r}$.

Proof. Without loss of generality, assume $S = \{s_1, \dots, s_n\}$ and $s = s_1$. Since $s \in S$ is initial, s is a c-singleton. The maximal cones $C(e)$ and $C(s)$ of $\mathcal{F}_c$ are therefore maximal cones of the Coxeter fan with extremal rays generated by $\{v_{s_1}, \dots, v_{s_n}\}$ and $\{s(v_{s_1}), v_{s_2}, \dots, v_{s_n}\}$. Since $\alpha_r = \sum_{i=1}^n \langle \alpha_r, \alpha_i \rangle v_i$, we have

$$s_1(v_{s_1}) = v_{s_1} - 2\frac{\langle \alpha_{s_1}, v_{s_1} \rangle}{\langle \alpha_{s_1}, \alpha_{s_1} \rangle} \alpha_{s_1} = -v_{s_1} + \sum_{i=2}^n \left(-2\frac{\langle \alpha_{s_1}, \alpha_{s_i} \rangle}{\langle \alpha_{s_1}, \alpha_{s_1} \rangle} \right) v_{s_i}.$$

In particular, $s(v_{s_1}) + v_{s_1} \in H_{s_1}$ and $\langle \alpha_{s_1}, \alpha_{s_i} \rangle \leq 0$ for $s_1 \neq s_i$. As $\boldsymbol{a}$ is a vertex of $Perm^{\boldsymbol{a}}(W)$, we conclude

$$\nu_{s_1(u_1)} = \langle \boldsymbol{a}, v_{s_1} \rangle > \langle \boldsymbol{a}, s_1(v_{s_1}) \rangle = -\nu_{s_1(v_{s_1})} + \sum_{i=2}^n b_i \nu_{s_i}$$

□

Some terminology and results due to N. Reading and D. Speyer are needed to prove Lemma 3.13 (and therefore to finish the proof of Theorem 3.4). To distinguish objects related to a Cambrian fan with respect to different Coxeter elements, we use the Coxeter element as index. For example, if we use the Coxeter element scs instead of c, then $C_{scs}(w)$ denotes the maximal cone that corresponds to the scs-sortable element w. If $s \in S$ is initial in c then $\mathcal{F}_{sc}$ is the sc-Cambrian fan for the Coxeter element sc of $W_{\langle s \rangle}$.

Lemma 3.10 ([23, Lemmas 4.1, 4.2])**.** *Let c be a Coxeter element and s initial in c.*

(i) Let $w \in W$ such that $\ell(sw) < \ell(w)$. Then w is c-sortable if and only if sw is scs-sortable.

(ii) Let $w \in W$ such that $\ell(sw) > \ell(w)$. Then w is c-sortable if and only if $w \in W_{\langle s \rangle}$ and w is sc-sortable.

Note that $\ell(sw) < \ell(w)$ if and only if the chamber $w(D)$ of the Coxeter arrangement corresponding to w lies above H_s. In this case, the maximal cone $C(w)$ of $\mathcal{F}_c$ is above H_s because w is minimal in its fibre $(\pi_\downarrow^c)^{-1}\pi_\downarrow^c(w) = [\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$ for the c-Cambrian congruence. On the other hand, if $\ell(sw) > \ell(w)$, then $w(D)$ is below H_s in the Coxeter arrangement. In this case, we know that the maximum element of the fibre $[\pi_\downarrow^c(w), \pi_c^\uparrow(w)]$ for w, and thus all of $C(w)$, is below H_s, by [23, Lemma 4.11]. It follows that the hyperplane H_s separates the cones of $\mathcal{F}$ into two families and it never intersects a maximal cone of $\mathcal{F}_c$ in its interior. For $\rho \in \mathcal{F}_c^{(1)}$ we define

$$\zeta_s(\rho) := \begin{cases} s(\rho) & \text{if } \rho \neq \rho_s, \\ -\rho_s & \text{otherwise.} \end{cases}$$

We abuse notation and consider ζ_s also as a map on the set of vectors generating the rays $\mathcal{F}_c^{(1)}$. The following lemma is a consequence of [23, Lemma 6.5] and [23, Theorem 1.1]. Compare also the comments after [23, Corollary 7.3], from which the last statement is taken.

Lemma 3.11 ([23])**.** *Let $s \in S$ be initial in the Coxeter element c. If $\rho_1, \dots, \rho_n$ are the extremal rays of the maximal cone $C(w) \in \mathcal{F}_c$ then $\zeta_s(\rho_1), \dots, \zeta_s(\rho_n)$ are the extremal rays of a maximal cone of $\mathcal{F}_{scs}$. If $\ell(sw) < \ell(w)$, then these extremal rays are the extremal rays of the maximal cone $C(sw)$ that corresponds to the scs-sortable element sw.*

Before we finish the proof of Theorem 3.4 with Lemma 3.13 we make an observation that will be useful also in Section 3.4.

Lemma 3.12. *Let c be a Coxeter element, $s \in S$ initial in c, and $w \in W_{\langle s \rangle}$ sc-sortable. Then the maximal cone $C(w) \in \mathcal{F}_c$ is spanned by $C_{sc}(w) \in \mathcal{F}_{sc}$ and the ray $\rho_s \in \mathcal{F}_c$.*

Proof. The ray ρ_s is the unique ray of $\mathcal{F}_c$ that is strictly below H_s by [23, Lemma 6.3]. From Lemma 3.10 it follows that $C(w)$ is below H_s and has ρ_s as an extremal ray. Hence $C(w)$ is spanned by ρ_s and a maximal cone $E(w) := C(w) \cap H_s \in \mathcal{A}_{\langle s \rangle}$.

Now consider all inversions t of w, that is, all reflections $H_t \in \mathcal{A}$ such that $C(w)$ is above H_t. Since $w \in W_{\langle s \rangle}$ we conclude $t \in W_{\langle s \rangle}$. Hence, the inversions of $E(w)$ and $C_{sc}(w)$ coincide and $E(w) = C_{sc}(w)$. $\square$

Lemma 3.13. *Let $w, w' \in W$ be c-sortable elements such that w covers w' in the lattice of c-sortable elements and $C(w) \cap C(w') \subset H_t$ is an $(n-1)$-dimensional cone of $\mathcal{F}_c$ for a reflection t. Let the extremal rays of $C := C(w)$ and $C' := C(w')$ be generated by $\{u_1, \dots, u_n\}$ and $\{u_1', u_2, \dots, u_n\}$.*

Then $u_1' + u_1 = \sum_{i=2}^n b_i u_i$ with $b_i \geq 0$ and $\nu_{u_1'} + \nu_{u_1} > \sum_{i=2}^n b_i \nu_{u_i}$.

Proof. The proof is an induction on the rank $n = |S|$ and the length $\ell(w)$.

If $|S| = 1$ then the result is clear, so assume that $S = \{s_1, \dots, s_n\}$ with $n > 1$ and $\ell(w) = 1$. Assume without loss of generality that $w = s_1$, and since w covers w', $w' = e$. If w is initial for c then we are done by Lemma 3.9. So assume that w is not initial for c. Then we have $u_i = v_{s_i}$, for $2 \leq i \leq n$, $u_1' = v_{s_1}$, and $u_1 = u$ for some $u \in \mathcal{R}$. Thus the maximal cones $C(e)$ and $C(w)$ are generated by $\{v_{s_1}, v_{s_2}, \dots, v_{s_n}\}$ and $\{u, v_{s_2}, \dots, v_{s_n}\}$. For sake of definiteness, suppose $s = s_2$ is initial in c. Then $C(e)$ and $C(w)$ are both below H_s. By Lemma 3.12, we have maximal cones $C_{sc}(e) = C(e) \cap H_s$ and $C_{sc}(w) = C(w) \cap H_s$ in the sc-Cambrian fan $\mathcal{F}_{sc}$ of $W_{\langle s \rangle}$ and these cones are generated by $\{v_{s_1}, v_{s_3}, \dots, v_{s_n}\}$ and $\{u, v_{s_3}, \dots, v_{s_n}\}$. So by induction on the rank of $|S|$, we obtain the claim with $b_2 = 0$.

For the induction, we assume that the claim is true whenever $\widetilde{w}$ is $\widetilde{c}$-sortable for a Coxeter group generated by $\widetilde{S}$ with $|\widetilde{S}| < |S|$ or $\widetilde{w}$ is a c-sortable element with $\ell(\widetilde{w}) < \ell(w)$.

Assume $w, w' \in W$ are c-sortable with $\ell(w) > 1$ and w covers w' in the lattice of c-sortable elements. Let $s \in S$ be initial in c. We split into cases based on the position of $C(w)$ and $C(w')$ relative to H_s. Note that $C(w)$ below H_s and $C(w')$ above H_s is not possible since w covers w' in the c-Cambrian lattice.

Case 1: Suppose $C(w)$ and $C(w')$ are above H_s. The ray ρ_s is strictly below H_s by [23, Lemma 6.3], so $v_s \notin \{u_1', u_1, \dots, u_n\}$. Moreover, we conclude from Lemma 3.11 that the maximal cones $C_{scs}(sw)$ and $C_{scs}(sw')$ in $\mathcal{F}_{scs}$ are generated by $\{s(u_1), \dots, s(u_n)\}$ and $\{s(u_1'), s(u_2), \dots, s(u_n)\}$ since $\ell(sw) < \ell(w)$, $\ell(sw') < \ell(w')$ and $w, w' > s$ in the right weak order. We have $C_{scs}(sw) \cap C_{scs}(sw') \subset H_{sts}$ because $C(w) \cap C(w') \subset H_t$. By induction on the length, we have

$$s(u_1) + s(u_1') = \sum_{i=2}^{n} b_i s(u_i) \quad \text{and} \quad \nu_{s(u_1)} + \nu_{s(u_1')} > \sum_{i=2}^{n} b_i \nu_{s(u_i)} \text{ with } b_i \geq 0.$$

Applying s to these (in)equalities yields

$$u_1 + u_1' = \sum_{i=2}^{n} b_i u_i \in H_s \quad \text{and} \quad \nu_{u_1} + \nu_{u_1'} > \sum_{i=2}^{n} b_i \nu_{u_i} \text{ with } b_i \geq 0,$$

since ν_u depends only on the orbit of u under the action of W.

Case 2: $C(w)$ and $C(w')$ are below H_s. Since w is c-sortable and $\ell(sw) > \ell(w)$, we have that $w \in W_{\langle s \rangle}$, and similarly for w'. The ray ρ_s is the only ray of $\mathcal{F}_c$ strictly below H_s by [23, Lemma 6.3], hence we may assume that $u_2 = v_s$. Now $\{u_1, u_3, \dots, u_n\}$ and $\{u_1', u_3, \dots, u_n\}$ generate the extremal rays of maximal cones $C_{sc}(\widetilde{w}), C_{sc}(\widetilde{w}') \subset H_s$ of the sc-Cambrian fan $\mathcal{F}_{sc}$ with $\widetilde{w}, \widetilde{w}' \in W_{\langle s \rangle}$. The claim follows by induction on the rank $|S|$.

Case 3: $C(w)$ is above H_s and $C(w')$ is below H_s. Hence $C(w)$ and $C(w')$ are separated by H_s, so we have $s = t$. Hence $u_1' = v_s$ (ρ_s is the only ray of $\mathcal{F}_c$ below H_s) and there is a maximal cone $C_{scs}(g)$ for some scs-sortable element $g \in W$ which is generated by the extremal rays $\zeta_s(u_1'), \zeta_s(u_2), \dots, \zeta_s(u_n)$. Now, observe that

$$\begin{aligned} \zeta_s(u_1) &= s(u_1), \\ \zeta_s(u_1') &= -u_1' = -v_s, \\ \text{and } \zeta_s(u_i) &= u_i \text{ for } 2 \leq i \leq n \end{aligned}$$

Thus the maximal cones $C_{scs}(g)$ and $C_{scs}(sw)$ have extremal rays generated by $-v_s, u_2, \dots, u_n$ and $s(u_1), u_2, \dots, u_n$. Moreover, $C_{scs}(g) \cap C_{scs}(sw) \subseteq H_s$.

We first show that $g = w$. By definition of $\mathcal{F}_c$, we have $sw(D) \subset C_{scs}(sw)$. From $C_{scs}(sw) \cap C_{scs}(g) \subseteq H_s$ we deduce that $w(D) \subset C_{scs}(g)$ or equivalently $w \in (\pi_{\downarrow}^{scs})^{-1}(g)$. Now $g > sw$ implies $h > sw$ for all $h \in (\pi_{\downarrow}^{scs})^{-1}(g)$. But $C(w)$ is above H_s, so $h < w$ implies $h \notin (\pi_{\downarrow}^{scs})^{-1}(g)$. Hence w is the minimal element of $[\pi_{\downarrow}^{scs}(g), \pi^{\uparrow}_{scs}(g)]$ and we have $w = g$.

Though $C_{scs}(w) \cap C_{scs}(sw) \subseteq H_s$, it is not possible to apply the induction hypothesis immediately, since the length $\ell(w)$ has not been reduced, but we claim that for any $z \in S$ initial in scs either $C_{scs}(w)$ and $C_{scs}(sw)$ are both above H_z or both below H_z. From $z \in S \setminus \{s\}$ we conclude that $v_z \in H_s$. We know that $u_2, \dots, u_n \in H_s$ and $u_1', u_1, s(u_1) \notin H_s$. So $v_z \in C_{scs}(w)$ if and only if $v_z \in$

$C_{scs}(sw)$. Since v_z is the only ray of $\mathcal{F}_{scs}$ below the hyperplane H_z, we have shown that $C_{scs}(w)$ and $C_{scs}(sw)$ are on the same side of H_z.

This implies that we are now in Case 1 or Case 2, so we first apply the argument of the relevant case to apply the induction hypothesis and conclude

$$\zeta_s(u_1) + \zeta_s(u_1') = \sum_{i=2}^{n} b_i u_i \in H_s \quad \text{with } b_i \geq 0.$$

Therefore

$$s(u_1 + u_1') = (\zeta_s(u_1) + \zeta_s(u_1')) + (v_s + s(v_s)) \in H_s$$

and $u_1 + u_1' = \sum_{i=2}^{n} b_i u_i \in H_t$ with $b_i \geq 0$, since $s = t$.

It remains to prove $\nu_{u_1} + \nu_{u_1'} > \sum_{i=2}^{n} b_i \nu_{u_i}$. By Lemma 3.8 it is sufficient to show that $\langle x(C(w')) - x(C(w)), u_1' \rangle > 0$.

Recall that $t = s \in S$. Pick a maximal chain in the c-Cambrian lattice

$$y_0 \lessdot y_1 \lessdot \ldots \lessdot y_p$$

with $y_0 = s$ and $y_p = w$. Then $s \leq y_i$ for $0 \leq i \leq p$, so $C(y_i)$ is above H_s for $0 \leq i \leq p$. So for the pair $\widetilde{w}' = y_{i-1}$ and $\widetilde{w} = y_i$ we have $z_i := x(C(y_i)) - x(C(y_{i-1})) = \mu_i \beta_i$ with $\mu_i > 0$ and $\beta_i \in \Phi^-$ by Lemma 3.8 and Case 1 above. Now $\langle \beta_i, v_s \rangle$ is the coefficient of the simple root α_s in the simple root expansion of β_i. Since β_i is a negative root, $\langle \beta_i, v_s \rangle \leq 0$. In particular we have

$$\langle x(C(y_{i-1})), v_s \rangle \geq \langle x(C(y_{i-1})), v_s \rangle + \langle z_i, v_s \rangle = \langle x(C(y_i)), v_s \rangle$$

for $1 \leq i \leq p$. Hence

$$\langle x(C(e)), v_s \rangle > \langle x(C(s)), v_s \rangle \geq \langle x(C(y_2)), v_s \rangle \geq \cdots \geq \langle x(C(w)), v_s \rangle,$$

where the first inequality is Lemma 3.9. As $u_1' = v_s$ we have

$$\langle x(C(e)), v_s \rangle = \nu_{v_s} = \nu_{u_1'} = \langle x(C(w')), v_s \rangle$$

Thus $\langle x(C(w')) - x(C(w)), u_1' \rangle > 0$. $\square$

3.4. **On integer coordinates.** Suppose that W is a Weyl group and that the root system Φ for W is crystallographic, that is, for any two roots $\alpha, \beta \in \Phi$ we have $s_\alpha(\beta) = \beta + \lambda\alpha$ for some $\lambda \in \mathbb{Z}$. The simple roots Δ span the lattice L and the fundamental weights v_s, $s \in S$, span a lattice L^* which is dual to L. For $\beta \in L$ and $v \in L^*$ we have $\langle \beta, v \rangle \in \mathbb{Z}$. In fact, $\beta \in L$ if and only if $\langle \beta, v \rangle \in \mathbb{Z}$ for all $v \in L^*$. For each ray $\rho \in \mathcal{F}_c$, we denote by $v_\rho \in L^*$ the lattice point closest to the origin.

Lemma 3.14. *Let Φ be a crystallographic root system for the Weyl group W and c a Coxeter element of W. The set $\{v_\rho \mid \rho$ an extremal ray of $C\}$ forms a basis of L^* for each maximal cone $C \in \mathcal{F}_c$.*

Proof. Let $C = C(w)$ denote the maximal cone of $\mathcal{F}_c$ for some c-sortable $w \in W$. The proof is by induction on $\ell(w)$ and the rank of W. Let s be initial in c and by Lemma 3.10 we have to distinguish two cases,

Suppose that $\ell(sw) < \ell(w)$. Then sw is scs-sortable and $C(w) = s\,(C_{scs}(sw))$. Since the simple reflection s preserves the lattice, the result follows by induction.

Suppose on the other hand that $\ell(sw) > \ell(w)$. Then the cone $C(w)$ lies below the hyperplane H_s and $w \in W_{\langle s \rangle}$ is sc-sortable. Let $C_{\langle s \rangle}(w)$ denote the maximal cone that corresponds to w in the Cambrian fan $\mathcal{F}_{\langle s \rangle} \subset H_s$ for $W_{\langle s \rangle}$. Then $C_{\langle s \rangle}(w) = C(w) \cap H_s$ by Lemma 3.12. The induction hypothesis implies that the extremal rays of $C_{\langle s\rangle}(w)$ form a basis for the lattice $L^*_{\langle s\rangle} \subset H_s$ and ρ_s is the unique extremal ray of $C(w)$ not contained in H_s by Lemma 3.12. Since the fundamental weights v_t, $t \in S$, span L^* it follows that L^* is spanned by v_s and $L^*_{\langle s\rangle} = L^* \cap H_s$. Hence, the extremal rays of $C(w)$ span L^*. □

Theorem 3.15. *Let Φ be a crystallographic root system for the Weyl group W and c a Coxeter element of W. Suppose that $\boldsymbol{a} \in L$. Then the vertex sets $V(Perm^{\boldsymbol{a}}(W))$ and $V(\mathsf{Asso}_c^{\boldsymbol{a}}(W))$ are contained in L.*

Proof. The result for the permutahedron is classic. Let $w \in W$ be c-sortable, $x(w)$ be the vertex of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ contained in the maximal cone $C(w) \in \mathcal{F}_c$, and ρ_i, $1 \le i \le n$ be the extremal rays of $C(w)$. Denote the lattice point on ρ_i closest to the origin by y_i. The point $x(w)$ satisfies $\langle x(w), y_i\rangle = c_i$ for some integer c_i since $\boldsymbol{a} \in L$. Because $\{y_i\}$, $1 \le i \le n$, is a basis of L^*, this set of equations for $x(w)$ has an integral solution. In other words, $x(w) \in L$. □

4. Observations and remarks

4.1. **Recovering the c-cluster complex from the c-singletons.** It is possible to obtain polytopal realizations of the c-cluster complex from the construction of generalized associahedra presented as follows. Suppose that we are given a W-permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$, a Coxeter element c, and the c-sorting word $\mathbf{w_0}$ of w_0. Then we can easily compute all c-singletons using the characterization given in Theorem 1.2. The associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ is now obtained from $\mathsf{Perm}^{\boldsymbol{a}}(W)$ by keeping all admissible inequalities for the permutahedron, that is all inequalities $\langle v, w(v_s)\rangle \le \langle \boldsymbol{a}, v_s\rangle$ for c-singleton w. We label the facet $\langle v, w(v_s)\rangle \le \langle \boldsymbol{a}, v_s\rangle$

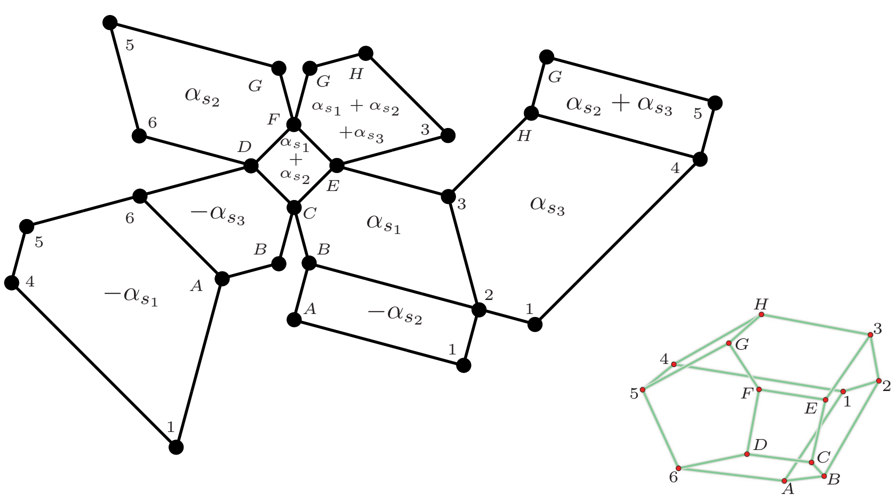

Figure 9. An unfolding of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_4)$ with $c = s_1 s_2 s_3$, the polar of the c-cluster complex. The 2-faces are labeled by replacing the labels $w(\rho_s)$ in Figure 7 by the almost positive root $\mathsf{Lr}_s(w)$.

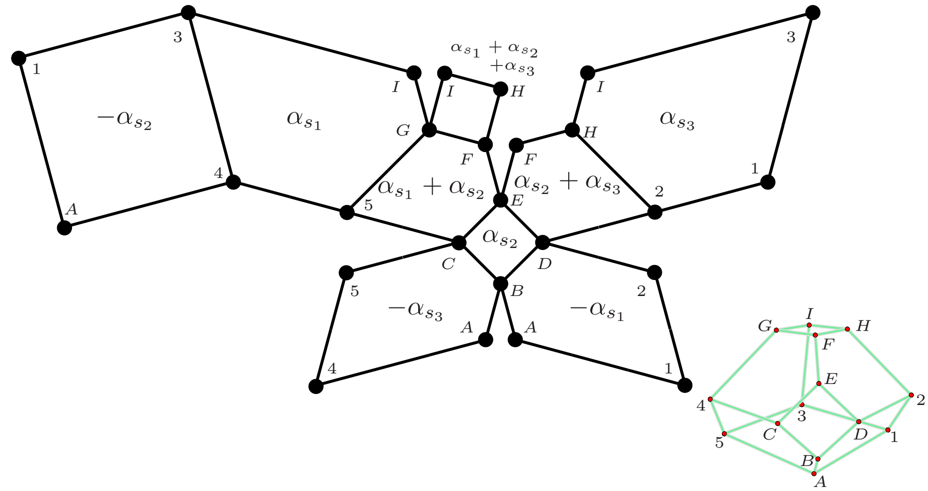

FIGURE 10. An unfolding of the associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_4)$ with $c = s_2s_1s_3$, the polar of the c-cluster complex. The 2-faces are labeled by replacing the labels $w(\rho_s)$ in Figure 8 by the almost positive root $\mathsf{Lr}_s(w)$.

of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ by the almost positive root $\mathsf{Lr}_s(w)$ and extend this labeling to the Hasse diagram of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ as follows: if a face f is the intersection of facets $F_1, \dots, F_k$ then assign f the union of the almost positive roots assigned to $F_1, \dots, F_k$. By Theorem 2.6, this labeling matches the labeling of the c-Cambrian fan by almost positive roots given by Reading and Speyer. Therefore, the opposite poset of this labeled Hasse diagram is the face poset of the c-cluster complex because it is the face poset of the c-Cambrian fan $\mathcal{F}_c$. The polar of $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ is therefore a polytopal realization of the c-cluster complex. In particular, a set of almost positive roots is c-compatible (see [21]) if and only if it can be obtained as the intersection of some facets of $\mathsf{Asso}_c^{\boldsymbol{a}}$ by the process described above.

We illustrate the recovery of the c-cluster complex for $W = S_4$. Figure 9 refers to the Coxeter element $c = s_1s_2s_3$ and Figure 10 refers to the Coxeter element $c = s_2s_1s_3$. We use the polar of the c-cluster complex for the illustration.

First consider the Coxeter element $c = s_1s_2s_3$. The facets are labeled by almost positive roots as indicated. The vertices correspond to clusters as follows:

$$
\begin{array}{ll}
A = \{-\alpha_{s_1}, -\alpha_{s_2}, -\alpha_{s_3}\}, & B = \{\alpha_{s_1}, -\alpha_{s_2}, -\alpha_{s_3}\} \\
C = \{\alpha_{s_1}, \alpha_{s_1}+\alpha_{s_2}, -\alpha_{s_3}\}, & D = \{\alpha_{s_2}, \alpha_{s_1}+\alpha_{s_2}, -\alpha_{s_3}\}, \\
E = \{\alpha_{s_1}, \alpha_{s_1}+\alpha_{s_2}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}\}, & F = \{\alpha_{s_2}, \alpha_{s_1}+\alpha_{s_2}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}\}, \\
G = \{\alpha_{s_2}, \alpha_{s_2}+\alpha_{s_3}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}\}, & H = \{\alpha_{s_3}, \alpha_{s_2}+\alpha_{s_3}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}\}, \\
1 = \{-\alpha_{s_1}, -\alpha_{s_2}, \alpha_{s_3}\}, & 2 = \{\alpha_{s_1}, -\alpha_{s_2}, \alpha_{s_3}\}, \\
3 = \{\alpha_{s_1}, \alpha_{s_1}+\alpha_{s_2}+\alpha_{s_3}, \alpha_{s_3}\}, & 4 = \{-\alpha_{s_1}, \alpha_{s_2}+\alpha_{s_3}, \alpha_{s_3}\}, \\
5 = \{-\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_2}+\alpha_{s_3}\}, & 6 = \{-\alpha_{s_1}, \alpha_{s_2}, -\alpha_{s_3}\}.
\end{array}
$$

Second consider the Coxeter element $c = s_2s_1s_3$. The facets are labeled by almost positive roots as indicated and the vertices correspond to clusters as follows:

$$
\begin{array}{ll}
A = \{-\alpha_{s_1}, -\alpha_{s_2}, -\alpha_{s_3}\}, & B = \{-\alpha_{s_1}, \alpha_{s_2}, -\alpha_{s_3}\} \\
C = \{\alpha_{s_1} + \alpha_{s_2}, \alpha_{s_2}, -\alpha_{s_3}\}, & D = \{-\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_2} + \alpha_{s_3}\}, \\
E = \{\alpha_{s_1} + \alpha_{s_2}, \alpha_{s_2}, \alpha_{s_2} + \alpha_{s_3}\}, & F = \{\alpha_{s_1} + \alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}, \alpha_{s_2} + \alpha_{s_3}\}, \\
G = \{\alpha_{s_1}, \alpha_{s_1} + \alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}\}, & H = \{\alpha_{s_3}, \alpha_{s_2} + \alpha_{s_3}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}\}, \\
I = \{\alpha_{s_1}, \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3}, \alpha_{s_3}\}, & 1 = \{-\alpha_{s_1}, -\alpha_{s_2}, \alpha_{s_3}\}, \\
2 = \{-\alpha_{s_1}, \alpha_{s_2} + \alpha_{s_3}, \alpha_{s_3}\}, & 3 = \{\alpha_{s_1}, -\alpha_{s_2}, \alpha_{s_3}\}, \\
4 = \{\alpha_{s_1}, -\alpha_{s_2}, -\alpha_{s_3}\}, & 5 = \{\alpha_{s_1}, \alpha_{s_1} + \alpha_{s_2}, -\alpha_{s_3}\}.
\end{array}
$$

4.2. **A conjecture about vertex barycentres.** J.-L. Loday mentions in [15] that F. Chapoton observed the following: the vertex barycentres of the permutahedron and associahedron coincide in the case of Loday's original realization of the (classical) type A associahedron. The first two authors observed the same phenomenon for the realizations of type A and B associahedra described in [9]. None of these observations have been proven so far. Checking numerous examples in GAP [25], we observed that the vertex barycentre of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ and $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ coincide for $\boldsymbol{a} = \sum_{s\in S} av_s$, $a > 0$. The cases include types A_n ($n \le 7$), B_n and D_n ($n \le 5$), F_4, H_3, H_4, and dihedral groups $I_2(m)$. The experiments can be summarized in the following conjecture.

Conjecture 4.1. *Let W be a Coxeter group and $c \in W$ a Coxeter element. Choose a real number $a > 0$ and set $\boldsymbol{a} = \sum_{s\in S} av_s$ to fix a realization of the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W)$. Then the vertex barycentres of $\mathsf{Perm}^{\boldsymbol{a}}(W)$ and $\mathsf{Asso}_c^{\boldsymbol{a}}(W)$ coincide.*

It is straightforward to prove this conjecture for a special family, the dihedral groups G_m of order $2m$. We outline the proof.

Let $m \ge 2$ be an integer. The dihedral group G_m of order $2m$ is the finite Coxeter group of type $I_2(m)$ generated by the two reflections s and t with st having order m. For any m, the action of G_m on $V = \mathbb{R}^2$ is essential and we identify $\mathbb{R}^2$ with the the complex numbers. If we define

$$v_s := \frac{1 + e^{i\frac{\pi}{m}}}{2} \quad \text{and} \quad v_t := \frac{1 + e^{-i\frac{\pi}{m}}}{2}$$

then G_m is generated by the reflections with respect to the hyperplanes spanned by v_s and v_t. We choose $\boldsymbol{a} = v_s + v_t = 1$ and follow our earlier notation where $M(w)$ denotes the point obtained by the action of $w \in G_m$ on $\boldsymbol{a}$. Then

$$M(w) = \begin{cases} e^{i\ell(w)\frac{\pi}{m}} & \text{if } \ell(sw) < \ell(w) \\ e^{-i\ell(w)\frac{\pi}{m}} & \text{if } \ell(sw) > \ell(w). \end{cases}$$

The convex hull of the points $M(w)$, $w \in G_m$, is the permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(G_m)$ which is a regular $2m$-gon. It is easy to verify that the origin is the vertex barycentre of $\mathsf{Perm}^{\boldsymbol{a}}(G_m)$.

We argue now for the the Coxeter element st; if $c = ts$, the reasoning is similar. The c-singletons are e and all $w \in G_m$ with $\ell(sw) < \ell(w)$. The generator t is the only c-sortable element which is not a c-singleton. Denote the intersection of the line through $M(e)$ and $M(t)$ and the line through $M(w_0)$ and $M(sw_0)$ by P. The associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(G_m)$ is the convex hull of the points $M(w)$, such that $w \in G_m$ is a c-singleton, and P. A straightforward computation yields

$$P = \frac{i \sin\left(\frac{\pi}{m}\right)}{\cos\left(\frac{\pi}{m}\right) - 1}.$$

and it is not hard to verify that

$$\sum_{\substack{w \in G_m \\ \text{not } c\text{-singleton}}} M(w) = \sum_{k=1}^{m-1} \left(e^{-i\frac{\pi}{m}}\right)^k = P,$$

so the vertex barycentres of $\mathsf{Perm}^{\boldsymbol{a}}(G_m)$ and $\mathsf{Asso}_c^{\boldsymbol{a}}(G_m)$ coincide.

4.3. **Recovering the realizations of** [9] **for types** A **and** B.

4.3.1. *Type A.* Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. The symmetric group S_n acts naturally on $\mathbb{R}^n$ by permutation of the coordinates. We set

$$\Delta := \{e_{i+1} - e_i \,|\, 1 \le i \le n-1\} \qquad \text{and} \qquad \Phi^+ := \{e_j - e_i \,|\, 1 \le i < j \le n\}.$$

Then $\Phi = \Phi^+ \cup (-\Phi^+)$ is a root system of type A_{n-1} with simple root system Δ. Moreover, we recall that the reflection group S_n acts essentially on

$$V := \mathbb{R}[\Delta] = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i = 0 \right\} \subset \mathbb{R}^n.$$

Let s_i be the simple reflection that maps the simple root $e_{i+1} - e_i$ to $e_i - e_{i+1}$. The dual basis Δ^* of Δ is described by

$$v_{s_i} := \frac{i-n}{n} \sum_{k=1}^{i} e_k + \frac{i}{n} \sum_{k=i+1}^{n} e_k \in V.$$

We choose $\boldsymbol{a} := \sum_{i=1}^{n-1} v_{\tau_i}$ and have $\boldsymbol{a} = \sum_{k=1}^{n} \left(k - \frac{n+1}{2}\right) e_k$.

There is a bijection between Coxeter elements $c \in S_n$ and orientations of the Coxeter graph of S_n: if s_i appears before s_{i+1} in a reduced expression of c then the edge between s_i and s_{i+1} is oriented from s_i to s_{i+1}. The orientation is from s_{i+1} to s_i if s_i appears after s_{i+1} in a reduced expression of c. Given an oriented Coxeter graph, we can apply the construction described earlier and obtain a permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(S_n)$ and an associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_n)$.

Consider the affine subspace $\mathcal{V} \subset \mathbb{R}^n$ that is a translate of V by $v_G = \frac{n+1}{2} \sum_{i=1}^n e_i$:

$$\mathcal{V} = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i = \frac{n(n+1)}{2} \right\}.$$

Translate $\mathsf{Perm}^{\boldsymbol{a}}(S_n) \subset V$ by v_G to obtain $\mathsf{Perm}^{\boldsymbol{a}}(S_n) + v_G \subset \mathcal{V}$. The vertices of $\mathsf{Perm}^{\boldsymbol{a}}(S_n) + v_G$ are the orbit of $\boldsymbol{a} + v_G = \sum_{i=1}^n i e_i$ under the action of S_n, in other words we have

$$M(w) = \sum_{i=1}^n w^{-1}(i) e_i$$

for $w \in S_n$. The permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(S_n) + v_G$ was used in [9] but the vertices were labeled according to $w \mapsto w^{-1}$.

Proposition 4.2. *Consider a Coxeter element $c \in S_n$ or equivalently an orientation of the Coxeter graph and let v_G and $\boldsymbol{a}$ be as above. The translated associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(S_n) + v_G$ is the associahedron Asso_c constructed in* [9].

Proof. In [9, Proposition 1.3], it was proved that the c-singletons are the common vertices of the permutahedron and the associahedron and that the normal fan of the latter is $\mathcal{F}_c$. In other words, the realization of the associahedron in [9] matches precisely the description of $\mathsf{Asso}_c^{\boldsymbol{a}}(S_n)$ given in Corollary 3.5. □

4.3.2. *Type B.* Consider the simple root system of type B given by

$$\Delta' := \{e_{n+1} - e_n\} \cup \{e_{i+1} - e_i + e_{2n+1-i} - e_{2n-i} \,|\, 1 \le i \le n-1\} \subset \mathbb{R}^{2n}.$$

If we set $V' := \mathbb{R}[\Delta']$ then V' is a n-dimensional subspace of $\mathbb{R}^{2n}$ which is contained in V, the span of the type A_{2n-1} root system as in 4.3.1. Denote the simple reflection that corresponds to $e_{n+1} - e_n$ by s_0 and the simple reflection that corresponds to $(e_{i+1} - e_i) + (e_{2n+1-i} - e_{2n-i})$ by s_{n-i}. The *hyperoctahedral group* W_n (or Coxeter group of type B_n) is generated by these reflections. It is easy to see that $V' = V \cap \bigcap_{i=1}^{n-1} V_i^B$ where $V_i^B := \{x \in \mathbb{R}^{2n} \,|\, x_i + x_{2n+1-i} = 0\}$. In particular we have $\boldsymbol{a} \in V'$.

The claim that $\boldsymbol{a}$ is in the open cone spanned by the fundamental weights of Δ' follows from the fact that the scalar product of $\boldsymbol{a}$ with any element of Δ' is strictly positive.

A Coxeter element $c \in W_n$ is related to an orientation of the Coxeter graph of W_n as in type A: If s_i appears before (resp. after) s_{i+1} in a reduced expression of c then the edge between s_i and s_{i+1} is oriented from s_i to s_{i+1} (resp. form s_{i+1} to s_i). A Coxeter element or an orientation of the Coxeter graph yields therefore a permutahedron $\mathsf{Perm}^{\boldsymbol{a}}(W_n)$ as described in Section 2.1 and an associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W_n)$. The orientation of the Coxeter graph of W_n determines a symmetric orientation of the Coxeter graph of S_{2n} (that is, of type A_{2n-1}), and thus a Coxeter element $\tilde{c}$ of S_{2n} and we have

$$\mathsf{Perm}^{\boldsymbol{a}}(W_n) = \mathsf{Perm}^{\boldsymbol{a}}(S_{2n}) \cap V' \quad \text{and} \quad \mathsf{Asso}_c^{\boldsymbol{a}}(W_n) = \mathsf{Asso}_{\tilde{c}}^{\boldsymbol{a}}(S_{2n}) \cap V'.$$

The following proposition is a direct consequence from the construction in [9].

Proposition 4.3. *Consider a Coxeter element $c \in W_n$ or equivalently an orientation of the Coxeter graph. Let v_G and $\boldsymbol{a}$ as above for type A. The translated associahedron $\mathsf{Asso}_c^{\boldsymbol{a}}(W_n) + v_G$ is the cyclohedron constructed in* [9] *that corresponds to the orientation of the Coxeter graph determined by c.*

Acknowledgements. The authors thank Nathan Reading and David Speyer for having made their results in [23] available to us during their work and for their helpful remarks. In particular, we thank David Speyer for having conjectured Theorem 3.1 while reading an earlier version of this article.

The authors wish to thank the Fields Institute, where much of this work was done, for its hospitality.

References

[1] N. Bergeron, C. Hohlweg, C. Lange and H. Thomas, *Isometry classes of generalized associahedra* (2007), 12 pages, arXiv:math.CO/0709.4421.

[2] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Springer, New York, 2005.

[3] A. Björner and M. Wachs, *Shellable nonpure complexes and posets II*, Trans. Amer. Math. Soc. **349** (1997), 3945–3975.

[4] R. Bott and C. Taubes, *On the self-linking of knots*, J. Math. Phys. **35** (1994), 5247–5287.

[5] F. Chapoton, S. Fomin and A. Zelevinsky, *Polytopal realizations of generalized associahedra*, Canad. Math. Bull. **45** (2003), 537–566.

[6] M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer, `CHEVIE` – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. Appl. Algebra Engrg. Comm. Comput. **7** (1996), 175–210.

[7] S. Fomin and A. Zelevinsky, *Y-systems and generalized associahedra*, Annals of Mathematics **158** (2003), 977–1018

[8] I. Gel'fand, M. Kapranov, and A. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[9] C. Hohlweg and C. Lange, *Realizations of the associahedron and the cyclohedron*, Discrete and Computational Geometry **37(4)** (2007), 517–543.

[10] C. Hohlweg, `http:www.lacim.uquam.ca/~hohlweg/cambrian.html`.

[11] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1990.

[12] J. Kamnitzer, *Mirković-Vilonen cycles and polytopes* (2005), 42 pages, `arXiv:math.AG/0501365`.

[13] C. Le Conte de Poly-Barbut, *Sur les treillis de Coxeter finis*, Math. Inf. Sci. Hum. **125** (1994), 41–57.

[14] C. Lee, *The associahedron and triangulations of the n-gon*, Europ. J. Combinatorics **10** (1989), 551–560.

[15] J.-L. Loday, *Realization of the Stasheff polytope*, Arch. Math. **83** (2004), 267–278.

[16] M. Markl, *Simplex, associahedron, and cyclohedron*, Contemp. Math. **227** (1999), 235–265.

[17] R. Marsh, M. Reineke and A. Zelevinsky, *Generalized associahedra via quiver representations*. Trans. Amer. Math. Soc. **355** (2003) no. 10, 4171–4186.

[18] A. Postnikov, *Permutohedra, associahedra, and beyond* (2004), 59 pages, `arXiv:math.CO/0507163`.

[19] N. Reading, *Lattice congruences, fans and Hopf algebras*, J. Combin. Theory Ser. A **110(2)** (2005), 237–273.

[20] N. Reading, *Cambrian lattices*, Adv. in Mathematics **205(2)** (2006), 313–353.

[21] N. Reading, *Clusters, Coxeter-sortable elements and noncrossing partitions*, Trans. Am. Math. Soc. **359(12)** (2007), 5931–5958.

[22] N. Reading, *Sortable elements and Cambrian lattices*, Algebra Universalis **56(3-4)** (2007), 411–437.

[23] N. Reading and D. Speyer, *Cambrian fans* (2006), to appear in J. Eur. Math. Soc., 40 pages, `arXiv:math.CO/0606201`.

[24] V. Reiner, *Equivariant fiber polytopes*, Documenta Math. **7** (2002), 113–132.

[25] M. Schönert et al., *GAP – Groups, Algorithms, and Programming – version 3 release 4 patchlevel 4"*. Lehrstuhl D für Mathematik, Rheinisch-Westfälische Technische Hochschule, Aachen, Germany, 1997.

[26] S. Shnider and S. Sternberg, *Quantum groups: from coalgebas to Drinfeld algebras*, International Press, 1994.

[27] R. Simion, *A type-B associahedron*, Adv. Appl. Math. **30** (2003), 2–25.

[28] R. Stanley, *Enumerative combinatorics, Vol.1*, Cambridge University Press, Cambridge, 1997.

[29] J. Stasheff, *Homotopy associativity of H-spaces I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–312.

[30] J. Stasheff, *From operads to "physically" inspired theories*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), 53–81, Contemp. Math., 202, Americ. Math. Soc. (1997).

[31] G. Ziegler, *Lectures on polytopes*, Springer-Verlag, New York, 1995.

(Christophe Hohlweg) Université du Québec à Montréal, LaCIM et Département de Mathématiques, C.P. 8888 Succ. Centre-Ville, Montréal, Québec, H3C 3P8, CANADA
E-mail address: hohlweg.christophe@uqam.ca
URL: http://www.lacim.uqam.ca/~hohlweg

(Carsten Lange) Freie Universität Berlin, Institut für Mathematik und Informatik, Arnimallee 3, 14195 Berlin, Germany
E-mail address: lange@math.tu-berlin.de

(Hugh Thomas) Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick, E3B 5A3, CANADA
E-mail address: hugh@math.unb.ca