SEPARABLY CLOSED FIELDS AND CONTRACTIVE ORE MODULES

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Abstract. We consider valued fields with a distinguished contractive map as valued modules over the Ore ring of difference operators. We prove quantifier elimination for separably closed valued fields with the Frobenius map, in the pure module language augmented with functions yielding components for a p-basis and a chain of subgroups indexed by the valuation group.

1. Introduction

Let K be a valued field of prime characteristic p, and let $Frob_p$ denote the Frobenius map $x \mapsto x^p$, and v the valuation map. In [21], Rohwer studied the additive structure $(K, +, Frob_p)$ in a formalism taking into account the valuation through the chain of subgroups $V_\delta = \{x : v(x) \geq \delta\}$, and he proved model-completeness for such models as $K = \mathbb{F}_p((T))$ and $K = \widetilde{\mathbb{F}}_p((T))$, $\widetilde{\mathbb{F}}_p$ being the algebraic closure of $\mathbb{F}_p$. We recall that the corresponding full theory of valued fields has been studied (see [1]), but is in general very far from being fully understood (see e.g. [9]), in particular for the above two examples. In [2], we investigated the additive theory of valued fields but with a distinguished isometry (at the opposite of the Frobenius map) and we could obtain results similar to Rohwer's, even at the level of quantifier elimination for such models as $K = \widetilde{\mathbb{F}}_p((T))$ with the isometry $\sigma(\sum a_i T^i) = \sum a_i^p T^i$. In contrast with Rohwer, our starting point does not address directly the structure of some specific classes of definable sets, but is in the spirit of classical elimination of quantifiers algorithms in the theory of modules. In this paper, we show that our methods can be applied to the Frobenius map for separably closed valued fields (Proposition 6.3), a case not covered by Rohwer (Lemma 4.16). In order to describe the theory of modules over the Ore ring of difference operators, we will use the formalism of λ-functions introduced by G. Srour ([22], see also [8], [12]), and follow the approach undertaken, for instance, in [5], [6] and [19]. Finally let us mention that new results have been obtained by G. Onay on the model theory of valued modules ([15]), both in the isometric case and the contractive case (the Frobenius map case).

We mostly use the notation of [2], with some slight modifications.

Date: June 4, 2015.

[1]Partially supported by NSERC.

[2]Research Director at the "Fonds de la Recherche Scientifique FNRS-FRS", partially funded by the FNRS-FRS Research Credit 14729555.

2. Rings of power series as modules

Let D be a ring with a distinguished endomorphism σ and let $A_0 := D[t;\sigma]$ the corresponding skew polynomial ring with the commutation rule $a.t = t.a^\sigma$ ([3] chapter 2). Recall that any element of A_0 can be written uniquely as $\sum_{i=0}^n t^i.a_i$, with $a_i \in D$ ([3] Proposition 2.1.1 (i)) and one has a degree function $deg : A_0 \setminus \{0\} \to \mathbb{N}$ sending $\sum_{i=0}^n t^i.a_i$ with $a_n \neq 0$, to $n \in \mathbb{N}$, and with the convention that $deg(0) = -\infty < \mathbb{N}$. Whenever we consider D as a right A_0-module, by interpreting scalar multiplication by t by the action of σ on D, we will denote it by $\mathcal{D}$.

In addition, we will assume that D is a right Ore domain, namely that D has no zero-divisors and for all nonzero a, b there are nonzero c, d such that $a.c = b.d$, and that σ is injective, which yields that A_0 has no zero-divisors ([3] Proposition 2.1.1 (ii)).

Under these assumptions, A_0 satisfies the *generalized right division algorithm*: for any $q_1(t),\ q_2(t) \in A_0$ with $deg(q_1) \geq deg(q_2)$, there exist $a \in D - \{0\},\ d \in \mathbb{N}$ and c, r in A_0 with $deg(r) < deg(q_2)$ such that $q_1.a^d = q_2.c + r$ (see e.g. [2], Lemma 2.2).

Since D is a right Ore domain, it has a right field of fractions K and we denote the extension of σ on K by the same letter.

In the following, we will always assume that D is a commutative ring (and so K will always be commutative); let $A := K[t;\sigma]$. Note that A is a principal right ideal domain (and so right Ore) ([3] Proposition 2.1.1 (iii)).

Let K^σ the subfield of K consisting of the image of K under σ. We fix a basis $\mathcal{C}$ of K viewed as a K^σ vector-space; we will call such basis a *σ-basis*.

Moreover we will assume that $\mathcal{C}$ can be chosen in D and that any element of D has a decomposition along that basis with coefficients in D^σ. This is the case for instance if K has characteristic p and σ is the Frobenius endomorphism.

For simplicity, we will assume that $\mathcal{C}$ is finite, that it contains 1 and we present $\mathcal{C}$ as a finite tuple of distinct elements $(1 = c_0, \cdots, c_{n-1})$. However the infinite-dimensional case is not essentially different (see [19]).

Later we will need both A_0 and A, but for the moment we will denote by the letter A a skew polynomial ring of the form $D[t;\sigma]$, where D satisfies the above hypothesis (which encompasses the case where D is a commutative field).

We will adopt the usual convention to denote by the corresponding script letter the structure and by a capital letter its domain. We will consider A-modules $\mathcal{M}$ which have a direct sum decomposition as follows : $M = \oplus_{i=0}^{n-1} M.tc_i$. We will add new unary function symbols $\lambda_i,\ i \in n = \{0, \cdots, n-1\}$ to the usual language of A-modules in order to ensure the existence of such decomposition in the class of A-modules we will consider. These functions will be additive and so we will stay in the setting of abelian structures (see for instance [20]).

Definition 2.1. Let $L_A := \{+, -, 0, \cdot a; a \in A\}$ be the usual language of A-modules. Let λ_i, $i \in n = \{0, \cdots, n-1\}$, be new unary function symbols. Let $\mathcal{L}_A = L_A \cup \{\lambda_i; i \in n\}$, and let T_σ be the following $\mathcal{L}_A$-theory:

(1) the L_A-theory of all right A-modules
(2) $\forall x\ \ (x = \sum_{i \in n} \lambda_i(x) \cdot tc_i)$

(3) $\forall x \forall (x_i)_{i\in n}(x = \sum_{i\in n} x_i \cdot tc_i \rightarrow \bigwedge_{i\in n} x_i = \lambda_i(x))$.

Note that $\mathcal{D}$ is a model ot T_σ, when viewed as an A-module as before.

We will need later that the functions λ_i, $i \in n$ are defined in any model of T_σ by the following $\mathcal{L}_A$-formula: $\lambda_i(x) = y$ iff $(\exists y_0 \cdots \exists y_{n-1}\ \ x = \sum_{j\in n} y_j \cdot tc_j$ and $y_i = y)$ iff $(\forall y_0 \cdots \forall y_{n-1}\ \ x = \sum_{j\in n} y_j \cdot tc_j \ \ \rightarrow \ \ y_i = y)$.

Such theories T_σ have been investigated in [5], [6] and [19], when D is a separably closed field of characteristic p and σ is the Frobenius endomorphism. Let us recall some of the terminology developed there.

Notation 2.2. An element $q(t)$ of A is *σ-separable* if $q(0) \neq 0$. In writing down an element of A, we will allow ourselves to either write it as q or $q(t)$ when stressing the fact that it is a polynomial in t.

In order to reduce divisibility questions to divisibility by separable polynomials, it is convenient to introduce the following notation.

Notation 2.3. (See Notation 3.2, Remark 2 and section 4 in [5].)

Given $q \in A$, we will define $\sqrt[\sigma]{q}$ and q^σ. First, for $a = \sum_i a_i^\sigma c_i \in D$, where the elements a_i belong to D and c_i's to $\mathcal{C}$, set $a^{1/\sigma} := \sum_i a_i c_i$. Observe that $(a^\sigma)^{1/\sigma} = a$, but unless $a \in A^\sigma$, $(a^{1/\sigma})^\sigma$ and a are distinct. Then, for $q = \sum_{j=0}^n t^j a_j \in A$ with $a_j \in D$, set $\sqrt[\sigma]{q} := \sum_{j=0}^n t^j a_j^{1/\sigma}$. We also define q^σ as $\sum_{i=0}^n t^j a_j^\sigma$, we have $tq^\sigma = qt$.

Iteration of $\sqrt[\sigma]{\ }$ m times is denoted by $\sqrt[\sigma^m]{\ }$. Let $a = \sum_{i=0}^{n-1} a_i^\sigma c_i \in D$, where $a_i \in D$ and $c_i \in \mathcal{C}$. Decompose each a_i along the basis $\mathcal{C}$, $a_i = \sum_{j=0}^{n-1} a_{ij}^\sigma c_j$, so $a_i^\sigma = \sum_{j=0}^{n-1} a_{ij}^{\sigma^2} c_j^\sigma$ and $a = \sum_{i,\, j=0}^{n-1} a_{ij}^{\sigma^2} c_j^\sigma c_i$. More generally, $a = \sum_{\bar{d}\in n^m} a_{\bar{d}}^{\sigma^m} c_{\bar{d}}$, where $\bar{d} := (d_1, \cdots, d_m) \in n^m$, $c_{\bar{d}} := c_{d_1}^{\sigma^{m-1}} \cdots c_{d_m}$.

Given $q \in A$, we write it as $q = \sum_i q_i c_i$ with $q_i = \sum_j t^j a_{ij}^\sigma$, $a_{ij} \in D$. Therefore, we have that $\sqrt[\sigma]{q_i} = \sum_j t^j a_{ij}$, so

$$tq = \sum_i \sqrt[\sigma]{q_i} tc_i.$$

Indeed, $\sum_i \sqrt[\sigma]{q_i} tc_i = \sum_i \sum_j t^j a_{ij} tc_i = \sum_i \sum_j t^{j+1} a_{ij}^\sigma c_i = tq$.

Similarly, $t^m q = \sum_{\bar{d}\in n^m} \sqrt[\sigma^m]{q}_{\bar{d}} t^m c_{\bar{d}}$.

For example, let F be a field of characteristic p, $D = F[x]$ and σ be the Frobenius map on D. We consider $\mathcal{D}$ as a module over $D[t;\sigma]$. The notion of σ-separable polynomials $q(t)$ coincides with the notion of p-polynomials $\sum_{i=0}^m a_i x^{p^{m-i}}$, with $a_m \neq 0$, introduced by O. Ore [17] (see also [5]), by making the identification $q(t)$ with $x.q(t)$ (recalling that $x.t = x^\sigma = x^p$). In case F is perfect, a σ-basis for $F[x]$ is $\{1, x, \cdots, x^{p-1}\}$. In general, let $\mathcal{B}$ be a p–basis of F. Then D^σ is equal to $F^\sigma[x^p]$ and there is a direct sum decomposition of D as $\oplus_{c_i\in\{\mathcal{B}, x.\mathcal{B}, \cdots, x^{p-1}.\mathcal{B}\}} D^\sigma \cdot c_i$.

We now assume that (K, v) is a valued commutative field with valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$ and residue field $\bar{K}$. Let $K^\times = K \setminus \{0\}$, we denote the value group $v(K^\times)$ by Γ.

We will set $\bar{a} = a + \mathfrak{m}_K$, the image of a under the residue map from $\mathcal{O}_K$ to $\bar{K}$. Moreover, as before K is endowed with an endomorphism σ which is (valuation) increasing on $\mathcal{O}_K$. This implies that σ is an isometry on the elements of valuation zero and strictly decreasing on the elements of negative valuation. In particular σ induces an endomorphism σ_v of $(\Gamma, +, \leq, 0)$ defined by

$$\sigma_v(v(a)) := v(\sigma(a)).$$

Note that σ_v is injective. We will denote the image by σ_v of an element $\gamma \in \Gamma$ either by γ^{σ_v} or by $\sigma_v(\gamma)$. In the example above where σ is the Frobenius map, we have $\sigma_v(\gamma) = p\gamma$. This action induced by σ on the value group makes it a *multiplicative ordered difference abelian group* in the terminology used by K. Pal ([18]), who investigated the model theory of such structures arising in the context of valued difference fields.

From now on, $A := K[t;\sigma]$, $A_0 = \mathcal{O}_K[t;\sigma]$. We extend the residue map to A_0 by sending $q(t) = \sum_j t^j a_j$ to $\bar{q}(t) := \sum_j t^j \bar{a}_j$. We denote by $\mathcal{I}$ the set of elements of A_0 which have at least one coefficient of valuation 0 (or equivalently $\bar{q}(t) \neq \bar{0}$). Note that unlike the case where σ is an isometry of K, one cannot extend the valuation v of $K^\times$ to $A^\times$ or to $A_0^\times$, but the product of two elements of $\mathcal{I}$ still belongs to $\mathcal{I}$. (To see this last property, let $g(t),\ h(t) \in \mathcal{I}$ with $g(t) = \sum_{j=0}^n t^j b_j$ and $h(t) = \sum_{k=0}^m t^k c_k$. Let j_0 (respectively k_0) be minimal such that $v(b_j) = 0$ (respectively $v(c_k) = 0$). Then the coefficient of $t^{j_0+k_0}$ in $g(t).h(t)$ has value zero.)

Definition 2.4. Let (K, σ, v) be a valued field endowed with an endomorphism σ. We will say that K is *separably σ-linearly closed* if any separable linear difference polynomial $q(x)$, namely $q(x)$ is of the form $p(x, x^\sigma, \cdots, x^{\sigma^n})$ with each x^{σ^i}, $0 \leq i \leq n$, occurring non-trivially with degree at most 1 and $\partial_x p(x, x^\sigma, \cdots, x^{\sigma^n}) \neq 0$, has a zero in K.

Proposition 2.5. *Assume that (K, v, σ) is separably σ-linearly closed. Then the σ-separable elements of $\mathcal{I}$ factor into linear factors belonging to $\mathcal{I}$.*

Proof: This follows from [16, chapter I, theorem 3], where the author dealt with p-polynomials.

First let us simply assume that $q(t) \in A$ with $q(0) \neq 0$. Write $q(t) := \sum_{i=0}^d t^{d-i} a_i$ with $a_0 \neq 0$ and $a_d \neq 0$. We apply the Euclidean algorithm in A and so for any $(t - f) \in A$, there exists $q_1(t)$ such that $q(t) = (t - f)q_1(t) + a$, for some $a \in K$ and $q_1(t) \in A$. We want to show that we can choose f such that $a = 0$. Write $q(t) = \sum_{i=0}^d t^{d-i} a_i$ and $q_1(t) = \sum_{j=0}^{d-1} t^{d-1-j} b_j$. Then we calculate $(t - f)q_1(t)$ and we express that $a = 0$. We obtain that f has to be a root of some polynomial in $K[x, x^\sigma, \cdots, x^{\sigma^{d-1}}]$ with coefficients $a_d, \cdots, a_0$. Namely, we get $a_0 = b_0$, $a_1 = b_1 - f^{\sigma^{d-1}} b_0$, $a_2 = b_2 - f^{\sigma^{d-2}}(a_1 + f^{\sigma^{d-1}} a_0)$, and finally $a_d = -f b_{d-1} = -f a_{d-1} - f f^\sigma a_{d-2} - \cdots - f f^\sigma f^{\sigma^2} \cdots f^{\sigma^{d-1}} a_0$. We consider the difference polynomial $p(x, x^\sigma, \cdots, x^{\sigma^n}) := x a_{d-1} + x x^\sigma a_{d-2} + \cdots + x x^\sigma x^{\sigma^2} \cdots x^{\sigma^{d-1}} a_0 + a_d$. Since $\partial_x p(x, x^\sigma, \cdots, x^{\sigma^n}) = a_{d-1} + x^\sigma a_{d-2} + \cdots + x^\sigma x^{\sigma^2} \cdots x^{\sigma^{d-1}} a_0 \neq 0$, we get a zero $f \in K - \{0\}$ of that polynomial (f is non zero since $a_d \neq 0$).

So, we have $q(t) = (t - f).q_1(t)$ with $b_0 \neq 0$ and $b_{d-1} \neq 0$ and so we can iterate the same process with $q_1(t)$. We finally obtain $q(t) = (t - f_1) \cdots (t - f_{d-1})(t.a_0 - f_d)$ with $f_i \in K - \{0\}$, $1 \leq i \leq d$.

Now let us assume that in addition $q(t) \in \mathcal{I}$. If $f_i \notin \mathcal{O}_K$, then write $(t - f_i) = ((tf_i^{-1}) - 1)f_i$, $1 \leq i \leq d-1$ and the last factor is written as $(t(a_0.f_d^{-1}) - 1).f_d$ if $a_0 f_d^{-1} \in \mathcal{O}_K$ or $(t - f_d.a_0^{-1}).a_d$ if $f_d a_0^{-1} \in \mathcal{O}_K$.

Proceeding successively, we obtain a factorization of $q(t)$ into linear factors of the form $(tf_i' - 1)$ or $(t - f_i')$ with $f_i' \in \mathcal{O}_K$, $1 \leq i \leq d$, together with a constant factor say $c \in K$. Since each of these linear factors belong to $\mathcal{I}$ and that $\mathcal{I}$ is closed under product, we get that $c \in \mathcal{O}_K \setminus \mathfrak{m}_K$. □

Corollary 2.6. *Assume K is a separably closed valued field of characteristic p and consider the skew polynomial ring $K[t;\sigma]$, where σ acts as the Frobenius. Let $q(t) \in \mathcal{I}$ be a σ-separable polynomial, then there exists a factorization of $q(t)$ into σ-separable linear factors belonging to $\mathcal{I}$.* □

Notation 2.7. For $q(t) \in A$ and $\mu \in K - \{0\}$, denote by $q^\mu(t)$ the element of A equal to $\mu.q(t).\mu^{-1}$. So if $q(t) = \sum_i t^i a_i$, $a_i \in K$, $q^\mu(t) = \sum_i t^i \mu^{\sigma^i} a_i \mu^{-1}$.

Note that if $q(t) \in A_0$ and $\mu \in \mathcal{O}_K$, then $q^\mu(t) \in A_0$.

3. Valued modules

We keep the same notation as in the previous section with a fixed (K, v, σ), $\Gamma = v(K^\times)$ endowed with the induced endomorphism σ_v, A the skew polynomial ring $K[t;\sigma]$ etc.

We will define the notion of σ-valued A-modules, or simply valued A-modules. Notions of valued modules occur in various places with many variations, see for instance [4] or [14](§2). The following generalizes the notion in [2].

Definition 3.1. (Cf. [11], [10], [2]) A *valued A-module* is a two-sorted structure $(M, (\Delta \cup \{+\infty\}, \leq, +\gamma; \gamma \in \Gamma), w)$, where M is an A-module, $(\Delta \cup \{+\infty\}, \leq)$ is a totally ordered set for which $+\infty$ is a maximum, $+\gamma$ is an action of $\gamma \in \Gamma$ on Δ, and w is a map $w : M \to \Delta \cup \{+\infty\}$ such that

(1) for all $\delta, \delta_1, \delta_2 \in \Delta$, if $\delta_1 \leq \delta_2$ then $\delta_1 + \gamma \leq \delta_2 + \gamma$, for each $\gamma \in \Gamma$, and $\delta + \gamma_1 < \delta + \gamma_2$, for each $\gamma_1 < \gamma_2 \in \Gamma$; [1]
(2) for all $m_1, m_2 \in M$, $w(m_1 + m_2) \geq \min\{w(m_1), w(m_2)\}$, and $w(m_1) = +\infty$ iff $m_1 = 0$;
(3) for all $m_1, m_2 \in M$, $w(m_1) < w(m_2)$ iff $w(m_1 \cdot t) < w(m_2 \cdot t)$;
(4) for all $m \in M - \{0\}$, $w(m \cdot \mu) = w(m) + v(\mu)$, for each $\mu \in K^\times$.

We denote the corresponding two-sorted language by L_w and the corresponding theory by T_w.

Taking $M = K$ and $w = v$, $\Delta = \Gamma$, then K is a valued A-module with t acting as σ and Γ acting on itself by translation.

From the axioms above, we deduce as usual the following properties : $w(m) = w(-m)$, and if $w(m_1) < w(m_2)$, then $w(m_1 + m_2) = w(m_1)$.

[1] This axiom should replace axiom (1) in the definition given in [2].

Note also, from axiom (1), that for each $m_1,\ m_2 \in M$ and $\mu \in K^\times$, if $w(m_1) \neq w(m_2)$ implies $w(m_1.\mu) \neq w(m_2.\mu)$.

Note that $(w(M), \leq)$ is a substructure of $(\Delta \cup \{+\infty\}, \leq)$, and that t induces an endomorphism τ on $(w(M), \leq)$ defined by $w(m.t) = \tau(w(m))$. It is well-defined since by axiom (3), if $w(m_1) = w(m_2)$ then $w(m_1 \cdot t) = w(m_2 \cdot t)$.

From now on, we will impose a growth condition on the action of t by introducing the following additional structure on Δ. This will induce in particular that the action of t on the corresponding class of valued A-modules will be *uniform*, with a compatibility condition between the action of (Γ, σ_v) and the action of τ.

Definition 3.2. Let $(\Delta, \leq, 0_\Delta, \tau, +\gamma; \gamma \in \Gamma)$ be a totally ordered set with a distinguished element 0_Δ, $+\gamma$ an action of $\gamma \in \Gamma$ on Δ, and τ a fixed endomorphism of $(\Delta, \leq)$.

We assume that for all $\delta, \delta_1, \delta_2 \in \Delta$, if $\delta_1 \leq \delta_2$ then $\delta_1 + \gamma \leq \delta_2 + \gamma$, for each $\gamma \in \Gamma$, and $\delta + \gamma_1 < \delta + \gamma_2$, for each $\gamma_1 < \gamma_2 \in \Gamma$.

The endomorphism τ satisfies the conditions, viz. : $\delta_1 < \delta_2 \to \tau(\delta_1) < \tau(\delta_2), \tau(0_\Delta) = 0_\Delta, \delta > 0_\Delta \to \tau(\delta) > \delta, \delta < 0_\Delta \to \tau(\delta) < \delta$, and finally a compatibility condition between the action of σ_v on Γ and the action of τ: for all $\gamma \in \Gamma$ we have $\tau(\delta + \gamma) = \tau(\delta) + \gamma^{\sigma_v}$.

We will sometimes write 0 instead of 0_Δ, for ease of notation.

Let us denote the corresponding language by $\mathcal{L}_{\Delta,\tau}$ and the corresponding theory by $T_{\Delta,\tau}$.

Let $L_{w,\tau} := L_w \cup \mathcal{L}_{\Delta,\tau}$; we will consider the the class $\Sigma_{w,\tau}$ of $L_{w,\tau}$-structures $(M, (\Delta \cup \{+\infty\}, \leq, \tau, 0_\Delta, +\gamma; \gamma \in \Gamma), w)$ satisfying the following properties :

(1) $(M, (\Delta \cup \{+\infty\}, \leq, +\gamma; \gamma \in \Gamma), w)$ is a valued A-module;
(2) $(\Delta, \leq, \tau, 0_\Delta, +\gamma; \gamma \in \Gamma)$ is a model of $T_{\Delta,\tau}$;
(3) $w(m \cdot t) = \tau(w(m))$.

Note that if $M \in \Sigma_{w,\tau}$, then if $w(m) > 0_\Delta$, then $w(m \cdot t) > w(m)$, if $w(m) = 0_\Delta$, then $w(m \cdot t) = w(m)$, and if $w(m) < 0_\Delta$, then $w(m \cdot t) < w(m)$. Moreover, letting $\Gamma^+ := \{\gamma \in \Gamma :\ \gamma > 0\}$, if $\gamma \in \Gamma^+$, then $\sigma_v(\gamma) > \gamma$. (Indeed, $0_\Delta + \gamma > 0_\Delta$, so $\tau(0_\Delta + \gamma) > 0_\Delta + \gamma$. By the compatibility condition, $\tau(0_\Delta + \gamma) = 0_\Delta + \sigma_v(\gamma)$ and so $\sigma_v(\gamma) > \gamma$.)

4. Abelian structures

In order to stay into the setting of abelian structures, we will use a less expressive language. This language was used by T. Rohwer while considering the field of Laurent series over the prime field $\mathbb{F}_p$ with the usual Frobenius map $y \mapsto y^p$ ([21]). Instead of the two-sorted structure (M, Δ, w), where $\mathcal{M}$ is a valued A-module, he considered the one-sorted abelian structure $(M, (M_\delta)_{\delta \in \Delta})$, where $M_\delta = \{x \in M : w(x) \geq \delta\}$. Similarly, given a valued A-module $\mathcal{M}$ with a direct sum decomposition $M = \oplus_{i=0}^{n-1} M.tc_i$, we will add the functions λ_i and consider the one-sorted abelian structure $(M, (\lambda_i)_{i \in n}, (M_\delta)_{\delta \in \Delta})$.

We will consider theories of abelian structures satisfying strong divisibility properties. The basic example is the separable closure of $\tilde{\mathbb{F}}_p((T))$. Note that this example is not covered by Rohwer, as we will indicate below, following Lemma 4.13.

Definition 4.1. Let $(\Delta, \leq, \tau, +, 0_\Delta, +\infty)$ be a model of $T_{\Delta,\tau}$. We set the language $\mathcal{L}_V := \mathcal{L}_A \cup \{V_\delta : \delta \in \Delta\}$, where V_δ is a unary predicate.

Let T_V be the $\mathcal{L}_V$-theory with the following axioms, with $\delta \in \Delta$:

(1) T_σ;
(2) $V_\delta(0)$;
(3) $\forall m\ (V_{\delta_2}(m) \rightarrow V_{\delta_1}(m))$, whenever $\delta_1 \leq \delta_2$, $\delta_1,\ \delta_2 \in \Delta$;
(4) $\forall m_1\ \forall m_2\ \ (V_\delta(m_1)\ \&\ V_\delta(m_2) \rightarrow V_\delta(m_1 + m_2))$;
(5) $\forall m\ \ (V_\delta(m) \rightarrow V_{\delta+v(\mu)}(m.\mu))$, where $\mu \in K^\times$;
(6) $\forall m\ (V_\delta(m) \leftrightarrow V_{\tau(\delta)}(m.t))$.

If (M, Δ, w) is a valued A-module with a direct sum decomposition $M = \oplus_{i\in n} M.tc_i$, and we let $\mathcal{M}_V = (M, +, 0, (.r)_{r\in A},\ (\lambda_i)_{i\in n}, (M_\delta)_{\delta\in\Delta})$, then $\mathcal{M}_V$ is a model of T_V, where V_δ is interpreted as M_δ.

The structure $\mathcal{M}_V$ is an abelian structure and one gets as in the classical case of (pure) modules that any formula is equivalent to a boolean combination of positive primitive formulas (p.p.) and index sentences (namely, sentences telling the index of two p.p.-definable subgroups of the domain of $\mathcal{M}_V$ in one another (see [19])) and this p.p. elimination is uniform in the class of such structures.

Note that the pure module theory of separably closed fields of characteristic p and fixed non-zero imperfection degree has quantifier elimination in the presence of the functions λ_i ([5]).

We want to axiomatize a class of abelian structures which contains the class of valued separably closed fields of characteristic p. Note that the theory of valued separably closed fields has been shown to be model-complete in the language of valued fields augmented with predicates expressing p-independence ([7]) and to admit quantifier elimination in the language of valued fields augmented with the λ_i functions ([13]).

In the remainder of the section, we will formalise certain properties of separably closed fields viewed as modules over the corresponding skew polynomial rings in order to axiomatize the class of modules we will be working with.

Notation 4.2. Let $\delta \in \Delta$, then we denote by $M_{\delta^+} = \{x \in M : w(x) > \delta\}$.

Corollary 4.3. *Assume that K is separably σ-linearly closed. Let $\mathcal{M}$ be a A-module with the following divisibility properties: given any $a \in \mathcal{O}_K \setminus \{0\}$, any $n_1, n_2 \in M_{0_\Delta}$, there exists $m_1, m_2 \in M_{0_\Delta}$ such that $m_1 \cdot (t - a) = n_1$, and $m_2 \cdot (ta - 1) = n_2$.*

Then given any σ-separable polynomial $q(t) \in \mathcal{I}$ and any element n of M_{0_Δ} there exists $m \in M_{0_\Delta}$ such that $m \cdot q(t) = n$. Moreover, if $n \in M_{0_\Delta} \setminus M_{0^+_\Delta}$, then there exists $m \in M_{0_\Delta} \setminus M_{0^+_\Delta}$ such that $m \cdot q(t) = n$.

Proof: By Proposition 2.5, it suffices to prove it for linear polynomials of the form $(t-a)$ or $(ta-1)$ with $a \in \mathcal{O}_K \setminus \{0\}$. Let $n \in M_{0_\Delta}$. Then by assumption, there exists $m \in M_{0_\Delta}$ such that $n = m\cdot(t-a)$ (respectively $n = m\cdot(ta-1)$). If $n \in M_{0_\Delta} \setminus M_{0^+_\Delta}$ and

$m \in M_{0^+_\Delta}$, then $\min\{w(m \cdot t), v(m \cdot a)\} > 0$ (respectively $\min\{w(m \cdot ta), v(m)\} > 0$), a contradiction. $\square$

In the following (see Lemma 4.8), under a further condition on Δ, we will show that given $n \in M$ and $q(t) \in \mathcal{I}$, if we can find an element $m \in M$ such that $n = m \cdot q(t)$, then we can find one whose value $w(m)$ can be determined in terms of the values of the coefficients of $q(t)$ and $w(n)$.

Definition 4.4. We will say that $(\Delta, \leq, 0_\Delta, \tau, +\gamma; \gamma \in \Gamma)$ is ordered linearly closed (o.l.-closed) if given any finite subset $\{\gamma_i \in \Gamma; 0 \leq i \leq d\}$, for any $\delta \in \Delta$ there exists $\mu \in \Delta$ such that $\delta = \min_{0 \leq i \leq d}\{\tau^i(\mu) + \gamma_i\}$.

Lemma 4.5. *Assume that τ is surjective on Δ. Then Δ is o.l.-closed.*

Proof: Let $q(t) \in A$, $q(t) = \sum_{i=0}^d t^i a_i$, and let $\{\gamma_i := v(a_i) : \ a_i \neq 0,\ 0 \leq i \leq d\}$. Given $\delta_0 \in \Delta$, let us show that there exists $\delta \in \Delta$ such that $\delta_0 = \min_{0 \leq i \leq d} \tau^i(\delta) + \gamma_i$.

Consider the functions f_i on Δ defined by $f_i(\delta) = \tau^i(\delta) + \gamma_i$, $0 \leq i \leq d$. Since τ is assumed here to be surjective, so is τ^i. Thus there exists $\delta_i \in \Delta$ such that $\delta_0 + (-\gamma_i) = \tau^i(\delta_i)$ and so $\delta_0 = f_i(\delta_i)$.

Each function f_i is strictly increasing: if $\delta_1 < \delta_2$, then $\tau^i(\delta_1) < \tau^i(\delta_2)$ and $\tau^i(\delta_1) + \gamma_i < \tau^i(\delta_2) + \gamma_i$. So the maximum μ of the δ_i's such that $f_i(\delta_i) = \delta_0$ is well-defined. Since $\delta_i \leq \mu$, we have that $\delta_0 = f_i(\delta_i) \leq f_i(\mu)$ (f_i is increasing), $0 \leq i \leq d$. So, $\delta_0 = \min_{0 \leq i \leq n} f_i(\mu)$, namely $\Upsilon(q, \delta_0) = \mu$. $\square$

Notation 4.6. Assume τ is surjective on Δ. Given $q(t) \in A$, $q(t) = \sum_{i=0}^d t^i a_i$, given δ and $\{\gamma_i := v(a_i) : \ a_i \neq 0,\ 0 \leq i \leq d\}$, by the above Lemma, there exists μ such that $\delta = \min\{\tau^i(\mu) + \gamma_i : \ 0 \leq i \leq d\}$. We will denote that μ by $\Upsilon(q, \delta)$. We also set $\Upsilon^{-1}(q, \mu) := \delta$.

As soon as Δ is o.l.-closed, the functions Υ^{-1} and Υ are well-defined and we have the following relationship between Υ^{-1} and Υ. Let $q(t) \in \mathcal{I}$, $\mu, \delta \in \Delta$, then $\Upsilon^{-1}(q(t), \Upsilon(q(t), \delta)) = \delta$ and $\Upsilon(q(t), \Upsilon^{-1}(q(t), \mu)) = \mu$. Moreover, for $m \in M$ we always have $w(m.q(t)) \geq \Upsilon^{-1}(q(t), w(m))$. And finally since each of the functions $\tau^i + \gamma_i$ are strictly increasing on Δ, $\Upsilon(q, \mu_2) \leq \mu_1 \leftrightarrow \mu_2 \leq \Upsilon^{-1}(q, \mu_1)$. This last equivalence implies that Υ is increasing and since it is injective, it is strictly increasing.

When the action of Γ is transitive on Δ, because of the compatibility condition on the two actions, whenever σ_v is surjective on Γ, τ is surjective on Δ and so Δ o.l.-closed. Note that if K is separably closed of characteristic p and σ is the Frobenius map, then $\sigma_v(\gamma) = p\gamma$, and $\Gamma = vK$ is divisible.

Definition 4.7. Assume that τ is surjective on Δ and that $(\Delta, \leq, 0_\Delta, \tau, +\gamma; \gamma \in \Gamma) \models T_{\Delta,\tau}$, then let T_V^+ be the following $\mathcal{L}_V$-theory:

(1) T_V,
(2) $\forall n\ (n \in V_0 \rightarrow (\exists m\ (m \in V_0 \,\&\, m \cdot q(t) = n)))$, for all $q(t) \in \mathcal{I}$, $q(t)$ σ-separable.

Lemma 4.8. *Let $\mathcal{M}$ be a valued A-module which is a model of T_V^+. Assume that Γ acts transitively on Δ. Let $q(t) \in \mathcal{I}$ be σ-separable, $\delta \in \Delta$ and let $\mu := \Upsilon(q, \delta)$. Then, for any $n \in M$ with $w(n) = \delta$, there exists $m \in M$ such that $m \cdot q(t) = n$ & $w(m) = \mu$. Moreover μ has the additional property that for any $m \in M$ with $w(m) = \mu$, $w(m \cdot q(t)) \geq \delta$.*

Proof: Let $q(t) = \sum_{i=0}^{d} t^i a_i \in \mathcal{I}$, with $a_0 \neq 0$. By axiom scheme (2) of T_V^+, for any $n_0 \in M$ with $w(n_0) = 0_\Delta$, there exists $m_0 \in M$ such that $m_0 \cdot q(t) = n_0$ and $w(m_0) = 0_\Delta$. Note that the axiom scheme (2) only gives us that $w(m_0) \geq 0_\Delta$, but since at least one coefficient of $q(t)$ has value 0, we get that $w(m_0)$ cannot be strictly bigger than 0_Δ. Since the action of Γ on Δ is transitive, there exist $k_\delta, k_\mu \in K$ such that $0_\Delta + v(k_\delta) = \delta$ and $0_\Delta + v(k_\mu) = \mu$.

Let $n \in M$ with $w(n) = \delta$ and consider the polynomial $\tilde{q}(t) := \sum_i t^i k_\mu^{\sigma^i} a_i k_\delta^{-1} = q^{k_\mu}(k_\mu k_\delta^{-1})$. Then by construction $\tilde{q}(t) \in \mathcal{I}$ and is still σ-separable.

Now $w(n \cdot k_\delta^{-1}) = 0$, so by hypothesis there exists $m_0 \in M_{0_\Delta}$ such that $m_0 \cdot \tilde{q}(t) = n \cdot k_\delta^{-1}$. So, $m_0 \cdot \sum_i t^i k_\mu^{\sigma^i} a_i = n$ and so $m_0 \cdot k_\mu q(t) = n$. Set $m := m_0 \cdot k_\mu$, we have $w(m) = \mu$ and $m \cdot q(t) = n$.

Moreover, if $w(m) = \mu$, then by the compatibility condition between τ and σ_v, we have $w(m \cdot q(t)) \geq \min_i \tau^i(w(m)) + v(a_i) \geq \delta$. □

Lemma 4.9. *Let $\mathcal{M}$ be a model of T_V^+ and assume that Γ acts transitively on Δ. Let $q(t) \in \mathcal{I}$ be σ-separable and $\delta \in \Delta$. Then, for any $n \in M_\delta$, there exists $m \in M_{\Upsilon(q,\delta)}$ such that $m \cdot q(t) = n$. Moreover $\Upsilon(q, \delta)$ is such that for any $m \in M_{\Upsilon(q,\delta)}$, $m \cdot q(t) \in M_\delta$.*

Proof: It follows from the proof of the above Lemma, replacing equalities of the form $w(m) = \delta \in \Delta$, by $m \in M_\delta$. □

The separable closure of a valued field of characteristic p is dense in its algebraic closure. This translates as follows in the case of models of T_V^+.

Lemma 4.10. *If $\mathcal{M}$ is a valued A-module which is a model of T_V^+, and if the action of Γ on Δ is transitive and satisfies the following $(\star\star)$: for all $\delta_1, \delta_2 \in \Delta$, there exists $\gamma \in \Gamma$ such that $\delta_1 + (\sigma_v(\gamma) - \gamma) = \delta_2$. Then for any δ and m with $w(m) \leq \delta$, there exists n such that $w(m - n \cdot t) = \delta$.*

Proof: W.l.o.g. we may assume that $\delta > w(m)$ (otherwise it suffices to choose $n = 0$). First choose $k \in K$ such that $w(m) + v(k^\sigma) < 0$.

By Lemma 4.8 M is $(t-1)$-divisible, so there exists $n \in M$ such that $m \cdot k^\sigma = n \cdot (t-1)$ and necessarily $w(n) < 0$. So $\tau(w(n)) < w(n)$ and therefore $w(n \cdot t) = w(m) + \sigma_v(v(k))$ and $w(n) = \tau^{-1}(w(m) + \sigma_v(v(k_\mu)))$.

We have that $m = n \cdot k^{-1} t - n \cdot k^{-\sigma}$.

We have $w(m - n \cdot k^{-1} t) = w(n) - v(k^\sigma) = \tau^{-1}(w(m)) + (v(k) - \sigma_v(v(k)))$. Now, by the extra assumption, there is k_1 such that $\tau^{-1}(w(m)) + (v(k_1) - \sigma_v(v(k_1))) = \delta$. It suffices to see that this forces $w(m) + v(k_1^\sigma) < 0$, so that the preceding discussion applies to k_1 as well, and we are done. But we have $w(m) + v(k_1^\sigma) = \tau(\delta + v(k_1^\sigma))$ and $w(m) + v(k_1^\sigma) < \delta + v(k_1^\sigma)$. So if $w(m) + v(k_1^\sigma) \geq 0$, we would get $0 < \delta + v(k_1^\sigma) <$

$\tau(\delta + v(k_1^\sigma))$, and then $w(m) + v(k_1^\sigma) < \tau(\delta + v(k_1^\sigma))$ which is absurd. Hence we must have $w(m) + v(k_1^\sigma) < 0$, as wanted. □

Remark 4.11. Note that in case the action of Γ on Δ is transitive, to meet the hypothesis $(\star\star)$, we can require that the action of $\sigma_v - 1$ is surjective on Γ.

Definition 4.12. Recall that $\Gamma^+ := \{\gamma \in \Gamma : \gamma \geq 0\}$. We will say that σ_v is 2-contracting on Γ if $\forall \gamma \in \Gamma^+\ \sigma_v(\gamma) \geq \gamma + \gamma$.

Lemma 4.13. *Suppose $\mathcal{M}$ is a model of T_V^+ and assume Γ acts transitively on Δ. If σ_v is 2-contracting on Γ, then for any $\delta \in \Delta$ and for all $m \in M$, there exists n such that $V_\delta(m - n \cdot t)$ holds.*

Proof: It follows from the proof of the above lemma, noting that we only need in this setting that $\forall \delta_1 \in \Delta \forall \delta_2 \in \Delta \exists\ \gamma \in \Gamma\ \delta_1 + (\sigma_v(\gamma) - \gamma) \geq \delta_2$. W.l.o.g., we may assume that $\delta_1 < \delta_2$. So given $\delta_1 < \delta_2 \in \Delta$, by transitivity of the action of Γ on Δ, we get that there exists $\tilde{\gamma} \in \Gamma^+$ such that $\delta_1 + \tilde{\gamma} = \delta_2$. Since σ_v is 2-contracting we get that $\sigma_v(\tilde{\gamma}) \geq \tilde{\gamma} + \tilde{\gamma}$. So $\delta_1 + \sigma_v(\tilde{\gamma}) \geq \delta_2 + \tilde{\gamma}$ and so $\delta_1 + \sigma_v(\tilde{\gamma}) - \tilde{\gamma} \geq \delta_2$. □

Definition 4.14. Assume that τ is surjective on Δ and that $(\Delta, \leq, 0_\Delta, \tau, +\gamma; \gamma \in \Gamma) \models T_{\Delta,\tau}$. Let T_V^{sep} be the $\mathcal{L}_V$-theory:

(1) T_V,
(2) $\forall n\ (n \in V_\delta \rightarrow (\exists m\ (m \in V_{\Upsilon(q,\delta)}\ \&\ m \cdot q(t) = n)))$, for all $q(t) \in \mathcal{I}$, $q(t)$ σ-separable and for all $\delta \in \Delta$.
(3) $\forall m\ \exists n\ \ V_\delta(m - n \cdot t)$, for all $\delta \in \Delta$.

In particular, if K is separably closed of characteristic p, then $\mathcal{K}$ viewed as a $K[t;\sigma]$-module with σ acting as the Frobenius map, is a model of T_V^{sep}.

More generally, if (Kv, σ) is separably σ-linearly closed and if σ_v is 2-contracting on Γ, then again $\mathcal{K}$ viewed as a A-module is a model of T_V^{sep}.

Further note that if Γ acts transitively on Δ and σ_v is 2-contracting on Γ, then a model of T_V^+ is a model of T_V^{sep} (by Lemmas 4.9 and 4.13).

We will prove in the next sections that T_V^{sep} eliminates quantifiers up to index sentences.

Before doing that, we now check that the basic example of the separable closure of $\tilde{\mathbb{F}}_p((T))$ is not covered by Rohwer (see [21], pp. 40-41), since it does not have a weak valuation basis.

Definition 4.15. Let $\mathcal{M} := (M, +, 0, .r; r \in A, \lambda_i, i \in n)$ be a model of T_σ. Then $\mathcal{M}$ is a *valued A-module with a weak σ-valuation basis* if there exists $r \in K$ such that for each $m \in M$ we have: $w(m) \leq \min_i\{w(\lambda_i(m) \cdot t) + v(c_i) + v(r) :\ c_i \in \mathcal{C}\}$. $(*)$

Lemma 4.16. *Let K be any valued separably closed field K of finite imperfection degree, then K does not have a weak σ-valuation basis, with σ the Frobenius endomorphism.*

Proof: By way of contradiction, let $c_1, c_2, \ldots$ be a linear basis of K over K^p and suppose that it is a weak σ-valuation basis and let δ be the corresponding $v(r)$. By adjusting δ and since $v(K^\times)$ is p-divisible, we may always assume that $c_1 = 1$. Let $\delta' \in \Gamma$ such that $\delta' > \{v(c_2), \delta, v(c_2)+\delta\}$. By Lemma 4.13 let a, b such that $c_2 = a^p + b$ with $v(b) \geq \delta'$. If $v(a^p) \neq v(c_2)$, then that would contradict the required inequality (*) for $a^p - c_2$. Otherwise, $v(a^p) = v(c_2)$, and again this contradicts (*) for $a^p - c_2$. $\square$

5. Special cases

In order to eliminate quantifiers in T_V^{sep}, we need some basic cases and reductions, which are treated in the following lemmas.

Our main tools will be axiom schemes (2) and (3) of T_V^{sep} and we will use Notation 4.6. We will treat the general case in the next section.

We will use the notation $u \cdot r \equiv_\delta m$ to mean that $V_\delta(u \cdot r - m)$ holds. We place ourselves in any model of T_V^{sep} and the b_i's that will occur in the systems are $\mathcal{L}_A$-terms in some tuple of variables $\bar{y}$.

Lemma 5.1. *Consider a system of the form*

$$\exists u \left\{ \begin{array}{lll} u \cdot t & \equiv_{\mu_1} & b_1 \\ u \cdot r & \equiv_{\mu_2} & b_2, \end{array} \right.$$

where $r \in \mathcal{I}$ is separable. Then this system is equivalent to a congruence of the form

$$b_1 \cdot r^\sigma \equiv_{\mu_3} b_2 \cdot t.$$

Proof: We distinguish two cases.

(*i*) $\Upsilon^{-1}(r^\sigma, \mu_1) \geq \tau(\mu_2)$. Then the system above is equivalent to

$$b_1 \cdot r^\sigma \equiv_{\tau(\mu_2)} b_2 \cdot t.$$

One implication is straightforward. For the reverse implication, since we are in a model of T_V^{sep}, by axiom (3), there exists u such that $u \cdot t \equiv_{\mu_1} b_1$. So $u \cdot tr^\sigma \equiv_{\Upsilon^{-1}(r^\sigma,\mu_1)} b_1 \cdot r^\sigma$. So $u \cdot rt \equiv_{\tau(\mu_2)} b_2 \cdot t$ and so $u \cdot r \equiv_{\mu_2} b_2$.

(*ii*) $\Upsilon^{-1}(r^\sigma, \mu_1) < \tau(\mu_2)$. Then the system above is equivalent to

$$b_1 \cdot r^\sigma \equiv_{\Upsilon^{-1}(r^\sigma,\mu_1)} b_2 \cdot t.$$

One implication is straighforward. For the reverse implication, choose μ such that $\Upsilon^{-1}(r^\sigma, \mu) \geq \tau(\mu_2)$ (and so $\mu > \mu_1$). Again by axiom (3), there exists u such that $u \cdot t \equiv_{\mu_1} b_1$. So $u \cdot tr^\sigma \equiv_{\Upsilon^{-1}(r^\sigma,\mu_1)} b_1 \cdot r^\sigma \equiv_{\Upsilon^{-1}(r^\sigma,\mu_1)} b_2 \cdot t$.

Since we are in a model of T^{sep}, by axiom (2), there exists u'' such that $u'' \cdot r^\sigma = (u \cdot r - b_2) \cdot t$ with $w(u'') \geq \mu_1$ (by definition of $\Upsilon^{-1}(r^\sigma, \mu_1)$). Let u' be such that $u' \cdot t \equiv_\mu u''$. Then $u' \cdot rt = u' \cdot tr^\sigma \equiv_{\tau(\mu_2)} (u \cdot r - b_2) \cdot t$, which implies that $(u - u') \cdot rt \equiv_{\tau(\mu_2)} b_2 \cdot t$ and so $(u - u') \cdot r \equiv_{\mu_2} b_2$, which finishes the proof since $(u - u') \cdot t \equiv_\mu u \cdot t - u'' \equiv_{\mu_1} b_1$. $\square$

Lemma 5.2. *Consider a system of the form*

$$\exists u \left\{ \begin{array}{lll} u \cdot r_1 & = & b_1 \\ u \cdot r_2 & \equiv_{\delta_2} & b_2, \end{array} \right.$$

where $r_1,\ r_2 \in \mathcal{I}$ are separable and assume that $deg(r_1) \geq deg(r_2)$. Then this system is equivalent to the following system

$$\exists u \left\{ \begin{array}{lll} u \cdot r_2 & = & b_2 \\ u \cdot r_3 & \equiv_\delta & b_1 \cdot \alpha - b_2 \cdot s, \end{array} \right.$$

where $\delta = \Upsilon^{-1}(r_1\alpha, \Upsilon(r_2, \delta_2))$, for some $\alpha \in \mathcal{O}_K$, and $deg(r_2) > deg(r_3)$.

Proof: By the generalized euclidean algorithm, there exists $\alpha \in \mathcal{O}_K$ such that $r_1\alpha = r_2 s + r_3$ with $deg(r_3) < deg(r_2)$.

Suppose u is a solution of the first system. Let u' be such that $u' \cdot r_2 = u \cdot r_2 - b_2$. We can find such u' with $w(u') \geq \Upsilon(r_2, \delta_2)$. So $(u-u') \cdot r_2.s + (u-u') \cdot r_3 = (u-u') \cdot r_1\alpha = b_1 \cdot \alpha - u' \cdot r_1\alpha$ i.e. $(u-u') \cdot r_3 = b_1 \cdot \alpha - b_2 \cdot s - u' \cdot r_1\alpha$.

Conversely, let u'' satisfy the second system. Then $u'' \cdot r_1\alpha = u'' \cdot r_2 s + u'' \cdot r_3 \equiv_\delta b_1 \cdot \alpha$. So let u''' be such that $u''' \cdot r_1 = u'' \cdot r_1 - b_1$ and we have to make sure that we can choose u''' such that $w(u''' \cdot r_2) \geq \delta_2$. In other words, $\Upsilon(r_1\alpha, \delta) = \Upsilon(r_2, \delta_2)$. □

Lemma 5.3. *Consider a system of the form*

$$\exists u \left\{ \begin{array}{lll} u \cdot r_1 & \equiv_{\delta_1} & b_1 \\ u \cdot r_2 & \equiv_{\delta_2} & b_2 \end{array} \right.$$

where $r_1,\ r_2 \in \mathcal{I}$ are separable and $\Upsilon(r_1, \delta_1) \leq \Upsilon(r_2, \delta_2)$. Then this system is equivalent to the following system

$$\exists u \left\{ \begin{array}{lll} u \cdot r_1 & \equiv_{\delta_1} & b_1 \\ u \cdot r_2 & = & b_2 \end{array} \right.$$

Proof: Indeed, we can choose u' such that $w(u') \geq \Upsilon(r_2, \delta_2)$ and such that $u' \cdot r_2 = b_2 - u \cdot r_2$ and so $(u + u') \cdot r_2 = b_2$. Moreover $w(u' \cdot r_1) \geq \delta_1$. □

Lemma 5.4. *Consider a system of the form*

$$\exists u \left\{ \begin{array}{lll} u \cdot r & = & b \\ u \cdot t & \equiv_\delta & b_1, \end{array} \right.$$

where $r \in \mathcal{I}$ is separable. Then this system is equivalent to the following system

$$\exists u \left\{ \begin{array}{lll} u \cdot r & \equiv_{\delta'} & b \\ u \cdot t & \equiv_\delta & b_1, \end{array} \right.$$

where δ' is chosen such that $\tau(\Upsilon(r, \delta')) \geq \delta$.

Proof: Let us show the non-trivial implication. Since r is separable, there exists u' such that $u' \cdot r = (u \cdot r - b)$ and we can choose such u' with $w(u') \geq \Upsilon(r, \delta')$. Since $\tau(\Upsilon(r, \delta')) \geq \delta$, we get that $w(u' \cdot t) \geq \delta$ and so $(u - u')$ is a solution of the first system. □

Lemma 5.5. *Consider a system of the form*

$$\exists u \left\{ \begin{array}{rll} u \cdot r_0 & = & b_0 \\ \bigwedge_{i=1}^{d} u \cdot t^n & \equiv_{\delta_i} & b_i, \end{array} \right.$$

where $r_0 \in \mathcal{O}_K$, $\bar{r}_0 \neq 0$. Then this system is equivalent to congruences of the following form

$$\begin{cases} b'_0 \cdot t^n & \equiv_\delta \quad b'_1 \cdot r_0^{\sigma^n} \\ \bigwedge_{i=2}^d b'_1 & \equiv_{\delta'_i} \quad b'_i, \end{cases}$$

where the b'_i are $\mathcal{L}_A$-terms in $\bar{y}$ obtained from the b_i's by scalar multiplication by elements of A.

Proof: We first proceed as in Lemma 5.4, replacing the equation $u \cdot r_0 = b_0$ by a congruence $u \cdot r_0 \equiv_{\delta'} b_0$ where δ' is chosen such that $\tau^n(\Upsilon(r_0, \delta')) \geq \{\delta_i : 1 \leq i \leq d\}$.

So it remains to consider a system of the form:

$$\exists u \begin{cases} u \cdot r_0 & \equiv_{\delta'} \quad b_0 \\ \bigwedge_{i=1}^d u \cdot t^n & \equiv_{\delta_i} \quad b_i, \end{cases}$$

Now we proceed as in Lemma 5.1. First we note that $u \cdot r_0 t^n \equiv_{\tau^n(\delta')} b_0 \cdot t^n$ is equivalent to $u \cdot r_0 \equiv_{\delta'} b_0$. We rewrite the first formula as $u \cdot t^n r_0^{\sigma^n} \equiv_{\tau^n(\delta')} b_0 \cdot t^n$.

We order the δ_i and w.l.o.g. assume that $\delta_1 \geq \max\{\delta_i : 1 \leq i \leq d\}$. Our system is then equivalent to:

$$\begin{cases} b_0 \cdot t^n & \equiv_{\Upsilon^{-1}(r_0^{\sigma^n}, \delta_1)} \quad b_1 \cdot r_0^{\sigma^n} \\ \bigwedge_{i=2}^d b_1 & \equiv_{\delta_i} \quad b_i, \end{cases}$$

Indeed, by axiom (3), there exists u such that $u \cdot t^n \equiv_{\delta_1} b_1$. So, if $\Upsilon^{-1}(r_0^{\sigma^n}, \delta_1) \geq \tau^n(\delta')$ $(\star)$, we get that $u \cdot t^n r_0^{\sigma^n} \equiv_{\Upsilon^{-1}(r_0^{\sigma^n}, \delta_1)} b_1 \cdot r_0^{\sigma^n}$ and so since $b_1 \cdot r_0^{\sigma^n} \equiv_{\Upsilon^{-1}(r_0^{\sigma^n}, \delta_1)} b_0 \cdot t^n$, we get by $(\star)$, that $u \cdot r_0 \equiv_{\delta'} b_0$.

Now assume that $\Upsilon^{-1}(r_0^{\sigma^n}, \delta_1) < \tau^n(\delta')$. So we choose δ'' such that $\Upsilon^{-1}(r_0^{\sigma^n}, \delta'') \geq \tau^n(\delta')$.

Again, by axiom (3), there exists u such that $u \cdot t^n \equiv_{\delta_1} b_1$. So we get

$$u \cdot t^n r^{\sigma^n} \equiv_{\Upsilon^{-1}(r_0^{\sigma^n}, \delta_1)} b_1 \cdot r^{\sigma^n} \equiv_{\Upsilon^{-1}(r_0^{\sigma^n}, \delta_1)} b_0 \cdot t^n.$$

By axiom (2), there exists u'' such that $u'' \cdot r^{\sigma^n} = (u \cdot r - b_0) \cdot t^n$ with $w(u'') \geq \delta_1$.

By axiom (3), there exists u' such that $u' \cdot t^n \equiv_{\delta''} u''$. Then $u' \cdot r t^n = u' \cdot t^n r^{\sigma^n} \equiv_{\tau^n(\delta')} (u \cdot r - b_0) \cdot t^n$, which implies that $(u - u') \cdot r t^n \equiv_{\tau^n(\delta')} b_0 \cdot t^n$ and so $(u - u') \cdot r \equiv_{\delta'} b_0$.

Since $w(u'') = \delta_1$, we may add to the other congruences $u'' = u' \cdot t^n$ without perturbing them. □

6. Quantifier elimination

We now prove that T_V^{sep} admits quantifier-elimination up to index sentences.

Notation 6.1. Let $\boldsymbol{d} = (d_1, \cdots, d_m) \in n^m$ be a m-tuple of natural numbers between 0 and $n-1$. We denote by $\lambda_{\boldsymbol{d}}^{(m)}$ the composition of the m λ-functions: $\lambda_{d_1} \circ \lambda_{d_2} \circ \cdots \circ \lambda_{d_m}$.

Lemma 6.2. *In any model of T_V, a system of equations $\bigwedge_{i=1}^d u \cdot r_i = t_i(\bar{y})$, where $t_i(\bar{y})$ is a $\mathcal{L}_A$-term and $r_i \in A$ with at least one r_i σ-separable, is equivalent to one equation of the form $u \cdot r = t(\bar{y})$, where $r \in A$ is separable together with a conjunction of atomic formulas in $\bar{y}$.*

Proof: We apply the Euclidean algorithm and do some bookeeping to check that we always keep a separable coefficient. Assume that r_1 is separable. Let us consider the system:

$$\left\{\begin{array}{rcl} u \cdot r_1 & = & t_1(\bar{y}) \\ u \cdot r_i & = & t_i(\bar{y}), \end{array}\right. \tag{1}$$

with $i \neq 1$.

If r_i is not separable and if $deg(r_1) \geq deg(r_i)$, then for some $r',\ r'' \in A$, we have $r_1 = r_i r' + r''$, then $r'' \neq 0$ and r'' is separable. So, the system is equivalent to:

$$\left\{\begin{array}{rcl} u \cdot r_i & = & t_i(\bar{y}) \\ u \cdot r'' & = & t_1(\bar{y}) - t_i(\bar{y}) \cdot r', \end{array}\right. \tag{2}$$

with $deg(r'') < deg(r_i)$ and r'' separable.

If r_i is not separable and if $deg(r_1) < deg(r_i)$, then for some $r',\ r'' \in A$, we have $r_i = r_1 s' + s''$, then either $s'' = 0$ and the system is equivalent to:

$$\left\{\begin{array}{rcl} u \cdot r_1 & = & t_1(\bar{y}) \\ t_1(\bar{y}) \cdot s' & = & t_i(\bar{y}), \end{array}\right. \tag{3}$$

or $s'' \neq 0$ and the system is equivalent to:

$$\left\{\begin{array}{rcl} u \cdot r_1 & = & t_1(\bar{y}) \\ u \cdot s'' & = & t_i(\bar{y}) - t_1(\bar{y}) \cdot s'. \end{array}\right. \tag{4}$$

If r_i is separable, then w.l.o.g. $deg(r_1) \geq deg(r_i)$. For some $r',\ r'' \in A$, we have $r_1 = r_i r' + r''$. Either $r'' = 0$ and the system is equivalent to:

$$\left\{\begin{array}{rcl} u \cdot r_i & = & t_i(\bar{y}) \\ t_1(\bar{y}) & = & t_i(\bar{y}) \cdot r', \end{array}\right. \tag{5}$$

or $r'' \neq 0$ and the system is equivalent to

$$\left\{\begin{array}{rcl} u \cdot r_i & = & t_i(\bar{y}) \\ u \cdot r'' & = & t_1(\bar{y}) - t_i(\bar{y}) \cdot r', \end{array}\right. \tag{6}$$

with r_i separable.

In each case, we showed that the system of two equations with the pair of coefficients (r_1, s_i) where r_1 separable, was equivalent with another system with a pair of coefficients consisting of a separable coefficient and such that the sum of the degrees of the coefficients decreased. If one of the coefficient is zero, we consider another equation, if applicable, of the conjunction. If both coefficients are nonzero, we repeat the procedure until either we considered all of the equations occurring in the conjunction, or one of the coefficient has degree zero which allows us to eliminate the variable u. $\square$

Proposition 6.3. *In T_V^{sep}, every $\mathcal{L}_V$ p.p. formula is equivalent to a positive quantifier-free formula.*

Proof: As usual, we proceed by induction on the number of existential quantifiers, so it suffices to consider a formula existential in just one variable $\exists u \phi(u, \mathbf{y})$, where $\phi(u, \mathbf{y})$ is a conjunction of atomic $\mathcal{L}_V$-formulas.

Note first that terms in u are L_A-terms in u, $\lambda_i(u)$, $i \in n^\ell$, for some $\ell \geq 1$, where λ_i denotes the composition of ℓ functions λ_j, $j \in n$ (see [5, Notation 3.3]). One uses the fact that the λ_i functions are additive and that $\lambda_i(u \cdot q(t))$ with $q(t) \in A$, can be expressed as an L_A-term in $\lambda_j(u)$, $j \in n$. Moreover since $u = \sum_{i \in n} \lambda_i(u) \cdot tc_i$, we may assume that the terms are terms in only the $\lambda_i(u)$, $i \in n^\ell$ (see [5] Lemma 3.2, and Notation 2.3).

Therefore we may replace the quantifier $\exists u$ by n^ℓ quantifiers $\exists u_{n^{\ell-1}} \cdots \exists u_0 \bigwedge_{i \in n^\ell} u_i = \lambda_i(u)$. We first tackle the quantifier $\exists u_0$ and for convenience, let us replace u_0 by u.

Since A is right Euclidean, we can always assume that we have at most one atomic formula involving u, of the form $u \cdot r_0(t) = t_0(\mathbf{y})$, where $t_0(\mathbf{y})$ is a $\mathcal{L}_A$-term.

Claim 6.4. We may assume that r_0 is separable.

Proof of Claim: Write $r_0 = t^m r'_0$, where $m \in \mathbb{N}$ and r'_0 separable. Express $r'_0 = \sum_{\boldsymbol{d} \in n^m} r'_{0\boldsymbol{d}} c_{\boldsymbol{d}}$ with the property that $r'_{0\boldsymbol{d}} \in A^{\sigma^m}[t;\sigma]$ e.g. $r'_{0\boldsymbol{d}} = \sum_j t^j a_{\boldsymbol{d}j}^{\sigma^m}$, with $a_{\boldsymbol{d}j} \in K$. Recall that $\sqrt[\sigma^m]{r'_{0\boldsymbol{d}}} = \sum_j t^j a_{\boldsymbol{d}j}$, so $t^m r'_0 = \sum_{\boldsymbol{d} \in n^m} \sqrt[\sigma^m]{r'_{0\boldsymbol{d}}} t^m c_{\boldsymbol{d}}$ (see Notation 2.3).

Using this equality, replace the atomic formula $u \cdot t^m r'_0 = t_0$ by the system

$$\bigwedge_{\boldsymbol{d} \in n^m} u \cdot \sqrt[\sigma^m]{r'_{0\boldsymbol{d}}} = \lambda_{\boldsymbol{d}}^{(m)}(t_0)$$

(see Notation 6.1). Note that for at least one tuple $\boldsymbol{d}$, $\sqrt[\sigma^m]{r'_{0\boldsymbol{d}}}$ is separable. So by Lemma 6.2, we may assume that we have just one equation with a separable coefficient together a conjunction of atomic formulas in $\bar{y}$. $\square$

Moreover, for any element $r(t) = \sum_j t^j a_j \in A$, there exists $\mu \in K$ such that $r(t)\mu \in \mathcal{I}$ (take $\mu := a_k^{-1}$, where $v(a_k) = \tilde{v}(r(t))$. So, we transform the atomic formula: $u \cdot r(t) = t(\mathbf{y})$ multiplying both sides by μ and we transform $u \cdot r(t) \equiv_\delta t(\mathbf{y})$, $\delta \in \Delta$, into $u \cdot r(t)\mu \equiv_{\delta+v(\mu)} t(\mathbf{y}) \cdot \mu$.

So we reduced ourselves to consider an existential formula of the form $\exists u\, \phi(u, \mathbf{y})$, where $\phi(u, \mathbf{y})$ is of the form

$$u \cdot r_0 = t_0(\mathbf{y}) \ \& \ \bigwedge_{k=1}^{m} u \cdot r_k \equiv_{\delta_k} t_k(\mathbf{y}) \ \& \ \theta(\mathbf{y})$$

with $r_k \in \mathcal{I}$, $\theta(\mathbf{y})$ a quantifier-free $\mathcal{L}_V$-formula, $t_k(\mathbf{y})$ are $\mathcal{L}_A$-terms, and $\delta_k \in \Delta$.

Note that in case $r_0 \neq 0$, we can always assume that $deg(r_0) > deg(r_k)$, for all k. Indeed, suppose that $deg(r_0) \leq deg(r_k)$, for some k, say $k = 1$. By the g.r. Euclidean algorithm in A_0, there exists $\mu \in \mathcal{O}_K$ such that $r_1\mu = r_0 r + r'_1$ with $deg(r'_1) < deg(r_0)$ and r, $r'_1 \in A_0$. So, we have that $u \cdot r_1\mu = u \cdot r_0 r + u \cdot r'_1 = t_0 \cdot r + u \cdot r'_1$, and we can replace $u \cdot r_1 \equiv_{\delta_1} t_1(\mathbf{y})$ by $u \cdot r'_1 + t_0(\mathbf{y}) \cdot r \equiv_{\delta_1 + v(\mu)} t_1(\mathbf{y}) \cdot \mu$.

First, we will assume that the equation present in $\phi(u, \mathbf{y})$, $u.r_0 = t_0(\mathbf{y})$, is non trivial, namely that $r_0 \neq 0$. We will concentrate on the system formed by this equation and the *congruences*. For ease of notation, we replaced $t_0(\mathbf{y})$ by b_0 and $t_i(\mathbf{y})$ by b_i. So consider a system of the form

$$\exists u \begin{cases} u \cdot r_0 & = & b_0 \\ \bigwedge_{i=1}^{d} u \cdot t^{n_i} r_i & \equiv_{\delta_i} & b_i, \end{cases} \tag{7}$$

where $r_0,\ r_i \in \mathcal{I}$, $r_0,\ r_i$ are σ-separable, $n_i \in \mathbb{N}$, $1 \leq i \leq d$.

We will call $\sum_{i=0}^{d} deg(r_i)$ the *separability degree* of that system, and we proceed by induction on that number.

We consider two cases : either there is $1 \leq i \leq d$ such that $n_i \geq 1$, or for all $1 \leq i \leq d$, $n_i = 0$. We will refer to the latter systems as *separable* systems, namely those for which r_0, r_i are σ-separable and $n_i = 0$ for all $1 \leq i \leq n$.

Case A: let $n_0 := \max\{n_i : 1 \leq i \leq d\}$ and suppose $n_0 \geq 1$. Then there exists δ such that the system (7) is equivalent to

$$\exists u \begin{cases} u \cdot r_0 & \equiv_{\delta} \quad b_0 \\ \bigwedge_{i=1}^{d} u \cdot t^{n_i} r_i & \equiv_{\delta_i} \quad b_i, \end{cases} \tag{8}$$

where δ is chosen such that $\Upsilon(r_0, \delta) \geq \max\{\tau^{-n_i}(\delta_i) : 1 \leq i \leq d\}$ (see Lemma 5.4). Then system (8) is equivalent to

$$\exists u \begin{cases} u \cdot r_0 t^{n_0} & \equiv_{\tau^{n_0}(\delta)} \quad b_0 \cdot t^{n_0} \\ \bigwedge_{i=1}^{d} u \cdot t^{n_i} r_i t^{n_0 - n_i} & \equiv_{\tau^{n_0 - n_i}(\delta_i)} \quad b_i \cdot t^{n_0 - n_i}, \end{cases} \tag{9}$$

We re-write system (9) as follows:

$$\exists u \begin{cases} u \cdot t^{n_0} r_0^{\sigma^{n_0}} & \equiv_{\tau^{n_0}(\delta)} \quad b_0 \cdot t^{n_0} \\ \bigwedge_{i=1}^{d} u \cdot t^{n_0} r_i^{\sigma^{n_0 - n_i}} & \equiv_{\tau^{n_0 - n_i}(\delta_i)} \quad b_i \cdot t^{n_0 - n_i}, \end{cases} \tag{10}$$

If all $r_i \in \mathcal{O}_K$, then we are done by Lemma 5.5. Otherwise we replace $u \cdot t^{n_0}$ by u_0 and we consider the following separable system of congruences, assuming that one of the $r_i \notin \mathcal{O}_K$:

$$\exists u_0 \begin{cases} u_0 \cdot r_0^{\sigma^{n_0}} & \equiv_{\tau^{n_0}(\delta)} \quad b_0 \cdot t^{n_0} \\ \bigwedge_{i=1}^{d} u_0 \cdot r_i^{\sigma^{n_0 - n_i}} & \equiv_{\tau^{n_0 - n_i}(\delta_i)} \quad b_i \cdot t^{n_0 - n_i}, \end{cases} \tag{11}$$

Suppose we can solve that system (see Lemma 5.3). Then by Lemma 4.13, there exists u such that $u \cdot t^{n_0} \equiv_{\delta_0} u_0$, where we can choose $\delta_0 \geq \max_{1 \leq i \leq d}\{\tau^{n_0}(\delta), \tau^{n_0 - n_i}(\delta_i)\}$.

Now we order the set $\{\Upsilon(r_i^{\sigma^{n_0 - n_i}}, \tau^{n_0 - n_i}(\delta_i)), \Upsilon(r_0^{\sigma^{n_0}}, \tau^{n_0}(\delta)) : 1 \leq i \leq d\}$ and we replace one of the congruences (the one corresponding to the maximum index) by the corresponding equation. Then we apply the g.r. Euclidean algorithm in order to obtain a system with separability degree strictly smaller than that of system (11).

Case B: suppose that for all $i, n_i = 0$. We have a system of the form

$$\exists u \begin{cases} u \cdot r_0 & = \quad b_0 \\ \bigwedge_{i=1}^{d} u \cdot r_i & \equiv_{\delta_i} \quad b_i, \end{cases} \tag{12}$$

We order the set $\{\Upsilon(r_i, \delta_i) : 1 \leq i \leq d\}$ and we show that system (12) is equivalent to another system where both the degrees of r_0 and of r_i, $1 \leq i \leq d$ have decreased. We proceed as in Lemma 5.2 with δ_2 replaced by the element of Δ realizing the maximum of the set above. Considering the equation with each of the congruences, we obtain a system where the coefficient appearing in the equation is separable and has degree strictly less than r_0 and the coefficients occurring in the congruences have each a strictly smaller degree but may no longer be separable. Then we use the previous Case A, to obtain an equivalent system where now all the coefficients of the congruences are separable and the degrees of the coefficients of both the equation and the congruences either stayed the same or have decreased. We obtain a system with strictly smaller separability degree.

Second, we will assume that there is no equation present in $\phi(u, \mathbf{y})$. So, we consider a system formed by *congruences* and for ease of notation, as before, we replace $t_i(\mathbf{y})$ by b_i. Consider a system of the form

$$\exists u \ \bigwedge_{i=1}^{d} u \cdot t^{n_i} r_i \ \equiv_{\delta_i} \ b_i, \tag{13}$$

where $r_i \in \mathcal{I}$, r_i is separable, $n_i \in \mathbb{N}$, $1 \leq i \leq d$.

Again we distinguish the two cases : either there is $1 \leq i \leq d$ such that $n_i \geq 1$, or for all $1 \leq i \leq d$, $n_i = 0$.

Case A': let $n_0 := \max\{n_i : 1 \leq i \leq d\}$ and suppose $n_0 \geq 1$. Then the system (13) is equivalent to

$$\exists u \ \bigwedge_{i=1}^{d} u \cdot t^{n_0} r_i t^{n_0 - n_i} \ \equiv_{\tau^{n_0-n_i}(\delta_i)} \ b_i \cdot t^{n_0-n_i} \tag{14}$$

First note that if all $r_i \in \mathcal{O}_K$, it implies since $r_i \in \mathcal{I}$, that $r_i^{-1} \in \mathcal{O}_K \cap \mathcal{I}$. In this case, w.l.o.g. we may assume that system (13) is of the form

$$\exists u \ \bigwedge_{i=1}^{d} u \cdot t^{n_i} \ \equiv_{\delta_i} \ b_i \tag{15}$$

System (15) is equivalent to the following system:

$$\exists u \ \bigwedge_{i=1}^{d} u \cdot t^{n_0} \ \equiv_{\tau^{n_0-n_i}(\delta_i)} \ b_i \cdot t^{n_0-n_i} \tag{16}$$

We order the elements $\{\tau^{n_0-n_i}(\delta_i) : \ 1 \leq i \leq n\}$. Let α be a permutation of $\{1, \cdots, n\}$ and suppose that $\tau^{n_0-n_{\alpha(1)}}(\delta_{\alpha(1)}) \leq \cdots \leq \tau^{n_0-n_{\alpha(n)}}(\delta_{\alpha(n)})$. We claim that system (16) is equivalent to:

$$\bigwedge_{i=1}^{d-1} b_{\alpha(i)} \cdot t^{n_0-n_{\alpha(i)}} \ \equiv_{\tau^{n_0-n_{\alpha(i)}}(\delta_{\alpha(i)})} \ b_{\alpha(i+1)} . t^{n_0-n_{\alpha(i+1)}} \tag{17}$$

We use Lemma 4.13 in order to find u such that $u \cdot t^{n_0} \equiv_{\tau^{n_0-n_j}(\delta_j)} b_j \cdot t^{n_0-n_j}$ and then we use the congruences.

Now assume that $r_i \notin \mathcal{O}_K$ for some i in system (14). Replace $u \cdot t^{n_0}$ by u_0 and consider the system :

$$\exists u_0 \ \bigwedge_{i=1}^{d} u_0 \cdot r_i^{\sigma^{n_0-n_i}} \ \equiv_{\tau^{n_0-n_i}(\delta_i)} \ b_i \cdot t^{n_0-n_i} \tag{18}$$

Suppose we can solve that system. Then by Lemma 4.13, there exists u such that $u \cdot t^{n_0} \equiv_{\delta_0} u_0$, where we can choose $\delta_0 \geq \max\{\tau^{n_0-n_i}(\delta_i) : \ 1 \leq i \leq d\}$.

Now we order the set $\{\Upsilon(r_i^{\sigma^{n_0-n_i}}, \tau^{n_0-n_i}(\delta_i)) : 1 \leq i \leq d\}$ and we replace one of the congruences by an equation, the one corresponding to the maximum index. So we are in the case of a separable system treated before.

Case B': supppose that for all $1 \leq i \leq d, n_i = 0$. We have the system

$$\exists u \ \bigwedge_{i=1}^{d} u \cdot r_i \equiv_{\delta_i} b_i \tag{19}$$

We order the set $\{\Upsilon(r_i, \delta_i) : 1 \leq i \leq d\}$ and we replace one of the congruences by an equation, the one corresponding to the maximum index. So we are in the case of a separable system treated before. □

Corollary 6.5. *In T_V^{sep}, any $\mathcal{L}_V$-formula is equivalent to a quantifier-free $\mathcal{L}_V$-formula up to index sentences.*

Recall that index sentences in particular tell us the sizes of the annihilators (of the separable) polynomials and the index of the subgroups $M_{\delta_1}.t^{n_1}/M_{\delta_2}.t^{n_2}$, with $\delta_1 < \delta_2$, $n_1, n_2 \in \mathbb{N}$. Also, the image of M by a L_A-term $u(x)$ with one free variable x is equal to $M.t^n$, for some $n \in \mathbb{N}$ and we can determine the n from the term $u(x)$, but since our language $\mathcal{L}_A$ contains λ functions, we need to consider terms in several variables.

In the next section, under the assumption that K is separably σ-linearly closed, we will show that if we add a list of axioms specifying the torsion to T_V^{sep}, then the torsion submodule is determined up to isomorphism (Corollary 7.10). Then in the last section, we will consider the class of torsion-free models of T_V^{sep} and we will show that any two elements are elementary equivalent (Corollary 8.3).

7. Torsion

In this section, we will work under the assumption that K is separably σ-linearly closed and that the action of Γ on Δ is transitive.

Let $\mathcal{M} \models T_V^{sep}$; denote by M_{tor} the submodule of M consisting of torsion elements. Note that $\mathcal{M}_{tor}$ is a $\mathcal{L}_A$-substructure of $\mathcal{M}$ ([5, Proposition 3.5]. Moreover by our quantifier elimination result (Proposition 6.3) M_{tor} is a pure submodule of M. So taking an ultrapower $\mathcal{M}^*$ of $\mathcal{M}$ which is $(|A| + \aleph_0)^+$-saturated, the corresponding ultrapower of $\mathcal{M}_{tor}$ is a direct summand of $\mathcal{M}^*$, namely $M^* = (M_{tor})^* \oplus M_{tf}$, where $\mathcal{M}_{tf}$ is a torsion-free A-module, an $\mathcal{L}_A$-substructure and a model of T_V^{sep}.

We will show that if we add to the theory T_V^{sep} a list of axioms specifying the torsion for each separable polynomial, then the submodule consisting of the torsion elements is unique up to isomorphism as an $\mathcal{L}$-substructure in any valued A-module, model of that extended theory.

We will show on one hand that we can determine all the valuations taken by the elements in the annihilator of a σ-separable polynomial belonging of $\mathcal{I}$ and on the other hand that given a non-zero element n of valuation δ and a separable polynomial $q(t) \in \mathcal{I}$, we can determine all the valuations taken by the elements m such that $m.q(t) = n$.

We will use axiom (2) of T_V^{sep} together with the factorization of such polynomials $q(t)$ into linear factors of the form $tb - 1$, $t - a$, c with $a, b, c \in \mathcal{O}_K$, $v(c) = 0, v(a) \geq 0, v(b) > 0$ (see Proposition 2.5).

Notation 7.1. Let $q(t) \in A$ and let $\mathcal{M}$ be an A-module, then denote $ann(q(t)) := \{m \in M : m \cdot q(t) = 0\}$.

Since we assume here that the action of Γ on Δ is transitive, the following Lemma is straightforward.

Lemma 7.2. *Let $\gamma \in \Gamma$, then there exists at most one $\delta \in \Delta$ such that $\tau(\delta) = \delta + \gamma$. Moreover if there exists δ_0 such that $\tau(\delta_0) = \delta_0 + \gamma$, then we have for $\delta > \delta_0$ that $\tau(\delta) > \delta + \gamma$ and for $\delta < \delta_0$ that $\tau(\delta) < \delta + \gamma$.* □

Notation 7.3. Let $\gamma \in \Gamma$ and suppose $\delta \in \Delta$ is such that $\tau(\delta) = \delta + \gamma$, then we will denote δ by $(\tau - 1)^{-1}(\gamma)$. In particular, $\tau((\tau - 1)^{-1}(\gamma)) = (\tau - 1)^{-1}(\gamma) + \gamma$.

Lemma 7.4. *Let $\mathcal{M}$ be a valued A-module and suppose that $\mathcal{M} \models T_V^{sep}$ and let $r(t) \in \mathcal{I}$ of degree 1.*

(1) *When $m \in ann(r(t))$, then $w(m)$ takes a unique value which can be expressed in terms of the values of the coefficients of $r(t)$.*

(2) *Let $n \in M - \{0\}$, then there exists $m \in M$ such that $n = m.r(t)$ and $w(m)$ can take at most two values which can be expressed in terms of $w(n)$ and the values of the coefficients of $r(t)$.*

Proof: We can restrict ourselves to consider $r(t)$ of the form $(t - a)$, or $(tb - 1)$ with $v(a) \geq 0$ and $v(b) > 0$.

(1) Suppose that $m \cdot (t - a) = 0$ with $m \neq 0$, then $m \cdot t = m \cdot a$ and so $\tau(w(m)) = w(m) + v(a)$. By Lemma 7.2, $w(m)$ is uniquely determined and we will use the above notation: $(\tau - 1)^{-1}(v(a))$.

Suppose now that $m \cdot (tb - 1) = 0$ and $m \neq 0$, then $m \cdot tb = m$ and so $\tau(w(m)) + v(b) = w(m)$. We denote $w(m)$ by $(\tau - 1)^{-1}(-v(b))$.

So in both cases, if there is such a non zero m, $w(m)$ can only take one value.

(2) Now let $n \in M$ with $w(n) = \delta \in \Delta$. By axiom (2) of T_V^{sep}, there exists m_0 such that $m_0 \cdot r(t) = n$ with $w(m_0) = \Upsilon(r(t), \delta)$ (and any other element m with $m \cdot r(t) = n$ differs from m_0 by an element of the annihilator of $r(t)$). Let us calculate explicitly $\Upsilon(r(t), \delta)$ in each case. Let $\Upsilon := \Upsilon(r(t), \delta)$.

First, let us consider the case $r(t) = (tb - 1)$.

Claim 7.5.
(i) If $\delta \geq (\tau - 1)^{-1}(-v(b))$, then $\Upsilon = \delta$.
(ii) If $\delta < (\tau - 1)^{-1}(-v(b))$, then $\Upsilon = \tau^{-1}(\delta - v(b))$.

Proof of Claim: By Lemma 7.2, there is at most one $\rho \in \Delta$ such that $\rho + (-v(b)) = \tau(\rho)$ and we have denoted such a ρ by $(\tau - 1)^{-1}(-v(b))$.

Moreover if $\rho' < (\tau-1)^{-1}(-v(b))$, then $\rho' > \tau(\rho') + v(b)$ and if $\rho' > (\tau-1)^{-1}(-v(b))$, then $\rho' < \tau(\rho') + v(b)$.

We compare δ, respectively Υ to $\tau(\delta) + v(b)$ (equivalently to $(\tau - 1)^{-1}(-v(b))$), respectively to $\tau(\Upsilon) + v(b)$.

Moreover, by definition Υ is such that $\delta = \min\{\Upsilon, \tau(\Upsilon) + v(b)\}$.

So, if $\Upsilon < \tau(\Upsilon) + v(b)$, then $\delta = \Upsilon$ and this corresponds to the case $\delta < \tau(\delta) + v(b)$ or equivalently to $\delta > (\tau - 1)^{-1}(-v(b))$.

If $\tau(\Upsilon) + v(b) < \Upsilon$, then $\delta = \tau(\Upsilon) + v(b)$ and so $\delta < \Upsilon$ and so $\tau(\delta) + v(b) < \tau(\Upsilon) + v(b) = \delta$, or equivalently $\delta < (\tau - 1)^{-1}(-v(b))$.

If $\tau(\Upsilon) + v(b) = \Upsilon$, then $\delta = \Upsilon$ and so $\delta = (\tau - 1)^{-1}(-v(b))$. □

Now suppose there exists $m \neq m_0$ such that $m \cdot r(t) = n$, equivalently assume we have $m_1 \in ann(r(t)) - \{0\}$. Then,

(i) If $\delta > (\tau - 1)^{-1}(-v(b))$, then $w(m_0 + m_1) = (\tau - 1)^{-1}(-v(b))$. So in this case we have two possible values for $w(m)$ with $m \cdot (tb - 1) = n$. In fact we have one element m_0 with $w(m_0) = \delta$ (and $m_0 \cdot r(t) = n$) and all the other elements m have value $(\tau - 1)^{-1}(-v(b))$.

(ii) If $\delta < (\tau-1)^{-1}(-v(b))$, then $\tau^{-1}(\delta - v(b)) < (\tau-1)^{-1}(-v(b))$. (Indeed, $\rho := (\tau-1)^{-1}(-v(b))$ is defined by: $\tau(\rho) + v(b) = \rho$. So we have to show that $\delta - v(b) < \tau(\rho)$, equivalently that $\delta < \tau(\rho) + v(b) = \rho$.)

So since $\Upsilon = \tau^{-1}(\delta - v(b))$, we have $w(m_0 + m_1) = \tau^{-1}(\delta - v(b))$ and in this case we have one possible value for $w(m)$ with $m.r(t) = n$.

(iii) If $\delta = (\tau-1)^{-1}(-v(b))$, then $w(m_0 + m_1) \geq (\tau-1)^{-1}(-v(b))$. Let us show that we have equality by way of contradiction.

Suppose that $w(m_0 + m_1) > (\tau-1)^{-1}(-v(b))$. By Lemma 7.2, this implies that $w(m_0 + m_1) < \tau(w(m_0 + m_1)) + v(b)$, then $w(n) = \delta = w(m_0 + m_1)$, a contradiction.

So, we get that $w(m_0 + m_1) = (\tau-1)^{-1}(-v(b)) = \Upsilon = w(m_0)$ and again in this case we have only one possible value for $w(m)$ with $m.r(t) = n$.

Second, let us consider the case $r(t) = (t - a)$.

Claim 7.6.

(i) If $\delta \geq (\tau-1)^{-1}(v(a))$, then $\Upsilon = \delta - v(a)$.

(ii) If $\delta < (\tau-1)^{-1}(v(a))$, then $\Upsilon = \tau^{-1}(\delta)$.

Proof of Claim: The proof is similar to the proof of the previous claim. But now, Υ is such that $\delta = \min\{\tau(\Upsilon), \Upsilon + v(a)\}$. As before, let m_0 such that $n = m_0 \cdot t - m \cdot a$ and $w(m_0) = \Upsilon$. So, we have that $\delta \geq \min\{\tau(\Upsilon), \Upsilon + v(a)\}$. We compare both Υ and δ to $\tau(\Upsilon) + v(a)$, respectively to $\tau(\delta) + v(a)$ and therefore also to $(\tau-1)^{-1}(v(a))$. □

Again any other solution m of $m \cdot (t + a) = n$ differs from m_0 by a non zero element m_1 of the annihilator of $r(t)$. Let us evaluate $w(m_0 + m_1)$.

(ia) If $(\tau-1)^{-1}(v(a)) < \delta < (\tau-1)^{-1}(v(a)) + v(a)$.

By the Claim, $w(m_0) = \delta - v(a)$, and so we have $w(m_0 + m_1) = \delta - v(a)$.

(ib) If $\delta > (\tau-1)^{-1}(v(a)) + v(a)$, then $w(m_0 + m_1) = (\tau-1)^{-1}(v(a)) = w(m_1)$.

(ic) If $\delta = (\tau-1)^{-1}(v(a)) + v(a)$, then $w(m_0 + m_1) \geq (\tau-1)^{-1}(v(a))$. Suppose that $w(m_0 + m_1) > (\tau-1)^{-1}(v(a))$, so $w(m_0 + m_1) > \delta - v(a)$. Since $(m_0 + m_1) \cdot (t + a) = n$, $w((m_0 + m_1) \cdot t) = \delta$. On the other hand, by Lemma 7.2, $\tau(w(m_0 + m_1)) > w(m_0 + m_1) + v(a) > \delta$, a contradiction. So, we also get in this case that $w(m_0 + m_1) = (\tau-1)^{-1}(v(a)) = w(m_1)$.

So in case (ia), we have two possible values depending on whether there is a non zero element in the annihilator of $r(t)$.

(ii) If $\delta < (\tau-1)^{-1}(v(a))$, then by the claim, $w(m_0) = \tau^{-1}(\delta)$; compare $\tau(w(m_0))$ to $\tau(w(m_1))(=\tau((\tau-1)^{-1}(v(a))))$.

We have $\tau((\tau-1)^{-1}(v(a))) = (\tau-1)^{-1}(v(a)) + v(a) > \delta$ (see Notation above) and so $w(m_0) < w(m_1)$ (τ respects $<$ on Δ) and so $w(m_0 + m_1) = w(m_0) = \tau^{-1}(\delta)$. So we have only one possible value.

(iii) If $\delta = (\tau-1)^{-1}(v(a))$, then $w(m_0) = \delta - v(a) = (\tau-1)^{-1}(v(a)) - v(a) < (\tau-1)^{-1}(v(a)) = w(m_1)$. Then $w(m_0 + m_1) = w(m_0) = \delta - v(a)$. So, again in this case we have only one possible value.

Note that in each case $w(m)$ can be expressed in terms of $w(n)$ and the values of the coefficients of $r(t)$. □

Proposition 7.7. *Let $\mathcal{M}$ be a valued A-module and suppose that $\mathcal{M} \models T_V^{sep}$, let $m \in M$ and let $q(t) \in \mathcal{I}$ of degree d. Then there is a finite subset $F_{q(t)} \subset \Delta$ of cardinality at most 2^{d-1} such that if $m \in ann(q(t)) - \{0\}$, then $w(m) \in F_{q(t)}$. (N.B. The elements of $F_{q(t)}$ whose values are taken by elements of $ann(q(t))$ only depend on the values of the coefficients of the factors of degree 1 of $q(t)$ and on which are the non-trivial annihilators in M.)*

Proof: We proceed by induction on d. For polynomials of degree 1, this is the content of Lemma 7.4. Let us assume $d \geq 2$. By hypothesis on K, $q(t) = r(t)q_1(t)$, where $r(t)$, $q_1(t) \in \mathcal{I}$ and $r(t)$ has degree 1. Now $m \cdot q(t) = 0$ is equivalent to $m \cdot r(t) = 0$ or $m \cdot r(t) \in ann(q_1(t)) - \{0\}$. Since degree of $q_1(t)$ is strictly less than d, we can apply the induction hypothesis and so we get at most 2^{d-2} possible values for the elements in $ann(q_1(t)) - \{0\}$. By axiom (2) of T_V^{sep}, for each $n \in ann(q_1(t)) - \{0\}$, there is an element m such that $m \cdot r(t) = n$ and by Lemma 7.4, for each of the values $w(n)$, we get at most 2 values for $w(m)$. □

Proposition 7.8. *Let $\mathcal{M}$ be a valued A-module and assume that $\mathcal{M} \models T_V^{sep}$. Let $q(t) \in \mathcal{I}$ of degree d and let $n \in M - \{0\}$. Then one can determine a finite set $G_{q(t)} \subset \Delta$ of cardinality at most 2^d such that $w(m) \in G_{q(t)}$ if and only if $m \cdot q(t) = n$, $m \in M$. Moreover, $G_{q(t)}$ only depends on $w(n), q(t)$ and on which are the non-trivial annihilators in M.*

Proof: We proceed by induction on $d \geq 1$. For polynomials of degree 1, this is the content of Lemma 7.4. So let us assume $d \geq 2$. By hypothesis on K, $q(t) = q_2(t)r(t)$, where $r(t)$ has degree 1. Now $m \cdot q(t) = n$ is equivalent to $m' \cdot r(t) = n$ and $m \cdot q_2(t) = m'$. By Lemma 7.4, given $w(n)$, we know that there is either one or two values for $w(m')$ with $m' \cdot r(t) = n$ depending on the respective positions of the values of the coefficients of $r(t)$ and $w(n)$, together whether $ann(r(t))$ is non-trivial. By axiom 2 of T_V^{sep}, there exists m such that $m \cdot q_2(t) = m'$. Then we apply the induction hypothesis to $q_2(t)$, so given each of these values for $w(m')$, the number of values of such m are bounded by 2^{d-1} (and we can determine the exact number which depends on the relative position on the chain Δ of the values of the coefficients and δ together with which are the non-trivial annihilators). □

Now we extend T_V^{sep} by specifying the torsion in our models. Note that in considering $ann(q(t))$, we may always assume that $q(t) \in \mathcal{I}$, also that annihilators are $Fix(\sigma)$-vector spaces. If $Fix(\sigma)$ is infinite then if we have two annihilators with one strictly included in the other, then the index is infinite ([2, Lemma 2.4 (in Corrigendum)]). So in this case we will add to the theory T_V^{sep} a list of axioms specifying which annihilators are non-trivial.

From now on let us assume $Fix(\sigma)$ is finite. For instance, in the case where K is a (valued) field of characteristic p and σ is the Frobenius endomorphism (or a power of it), then $Fix(\sigma)$ is finite. We will specify the torsion as follows.

Definition 7.9. Let T_{tor} be the theory of A-modules together with the following scheme of axioms, for each element $q \in \mathcal{I}$ of degree d: there exist exactly d elements $x_1, \cdots, x_d$ which are linearly independent over $Fix(\sigma)$ and such that $x_i \cdot q = 0$, for all $1 \leq i \leq d$.

We will consider now the theory $T_V^{sep} \cup T_{tor}$. A model of that theory is for instance the separable closure of K.

Corollary 7.10. *Let $\mathcal{M}, \mathcal{N}$ be two valued A-modules, models of $T_V^{sep} \cup T_{tor}$ containing, respectively, isomorphic $\mathcal{L}_V$-structures $\mathcal{M}_0$, $\mathcal{N}_0$. Then we may extend this partial isomorphism to a minimal submodel $\mathcal{M}_0^{sep}$ of $\mathcal{M}$ containing $\mathcal{M}_0$.*

Proof: We follow the proof of [5, Proposition 5.8] and we apply Proposition 7.8. In particular, a main ingredient in [5, Proposition 5.8] is the following (see [5, Lemma 5.1]). Let $\mathcal{N}$ be an $\mathcal{L}_A$-structure and $\mathcal{N}_0$ a substructure of $\mathcal{N}$. Let $u \in N - N_0$ and assume that $u \cdot q \in N_0$ for some $q \in \mathcal{I}$. Then there is a unique (up to multiplication by elements of $Fix(\sigma)$) element q_u of $\mathcal{I}$ such that $u \cdot q_u \in N_0$. □

8. Torsion-free models of T_V^{sep}

In this section, we show that the theory of torsion-free models of T_V^{sep} is complete, and we specify the other completions of T_V^{sep} (see Corollary 8.3).

Let $\mathcal{M} \models T_V^{sep}$ and assume that $\mathcal{M}$ is a torsion-free A-module. Given G_0, G_1 two p.p. definable subgroups of $\mathcal{M}$, we wish to determinate the index of $[G_0 : G_1]$.

Here we will assume that the residue field $\bar{K}$ is infinite, which is the case if K is separably closed of characteristic p and finite (non-zero) imperfection degree, and so $\bar{K}$ is algebraically closed (and so infinite) and that σ acts on K as the Frobenius, or more generally if (K, σ) is separably σ-linearly closed.

In the case where the map w is surjective and the action of Γ is transitive on Δ, we note that certain p.p. definable subgroups have infinite index. For instance, the index $[M_{0_\Delta} : M_{0_\Delta^+}]$ is infinite, so there is an element $a \in M$ with $w(a) = 0$ and multiplying a by elements of $\mathcal{O}_K - \mathfrak{m}_K$ with different images in $\bar{K}$, we get elements of M_{0_Δ} in distinct cosets modulo $M_{0_\Delta^+}$ (and so a fortiori modulo any M_δ with $\delta > 0$). Since the action of Γ is transitive on Δ, we get the indices $[M_\delta : M_{\delta^+}]$ are also infinite, for any $\delta \in \Delta$. Also the index of $M.t$ in M is infinite (this follows from the fact that K^σ is infinite) ([6, Proposition 3.2]) and the index of $M.t^{m+1}$ in $M.t^m$ as well.

Now let us consider the general case.

Proposition 8.1. *Let $\mathcal{M}$ be a model of T_V^{sep} and assume that $\mathcal{M}$ is a torsion-free A-module. Then the index of any two p.p. definable subgroups $G_1 \subsetneqq G_0$ of M is infinite.*

Proof: By the positive q.e. result (see Proposition 6.3), a p.p. definable subgroup of $\mathcal{M}$ is defined by a positive quantifier-free formula of the following kind: $\bigwedge_i t_i(u) = 0$ & $\bigwedge_j t_j(u) \equiv_{\delta_j} 0$, where t_i, t_j are $\mathcal{L}_A$-terms and u one free variable. As noted before a $\mathcal{L}_A$-term $t(u)$ is a L_A-term in u, $\lambda_i(u)$, $i \in n^\ell$, for some $\ell \geq 1$. Using the decomposition $tq = \sum_i \sqrt[\sigma]{q_i} tc_i$ (see Notation 2.2), we may assume that each equation contains a separable coefficient. (This process increases the number of equations but decreases the degree of the coefficients, so it will eventually terminate.) Note that if we have two equations containing a separable coefficient for $\lambda_i(u)$, say q_0, q_1, then by multiplying by an element of $K^\times$, we may assume that $q_0 \in \mathcal{I}$ (respectively $q_1 \in \mathcal{I}$). By the Ore property of A_0, there exist q_0', q_1' such that $q_0 q_1' = q_1 q_0'$ and note that we may choose $q_0', q_1' \in \mathcal{I}$. So we may just keep one equation with $\lambda_i(u)$ and assume that it has a separable coefficient. Continuing like that we get a lower triangular $d \times k$-matrix D with non zero separable coefficients on its diagonal. Recall that such matrix was called *lower triangular separable* (l.t.s.) of co-rank ℓ in [5, Definition 9] where $d-\ell$ is the number of non-zero (or equivalently separable) elements on its diagonal. We will call the corresponding system a l.t.s. system.

So we may assume that G_0 (respectively G_1) is defined by a l.t.s. system of co-rank $\ell \leq d$ (respectively $\ell_1 \leq \ell$) equations (on the same variables) with in addition congruences conditions.

Set $u_{d-\ell} := \lambda_m(u)$, where $\lambda_m(u)$ is the variable multiplied by the $(d-\ell, d-\ell)$-coefficient of D. Let δ_i, $i \in m$, be the elements of Δ occurring in the congruences.

Since $G_1 \subseteq G_0$, we may assume that G_1 is defined by a l.t.s. system of co-rank $\ell_1 \leq \ell$ and with the same first $d-\ell$ equations.

Claim 8.2. Let $q \in \mathcal{I}$, then in any torsion-free model of T_V^{sep}, we have the following equivalence: $u \equiv_\delta 0 \leftrightarrow u \cdot t^n q \equiv_{\Upsilon^{-1}(q,\tau^n(\delta))} 0$.

Proof of Claim: By definition of Υ^{-1} (see Notation 4.6), we get in any valued A-module, that $u \equiv_\delta 0 \to u \cdot t^n q \equiv_{\Upsilon^{-1}(q,\tau^n(\delta))} 0$. Now assume that $w(u \cdot t^n q) \geq \Upsilon^{-1}(q, \tau^n(\delta))$, since $q \in \mathcal{I}$, there exists u' with $w(u') \geq \tau^n(\delta)$ (Υ is increasing, see Lemma 4.5) such that $u' \cdot q = u \cdot t^n q$. Since $\mathcal{M}$ is torsion-free, $u' = u \cdot t^n$ and so $w(u) \geq \delta$. $\square$

So if we have a l.t.s. system of co-rank $\ell \leq d$, with $d \geq 1$ and if u_m, $1 \leq m \leq d-\ell$, occurs in an equation with a separable coefficient and if it also occurs in a congruence condition, we may replace it in the congruences conditions in terms of the other variables (using the Ore property of A_0 and the Claim above). So w.l.o.g., we may assume that in the congruences none of the variables u_m, $1 \leq m \leq d-\ell$ occur.

First we assume that $\ell_1 < \ell$, so we have at least one more equation in G_1 of the form $u_{d-\ell+1} \cdot q_{d-\ell+1} + \cdots = 0$ $(\star)$. Denote by $\delta(G_0)$ the minimum $\delta \in \Delta$ such that any tuple of elements each in M_δ satisfies the congruence conditions appearing in the definition of G_0 and denote by χ the conjunction of these congruences.

Take $\overrightarrow{u} = (u_{d-\ell+1}, \overrightarrow{u_0})$ satisfying χ and verify whether $\overrightarrow{u}$ satisfies equation $(\star)$. If it does, add to $u_{d-\ell+1}$ a non-zero element $u_{\delta,1} \in M_{\delta(G_0)}$. So the tuple $(u_{n-\ell+1} + u_{\delta,1}, \overrightarrow{u_0})$ still satisfies χ (since $(u_{\delta,1}, \overrightarrow{0})$ satisfies χ) but no longer equation $(\star)$ ($u_{d-\ell+1}$ was uniquely determined in terms of $\overrightarrow{u_0}$ since we are in a torsion-free module). Since the system is l.t.s. and $\mathcal{M} \models T_V^{sep}$, we may find $(u_1, \cdots, u_{d-\ell})$ such that $(u_1, \cdots, u_{d-\ell}, u_{d-\ell+1} + u_{\delta,1}, \overrightarrow{u_0}) \in G_0$, then choose in $M_{\delta(G_0)}$ a non zero $u_{\delta,2} \neq u_{\delta,1}$. The tuple $(u_{d-\ell+1} + u_{\delta,2}, \overrightarrow{u_0})$ still satisfies χ but no longer equation $(\star)$. Again we may complete this tuple to $(u_1', \cdots, u_{d-\ell}', u_{d-\ell+1} + u_{\delta,2}, \overrightarrow{u_0}) \in G_0$, and so $(u_1' - u_1, \cdots, u_{d-\ell}' - u_{d-\ell}, u_{\delta,2} - u_{\delta,1}, \overrightarrow{0}) \in G_0 - G_1$, (since $(u_{\delta,2} - u_{\delta,1}, \overrightarrow{0})$ does not satisfy equation $(\star)$) and we may continue infinitely often since $M_{\delta(G_0)}$ is infinite.

Then assume that $\ell_1 = \ell$ and so that the l.t.s. system of equations occurring in the definition of G_0 and G_1 is the same.

Then using the Claim 8.2, we triangularize further the system of congruences as follows.

Suppose we have the following system of two congruences with $r_1, r_2 \in \mathcal{I}$ and $\delta_1 \geq \delta_2 \in \Delta$, occurring in the definition of G_0 and for simplicity re-index the variables by $u_0, u_1, \cdots$.

$$\begin{cases} u_0 \cdot r_1 + u_1 \cdot r_3 + \cdots & \equiv_{\delta_1} \quad 0 \\ u_0 \cdot r_2 + u_1 \cdot r_4 + \cdots & \equiv_{\delta_2} \quad 0 \end{cases} \tag{20}$$

By the right Ore property of A_0, there exist $q_1, q_2 \in A_0$ (and we may assume that they belong to $\mathcal{I}$) such that $r_1 q_2 = r_2 q_1$.

Assume $\Upsilon^{-1}(q_2, \delta_1) \leq \Upsilon^{-1}(q_1, \delta_2)$ (the other case is similar). Then system (20) is equivalent to:

$$\begin{cases} u_0 \cdot r_2 + u_1 \cdot r_4 + \cdots & \equiv_{\delta_2} \quad 0 \\ u_1 \cdot (r_3 q_2 - r_4 q_1) + \cdots & \equiv_{\Upsilon^{-1}(q_2,\delta_1)} \quad 0 \end{cases} \tag{21}$$

Given a solution of the system (21), we multiply the first equation by q_1 and get $u_0 \cdot r_2 q_1 + u_1 \cdot r_4 q_1 + \cdots \equiv_{\Upsilon^{-1}(q_1,\delta_2)} 0$ and add the second equation to get $u_0 \cdot r_1 q_2 + u_1 \cdot r_3 q_2 + \cdots \equiv_{\Upsilon^{-1}(q_2,\delta_1)} 0$. This last equation is equivalent by Claim 8.2 to $u_0 \cdot r_1 + u_1 \cdot r_3 + \cdots \equiv_{\delta_1} 0$.

So, w.l.o.g. we may assume that we have the following system of two congruences with $r_1, r_2 \in \mathcal{I}$, the first one being the last one occurring in the definition of G_0 and the second one being the first one in G_1 and for simplicity re-index the variables by $u_0, u_1, \cdots$.

$$\begin{cases} u_0 \cdot r_1 + u_1 \cdot r_3 + \cdots & \equiv_{\delta_1} \quad 0 \\ u_1 \cdot r_4 + \cdots & \equiv_{\delta_2} \quad 0 \end{cases} \tag{22}$$

First choose $(u_1, \overrightarrow{0})$ such that the second equation does not hold (take $u_1 \notin M_{\Upsilon^{-1}(r_4,\delta_2)}$). Then choose u_0 such that $u_0 \cdot r_1 + u_1 \cdot r_3 + \cdots = 0$, which is possible whenever r_1 is separable and in the case r_1 is of the form $t^m.r_1'$ with $r_1' \in \mathcal{I}$ separable and $m \geq 1$, we proceed as follows. First choose, u_0' such that $u_0' \cdot r_1' + u_1 \cdot r_3 + \cdots = 0$ holds and then u_0 with $w(u_0 - u_0') \geq \delta_1$ (see Lemma 4.13). So in both cases, $(u_0, u_1, \overrightarrow{0})$ satisfies (22) and also the conjunction χ of congruences occurring in the definition of G_0 but doesn't satisfy those occurring in the definition of G_1. Then continue to solve the system in order to get an element in G_0 using the fact we have put the system in triangular form. Finally choose infinitely many such u_1 not congruent modulo $M_{\Upsilon^{-1}(r_4,\delta_2)}$. □

Corollary 8.3. *Assume that K is separably σ-linearly closed and that the action of Γ on Δ is transitive. Then, the theory of torsion-free (as A-modules) models of T_V^{sep} is complete.*

Proof: For the first statement, apply Proposition 6.5 and Proposition 8.1. □

Corollary 8.4. *Under the same hypotheses on K, Δ and Γ, let $\mathcal{M}$, $\mathcal{N}$ be two valued A-modules. Suppose that their $\mathcal{L}_V$-reducts are models of $T_V^{sep} \cup T_{tor}$ and that $Fix(\sigma)$ is finite. Then $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent as $\mathcal{L}_V$-structures.*

Suppose that $Fix(\sigma)$ is infinite, and let T_{an} be any extension of T_V^{sep} containing the list of axioms telling for each separable $q \in \mathcal{I}$ whether the annihilator of that element *is trivial or not. Suppose that the $\mathcal{L}_V$-reduct of $\mathcal{M}$ respectively $\mathcal{N}$ are models of T_{an}, then they are elementarily equivalent.*

Proof: For the first statement, apply Proposition 6.5, Corollary 7.10 and Proposition 8.1. For the second one, apply Proposition 6.5 and Proposition 8.1. □

Acknowledgments: We would like to thank the referee for his/her useful remarks.

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