Ann. Sci. Math. Québec **xx** (xxxx), no x, x–x

PANORAMA OF P-ADIC MODEL THEORY

LUC BÉLAIR

RÉSUMÉ. Nous donnons un aperçu des développements de la théorie des modèles des nombres p-adiques depuis les travaux de Denef sur la rationalité de séries de Poincaré, par une revue de la bibliographie.

ABSTRACT. We survey the literature in the model theory of p-adic numbers since Denef's work on the rationality of Poincaré series.

« Je serai trop récompensé de mon travail, si je puis me flatter de contribuer aux progrès des jeunes Géomètres. La gloire des inventeurs est plus brillante sans doute ; mais serait-on Citoyen, si l'on ne préférait la satisfaction d'être utile à l'honneur d'être admiré ? » Bougainville, *Traité du calcul intégral*, 1754.

1. Introduction

Macintyre [164] described vividly the first twenty years of the model-theoretical study of the field of p-adic numbers since the breakthroughs by Ax-Kochen and Ershov around 1964. In this paper I will survey the literature of p-adic model theory since Denef's work [9] on the rationality of Poincaré series. I will try to start roughly where Macintyre's survey paper left off (which is more or less [9]), and not intersect with it.

In writing this paper, there was the issue of delimiting the subject matter. I have interpreted "p-adic" in a somewhat narrow sense : I have tried to restrict to results which touched on directly and/or specifically the classical p-adic field $\mathbb{Q}_p$.

The bibliography is fairly extensive, without being exhaustive, even though we feel most of the literature would be covered through cross-references. After some reflexion, it seemed of interest to collect references up to 1984, which (almost) corresponds to the time period covered by Macintyre's paper, and again we do not claim to be exhaustive. We have listed that time period separately, taking 1984 as a somewhat arbitrary dividing point, even though it coincides with Denef's famous paper.

We will recall some basic facts and fundamental results, as they will be useful to enlighten some of our later remarks.

I think it is fair to say that Denef's 1984 paper is probably the most influential work during the time period under study, and is a landmark in the coming of age of model

Reçu le xxxxx et, sous forme définitive, le xxxxx.

theory in its relation with the rest of mathematics in the intervening years. And the work which grew out of that paper is still a main influence.

The mainstrean of model theory now flows through geometrical stability theory. The field $\mathbb{Q}_p$ is certainly not a stable structure, but it retains nevertheless (like $\mathbb{R}$) a property of stable structures : a set definable with parameters taken from an elementary extension is already definable with parameters from $\mathbb{Q}_p$ [84]. It seems that the development of a finer analysis of forking has brought many unstable structures within the reach of its methods, e.g. structures without the independence property, to which the field $\mathbb{Q}_p$ belongs. These ideas have already emerged into a deeper analysis of the model theory of algebraically closed valued fields (see [133]), and had an impact on $\mathbb{Q}_p$ (see [138]). No doubt there is more to come for $\mathbb{Q}_p$ in this direction.

We now summarize the organization of the paper. In section 2 we recall some basic facts about the p-adic numbers. In section 3 we recall the fundamental results for the model theory of $\mathbb{Q}_p$, namely the axiomatization of its first-order theory, Macintyre's quantifier elimination and Denef's cell decomposition. In section 4, we review some key features in the definability theory : angular components, the independence property, imaginaries, p-minimality, and definable groups. Section 5 is devoted to p-adic and motivic integration, section 6 to geometry in a broad sense and section 7 to algorithmics. We have gathered in section 8 a certain number of other topics.

We will use boldface notation for tuples, e.g. $\mathbf{x} = (x_1, \ldots, x_n)$. We will denote a valued field by a couple (K, v) where K is the underlying field and v the valuation map, we denote its value group by vK, its valuation ring by $\mathcal{O}_K$ and its residue field by $\bar{K}$. The natural residue map from $\mathcal{O}_K$ to $\bar{K}$ is denoted by $^-$, so the residue of x is $\bar{x}$.

Experience has shown that multisorted languages allow to bring out the basic intuition that a valued field is somewhat an extension of the residue field by the valued group, through many precise forms of elimination (of quantifiers, or else) of the base field sort *relative to other sorts*. The simplest case is relative to the residue field and value group. We will denote by L the basic language of rings $(+, -, \cdot, 0, 1)$, and by L_{vf} the basic 2-sorted language of valued fields, with a sort for the base field (construed as a ring), a sort for the value group (construed as an ordered group), and a symbol for the valuation map. Let $L_{\mathcal{O}}$ be the language obtained from L by adding a unary predicate $\mathcal{O}$ for the valuation ring, and let L_{Div} be the language obtained from L by adding a binary predicate Div for a " divisibility relation", interpreted as $Div(x, y) \leftrightarrow v(x) \leq v(y)$. Let L_v be the 3-sorted language with a sort for the base field, a sort for the value group and a sort for the residue field, and a symbol for the valuation map and the residue map.

For model theory we refer to the books of Marker [167] and Poizat [183]. For valuations we refer to Engler and Prestel's book [110]. For various basic properties of $\mathbb{Q}_p$ we refer to [187].

2. The p-adic numbers

2.1. Hensel's numbers

Let p be a prime number. The p-adic numbers were introduced by Kurt Hensel at the end of the 19th century, with the aim of transposing to number theory the methods

of power series expansions used in the theory of functions of a complex variable (see [193], [210]). The p-adic numbers form a field, usually denoted by $\mathbb{Q}_p$, and can be described in many ways. Fix p and consider the ring formed by infinite sums of the form $a_0 + a_1 p + \ldots + a_n p^n + \ldots$, with $a_i \in \mathbb{Z}, 0 \leq a_i < p$, and where addition and multiplication are performed *in base* p in the natural way. It turns out to be an integral domain of characteristic 0. It is the *ring of* p*-adic integers*, denoted by $\mathbb{Z}_p$. Note that any positive integer $1, 2, 3, \ldots$ can be written as such a *finite* sum ; in contrast we have $-1 = (p-1) + (p-1)p + \ldots + (p-1)p^n + \ldots$. One checks that e.g. $1-p$ is invertible in $\mathbb{Z}_p$, namely $(1-p)^{-1} = 1 + p + p^2 + p^3 + \ldots + p^n + \ldots$. The field $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$. One sees that for any positive integer n the quotient ring $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is canonically isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, the ring of integers modulo p^n. In particular $\mathbb{Z}_p/p\mathbb{Z}_p$ is canonically isomorphic to the finite field $\mathbb{F}_p$ with p elements, so the principal ideal $p\mathbb{Z}_p$ is maximal. In fact, as in the power series ring over a field, a p-adic integer $a_0 + a_1 p + \ldots + a_n p^n + \ldots$ is invertible in $\mathbb{Z}_p$ if and only if $a_0 \neq 0$. In other words $p\mathbb{Z}_p$ coincides with the noninvertible elements [1]. In particular note that any nonzero p-adic integer a can be expressed as $a = p^k u$ where $k \in \mathbb{N}$ and $u \in \mathbb{Z}_p$ is invertible, so that any p-adic number $x \in \mathbb{Q}_p$ can expressed as $x = p^N u$ with $N \in \mathbb{Z}$ and $u \in \mathbb{Z}_p$ as above, in other words $x = a_N p^N + a_{N+1} p^{N+1} + \ldots, a_N \neq 0$. We define the map $v_p : \mathbb{Q}_p \setminus \{0\} \to \mathbb{Z}$ as follows, for $x = a_N p^N + a_{N+1} p^{N+1} + \ldots, a_N \neq 0$, we set $v_p(x) = N$. For $a \in \mathbb{Z}$, $v_p(a)$ is the exponent of the highest power of p which divides a. The (valuation) map v_p has the following properties : $v_p(1) = 0$, $v_p(xy) = v_p(x) + v_p(y)$, $v_p(x+y) \geq \min(v_p(x), v_p(y))$. It can be used to define a norm $|\quad|_p$ on $\mathbb{Q}_p$, namely $|x|_p = p^{-v_p(x)}$. It is a key feature that this norm is *nonarchimedean* : $|(x+y)|_p \leq \max(|x|_p, |y|_p))$.

One checks that $(\mathbb{Q}_p, |\quad|_p)$ is the Cauchy completion of $\mathbb{Q}$ with respect to this norm, and that $\mathbb{Z}_p$ is the projective limit of the system of natural projections $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$, $x + p^n\mathbb{Z} \mapsto x + p^m\mathbb{Z}$, when $n \geq m, n, m \in \mathbb{N}$. We get a topological normed field. The unit ball is $\mathbb{Z}_p$, which is not only closed but also open. This makes $\mathbb{Q}_p$ totally disconnected. It is also the case that the $|\quad|_p$ topology is the same as the topology inherited from the discrete topology on the $\mathbb{Z}/p^n$ via the projective limit, which makes $\mathbb{Z}_p$ compact. So $(\mathbb{Q}_p, |\quad|_p)$ is locally compact. It is one of the classical *local fields*, the others being the finite extensions of the $\mathbb{Q}_p$'s with the induced norm, $\mathbb{R}$ and $\mathbb{C}$ with their standard norm, and the Laurent series $\mathbb{F}((T))$, with $\mathbb{F}$ a finite field, with *the order at* 0 norm.

Example 2.1. Let $p = 3$, then 2^{3^n} is a Cauchy sequence and it converges to -1. In general for $0 < j \leq p-1, j \in \mathbb{N}$, j^{p^n} is a Cauchy sequence and it converges to a $(p-1)$-th root of 1.

It turns out that to solve polynomial equations in the p-adic numbers, Newton's tangent method can be used systematically. This takes the form of the famous "Hensel's lemma", of which we state the following form : let $f(X) \in \mathbb{Z}_p[X]$ and $a \in \mathbb{Z}_p$ such that $v_p(f(a)) > 2v(f'(a))$, then there exists $x \in \mathbb{Z}_p$ such that $f(x) = 0$ and $v_p(x-a) > v_p(f'(a))$. For example, fix any positive integer $n \geq 2$ and let $b \in \mathbb{Q}_p$ such

1. And hence is the only maximal ideal, so $\mathbb{Z}_p$ is a so called *local ring*.

that $v_p(b) = 0$ and consider the equation $X^n - b = 0$. By Hensel's lemma, this equation has a solution if and only if it has a solution in the residue ring $\mathbb{Z}_p/p^{2v_p(n)+1}\mathbb{Z}_p = \mathbb{Z}/p^{2v_p(n)+1}\mathbb{Z}$. A simple calculation insures that this is enough to see that the multiplicative group of n-th powers has finite index in $\mathbb{Q}_p^\times$, with a set of representatives among the (ordinary) integers.

It is instructive to note that Hensel's lemma makes $\mathbb{Z}_p$ algebraically definable in $\mathbb{Q}_p$, as follows [2] :

$$\mathbb{Z}_p = \{y \in \mathbb{Q}_p : \exists t \in \mathbb{Q}_p, t^2 = 1 + p^3y^4\} \tag{1}$$

Interestingly, uniform $\forall\exists$ or $\exists\forall$ algebraic definitions of $\mathbb{Z}_p$ in $\mathbb{Q}_p$ and $\mathbb{F}_p[[T]]$ in $\mathbb{F}_p((T))$ which work for any p and the ring of integral elements in any finite extensions are now known (see [69]).

Let U_p be the group of units in $\mathbb{Z}_p$, and let $\mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$. Then $\mathbb{Q}_p^\times/U_p$ is isomorphic to the group $p^{\mathbb{Z}}$, which is order-isomorphic to the additive group of the integers, and the map v_p is essentially given by the quotient map $\mathbb{Q}_p^\times \to \mathbb{Q}_p^\times/U_p$. The subring $\mathbb{Z}_p$ of $\mathbb{Q}_p$ has the property that for any $x \in \mathbb{Q}_p$, $x \in \mathbb{Z}_p$ or $x^{-1} \in \mathbb{Z}_p$ [3], which is enough to ensure that the divisibility relation in $\mathbb{Z}_p$ makes the quotient group $\mathbb{Q}_p^\times/U_p$ into an ordered abelian group, whose order is in fact the same as above. By the formula (1) above, any elementary extension of $\mathbb{Q}_p$, or elementary equivalent field, will carry a definable subring mimicking these features of $\mathbb{Z}_p$, which will make it a field *with a valuation*, i.e. a valued field.

2.2. Analogy with other classical local fields

One of the main driving force in studying $\mathbb{Q}_p$ is the pursuit of analogies with the other classical local fields. This is the main theme in [164], to which we refer. In the case of analogies with the Laurent series $\mathbb{F}_p((T))$, maybe the most fruitful model-theoretically is the search for uniformity in p, which takes the form of passing to nonprincipal ultraproducts of the $\mathbb{Q}_p$ over the prime numbers p. We know from the Ax-Kochen-Ershov analysis that they are elementary equivalent as valued fields to the corresponding ultraproducts of the $\mathbb{F}_p((T))$. The analogies with the real numbers $\mathbb{R}$ are multifaceted. For example, any infinite subset of $\mathbb{Q}_p$ definable in the ring language has non-empty interior. In the period under study, one might say that the analogy with $\mathbb{C}$ has emerged, through the analysis of imaginaries in algebraically closed valued fields.

3. Basic fundamental results

3.1. Axioms, p-adically closed fields

A valued field is called p-adically closed if it satisfies the following axioms : it is of characteristic zero with $v(p)$ the least positive element in the value group, henselian [4], with residue field the prime field of characteristic p, and with value group a $\mathbb{Z}$-group. Recall that $\mathbb{Z}$-groups are axiomatized as abelian ordered groups with a smallest positive element 1 (here 1 = valuation of the prime p) and satisfying the axiom scheme

2. This formula has the elegance of working for all p. E.g., for p odd, $\exists t(t^2 = 1 + py^2)$ would do.
3. I.e. it is a valuation ring.
4. I.e. it satisfies Hensel's lemma, the valuation ring replacing $\mathbb{Z}_p$.

$\forall x \exists y \vee_{0 \leq r < n} (x = n.y + r.1)$. We know from the work of Ax-Kochen and Ershov that the first-order theory of the structure $(\mathbb{Q}_p, v_p)$ is axiomatized by these axioms. The axioms of p-adically closed fields are therefore complete, and decidable.

We will denote by pCF this set of axioms [5], without fixing the exact language of valued fields. The appropriate language should be made precise by context in further discussions.

In a remarkable paper, Koenigsmann ([147]), and independently for odd p Efrat [105] and Pop (see [147]), characterized the first-order L-theory of $\mathbb{Q}_p$ by its absolute Galois group [6] : a field is elementary equivalent to $\mathbb{Q}_p$ if and only if its absolute Galois group is isomorphic to that of $\mathbb{Q}_p$. This is still another, beautiful, analogy with $\mathbb{R}$. It was used in [149].

Note that a field elementary equivalent to a finite extension of $\mathbb{Q}_p$ is not necessarily a finite extension of a p-adically closed field [47].

3.2. Macintyre's elimination of quantifiers

We recall that elimination of quantifiers was already considered and proved in Ax and Kochen's third paper, using a cross-section map. This is (indeed) a section of the valuation map, from the value group to the multiplicative group, which materializes in $\mathbb{Q}_p$ as $\mathbb{Z} = v\mathbb{Q}_p \to \mathbb{Q}_p^\times, m \mapsto p^m$. It does not look definable in $L_{\mathcal{O}}$, and indeed it is not ([22]) [7]. Macintyre found the key definabe expansion of $L_{\mathcal{O}}$ (or L) for elimination of quantifiers.

Definition 3.1. A semi-algebraic formula is a boolean combination of formulas $\varphi(\mathbf{x})$ of the form $\varphi(\mathbf{x}) : \exists y(f(\mathbf{x}) = y^n)$, where $f(\mathbf{x}) \in \mathbb{Z}[\mathbf{X}]$, and $n \in \mathbb{N}, n \geq 2$.

Definition 3.2. Let K be a p-adically closed field. A subset of K^m is called semi-algebraic (with parameters) if it is the solution set of some semi-algebraic formula (with parameters).

Theorem 3.3 (Macintyre's theorem). *Let K be a p-adically closed field. Let S be a semi-algebraic subset of K^{m+1}, then the set $\{\mathbf{x} \in K^m : \exists y \in K \text{ s.t. } (\mathbf{x}, y) \in S\}$ is semi-algebraic.*

It follows from this theorem that if we add to the language of valued fields $L_{\mathcal{O}}$ new unary predicates P_n for each positive integer $n \geq 2$ with the interpretation $P_n(x) \iff \exists y(x = y^n)$, then we get quantifier elimination, in fact uniformly in p-adically closed fields. Let $L_{\mathcal{O}}(P_\omega)$ denote this new language.

Theorem 3.4 (Macintyre's elimination of quantifiers). *The theory pCF admits elimination of quantifiers in the language $L_{\mathcal{O}}(P_\omega)$.*

Note that in pCF, as we observed (formula (1)), the valuation ring is defined by the purely algebraic atomic formula $P_2(1 + p^3x^4)$. Note also that in the discussion above we can replace $L_{\mathcal{O}}$ by L_{Div}, and get elimination in $L_{Div}(P_\omega)$.

5. Note that the universal part of pCF, $(pCF)_\forall$, was axiomatized in [40], [190].

6. I.e. the Galois group of its algebraic closure over itself. Note that in this case, it is topologically finitely generated.

7. This follows from the fact already mentioned that any infinite L-definable subset of $\mathbb{Q}_p$ has non empty interior.

We recall the analogy of Macintyre's theorem with Chevalley's theorem in algebraic geometry, taking algebraically closed instead of p-adically closed, and constructible instead of semi-algebraic (see e.g. [41]). The parallel between Chevalley's theorem and Tarski's elimination of quantifiers in algebraically closed fields is by now very well known. Joyal [141] might have been the first occurence in the literature to point it out.

Recently Denef [88] gave a purely algebraic geometric proof of Macintyre's theorem (for $\mathbb{Q}_p$), based on *monomialization of morphisms*.

3.3. Denef's cell decomposition

At the heart of the Denef's school of p-adic integration, is the method of cell decomposition. Denef explained very well that he was inspired by Cohen's procedure to eliminate quantifiers [7] [8]. The original cell decomposition has now been refined, generalized and applied in ever increasing sophisticated context, up to motivic integration.

Cell decomposition is a finer analysis of definable sets, in terms of well behaved definable functions. Definable sets partition into cells. On can view a cell as a set where one should be able to perform integration through iterated integrals (Fubini's), more or less like in calculus [9]. Cells have good geometrical properties.

We give two basic precise statements, which we take from the lucid [87].

3.3.1. Cells and cell decomposition. The following *semi-algebraically commensurable* is not standard terminology [10], but we find it somewhat convenient.

Definition 3.5. Let K be a p-adically closed field. A function $f : K^m \to K$ is called *semi-algebraically commensurable* (with parameters) if for every semi-algebraic subset $S \subseteq K \times K^r$ (with parameters) the set

$$\{(\mathbf{x}, \mathbf{y}) \in K^{m+r} : (f(\mathbf{x}), \mathbf{y}) \in S\}$$

is semi-algebraic (with parameters). In other words, the pullback function $(f \times 1_{K^r})^*$ sends semi-algebraic sets to semi-algebraic sets, for every identity map $1_{K^r} : K^r \to K^r$.

A semi-algebraically commensurable function f is always semi-algebraic, i.e. its graph Γ_f is semi-algebraic. Indeed, Γ_f can be expressed as

$$\Gamma_f = (f \times 1_K)^*(\Delta_{K^2})$$

where Δ_{K^2} is the diagonal $\Delta_{K^2} = \{(z, y) \in K^2 : z = y\}$. It follows from Macintyre's theorem that a semi-algebraic function is semi-algebraically commensurable, for if Γ_f and $S \subseteq K \times K^r$ are semi-algebraic sets, then so is

$$S_1 = \{(\mathbf{x}, y, \mathbf{z}) \in K^{m+1+r} : y = f(\mathbf{x}) \ \& \ (y, \mathbf{z}) \in S\}$$

and

$$(f \times 1_{K^r})^*(S) = \{(\mathbf{x}, \mathbf{z}) \in K^{m+r} : \exists y \in K \ s.t. \ (\mathbf{x}, y, \mathbf{z}) \in S_1\}$$

But it will be necessary to temporarily distinguish the two when we deduce Macintyre's theorem from cell decomposition. So we temporarily stick to the two names.

8. It might be worthwhile to recall that Cohen was a trained analyst.
9. In the context of o-minimality, this remark is literally true.
10. In his paper [87], Denef uses (ephemerally, so to speak (cf. further below)) "semi-algebraic function" for what we call here "semi-algebraically commensurable".

Definition 3.6. Let K be a p-adically closed field. A *cell* in $K^m \times K$ is a set of the form

$$\{(\mathbf{x}, t) \in K^m \times K : \mathbf{x} \in C, \text{ and } v(a_1(\mathbf{x})) \,\square_1\, v(t - c(\mathbf{x})) \,\square_2\, v(a_2(\mathbf{x}))\}$$

where $a_1(\mathbf{x}), a_2(\mathbf{x}), c(\mathbf{x}), h_i(\mathbf{x})$ are semi-algebraically commensurable functions of $\mathbf{x}$, C is a semi-algebraic subset of K^m, $\nu_i \in \mathbb{N}$ and each of $\square_1, \square_2$ denotes either $\leq, <$ or no condition. The function $c(\mathbf{x})$ is called *a center* of the cell.

Theorem 3.7 (Cell decomposition theorem I). *Let t be a variable and $\mathbf{x} = (x_1, \ldots, x_m)$. Let $f(\mathbf{x}, t)$ be a polynomial in t with coefficients which are semi-algebraically commensurable functions of $\mathbf{x}$. Then there exists a finite partition of $K^m \times K$ into cells A, such that each such cell A has a center $c(\mathbf{x})$ such that the following holds :*

if we write $f(\mathbf{x}, t)$ as a polynomial in $t - c(\mathbf{x})$

$$f(\mathbf{x}, t) = a_0(\mathbf{x}) + a_1(\mathbf{x})(t - c(\mathbf{x}) + \cdots + a_i(\mathbf{x})(t - c(\mathbf{x})^i + \cdots$$

then

$$v(f(\mathbf{x}, t)) - \min_i v(a_i(\mathbf{x})(t - c(\mathbf{x})^i)$$

is bounded on A.

Theorem 3.8 (Cell decomposition theorem II). *Let t be a variable and $\mathbf{x} = (x_1, \ldots, x_m)$. Let $f_i(\mathbf{x}, t), i = 1, \ldots, r$ be polynomials in t with coefficients which are semi-algebraically commensurable functions of $\mathbf{x}$. Let $n \in \mathbb{N}, n > 0$, be fixed. Then there exists a finite partition of $K^m \times K$ into cells A, such that each such cell A has a center $c(\mathbf{x})$ such that for all $(\mathbf{x}, t) \in A$ we have*

$$f_i(\mathbf{x}, t) = u_i(\mathbf{x}, t)^n h_i(\mathbf{x})(t - c(\mathbf{x}))^{\nu_i}, \text{ for } i = 1, \ldots, r$$

with $v(u_i(\mathbf{x}, t)) = 0, h_i(\mathbf{x})$ a semi-algebraically commensurable function of $\mathbf{x}$, and $\nu_i \in \mathbb{N}$.

3.3.2. Quantifier elimination from cell decomposition. In order to have a glimpse at the power of cell decomposition, we will deduce quantifier elimination from it. We roughly reproduce the proof from [87], where the above cell decompositions are proved directly without Macintyre's theorem. So as we mentioned, the proof uses the distinction between semi-algebraically commensurable functions and functions with semi-algebraic graphs, and after the fact, implies that the two coincide.

Theorem 3.9 (Macintyre's theorem). *Let S be a semi-algebraic subset of K^{m+1}, then the set $\{\mathbf{x} \in K^m : \exists y \in K \text{ s.t. } (\mathbf{x}, y) \in S\}$ is semi-algebraic.*

Proof. We may suppose that S is the set of all $(\mathbf{x}, t) \in K^m \times K$ satisfying

$$f_i(\mathbf{x}, t) \text{ is (is not) an } n_i\text{-th power }, i = 1, \ldots, k.$$

where the f_i are polynomials over K, and $n_i \in \mathbb{N}, n_i \geq 2$. Let n be a common multiple of the n_i. By the Cell decomposition theorem II, we get that S is a disjunction of sets of the form

$$v(a_1(\mathbf{x})) \,\square_1\, v(t - c(\mathbf{x})) \,\square_2\, v(a_2(\mathbf{x})), \text{ and } \mathbf{x} \in C \text{ and}$$

$$h_i(\mathbf{x})(t - c(\mathbf{x}))^{\nu_i} \text{ is (is not) an } n_i\text{-th power, } i = 1, \ldots, k.$$

where $a_1(\mathbf{x}), a_2(\mathbf{x}), c(\mathbf{x}), h_i(\mathbf{x})$ are semi-algebraically commensurable functions of $\mathbf{x}$, C is a semi-algebraic subset of K^m, $\nu_i \in \mathbb{N}$ and each of $\square_1, \square_2$ denotes either $\leq, <$ or

no condition. By making a disjunction over all possible n-th power residues for $h_i(\mathbf{x})$ and $t - c(\mathbf{x})$, we obtain that S is a disjunction of sets of the form

$$v(a_1(\mathbf{x})) \,\square_1\, v(t - c(\mathbf{x})) \,\square_2\, v(a_2(\mathbf{x})), \text{ and } \mathbf{x} \in C \text{ and}$$

$$(t - c(\mathbf{x})) = \rho \cdot (\text{nonzero } n\text{-th power})$$

where $\rho \in K$. Hence, we may assume that S is of the form

$$v(a_1(\mathbf{x})) \leq v(t - c(\mathbf{x})) \leq v(a_2(\mathbf{x})), \text{ and}$$

$$(t - c(\mathbf{x})) = \rho \cdot (\text{nonzero } n\text{-th power})$$

where $\rho \neq 0$. Then, the set to be proven semi-algebraic is that of all $\mathbf{x} \in K^m$ such that

$$\exists \ell \in \mathbb{Z}, v(a_1(\mathbf{x})) \leq \ell \leq v(a_2(\mathbf{x})) \text{ and } l \equiv v(\rho) \text{mod } n.$$

This condition is equivalent to

$$(*) \exists \ell \in \mathbb{Z}, \frac{v(a_1(\mathbf{x})\rho^{-1})}{n} \leq \ell \leq \frac{v(a_2(\mathbf{x})\rho^{-1})}{n}.$$

If $v(a_1(\mathbf{x})\rho^{-1}) \equiv 0 \text{ mod } n$, then $(*)$ is equivalent to

$$v(a_1(\mathbf{x})\rho^{-1}) \leq v(a_2(\mathbf{x})\rho^{-1})$$

If $v(a_1(\mathbf{x})\rho^{-1}) \equiv \gamma \text{ mod } n$, with $0 < \gamma < n$, then $(*)$ is equivalent to

$$v(a_1(\mathbf{x})\rho^{-1}) + n - \gamma \leq v(a_2(\mathbf{x})\rho^{-1})$$

But these last two conditions are semi-algebraic, and we are done. $\square$

4. Definability

4.1. The angular component formalism

Another key function in $\mathbb{Q}_p$ is the so called *angular component map*,

$$ac : \mathbb{Q}_p^\times \to \mathbb{F}_p^\times, \;\; ac(x) = \overline{xp^{-v_p(x)}}$$

as well as its higher order versions for each positive integer $n \geq 1$,

$$ac_n : \mathbb{Q}_p^\times \to (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^\times, \;\; ac_n(x) = res_n(xp^{-v_p(x)})$$

They are group homomorphisms and $ac_n(u) = u \mod p^{n+1}$, if $v_p(u) = 0$.

We should readily emphasize that these maps are definable in L_{vf} ([87]), essentially because the residue rings $\mathbb{Z}_p/p^m\mathbb{Z}_p$ are finite. So they are really part of the theory pCF. Note that for non zero $x \in \mathbb{Q}_p$, $P_n(x)$ holds iff $v_p(x)$ is divisible by n and $ac_m(x)$ is an n-th power for $m = 2v_p(n)$.

But it took a while before the importance of these functions emerged in the definability theory of $\mathbb{Q}_p$ [11]. Their importance emerged in the Denef school. The map ac first occurs in Denef's work in his paper [87] where he proves Macintyre's elimination of quantifiers from cell decomposition. It plays a crucial role in the study of uniformity in p by Pas [173, 174], and the language of quantifer elimination they yield is often referred to as Pas's language (or Denef-Pas).

11. And other valued fields.

Van den Dries (e.g. [96]) showed the technical flexibility of these functions in model theoretical methods and he used them in [97] in a context of analytic functions. They are axiomatized in a valued field (K, v) as homomorphisms $ac_I : K^\times \to (\mathcal{O}_K/I)^\times$, where I is an ideal of $\mathcal{O}_K$, such that $ac_I(x) = x \mod I$ when $v(x) = 0$. They are also called angular component map modulo I. Taking I the maximal ideal of $\mathcal{O}_K$ we get what is usually called an angular component map.

Let $L_{v,ac}$ be the language of valued fields obtained from L_v by adding a function symbol for an angular component map, and let L_{vf,ac_ω} be the similar language obtained by adding function symbols ac_n and appropriate sorts and transition maps for the projective systems of higher order angular component maps for the ideals $p^{n+1}\mathcal{O}_K$. Pas (ibid.) proved the following relative quantifier elimination results : (1) the theory of henselian valued fields of residual characteristic 0 admits elimination of base field quantifiers in the language $L_{vf,ac}$, (2) the theory of henselian valued fields of characteristic 0 with $v(p)$ the least positive element in the value group, admits elimination of base field quantifiers in the language L_{vf,ac_ω}.

The key feature is that not only elimination of base field quantifiers is relative [12] to the value group and residue rings, but also that the interplay between variables of different sorts seems minimal, whence often optimal. In Pas's work, it allows to control the dependence on p and on the residue fields. A simpler example (but see also the next section 4.2) is that it allows to axiomatize types and count coheirs in a clearcut way in the general cases ([46]), rendering transparent the initial analysis of Delon [8] (see also [85]) of the independence property, and allowing the similar analysis in the mixed characteristic case ([45]).

One sees that a cross-section π yields an angular component map just as in the p-adic case, $x \mapsto \overline{x\pi(v(x))^{-1}}$, but angular components are strictly weaker than cross-sections. For example, in the p-adic case, Scowcroft [200] has constructed p-adically closed fields without sections of the p-adic valuation map. The situation is somewhat enlightened by the observation that an angular component map corresponds in a natural way to a splitting of the lower exact sequence in the following commutative diagram :

$$\begin{array}{ccccccccc}
1 & \to & U_K & \to & K^\times & \xrightarrow{v} & vK & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \bar{K}^\times & \to & K^\times/1+\mathfrak{m}_K & \to & vK & \to & 0
\end{array}$$

where U_K is the group of units of $\mathcal{O}_K$, $\mathfrak{m}_K$ the ideal of nonunits of $\mathcal{O}_K$, and $1 + \mathfrak{m}_K$ the subgroup of U_K of elements $1 + x, x \in \mathfrak{m}_K$. Neither of these exact sequences need split : it is well known there are valued fields without cross-sections, and for angular component maps see [175]. Since U_K is a pure subgroup of $K^\times$, we see that an angular component (resp. cross-section) map exists as soon as $\bar{K}^\times$ (resp. $K^\times$) is pure-injective, and it follows that every valued field has an elementary extension with an angular component map (see ibid.).

12. A phenomenon already emphasized by Cohen [7]. It would be taken up and studied in [31, 32], [37] and many more since then.

The exact sequence that we linked to angular component maps, or rather the group $K^\times/1+\mathfrak{m}_K$, is a classical object in number theory (see e.g. the *multiplicative congruences* in [135], chap. 15-§.1, chap. 23-§.2.). It is also the underlying structure of Krasner's *corpoïde* ([151]). It appears implicitely in the work around the Ax-Kochen-Ershov analysis up to Delon's thesis. Poizat [13] had already observed in the 1970's the relevance of this structure for the model theory of valued fields. Later Basarab [37] made the same observation in a broad context and obtained an elimination of base field quantifiers relative to that structure and its higher order versions, in particular for pCF. This analysis was made more precise in [38] and [152]. This structure appeared more recently in diffferent contexts where it is often called *RV-structure*, e.g. with some finer analysis in [119], or in motivic integration [136].

4.2. The independence property

Matthews [169] showed that p-adically closed fields do not have the independence property [14]. In contrast, we now know that the field of Laurent series $\mathbb{F}_p((T))$ does have the independence property ([144]).

Karpinski and Macintyre [145] have computed some bounds related to the absence of the independence property in $\mathbb{Q}_p$, in connection with Vapnik-Chervonenkis dimension. There is recent work along these lines by Aschenbrenner et al. [36].

The development of a finer analysis of forking in geometric stability theory has triggered a renewed interest in theories without the independence property (see e.g. [35]), also called NIP theories [15]. They are now somewhat considered as "tame" settings. This should shed some new lights on $\mathbb{Q}_p$. We first mention Shelah's result that p-adicaly closed fields are « strongly dependent » [202]. A theory T is strongly dependent if there are no $\mathfrak{M} \models T$, formulas $\phi_i(\mathbf{x}, \mathbf{y})$ for $i \in \omega$, and elements $\mathbf{a}_i^j \in M$ so that for each function $\eta : \omega \to \omega$ the type $\{\phi_i(\mathbf{x}, \mathbf{a}_i^j) : i, j \in \omega, \eta(i) = j\} \cup \{\neg\phi_i(\mathbf{x}, \mathbf{a}_i^j) : i, j \in \omega, \eta(i) \neq j\}$ is consistent. Berenstein, Dolich and Onshuus showed in [51] that the theory of dense pairs of p-adically closed fields is also « strongly dependent », in particular it does not have the independence property (see also [55]). The activity around NIP theories is still flourishing (see [140]).

This reassessment has produced simpler arguments for NIP for valued fields, which we will sketch here for the p-adics (see e.g. [202], or [204]). It is based on some simple observations about indiscernible sequences and exploits relative quantifier elimination with angular components. The first observation is that in a total order, an indiscernible sequence is either strictly monotone, or constant. In particular, in a $\mathbb{Z}$-group, because indiscernibles will be congruent $\mod n$ for all n, an indiscernible sequence (γ_i) is either strictly monotone and moreover $|\gamma_i - \gamma_{i+1}|$ is infinite, or it is constant. A second observation is about angular component maps ac_n : if $v(x) < v(y)$ and $|v(x) - v(y)|$ is infinite, then $ac_n(x+y) = ac_n(x)$, and if $v(x+y) = v(x) = v(y)$, then $ac_n(x+y) = ac_n(x) + ac_n(y)$.

13. Unpublished notes.

14. In most theories of valued fields which satisfy classical Ax-Kochen-Ershov theorems, the independence property is reduced to that of the residue field (see [45]).

15. Now often called "dependent theories", a quite unfortunate choice of terminology.

Lemma 4.1. *Let K be p-adically closed and $f(X) = \sum a_i X^i \in K[X]$. Let $(x_n)_{n\in\mathbb{N}}$ a sequence of indiscernible elements from K, and $f \in K[X]$ and $m \in \mathbb{N}$. Then there exists a sequence of indiscernibles from $vK, (\alpha_n)_{n\in\mathbb{N}}$, and a sequence of indiscernibles from $\mathcal{O}_K/(p^{m+1})$, $(b_n)_{n\in\mathbb{N}}$, depending only on $(x_n)_{n\in\mathbb{N}}$, and there exist $n_0, \ell \in \mathbb{N}, \gamma \in vK, g \in \mathcal{O}_K/(p^{m+1})[X]$, such that for all $n \geq n_0$, we have $v(f(x_n)) = \ell\alpha_n + \gamma$ and $ac_m(f(x_n)) = g(b_n)$.*

Proof. If the sequence $v(x_i)$ is strictly monotone, the desired conclusion follows directly from the preceding observations. In case the sequence $v(x_i)$ is constant, consider the polynomial $R(y)$ such that $f(x+y) = f(x) + R(y)$. Let $y_i = x_i - x_0$, so for all i, $f(x_i) = f(x_0) + R(y_i)$. If the sequence $v(y_i)$ is strictly monotone, we can conclude again using the polynomial $f(x_0) + R(y)$ and the sequence (y_i). We are left with the case $v(y_i)$ is also constant, say $v(y_i) = v_0$. Since the x_i are indiscernibles we have $v(y_i - y_j) = v(x_i - x_j) = v_0$, for all $i \neq j$. By the preceding observations, $ac(y_i - y_j) = ac(y_i) - ac(y_j)$, and since ac never takes the value 0 we have $ac(y_i) \neq ac(y_j)$, for $i \neq j$, but this is impossible in the finite residue field $\mathbb{F}_p$. So we are done. $\square$

Recall that it suffices to check for the independence property for formulas of the form $\varphi(x, \boldsymbol{y})$. Recall also the characterization through indiscernible sequences : a formula $\phi(x, \boldsymbol{y})$ has the independence property iff there exists an indiscernible sequence $(a_i)_{i\in\mathbb{N}}$ and some $\boldsymbol{b}$ in some model $\mathfrak{M}$ such that $\mathfrak{M} \models \phi(a_{2i}, \boldsymbol{b}) \wedge \neg\phi(a_{2i+1}, \boldsymbol{b})$, for all i.

By quantifier elimination in the Pas language and the fact that formulas without the independence property are closed under boolean combinations, it suffices to check for formulas of the following form :

(1) $f(x, \boldsymbol{y}) = 0$, f a polynomial over the base p-adically closed K ;

(2) $\varphi(x, \boldsymbol{\tau}(\boldsymbol{y}))$, φ a formula over $\mathbb{Z}/(p^{m+1})$, $\boldsymbol{\tau}$ suitable terms ;

(3) $\Phi(x, \boldsymbol{\tau}(\boldsymbol{y}))$, Φ a formula in the theory of $\mathbb{Z}$-groups, $\boldsymbol{\tau}$ suitable terms ;

(4) $\varphi(ac_m(f_1(x, \boldsymbol{y}_1)), \ldots ac_m((f_n(x, \boldsymbol{y}_1)), \boldsymbol{y}_2)$, φ a formula over $\mathbb{Z}/(p^{m+1})$, $f_1, \ldots f_n$ polynomials over K, $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2)$;

(5) $\Phi(v(f_1(x, \boldsymbol{y}_1)), \ldots v((f_n(x, \boldsymbol{y}_1)), \boldsymbol{y}_2)$, Φ a formula in the theory of $\mathbb{Z}$-groups, $f_1, \ldots f_n$ polynomials over K, $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2)$.

Now, case (1) can be checked directly not to have the independence property ; in case (2) one can remark that a finite structure can not have the independence property ; in case (3) it is known that the theory of $\mathbb{Z}$-groups does not have the independence property [16] ; case (4) and (5) reduces respectively to case (2) and (3) by the preceding lemma.

4.3. Imaginaries

Recall that a complete theory (of rings) has elimination of imaginaries if for every definable equivalence relation $E(\mathbf{x}, \mathbf{y})$ on n-tuples there exists a definable function f such that $E(\mathbf{x}, \mathbf{y}) \leftrightarrow f(\mathbf{x}) = f(\mathbf{y})$ holds. The equivalence relation of having the same valuation is already problematic in the language $L_{\mathcal{O}}$, for no such definable function can exist as any definable subset of $\mathbb{Q}_p^m$ is either finite or uncountable [17]. The relation

16. As any theory of ordered abelian groups [122].
17. For $m = 1$, follows from nonemptiness of interior of infinite subsets of $\mathbb{Q}_p$.

$x \simeq y \leftrightarrow v(x-y) \geq 0$ is also problematic for the same reason. Adding a sort for the value group and the valuation map directly remedies the former, but even also adding a sort for the quotient set of $\simeq$ is not enough (see [201]).

The analysis of imaginaries turns out to be quite complicated. Elimination of imaginaries for p-adically closed fields in a definable expansion is due to Hrushovski-Martin [138]. They deduced it from the fundamental result on algebraically closed valued fields [132], where one goes through a careful look at the behavior of types. Besides the basic imaginary sorts of the value group ($K^\times/U$) and residue field ($\mathcal{O}_K/\max\mathcal{O}_K$), one has to consider translates of free $\mathcal{O}_K$-modules in affine space. For each $n \in \mathbb{N}$, let S_n be the additional sort defined by the quotient set $S_n(K) = GL_n(K)/GL_n(\mathcal{O}_K)$. Note that, the value group can be identified with S_1, our good friend $K^\times/1+\mathfrak{m}_K$ can be identified with the image in S_2 of the matrices $\begin{bmatrix} px & x \\ 0 & 1 \end{bmatrix}, x \in K^\times$, and the residue field can be identified with the fiber above 1 of the natural map $K^\times/1+\mathfrak{m}_K \to S_1$. Then pCF eliminates imaginaries in the language of rings L augmented with the sorts S_n.

Scowcroft [201] [18] has a small language eliminating "one variable" imaginaries, i.e. coming from an equivalence relation on the underlying base field ; see also [119] in this vein. See [166] for the dismal situation in languages including a cross-section and dealing with the above $\simeq$ by adding the new function $T(\sum_{n\geq N} a_n p^n) = \sum_{N\leq n\leq 0} a_n p^n$.

4.4. Some p-minimality

Time has shown the spectacular success of the concept of o-minimality for the field of real numbers (see e.g. [211]). It was natural to look for analogues in the context of the p-adic and other valued fields. Haskell and Macpherson [134] introduced a reasonably natural notion for the p-adics which any analogue should encompass. Let L' be a language expanding the elimination language $L_{Div}(P_\omega)$, and $\mathfrak{K} = (K, v, \ldots)$ an L'-structure which is a valued field, then $\mathfrak{K}$ is said to be P-minimal [19] if K is of characteristic zero with $v(p)$ the least positive element in the value group, with residue field the prime field of characteristic p, and with value group a $\mathbb{Z}$-group, and if for every $\mathfrak{M}$ elementary equivalent to $\mathfrak{K}$, every definable subset of M is quantifier-free definable by a formula in $L_{Div}(P_\omega)$. The authors prove some basic results showing how the analogy goes through, or not. For example, there is a good theory of dimension, but no readily available cell decomposition nor a good "monotonicity theorem" for unary definable functions. They give (shown in [98]) the example of an expansion of $\mathbb{Q}_p$ by some analytic functions, namely essentially the structure studied in [89] (see section 6 below). In [171] Mourgues show a cell decomposition theorem for P-minimal structures which have definable Skolem functions. There are some limitations on the expansion of the value group (see [58], and [57] for some further questions).

This notion is used in [98] to show that if $S \subset \mathbb{Z}_p^{m+1}$ is a subanalytic set (see section 6 below), then there is a semialgebraic set $S' \subset \mathbb{Z}_p^{m'+1}$ such that for each $x \in Z_p^m$ there is $x \in Z_p^{m'}$ such that we get equality of the fibers $S_x = S'_{x'}$. This generalizes a result of [89] saying that p-adic subanalytic subsets of $\mathbb{Z}_p$ are semialgebraic.

18. See the conclusion of the paper.
19. The P is meant to refer to the P_n predicates.

The bounds of [145] related to the absence of the independence property of $\mathbb{Q}_p$ in connection with Vapnik-Chervonenkis dimension, are proved in the context of P-minimal theories.

Cluckers and Loeser [73] have introduced the very general framework of *b-minimality* covering the above P-minimality in the presence of definable Skolem functions. It takes a form of cell decomposition as an axiom, but the underlying structure need not be a field, and the notion also covers o-minimality. See [52] for an example with a *nonstandard* analytic structure on the p-adically closed field of power series $\mathbb{Q}_p((T^{\mathbb{Q}}))$.

Recently, Cluckers and Leenknegt [69] point out a minimal language for "p-adic minimality" which would parallel the plain order language in o-minimality. The only predicates in this language, call it L_p, are the ternary $R_{n,m}$, for each positive integers n, m, and defined in $\mathbb{Q}_p$ as follows : $R_{n,m}(x, y, z)$ iff $y - x = zp^{jn}(1 + p^m h)$ for some $j \in \mathbb{Z}$ and some $h \in \mathbb{Z}_p$. They show that subanalytic subsets of $\mathbb{Q}_p$ are L_p-definable, in particular semi-algebraic subsets. By going to the definable expansion of L_p by the predicates $D_k(x, y, z)$ iff $z \neq y \wedge v_p(x - y) < v_p(z - y) + k$, for each $k \in \mathbb{Z}$, one has quantifier elimination in $\mathbb{Q}_p$.

4.5. Definable groups

Definable groups in $\mathbb{Q}_p$ have been analyzed or studied in [181], [182], [139]. The analysis in general p-adically closed fields is still lacking (perhaps [178] would be helpful). Definable groups in $\mathbb{Q}_p$ can be equipped definably with the structure of a p-adic Lie group [181]. The analysis was pushed further in [139], where it is shown that for each such group G, there is an algebraic group H and a Nash isomorphism between neighborhoods of the identity of G and $H(\mathbb{Q}_p)$. A Nash function is a definable analytic function from an open definable subset of $\mathbb{Q}_p^n$ into $\mathbb{Q}_p$. So even though multiplication on G is given by a Nash function, locally G is definably isomorphic to a group where multiplication is given by a rational function.

Definable fields in $\mathbb{Q}_p$ are just its finite extensions ([181]).

There is a standing conjecture on groups definable in saturated P-minimal expansions of p-adically closed fields [172], which is a p-adic analogue of a (now established) conjecture about o-minimal groups related to "standard part maps". We roughly state half of the conjecture. In a saturated expansion M of a p-adically closed field, for a definable group G, G° denotes the intersection of all definable subgroups of G of finite index, and $G^{\circ\circ}$ the intersection of all type-definable subgroups of G of index smaller than the cardinality of M, and $G^{\circ\circ} \subseteq G^\circ$. The conjecture states that if G is a group definable in a saturated P-minimal expansion of a p-adically closed field, then there is an open definable subgroup H of G such that H/H^{00} equipped with the logic topology is a compact p-adic Lie group whose dimension as a p-adic manifold equals the P-minimal dimension of H (and G). The full conjecture implies that $G^{\circ\circ} = G^\circ$. Here is a sample from the paper. Suppose G is semi-algebraic, defined over $\mathbb{Q}_p$ and such that $G(\mathbb{Q}_p)$ is a compact p-adic Lie group, we then have the "standard part map" $st : G \to G(\mathbb{Q}_p)$, and the authors show that the group $G^{\circ\circ}$ coincides with $\ker(st)$, and st induces a homeomorphism between $G/G^{\circ\circ}$ (with its logic topology) and the p-adic Lie group $G(\mathbb{Q}_p)$. The following was raised [20] in [182] : is any open (not necessarily

20. To our knowledge, still open.

compact) subgroup of a semialgebraic p-adic Lie group semialgebraic ?

5. Integration

Denef's paper "The rationality of the Poincaré series..." showed how one could exploit the flexibility of the notion of definable sets and functions with appropriate quantifier elimination, coupled with a finer analysis of these (through cell decomposition), in order to evaluate p-adic integrals and obtain some precise information on the result. Denef pointed out in [86] that the interest of his method is not only its "elementary" nature (compared with resolution of singularities), but that it gives *more* information on the result : "This method has the advantage that one can easily control the dependence on parameters (see paragraph 3) which I don't know how to do with the resolution of singularities method." (ibid., p. 31).

Let $\varphi(\mathbf{x})$ be a L-formula anf for each $n \in \mathbb{N}$ let $\tilde{N}_{n,\varphi}$ be the cardinal of the set $\{\mathbf{a} \in (\mathbb{Z}/p^n\mathbb{Z})^m \ : \ \varphi(\mathbf{a})$ is true in $\mathbb{Z}/p^n\mathbb{Z}\}$ and $N_{n,\varphi}$ the cardinal of the set $\{\mathbf{a} \mod p^n \ : \ \mathbf{a} \in \mathbb{Z}_p^m$ and $\varphi(\mathbf{a})$ is true in $\mathbb{Z}_p\}$ and set $\tilde{P}_{\varphi,p}(T) = \sum_{n=0}^{\infty} \tilde{N}_{n,\varphi} T^n, \quad P_{\varphi,p}(T) = \sum_{n=0}^{\infty} N_{n,\varphi} T^n$. Denef proved that both power series $\tilde{P}_{\varphi,p}(T), P_{\varphi,p}(T)$ are in fact rational functions [21]. For $\varphi(\mathbf{x})$ a polynomial system, the rationality of $P_{\varphi,p}(T)$ was a question of Serre and Oesterlé, and that of $\tilde{P}_{\varphi,p}(T)$ was a result of Igusa.

The rationality of the power series reduces to showing the rationality of related p-adic integrals. Here is a sample result.

Theorem 5.1 (Denef). *Let S be a L-definable subset of $\mathbb{Q}_p^m$, which is contained in a compact subset. Let $h(\mathbf{x})$ be a definable function from $\mathbb{Q}_p^m$ to $\mathbb{Q}_p$ such that $|h(\mathbf{x})|_p$ is bounded on S. Suppose that $v_p(h(\mathbf{x})) \in \mathbb{Z} \cup \{\infty\}$, for all $\mathbf{x} \in S$. Then $Z_S(s) = \int_S |h(\mathbf{x})|_p^s d\mathbf{x}, \ \ (s \in \mathbb{R}, s > 0)$ is a rational function of p^{-s}.*

We will not go into the details of Denef's method, but refer to his very lucid [86].

The key step in Denef's method is cell decomposition. It allows separation of variables and integration one variable at a time. We already mentioned the control it allows on the dependence on certain parameters.

Denef's method was extended by himself and van den Dries to the larger definable category of subanalytic sets and functions in [89], where they prove similar rationality results (see section 6).

The study of the dependence on p and under unramified extensions of Denef's integrals, using cell decomposition, was taken up successfully by Pas [173, 174] and Macintyre [165] independently. E. g. writing $Z_S(T,p)$ for the above $Z_S(s)$ as a function of $T = p^{-s}$ and to stress the dependence on p, they show that there is a polynomial $Q(X,T) \in \mathbb{Z}[X,T]$ and a natural number d so that for all sufficiently large primes p, we have $Z_S(T,p) = R_p(T)/Q(p,T)$ where R_p is a polynomial of degree d. It is in this work of Pas (see also [176]) that the importance of angular component maps emerged

21. It might be worthwhile to recall that a formal power series is a rational function iff the sequence of its coefficients satisfy a linear recurrence relation.

distinctly. He proves a version of cell decomposition for (unramified) henselian valued fields of characteristic zero with angular components, yielding the quantifier elimination theorems we mentioned in section 4.1.

Cluckers and Denef [63] applied the method to *orbital integrals*.

Now, p-adic methods and integrals are used in other contexts. Denef's method was used (early on) to study subgroup growth, and for this we refer to the book of Lubotzky and Segal [161] (see also [138]). For a group G, e.g. finitely generated, one is interested in the sequences of integers $a_n(G)$ and $s_n(G)$ giving respectively the number of subgroups of index n, or of index at most n.

In [90], Denef and Loeser introduced *motivic integration* in the context of definable sets in henselian discretely valued fields. This work was developed further since by them and others. In particular in [91] they developed "arithmetic" motivic integration, which enabled them to make a precise connection to p-adic integration.

In the remainder of this section we will try to give some insights into this vast and deep subject, which is not yet familiar to many model theorists. We will try to focus on the idea of "*enhancing uniformity*". For more appropriate introductions, we refer to the surveys and introductory accounts by Denef-Loeser themselves [92], Gordon-Yaffe [121], Hales [128], Yin [212] and the lectures by T. Scanlon [196] aimed at model theorists.

According to Deligne [82], motives were first introduced by Grothendieck in an attempt to find an explanation for the "family likeness" [22] of the étale l-adic cohomology groups of an algebraic variety over a field of nonzero characteristic p, when l varies over the primes $l \neq p$. In characteristic zero, for a an algebraic variety X over a subfield k of $\mathbb{C}$, an explanation would be given by comparison isomorphisms between the ("usual") integer valued topological cohomology groups $H^i(X(\mathbb{C}), \mathbb{Z})$ and the ℓ-adic étale cohomology groups $H^i_{ét}(X, \mathbb{Z}_l)$:

$$H^i(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_l \xrightarrow{\sim} H^i_{ét}(X, \mathbb{Z}_l)$$

The theory of motives would be an attempt to find a substitute for the $H^i(X(\mathbb{C}), \mathbb{Z})$ (nonexistent in characteristic p).

Kontsevich initially introduced motivic integration as an analogue of p-adic integration over $K[[T]]$-rational points of algebraic variety and taking values in a ring (of motives) constructed from the category of varieties. Let k be a field of characteristic zero and denote by $\mathcal{M}$ the Grothendieck ring of algebraic varieties over k. It is the ring generated by symbols $[S]$, for S an algebraic variety over k, with the relations $[S] = [S']$ if S is isomorphic to S', $[S] = [S \setminus S'] + [S']$ if S' is closed in S and $[S \times S'] = [S][S']$. Note that, for S an algebraic variety over k, the mapping $S' \mapsto [S']$ from the set of closed subvarieties of S extends uniquely to a mapping $W \mapsto [W]$ from the set of constructible subsets of S to $\mathcal{M}$, satisfying $[W \cup W'] = [W] + [W'] - [W \cap W']$. Let $\mathbb{L} = [\mathbb{A}^1_k]$ and $\mathcal{M}_{loc} = \mathcal{M}[\mathbb{L}^{-1}]$. Let T be an indeterminate and denote by $\mathcal{M}[T]_{loc}$ the subring of $\mathcal{M}_{loc}[[T]]$ generated by $\mathcal{M}_{loc}[T]$ and the series $(1 - \mathbb{L}^a T^b)^{-1}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N} \setminus \{0\}$. Let X be an algebraic variety over k. For each $n \in \mathbb{N}$ there is a scheme $\mathcal{L}_n(X)$ such that for any k-algebra R the set of R-rational points of $\mathcal{L}_n(X)$ can

22. "Air de famille", ibid.

be canonically identified with the set of $R[t]/t^{n+1}R[t]$-rational points of X. One gets the scheme $\mathcal{L}(X)$ defined as the projective limit $\mathcal{L}(X) = \varprojlim_n \mathcal{L}_n(X)$. As above, the R-rational points of $\mathcal{L}(X)$ can be identified with the $R[[t]]$-rational points of X. Let $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ denote the projection in this limit, which corresponds to truncation at the level of the rational points above. Kontsevich introduced the completion $\widehat{\mathcal{M}}$ of $\mathcal{M}_{loc}$ with respect to the filtration $F^m\mathcal{M}_{loc}$, where $F^m\mathcal{M}_{loc}$ is the subgroup of $\mathcal{M}_{loc}$ generated by $\{[S]\mathbb{L}^{-i} : i - \dim S \geq m\}$, and defined, for smooth X, a motivic integration on $\mathcal{L}(X)$ with values into $\widehat{\mathcal{M}}$.

Denef and Loeser proved in [90] that the power series

$$P(T) = \sum_{n=0}^{\infty} [\pi_n(\mathcal{L}(X))]T^n$$

considered as an element of $\mathcal{M}_{loc}[[T]]$, is rational and belongs to $\mathcal{M}[T]_{loc}$. This is an analogue to Denef's rationality of p-adic Poincaré series $P_{\varphi,p}(T)$ described before, and the proof goes, very crudely, along similar lines using (Pas) cell decomposition, some desingularization, and a suitable extension of Kontsevitch motivic integration.

In their paper [91], Denef and Loeser develop another version of motivic integration, "arithmetic" motivic integration, and succeed in proving a similar rationality result, but *which specializes for almost all* p to the rationality of the p-adic $P_{\varphi,p}(T)$! This refines, in a spectacular way, the previous uniformity result of Pas/Macintyre.

As we mentioned, p-adic methods and integrals are used in many contexts, and the new motivic integration(s) offered new opportunities. See [194] for applications to some infinite dimensional Lie algebras. Hales [127] proposed to investigate the representation of groups and orbital integrals. One can see e.g. [94] as an example of the issue of certain objects falling into the *definable* category.

In [74], Cluckers and Loeser give a new version of motivic integration, now subsuming the two previous theories of Denef and Loeser, and they apply it to p-adic integrals to extend the classical transfer principle of Ax-Kochen-Ershov between the fields of p-adic numbers $\mathbb{Q}_p$ and power series over prime finite fields $\mathbb{F}_p((t))$. We refer to their previous [72] for an overview, but we will give a basic illustration from section 2 of that paper. Assume φ is a formula with m free variables. For a field K let $h_\varphi(K)$ be the set of points $(x_1, \ldots, x_m)$ in K^m such that φ is true. When $m = 0$, φ is a sentence and $h_\varphi(K)$ is either the one point set if φ is true, or the empty set if φ is false. Denef and Loeser showed that given φ, there exists a virtual motive M_φ canonically attached to φ such that for almost all prime p, the volume of $h_\varphi(\mathbb{Q}_p)$ is finite if and only if the volume of $h_\varphi(\mathbb{F}_p((t)))$ is finite, and in such a case they are both equal to the number of points of M_φ in $\mathbb{F}_p$. In particular, when $m = 0$, the validity of φ is controlled by M_φ. Originally, M_φ lies in a certain completion of a subring of the Grothendieck ring of Chow motives with rational coefficients tensored by the rationals. Call R this tensored subring. The new motivic foundations allow one to avoid the completion, and to choose M_φ from the localization of R obtained by inverting the Lefschetz motive $\mathbb{L}$ and $1 - \mathbb{L}^{-n}$, for $n > 0$. The authors get similar results for more general integrals than measures. Amazingly, their transfer theorems allow the transfer to p-adic fields of the "fundamental lemma" from Langland's program, see [65]. For more motivic integration à la Cluckers-Loeser see [71], [75], [76]. Hrushovski and Kazhdan [136] have a theory of integration with

a similar scope, but on different foundations (see also the expository [212]) : e.g. the main thrust is a geometric approach "à la Weil" concerning the base field, the basic formalism for valued fields is the RV-structure (see section 4.1), dimension relies on the framework from [133] and the residue field and value group are orthogonal ([133] again), which has an effect at the level of definable subsets of the valued base field.

6. Geometry

It does not seem to be clear yet what is meant exactly by "p-adic geometry", in contrast with say "real algebraic geometry" or "real analytic geometry". "P-adic geometry" usually takes place in the completion of the algebraic closure of $\mathbb{Q}_p$ and is often called "rigid geometry" (sometimes "rigid analytic"). We will stick here to the convention of this paper and stay inside $\mathbb{Q}_p$.

E. Robinson [189] has introduced a close p-adic analogue to the real spectrum of real (semi-) algebraic geometry (see also [53, 39], but in particular [53] or [41] for a hands-on treatment). But it did not have the success of its real analogue. As a topological space, the p-adic spectrum $pSpec(\mathbb{Q}_p[\mathbf{X}])$ of the polynomial ring $\mathbb{Q}_p[\mathbf{X}] = \mathbb{Q}_p[X_1, \ldots, X_n]$ can be viewed as the space of n-types of pCF in the language $L_{\mathcal{O}}(P_\omega)$ but with the topology generated by the formulas defining open semi-algebraic sets (in the valuation topology). It is a spectral space [23] and there is a dense embedding $\mathbb{Q}_p^n \hookrightarrow pSpec(\mathbb{Q}_p[\mathbf{X}])$. It produces a good notion of dimension for semi-algebraic sets through chains of specializations, which coincides with the notion introduced by van den Dries and Scowcroft [99] through a version of cell decomposition and projections on affine spaces, and with the general setting of van den Dries for dimension of definable sets in henselian valued fields of characteristic zero [95] (see e.g. [44] or [205] for the equivalence of these dimensions). In the end, the dimension of a semi-algebraic set coincides with the dimension of its Zariski closure in algebraic geometry. Scowcroft [199] has pointed out that there is also a reasonable dimension in the presence of a cross-section. The p-adic spectrum is used in [49] to study the continuous dependence on parameters in Kochen's p-adic analogue of Hilbert's seventeenth problem. It appears also in [43, 44], and implicitly in [180], to study rings of definable continuous functions. It is used in [143] to get an analytic version of the p-adic analogue of Hilbert's seventeenth problem. In the nonarchimedean context (of algebraically closed valued fields), it seems that the idea of compactifying varieties by appropriate points has succeeded through "Berkovich spaces" (see [101]), which can be closely related to suitable spaces of types coming from geometrical stability theory (see [137]).

Cluckers [56] has proved the remarkable fact that given two semi-algebraic sets, there is a definable bijection between them if and only if they have the same dimension (as above). This insight has been used in [68] to give an alternative proof of Denef's results on the rationality of Poincaré series, which uses a minimum of integration.

Euler characteristics and Grothendieck rings of a first-order structure were introduced to investigate such questions as how much of finite combinatorics carries over in

23. I.e. it is T_0, quasicompact with a basis of compact-open sets closed under intersection, and every closed irreducible set is the closure of a unique point ; or equivalently, homeomorphic to the prime spectrum of a commutative ring.

a definable setting, see [150]. For example, the Grothendick ring of the field $\mathbb{C}$ is the Grothendieck ring of varieties $\mathcal{M}$ described in section 5, extended to constructible sets. We now know [67] that the Grothendieck ring of $\mathbb{Q}_p$ is trivial, which is an indication of limitations in the above question, e.g. they give a definable bijection between $\mathbb{Z}_p$ and $\mathbb{Z}_p \setminus \{0\}$, in constrast with say the intervals $[0,1]$ and $(0,1]$ in $\mathbb{R}$ where there is no such bijection. This is closely related to the previous result of Cluckers. Cluckers and Edmundo have a « relative » version of these notions [64] where the problem for $\mathbb{Q}_p$ is not yet settled.

Darnière has some work on the lattice of closed semi-algebraic sets in n-space [80].

Though not definable, the genus of an algebraic p-adic function field over $\mathbb{Q}_p$ is first-order elementary in the class of such function fields [48]. It is known that a p-adically closed field K is existentially definable in a function field F/K [203] (see also [148]). In fact, we now know that one can code Hilbert's tenth problem in the existential theory of those function fields, see section 7 below.

In his famous paper [9], Denef answered a question posed by Serre and Oesterlé about the rationality of Poincaré series associated to p-adic algebraic varieties. Serre and Oesterlé had posed similar questions about closed analytic subsets of $\mathbb{Z}_p^m$. In still another remarkable paper [89], Denef this time with van den Dries, transposed his methods to the setting of p-adic analytic functions. But to carry out their program, they had to first lay the foundations and develop a p-adic analogue of the theory of real semi- and subanalytic sets, and of the appropriate first-order elimination theory. At the same time this new model-theoretical outlook also feed back into real analytic geometry, e.g. with shorter proofs or more explicit versions of known results.

Roughly, the analytic elimination theory is based on the idea of using a Weierstrass factorization theorem for analytic functions to reduce to the semi-algebraic context. The key first-order language L_{an}^D consists of the following : (i) unary predicates $P_n, n \geq 1$ to denote the set of n-th powers ; (ii) a binary function symbol D to denote « truncated division », i.e. $D(x,y) = x/y$, if $y \neq 0$ and $v(x) \geq v(y)$, and $D(x,y) = 0$ otherwise [24] ; (iii) a m-ary function symbol for each formal power series in $\mathbb{Z}_p[[X_1, \ldots, X_m]]$ with coefficients converging to 0. They show that the first-order theory of $\mathbb{Z}_p$ admits quantifier elimination in this language. The setting of the geometric theory is p-adic analytic manifolds, but we will stick to affine space to give a rough idea. A subset $S \subseteq \mathbb{Q}_p^n$ is called semi-analytic at the point $\mathbf{x} \in \mathbb{Q}_p^n$ if $\mathbf{x}$ has an open neighborhood U such that $U \cap S$ is a finite union of sets of the form

$$\{\mathbf{y} \in U : f(\mathbf{y}) = 0, g_1(\mathbf{y}) \in \mathbb{Q}_p^{\times, n_1}, \ldots, g_k(\mathbf{y}) \in \mathbb{Q}_p^{\times, n_k}\}$$

where $f, g_1, \ldots, g_k$ are analytic functions on U, and $n_1, \ldots, n_k$ are positive integers and $\mathbb{Q}_p^{\times, n_i}$ denote the nonzero n_i-th powers. The set S is called semi-analytic if it is semi-analytic at each point of $\mathbb{Q}_p^n$. The set S is called subanalytic at the point $\mathbf{x} \in \mathbb{Q}_p^n$ if there is an open neihborhood U of $\mathbf{x}$ and a semi-analytic subset $S' \subseteq U \times \mathbb{Z}_p^N$ for some N, such that $U \cap S = \pi(S')$, where $\pi : U \times \mathbb{Z}_p^N \to U$ is the projection map. The set S is called subanalytic if it is subanalytic at each point of $\mathbb{Q}_p^n$. The set S is

24. The function D is reminescent of some artefacts when blowing up points in algebraic geometry. A relationship between desingularization and model theory is pointed out by the paper of Denef and Schoutens [93] on the existential theory of the field $\mathbb{F}_p((T))$.

called closed analytic if it is closed and each $\mathbf{x} \in S$ has an open neighborhood U with $U \cap S = \{\mathbf{y} \in U : f(\mathbf{y}) = 0\}$ for some analytic function f on U. Sample results : subanalytic subsets of $\mathbb{Z}_p^M$ coincide with the L_{an}^D-definable ones ; subanalytic subsets of $\mathbb{Z}_p^2$ are semi-analytic. They show that if $S \subset \mathbb{Z}_p^M$ is subanalytic and $N_{n,S}$, for $n \in \mathbb{N}$, denote the number of elements in the set $\{\mathbf{x} \mod p^n : \mathbf{x} \in S\}$, then the Poincaré series $P_S(T) = \sum_{n=0} N_{n,S} T^n$ is rational.

Interestingly, this work has consequences on the complexity of solutions to polynomial systems. Consider a polynomial system $F_i(x_1, \ldots, x_n) = 0, i = 1, \ldots, k$, where F_i involves only m_i monomials with nonzero coefficients and $\sum_i m_i = M$. A result of Khovanskii says that over $\mathbb{R}$ there is a bound $b(n, M)$ such that if such a system has finitely many solutions in $\mathbb{R}^n$ then it has at most $b(n, M)$ such solutions. Denef and van den Dries show that for each prime p there is a bound $\beta_p(n, M)$ such that if such a system has finitely many solutions in $\mathbb{Z}_p^n$ satisfying $x_i \not\equiv 0 \mod p$ then it has at most $\beta_p(n, M)$ such solutions. Lipshitz [157] has shown that the bound $\beta_p(n, M)$ can be taken independent of p. For further similar results see [192], [208].

The paper of Denef and van den Dries has been influential and triggered work in the same vein on non-archimedean analytic functions (see e.g [158]). See [160], [191], [59], [61], [62], [66], [70] for further work on p-adic analytic and subanalytic sets and functions. In particular [70] covers many redutcs of $\mathbb{Q}_p$ betwen the semi-algebraic and the subanalytic structures, and [62] deals with finer geometrical properties.

The paper of Denef and van den Dries is set up for analytic functions on the p-adic integers, the natural domain for the p-adic exponential function. For $x \in \mathbb{Q}_p$, the usual exponential series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges only if $v_p(x) \geq 1$. Note that, $(1+p)^x$ makes perfect (usual) sense when $x \in \mathbb{N}$, so that one gets by continuity a bona fide function $E_p(x) = (1+p)^x : \mathbb{Z}_p \to \mathbb{Z}_p$, such that $E_p(0) = 1, E_p(x + y) = E_p(x)E_p(y)$, and in fact $E_p(x) = e^{x \log(1+p)}$. In parallel with the work motivated by Tarski's problem on the real exponential function and which would culminate in Wilkie's proof of model-completeness and o-minimality, there was some work on the natural p-adic exponential E_p initiated by Macintyre [23, 163]. It follows from [98] that the structure of the exponential ring $(\mathbb{Q}_p, \mathbb{Z}_p, E_p)$ is P-minimal. Macintyre has some unpublished notes on (effective) model-completeness for $(\mathbb{Z}_p, E_p)$.

The tree-like structure of p-adic numbers has not been scrutinized much by model theorists. Halupczok's paper [129] seems to be a first step. A tree is associated to a semi-algebraic set as follows. Suppose that X is a semi-algebraic subset of $\mathbb{Q}_p^n$ and for each integer $\ell \geq 0$, let X_ℓ be the image of $X \cap \mathbb{Z}_p^n$ under the projection $\mathbb{Z}_p^n \to (\mathbb{Z}/p^\ell\mathbb{Z})^n$. The disjoint union $T(X) = \cup_{\ell \geq 0} X_\ell$ carries a tree structure defined by the projections $(\mathbb{Z}/p^{\ell+1}\mathbb{Z})^n \to (\mathbb{Z}/p^\ell\mathbb{Z})^n$. The main conjecture is that there are prescribed combinatorial restrictions on $T(X)$ related to the p-adic dimension of X ("$T(X)$ is a tree of level $\dim X$").

7. Algorithmics

Algorithmics is an aspect of the p-adics where some basic questions are stil unanswered.

The precise complexity of the decision problem for pCF is still unknown. In particular, it is still not known if pCF has an elementary recursive decision procedure. In that direction, [108] is still the most precise analysis. We recall that because of the value group, the complexity is at least that of the theory of $\mathbb{Z}$-groups (Presburger arithmetic) which is known to be doubly exponential. There are results for the restricted class of linear L_{Div}-formulas, i.e. where no multiplication occurs, giving (roughly) a singly exponential procedure, see [207].

Any finite subtheory of pCF is undecidable [34].

Hodgson observed in [15] that the additive group of $\mathbb{Q}_p$ (or $\mathbb{Z}_p$), as well as the multiplicative group, is decidable by finite automaton. But there seems to be no hope to get to the ring structure in this way (see [188]).

The decidability of the ring of p-adic algebraic integers is treated in [186], [77], and [113] (see also [78]), and there is related work in [197]. The decidability of the ring of totally p-adic algebraic integers is treated in [79]. This is in contradistinction with the result of Julia Robinson that the ring of totally real algebraic integers is undecidable.

For a fixed p, the decidability of the first-order theory of *all* finite extensions of $\mathbb{Q}_p$ is an open problem.

We know from [170] and [109] that the existential theory of a p-adic function field of an algebraic variety over $\mathbb{Q}_p$ is undecidable, when p is odd. Kim and Roush ([146]) had proven the case of the field of rational functions $\mathbb{Q}_p(X)$ when p is odd. Degroote and Demeyer [81] have recently given a new and shorter proof, for all p, of the case of $\mathbb{Q}_p(X)$, by relying more on the theory of quadratic forms. It is still open if any of the above hold for p-adically closed base fields with a non-archimedean value group, even for the full elementary theory. It would be interesting to have for $\mathbb{Q}_p(X)$, a proof of undecidability somewhat closer to the situation over $\mathbb{R}$, where one can rely on strong representation theorems by sums of squares (see [179]). If possible, a similar proof over the p-adics, even for the full elementary theory, would probably require some new insight, e.g. perhaps on bounds in Kochen's analogue of Hilbert's 17th problem (on this see [185]).

Scowcroft has observed that definable Skolem functions are effective [198].

For fixed $n \geq 1$, it is not known if the ring of continuous definable functions $f : \mathbb{Q}_p^n \to \mathbb{Q}_p$ is decidable or not. Note that in the case of $\mathbb{R}$, it is known [25] to be undecidable for $n > 1$, but the case $n = 1$ is still open. It is known that the ring of all continuous functions $f : \mathbb{Q}_p^n \to \mathbb{Q}_p$ is undecidable [6].

The interest for computer packages [26] to perform calculations in $\mathbb{Q}_p$ has triggered interest in effective algebraic methods in $\mathbb{Q}_p$, see for example [120]. In this vein, some hands-on methods of Krasner have found a new life in [177]. Such methods are often related to definability issues and would probably be worth looking into.

A somewhat simple $\mathbb{Q}_p$ analogue of Sturm's algorithm to determine the number of roots of a real polynomial in a given interval was spelled out by T. Sturm and Weispfenning in [208]. They provide algorithms for counting and isolating all p-adic roots

25. By transposing the arguments in [162].
26. E.g. PARI, MAPLE PADIC, MAGMA, SAGE, ...

of monic polynomials over the rationals by appropriate p-adic balls. The method is polynomial space in all input data including the prime p. But one is still puzzled by the disarming simplicity of Sturm's algorithm in the reals, and its strong consequences on complexity of the decision problem (see [108]).

8. Other topics

Some constructive aspects of $\mathbb{Q}_p$, in the sense of intuitionistic logic, have been studied in [131].

There is some work covering pairs of p-adically closed fields [83, 155, 156], and the already mentioned [51] showing the absence of the independence property for dense pairs of p-adically closed fields.

There is a well developed p-adic (as well as real) analogue of pseudo-algebraically closed fields (PAC), for this see [106, 102, 103, 104, 153, 154, 111, 112, 114, 115, 116, 184, 130]. This is also related to the above decidability results for rings of p-adic algebraic integers.

There is some work on additive reducts of the ring structure on $\mathbb{Q}_p$ [168], [159], but the situation is far less clear than for $\mathbb{R}$.

The papers [209], [126] make a connection with differential fields, but where the derivative(s) have little interaction with the valuation.

What we could call *p-adic algebra* is rather less known and developed than real algebra and seems more intricate. There are papers going in various directions, e.g. [25], [43, 44], [49], [123, 124], [118], [206]. There is some interesting ongoing work by Guzy and Tressl to work out a p-adic analogue of the real closed rings of N. Schwartz (see [125]).

Scanlon [195] has some results, using model theory, on the dynamics of analytic maps on the unit disc in a complete discretely valued field.

The function $\delta_p(x) = \frac{1}{p}(x - x^p)$ is a plain polynomial function, which appears indirectly in Kochen's p-adic analogue of Hilbert's 17th problem inside his γ-function : $\gamma(x) = \frac{1}{p}\frac{x^p - x}{(x-x^p)^2-1} = \frac{\delta_p(x)}{(p\delta_p(x))^2-1}$. But in view of the role of δ_p in the fundamentals of the p-adic numbers viewed as Witt vectors over $\mathbb{F}_p$ ([142], see also [50]), and the fact that this is the restriction of the p-derivation used in the work of Buium (see [54]), one wonders if it could shed some new light on the model theory of $\mathbb{Q}_p$.

Acknowledgements. Je remercie l'Équipe de logique mathématique CNRS de l'Université Paris Diderot – Paris 7 pour son hospitalité en 2009-2010. I thank the following people for useful remarks : É. Bouscaren, R. Cluckers, F. Delon, J. Demeyer, D. Haskell, G. Leloup, A. Macintyre, L. Moret-Bailly, P. Scowcroft. Partially supported by NSERC.

REFERENCES FOR P-ADIC MODEL THEORY UP TO 1984

[1] J. Ax and S. Kochen, *Diophantine problems over local fields I-II*, *Amer. J. Math.* **87** 1965 605-648 ; —— *III*, *Ann. of Math.* **83** (1966) 439-456.

[2] Ş. A. Basarab, *Towards a general theory of formally* $\mathfrak{p}$*-adic fields*, Manuscripta Math. **30** (1980), no. 3, 279-327.

[3] Ş. A. Basarab, *Extension of places and contraction properties for function fields over* $\mathfrak{p}$*-adically closed fields* J. Reine Angew. Math. **326** (1981), 54-78.

[4] B. J. Birch and K. McCann, *A criterion for the* p*-adic solubility of diophantine equations*, Quat. J. Math. **18** (1967) 59-63.

[5] S. Brown, *Bounds on transfer principles for algebraically closed and complete discretely valued fields*, Memoirs of the Am. Math. Soc. no 204, 1978, 92 pp.

[6] G. Cherlin, *Rings of continuous functions : decision problems*, in Model theory of algebra and arithmetic (Proc. Conf., Karpacz, 1979), pp. 44-91, Lecture Notes in Math., no 834, Springer, Berlin, 1980.

[7] P. J. Cohen, *Decision procedures for real and* p*-adic fields*, Comm. on pure and applied math. **XXII** (1969) 131-151.

[8] F. Delon, *Types sur* $\mathbb{C}((X))$, in Groupe d'Études de théories stables, année 78-79, Secrétariat mathématique, Institut Henri-Poincaré, Université Pierre et Marie Curie, 1981.

[9] J. Denef, *The rationality of the Poincaré series associated to the* p*-adic points of a variety*, Inv. Math. **77** (1984) 154-166.

[10] L. van den Dries, *Quantifier elimination for linear formulas over ordered and valued fields*, in Proceedings of the Model Theory Meeting (Univ. Brussels, Brussels/Univ. Mons, Mons, 1980), Bull. Soc. Math. Belg. Sér. B **33** (1981), no. 1, 19-31.

[11] L. van den Dries, *A specialization theorem for* p*-adic power series converging on the closed unit disc*, J. Algebra **73** (1981) 613-623.

[12] L. van den Dries, *Some applications of a model-theoretic fact to (semi)-algebraic geometry*, Indag. Math. **44** (1982) 397-401.

[13] L. van den Dries, *Algebraic theories with definable Skolem functions*, J. Symb. Logic **49** (1984) 625-629.

[14] Y. Ershov, *On elementary theories of local fields*, Algebra i Logika **4** (1965) 5-30.

[15] B. Hodgson, *Décidabilité par automate fini*, Ann. Sci. Math. Québec **7** (1983), no. 1, 39-57.

[16] M. Jarden and P. Roquette, *The Nullstellensatz over* $\mathfrak{p}$*-adically closed fields*, J. Math. Soc. Japan **32** (1980), no 3, 425-460. (MR0575400)

[17] M. Jarden, *Transfer principles for finite and* p*-adic fields*, Nieuw Arch. Wisk. (3) **28** (1980), no 2, 139-158.

[18] U. Kiehne, *Bounded products in the theory of valued fields*, J. Reine Angew. Math. **305** (1979) 9-36.

[19] S. Kochen, *Integer valued rational functions over* p*-adic numbers. A* p*-adic analogue of the theory of real fields*, in Number Theory (Houston, 1967), 57-73, Proceedings Symposia in Pure Math. **XII**, Amer. Math. Soc., 1969.

[20] S. Kochen, *The model theory of local fields*, in Logic Conference – Kiel 1974, Lecture Notes in Math., no 499, Springer, 1975.

[21] A. Macintyre, *Dense embeddings. I : A theorem of Robinson in a general setting*, in Model Theor. Algebra, Mem. Tribute Abraham Robinson, pp. 200-219, Lect. Notes Math., no 498, Springer, 1975.

[22] A. Macintyre, *On definable subsets of p-adic fields*, J. Symbolic Logic **41** (1976) 605-610.

[23] A. Macintyre, *Decision problems for exponential rings : the p-adic case*, in Foundations of computation theory (Borgholm, 1983), 285-289, Lecture Notes in Comput. Sci. no 158, Springer, 1983.

[24] A. Macintyre, K. McKenna and L. van den Dries, *Elimination of quantifiers in algebraic structures*, Adv. in Math. **47** (1983) 74-87.

[25] K. McKenna, *Some Diophantine Nullstellansätze*, in Model theory of algebra and arithmetic – Proc. Conf. Karpacz 1979, 228-247, Lecture Notes in Math., no 834, Springer, 1980.

[26] A. Nerode, *A decision procedure for p-adic integral zeros of diophantine equations*, Bull. Amer. Math. Soc. **69** (1963) 513-517.

[27] A. Prestel and P. Roquette, *Formally p-adic fields*, Lecture Notes in Math., no 1050, Springer, 1984.

[28] J. Robinson, *The decision problem for fields*, in Theory of Models (Proc. 1963 Internat. Sympos. Berkeley), 299-311, North-Holland, 1965.

[29] P. Roquette, *A criterion for rational places over local fields*, J. Reine Angew. Math. **292** (1977) 90-108.

[30] F. Taha, *Algèbres simples centrales sur les corps ultra-produits de corps p-adiques* in : Paul Dubreil and Marie-Paule Malliavin Algebra Seminar 34th Year (Paris, 1981), 89-128, Lecture Notes in Math. no 924, Springer, 1982.

[31] V. Weispfenning, *On the elementary theory of Hensel fields*, Ann. Math. Logic **10** (1976) 59-93.

[32] V. Weispfenning, *Quantifier elimination and decision procedures for valued fields*, in Models and sets (Aachen, 1983), 419-472, Lecture Notes in Math., no 1103, Springer, 1984.

[33] W. H. Wheeler, *Model complete theories of formally real fields and formally p-adic fields*, J. Symbolic Logic **48** (1983), no 4, 1130-1139.

[34] M. Ziegler, *Einige unentscheidbare Körpertheorien*, Enseign. Math. (2) 28 (1982), no. 3-4, 269-280.

OTHER REFERENCES

[35] H. Adler, *Introduction to theories without the independence property*, to appear in Arch. Math. Logic.

[36] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson and S. Starchenko, *Bounds on VC density of formulae in some NIP theories*, in Oberwolfach report 01 2010, Model theory around valued fields and dependent theories, abstracts.

[37] S. Basarab, *Relative elimination of quantifiers for henselian valued fields*, Ann. Pure Appl. Logic **53** (1991) 51-74.

[38] S. Basarab and F.-V. Kuhlmann, *An isomorphism theorem for Henselian algebraic extensions of valued fields* Manuscripta Math. **77** (1992) 113-126.

[39] L. Bélair, *Spectres p-adiques en rang fini*, C. R. Acad. Sci. Paris **305** (1987) 1-4.

[40] L. Bélair, *Substructures and uniform elimination in p-adic fields*, Annals of Pure and Applied Logic **39** (1988) 1-17.

[41] L. Bélair, *Le théorème de Macintyre, un théorème de Chevalley p-adique*, Ann. Sci. Math. Québec **14** (1990) 109-120.

[42] L. Bélair, *Spectre p-adique : aspects topologiques et géométriques*, in Séminaire sur les Structures Algébriques Ordonnées, vol. II, pp. 151-163, Publ. Math. Univ. Paris VII, no 33, Univ. Paris VII, 1990.

[43] L. Bélair, *Anneaux p-adiquement clos et anneaux de fonctions définissables*, J. Symbolic Logic **56** (1991) 539-553.

[44] L. Bélair, *Anneaux de fonctions p-adiques*, J. Symbolic Logic **60** (1995) 484- 497.

[45] L. Bélair, *Types dans les corps valués munis d'applications coefficients*, Illinois J. Math. **43** (1999) 410-425.

[46] L. Bélair and M. Bousquet, *Types dans les corps valués*, C. R. Acad. Sci Paris série I **323**(1996) 841-846.

[47] L. Bélair, L. van den Dries, and A. Macintyre, *Elementary equivalence and codimension in p-adic fields*, Manuscripta Math. **62** (1988) 219-225.

[48] L. Bélair and J.-L. Duret, *Définissabilité dans les corps de fonctions p-adiques*, J. Symbolic Logic **56** (1991) 783-785.

[49] L. Bélair and D. Haskell, *Dependence on parameters in analogues of Hilbert's seventeenth problem*, Comm. Algebra **24** (1996) 2407-2420.

[50] L. Bélair, A. Macintyre and T. Scanlon, *Model theory of the Frobenius on the Witt vectors*, Amer. J. Math. **129** (2007), no. 3, 665-721.

[51] A. Berenstein, A. Dolich and A. Onshuus, *The independence property in generalized dense pairs of structures*, J. Symbolic Logic **76** (2011), no 2, 391-404.

[52] A. Bleybel, *p-adically closed fields with nonstandard analytic structure*, J. Symbolic Logic **75** (2010) 802-816.

[53] L. Bröcker and J. H. Schinke, *On the L-adic spectrum*, Schriftenreihe des Mathematischen Instituts der Universität Münster, 2. Serie, no 40, Universitüt Münster, Mathematisches Institut, Münster, 1986, iv+45 pp.

[54] A. Buium, *Arithmetic differential equations*, Mathematical Surveys and Monographs, 118. American Mathematical Society, Providence, RI, 2005. xxxii+310 pp.

[55] A. Chernikov and P. Simon, *Externally definable sets and dependent pairs*, preprint 2010, arXiv.10074468v1 .

[56] R. Cluckers, *Classification of semi-algebraic p-adic sets up to semi-algebraic bijection*, J. reine angew. Math. 540 (2001) 105-114.

[57] R. Cluckers, *Model theory of valued fields*, in Model theory and applications, pp. 73-85 , Quad. Mat., **11**, Aracne, 2002.

[58] R. Cluckers, *Presburger sets and p-minimal fields*, J. Symbolic Logic **68** (2003) 153-162.

[59] R. Cluckers, *Analytic p-adic cell decomposition and integrals*, Trans. Amer. Math. Soc. **356** (2004) 1489-1499.

[60] R. Cluckers, *Parametrized local zeta functions*, in Logic Colloquium 2004, pp. 84-92, Lect. notes in logic , no 29, Ass. Symbolic Logic, 2008.

[61] R. Cluckers, G. Comte and F. Loeser, *Lipschitz continuity properties for p-adic semi-algebraic and subanalytic functions*, Geom. Funct. Anal. 20 (2010) 68-87.

[62] R. Cluckers, G. Comte and F. Loeser, *Local metric properties and regular stratifications of p-adic definable sets*, to appear in Comment. Math. Helv., arXiv :0910.0799 .

[63] R. Cluckers and J. Denef, *Orbital integrals for linear groups* J. Inst. Math. Jussieu **7** (2008), no. 2, 269-289.

[64] R. Cluckers and M. Edmundo, *Integration of positive constructible functions against Euler characteristic and dimension.* J. Pure Appl. Algebra **208** (2007), no. 2, 691-698.

[65] R. Cluckers, T. Hales and F. Loeser, *Transfer principle for the fundamental lemma*, in On the Stabilization of the Trace Formula, ed. by L. Clozel et al., Int. Press of Boston, 2011.

[66] R. Cluckers and I. Halupczok, *Approximations and Lipschitz continuity in p-adic semi-algebraic and subanalytic geometry*, preprint 2011, arXiv :1009.3414

[67] R. Cluckers and D. Haskell, *Grothendieck rings of $\mathbb{Z}$-valued fields*, Bull. Symb. Log. **7** (2001) 262-269.

[68] R. Cluckers and E. Leenknegt, *Rectilinearization of semi-algebraic p-adic sets and Denef's rationality of Poincaré series*, J. Number Theory **128** (2008), no. 7, 2185-2197.

[69] R. Cluckers and E. Leenknegt, *A version of p-adic minimality*, J. Symbolic Logic **77** (2012) 621-630.

[70] R. Cluckers and L. Lipshitz, *Fields with analytic structures*, J. Eur. Math. Soc. **13** (2011) , no 4, 1147-1223.

[71] R. Cluckers, L. Lipshitz and Z. Robinson, *Analytic cell decomposition and analytic motivic integration*, Ann. Scient. Éc. Norm. Sup., 4e série **39** (2006) 535-568.

[72] R. Cluckers and F. Loeser, *Ax-Kochen-Ershov theorems for p-adic integrals and motivic integration*, in Geometric methods in algebra and number theory, pp. 109-137, Birkhäuser, 2005.

[73] R. Cluckers and F. Loeser, *b-minimality*, J. Math. Log. **7** (2007) 195-227.

[74] R. Cluckers and F. Loeser, *Constructible motivic functions and motivic integration*, Invent. Math. **173** (2008) 23-121.

[75] R. Cluckers and F. Loeser, *Constructible exponential functions, motivic Fourier transform and transfer principle*, Ann. Math. **171** (2010) 1011-1065.

[76] R. Cluckers and F. Loeser, *Motivic integration for all residue characteristic for henselian discretely valued fields of characterisitic zero*, preprint 2011.

[77] L. Darnière, *Nonsingular Hasse principle for rings*, J. Reine Angew. Math. **529** (2000) 75-100.

[78] L. Darnière, *Decidability and local-global principles, in Hilbert's tenth problem : relations with arithmetic and algebraic geometry (Ghent, 1999*), pp. 145-167, Contemp. Math., **270**, Amer. Math. Soc., 2000.

[79] L. Darnière, *Pseudo-algebraically closed rings* Manuscripta Math. **105** (2001), no. 1, 13-46.

[80] L. Darnière, *Model-completions of scaled lattices*, preprint 2005.

[81] C. Degroote and J. Demeyer, *Hilbert's Tenth Problem for rational function fields over p-adic fields*, preprint 2011, arXiv :1106.4912v1 .

[82] P. Deligne, *Quelques idées maîtresses de l'oeuvre de A. Grothendieck*, in Matériaux pour l'histoire des mathématiques au XXe siècle (Nice, 1996), pp. 11-19, Séminaires et Congrès, no 3, Société Mathématique de France, 1998.

[83] F. Delon, *Extensions séparées et immédiates de corps valués*, J. Symbolic Logic **53** (1988), no. 2, 421-428.

[84] F. Delon, *Définissabilité avec paramètres extérieurs dans $\mathbb{Q}_p$ et $\mathbb{R}$*, Proc. Amer. Math. Soc. **106** (1989) 193-198.

[85] F. Delon, *Corps avec ou sans la propriété d'indépendance*, in Séminaire de structures algébriques ordonnées (1995-1996), Prépublications de l'Équipe de logique mathématique,no 61, Université Paris Diderot - Paris 7, Paris, 1997.

[86] J. Denef, *On the evaluation of certain p-adic integrals*, in Séminaire de théorie des nombres – Paris 1983-84, pp. 25-47, Progress in Math., no 59, Birkhauser, 1985.

[87] J. Denef, *P-adic semi-algebraic sets and cell decomposition*, J. Reine Angew. Math. **369** (1986) 154-166.

[88] J. Denef, *Monomialization of morphisms and p-adic quantifier elimination*, to appear in Proc. Am. Math. Soc. .

[89] J. Denef and L. van den Dries, *P-adic and real subanalytic sets*, Ann. of Math. (2) **128** (1988) 79-138.

[90] J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201-232.

[91] J. Denef and F. Loeser, *Definable sets, motives and p-adic integrals*, J. Am. Math. Soc. **14** (2001) 429-469.

[92] J. Denef and F. Loeser, *Motivic integration and the Grothendieck group of pseudo-finite fields*, in Proceedings of the international congress of mathematicians, ICM 2002, Beijing, China, August 20-28, 2002. Vol. II : Invited lectures, pp. 13-23, Beijing : Higher Education Press ; Singapore : World Scientific, 2002.

[93] J. Denef and H. Schoutens, *On the decidability of the existential theory of $\mathbb{F}_p[[t]]$*, in Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), pp. 43-60, Fields Inst. Commun., no 33, Amer. Math. Soc., 2003.

[94] J. M. Diwadkar, *Nilpotent conjugacy classes in p-adic Lie algebras : the odd orthogonal case*, Can. J. Math. **60** (2008) 88-108.

[95] L. van den Dries, *Dimension of definable sets, algebraic boundedness and Henselian fields*, Ann. Pure Appl. Logic **45** (1989), no. 2, 189-209.

[96] L. van den Dries, *The logic of local fields*, lecture notes from Logic, local fields and subanalytic sets / AMS-IMS-SIAM Joint Summer Research Conference, Amherst, June 28 - July 4, 1990.

[97] L. van den Dries, *Analytic Ax-Kochen-Ershov theorems*, Contemp. Math. **131** (1992), 379-398.

[98] L. van den Dries, D. Haskell, D. Macpherson, *One-dimensional p-adic subanalytic sets*, J. London Math. Soc. (2) **59** (1999) 1-20.

[99] L. van den Dries and P. Scowcroft, *On the structure of semialgebraic sets over p-adic fields*, J. Symbolic Logic **53** (1988) 1138-1164.

[100] D. P. Dubhashi, *Quantifier elimination in p-adic fields*, Comput. J. **36** (1993) 419-426.

[101] A. Ducros, *Espaces analytiques p-adiques au sens de Berkovich*, in Séminaire Bourbaki, Vol. 2005/2006, Exp. No. 958, 137-176, Astérisque **No. 311**, Soc. Math. de France, 2007.

[102] I. Efrat, *The elementary theory of free pseudo p- adically closed fields of finite corank*, J. Symbolic Logic **56** (1991) 484- 496.

[103] I. Efrat, *On the model companion for e-fold p-adically valued fields*, Manuscripta Math. **73** (1991), 359-371.

[104] I. Efrat, *Undecidability of pseudo p-adically closed fields*, Arch. Math. (Basel) **58** (1992) 444-452.

[105] I. Efrat, *A Galois-theoretic characterization of p-adically closed fields*, Israel J. Math **91** (1995) 273-284.

[106] I. Efrat and M. Jarden, *Free pseudo p-adically closed fields of finite corank*, J. Algebra **133** (1990) 132-150.

[107] L. Egidi, *The complexity of the theory of p-adic numbers*, in 34th Annual Symposium on Foundations of Computer Science (Palo Alto, CA, 1993), pp. 412–421, IEEE Comput. Soc. Press, Los Alamitos, CA, 1993.

[108] L. Egidi, *A quantifier elimination for the theory of p-adic numbers*, Comput. Complexity **7** (1998) 205-263.

[109] K. Eisenträger, *Hilbert's tenth problem for function fields of varieties over number fields and p-adic fields*, J. Algebra **310** (2007), no. 2, 775-792.

[110] A. J. Engler and A. Prestel, *Valued fields*, Springer, 2005.

[111] Y. L. Ershov, *Characterization of Kochen rings of PC_M-fields*, Sov. Math., Dokl. **43** (1991), n 1, 26-29 ; translation from Dokl. Akad. Nauk SSSR **316** (1991), No.1, 33-36.

[112] Y. L. Ershov, *PCp-fields with universal Galois group*, Siberian Advances in Mathematics. Siberian Adv. Math. **1** (1991), no. 4, 1-26.

[113] Y. L. Ershov, *Near regularly-Prüfer rings*, Sib. Adv. Math. **9** (1999) 1-45.

[114] Y. L. Ershov, *On amazing extensions of the field of rational numbers*, Dokl. Math. **62** (2000) 8-9.

[115] Y. L. Ershov, *Multi-valued fields*, Translated from the Russian original by Tamara Rozhkovskaya. Siberian School of Algebra and Logic. Kluwer Academic/Consultants Bureau, 2001.

[116] Y. L. Ershov, *Multiply normed fields. II.* Algebra Logika **41** (2002), no. 6, 682-712, 758 ; translation in Algebra Logic **41** (2002), no. 6, 374-390.

[117] A. Fehm and W.-D. Geyer, *A note on defining transcendentals in function fields*, J. Symb. Logic **74** (2009) 1206-1210.

[118] A. Fekak and A. Srhir, *On the p-adic algebra and its applications*, Int. Math. Forum **4** (2009), no. 25-28, 1267-1280.

[119] J. Flenner, *Relative decidability and definability in henselian valued fields*, J. Symb. Logic **76** (2011) 1240-1260.

[120] D. Ford, S. Pauli and X.-F. Roblot, *A fast algorithm for polynomial factorization over $\mathbb{Q}_p$*, J. Théor. Nombres Bordeaux **14** (2002), no. 1, 151-169.

[121] J. Gordon and Y. Yaffe, *An overview of arithmetic motivic integration*, preprint 2008, arXiv :0811.2160 .

[122] Y. Gurevich and P. H. Schmitt, *The theory of ordered abelian groups does not have the independence property*, Trans. Amer. Math. Soc. **284** (1984) no. 1, 171-182.

[123] N. Guzy, *p-convexly valued rings*, J. Pure Appl. Algebra **199** (2005), no. 1-3, 111-131.

[124] N. Guzy, *Henselian residually p-adically closed fields*, J. Pure Appl. Algebra **209** (2007), no. 2, 461-476.

[125] N. Guzy and M. Tressl, *p-adically closed rings*, in Séminaire de structures algébriques ordonnées (2007-2008), Prépublications de l'Équipe de logique mathématique, no 81, Université Paris Diderot - Paris 7, Paris, 2009.

[126] N. Guzy and F. Point, *Topological differential fields*, Ann. Pure Appl. Logic **161** (2010) 570-598.

[127] T. C. Hales, *Can p-adic integrals be computed ?*, in Contributions to automorphic forms, geometry, and number theory, pp. 313-329, Johns Hopkins Univ. Press, 2004.

[128] T. C. Hales *What is motivic measure ?* Bull. Amer. Math. Soc. (N.S.) **42** (2005), no. 2, 119-135.

[129] I. Halupczok, *Trees of definable sets over the p-adics*, J. Reine Angew. Math. **642** (2010) 157-196.

[130] D. Haran, M. Jarden and F. Pop, *Projective group structures as absolute Galois structures with block approximation*, Mem. Amer. Math. Soc. **189** (2007), no. 884, vi+56 pp.

[131] D. Haskell, *A transfer theorem in constructive p-adic algebra*, Ann. Pure Appl. Logic **58** (1992) 29-55.

[132] D. Haskell, E. Hrushovski et D. Macpherson, *Definable sets in algebraically closed valued fields : elimination of imaginaries*, J. Reine Angew. Math. **597** (2006) 175-236.

[133] D. Haskell, E. Hrushovski and D. Macpherson, *Stable domination and independence in algebraically closed valued fields*, Lecture Notes in Logic, 30, Association for Symbolic Logic, Chicago, IL / Cambridge University Press, Cambridge, 2008.

[134] D. Haskell and D. Macpherson, *A version of o-minimality for the p-adics*, J. Symbolic Logic **62** (1997) 1075-1092.

[135] H. Hasse, *Number theory*, Springer, 1980.

[136] E. Hrushovski and D. Kazhdan, *Integration in valued fields*, in Algebraic geometry and number theory, pp. 261-405, Birkhäuser, 2006.

[137] E. Hrushovski and F. Loeser, *Non-archimedean tame topology and stably dominated types*, preprint 2011.

[138] E. Hrushovski and B. Martin, *Zeta functions and definable equivalence relations*, preprint 2006, arXiv :math/0701011v1 .

[139] E. Hrushovski and A. Pillay, *Groups definable in local fields and pseudo-finite fields*, Israel J. Math. **85** (1994) 203-262.

[140] E. Hrushovski and A. Pillay, *On NIP and invariant measures*, J. European Math. Soc. **13** (2011) 1005-1061.

[141] A. Joyal, *Les théorèmes de Chevalley-Tarski et remarques sur l'algèbre constructive*, Cahiers Topologie Géom. Différentielle **16** (1975) 256-258.

[142] A. Joyal, *δ-anneaux et vecteurs de Witt*, C. R. Math. Rep Acad. Sci. Canada **VII** (1985) 177-182.

[143] T. Kalthoff, *Rein K-analytische Räume [Purely K-analytic spaces]*, Schriftenreihe des Mathematischen Instituts der Universitüt Münster. 3. Serie, no 27, Univ. Münster, Münster, 2000, iv+115 pp.

[144] I. Kaplan, T. Scanlon and F. Wagner, *Artin-Schreier extensions in dependent and simple fields*, to appear in Israel J. Math.

[145] M. Karpinski and A. Macintyre, *Approximating volumes and integrals in o-minimal and P-minimal theories*, in Connections between model theory and algebraic and analytic geometry , pp. 149-177, Quad. Mat., no 6, Aracne, 2000.

[146] K. H. Kim and F. W. Roush, *Diophantine unsolvability over p-adic function fields*, J. Algebra **176** (1995), no. 1, 83-110.

[147] J. Koenigsmann, *From p-rigid elements to valuations (with a Galois-characterization of p-adic fields). With an appendix by Florian Pop.*, J. Reine Angew. Math. **465** (1995), 165-182.

[148] J. Koenigsmann, *Defining transcendentals in function fields*, J. Symb. Logic **67** (2002) 947-956. (N.B. See [117] for some corrections.)

[149] J. Koenigsmann, *On the 'section conjecture' in anabelian geomety*, J. Reine Angew. Math. **588** (2005), 221–235.

[150] J. Krajíček and T. Scanlon, *Combinatorics with definable sets : Euler characteristics and Grothendieck rings*, Bull. Symbolic Logic **6** (2000), no. 3, 311-330.

[151] M. Krasner, *Une généralisation de la notion de corps - corpoïde. Un corpoïde remarquable de la théorie des corps valués*, C. R. Acad. Sci. Paris **219** (1944) 345-347.

[152] F.-V. Kuhlmann, *Quantifier elimination for Henselian fields relative to additive and multiplicative congruences*, Israel J. Math. **85** (1994), no. 1-3, 277-306.

[153] U.-M. Künzi, *Corps multiplement pseudo-p-adiquement clos*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 4, 205-208.

[154] U.-M. Kyuntsi (Künzi), *Decidable theories of pseudo-p-adically closed fields*, Algebra i Logika 28 (1989), no. 6, 643-669, 743 ; translation in Algebra and Logic **28** (1989), no. 6, 421-438 (1990).

[155] G. Leloup, *Théories complètes de paires de corps valués henseliens*, J. Symbolic Logic **55** (1990), 323-339.

[156] G. Leloup, *Properties of extensions of algebraically maximal fields*, Collect. Math. **54** (2003), no. 1, 53-72.

[157] L. Lipshitz, *p-adic zeros of polynomials*, J. Reine Angew. Math. **390** (1988) 208-214.

[158] L. Lipshitz and Z. Robinson, *Uniform properties of rigid subanalytic sets* Trans. Amer. Math. Soc. 357 (2005), no. 11, 4349-4377.

[159] N. Liu, *Semilinear cell decomposition*, J. Symbolic Logic **59** (1994) 199-208.

[160] N. Liu, *Analytic cell decomposition and the closure of p-adic semianalytic sets*, J. Symbolic Logic **62** (1997) 285-303.

[161] A. Lubotzky and D. Segal, *Subgroup growth*, Birkhäuser, 2003.

[162] A. Macintyre, *On the elementary theory of Banach algebras*, Ann. Math. Logic **3** (1971), no. 3, 239-269.

[163] A. Macintyre, *Effective determination of the zeros of p-adic exponential functions*, in Topics in the theory of computation (Borgholm, 1983), pp. 103-109, North-Holland, 1985.

[164] A. Macintyre, *Twenty years of p-adic model theory*, in Logic Colloquium 84 (Manchester), 121-153, North Holland, 1986.

[165] A. Macintyre, *Rationality of p-adic Poincaré series : uniformity in p*, Ann. Pure Appl. Logic 49 (1990) 31-74.

[166] A. Macintyre and P. Scowcroft, *On the elimination of imaginaries from certain valued fields*, Ann. Pure Appl. Logic **61** (1993) 241-276.

[167] D. Marker, *Model Theory : An Introduction*, Springer, 2002.

[168] D. Marker, Y. Peterzil and A. Pillay, *Additive reducts of real closed fields*, J. Symbolic Logic **57** (1992) 109-117.

[169] L. Matthews, *The independence property in unstable algebraic structures I : p-adically closed fields*, preprint, 1993.

[170] L. Moret-Bailly, *Elliptic curves and Hilbert's tenth problem for algebraic function fields over real and p-adic fields*, J. Reine Angew. Math. (2005) 587, 77-143.

[171] M.-H. Mourgues, *Cell decomposition for P-minimal fields*, MLQ Math. Log. Q. **55** (2009) 487-492.

[172] A. Onshuus and A. Pillay, *Definable groups and compact p-adic Lie groups*, J. Lond. Math. Soc. (2) **78** (2008), no. 1, 233-247.

[173] J. Pas, *Uniform cell decomposition and local zeta function*, J. Reine Angew. Math. **399** (1989) 137-172.

[174] J. Pas, *Cell decomposition and local zeta functions in a tower of unramified extensions of a p-adic field*, Proc. London Math. Soc. **65** (1990) 37-67.

[175] J. Pas, *On the angular component modulo p*, Journal of Symbolic Logic **46** (1990) 1125-1129.

[176] J. Pas, *Local zeta functions and Meuser's invariant functions*, J. Number Theory **38** (1991), no. 3, 287–299.

[177] S. Pauli et X.F. Roblot, *On the computation of all extensions of a p-adic field of a given degree*, Math. Comp. **70** (2001), no. 236, 1641-1659.

[178] D. Perrin, *Groupes henséliens*, Bull. Soc. Math. France **104** (1976), no. 4, 369-381.

[179] T. Pheidas and K. Zahidi, *Decision problems in algebra and analogues of Hilbert's tenth problem* in Model theory with applications to algebra and analysis. Vol. 2, 207-235, London Math. Soc. Lecture Note Ser., no 350, Cambridge Univ. Press, Cambridge, 2008.

[180] A. Pillay, *Sheaves of continuous definable functions*, J. Symbolic Logic **53** (1988), no. 4, 1165-1169.

[181] A. Pillay, *On fields definable in Q_p*, Arch. Math. Logic **29** (1989) 1-7.

[182] A. Pillay, *An application of model theory to real and p-adic algebraic groups*, J. Algebra **126** (1989) 139-146.

[183] B. Poizat, *A course in model theory*, Springer, 2000.

[184] F. Pop, *Classically projective groups and pseudo classically closed fields*, in Valuation theory and its applications, Vol. II (Saskatoon, SK,1999), pp. 251-283, Fields Inst. Commun., no 33, Amer. Math. Soc., 2003.

[185] A. Prestel and C. Ripoll, *Integral-valued rational functions on valued fields*, Manuscripta Math. **73** (1991) 437-452.

[186] A. Prestel and J. Schmid, *Decidability of the rings of real algebraic and p-adic algebraic integers*, J. Reine Angew. Math. **414** (1991)141-148.

[187] A. M. Robert, *A course in p-adic analysis*, Springer-Verlag, New York, 2000.

[188] S. Rabin, *Automata presenting structures : a survey of the finite string case*, Bull. Symbolic Logic **14** (2008) 169-209.

[189] E. Robinson, *The p-adic spectrum*, J. Pure Applied Alg. **40** (1986) 281-297.

[190] E. Robinson, *The geometric theory of p-adic fields*, J. Algebra **110** (1987) 158-172.

[191] Z. Robinson, *Flatness and smooth points of p-adic subanalytic sets*, Joint AILA-KGS Model Theory Meeting (Florence, 1995), Ann. Pure Appl. Logic **88** (1997) 217-225.

[192] J. M. Rojas, *Arithmetic multivariate Descartes' rule*, Amer. J. Math. **126** (2004) 1-30.

[193] P. Roquette, *History of valuation theory. Part I*, in Valuation theory and its applications, Vol I – Proceedings of the first international conference on valuation theory (Saskatoon, SK, 1999), pp. 291-355, F.-W. Kuhlmann et al. eds., Fields Inst. Commu., no 32, Amer. Math. Soc., 2002. (See http ://www.rzuser.uni-heidelberg.de/ ci3/ for an update.)

[194] M. du Sautoy and F. Loeser, *Motivic zeta functions of infinite-dimensional Lie algebras*, Selecta Math. (N.S.) **10** (2004) 253–303.

[195] T. Scanlon, *Analytical relations on a dynamical orbit*, preprint 2008.

[196] T. Scanlon, *Motivic integration : an outsider's tutorial*, slides from the lectures given at New Directions in the Model Theory of Fields, Durham, 2009, http ://math.berkeley.edu/ scanlon/papers/papers.html .

[197] J. Schmid, *Existentially closed fields with holomorphy rings*, Arch. Math. Logic **36** (1997) 127-135.

[198] P. Scowcroft, *A note on definable Skolem functions*, J. Symbolic Logic **53** (1988) 905-91.

[199] P. Scowcroft, *More on definable sets of p-adic numbers*, J. Symbolic Logic **53** (1988) 912-920.

[200] P. Scowcroft, *Cross-sections for p-adically closed fields*, J. Algebra **183** (1996) 913-928.

[201] P. Scowcroft, *More on imaginaries in p-adic fields*, J. Symbolic Logic **62** (1997) 1-13.

[202] S. Shelah, *Strongly dependent theories*, SH863, available at shelah.logic.at/ .

[203] A. Shlapentokh, *First-order definitions of rational functions and S-integers over holomorphy rings of algebraic functions of characteristic 0*, Ann. Pure Appl. Logic **136** (2005), no. 3, 267-283.

[204] P. Simon, Théories NIP, mémoire de master2, Univ. Paris sud, 2008.

[205] A. Srhir, *A new characterization of the p-adic spectrum*, Int. J. Algebra **2** (2008), no. 1-4, 53-63.

[206] A. Srhir, *P-adic ideals of p-rank d and the p-adic Nullstellensatz*, J. Pure Appl. Algebra **180** (2003) 299-311.

[207] T. Sturm, *Linear problems in valued fields* J. Symbolic Comput. **30** (2000), no. 2, 207-219.

[208] T. Sturm and V. Weispfenning, *P-adic root isolation*, RACSAM - Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **98** (2004) 239-258.

[209] M. Tressl, *The uniform companion for large differential fields of characteristic* 0, Trans. Am. Math. Soc. **357** (2005) 3933-3951.

[210] P. Ullrich, *The genesis of Hensel's p-adic numbers*, in Charlemagne and his heritage : 1200 years of civilization and science in Europe (Aachen 1995), Vol. 2 Mathematical arts, 163-178, ed. P. L. Butzer et al. , Brepols Publishers, 1998.

[211] A. J. Wilkie, *o-minimal structures*, in Séminaire Bourbaki. Vol. 2007/2008, Exp. No. 985, 131-142, Astérisque, **326**, Soc. Math. de France, 2009 [2010].

[212] Y. Yin, *Grothendieck homomorphisms in algebraically closed valued fields*, preprint 2010. arXiv :0809.0473v1

L. BÉLAIR, DÉP. DE MATHÉMATIQUES / LACIM, UQAM, C.P. 8888 SUCC. CENTREVILLE, MONTRÉAL (QUÉBEC), H3C 3P8
belair.luc@uqam.ca