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Contents

RÉSUMÉ

Le thème principale de cette thèse est l'étude des métriques presque-kählériennes extrémales compatibles sur une variété symplectique compacte.

Nous allons généraliser les notions d'invariant de Futaki et du champ de vecteurs extrémal sur une variété kählérienne compacte au cas presque-kählérien. Nous allons montrer la périodicité du champ de vecteurs extrémal quand la forme symplectique représente une classe cohomologique entière modulo torsion.

Nous donnerons une formule explicite de la courbure scalaire hermitienne en coordonnées de Darboux. Ceci nous permettra, en dimension 4, de construire des exemples de métriques strictement presque-kählériennes qui satisfont l'égalité dans les estimations de LeBrun.

Nous allons étudier la stabilité sous déformations des métriques presque-kählériennes extrémales en dimension 4. Étant donné un chemin lisse de métriques presque-kählériennes compatibles avec une forme symplectique fixe, tel que au temps zéro la métrique est kählérienne et extrémale, nous prouverons, pour un temps assez petit et sous une certaine condition, l'existence d'une famille de métriques presque-kählériennes extrémales, compatibles avec la même forme symplectique, telle que chaque structure presque-complexe induite est difféomorphe à celle induite par le chemin. En particulier, le difféomoprhisme est l'identité au temps zéro.

Sur une variété torique, nous allons discuter de l'unicité et la stabilité des métriques presque-kählériennes extrémales invariantes par un tore dans l'orbite 'complexifié' par l'action du groupe des hamiltoniens.

ABSTRACT

The principal subject of this thesis is the study of compatible extremal almost-Kähler metrics on a symplectic compact manifold.

We generalize the notions of Futaki invariant and extremal vector field of a compact Kähler manifold to the general almost-Kähler case and show the periodicity of the extremal vector field when the symplectic form represents an integral cohomology class modulo torsion. We also give an explicit formula for the hermitian scalar curvature in Darboux coordinates, which allows us to obtain examples of non-integrable extremal almost-Kähler metrics saturating LeBrun's estimates.

Given a path of almost-Kähler metrics compatible with a fixed symplectic form on a compact 4-manifold, such that at time zero the almost-Kähler metric is an extremal Kähler one, we prove, for a short time and under a certain hypothesis, the existence of a smooth family of extremal almost-Kähler metrics compatible with the same symplectic form, such that at each time the induced almost-complex structure is diffeomorphic to the one induced by the path.

On a toric symplectic manifold, we discuss the uniqueness and stability under deformations of compatible extremal almost-Kähler metrics invariant by the torus in the 'complexified orbit' of the action of the hamiltonian group.

Introduction

The central subject of this thesis is the study of compatible riemannian metrics on a given compact symplectic manifold (M, ω) of dimension $2n$. Donladson [15] showed how one can use a (formal) framework of the GIT [20] in order to define natural representatives in the space AK_ω of all compatible almost-Kähler metrics on (M, ω), which we call *extremal almost-Kähler metrics*.

Motivated by straightforward analogy with the theory of the *extremal Kähler metrics* introduced by Calabi [11], we are interested in differential-geometric properties of *non-integrable* extremal almost-Kähler metrics, their existence and uniqueness, stability under deformations, etc.

Recall that an almost-complex structure J is compatible with ω if the tensor field $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ defines a Riemannian metric on M; in this case, (J, g) is referred to as an ω-compatible almost-Kähler metric on (M, ω). Any such metric defines, in a canonical way, a hermitian connection ∇ on the complex tangent bundle $(T(M), J, g)$. Taking trace and contracting the curvature of ∇ by ω, one obtains the hermitian scalar curvature s^∇ of (J, g).

It is well-known [15, 21] that the space of all ω-compatible almost-Kähler structures, here denoted by AK_ω, is a contractible Fréchet manifold endowed with a formal Kähler structure. The infinite dimensional group $\mathrm{Ham}(M, \omega)$ of hamiltonian symplectomorphisms naturally acts on AK_ω and a crucial observation of Donaldson [15] (generalizing [21] to the non-integrable almost-Kähler case) is that this action is hamiltonian with moment map $\mu : AK_\omega \to (\mathrm{Lie}(\mathrm{Ham}(M, \omega))^*$ given by $\mu_J(f) = \int_M s^\nabla f \frac{\omega^n}{n!}$, where f is any smooth function with zero integral on M, viewed also as an element of the Lie algebra of $\mathrm{Ham}(M, \omega)$. As already observed in [8], this interpretation of s^∇ immediately implies that the critical points of the functional $J \mapsto \int_M \left(s^\nabla\right)^2 \frac{\omega^n}{n!}$ over AK_ω are the almost-Kähler metrics for which the symplectic gradient of the hermitian scalar curvature is an infinitesimal isometry of the almost-complex structure J. We refer to the almost-Kähler metrics verifying the above condition as extremal almost-Kähler metrics. In particular, extremal Kähler metrics in the sense of Calabi [11] and almost-Kähler metrics with constant hermitian scalar curvature are extremal.

The above formal picture, restricted to the subspace of diffeomorphic integrable ω-compatible almost-Kähler structures, gives many insights in the theory of extremal Kähler metrics, where the leading conjectures are derived by a considerable scope of analogy with the well-established GIT in the finite dimensional case [20]. It also suggests that extremal almost-Kähler metrics would provide a natural extension of the theory of extremal Kähler metrics to the non-integrable case. In fact, this link has already become explicit in the toric case [16], where the existence of an extremal Kähler metric is conjecturally equivalent to the existence of (infinitely many) non-integrable extremal almost-Kähler metrics; this link was also used in [6] to find an *explicit* criterion to test K-stability (and therefore (non) existence of extremal Kähler metrics) on projective plane bundles over a curve and construct explicit examples of extremal almost-Kähler metrics. Besides the now appealing motivation, a systematic study of extremal almost-Kähler metric is still to come (see however [8, 31]).

The GIT formal picture suggests the existence and the uniqueness of an extremal almost-Kähler metric, modulo the action of $\mathrm{Ham}(M,\omega)$, in each 'stable complexified' orbit of the action of $\mathrm{Ham}(M,\omega)$. However, in this formal infinite dimensional setting, a natural complexification of $\mathrm{Ham}(M,\omega)$ does not exist. When $H^1(M,\mathbb{R}) = 0$, an identification of the 'complexified' orbit of a Kähler metric $(J,g) \in AK_\omega$ is given by considering all Kähler metrics $(J,\tilde{g})$ in the Kähler class $[\omega]$ and applying Moser's Lemma [15]. In this setting, Fujiki–Schumacher [22] and LeBrun–Simanca [36] showed, in the abscence of holomorphic vector fileds, that the existence of an extremal Kähler metric is an open condition on the space of such orbits. Moreover, Apostolov–Calderbank–Gauduchon–Friedman [6] generalized this result by fixing a maximal torus T in the reduced automorphism group of (M,J) and considering T-invariant ω-compatible Kähler metrics. In general, for an almost-Kähler metric, a description of these 'complexified' orbits is not avaible, see however [16] and chapter 4 for the toric case. Nevertheless, the formal picture suggests that the existence of an extremal Kähler metric should persist for smooth almost-Kähler metrics close to an extremal one.

In chapter 1, we introduce the necessary background of almost-Kähler geometry with special attention to *holomorphic vector fields* on almost-Kähler manifolds. In particular, we obtain Bochner formulae involving the hermitian Ricci form ρ^∇ and the so-called $*$-Ricci form ρ^* and derive some vanishing results. We emphasise the importance of the canonical hermitian connection which induces a Cauchy–Riemann operator on $T^{1,0}(M)$. We shall compute the momentum map of a hamiltonian S^1-action with respect to the hermitian Ricci form ρ^∇. We give also an alternative proof to the known result that, on a compact symplectic manifold (M, ω) satisfying the *hard Lefschetz condition*, a symplectic S^1-action is hamiltonian if and only if it has fixed points.

In chapter 2, we generalize the notion of the Futaki invariant and extremal vector field to the general almost-Kähler case. This amounts to the observation that fixing a compact subgroup $G \subset \mathrm{Ham}(M, \omega)$ and considering G-invariant ω-compatible almost-Kähler structures (g, J), the L^2-projection of the hermitian scalar curvature to the finite dimensional space of hamiltonians of elements of $\mathrm{Lie}(G)$ is independent of (g, J). This fact, which easily follows from the formal picture described above and which is certainly known to experts (see e.g. [6, 25]), is established by a direct argument. By taking G be a maximal torus $T \subset \mathrm{Ham}(M, \omega)$, we show that the projection of the hermitian scalar curvature of any T-invariant compatible almost-Kähler metric defines a Killing potential, z_ω^T, which must coincide with the hermitian scalar curvature of any T-invariant extremal almost-Kähler metric (should it exist). We call the vector field $Z_\omega^T = grad_\omega z_\omega^T$ the *extremal vector field* relative to T. Its non-vanishing is an obstruction to the existence of T-invariant almost-Kähler metrics of constant hermitian scalar curvature. Noting that Z_ω^T doesn't change under a T-invariant isotopy of ω, it naturally generalizes the extremal vector field introduced [23] in the Kähler context. We prove the periodicity of Z_ω^T when $\left[\frac{\omega}{2\pi}\right]$ is integral modulo torsion.This extends the corresponding result in the Kähler case [44], but we need to adapt the 'symplectic' approach of [24] which relies on the localization formula [19]. So, we claim the following

Theorem. *Assume that* $\left[\frac{\omega}{2\pi}\right]$ *is integral modulo torsion. Then, there exist a positive integer* ν *such that* $\exp(2\pi\nu Z_\omega^T) = 1$.

Still in chapter 2, we give an explicit (local) formula of the hermitian scalar curvature in Darboux coordinates which allows us to recast the expressions of the hermitian Ricci form and the scalar curvature in the toric case [1, 16]. We then specialise to the 4-dimensional case and construct infinite dimensional families of non-integrable extremal almost-Kähler metrics by using local toric symmetry; this allows us to obtain examples of non-integrable Hermitian-Einstein almost-Kähler metrics saturating LeBrun's estimates [35].

In chapter 3, we consider the 4-dimensional case where one can introduce a notion of almost-Kähler potential related to the one defined by Weinkove [50, 51]. In the spirit of [22, 36], we then apply the Banach Implicit Function Theorem in order to construct a path of extremal T-invariant ω-compatible almost-Kähler metrics, where $T \subset \mathrm{Ham}(M, \omega)$ is a maximal torus. The main technical problem is the regularity of a family of Green operators involved in the definition of the almost-Kähler potential. Using a Kodaira–Spencer result [33, 34], one can resolve this problem if we suppose that the dimension of g_t-harmonic J_t-anti-invariant 2-forms, denoted by $h^-_{J_t}$ (see [18]), satisfies the condition $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ along the path $(J_t, g_t) \in AK^T_\omega$ in the space of T-invariant ω-compatible almost-Kähler metrics. So, we claim the following

Theorem. *Let (M, ω) be a 4-dimensional compact symplectic manifold and T a maximal torus in $Ham(M, \omega)$. Let (J_t, g_t) be any smooth family of almost-Kähler metrics in AK^T_ω such that (J_0, g_0) is an extremal Kähler metric. Suppose that $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$. Then, there exists a smooth family $(\tilde{J}_t, \tilde{g}_t)$ of extremal almost-Kähler metrics in AK^T_ω, defined for sufficiently small t, with $(\tilde{J}_0, \tilde{g}_0) = (J_0, g_0)$ and such that $\tilde{J}_t$ is equivariantly diffeomorphic to J_t.*

The condition $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ is satisified in the following cases: when J_t are integrable almost-complex structures for each t, by a well-known result of Kodaira (see [33, 34]), and when $b^+(M) = 1$ so $h^-_{J_t} = 0$ for each t. The latter condition is satisfied when (M, ω) admits a non trivial torus in $\mathrm{Ham}(M, \omega)$ [29].

Kim and Sung [31] showed that, in any dimension, if one starts with a Kähler metric of constant scalar curvature with no holomorphic vector fields, one can construct infinite dimensional families of almost-Kähler metrics of constant hermitian scalar curvature which concide with the initial metric away from an open set. Similar existence result are presented in this thesis when the initial Kähler metric is locally toric.

In chapter 4, we discuss some avenues for further research. In particular, in the toric case, we discuss the uniqueness and the stability of extremal almost-Kähler metrics invariant by the torus in the 'complexified' orbit for the action of $\mathrm{Ham}(M, \omega)$. We speculate how to prove an analogue of the above Theorem when the initial almost-Kähler metric (J_0, g_0) is Hermitian-Einstein.

Chapter I

PRELIMINARIES

1.1 Almost-Kähler metrics

An *almost-Kähler metric* on a $2n$-dimensional *symplectic* manifold (M, ω) is induced by an *almost-complex structure* J which is *ω-compatible* in the sense that the tensor field

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

is symmetric and positive definite and thus it defines a Riemannian metric on M. The fact that the tensor field g is symmetric implies that ω is *J-invariant* i.e. $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$. If, additionally, the almost complex structure J is *integrable*, then we have a *Kähler metric* on M. By the Newlander–Nirenberg theorem [43], the almost-complex structure J is integrable if and only if the *Nijenhuis tensor* N

$$4N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$$

is identically zero; here $[\cdot, \cdot]$ stands for the Lie bracket. The Nijenhuis tensor may be expressed also as

$$4N(X, Y) = J(D^g_Y J)X - J(D^g_X J)Y - (D^g_{JY} J)X + (D^g_{JX} J)Y, \tag{1.1}$$

where D^g is the Levi-Civita connection with respect to the metric g and X, Y are vector fields on M. We deduce from (1.1) that $D^g J = 0$ implies that the Nijenhuis tensor vanishes.

Moreover, J is integrable if and only if $\mathfrak{L}_{JX} J = J\mathfrak{L}_X J$ for any vector field X, where $\mathfrak{L}$ stands for the Lie derivative. Indeed, a direct computation shows that

$$\mathfrak{L}_{JX} J - J\mathfrak{L}_X J = 4N(X, \cdot). \tag{1.2}$$

The complexified tangent bundle splits as

$$T(M) \otimes \mathbb{C} = T^{1,0}(M) \oplus T^{0,1}(M),$$

where $T^{1,0}(M)$ (resp. $T^{0,1}(M)$) corresponds to the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$) under the $\mathbb{C}$-linear action of J. The complex vector bundle $(T(M), J)$ is identified with $T^{1,0}(M)$ via the map $X \mapsto X^{1,0} = \frac{1}{2}\left(X - \sqrt{-1}JX\right)$ (resp. $X^{0,1} = \frac{1}{2}\left(X + \sqrt{-1}JX\right)$). The almost-complex structure J also induces a decomposition of the complexified cotangent bundle

$$T^*(M) \otimes \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$$

where $\Lambda^{1,0}(M)$ (resp. $\Lambda^{0,1}(M)$) is the annihilator of $T^{0,1}(M)$ (resp. $T^{1,0}(M)$). The almost-complex structure J acts on the cotangent bundle $T^*(M)$ by $(J\alpha)(X) = -\alpha(JX)$. This action is extended to any section ψ of the vector bundle $\Lambda^p(M)$ of (real) p-forms by $J\psi(X_1, \cdots, X_p) = (-1)^p\psi(JX_1, \cdots, JX_p)$.

Any section ψ of $\otimes^2 T^*(M)$ (and therefore of $T^*(M) \otimes T(M)$ which is identified to $\otimes^2 T^*(M)$ via the metric) admits an orthogonal splitting $\psi = \psi^{J,+} + \psi^{J,-}$, where $\psi^{J,+}$ is the J-invariant part and $\psi^{J,-}$ is the J-anti-invariant part, given by

$$\psi^{J+}(\cdot,\cdot) = \frac{1}{2}\left(\psi(\cdot,\cdot) + \psi(J\cdot, J\cdot)\right) \text{ and } \psi^{J,-}(\cdot,\cdot) = \frac{1}{2}\left(\psi(\cdot,\cdot) - \psi(J\cdot, J\cdot)\right).$$

In particular, the bundle of 2-forms decomposes under the action of J

$$\Lambda^2(M) = \mathbb{R}\,.\,\omega \oplus \Lambda_0^{J,+}(M) \oplus \Lambda^{J,-}(M), \tag{1.3}$$

where $\Lambda_0^{J,+}(M)$ is the subbundle of the *primitive* J-invariant 2-forms (i.e. 2-forms pointwise orthogonal to ω) and $\Lambda^{J,-}(M)$ is the subbundle of J-anti-invariant 2-forms. Hence, the subbundle of primitive 2-forms $\Lambda_0^2(M)$ admits the splitting

$$\Lambda_0^2(M) = \Lambda_0^{J,+}(M) \oplus \Lambda^{J,-}(M).$$

The fact that ω is closed implies the following identities (see [32])

$$g((D^g_X J)Y, Z) + g((D^g_Y J)Z, X) + g((D^g_Z J)X, Y) = 0, \tag{1.4}$$

$$\left(D^g_X\omega\right)(Y,Z) = 2g\left(JX, N(Y,Z)\right), \tag{1.5}$$

where D^g is the Levi-Civita connection with respect to the Riemannian metric g and X, Y, Z are any vector fields. Since N is a J-anti-invariant 2-form with values in $T(M)$, it follows from (1.5) that

$$D^g_{JX}J = -JD^g_X J. \tag{1.6}$$

Moreover, we readily deduce from the relation (1.5) that the Nijenhuis tensor is identically zero if and only if ω (or equivalently J) is D^g-parallel.

The *codifferential* δ^g stands for the formal adjoint of D^g with respect to the metric g when it is applied to sections of $\otimes^p T^*(M)$. More explicitely, δ^g acts on any section ψ of $\otimes^p T^*(M)$ in the following way

$$(\delta^g\psi)(X_1,\cdots,X_{p-1}) = -\sum_i (D^g_{e_i}\psi)(e_i, X_1,\cdots,X_{p-1}),$$

where $\{e_i\}$ is a local orthonormal frame. In particular, δ^g is the formal adjoint of the exterior derivative d when it is applied to p-forms. From the relation (1.6), it follows that the symplectic form ω is δ^g-closed i.e. $\delta^g\omega = 0$ and thus g-harmonic i.e. $\Delta^g\omega = 0$, where $\Delta^g := \delta^g d + d\delta^g$ is the (Riemannian) *Laplacian* operator acting on p-forms. On any 2-form ψ, the operators Δ^g and the *Rough* Laplacian $(D^*D)^g := \delta^g D^g$ are related by the following *Weintzenböck–Bochner formula* (see e.g. [8, 10, 25])

$$\begin{aligned}(\Delta^g - (D^*D)^g)\psi &= \left(r^g(\Psi\cdot,\cdot) - r^g(\cdot,\Psi\cdot)\right) - R^g(\psi) && (1.7)\\ &= \frac{2(n-1)}{n(2n-1)}s_g\psi - 2W^g(\psi) + \frac{n-2}{n-1}\left(r^g_0(\Psi\cdot,\cdot) - r^g_0(\cdot,\Psi\cdot)\right), && (1.8)\end{aligned}$$

where R^g stands for the Riemannian curvature operator, r^g for the Ricci tensor defined as the trace of R^g, s_g for the (Riemannian) scalar curvature defined as the trace of r^g, $r^g_0 = r^g - \frac{s_g}{2n}g$ for the trace-free part of the Ricci tensor, W^g for the Weyl tensor and Ψ for the skew-symmetric endomorphism defined by $g(\Psi\cdot,\cdot) = \psi(\cdot,\cdot)$.

From the exterior derivative d, we can define the *twisted exterior differential* $d^c = (-1)^p JdJ$ acting on p-forms (in particular $d^c f = Jdf$ for a smooth function f). A direct computation shows the following relation for any smooth function f (see e.g. [25])

$$dd^c f + d^c df = 4d^c f\left(N(\cdot,\cdot)\right). \tag{1.9}$$

It follows that the almost-complex structure J is integrable if and only if d and d^c anticommute.

We denote by Λ_ω the contraction by the symplectic form ω, defined for a p-form ψ by $\Lambda_\omega(\psi) = \frac{1}{2}\sum_{i=1}^{2n}\psi\left(e_i, Je_i, \cdots\right)$, where $\{e_i\}$ is a local J-adapted orhonormal frame. As noticed by Gauduchon [25] and Merkulov [41], the commutator of Λ_ω and d is equal to

$$[\Lambda_\omega, d] = -\delta^c, \tag{1.10}$$

where $\delta^c = (-1)^p J\delta^g J$ is the *twisted codifferential* acting on p-forms; here δ^g is the the formal adjoint of d with respect to the metric g. The operator Λ_ω commutes with J, so the relation (1.10) implies

$$[\Lambda_\omega, d^c] = \delta^g. \tag{1.11}$$

It follows from (1.11) that on any almost-Kähler manifold we have [25]

$$\delta^g d^c + d^c \delta^g = 0. \tag{1.12}$$

The (Riemannian) *Hodge operator* $*_g : \Lambda^p(M) \to \Lambda^{2n-p}(M)$ is defined to be the unique isomorphism such that $\psi_1 \wedge (*_g\,\psi_2) = g(\psi_1, \psi_2)\,\frac{\omega^n}{n!}$, for any p-forms ψ_1, ψ_2. Then, the codifferential δ^g, defined as the formal adjoint of the exterior derivative d with respect to g, is related to d by the relation [10, 25]

$$\delta^g = -*_g\, d\, *_g\,.$$

It follows that

$$d = *_g\, \delta^g\, *_g\,. \tag{1.13}$$

1.1.1 The hermitian connection

Let (M, ω) be a symplectic manifold and (J, g) an ω-compatible almost-Kähler metric. The canonical hermitian connection ∇ corresponding to J on the complex tangent bundle $(T(M), J, g)$ is defined by (see e.g. [26, 39])

$$\nabla_X Y = D^g_X Y - \frac{1}{2} J\left(D^g_X J\right) Y,$$

where X, Y are vector fields on M. The connection ∇ preserves J and g (i.e. $\nabla J = \nabla g = 0$) and its torsion T^∇, defined by $T^\nabla_{X,Y} = \nabla_X Y - \nabla_Y X - [X,Y]$, is J-anti-invariant. It also induces the Cauchy–Riemann operator $\bar\partial$ on $T^{1,0}(M)$ (see [25]), where we recall $(\bar\partial Y^{1,0})(X^{0,1}) := \left[X^{0,1}, Y^{1,0}\right]^{1,0}$. Indeed,

Proposition 1.1 *For any vector fields X, Y on M*

$$\nabla_{X^{0,1}} Y^{1,0} = \left[X^{0,1}, Y^{1,0}\right]^{1,0}.$$

Proof. Using the above defintion of ∇ and the relation (1.6), we have

$$\begin{aligned}
\nabla_{X^{0,1}} Y^{1,0} &= D^g_{X^{0,1}} Y^{1,0} - \frac{1}{2} J \left(D^g_{X^{0,1}} J\right) Y^{1,0} \\
&= \frac{1}{4} D^g_{X+\sqrt{-1}JX} \left(Y - \sqrt{-1}JY\right) - \frac{1}{8} J \left(D^g_{X+\sqrt{-1}JX} J\right) \left(Y - \sqrt{-1}JY\right) \\
&= \frac{1}{4}\Big(D^g_X Y + D^g_{JX}(JY)\Big) + \frac{1}{4}\sqrt{-1}\Big(D^g_{JX} Y - D^g_X(JY)\Big) \\
&= \frac{1}{4}\left(Id - \sqrt{-1}J\right)\Big(D^g_X Y + D^g_{JX}(JY)\Big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left[X^{0,1}, Y^{1,0}\right]^{1,0} &= \frac{1}{4}\left[X + \sqrt{-1}JX, Y - \sqrt{-1}JY\right]^{1,0} \\
&= \frac{1}{8}\left(Id - \sqrt{-1}J\right)\left[X + \sqrt{-1}JX, Y - \sqrt{-1}JY\right] \\
&= \frac{1}{8}\left(Id - \sqrt{-1}J\right)\Big([X,Y] + [JX, JY] + J[JX, Y] - J[X, JY]\Big) \\
&= \frac{1}{8}\left(Id - \sqrt{-1}J\right)\Big(D^g_X Y - D^g_Y X + D^g_{JX}(JY) - D^g_{JY}(JX) \\
&\quad + JD^g_{JX} Y - JD^g_Y(JX) - JD^g_X(JY) + JD^g_{JY} X\Big) \\
&= \frac{1}{8}\left(Id - \sqrt{-1}J\right)\Big(D^g_X Y - D^g_Y X + D^g_{JX}(JY) + J(D^g_Y J)X - J(D^g_{JY} X) \\
&\quad + JD^g_{JX} Y + D^g_Y X - J(D^g_Y J)X + D^g_X Y + (D^g_{JX} J)Y + JD^g_{JY} X\Big) \\
&= \frac{1}{4}\left(Id - \sqrt{-1}J\right)\Big(D^g_X Y + D^g_{JX}(JY)\Big).
\end{aligned}$$

The proposition follows. □

1.1.2 Ricci forms

Denote by R^∇ the curvature of the canonical hermitian connection ∇. Then, the *hermitian Ricci form* ρ^∇ is the trace of $R^\nabla_{X,Y}$ viewed as an anti-hermitian linear operator of $(T(M), J, g)$, i.e.

$$\rho^\nabla(X,Y) = -tr(J \circ R^\nabla_{X,Y}).$$

Hence, the 2-form ρ^∇ is a closed (real) 2-form and it is a deRham representative of $2\pi c_1(T(M), J)$ in $H^2(M, \mathbb{R})$, where $c_1(T(M), J)$ is the first (real) Chern class. Moreover, $\sqrt{-1}\rho^\nabla$ is the curvature of the induced hermitian connection on the anti-canonical bundle $K_J^{-1}(M) = \Lambda^{n,0}(M)$ (equipped with the hermitian structure induced by g).

The following useful observation appears in [49, 50] and was communicated to me by T. Drăghici

Lemma 1.2 *Suppose that ω and $\tilde{\omega}$ are symplectic forms compatible with the same almost-complex structure J and satisfy $\tilde{\omega}^n = e^F \omega^n$ for some smooth real-valued function F on M, then*

$$\tilde{\rho}^\nabla = -\frac{1}{2} dJdF + \rho^\nabla, \tag{1.14}$$

where $\tilde{\rho}^\nabla$ is the hermitian Ricci form of the almost-Kähler metric $(J, \tilde{g})$ (here $\tilde{g}(\cdot,\cdot) = \tilde{\omega}(\cdot, J\cdot)$ is the induced Riemannian metric).

Proof. Let $\tilde{h}$ (resp h) and $\tilde{\nabla}$ (resp ∇) be the induced hermitian inner product and hermitian connection on $K_J^{-1}(M) = \Lambda^{n,0}(M)$ by the almost-Kähler structure $(\tilde{\omega}, J, \tilde{g})$ (resp. (ω, J, g)) Then, there exists a complex-valued 1-form α such that $\tilde{\nabla}_X \Psi = \nabla_X \Psi + \alpha(X)\Psi$ for any smooth section Ψ of $K_J^{-1}(M)$. Proposition 1.1 implies that $\alpha(X^{0,1}) \equiv 0$ and thus $\alpha = \xi + \sqrt{-1}J\xi$, where ξ is a real-valued 1-form. As the connections $\tilde{\nabla}$ and ∇ are hermitian, we have the following two equalities

$$\tilde{h}(\tilde{\nabla}_X\Psi, \Psi) = X.|\Psi|^2_{\tilde{h}} - \tilde{h}(\Psi, \tilde{\nabla}_X\Psi) \tag{1.15}$$

$$h(\nabla_X\Psi, \Psi) = X.|\Psi|^2_h - h(\Psi, \nabla_X\Psi). \tag{1.16}$$

Note that up to a constant $\tilde{h}(\Psi, \Psi) = \frac{(\Psi \wedge \bar{\Psi})}{\tilde{\omega}^n} = e^{-F} h(\Psi, \Psi)$. The equality (1.15) becomes

$$e^{-F} h(\nabla_X \Psi, \Psi) + e^{-F} \alpha(X) |\Psi|_h^2 \;=\; X.(e^{-F} |\Psi|_h^2) - e^{-F} h(\Psi, \nabla_X \Psi) - e^{-F} \overline{\alpha(X)} |\Psi|_h^2,$$

where $\overline{\alpha(X)}$ is the complex conjugate of $\alpha(X)$. Using (1.16), we simplify the latter equality and we obtain

$$e^{-F}(\alpha(X) + \overline{\alpha(X)}) = X.(e^{-F}).$$

This implies that $2\xi = d(\log(e^{-F})) = -dF$ and hence $\alpha = \xi + \sqrt{-1} J\xi = \frac{1}{2}\left(-dF - \sqrt{-1} JdF\right)$. The lemma follows. $\square$

We consider also $\rho^* = R(\omega)$, the image of the symplectic form ω by the (Riemannian) curvature operator R. The 2-form ρ^* is called the $*$-*Ricci form.* We have the following relation between these Ricci-type tensors (see [8])

$$\rho^{\nabla}(X, Y) = \rho^*(X, Y) - \frac{1}{4} tr(J D^g_X J \circ D^g_Y J). \tag{1.17}$$

In the Kähler case (i.e. when $D^g \omega = D^g J = 0$), we readily deduce from the relation (1.17) that $\rho^{\nabla} = \rho^* = \rho$, where ρ is the usual Ricci form defined by $\rho(\cdot, \cdot) = r^g(J\cdot, \cdot)$, where r^g is the Ricci tensor. Note that neither ρ^{∇} nor ρ^* is J-invariant in general. In fact, by (1.17) and (1.6), ρ^{∇} is J-invariant if and only if ρ^* is.

From the Weitzenböck–Bochner formula (1.7), we deduce the following relation between ρ^* and the (Riemannian) Ricci tensor r (see [8])

$$2\rho^*(X, Y) = \left(r^g(JX, Y) - r^g(X, JY)\right) + \left((D^* D)^g \omega\right)(X, Y).$$

Hence, ρ^{∇} is J-invariant if and only if $(D^* D)^g \omega$ is.

1.1.3 The decomposition of the curvature in dimension 4

In dimension $2n = 4$, the bundle of 2-forms decomposes as

$$\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M),$$

where $\Lambda^{\pm}(M)$ correspond to the eigenvalue (± 1) under the action of the (Riemannian) Hodge operator $*_g$. This decomposition is related to the splitting (1.3) as follows

$$\Lambda^+(M) = \mathbb{R} \,.\, \omega \oplus \Lambda^{J,-}(M) \text{ and } \Lambda^-(M) = \Lambda_0^{J,+}(M). \tag{1.18}$$

The (Riemannian) curvature R^g, viewed as a (symmetric) linear map of $\wedge^2(M)$, decomposes as follows

$$R^g = \begin{pmatrix} W^+ + \frac{s_g}{12} Id|_{\wedge^+(M)} & \frac{1}{2}\widetilde{r_0}|_{\wedge^-(M)} \\ \frac{1}{2}\widetilde{r_0}|_{\wedge^+(M)} & W^- + \frac{s_g}{12} Id|_{\wedge^-(M)} \end{pmatrix}$$

where $W^{\pm}$ are symmetric trace-free endomorphism of $\Lambda^{\pm}(M)$ respectively acting trivially on $\Lambda^{\mp}(M)$ and $\widetilde{r_0}$ is the (symmetric) operator defined on $\Lambda^2(M)$ as $\widetilde{r_0}(X \wedge Y) = r_0^g(X, \cdot) \wedge Y + X \wedge r_0^g(Y, \cdot)$; here $r_0^g = r^g - \frac{s_g}{4} g$ is the trace-free part of the Ricci tensor r^g and s_g is the Riemannian scalar curvature (for more details see [5]). The tensor W^+ (resp. W^-) is called the *selfdual Weyl tensor* (resp. *anti-selfdual Weyl tensor*) .

1.2 Holomorphic vectors fields on an almost-Kähler manifold

In this section, we study (pseudo-)holomorphic vector fields on an almost-Kähler manifold (M^{2n}, ω, J, g). A (real) vector field X is said *holomorphic* if $\mathfrak{L}_X J = 0$, where $\mathfrak{L}$ denotes the Lie derivative. It is equivalent to say that $[X, JY] = J[X, Y]$ for any vector field Y. Since $\mathfrak{L}_{[X,Y]} = [\mathfrak{L}_X, \mathfrak{L}_Y]$, the space of holomorphic vector fields is a Lie algebra.

When J is integrable, we readily infer from (1.2) that X is holomorphic if and only if JX is holomorphic. Hence, the space of holomorphic vector fields is a complex Lie algebra. Moreover, when M is compact, it is a finite dimensional space. Indeed, the space of holomorphic vector fields are the kernel of the (elliptic) Cauchy–Riemann operator. Furthemore, X is holomorphic if and only $X^{1,0} = \frac{1}{2}\left(X - \sqrt{-1}JX\right)$ is holomorphic as a section of the vector bundle $T^{1,0}(M)$ i.e. the components of $X^{1,0}$ are holomorphic functions.

Lemma 1.3 *On an almost-Kähler manifold (M^{2n}, ω, J, g), X is a holomorphic vector field if and only if*

$$(D^g X)^{J,-} = -\frac{1}{2} D^g_{JX} J. \tag{1.19}$$

Proof. For any vector field X, we have $\mathfrak{L}_X J = D^g_X J - [D^g X, J]$. In particular, if X is holomorphic then $D^g_X J = [D^g X, J]$. On the other hand $J[D^g X, J] = 2(D^g X)^{J,-}$. The equality (1.19) followns from (1.6). □

Note that, when J is integrable, $(D^g X)^{J,-} = 0$, so $D^g X$ is J-anti-invariant.

Corollary 1.4 *Let X be a holomorphic vector field on almost-Kähler manifold (M^{2n}, ω, J, g) and $\alpha = X^{\flat_g}$ the dual of X with respect to the metric g. Then, we have*

$$\begin{aligned} \delta^g \delta^g (D^g \alpha)^{J,-} &= 0, && (1.20) \\ dd^c \alpha &= 0. && (1.21) \end{aligned}$$

Proof. By Lemma 1.3, $(D^g \alpha)^{J,-}$ is anti-symmetric and hence we obtain (1.20). Since X is holomorphic, a direct computation shows that $\mathfrak{L}_X \omega$ is a J-invariant 2-form. Indeed, for any vector fields Y, Z

$$\begin{aligned} (\mathfrak{L}_X \omega)(JY, JZ) &= X.(\omega(JY, JZ)) - \omega(\mathfrak{L}_X(JY), JZ) - \omega(JY, \mathfrak{L}_X(JZ)) \\ &= X.(\omega(Y, Z)) - \omega(J\mathfrak{L}_X Y, JZ) - \omega(JY, J\mathfrak{L}_X Z) \\ &= (\mathfrak{L}_X \omega)(Y, Z). \end{aligned}$$

This means that $dJ\alpha$ is J-invariant. The equation (1.21) follows. □

When $\alpha = df$ for some smooth function f, the equation (1.20) becomes $\delta^g \delta^g (D^g df)^{J,-} = 0$. When M is compact, this is equivalent to the equation $(D^g df)^{J,-} = 0$. In the Kähler case, this implies that $grad_\omega f = Jgrad\, f$ is a hamiltonian Killing vector field. We deduce then that, on a compact Kähler manifold, the kernel of the so-called *Lichnerowicz operator* $L(f) = -2\delta^g \delta^g (D^g df)^{J,-}$ consists of the hamiltonian Killing vector fileds potentials (for more details see [10, 25]).

When J is integrable, using the known *dd^c-Lemma* [10, 25], the equation (1.21) implies that the *Hodge decomposition* of the dual $\alpha = X^{\flat_g}$ of a holomorphic vector field X via the metric g is given by

$$\alpha = \alpha_H + df + d^c g,$$

where α_H is a g-harmonic 1-form and f, g smooth functions on M.

Lemma 1.5 *Let X be a holomorphic vector field on a compact almost-Kähler manifold (M^{2n}, ω, J, g) and $\alpha = X^{\flat_g}$ the dual of X with respect to the metric g. Then, the following three statements are equivalent:*

1. X *is* D^g*-parallel i.e.* $D^gX = 0$.

2. α *is* g*-harmonic.*

3. $d\alpha = d^c\alpha = 0$

Proof. In the proof, we use the fact that for any 1-form ξ which verifies $d\xi = 0$, we have by a direct computation that

$$d^c\xi = D^g_{\xi^{\sharp_g}}\omega + 2(D^g\xi)^{J,+}_{J\cdot,\cdot}, \tag{1.22}$$

where $\sharp_g$ stands for the isomorphism between $T^*(M)$ and $T(M)$ induced by g^{-1}.

The statement (1) implies obviously (2). Now, we claim that (2) implies (1). Indeed, since $d\alpha = 0$, D^gX is symmetric and thus $(D^gX)^{J,-}$ is. But by hypothesis X is holomorphic, so by Lemma 1.3 $(D^gX)^{J,-}$ is anti-symmetric and hence $(D^gX)^{J,-} = D^g_X J = 0$. On the other hand, using (1.11), (1.21) and the fact that $\delta^g\alpha = 0$, we obtain $\delta^g dJ\alpha = -d^c(dJ\alpha,\omega) = -d^c\delta^g(\omega(JX,\cdot)) = d^c\delta^g\alpha = 0$. This implies that $dJ\alpha = 0$. From the above arguments and (1.22), we conclude that $(D^g\alpha)^{J,+} = 0$ and the claim follows.

Let show that (1) implies (3). The vector field X is parallel so $d\alpha = 0$ and $(D^gX)^{J,-} = -\frac{1}{2}D^g_{JX}J = 0$. Thus, using (1.19) and (1.22), we obtain that $d^c\alpha = D^g_X\omega = 0$. We claim also that (3) implies (1). Indeed, $d\alpha = d^c\alpha = 0$ implies by (1.22) that $D_X\omega = -2(D^g\xi)^{J,+}_{J\cdot,\cdot} = 0$. But X isholomorphic and thus $(D^gX)^{J,-} = -\frac{1}{2}D^g_{JX}J = 0$. Consequently, $(D^gX)^{J,-} = (D^gX)^{J,+} = 0$ so the claim follows. □

Remark. The statement (3) implies (2) without the assumption that X is holomorphic . Indeed, if we suppose that $d^c\alpha = 0$, then by the relation (1.11) we have $\delta^g\alpha = [\Lambda_\omega, d^c]\,\alpha = 0$. The question now is: does (2) implies (3) without the hypothesis that X is holomorphic? The answer is positive in the Kähler case (see [25]). Indeed, when the almost-complex J is integrable then $\Delta^g = \Delta^c$, where Δ^c is the *twisted Laplace operator* defined by $\Delta^c = \delta^c d^c + d^c\delta^c$. This follows from (1.10), (1.11) and the fact that $dd^c + d^c d = 0$ (since J is integrable). So, $\Delta^c\alpha = \Delta^g\alpha = 0$ and thus $d^c\alpha = 0$. We conclude that, on any compact complex manifold (M, J), the space of g-harmonic 1-forms is independant of the Kähler metric and closed under the action of J.

In the general almost-Kähler case, we have $\Delta^g - \Delta^c = [\Lambda_\omega, (dd^c + d^c d)]$. The operators Δ^g and Δ^c have the same *principal symbol*, so the difference is of order less than 2. Indeed, the operator $dd^c + d^c d$ is essentially given by the Nijenhuis tensor (recall the relation (1.9)).

Lemma 1.6 *Let α be a 1-form. We have*

$$\delta^g(D^g\alpha)^{J,+} - \delta^g(D^g\alpha)^{J,-} = \rho^*\left(\alpha^{\sharp_g}, J\cdot\right) - \sum_{i=1}^{2n}(D^g_{Je_i}\alpha)\left((D^g_{e_i}J)(\cdot)\right),$$

where $\{e_i\}$ is a local J-adapted orhonormal frame, δ^g the formal adjoint of the Levi-Civita connection D^g with respect to the metric g.

Proof. By using the fact that $\delta^g J = -\sum_{i=1}^{2n}(D^g_{e_i}J)(e_i) = 0$, we have

$$\begin{aligned}
\left(\delta^g(D^g\alpha)^{J,+} - \delta^g(D^g\alpha)^{J,-}\right)(X) &= -\sum_{i=1}^{2n} D^g_{e_i}\left((D^g\alpha)^{J,+} - (D^g\alpha)^{J,-}\right)(e_i, X)\\
&= -\sum_{i=1}^{2n}\Big[D^g_{e_i}\left[(D^g_{Je_i}\alpha)(JX)\right] - (D^g_{D^g_{e_i}(Je_i)}\alpha)(JX) - (D^g_{Je_i}\alpha)(JD^g_{e_i}X)\Big]\\
&- (D^g_{Je_i}\alpha)(JD^g_{e_i}X)\Big]\\
&= -\sum_{i=1}^{2n}\Big[\left(D^g_{e_i}(D^g_{Je_i}\alpha)\right)(JX) - (D^g_{D^g_{e_i}(Je_i)}\alpha)(JX)\\
&+ (D^g_{Je_i}\alpha)\left((D^g_{e_i}J)(X)\right)\Big]\\
&= -\sum_{i=1}^{2n}(D^2_{e_i,Je_i}\alpha)(JX) - \sum_{i=1}^{2n}(D^g_{Je_i}\alpha)\left((D^g_{e_i}J)(X)\right)\\
&= \frac{1}{2}\sum_{i=1}^{2n}(R_{e_i,Je_i}\alpha)(JX) - \sum_{i=1}^{2n}(D^g_{Je_i}\alpha)\left((D^g_{e_i}J)(X)\right)\\
&= \rho^*_{\alpha^{\sharp_g},JX} - \sum_{i=1}^{2n}(D^g_{Je_i}\alpha)\left((D^g_{e_i}J)(X)\right). \square
\end{aligned}$$

Corollary 1.7 *Let α be a 1-form such that $X = \alpha^{\sharp_g}$ is holomorphic vector field. Then*

$$\delta^g(D^g\alpha)^{J,+} - \delta^g(D^g\alpha)^{J,-} = \rho^\nabla(X, J\cdot).$$

Proof. Combining Lemmas 1.3 and 1.6 and using the relations (1.4) and (1.17), we have

$$\begin{aligned}
\left(\delta^g(D^g\alpha)^{J,+} - \delta^g(D^g\alpha)^{J,-}\right)(Y) &= \rho^*_{X,JY} - \sum_{i=1}^{n}(D^g_{Je_i}\alpha)\Big((D^g_{e_i}J)(Y)\Big)\\
&= \rho^{\nabla}_{X,JY} + \frac{1}{4}tr(JD^g_XJ\circ D^g_{JY}J) - \sum_{i=1}^{2n}(D^g_{Je_i}\alpha)\Big((D^g_{e_i}J)(Y)\Big)\\
&= \rho^{\nabla}_{X,JY} + \frac{1}{4}tr(JD^g_XJ\circ D^g_{JY}J) - \sum_{i=1}^{2n}\left((D^g\alpha)^{J,-}_{Je_i}\right)\Big((D^g_{e_i}J)(Y)\Big)\\
&= \rho^{\nabla}_{X,JY} + \frac{1}{4}tr(JD^g_XJ\circ D^g_{JY}J) + \frac{1}{2}\sum_{i=1}^{2n}g\Big((D^g_{JX}J)(Je_i),(D^g_{e_i}J)(Y)\Big)\\
&= \rho^{\nabla}_{X,JY} + \frac{1}{4}tr(JD^g_XJ\circ D^g_{JY}J) - \frac{1}{2}\sum_{i=1}^{2n}g\Big((D^g_XJ)(e_i),(D^g_{e_i}J)(Y)\Big)\\
&= \rho^{\nabla}_{X,JY} + \frac{1}{4}tr(JD^g_XJ\circ D^g_{JY}J)\\
&\quad -\frac{1}{2}\sum_{i,k=1}^{2n}\Big((D^g_XJ)(e_i,e_k)\otimes(D^g_{e_i}J)(Y,e_k)\Big)\\
&= \rho^{\nabla}_{X,JY} + \frac{1}{4}tr(JD^g_XJ\circ D^g_{JY}J)\\
&\quad -\frac{1}{4}\sum_{i,k=1}^{2n}\Big((D^g_XJ)(e_i,e_k)\otimes\Big[(D^g_{e_i}J)(Y,e_k) - (D^g_{e_k}J)(Y,e_i)\Big]\Big)\\
&= \rho^{\nabla}_{X,JY} + \frac{1}{4}tr(JD^g_XJ\circ D^g_{JY}J)\\
&\quad -\frac{1}{4}\sum_{i,k=1}^{2n}\Big((D^g_XJ)(e_i,e_k)\otimes(D^g_YJ)(e_i,e_k)\Big)\\
&= \rho^{\nabla}_{X,JY} + \frac{1}{4}tr(JD^g_XJ\circ D^g_{JY}J) - \frac{1}{4}\sum_{i=1}^{2n}g\Big((D^g_XJ)(e_i),(D^g_YJ)(e_i)\Big)\\
&= \rho^{\nabla}_{X,JY} - \frac{1}{4}tr(D^g_XJ\circ D^g_YJ) + \frac{1}{4}\sum_{i=1}^{2n}g\Big((D^g_XJ)(D^g_YJ)e_i,e_i\Big)\\
&= \rho^{\nabla}_{X,JY} - \frac{1}{4}tr(D^g_XJ\circ D^g_YJ) + \frac{1}{4}tr(D^g_XJ\circ D^g_YJ)\\
&= \rho^{\nabla}_{X,JY}. \ \square
\end{aligned}$$

Corollary 1.8 *Let (M^{2n},ω,J,g) be a compact almost-Kähler manifold. Suppose that the tensor $(\rho^*)^{J,+}(\cdot,J\cdot)$ is negative-definite everywhere. Then, there is no non-trivial holomorphic vector field on M.*

Proof. Let α be the dual (by the metric g) of a holomorphic vector field X. By Lemma 1.6, we have

$$\left(\delta^g(D^g\alpha)^{J,+} - \delta^g(D^g\alpha)^{J,-}\right)(X) = \rho^*(X, JX) - \sum_{i=1}^{2n}(D^g_{Je_i}\alpha)\left((D^g_{e_i}J)(X)\right). \tag{1.23}$$

Using Lemma 1.3, we simplify the second term of the right hand side of (1.23) as in the proof of Corollary 1.7

$$\begin{aligned}
\sum_{i=1}^{n} g\Big((D^g_{Je_i}\alpha), (D^g_{e_i}J)(X)\Big) &= \sum_{i=1}^{2n} g\Big((D^g\alpha)^{J,-}_{Je_i}, (D^g_{e_i}J)(X)\Big) \\
&= -\frac{1}{2}\sum_{i=1}^{2n} g\Big((D^g_{JX}J)(Je_i), (D^g_{e_i}J)(X)\Big) \\
&= \frac{1}{2}\sum_{i=1}^{2n} g\Big((D^g_X J)(e_i), (D^g_{e_i}J)(X)\Big) \\
&= \frac{1}{2}\sum_{i,k=1}^{2n} \Big((D^g_X J)(e_i, e_k) \otimes (D^g_{e_i}J)(X, e_k)\Big) \\
&= \frac{1}{4}\sum_{i,k=1}^{2n} \Big((D^g_X J)(e_i, e_k) \otimes \left[(D^g_{e_i}J)(X, e_k) - (D^g_{e_k}J)(X, e_i)\right]\Big) \\
&= \frac{1}{4}\sum_{i,k=1}^{2n} \Big((D^g_X J)(e_i, e_k) \otimes (D^g_X J)(e_i, e_k)\Big) \\
&= \frac{1}{4}\sum_{i=1}^{2n} \Big((D^g_X J)(e_i), (D^g_X J)(e_i)\Big).
\end{aligned}$$

Integrating (1.23), we obtain

$$\left|(D^g\alpha)^{J,+}\right|^2_{L^2} - \left|(D^g\alpha)^{J,-}\right|^2_{L^2} = \left(\int_M \rho^*(X, JX)\frac{\omega^n}{n!}\right) - \frac{1}{4}\left|D^g_X J\right|^2_{L^2}.$$

Because of Lemma 1.3, $\left|(D^g\alpha)^{J,-}\right|^2_{L^2} = \frac{1}{4}\left|D^g_X J\right|^2_{L^2}$. Then, $\left|(D^g\alpha)^{J,+}\right|^2_{L^2} = \left(\int_M \rho^*(X, JX)\frac{\omega^n}{n!}\right)$. The corollary follows. □

Lemma 1.9 *Let (M^{2n}, ω, J, g) be an almost-Kähler manifold. If X is a hamiltonian Killing vector field, then*

$$-\frac{1}{2}d\Delta^g f^X = \rho^\nabla(X, \cdot), \tag{1.24}$$

where f^X is the momentum of X with respect to ω (i.e. $\omega(X, \cdot) = -df^X$) and Δ^g is the Riemannian Laplacian with respect to g.

Proof. By hypothesis, $X = (d^c f)^{\sharp_g}$ is a Killing vector field then $D^g d^c f = \frac{1}{2} dd^c f$. So, we have $\delta^g D^g d^c f = \delta^g (D^g d^c f)^{J,+} + \delta^g (D^g d^c f)^{J,-} = \frac{1}{2}\delta^g dd^c f$. Combining this with Corollary 1.7 applied for the holomorphic vector field X, we obtain

$$2\delta^g (D^g d^c f)^{J,-} = \frac{1}{2}\delta^g dd^c f - \rho^\nabla(X, J\cdot). \tag{1.25}$$

On the other hand (by Lemma 1.3) $(D^g d^c f)^{J,-} = \frac{1}{2} D^g_{(df)^{\sharp_g}}\omega$. Then, using the relations (1.5) and (1.9), we obtain $(D^g d^c f)^{J,-} = \frac{1}{4}(dd^c f + d^c df)$ and therefore

$$\delta^g (D^g d^c f)^{J,-} = \frac{1}{4}\delta^g (dd^c f + d^c df). \tag{1.26}$$

Combining (1.26) with (1.25), we obtain, using the relation (1.12)

$$\begin{aligned} \frac{1}{2}\delta^g (dd^c f + d^c df) &= \frac{1}{2}\delta^g dd^c f - \rho^\nabla(X, J\cdot) \\ \frac{1}{2}\delta^g d^c df &= -\rho^\nabla(X, J\cdot) \\ -\frac{1}{2} d^c \delta^g df &= -\rho^\nabla(X, J\cdot) \\ -\frac{1}{2} d\Delta^g f &= \rho^\nabla(X, \cdot). \end{aligned}$$

□

Corollary 1.10 *Let (M^{2n}, ω, J, g) be a compact almost-Kähler manifold. Suppose that the tensor $(\rho^\nabla)^{J,+}(\cdot, J\cdot)$ is negative-semidefinite everywhere. Then, there is no non-trivial hamiltonian Killing vector field on M.*

Proof. Suppose that $X = (d^c f)^{\sharp_g}$ is a hamiltonian Killing vector field. By Lemma 1.9, we have

$$-\frac{1}{2} d\Delta^g f\left((df)^{\sharp_g}\right) = \rho^\nabla\left((d^c f)^{\sharp_g}, (df)^{\sharp_g}\right).$$

By integrating, we obtain $\frac{1}{2}\left|\Delta^g f\right|^2_{L^2} = \left(\int_M \rho^\nabla\left((df)^{\sharp_g}, (d^c f)^{\sharp_g}\right)\frac{\omega^n}{n!}\right)$. If $\Delta^g f$ is identically zero on a compact Riemannian manifold, f must be constant and thus $X = 0$. The corollary follows. □

1.2.1 Killing vector fields with zeros

Given a $2n$-dimensional compact connected symplectic manifold (M,ω), it is well-known [40, 45] that if the map $\wedge\omega^{n-1} : H^1(M,\mathbb{R}) \to H^{2n-1}(M,\mathbb{R})$ is an isomorphism, then a symplectic S^1-action on (M,ω) is hamiltonian if and only if it has fixed points. We give here an alternative argument of this statement based on the *Hodge decomposition.*

A $2n$-dimensional symplectic manifold (M,ω) is said to satisfy the *hard Lefschetz condition* if the map $\wedge\omega^k : H^{n-k}(M,\mathbb{R}) \to H^{n+k}(M,\mathbb{R})$ is an isomorphism for any $k \leqslant n$. For example, Kähler manifolds satisfy this condition. We deduce the following for manifolds satisfying such condition

Lemma 1.11 *Let (M^{2n},ω,J,g) be a compact almost-Kähler manifold wich satisfy the hard Lefschetz condition and α be a 1-form which is δ^g-exact. Then, $\alpha = d^c f$ for some function f if and only if $d^c\alpha = 0$.*

Proof. Let $\beta = J\alpha$. Since $\alpha = \delta^g\phi$ for some 2-form ϕ then $\beta = \delta^c\psi$ where $\psi(\cdot,\cdot) = \phi(J\cdot,J\cdot)$. Hence, β is δ^c-exact and d-closed since $d^c\alpha = 0$. By the result of Merkulov [41, Porposition 1.4], $\beta = d\delta^c\xi$ for some 1-form ξ. It follows that $\alpha = d^c f$, where $f = -\delta^c\xi$. □

Proposition 1.12 *[40, 45] Let X be a Killing vector field on a compact almost-Kähler manifold (M^{2n},ω,J,g) which satisfies the Hard Lefschetz condition. Then, X admits zeros if and only if X is a hamiltonian Killing vector field.*

Proof. X is a Killing vector field with respect to the metric g. Let $\alpha = X^{\flat_g}$ be the dual of X with respect to g. Then, α admits the following Hodge composition $\alpha = \alpha_H + \delta^g\phi$, where α_H is the g-harmonic part of α. Moreover, $\mathfrak{L}_X\alpha_H = 0$, where $\mathfrak{L}$ is the Lie derivative (see [25]). Indeed, Killing vector fields preserve the g-harmonic p-forms. On the other hand, by *Cartan formula*, $\mathfrak{L}_X\alpha_H = d(\alpha_H(X))$, so $(\alpha_H(X))$ is constant. Furthermore, $\int_M (\alpha_H(X))\frac{\omega^n}{n!} = \langle\alpha_H,\alpha\rangle_{L^2} = \langle\alpha_H,\alpha_H + \delta^g\phi\rangle_{L^2} = \langle\alpha_H,\alpha_H\rangle_{L^2}$. It follows that $\langle\alpha_H,\alpha_H\rangle_{L^2}$ is constant. But X admits zeros, it implies that $\alpha_H = 0$. We deduce that $\alpha = \delta^g\phi$ so α is δ^g-exact. Moreover, $\mathfrak{L}_X\omega = dJ\alpha = 0$ since X is Killing. The claim follows from Lemma 1.11. □

1.2.2 A localization formula

Let (M^{2n}, ω) be a compact symplectic manifold endowed with a hamiltonian S^1-action. Let X be the generator of this action with a momentum f^X i.e. $\omega(X, \cdot) = -df^X$. Thus, X is a hamiltonian Killing vector field with respect to some compatible almost-Kähler metrics. The fixed points of the action form a finite union of connected symplectic submanifolds [40] $N_1, \cdots, N_\gamma$ such that f^X is constant on each N_j. For each N_j, the normal bundle E_j splits into complex line bundles $E_j = L_1^j \oplus \cdots \oplus L_{m_j}^j$ on which S^1 acts with integer weights $k_1^j, \cdots, k_{m_j}^j$. Then, we have the following formula (see [19, 40])

$$\int_M e^{-hf^X} \frac{\omega^n}{n!} = \sum_{j=0}^{\gamma} \left(\int_{N_j} \prod_{i=1}^{m_j} \frac{1}{c_1(L_i^j) + k_i^j h} \right) e^{-hf^X(N_j)}, \tag{1.27}$$

for every $h \in \mathbb{C}$. Here, $c_1(\cdot)$ denote the first Chern class and we take the formal inverse

$$\frac{1}{c_1(L_i^j) + k_i^j h} = \frac{1}{k_i^j h} \sum_{r=1}^{(dim N_j)/2} \left(-\frac{c_1(L_i^j)}{k_i^j h} \right)^r .$$

Chapter II

EXTREMAL ALMOST-KÄHLER METRICS

For this chapter, (M, ω) is a compact and connected symplectic manifold of dimension $2n$. We denote by $[\omega]$ the deRham cohomology class of ω. Any ω-compatible almost-complex structure J is identified with the induced Riemannian metric.

2.1 Extremal almost-Kähler metrics

2.1.1 Hermitian scalar curvature as a moment map.

We define the *hermitian scalar curvature* s^∇ of an almost-Kähler structure (ω, J, g) as the trace of ρ^∇ with respect to ω, i.e.

$$s^\nabla \omega^n = 2n \left(\rho^\nabla \wedge \omega^{n-1}\right), \tag{2.1}$$

or, in equivalent way,

$$s^\nabla = 2\Lambda_\omega \rho^\nabla. \tag{2.2}$$

Denote by AK_ω the Fréchet space of ω-compatible almost-complex structures. The space AK_ω comes naturally equipped with a formal Kähler structure $(\mathbf{\Omega}, \mathbf{J})$ (described first by Fujiki in [21]). More precisely, the tangent space $T_J(AK_\omega)$ at a point J is identified with the space of g-symmetric, J-anti-invariant fields of endomorphisms of $T(M)$ (where (ω, J, g) is the corresponding almost-Kähler metric). Then, for $A, B \in T_J(AK_\omega)$, the Kähler form $\mathbf{\Omega}$ is given by $\mathbf{\Omega}_J(A, B) = \int_M tr\,(J \circ A \circ B) \frac{\omega^n}{n!}$ while the $\mathbf{\Omega}$-compatible (integrable) almost-complex structure $\mathbf{J}$ is defined by $\mathbf{J}_J X = J \circ X$.

Let $\mathrm{Ham}(M,\omega)$ be the group of hamiltonian symplectomorphisms of (M,ω). The Lie algebra of $\mathrm{Ham}(M,\omega)$ is identified with the space of smooth functions on M with zero mean value and the *Poisson bracket* defined by $\{f_1,f_2\} = \omega(grad_\omega f_1, grad_\omega f_2)$; it is also equipped with an equivariant inner product, given by the L^2-norm with respect to $\frac{\omega^n}{n!}$.

The group $\mathrm{Ham}(M,\omega)$ acts naturally on AK_ω by $\gamma_* J\gamma_*^{-1}$, where $\gamma \in \mathrm{Ham}(M,\omega)$ and $J \in AK_\omega$. For a vector field $Z \in \mathrm{Lie}(\mathrm{Ham}(M,\omega))$, the induced vector field $\hat{Z}$ by the action on AK_ω is given at a point J by $\hat{Z}(J) = -\mathfrak{L}_Z J$, where $\mathfrak{L}$ is the Lie derivative. A key observation, made by Fujiki [21] in the integrable case and by Donaldson [15] in the general almost-Kähler case, asserts that this natural action of $\mathrm{Ham}(M,\omega)$ on AK_ω is hamiltonian with momentum given by the hermitian scalar curvature s^∇. More precisely, the moment map is

$$\mu_J(f) = \int_M s^\nabla f \frac{\omega^n}{n!} \tag{2.3}$$

where s^∇ is the hermitian scalar curvature of the induced almost-Kähler metric (ω, J, g). The square-norm of the hermitian scalar curvature defines a functional on AK_ω

$$J \mapsto \int_M (s^\nabla)^2 \frac{\omega^n}{n!}. \tag{2.4}$$

Definition 2.1 *The critical points of the functional (2.4) are called extremal almost-Kähler metrics .*

The functional (2.4) corresponds to the *square-norm function* of the moment map. This observation was used [8] to deduce that the critical points of (2.4) are precisely the almost-Kähler metrics g for which the symplectic gradient of their hermitian scalar curvature $grad_\omega s^\nabla = J grad\, s^\nabla$ is a Killing vector field of g; as it is hamiltonian, this is also equivalent to being holomorphic with respect to J. Indeed, it follows from [20] that a point x_0 is critical for the square norm function of the momentum map if and only if the image of x_0 by the moment map belongs to the Lie subalgebra corresponding to the stabilizer of x_0 by the action (where the Lie algebra is identified with its dual vector space via the inner product). In our context, this implies that if J is a critical point of (2.4) then $\mu(J) = s^\nabla$ is an element of the Lie subalgebra of $\mathrm{Lie}(\mathrm{Ham}(M,\omega))$ (by replacing s^∇ with its zero integral part) corresponding to the stabilizer of

J. Hence, $\mathfrak{L}_{grad_\omega s^\nabla} J = 0$. This precisely means that $grad_\omega s^\nabla$ is a hamiltonian Killing vector field.

Proposition 2.2 *A metric g is a critical point of (2.4) if and only if $grad_\omega s^\nabla$ is a Killing vector field with respect to g.*

Proof. We reproduce here a direct verification made by Gauduchon [25] using Mohsen formula [42]. The Mohsen formula states that, for a path $J_t \in AK_\omega$, the first variation of the connection 1-forms α_t of the hermtian connections on $K_{J_t}^{-1}(M)$, induced by the canonical hermitian connections ∇_t corresponding to J_t, is given by

$$\frac{d}{dt}\alpha_t = \frac{1}{2}\delta^{g_t}\dot{J},$$

where δ^{g_t} is the codifferential with respect to the metric $g_t(\cdot,\cdot) = \omega(\cdot, J_t\cdot)$ and $\dot{J} = \frac{d}{dt}J_t$. Therefore, by definition, $\frac{d}{dt}\rho^{\nabla_t} = -\frac{1}{2}d\delta^{g_t}\dot{J}$. Hence, by (2.2) and (1.10), we obtain $\frac{d}{dt}s^{\nabla_t} = -\Lambda_\omega d\delta^{g_t}\dot{J} = \delta^c_t\delta^{g_t}\dot{J} = -\delta^{g_t}J_t\delta^{g_t}\dot{J}$; here δ^c_t is the twisted codifferential with respect to J_t. Therefore,

$$\frac{d}{dt}s^{\nabla_t} = -\delta^{g_t}J_t\delta^{g_t}\dot{J}. \tag{2.5}$$

Using (2.5), we compute the first derivative of (2.4) in the direction of $\dot{J}$

$$\frac{d}{dt}\left(\int_M (s^{\nabla_t})^2\frac{\omega^n}{n!}\right) = 2\int_M s^{\nabla_t}\left(-\delta^{g_t}J_t\delta^{g_t}\dot{J}\right)\frac{\omega^n}{n!} = 2\int_M g_t\left(D^{g_t}d^c_t s^{\nabla_t}, \dot{J}\right)\frac{\omega^n}{n!},$$

where d^c_t is the twisted exterior differential corresponding to J_t. However, $\dot{J} = \frac{d}{dt}J_t$ is a g_t-symmetric, J_t-anti-invariant endomorphism of $T(M)$. Hence, J is a critical point if and only if the symmetric, J-anti-invariant part of $D^g d^c s^\nabla$ is identically zero. On the other hand, for any vector field X preserving ω, we have

$$\begin{aligned} 0 &= (\mathfrak{L}_X\omega)(JY, Z) \\ &= -(\mathfrak{L}_X g)(Y, Z) + g((\mathfrak{L}_X J)JY, Z) \\ &= -g((D^g X)Y, Z) - g(Y, (D^g X)Z) + g((\mathfrak{L}_X J)JY, Z). \end{aligned}$$

It follows that the symmetric part of $D^g d^c s^\nabla$ is already J-anti-invariant. Hence, J is a critical point if and only if $D^g d^c s^\nabla$ is skew-symmetric which means that the symplectic gradient $grad_\omega s^\nabla$ is a Killing vector field. $\square$

Remark. On a complex manifold (M, J), the Calabi problem [11] consist in studying the Calabi functional given by the L^2-norm of the scalar curvature of the Kähler metrics whose Kähler form belongs to a fixed Kähler class $\Omega = [\omega]$. It turns out that the critical points of the Calabi functional, called *Calabi extremal Kähler metrics* , are the Kähler metrics for which the symplectic gradient of the scalar curvature is a Killing vector field. The extremal almost-Kähler metrics thus appear as a natural extension of the Calabi extremal Kähler metrics to the non-integrable case. Indeed, since any two Kähler forms in a fixed Kähler class Ω are isotopic, the Kähler metrics in $\Omega =[\, \omega]$ are embedded, via Moser's lemma [40], in the space of ω-compatible integrable almost-complex structures K_ω. □

2.1.2 The extremal vector field

We fix a compact group G in the (infinite dimensional) group $\mathrm{Ham}(M, \omega)$ of hamiltonian symplectomorphisms of (M, ω). Denote by $\mathfrak{g}_\omega \subset C^\infty(M)$ the finite dimensional space of smooth functions which are hamiltonians with zero mean value of elements of $\mathfrak{g} = \mathrm{Lie}(G)$. It is well-known that $\mathfrak{g}_\omega$ has a Lie algebra structure given by the Poisson bracket and $\mathfrak{g}_\omega$ is isomorphic to $\mathfrak{g}$. Denote by Π_ω the L^2-orthogonal projection of a smooth function on $\mathfrak{g}_\omega$ with respect to the volume form $\frac{\omega^n}{n!}$. Let AK^G_ω be the space of ω-compatible G-invariant almost-complex structures. As AK_ω is contractible, it is also connected; the same is true for AK^G_ω by taking the average of a path of metrics in AK_ω over G.

In this context, the following remark, generalizing [6, Lemma 2] suggests the definition of an *extremal vector field* of AK^G_ω:

Lemma 2.3 *Let J_t be a smooth family of almost-complex structures compatible with the fixed symplectic form ω, which are invariant under a compact group G of symplectomorphisms acting in a hamiltonian way on the compact symplectic manifold (M, ω). Then, the L^2-orthogonal projection of the hermitian scalar curvature s^{∇_t} of (ω, J_t, g_t) to $\mathfrak{g}_\omega$ is independent of t.*

Proof. By definition, any $f \in \mathfrak{g}_\omega$ defines a vector field $X = grad_\omega f$ which is in $\mathfrak{g}$, and is therefore Killing with respect to any of the metrics $g_t(\cdot, \cdot) = \omega(\cdot, J_t \cdot)$ in AK^G_ω. To prove our claim, we have to show that $\int_M f s^{\nabla_t} \frac{\omega^n}{n!}$ is independent of t. Using the relation (2.5), we obtain

$$\frac{d}{dt}\left(\int_M f s^{\nabla_t} \frac{\omega^n}{n!}\right) = \int_M \left(-\delta^{g_t} J_t \delta^{g_t} \dot{J}\right) f \frac{\omega^n}{n!} = \int_M g_t(\dot{J}, D^{g_t} d_t^c f) \frac{\omega^n}{n!}.$$

The fact that $X = grad_\omega f$ is Killing implies that $D^{g_t} d_t^c f$ is an anti-symmetric tensor. The result follows if we recall that $\dot{J}$ is a g_t-symmetric endomorphism of $T(M)$. □

Remark. The above lemma can also be viewed as a consequence from the fact that s^∇ is the momentum map for the action of $\mathrm{Ham}(M,\omega)$. Indeed, consider a Lie subgroup $\mathbb{G} \subset \mathbb{H}$ where $\mathbb{H}$ is a Lie group (equipped with a bi-invariant metric) acting in a hamiltonian way on a symplectic manifold with moment map $\mu^{\mathbb{H}}$. Let N be a $\mathbb{G}$-invariant connected subspace and denote by $\mu^{\mathbb{G}}$ the projection (with respect to the inner product) of the image of $\mu^{\mathbb{H}}$ on the dual of the Lie subalgebra of $\mathbb{G}$. Since N is $\mathbb{G}$-invariant, the differential of $\mu^{\mathbb{G}}|_N$ (restriction of $\mu^{\mathbb{G}}$ to N) is zero. Therefore, $\mu^{\mathbb{G}}|_N$ is constant. In our case, $\mathbb{H} = \mathrm{Ham}(M,\omega), \mathbb{G} = G, \mathrm{Lie}(G) \cong \mathfrak{g}_\omega, N = AK_\omega^G$ and $\mu^{\mathbb{H}}$ is given by (2.3). Lemma 2.3 is equivalent to the fact that $\mu^{\mathbb{G}}|_N$ is constant. □

Given any $J \in AK_\omega^G$, we define $z_\omega^G := \Pi_\omega s^\nabla$, where s^∇ is the hermitian scalar curvature of (ω, J, g). This z_ω^G is independant of J by Lemma 2.3. Let $G = T$ be a maximal torus in $\mathrm{Ham}(M,\omega)$. We obtain the following lemma

Lemma 2.4 *For any $J \in AK_\omega^T$, the almost-Kähler metric (ω, J, g) is extremal if and only if*

$$\mathring{s}^\nabla = z_\omega^T,$$

where $\mathring{s}^\nabla$ is the integral zero part of the hermitian scalar curvature s^∇ of (ω, J, g) given by $\mathring{s}^\nabla = s^\nabla - \frac{\int_M s^\nabla \omega^n}{\int_M \omega^n}$.

Proof. Let g be such an extremal metric, then $X = grad_\omega s^\nabla$ is a Killing field which is invariant by T. Denote by $\Xi = span\{X, \mathfrak{t}\}$ where $\mathfrak{t} = \mathrm{Lie}(T)$, then the closure of $\{\exp \Xi\}$ in the (compact) isometry group is a compact torus which contains T. By the maximality of the torus, we have $X \in \mathfrak{t}$. The other direction is obvious. □

Definition 2.5 *The vector field $Z_\omega^T := grad_\omega z_\omega^T$ is called the extremal vector field relative to T.*

Remark. The vector field Z^T_ω is invariant under isotopy of ω: let ω_t an isotopy of T-invariant symplectic forms in $[\omega]$ with $\omega_0 = \omega$, i.e.

$$\omega_t = \omega_0 + d\sigma_t,\ 0 \leqslant t \leqslant 1,$$

where σ_t is a T-invariant 1-form. The flow Φ_t of $X_t = -\sigma_t^{\sharp_{\omega_t}}$ ($\sharp_{\omega_t}$ stands for the isomorphism between $T^*(M)$ and $T(M)$ via ω_t) verifies $(\Phi_t)^*\omega_t = \omega_0$ so $(\Phi_t)^*(Z^T_{\omega_t}) = Z^T_{\omega_0}$. On the other hand, as the vector field X_t is T-invariant, $(\Phi_t)^*(Z^T_{\omega_t}) = Z^T_{\omega_t}$.

It follows that the introduced extremal vector field Z^T_ω coincides with the one defined by Futaki and Mabuchi in the Kähler case [23]. Indeed, in a fixed T-invariant Kähler class, any two Kähler forms ω_1 and ω_2 are isotopic. Therefore, $Z^T_{\omega_1} = Z^T_{\omega_2}$. □

Remark. The assumption that $T \subset \mathrm{Ham}(M, \omega)$ is a maximal torus is used only in Lemma 2.4. Indeed, the arguments in Lemma 2.3 show that $z^T_\omega = \Pi^T_\omega s^\nabla$ is independent of (J, g) for any torus $T \subset \mathrm{Ham}(M, \omega)$ and the above Remark still holds true for the corresponding vector field $Z^T_\omega = grad_\omega z^T_\omega$. □

Denote by $\mathfrak{M}^T_\omega$ the set all T-invariant almost-Kähler metrics induced by T-invariant symplectic forms isotopic to ω. The extremal vector field Z^T_ω is an obstruction to the existence of metrics of constant hermitian scalar curvature in $\mathfrak{M}^T_\omega$. We recall that the space $\mathfrak{M}^T_\omega$ is related (via Moser's lemma) to the space AK^T_ω.

Corollary 2.6 *If $\mathfrak{M}^T_\omega$ contains a metric with constant hermitian scalar curvature, then $Z^T_\omega = 0$. Conversely, If $Z^T_\omega = 0$, any extremal metric in $\mathfrak{M}^T_\omega$ is of constant hermitian scalar curvature.*

Proof. We readily deduce from the defintion of Z^T_ω the first assertion of the corollary. Now, if $Z^T_\omega = 0$, then $z^T_{\tilde{\omega}} = 0$ for any $\tilde{\omega}$ isotopic to ω. In particular, any extremal metric in $\mathfrak{M}^T_\omega$ is of constant hermitian scalar curvature. □

Corollary 2.7 *If $\mathfrak{M}^T_\omega$ contains an extremal metric (J, g) such that the tensor $(\rho^\nabla)^{J,+}(\cdot, J\cdot)$ is negative-semidefinite everywhere, then $Z^T_\omega = 0$.*

Proof. This is a direct consequence of Corollary 1.10 and Corollary 2.6. □

2.1.3 Periodicity of the extremal vector field

In this section, we show the periodicity of the introduced vector field Z_ω^T relative to a fixed maximal torus T in $\mathrm{Ham}(M,\omega)$ when $\left[\frac{\omega}{2\pi}\right]$ is integral modulo torsion.

Theorem 2.8 *Assume that $\left[\frac{\omega}{2\pi}\right]$ is integral modulo torsion. Then, there exist a positive integer ν such that* $\exp(2\pi\nu Z_\omega^T) = 1$.

Proof. As in the previous section, we denote by $\mathfrak{t}_\omega \subset C^\infty(M)$ the finite dimensional space of smooth functions which are hamiltonians with zero mean value of elements of $\mathfrak{t} = \mathrm{Lie}(T)$.

In the symplectic context, Futaki and Mabuchi defined in [24] a bilinear symmetric form Φ : $\mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$

$$\Phi(X,Y) = \frac{1}{(2\pi)^2}\int_M f^X f^Y \frac{\omega^n}{(2\pi)^n}, \tag{2.6}$$

where $f^X, f^Y \in \mathfrak{t}_\omega$ momentums of X, Y.

By hypothesis, $\left[\frac{\omega}{2\pi}\right]$ is an integral cohomology class modulo torsion. Under this condition, Futaki and Mabuchi showed [24] that if X, Y are generators of S^1-actions (i.e. $\exp(X) = \exp(Y) = 1$), then $\Phi(X,Y) \in \mathbb{Q}$. To prove our claim, we have to show that $2\pi\Phi(X_i, Z_\omega^T) \in \mathbb{Q}$ for all $1 \leqslant i \leqslant k$, where Z_ω^T is the extremal vector field relative to T and $X_1, \cdots, X_k$ are generators of the torus action. This would imply $2\pi Z_\omega^T \in \sum_{i=1}^k \mathbb{Q}X_i$, so $\exp\left(2\pi\nu Z_\omega^T\right) = 1$ for some positive integer ν.

To show our claim, we recall Nakagawa's modified version [44] of Tian's formula [48]

$$\begin{aligned}\frac{(n+1)!2^{n-1}n!}{2\pi}\int_M f\mathring{s}^\nabla \frac{\omega^n}{(2\pi)^n} = \sum_{j=0}^n(-1)^j\binom{n}{j}\int_M\Bigg[&\left(\frac{1}{2}\frac{\Delta^g f}{2\pi}+\frac{\rho^\nabla}{2\pi}\right)\\ &+(N+n-2j)\left(\frac{f}{2\pi}+\frac{\omega}{2\pi}\right)\Bigg]^{n+1}\\ &-\left(N+\frac{n\mu}{n+1}\right)2^n(n+1)!\int_M\left(\frac{f}{2\pi}+\frac{\omega}{2\pi}\right)^{n+1},\end{aligned} \tag{2.7}$$

where f is a smooth function on M, $\mathring{s}^\nabla$ is the zero integral part of the hermitian scalar curvature s^∇ of (ω, J, g), ρ^∇ is the hermitian Ricci form, $\mu = \frac{\int_M \rho^\nabla\omega^{n-1}}{\int_M\omega^n}$, N is an integer and Δ^g is the

Riemannian Laplacian with respect to g. The formula (2.7) takes in account the normalization (2.1) and is a direct consequence of the fact that $\int_M (\Delta^g f)\,\omega^n = 0$ and the identities

$$\sum_{j=0}^{n}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} (n-2j)^k = 0 \ \text{ if } k < n \text{ or } k = n+1,$$
$$\sum_{j=0}^{n}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} (n-2j)^n = 2^n n!.$$

As Φ is non-degenerate and $2\pi\Phi(X, Z^T_\omega) = \frac{1}{2\pi}\int_M f^X z^T_\omega \frac{\omega^n}{(2\pi)^n} = \frac{1}{2\pi}\int_M f^X \mathring{s}^\nabla \frac{\omega^n}{(2\pi)^n}$, we thus reduced the problem to show that the right hand side of (2.7) is rational when $f = f^X \in \mathfrak{t}_\omega$ is the momentum with respect to ω of a hamiltonian vector field X generating an S^1-action. We have then essentially two integrals in the right hand side of (2.7) which are $\int_M f^X \omega^n$ and $\frac{1}{2\pi}\int_M \left(\frac{1}{2}\Delta^g f^X + (N+n-2j) f^X\right)\widetilde{\omega}^n$ with $\widetilde{\omega} = \frac{\rho^\nabla}{2\pi} + (N+n-2j)\,\frac{\omega}{2\pi}$.

In order to compute the latter integral, we will use the localization formula (1.27). Since M is compact, we consider an integer N large enough such that $\widetilde{\omega}$ is symplectic. By Lemma 1.9, the vector field X verifies

$$2\pi\widetilde{\omega}(X,\cdot) = -d\left(\frac{1}{2}\Delta^g f^X + (N+n-2j)f^X\right).$$

On the other hand, Futaki and Mabuchi showed [24] that if $\int_M f^X\omega^n = 0$ then $f^X(p) \in 2\pi\mathbb{Q}$ for any fixed point $p \in M$ of the S^1-action. Moreover, we have $\left(\Delta^g f^X\right)(p) = -tr\left(D^g df^X\right)_p = tr\left(JD^g X\right)_p$. Note that $\left(JD^g X\right)_p$ is a symmetric hermitian endomorphism which is the generator of the induced linear S^1-action on $T_p(M)$ (see e.g. [40]). This implies that the trace $tr\left(JD^g X\right)_p \in 2\pi\mathbb{Z}$ and therefore $\left(\Delta^g f^X\right)(p) \in 2\pi\mathbb{Z}$. By (1.27) and using the power series of the exponential function in h, we obtain

$$\frac{1}{2\pi}\int_M \left(\frac{1}{2}\Delta^g f^X + (N+n-2j)f^X\right)\widetilde{\omega}^n \in \mathbb{Q}.$$

This concludes the proof. $\square$

2.1.4 The Hermitian-Einstein condition

Definition 2.9 *An almost-Kähler metric (ω, J, g) is called Hermitian-Einstein if the hermitian Ricci form ρ^∇ is a (constant) multiple of the symplectic form ω, i.e.*

$$\rho^\nabla = \frac{s^\nabla}{2n}\omega,$$

so the hermitian scalar curvature s^∇ is constant.

Corollary 2.10 *If $c_1(M,\omega)$ is a multiple of $[\omega]$ and $Z_\omega^T = 0$, then an extremal almost-Kähler metric (J,g) in $\mathfrak{M}_\omega^T$ is Hermitian-Einstein if and only if ρ^∇ is J-invariant.*

Proof. Let $(\widetilde{\omega}, J, g)$ be an extremal almost-Kähler metric in $\mathfrak{M}_\omega^T$. By Corollary 2.6 and since $Z_\omega^T = 0$, we deduce that the hermitian scalar curvature s^∇ of $(\widetilde{\omega}, J, g)$ is constant. To prove our claim, it is enough to show that ρ^∇ is co-closed (and therefore harmonic), i.e. $\delta^g \rho^\nabla = 0$, where δ^g is the codifferential with respect to g. Indeed, as ρ^∇ and $\widetilde{\omega}$ be then two harmonic forms representing the same cohomology class up to a multiple, Hodge theory implies there are equal up to the same multiple.

Denote by d^c the twisted exterior differential with respect to J and $\Lambda_{\widetilde{\omega}}$ the contraction by $\widetilde{\omega}$. By hypothesis ρ^∇ is J-invariant, then $d^c\rho^\nabla = 0$. Now, using the relations (1.11) and (2.2), we have $\delta^g\rho^\nabla = [\Lambda_{\widetilde{\omega}}, d^c]\,\rho^\nabla = -d^c\Lambda_{\widetilde{\omega}}\rho^\nabla = -\frac{1}{2}d^c s^\nabla = 0$ since s^∇ is constant. Therefore, ρ^∇ is co-closed. The other direction is obvious. □

2.2 Explicit formula for the hermitian scalar curvature

Let (M^{2n}, ω, J, g) be an almost-Kähler manifold. By Darboux theorem, there exist coordinates $\{z_i, t_i\}$ defined on an open set U such that the symplectic form ω has the form $\omega = \sum_{i=1}^n dz_i \wedge dt_i$ on U; $\{z_i, t_i\}$ are called the Darboux coordinates. In this section, we give an explicit formula of the hermitian scalar curvature s^∇ of (ω, J, g) in terms of the coordinates $\{z_i, t_i\}$. On U, the metric g is of the form,

$$g = \sum_{i,j=1}^n G_{ij}(z,t)dz_i \otimes dz_j + H_{ij}(z,t)dt_i \otimes dt_j + P_{ij}(z,t)dz_i \odot dt_j,$$

where $G = (G_{ij}), H = (H_{ij})$ are symmetric positive-definite matrix-valued functions which satisfy the compatibility conditions $GH - P^2 = Id$ and $HP = {}^tPH$ (where ${}^tP$ denote the transpose of $P = (P_{ij})$). We define a local section Φ of the anti-canonical bundle $K_J^{-1}(M)$ by $\Phi := (K_1^{\flat_g} - \sqrt{-1}JK_1^{\flat_g}) \wedge \cdots \wedge (K_n^{\flat_g} - \sqrt{-1}JK_n^{\flat_g})$ where $K_i = \frac{\partial}{\partial t_i}$ and $\flat$ stands for the isomorphism between $T(M)$ and $T^*(M)$ via g. Let ϕ and ψ the real n-forms such that $\Phi = \phi + \sqrt{-1}\psi$ (in fact ψ and ϕ are related in the following way $\psi(JX_1, \cdots, X_n) = \phi(X_1, \cdots, X_n)$). We still denote by ∇ the hermitian connection induced on $K_J^{-1}(M)$ by the canonical hermtian connection . For any vector field X, we have

$$\nabla_X \phi = a(X)\phi + b(X)\psi$$

for some real 1-forms a, b (we can deduce that $a = \frac{1}{2} d \ln |\phi|^2$). Moreover, the hermitian Ricci form ρ^∇ is given by $\rho^\nabla = db$. Now, span $\{K_1, \cdots, K_n\}$ is Lagrangian. Hence, $\Phi(K_1, \cdots, K_n) = \phi(K_1, \cdots, K_n) = \det g(K_i, K_j) = \det H$ which implies that

$$(\nabla_X \Phi)(K_1, \cdots, K_n) = (a(X) - ib(X))\Phi(K_1, \cdots, K_n) = (a(X) - ib(X)) \det H.$$

Let $\beta(X) := (\nabla_X \Phi)(K_1, \cdots, K_n)/\det H$, then $\beta = trace\left(H^{-1} \circ B\right)$ where $B(X) = \left(g(\nabla_X K_i, K_j) + \sqrt{-1}g(\nabla_X K_i, JK_j)\right) = \left(2g(\nabla_X K_i^{1,0}, K_j)\right)$. In particular,

$$B(X^{0,1}) = \left(2g(\nabla_{X^{0,1}} K_i^{1,0}, K_j)\right). \tag{2.8}$$

Using Proposition 1.1, we can compute, via (2.8), the imaginary part of B and therefore ρ^∇ and s^∇. Indeed, denote by $H^{-1} = (H^{ij})$ the inverse of $H = (H_{ij})$, $G_{ij,k} = \frac{\partial G_{ij}}{\partial z_k}$, $G_{ij}^{,k} = \frac{\partial G_{ij}}{\partial t_k}$ etc and let $X = \sum_k \alpha_k \frac{\partial}{\partial z_k} + \beta_k \frac{\partial}{\partial t_k}$, so $X^{\flat_g} = \sum_{k,l} (\alpha_k G_{kl} + \beta_k P_{lk})\, dz_l + (\alpha_k P_{kl} + \beta_k H_{kl})\, dt_l$ and then

$$JX = \sum_{k,l} - (\alpha_k P_{kl} + \beta_k H_{kl}) \frac{\partial}{\partial z_l} + (\alpha_k G_{kl} + \beta_k P_{lk}) \frac{\partial}{\partial t_l}.$$

Let $B_{ij} = \sum_n a_n dz_n + b_n dt_n + \sqrt{-1}\,(c_n dz_n + d_n dt_n)$, then

$$\begin{aligned}
2B_{ij}(X^{0,1}) &= B_{ij}(X + \sqrt{-1}JX) \\
&= \left(\sum_n a_n dz_n + b_n dt_n\right)(X) - \left(\sum_n c_n dz_n + d_n dt_n\right)(JX) \\
&+ \sqrt{-1}\left[\left(\sum_n a_n dz_n + b_n dt_n\right)(JX) + \left(\sum_n c_n dz_n + d_n dt_n\right)(X)\right] \qquad (2.9)\\
&= \sum_{k,l}(a_k + P_{kl}c_l - G_{kl}d_l)\alpha_k + (b_k + H_{kl}c_l - P_{lk}d_l)\beta_k \\
&+ \sqrt{-1}\left[(c_k - P_{kl}a_k + G_{kl}b_k)\,\alpha_k + (d_k - H_{kl}a_k + P_{lk}b_k)\,\beta_k\right].
\end{aligned}$$

On the other hand, by the equality (2.8)

$$-\mathcal{I}m\left(B_{ij}(X^{0,1})\right) = \frac{1}{4}g\left(J\left[JX, JK_i\right] + J\left[X, K_i\right] + \left[X, JK_i\right] - \left[JX, K_i\right], K_j\right), \qquad (2.10)$$

where $\mathcal{I}m$ denote the imaginary part. Now, we compute the r.h.s of (2.10). For $K_i = \frac{\partial}{\partial t_i}$, we have $JK_i = \sum_k -H_{ik}\frac{\partial}{\partial z_k} + P_{ki}\frac{\partial}{\partial t_k}$ so

$$\begin{aligned}
[JX, K_i] &= \sum_{k,l}(\alpha_k P_{kl} + \beta_k H_{kl})^{,i}\frac{\partial}{\partial z_l} - (\alpha_k G_{kl} + \beta_k P_{lk})^{,i}\frac{\partial}{\partial t_l}, \\
[X, JK_i] &= \sum_{k,l}\left(-\alpha_l H_{ik,l} - \beta_l H_{ik}^{,l} + H_{il}\alpha_{k,l} - P_{li}\alpha_k^{,l}\right)\frac{\partial}{\partial z_k} + \left(\alpha_l P_{ki,l} + \beta_l P_{ki}^{,l} + H_{il}\beta_{k,l} - P_{li}\beta_k^{,l}\right)\frac{\partial}{\partial z_k}
\end{aligned}$$

We have also $[X, K_i] = -\sum_k \alpha_k^{,i}\frac{\partial}{\partial z_k} + \beta_k^{,i}\frac{\partial}{\partial t_k}$, that implies

$$\begin{aligned}
[X, K_i]^{\flat_g} &= -\sum_{k,l}\left(G_{kl}\alpha_k^{,i} + P_{lk}\beta_k^{,i}\right)dz_l + \left(P_{kl}\alpha_k^{,i} + H_{kl}\beta_k^{,i}\right)dt_l \\
J\left[X, K_i\right] &= \sum_{k,l}\left(P_{kl}\alpha_k^{,i} + H_{kl}\beta_k^{,i}\right)\frac{\partial}{\partial z_l} - \left(G_{kl}\alpha_k^{,i} + P_{lk}\beta_k^{,i}\right)\frac{\partial}{\partial t_l}.
\end{aligned}$$

For the last term

$$\begin{aligned}
[JX, JK_i] &= \sum_{k,l,m}\Big[\left(\alpha_k P_{kl} + \beta_k H_{kl}\right)H_{im,l} - \left(\alpha_k G_{kl} + \beta_k P_{lk}\right)H_{im}^{,l} \\
&- \left(\alpha_k P_{km} + \beta_k H_{km}\right)_{,l}H_{il} + \left(\alpha_k P_{km} + \beta_k H_{km}\right)^{,l}P_{li}\Big]\frac{\partial}{\partial z_m} \\
&+ \Big[-\left(\alpha_k P_{kl} + \beta_k H_{kl}\right)P_{mi,l} + \left(\alpha_k G_{kl} + \beta_k P_{lk}\right)P_{mi}^{,l} \\
&+ \left(\alpha_k G_{km} + \beta_k P_{mk}\right)_{,l}H_{il} - \left(\alpha_k G_{km} + \beta_k P_{mk}\right)^{,l}P_{li}\Big]\frac{\partial}{\partial t_m}.
\end{aligned}$$

Let $[JX, JK_i] = \sum_{k,l,m} a_{klm}\frac{\partial}{\partial z_m} + b_{klm}\frac{\partial}{\partial t_m}$. Then,

$$J[JX, JK_i] = \sum_{k,l,m,n} -\left(P_{mn}a_{klm} + H_{mn}b_{klm}\right)\frac{\partial}{\partial z_n} + \left(G_{mn}a_{klm} + P_{nm}\right)\frac{\partial}{\partial t_n}.$$

Hence,

$$\begin{aligned}
-4\mathcal{I}m\left(B_{ij}(X^{0,1})\right) &= \sum_{k,l,m,n} -P_{nj}\left(P_{mn}a_{klm} + H_{mn}b_{klm}\right) + H_{nj}\left(G_{mn}a_{klm} + P_{nm}b_{klm}\right) \\
&+ \sum_{k,m} P_{mj}\left(P_{km}\alpha_k^{;i} + H_{km}\beta_k^{;i}\right) - H_{mj}\left(G_{km}\alpha_k^{;i} + P_{mk}\beta_k^{;i}\right) \\
&+ \sum_{k,n} P_{kj}\left(-\alpha_n H_{ik,n} - \beta_n H_{ik}^{;n} + H_{in}\alpha_{k,n} - P_{ni}\alpha_k^{;n}\right) \\
&+ \sum_{k,n} H_{kj}\left(\alpha_n P_{ki,n} + \beta_n P_{ki}^{;n} + H_{in}\beta_{k,n} - P_{ni}\beta_k^{;n}\right) \\
&+ \sum_{k,l} -P_{lj}\left(P_{kl}\alpha_k + H_{kl}\beta_k\right)^{;i} + H_{lj}\left(G_{kl}\alpha_k + P_{lk}\beta_k\right)^{;i}.
\end{aligned}$$

Using the compatibility condition ($GH - P^2 = I$ and $HP = {}^tPH$ where ${}^tP$ denote the transpose of P), we can simplify the latter expression so we obtain

$$\begin{aligned}
-4\mathcal{I}m\left(B_{ij}(X^{0,1})\right) &= \sum_{k,l}\Big[P_{kl}H_{ij,l} - G_{kl}H_{ij}^{;l} - P_{kj,l}H_{il} + P_{kj}^{;l}P_{li} \\
&- P_{lj}H_{il,k} + H_{lj}P_{li,k} - P_{lj}P_{kl}^{;i} + H_{lj}G_{kl}^{;i}\Big]\alpha_k \qquad (2.11)\\
&+ \Big[H_{kl}H_{ij,l} - P_{lk}H_{ij}^{;l} - H_{kj,l}H_{il} + H_{kj}^{;l}P_{li} \\
&- P_{lj}H_{il}^{;k} + H_{lj}P_{li}^{;k} - P_{lj}H_{kl}^{;i} + H_{lj}P_{lk}^{;i}\Big]\beta_k.
\end{aligned}$$

Comparing (2.9) with (2.11), we have

$$\begin{aligned}
A_{ij} &= -\mathcal{I}m(B_{ij}) \\
&= \frac{1}{2}\sum_{k,l}\Big[-P_{kj,l}H_{il} + P_{kj}^{;l}P_{li} - P_{lj}H_{il,k} \\
&+ H_{lj}P_{li,k} - P_{lj}P_{kl}^{;i} + H_{lj}G_{kl}^{;i}\Big]dz_k \\
&+ \Big[-H_{kj,l}H_{il} + H_{kj}^{;l}P_{li} - P_{lj}H_{il}^{;k} \\
&+ H_{lj}P_{li}^{;k} - P_{lj}H_{kl}^{;i} + H_{lj}P_{lk}^{;i}\Big]dt_k.
\end{aligned}$$

$$
\begin{aligned}
b &= \sum_{i,j} H^{ij} A_{ij} \\
&= \frac{1}{2}\sum_{k,i}\left(-P_{ki,i} + P_{ii,k} + G_{ki}^{,i}\right) dz_k \\
&+ \frac{1}{2}\sum_{i,j,k,l} H^{ij}\left[P_{kj}^{,l} P_{li} - P_{lj} H_{il,k} - P_{lj} P_{kl}^{,i}\right] dz_k \\
&+ \frac{1}{2}\sum_{k,i}\left(P_{ik}^{,i} + P_{ii}^{,k} - H_{ki,i}\right) dt_k \\
&+ \frac{1}{2}\sum_{i,j,k,l} H^{ij}\left[H_{kj}^{,l} P_{li} - P_{lj} H_{il}^{,k} - P_{lj} H_{kl}^{,i}\right] dt_k.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\rho^{\nabla} =& \frac{1}{2}\sum_{k,i,l}\left(P_{ki,i}^{,l} - G_{ki}^{,il} + P_{il,k}^{,i} - H_{li,ik}\right) dz_k \wedge dt_l \\
&+ \frac{1}{2}\sum_{i,j,k,l,r}\left[\left(H^{ij} P_{lj} H_{il,k}\right)^{,r} - \left(H^{ij} P_{lj} H_{il}^{,r}\right)_{,k}\right] dz_k \wedge dt_r \\
&+ \frac{1}{2}\sum_{k,i,l}\left(P_{ki,il} - P_{ii,kl} - G_{ki,l}^{,i}\right) dz_k \wedge dz_l + \frac{1}{2}\sum_{i,j,k,l,r}\left(H^{ij} P_{lj} H_{il,k}\right)_{,r} dz_k \wedge dz_r \\
&+ \frac{1}{2}\sum_{k,i,l}\left(P_{il}^{,ik} + P_{ii}^{,lk} - H_{li,i}^{,k}\right) dt_k \wedge dt_l - \frac{1}{2}\sum_{i,j,k,l,r}\left(H^{ij} P_{lj} H_{il}^{,r}\right)^{,k} dt_k \wedge dt_r.
\end{aligned} \tag{2.12}
$$

$$
s^{\nabla} = \sum_{i,j}\left(-G_{ij}^{,ij} - H_{ij,ij} + P_{ij,j}^{,i} + P_{ji,i}^{,j}\right) + \sum_{i,j,k,l}\left(H^{ij} P_{lj} H_{il,k}\right)^{,k} - \sum_{i,j,k,l}\left(H^{ij} P_{lj} H_{il}^{,k}\right)_{,k}. \tag{2.13}
$$

2.2.1 The toric case

A $2n$-dimensional *toric* symplectic manifold (M, ω) is a symplectic manifold equipped with an effective hamiltonian action of an n-dimensional torus T. It is generated by a family of hamiltonian vector fileds $\{K_1, \cdots, K_n\}$ which are linearly independant on a dense open set M° and satisfy the condition $\omega(K_i, K_j) = 0$ for all i, j. It is well-known that the symplectic form ω has the following local expression $\omega = \sum_{i=1}^n dz_i \wedge dt_i$, where z_i is the momentum coordinate and t_i is a local coordinate such that $K_i = \frac{\partial}{\partial t_i}$. So, any ω-compatible T-invariant almost-Kähler metric g has the local expression

$$g = \sum_{i,j=1}^{n} \Big(G_{ij}(z) dz_i \otimes dz_j + H_{ij}(z) dt_i \otimes dt_j + P_{ij}(z) dz_i \odot dt_j \Big). \tag{2.14}$$

An almost-Kähler manifold (M, ω, g) is called *locally toric* if in local coordinates the symplectic form is written as $\omega = \sum_{i=1}^{n} dz_i \wedge dt_i$ and the almost-Kähler metric g has the form (2.14).

We deduce from (2.12) and (2.13) the expressions of the hermitian Ricci form ρ^{∇} and the hermitian scalar curvature s^{∇}

$$\begin{aligned} \rho^{\nabla} =& \frac{1}{2} \sum_{i,k,l} -H_{li,ik} dz_k \wedge dt_l + \frac{1}{2} \sum_{k,i,l} (P_{ki,il} - P_{ii,kl})\, dz_k \wedge dz_l &(2.15)\\ &+ \frac{1}{2} \sum_{i,j,k,l,r} \left(H^{ij} P_{lj} H_{il,k} \right)_{,r} dz_k \wedge dz_r, \\ s^{\nabla} =& - \sum_{i,j}^{n} H_{ij,ij}, &(2.16) \end{aligned}$$

generalizing the formula of s^{∇} given by Abreu [1] in the integrable case and rediscovering the expression found by Donaldson [16].

Now, we suppose that the (lagrangian) g-orthogonal distribution to the T-orbits is involutive. This condition is automatically satisfied in the Kähler case. Then, using Frobenius' theorem, there exist local coordinates $\{t_i\}$ such that $\{dt_1, \cdots, dt_n\}$ spann the annihilator of the orthogonal distribution to the T-orbits and $\omega = \sum_{i=1}^{n} dz_i \wedge dt_i$. Any such ω-compatible T-invariant almost-Kähler metric (ω, g) has the local expression

$$g = \sum_{i,j=1}^{n} \Big(G_{ij}(z) dz_i \otimes dz_j + H_{ij}(z) dt_i \otimes dt_j \Big) \text{ and } \omega = \sum_{i=1}^{n} dz_i \wedge dt_i. \tag{2.17}$$

Therefore, the expression (2.15) of ρ^{∇} further simplifies

$$\rho^{\nabla} = -\frac{1}{2} \sum_{i,k,l} H_{li,ik} dz_k \wedge dt_l.$$

Remark. We can deform an ω-compatible toric metric of the diagonal form (2.17) by considering $H^{\epsilon} = H + \epsilon U$ and $G^{\epsilon} = (H^{\epsilon})^{-1}$ for some non-negative definite matrix-valued function

$U = (U_{ij})$, say with compact support on an open set and ϵ small enough. The fact that the right hand side of (2.16) is an under-determined linear differential operator implies that there are infinite dimensional families of ω-compatible extremal almost-Kähler metrics around an extremal almost-Kähler metric of the form (2.17). In particular, infinite dimensional families of non-integrable extremal almost-Kähler metrics do exist around any extremal Kähler toric metric. $\square$

Lemma 2.11 *Let (M, ω, J_0, g_0) be an almost-Kähler 4-manifold such that (ω, g_0) have the form (2.17) on some open set V of M. Then, there exists an infinite dimensional family of almost-Kähler metrics $(\omega, J_\epsilon, g_\epsilon)$, defined for a sufficiently small ϵ, such that $\rho^{\nabla_\epsilon} = \rho^{\nabla_0}$ for all ϵ; here ∇_ϵ is the canonical hermitian connection corresponding to J_ϵ. Moreover, if (ω, J_0, g_0) is Kähler, we obtain an infinite dimensional family of non-integrable ω-compatible almost-Kähler metrics.*

Proof. let $H^\epsilon_{ij}(z) = H_{ij}(z) + \epsilon U_{ij}(z)$, where ϵ is a real, $U_{ij} = f_{ij}(z_1)h_{ij}(z_2)$ with $f_{ij} = f_{ji}, h_{ij} = h_{ji}$. Now, the condition $\rho^{\nabla_\epsilon} - \rho^{\nabla_0} = 0$ gives the following system of O.D.E.'s

$$\sum_{k=1}^{2} (f_{ik}(z_1)h_{ik}(z_2))_{,kj} = 0, \tag{2.18}$$

which reduces to the relations $f'_{12} = \alpha f_{22}$, $f'_{11} = \beta f_{12}$, $h'_{12} = -\beta h_{11}$ and $h'_{22} = -\alpha h_{12}$ for some (real) constants α, β which we assume non-zero. Choosing f_{11} and h_{22} arbitrary, the above relations determine the remaining functions $f_{12}, f_{22}, h_{11}, h_{12}$. In particular, for f_{11} and h_{22} with compact support on V, U_{ij} has a compact support on V. This ensures that for a sufficiently small ϵ, $H^\epsilon = (H^\epsilon_{ij})$ is positive-definite. Letting $G^\epsilon = (H^\epsilon)^{-1}$, we obtain the ω-compatible metric

$$g_\epsilon = \begin{cases} \sum_{i,j=1}^{2} G^\epsilon_{ij}(z)dz_i \otimes dz_j + H^\epsilon_{ij}(z)dt_i \otimes dt_j & \text{on } V, \\ g_0 & \text{elsewhere.} \end{cases}$$

One can check directly that for a generic choice of f_{11} and h_{22}, the corresponding almost-complex structure J_ϵ is non-integrable. Indeed, to detect whether J_t is non-integrable for $t \neq 0$

when J_0 is integrable, we compute

$$4N_{J_t}(\frac{\partial}{\partial t_i},\frac{\partial}{\partial z_j}) = -\sum_{k,l=1}^{2}\left[H^t_{ik}G^t_{jl,k} + H^t_{ik,j}G^t_{kl}\right]\frac{\partial}{\partial t_l},$$

$$4\frac{d}{dt}N_{J_t}(\frac{\partial}{\partial t_i},\frac{\partial}{\partial z_j})|_{t=0} = -\sum_{k,l=1}^{2}\left[U_{ik}G_{jl,k} + H_{ik}(GUG)_{jl,k} + U_{ik,j}G_{kl} + H_{ik,j}(GUG)_{kl}\right]\frac{\partial}{\partial t_l}. \tag{2.19}$$

For example. for $i = j = l = 1$, it gives the following coefficient in (2.19)

$$\begin{aligned}
&\frac{1}{\alpha\beta}f_{11}h''_{22}(G_{11,1} + 2H_{11}G_{11}G_{11,1} + 2H_{12}G_{11}G_{11,2} + H_{11,1}G_{11}G_{11} + H_{12,1}G_{11}G_{12})\\
&+\frac{1}{\alpha\beta}f'_{11}h'_{22}(G_{11,2} + H_{11}G_{11,1}G_{12} + H_{11}G_{11}G_{12,1} + 2H_{11}G_{12}G_{12,1} + H_{12}G_{11,2}G_{12}\\
&+H_{12}G_{11}G_{12,2} + 2H_{12}G_{12}G_{12,2} + H_{11,1}G_{11}G_{12} + H_{11,1}G_{12}G_{12} + H_{12,1}G_{12}G_{12} + H_{12,1}G_{12}G_{22})\\
&+\frac{1}{\alpha\beta}f'_{11}h''_{22}(H_{11}G_{11}G_{11} + G_{11} - H_{12}G_{11}G_{12} + H_{12}G_{12}G_{12})\\
&-\frac{1}{\alpha\beta}f''_{11}h'_{22}(H_{11}G_{11}G_{12} + H_{11}G_{12}G_{12} + G_{12} - H_{12}G_{12}G_{12})\\
&+\frac{1}{\alpha\beta}f''_{11}h_{22}(2H_{11}G_{12}G_{12,1} + 2H_{12}G_{12}G_{12,2} + H_{11,1}G_{12}G_{12} + H_{12,1}G_{22}G_{12})\\
&+\frac{1}{\alpha\beta}f'''_{11}h_{22}(H_{11}G_{12}G_{12}).
\end{aligned}$$

Even, if the (G_{ij}) and (H_{ij}) are constants with $G_{12} = 0$, for a generic f_{11} and h_{22} this coefficient *does not* vanish. □

Corollary 2.12 *Let (M,ω,J,g) be a Kähler-Einstein (complex) surface which is locally toric. Then, ω admits an infinite dimensional family of non-integrable, ω-compatible almost-complex structures inducing Hermitian-Einstein almost-Kähler metrics.*

Example. The corollary applies to the toric Kähler-Einstein surfaces $\mathbb{CP}^2$, $\mathbb{CP}^1\times\mathbb{CP}^1$ and $\mathbb{CP}^1\#3\overline{\mathbb{CP}^2}$, but also to the locally symmetric Hermitian-Einstein spaces $\mathbb{CH}^2/\Gamma$ and $\left(\mathbb{CH}^1\times\mathbb{CH}^1\right)/\Gamma$. □

2.2.2 Extremal almost-Kähler metrics saturating LeBrun's estimates

We recall now a (weak version) of LeBrun's result in [35, Proposition 2.2].

Proposition 2.13 *Let (M,ω) be a compact symplectic 4-manifold and $g \in AK_\omega$ be a ω-compatible almost-Kähler metric. Then*

$$V^{\frac{1}{3}}\left(\int_M \left|\left(\frac{2}{3}s_g + 2e\right)_-\right|^3 \frac{\omega^2}{2}\right)^{\frac{2}{3}} \geq 32\pi^2 \left|c_1^+\right|_{L^2}, \tag{2.20}$$

where $V = \int_M \frac{\omega^2}{2}$ is the total volume of (M^4,ω), $e(x)$ is the lowest eignevalue of W^+ at x, for any real-valued function f on M : $f_-(x) = \min(f(x), 0)$ and c_1^+ denotes the self dual part of the g-harmonic 2-form representing $c_1(M,\omega)$.

Moreover, the equality holds in (2.20) if and only if g is an extremal almost-Kähler metric with negative constant hermitian scalar curvature s^∇ and at each point ω is an eigenform of W^+ correponding to its lowest eigenvalue.

Corollary 2.12, applied to the (complex) surfaces $\mathbb{CH}^2/\Gamma$ and $\left(\mathbb{CH}^1 \times \mathbb{CH}^1\right)/\Gamma$, provides examples of non-integrable extremal almost-Kähler metrics saturating the inequality (2.20). Indeed, for a Hermitian-Einstein almost-Kähler metric, ρ^∇ is J-invariant and therefore $\rho^* = R(\omega)$ is (see (1.17)). Since $R(\omega) = W^+(\omega) + \frac{s_g}{12}\omega$, we deduce that ω belongs to the eigenspace of W^+ if and only if ρ^∇ is J-invariant. Recall that in the Kähler case the Riemannian curvature R^g and $\widetilde{r_0}$ act trivially on $\wedge^{J,-}(M)$, and the selfdual Weyl tensor W^+ decomposes as

$$W^+ = \begin{pmatrix} \frac{s}{6} & 0 & 0 \\ 0 & -\frac{s}{12} & 0 \\ 0 & 0 & -\frac{s}{12} \end{pmatrix}.$$

For the Kähler-Einstein metric on $\mathbb{CH}^2/\Gamma$ and $\left(\mathbb{CH}^1 \times \mathbb{CH}^1\right)/\Gamma$, $s_g = s^\nabla$ is negative and thus ω belongs to the lowest eigenspace of W^+. Then, by Corollary 2.12, we obtain an infinite dimensional family of non-integrable Hermitian-Einstein almost-Kähler metrics whose the almost-Kähler form belongs to the lowest eigenspace of W^+ with non-positive constant hermitian scalar curvature s^∇. Other examples of non-integrable almost-Kähler metrics saturating the inequality (2.20) appear in [47].

Chapter III

DEFORMATIONS OF EXTREMAL ALMOST-KÄHLER METRICS

As discussed in the introduction, the GIT formal picture in [15] suggests the existence and the uniqueness of an extremal almost-Kähler metric, modulo the action of $\mathrm{Ham}(M,\omega)$, in each 'stable complexified' orbit of the action of $\mathrm{Ham}(M,\omega)$. Moreover, the existence of extremal almost-Kähler metric is an open condition on the space of such orbits. In this chapter, we consider the 4-dimensional case where one can introduce a notion of almost-Kähler potential related to the one defined by Weinkove [50, 51]. In the spirit of [22, 36], we shall apply the Banach Implicit Function Theorem for the hermitian scalar curvature of T-invariant ω-compatible almost-Kähler metrics where T is a maximal torus in $\mathrm{Ham}(M,\omega)$. The main technical problem is the regularity of a family of Green operators involved in the definition of the almost-Kähler potential. Using a Kodaira–Spencer result [33, 34], one can resolve this problem if we suppose that the dimension of g_t-harmonic J_t-anti-invariant 2-forms, denoted by $h^-_{J_t}$ (see [18]), satisfies the condition $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ along the path $(J_t, g_t) \in AK^T_\omega$ in the space of T-invariant ω-compatible almost-Kähler metrics. So, we claim the following

Theorem 3.1 *Let (M,ω) be a 4-dimensional compact symplectic manifold and T a maximal torus in $Ham(M,\omega)$. Let (J_t, g_t) be any smooth family of almost-Kähler metrics in AK^T_ω such that (J_0, g_0) is an extremal Kähler metric. Suppose that $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$. Then, there exists a smooth family $(\tilde{J}_t, \tilde{g}_t)$ of extremal almost-Kähler metrics in AK^T_ω, defined for sufficiently small t, with $(\tilde{J}_0, \tilde{g}_0) = (J_0, g_0)$ and such that $\tilde{J}_t$ is equivariantly diffeomorphic to J_t.*

Remark.

(i) The condition that $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$ is satisfied in the following cases:

1. When J_t are integrable almost-complex structures for each t. Then, $h^-_{J_t} = 2h^{2,0}(M, J_t) = b^+(M) - 1$ by a well-known result of Kodaira [9]. On the other hand, it is unknown whether or not, for a ω-compatible *non-integrable* almost-complex J on a compact 4-dimensional symplectic manifold M with $b^+(M) \geq 3$, the equality $h^-_J = b^+(M) - 1$ is possible (see [18]).

2. When $b^+(M) = 1$, $h^-_{J_t} = 0$ for each t. This condition is satisfied when (M, ω) admits a non trivial torus in $\mathrm{Ham}(M, \omega)$ [29].

(ii) Theorem 3.1 holds under the weaker assumption that the torus $T \subset \mathrm{Ham}(M, \omega)$ is maximal in $\mathrm{Ham}(M, \omega) \cap \mathrm{Isom}_0(M, g_0)$, where $\mathrm{Isom}_0(M, g_0)$ denotes the connected component of the isometry group of the initial metric g_0. By a known result of Calabi [12], any extremal Kähler metric is invariant under a maximal connected compact subgroup of $\mathrm{Ham}(M, \omega) \cap \widetilde{\mathrm{Aut}}(M, J_0)$, where $\widetilde{\mathrm{Aut}}(M, J_0)$ is the reduced automorphism group of (M, J_0). Hence, Theorem 3.1 generalizes the result of [22, 36] in the 4-dimensional case.

(iii) It was kindly pointed out to us by T. Drăghici that using a recent result of Donaldson and Remarks (i) and (ii) above, one can further extend Theorem 3.1 in the case when $b^+(M) = 1$ as follows: Let (M, ω_0, J_0, g_0) be a compact 4-dimensional extremal Kähler manifold with $b^+(M) = 1$ and T be a maximal torus in $\mathrm{Ham}(M, \omega) \cap \mathrm{Isom}_0(M, g_0)$. Then, for any smooth family of T-invariant almost-complex structures $J(t)$ with $J(0) = J_0$, $J(t)$ is compatible with an extremal almost-Kähler metric g_t for $t \in (-\epsilon, \epsilon)$. Indeed, as $J(t)$ are tamed by ω_0 for $t \in (-\epsilon, \epsilon)$ and $b^+(M) = 1$, one can use the openess result of Donaldson [17, Proposition 1] (see also [18, Sec. 5]) to show that there exists a smooth family of $J(t)$-invariant symplectic forms ω_t with $[\omega_t] = [\omega_0]$ (for eventually smaller ϵ). Averaging ω_t over the compact group T and using the equivariant Moser Lemma, we obtain a family J_t of T-invariant ω_0-compatible almost-complex structures such that J_t is T-equivariantly diffeomorphic to $J(t)$. We can then apply Theorem 3.1 to produce compatible extremal metrics. □

3.1 Almost-Kähler potentials in dimension 4

Let (M, ω) be a compact symplectic manifold of dimension $2n = 4$ and (J, g) a ω-compatible almost-Kähler metric. In order to define the almost-Kähler potentials, we consider the following second order linear differential operator [38] on the smooth sections $\Omega^{J,-}(M)$ of the bundle of J-anti-invariant 2-forms.

$$\begin{array}{rccc} E: & \Omega^{J,-}(M) & \longrightarrow & \Omega^{J,-}(M) \\ & \psi & \longmapsto & (d\delta^g\psi)^{J,-}, \end{array}$$

where δ^g is the codifferential with respect to the metric g.

Lemma 3.2 *E is a self-adjoint strongly elliptic linear operator with kernel the g-harmonic J-anti-invariant 2-forms.*

Proof. The principal symbol of E is given by the linear map $\sigma(E)_\xi(\psi) = -\frac{1}{2}|\xi|^2\psi$, $\forall \xi \in T^*_x(M), \psi \in \Omega^{J,-}(M)$. So, E is a self-adjoint elliptic linear operator with respect to the global inner product $\langle \cdot, \cdot \rangle = \int_M g(\cdot, \cdot)\frac{\omega^2}{2}$. Now, let $\psi \in \Omega^{J,-}(M)$ and suppose that $E(\psi) = 0$. Then, $0 = \langle (d\delta^g\psi)^{J,-}, \psi \rangle = \langle d\delta^g\psi, \psi \rangle = \langle \delta^g\psi, \delta^g\psi \rangle$ which means that $\delta^g\psi = 0$. It follows from (1.18) and since ψ is J-anti-invariant that $*_g\psi = \psi$. Using the relation (1.13), we obtain $d\psi = *_g\delta^g *_g \psi = *_g\delta^g\psi = 0$. Hence, $d\psi = \delta^g\psi = 0$ and thus ψ is a g-harmonic J-anti-invariant 2-form. □

Corollary 3.3 *For $f \in C^\infty(M, \mathbb{R})$, there exist a unique $\psi_f \in \Omega^{J,-}(M)$ orthogonal to the kernel of E such that $(d\delta^g\psi_f)^{J,-} = (dJdf)^{J,-}$.*

Proof. For a smooth real-valued function $f \in C^\infty(M, \mathbb{R})$ and any α in the kernel of E, we have $\langle (dJdf)^{J,-}, \alpha \rangle = \langle dJdf, \alpha \rangle = \langle Jdf, \delta^g\alpha \rangle = 0$. By a standard result of elliptic theory [10, 52] and since E is self-adjoint, there exist a smooth section $\psi_f \in \Omega^{J,-}(M)$ such that

$E(\psi_f) = (dJdf)^{J,-}$. Moreover, ψ_f is unique if one requires ψ_f be orthogonal to the kernel of E. $\square$

From Corollary 3.3, it follows that, for $f \in C^\infty(M, \mathbb{R})$, the symplectic form $\omega_f = \omega + d(Jdf - \delta^g \psi_f)$ is a J-invariant closed 2-form. Then, the function f is called an *almost-Kähler potential* if the induced symmetric tensor $g_f(\cdot,\cdot) := \omega_f(\cdot, J\cdot)$ is a Riemannian metric. This notion of almost-Kähler potential is closely related but different (in general) from the one defined by Weinkove in [51]. More precisely, if the almost-complex structure J is compatible with a symplectic form $\tilde{\omega}$ which is cohomologous to ω i.e. $\tilde{\omega} - \omega = d\alpha$ (for some 1-form α), then the almost-Kähler potential defined by Weinkove is given by the function $\tilde{f}$ which is uniquely determined (up to the addition of constant) by the Hodge decomposition of α with respect to the (self-adjoint elliptic) twisted Laplace operator $\tilde{\Delta}^c = J\Delta^{\tilde{g}}J^{-1}$, where $\Delta^{\tilde{g}}$ is the (Riemannian) Laplace operator with respect to the induced metric $\tilde{g}(\cdot,\cdot) = \tilde{\omega}(\cdot, J\cdot)$. In other words, we have the decomposition $\alpha = \alpha_{H^c} + \tilde{\Delta}^c\tilde{\mathbb{G}}\,\alpha$, where $\mathbb{G}$ is the *Green operator* associated to $\tilde{\Delta}^c$ and α_{H^c} is the harmonic part of α with respect to $\tilde{\Delta}^c$. Thus, $\tilde{f} = -\delta^{\tilde{g}} J\,\tilde{\mathbb{G}}\alpha$, where $\delta^{\tilde{g}}$ is the codifferential with respect to the metric $\tilde{g}$.

Note that $(dJdf)^{J,-} = D^g_{(df)^{\sharp_g}}\omega$ (see e.g. [25]), where $\sharp_g$ stands for the isomorphism between $T^*(M)$ and $T(M)$ induced by g^{-1}. Hence, in the Kähler case, $(dJdf)^{J,-} = 0$ which implies that $\psi_f = 0$ and thus this almost-Kähler potential coincides with the usual Kähler one.

3.2 Proof of Theorem 3.1

Let (M, ω) be a compact and connected symplectic manifold of dimension $2n = 4$ and $J_t \in AK_\omega$ be a smooth path of ω-compatible almost-complex structures. We define the following family of differential operators associated to J_t

$$\begin{aligned} P_t : \ \Omega^2_0(M) &\longrightarrow \Omega^2_0(M) \\ \psi &\longmapsto \tfrac{1}{2}\Delta^{g_t}\psi - \tfrac{1}{4}g_t(\Delta^{g_t}\psi, \omega)\omega, \end{aligned}$$

where $\Omega^2_0(M)$ is the space of smooth sections of the bundle $\Lambda^2_0(M)$ of primitive 2-forms (pointwise orthogonal to ω) and Δ^{g_t} is the (Riemannian) Laplacian with respect to the metric $g_t(\cdot,\cdot) = \omega(\cdot, J_t\cdot)$ (here we use the convention $g_t(\omega, \omega) = 2$).

One can easily check that P_t preserves the decomposition

$$\Omega^2_0(M) = \Omega_0^{J_t,+}(M) \oplus \Omega^{J_t,-}(M).$$

Furthermore,

$$P_t|_{\Omega^{J_t,-}(M)}(\psi) = (d\delta^{g_t}\psi)^{J_t,-} \text{ and } P_t|_{\Omega_0^{J_t,+}(M)}(\psi) = \frac{1}{2}\Delta^{g_t}\psi.$$

It follows that the kernel of P_t consists of primitive harmonic 2-forms which splits as anti-selfdual and J_t-anti-invariant ones so we have

$$\dim \ker(P_t) = b^-(M) + h^-_{J_t},$$

where $h^-_{J_t}$ is introduced by Drăghici–Li–Zhang in [18].

Moreover, $P_t - \frac{1}{2}\Delta^{g_t}$ is a linear differential operator of order 1. Indeed, using the Weintzenböck–Bochner formula (1.7), a direct computation shows that

$$\begin{aligned}\left(P_t - \frac{1}{2}\Delta^{g_t}\right)(\psi) &= \frac{1}{2}\left[\frac{1}{2}\delta^{g_t}\left(D^{g_t}\omega(\psi)\right) - \frac{1}{2}g_t(D^{g_t}\psi, D^{g_t}\omega)\right.\\ &+ \left.\frac{s_{g_t}}{6}g_t(\omega,\psi) - W^{g_t}(\omega,\psi)\right]\omega,\end{aligned}$$

where W^{g_t} stands for the Weyl tensor (see e.g. [10]), D^{g_t} (resp. δ^{g_t}) for the Levi-Civita connection (resp. the codifferential) with respect to the metric g_t and s_g for the Riemannian scalar curvature defined as the trace of the Ricci tensor.

The operator P_t is a self-adjoint strongly elliptic linear operator of order 2. We obtain then a family of Green operators $\mathbb{G}_t$ associated to P_t. If $h^-_{J_t} = h^-_{J_0} = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$, then $\mathbb{G}_t : \Omega^2_0(M) \to \Omega^2_0(M)$ is C^∞ *differentiable* in $t \in (-\epsilon, \epsilon)$ [33, 34], meaning that $\mathbb{G}_t(\psi_t)$ is a smooth family of sections of $\Lambda^2_0(M)$ for any smooth sections ψ_t.

To show Theorem 3.1, we need to extend $\mathbb{G}_t$ to Sobolev spaces $W^{k,p}(M, \Lambda^2_0(M))$ defined in the following way [52]: let $\{U_i, \rho_i\}$ a finite trivializing cover of the bundle $\Lambda^2_0(M)$, $\{\tilde{\rho}_i\}$ the induced map identifying smooth sections of $\Lambda^2_0(M)$ restricted to U_i to smooth functions on an open set of $\mathbb{R}^{2n}$ and $\{\varphi_i\}$ a partition of unity subordinate to $\{U_i\}$. For a smooth section $\psi \in \Omega^2_0(M)$, we define $\|\psi\|^{\Lambda^2_0(M)}_{p,k} = \sum_i \|\tilde{\rho}_i\varphi_i\psi\|_{p,k}$, where $\|\cdot\|_{p,k}$ is the Sobolev norm involving derivatives up to k. We define $W^{k,p}(M, \Lambda_0^2(M))$ as the completion of $\Omega_0^2(M)$ with respect to the norm $\|\cdot\|_{p,k}^{\Lambda_0^2(M)}$ (in fact, the topology of $W^{k,p}(M, \Lambda_0^2(M))$ does not depend on the partition of unity).

Lemma 3.4 *Let $\mathbb{G}_t : \Omega_0^2(M) \to \Omega_0^2(M)$ the family of the above Green operators associated to P_t and suppose that $h_{J_t}^- = h_{J_0}^- = b^+(M) - 1$ for $t \in (-\epsilon, \epsilon)$. Then, the extension of $\mathbb{G}_t$ to Sobolev spaces, still denoted by $\mathbb{G}_t$, defines a C^1 map $\mathbb{G} : (-\epsilon, \epsilon) \times W^{p,k}(M, \Lambda_0^2(M)) \to W^{p,k+2}(M, \Lambda_0^2(M))$*

Proof. Denote by Π_t the L^2-orthogonal projection to the kernel of P_t with respect to $\langle\cdot,\cdot\rangle_{L^2_{g_t}} = \int_M g_t(\cdot,\cdot)\omega^n$. We claim that $\mathbb{G}_t \circ \Pi_0$ and $\Pi_0 \circ \mathbb{G} : (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda_0^2(M)) \to W^{k+2,p}(M, \Lambda_0^2(M))$ are C^1 maps. Indeed, let $\{\psi_0^i\}$ be an orthonormal basis of the kernel of P_0 with respect to $\langle\cdot,\cdot\rangle_{L^2_{g_0}}$. Note that ψ_0^i are smooth since P_0 is elliptic. Then, we have

$$\begin{aligned}
(\mathbb{G}_t \circ \Pi_0)(\psi) &= \sum_i \langle \psi, \psi_0^i \rangle_{L^2_{g_0}} \mathbb{G}_t(\psi_0^i). \\
(\Pi_0 \circ \mathbb{G}_t)(\psi) &= \sum_i \langle \mathbb{G}_t(\psi), (\psi_0^i)^{J_0,+} + (\psi_0^i)^{J_0,-} \rangle_{L^2_{g_0}} \psi_0^i \\
&= \sum_i \left(\int_M -\mathbb{G}_t(\psi) \wedge (\psi_0^i)^{J_0,+} + \mathbb{G}_t(\psi) \wedge (\psi_0^i)^{J_0,-} \right) \psi_0^i \\
&= \sum_i \left(\int_M -\mathbb{G}_t(\psi) \wedge \left((\psi_0^i)^{J_0,+}\right)^{J_t,+} - \mathbb{G}_t(\psi) \wedge \left((\psi_0^i)^{J_0,+}\right)^{J_t,-} \right. \\
&+ \left. \mathbb{G}_t(\psi) \wedge \left((\psi_0^i)^{J_0,-}\right)^{J_t,+} + \mathbb{G}_t(\psi) \wedge \left((\psi_0^i)^{J_0,-}\right)^{J_t,-} \right) \psi_0^i \\
&= \sum_i \left[\left\langle \psi, \mathbb{G}_t\left(\left((\psi_0^i)^{J_0,+}\right)^{J_t,+} \right) \right\rangle_{L^2_{g_t}} - \left\langle \psi, \mathbb{G}_t\left(\left((\psi_0^i)^{J_0,+}\right)^{J_t,-} \right) \right\rangle_{L^2_{g_t}} \right. \\
&- \left. \left\langle \psi, \mathbb{G}_t\left(\left((\psi_0^i)^{J_0,-}\right)^{J_t,+} \right) \right\rangle_{L^2_{g_t}} + \left\langle \psi, \mathbb{G}_t\left(\left((\psi_0^i)^{J_0,-}\right)^{J_t,-} \right) \right\rangle_{L^2_{g_t}} \right] \psi_0^i.
\end{aligned}$$

In the latter equality, we used the fact that $\mathbb{G}_t$ is self-adjoint with respect to $L^2_{g_t}$. The claim follows from the result of Kodaira–Spencer [33, 34].

Denote by $W^{k,p}(M, \Lambda_0^2(M))^\perp$ the space of 2-forms in $W^{k,p}(M, \Lambda_0^2(M))$ which are orthogonal to the kernel of P_0 with respect to $L^2_{g_0}$ and consider the map

$$\begin{aligned}
\Phi : \quad (-\epsilon, \epsilon) \times W^{k+2,p}(M, \Lambda_0^2(M))^\perp &\longrightarrow (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda_0^2(M))^\perp \\
(t, \psi) &\longmapsto (t, (Id - \Pi_0)P_t(\psi)),
\end{aligned}$$

Clearly, the map Φ is of class C^1 and its differential at $(0, \psi)$ is an isomorphism so by the *inverse function theorem* for Banach spaces there exist a neighboorhood V of $(0, \psi)$ such that $\Phi|_V$ admits an inverse of class C^1. By the The Kodaira–Spencer result [33, 34], the map $\Pi : (-\epsilon, \epsilon) \times W^{k,p}(M, \Lambda_0^2(M)) \to W^{k,p}(M, \Lambda_0^2(M))$ is C^1 and thus the map $P_t(Id - \Pi_0)G_t(Id - \Pi_0) = (Id - \Pi_t)(Id - \Pi_0) - P_t(\Pi_0 G_t)(Id - \Pi_0)$ is clearly C^1 since it is a composition of such operators. Then, the map

$$\begin{aligned} \Phi|_V^{-1}(t, (Id - \Pi_0)P_t(Id - \Pi_0)\mathbb{G}_t(Id - \Pi_0)) &= (t, (Id - \Pi_0)\mathbb{G}_t(Id - \Pi_0)) \\ &= (t, \mathbb{G}_t - \Pi_0\mathbb{G}_t - \mathbb{G}_t\Pi_0 + \Pi_0\mathbb{G}_t\Pi_0) \end{aligned}$$

is C^1 and hence $\mathbb{G}_t$ is C^1. $\square$

Proof of Theorem 3.1. Let (M, ω) be a 4-dimensional compact and connected symplectic manifold and T a maximal torus in $\mathrm{Ham}(M, \omega)$. Let (J_t, g_t) a smooth family of ω-compatible almost-Kähler metrics in AK_ω^T such that (J_0, g_0) is an extremal Kähler metric.

Following [36], we consider the almost-Kähler deformations

$$\omega_{t,f} = \omega + d(J_t df - \delta^{g_t}\psi_f^t),$$

where f belongs to the Fréchet space $\widetilde{C}_T^\infty(M, \mathbb{R})$ of T-invariant smooth functions (with zero integral), which are L^2-orthogonal, with respect to $\frac{\omega^2}{2}$, to $\mathfrak{t}_\omega$ and where the 2-form ψ_f^t is given by Corollary 3.3.

Let $\mathcal{U}$ be an open set in $\mathbb{R} \times \widetilde{C}_T^\infty(M, \mathbb{R})$ containing $(0, 0)$ such that the symmetric tensor $g_{t,f}(\cdot, \cdot) := \omega_{t,f}(\cdot, J_t \cdot)$ is a Riemannian metric.

By possibly replacing $\mathcal{U}$ with a smaller open set, we may assume as in [36] that the kernel of the operator $(Id - \Pi_\omega^T) \circ (Id - \Pi_{\omega_{t,f}}^T)$ is equal to the kernel of $(Id - \Pi_{\omega_{t,f}}^T)$. Indeed, let $\{X_1, \cdots, X_n\}$ be a basis of $\mathfrak{t} = Lie(T)$. Then, the corresponding hamiltonians with zero mean value $\{\xi_\omega^1, \cdots, \xi_\omega^n\}$ resp. $\{\xi_{\omega_{t,f}}^1, \cdots, \xi_{\omega_{t,f}}^n\}$, with respect to ω resp. $\omega_{t,f}$, form a basis of $\mathfrak{t}_\omega$ resp. $\mathfrak{t}_{\omega_{t,f}}$. Let $\{\tilde{\xi}_\omega^1, \cdots, \tilde{\xi}_\omega^n\}$ resp. $\{\tilde{\xi}_{\omega_{t,f}}^1, \cdots, \tilde{\xi}_{\omega_{t,f}}^n\}$ the corresponding orthonormal basis obtained by the Gram–Schmidt procedure. Since $\det\left[\left\langle \tilde{\xi}_\omega^i, \tilde{\xi}_{\omega_{t,f}}^j \right\rangle\right]$ defines a continuous function on $\mathcal{U}$, then we may suppose that $\det\left[\left\langle \tilde{\xi}^i_\omega, \tilde{\xi}^j_{\omega_{t,f}}\right\rangle\right] \neq 0$ on an eventually smaller open set than $\mathcal{U}$ (here $\langle\cdot,\cdot\rangle$ denotes the L^2 product with respect to the volume form $\frac{\omega^2_{t,f}}{2}$). So, if $u \in ker\left((Id - \Pi^T_\omega)\circ(Id - \Pi^T_{\omega_{t,f}})\right)$ then $v \in \mathfrak{t}_\omega \cap (\mathfrak{t}_{\omega_{t,f}})^{\perp_{g_{t,f}}}$, where $v = (Id - \Pi^T_{\omega_{t,f}})u$. But the hypothesis $\det\left[\left\langle \tilde{\xi}^i_\omega, \tilde{\xi}^j_{\omega_{t,f}}\right\rangle\right] \neq 0$ implies that $v \equiv 0$ and then $ker\left((Id - \Pi^T_\omega)\circ(Id - \Pi^T_{\omega_{t,f}})\right) = ker(Id - \Pi^T_{\omega_{t,f}})$.

We then consider the map:

$$\begin{array}{rccl} \Psi: & \mathcal{U} & \longrightarrow & \mathbb{R}\times\widetilde{C}^\infty_T(M,\mathbb{R}) \\ & (t,f) & \longmapsto & \left(t, (Id - \Pi^T_\omega)\circ(Id - \Pi^T_{\omega_{t,f}})(\mathring{s}^{\nabla_{t,f}})\right), \end{array}$$

where $\mathring{s}^{\nabla_{t,f}}$ is the zero integral part of the hermitian scalar curvature $s^{\nabla_{t,f}}$ of $(J_t, g_{t,f})$.

It follows from Lemma 2.4 that $\Psi(t,f) = (t,0)$ if and only if $(J_t, g_{t,f})$ is an extremal almost-Kähler metric. In particular, $\Psi(0,0) = (0,0)$.

Let $\alpha_{t,f} = J_t df - \delta^{g_t}\psi^t_f = J_t df - \delta^{g_t}\mathbb{G}_t\left((dJ_t df)^{J_t,-}\right) = J_t df - \delta^{g_t}\mathbb{G}_t(D^{g_t}_{df^{\sharp_{g_t}}}\omega)$, where $\mathbb{G}_t$ is the Green operator associated to the elliptic operator $P_t : \Omega^{J_t,-}(M) \longrightarrow \Omega^{J_t,-}(M)$. In order to extend the map Ψ to Sobolev spaces, we give an explicit expression of $(Id - \Pi^T_{\omega_{t,f}})(s^{\nabla_{t,f}})$. A direct computation using (1.14) shows that

$$s^{\nabla_{t,f}} = \Delta^{g_{t,f}}F_{t,f} + g_{t,f}(\rho^{\nabla_t}, \omega_{t,f}), \tag{3.1}$$

where $F_{t,f} = \log\left(\frac{1}{2}\left((1 + g_t(d\alpha_{t,f},\omega))^2 + 1 - g_t(d\alpha_{t,f}, d\alpha_{t,f})\right)\right)$ satisfying the relation $\omega^2_{t,f} = e^{F_{t,f}}\omega^2$. Then

$$(Id - \Pi^T_{\omega_{t,f}})(s^{\nabla_{t,f}}) = \Delta^{g_{t,f}}F_{t,f} + g_{t,f}(\rho^{\nabla_t}, \omega_{t,f}) - \sum_j \left\langle s^{\nabla_{t,f}}, \tilde{\xi}^j_{\omega_{t,f}}\right\rangle \tilde{\xi}^j_{\omega_{t,f}}. \tag{3.2}$$

Let $\widetilde{W}^{p,k}_T$ be the completion of $\widetilde{C}^\infty_T(M,\mathbb{R})$ with respect to the Sobolev norm $\|\cdot\|_{p,k}$ involving derivatives up to order k. We choose p,k such that $pk > 2n$ and the corresponding Sobolev space $\widetilde{W}^{p,k}_T \subset C^3_T(M,\mathbb{R})$ so that all coefficients are $C^0_T(M,\mathbb{R})$. Since $\widetilde{W}^{p,k}_T$ form an algebra relative to the standard multiplication of functions [3], we dedudce from the expression (3.2) that the extension of Ψ to the Sobolev completion of $\widetilde{C}^\infty_T(M,\mathbb{R})$ is a map $\Psi^{(p,k)} : \widetilde{\mathcal{U}} \subset \mathbb{R}\times\widetilde{W}^{p,k+4}_T \longrightarrow \mathbb{R}\times\widetilde{W}^{p,k}_T$.

Clearly $\Psi^{(p,k)}$ is a C^1 map (in a small enough open around $(0,0)$). Indeed, it is obtained by a composition of C^1 maps by Lemma 3.4 and (3.2).

As in [36] and using Lemma 2.3, the differential of $\Psi^{(p,k)}$ at $(0,0)$ is given by

$$\left(\mathbf{T}_{(0,0)}\Psi^{(p,k)}\right)(t,f) = \left(t, t\delta^{g_0}\delta^{g_0}h - 2\delta^{g_0}\delta^{g_0}(D^{g_0}df)^{J_0,-}\right),$$

where $h = \frac{d}{dt}|_{t=0}\, g_t$.

The operator $L := \frac{\partial\Psi}{\partial f}|_{(0,0)}$ given by $L(f) = -2\delta^{g_0}\delta^{g_0}(D^{g_0}df)^{J_0,-}$ is called the Lichnerowicz operator (see chapter 1). It is a 4-th order self-adjoint T-invariant elliptic linear operator leaving invariant $(\mathfrak{t}_\omega)^\perp$ since $L(f) = 0$ for any $f \in \mathfrak{t}_\omega$. By a known result of the elliptic theory [10, 52], we obtain the L^2-orthogonal splitting $\widetilde{C}^\infty_T(M,\mathbb{R}) = \ker(L) \oplus \operatorname{Im}(L)$. Following the argument in [6, Lemma 4], any $f \in \ker(L)$ gives rise to a Killing vector field in the centralizer of $\mathfrak{t} = \operatorname{Lie}(T)$. By the maximality of the torus T, $f \in \mathfrak{t}_\omega$. It follows that L is an isomorphism of $\widetilde{C}^\infty_T(M,\mathbb{R})$

The natural extension of L to $\widetilde{W}^{p,k+4}_T$ is defined in the following way: for a Cauchy sequence $\{f_n\}$ which converges to $f \in \widetilde{W}^{p,k+4}_T$, we define $L(f) := \lim L(f_n)$ (it is easily seen that the limit exists and is independent of the choice of the sequence since L is continuous). Then, L is also an isomorphism from $\widetilde{W}^{p,k+4}_T$ to $\widetilde{W}^{p,k}_T$. Indeed, $\widetilde{W}^{p,k+4}_T \subseteq \widetilde{W}^{p,k}_T$ and L is surjective. More explicitely, for any given $f \in \widetilde{W}^{p,k}_T$, there exist a Cauchy sequence $\{f_n\} \in \widetilde{C}^\infty_T(M,\mathbb{R})$ which converges to f. Then, there exist a sequence $\{g_n\} \in \widetilde{C}^\infty_T(M,\mathbb{R})$ such that $L(g_n) = f_n$. Using the Sobolev estimates for an elliptic operator (with zero kernel) [10, 52], we have the following inequality $\|g_n - g_m\|_{p,k+4} \leqslant c\,\|L(g_n - g_m)\|_{p,k} = c\,\|f_n - f_m\|_{p,k}$, where c is a constant and $\|\cdot\|_{p,k}$ is the Sobolev norm. So $\{g_n\}$ is also a Cauchy sequence and thus $L(g) := \lim L(g_n) = \lim f_n = f$.

Thus, $\mathbf{T}_{(0,0)}\Psi^{(p,k)}$ is an isomorphism from $\mathbb{R}\oplus\widetilde{W}^{p,k+4}_T$ to $\mathbb{R}\oplus\widetilde{W}^{p,k}_T$. In fact, if $\left(\mathbf{T}_{(0,0)}\Psi^{(p,k)}\right)(t,f) = 0 \oplus 0$ then $L(f) = 0$ which means that $f \in \mathfrak{t}_\omega$ but $\widetilde{W}^{p,k+4}_T$ is the completion of $\widetilde{C}^\infty_T(M,\mathbb{R})$ so $f \equiv 0$. This proves the injectivity of $\mathbf{T}_{(0,0)}\Psi^{(p,k)}$. Now, in order to show the surjectivity of $\mathbf{T}_{(0,0)}\Psi^{(p,k)}$, we have to solve the equation $\left(\mathbf{T}_{(0,0)}\Psi^{(p,k)}\right)(t,f) = (t_0, f_0)$. We thus obtain $t = t_0$ and $L(f) = f_0 + t_0\delta^{g_0}J_0\delta^{g_0}\dot{J}$. Since, the r.h.s of the latter equation belongs to $\widetilde{W}^{p,k}_T$

and L is an isomorphism from $\widetilde{W}_T^{p,k+4}$ to $\widetilde{W}_T^{p,k}$, we have a solution and thus $\mathbf{T}_{(0,0)}\Psi^{(p,k)}$ is an isomorphism.

It follows from the *inverse function theorem* for Banach manifolds that $\Psi^{(p,k)}$ determines an isomorphism from an open neighbourhood V of $(0,0)$ to an open neighbourhood of $(0,0)$. In particular, there exists $\mu > 0$ such that for $|t| < \mu$, $\Psi^{(p,k)}\left(\Psi^{(p,k)}|_V^{-1}(t,0)\right) = (t,0)$. By Sobolev embedding, we can choose a k large enough, such that $\widetilde{W}_T^{p,k+4} \subset \widetilde{C}_T^6(M,\mathbb{R})$. Thus, for $|t| < \mu$, $(J_t, g_{\Psi^{(p,k)}|_V^{-1}(t,0)})$ is an extremal almost-Kähler metric of regularity at least C^4 (so we ensure, in this case, that $grad_\omega s^{\nabla_{t,f}}$ is of regularity C^1).

The extremal vector field $Z^T_{\omega_{t,f}} = Z^T_\omega$ is smooth for any almost-Kähler metric $(J_t, g_{t,f})$. In particular, for an extremal almost-Kähler metric $(J_t, g_{t,f})$ of regularity C^4, the dual $ds^{\nabla_{t,f}}$ of Z^T_ω with respect to $\omega_{t,f}$ is of regularity C^4, then the hermitian scalar curvature $s^{\nabla_{t,f}}$ of $(J_t, g_{t,f})$ is of regularity C^5. From (3.1), it follows that the hermitian scalar curvature is given by the pair of equations

$$s^{\nabla_{t,f}} - g_{t,f}(\rho^{\nabla_t}, \omega_{t,f}) = \Delta^{g_{t,f}}(u), \tag{3.3}$$

$$e^u = \frac{\omega^2_{t,f}}{\omega^2}. \tag{3.4}$$

From (3.3), using the ellipticity [10] of the (Riemannian) Laplacian $\Delta^{g_{t,f}}$ and since the l.h.s of (3.3) is of Hölder class $C^{3,\beta}$ for any $\beta \in (0,1)$, it follows that u is of class $C^{5,\beta}$. Following [17, 51], the linearisation of the equation (3.4) $(\omega + d\alpha)\wedge d\dot{\alpha} = 0$ together with the constraints $\delta^{g_t}\dot{\alpha} = 0$ and $(d\dot{\alpha})^{J_t,-} = 0$ form a linear elliptic system in $\dot{\alpha}$. Elliptic theory [4, 10] ensures that the almost-Kähler metric $g_{t,f}$ is of class $C^{5,\beta}$ as the volume form and we can prove that any extremal almost-Kähler metric of regularity C^4 is smooth by a bootstraping argument (in the Kähler case see [36]).

We obtain then a smooth family of T-invariant extremal almost-Kähler structures $(J_t, \omega_t = \omega + d\alpha_t)$ defined for $|t| < \mu$. The main theorem follows from the Moser Lemma [40]. □

Chapter IV

FURTHER DIRECTIONS

4.1 Uniqueness of extremal almost-Kähler metrics in the toric case

Let (M, ω) be a compact symplectic $2n$-manifold equipped with an effective hamiltonian action of an n-dimensional torus T. Let $z : M \to \Delta \subset \mathfrak{t}^*$ be the moment map, where Δ is a convex polytope in $\mathfrak{t}^*$ the dual of $\mathfrak{t} = \mathrm{Lie}(T)$ (see [13]). Denote by $u = \{u_1, \cdots, u_d\}$ the normals to the polytope. Using the same notations as in chapter 2, any ω-compatible T-invariant almost-Kähler metric has the expression

$$g = \sum_{i,j=1}^{n} \Big(G_{ij}(z) dz_i \otimes dz_j + H_{ij}(z) dt_i \otimes dt_j + P_{ij}(z) dz_i \odot dt_j \Big) \text{ and } \omega = \sum_{i=1}^{n} dz_i \wedge dt_i \quad (4.1)$$

on the dense open set $\mathring{M} = \mu^{-1}(\mathring{\Delta})$ where the torus action is free ($\mathring{\Delta}$ is the interior of Δ). Here z_i is the momentum corrdinate and t_i is an angle coordinate introduced with respect to the canonical Kähler strucuture on (M, ω) [28]. The matrices G and H are symmetric positive definite which satisfy the compatibility condition $GH - P^2 = Id$ and $HP = {}^tPH$ (where ${}^tP$ denote the transpose of P).

Now, we recall the boundary conditions obtained in [7, Proposition 1] in the case when the (lagrangian) g-orthogonal distribution to the T-orbits is involutive. This is the case when $P = 0$ i.e.

$$g = \sum_{i,j=1}^{n} \Big(G_{ij}(z) dz_i \otimes dz_j + H_{ij}(z) dt_i \otimes dt_j \Big) \text{ and } \omega = \sum_{i=1}^{n} dz_i \wedge dt_i. \quad (4.2)$$

Proposition 4.1 *A symmetric positive definite-valued function H on $\mathring{\Delta}$ comes from a T-invariant ω-compatible almost-Kähler metric of the form (4.2) if and only if*

1. *$H_{ij}(z)$ are the restrictions of smooth functions on Δ.*

2. *For any point p on any codimension one face F_i with inward normal u_i, we have*

$$H_p(u_i, \cdot) = 0 \textit{ and } (dH)_p(u_i, u_j) = 2\delta_{ij}u_i,$$

   where δ_{ij} is the Kronecker function.

3. *For any point p in the interior of a face F, $H_p(\cdot, \cdot)$ is a positive definite bilinear form on $\mathbb{R}^n/Span\{u_{i_1}, \cdots, u_{i_k}\}$, where $u_{i_1}, \cdots, u_{i_k}$ are the inward normals to the all codimension one faces containing F.*

4.1.1 Action of the Hamiltonian group

Following [14, 16], at each point, there is a correspondance between between the space AK_ω^T of ω-compatible T-invariant almost-Kähler metrics and the *Siegel upper half space* which is the space of complex symmetric matrices with positive definite imaginary part. More explicitely, we identify $(J, g) \in AK_\omega^T$ of the form (4.1) with $Z = -H^{-1}\,{}^tP - \sqrt{-1}H^{-1}$. So, elements of AK_ω^T are represented by maps from the interior of the polytope to the Siegel upper half space. Moreover, the Siegel upper half space is diffeomorphic to the homogeneous space $Sp(2n)/U(n)$ which has a Kähler structure [46].

We consider the action of elements of $\mathrm{Ham}(M, \omega)$ on AK_ω^T which commute with T. These elements can be identified with real-valued function $f = f(z)$. The action of a function f on a point is given by $(z, t) \mapsto (z, t + grad\, f)$. The expression $t + grad\, f$ has a sens since at a point p, $t(p) \in \mathfrak{t} \simeq \mathbb{R}^n$ and $(grad\, f)_p \in \mathbb{R}^n$. This induces an action on AK_ω^T given by $Z \mapsto Z + Hess\, f$, where $Hess\, f = (\frac{\partial^2 f}{\partial z_i \partial t_i})$. Furthermore, we can consider the formal complexification of this action mapping $Z \mapsto Z + Hess\, F$, where $F = f + \sqrt{-1}h$ is a complex-valued function. We remark here that this complexified action is local: the condition that the imiginary part is positive definite could be violated.

4.1.2 The extremal problem

Recall that the hermitian scalar curvature s^{∇} of an ω-compatible almost-Kähler metric of the form (4.1) is given by (see (2.16))

$$s^{\nabla} = -\sum_{i,j} \frac{\partial^2 H_{ij}}{\partial z_i \partial z_j}.$$

The *extremal affine function* $\zeta_{(\Delta,u)}$ is the L^2-projection of the hermtian scalar of s^{∇} of an almost-Kähler metric (J, g) to the finite dimensional space of affine linear function on Δ. Then, the extremal affine function $\zeta_{(\Delta,u)}$ is independant of (J, g) and is determined only by the polytope. Indeed, choosing a basis $\{e_1, \cdots, e_n\}$ of $\mathfrak{t} = \mathrm{Lie}(T)$ gives a basis $\{z_0 = 1, z_1 = \langle e_1, \cdot \rangle, \cdots, z_n = \langle e_n, \cdot \rangle\}$ of affine linear functions. Then, the extremal affine function is given by $\zeta_{(\Delta,u)} = \sum_{i=0}^{n} \zeta_i z_i$, where $(\zeta_0, \cdots, \zeta_n)$ is the unique solution of the linear system of $n+1$ equations

$$\sum_{j=0}^{n} A_{ij} \zeta_j = B_i, \qquad i = 0, \cdots, n, \tag{4.3}$$

where $A_{ij} = \int_{\Delta} z_i z_j dv$ and $B_i = 2 \int_{\partial\Delta} z_i d\nu$; here $dv = dz_1 \wedge \cdots \wedge dz_n$ and $d\nu$ is defined by the equality $u_j \wedge d\nu = -dv$ on any codimension one face with inward normal u_j. In fact, the system (4.3) is just given by the equations $\int_{\Delta} \zeta_{(\Delta,u)} z dv = \int_{\Delta} s^{\nabla} z_i dv$. Indeed, integrating by part and using the boundary condition $\int_{\Delta} s^{\nabla} z_i dv = 2 \int_{\partial\Delta} z_i d\mu$.

We define the *relative Futaki functional* over functions on Δ by

$$\mathcal{L}(f) = \int_{\partial\Delta} f d\nu - \frac{1}{2} \int_{\Delta} f \zeta_{(\Delta,u)} dv.$$

By defintion of $\zeta_{(\Delta,u)}$, $\mathcal{L}(f) = 0$ for any affine linear function f.

Problem: (for more details see [6, 14, 16, 37]) Given a polytope (Δ, u), is there an almost-Kähler metric (J, g) (or a symmetric positive definite-valued function H) such that

$$s^{\nabla} = -\sum_{i,j} \frac{\partial^2 H_{ij}}{\partial z_i \partial z_j} = \zeta_{(\Delta,u)}.$$

In [16], Donaldson claims the following conjecture:

Conjecture 1 *Let (Δ, u) be a polytope. There exist a solution to the extremal problem if and only if $\mathcal{L}(f) \geqslant 0$ for any piecewise linear convex function f on Δ and is equal to zero if and only if f is an affine linear function.*

Donaldson proved this conjecture for $n = 2$ (polygons) when $\zeta_{(\Delta,u)}$ is constant (using the continuity method). Moreover, Zhou–Zhu [53] proved that the existence of a solution of the extremal problem implies $\mathcal{L}(f) \geqslant 0$ for any piecewise linear convex function f on Δ and is zero if and only if f is an affine linear function.

4.1.3 Uniqueness of the extremal almost-Kähler metric

In the integrable case, Guan [27] showed the uniqueness, up to automorphisms, of compatible extremal Kähler toric metrics on a compact symplectic manifold. In this section, we show the following

Proposition 4.2 *There exists a unique extremal ω-compatible T-invariant extremal almost-Kähler toric metric, up to the action of of elements of $Ham(M,\omega)$ commuting with T, in the orbit of the 'complexified' action of elements of $Ham(M,\omega)$ commuting with T.*

Proof. We fix an initial almost-Kähler metric $(J_0, g_0) \in AK^T_\omega$. We consider the following *Mabuchi functional* on the orbit of (J_0, g_0) under the 'formal complexified' action of elements Ham(M,ω) commuting with T:

$$\mathcal{M}(H) := \mathcal{M}(h) = 2\mathcal{L}(h) - \int_\Delta \log\det H^{-1} dv,$$

where H is associated with the almost-Kähler metric (J, g) in the orbit of (J_0, g_0), so $-H^{-1} = -H_0^{-1} + Hess\, h$.

Now, suppose that H_0 and H_1 are associated with two extremal almost-Kähler metrics in the same formal complexified orbit so $-H_1^{-1} = -H_0^{-1} + Hess\, \tilde{h}$. We consider the path $H_t^{-1} = H_0^{-1} - tHess\, \tilde{h}$. Using the identity $d(\log\det A) = tr(A^{-1}dA)$ for any matrix A, we compute

$$(d\mathcal{M})_{H_t}(\dot{H}_t^{-1}) = 2\mathcal{L}(\tilde{h}) - \int_\Delta tr(H_t \circ Hess\, \tilde{h}) dv,$$

where $\dot{H}_t^{-1} = \frac{d}{dt}H_t^{-1}$. Integrating by part and using the boundary condition, we have

$$(d\mathcal{M})_{H_t}(\dot{H}_t^{-1}) = \int_\Delta \left(\xi_{(\Delta,u)} - \sum \frac{\partial^2 H_{tij}}{\partial z_i \partial z_j}\right) \tilde{h}\, dv.$$

This implies that $(d\mathcal{M})_{H_0} = (d\mathcal{M})_{H_1} = 0$ since H_0 and H_1 represent extremal almost-Kähler metrics. Now,

$$\begin{aligned} \frac{d^2}{dt^2}\mathcal{M}(H_t) &= \frac{d}{dt}((d\mathcal{M})_{H_t}(\dot{H}_t^{-1})) \\ &= -\int_\Delta tr(\dot{H}_t \circ Hess\,\tilde{h})dv \\ &= \int_\Delta tr(H_t \circ Hess\,\tilde{h} \circ H_t \circ Hess\,\tilde{h})dv \end{aligned}$$

The matrix H_t is positive definite and the matrix $Hess\,\tilde{h}$ is symmetric. This implies that the product $H_t \circ Hess\,\tilde{h}$ is *diagonalizable*, so $tr(H_t \circ Hess\,\tilde{h} \circ H_t \circ Hess\,\tilde{h}) \geqslant 0$. Hence, $\mathcal{M}$ is convex along the path H_t, so $\frac{d^2}{dt^2}\mathcal{M}(H_t) = 0$ since $(d\mathcal{M})_{H_0} = (d\mathcal{M})_{H_1} = 0$. We deduce that $H_t \circ Hess\,\tilde{h} = 0$. But H_t is positive definite, so $Hess\,\tilde{h} = 0$ and thus $H_0 = H_1$. □

4.1.4 Stability of toric extremal metrics

We can look for an analogue of Theorem 3.1 in the toric case. Indeed, Suppose that H_t is a path of positive definite matrix representing smooth ω-compatible T-invariant almost-Kähler metrics such that H_0 represents an extremal almost-Kähler metric . We can wonder if there exist a family of functions h_t such that $\widetilde{H}_t = (H_t^{-1} + Hess\,h_t)^{-1}$ represent ω-compatible T-invariant extremal almost-Kähler metrics. This is a question we plan to investigate in a near future.

4.2 Stability under deformations of Hermtian-Einstein metrics.

Let (M, ω) be a compact symplectic manifold of dimension $2n = 4$ and T a maximal torus in $\mathrm{Ham}(M, \omega)$. Denote by $\mathfrak{t}_\omega \subset C^\infty(M)$ the finite dimensional space of smooth functions which are hamiltonians with zero mean value of elements of $\mathfrak{t} = \mathrm{Lie}(T)$. To state an analogue of Theorem 3.1 when the metric at time zero is Hermitian-Einstein with negative hermitian scalar curvature, we consider, for an ω-compatible almost-Kähler metric (J, g), the differential linear operator

$$\begin{aligned} Q : \Omega^{J,-}(M) &\rightarrow \Omega^{J,-}(M) \\ \psi &\mapsto (d\delta^c\psi)^{J,-}, \end{aligned}$$

where $\Omega^{J,-}(M)$ is the space of smooth sections of the bundle $\Lambda^{J,-}(M)$ of J-anti-invariant 2-forms and δ^c is the twisted codifferential with respect to J.

The operator Q is elliptic. Indeed, the principal symbol is given by

$$\sigma(Q)_\xi(\psi) = \frac{1}{2}\left(-\xi \wedge J\iota_\xi(J\psi) - J\xi \wedge \iota_\xi(J\psi)\right),$$

$\forall \xi \in T^*_x(M), \psi \in \Omega^{J,-}(M)$, where ι_ξ denotes the contraction by ξ. If we consider a basis $\{\varphi = e_1 \wedge e_2 - Je_1 \wedge Je_2, J\varphi = e_1 \wedge Je_2 + Je_1 \wedge e_2\}$ of $\Omega^{J,-}_p(M)$, where $\{e_1, e_2, Je_1, Je_2\}$ is an orthonormal basis of $T^*_p(M)$, then a direct computation shows that $\sigma(Q)_\xi(\varphi) = \frac{1}{2}|\xi|^2\varphi$ and $\sigma(Q)_\xi(J\varphi) = \frac{1}{2}|\xi|^2 J\varphi$. Note that Q is is not self-adjoint. However, since $\langle Q(\phi), \psi\rangle_{L^2} = -\langle \phi, Q(\psi)\rangle_{L^2}$, one can instead consider the self-adjoint operator $-Q^2$.

Let $(J_t, g_t) \in AK^T_\omega$ be a smooth path of ω-compatible T-invariant almost-Kähler metrics such that (J_0, g_0) is Hermitian-Einstein and we consider the familty of operators $Q_t : \Omega^{J_t,-}(M) \rightarrow \Omega^{J_t,-}(M)$ associated to (J_t, g_t). Suppose that $\mathrm{Ker}(Q_t) = \mathrm{Ker}(Q_t^2) = 0$ for $|t| < \epsilon$. By elliptic theory [33, 34], it is enough to assume that $\mathrm{Ker}(Q_0) = 0$. Then, for any smooth function f, one can find (for $|t| < \epsilon$) a smooth J_t-anti-invariant 2-form ψ^f_t such that $(dJ_tdf)^{J_t,-} = \left(d\delta^c_t\psi^f_t\right)^{J_t,-}$, where δ^c_t is the twisted codifferential with respect to J_t. Now, we consider the almost-Kähler deformations $\omega_{t,f} = \omega + d\left(J_tdf - \delta^c_t\psi^f_t\right)$. We assume furthermore that the map $(t, f) \mapsto \psi(t, f) := \psi^f_t$ is C^1. For this to hold, one need to establish an analogue of Lemma 3.4. The linearisation of the reduced hermitian scalar curvature is just an operator which depends only on f. More explicitely,

$$L(f) := \frac{\partial(Id - \Pi^T_{\omega_{t,f}})(s^{\nabla_{t,f}})}{\partial f}|_{(0,f)} = -(\Delta^{g_0})^2 f + \frac{s^\nabla}{2}\Delta^{g_0} f,$$

where Δ^{g_0} is the Riemannian Laplacian with respect to the metric g_0 and s^∇ is the hermitian scalar curvature of (J_0, g_0). Moreover, the operator L is self-adjoint elliptic leaving invariant $(\mathfrak{t}_\omega)^\perp$ since $L(f) = 0$ for any $f \in \mathfrak{t}_\omega$. Indeed, if $X = Jgrad f$ is a hamiltonian Killing vector field, then by Lemma 1.9, the function f verifies $-\frac{1}{2}d\Delta^{g_0}f = \rho^{\nabla}(X,\cdot)$. As the almost-Kähler metric (J_0, g_0) is Hermitian-Einstein, $-\frac{1}{2}d\Delta^{g_0}f = \frac{s^{\nabla}}{4}\omega(X,\cdot) = -\frac{s^{\nabla}}{4}df$ and hence $(\Delta^{g_0})^2 f = \frac{s^{\nabla}}{2}\Delta^{g_0}f$. We deduce that $(Id - \Pi^T_{\omega})L(f) = L(f)$.

Furthermore, the operator L is an isomorphism of $(\mathfrak{t}_{\omega})^{\perp}$ if $s^{\nabla} \leqslant 0$. Indeed, suppose that $L(f) = -(\Delta^{g_0})^2 f + \frac{s^{\nabla}}{2}\Delta^{g_0}f = 0$, then $\Delta^{g_0}(\Delta^{g_0}f) = \frac{s^{\nabla}}{2}\Delta^{g_0}f$. By hypothesis $s^{\nabla} \leqslant 0$, it implies that $\Delta^{g_0}f = 0$ so $f \equiv 0$ if we consider functions we zero mean value.

Thus, modulo the regularity of ψ^f_t, and assuming the vanishing of dim Ker (Q_0), we expect the stability theorem will hold starting with a not necessary integrable Hermitian-Einstein almost-Kähler metric.

4.3 Further remarks and questions

(1) In [16], it is shown that in the toric case the existence of an extremal almost-Kähler metric is closely related (and conjecturally equivalent) to the existence of an extremal Kähler metric; this was further generalized to certain toric bundles in [6]. It will be interesting to establish a similar link in general, assuming that there are integrable complex structures in AK^T_{ω}.

(2) A well-know result of Calabi [12] states that any extremal Kähler metric (ω, g) on a compact complex manifold (M, J) is invariant under a maximal connected compact subgroup G of $\mathrm{Aut}_0(M, J) \cap \mathrm{Ham}(M, \omega)$. It will be desirable to know whether or not such a G (or a maximal torus in it) is also maximal as a compact subgroup of $\mathrm{Ham}(M, \omega)$? More generally, one would like to know whether or not an extremal almost-Kähler metric is necessarily invariant under a maximal torus in $\mathrm{Ham}(M, \omega)$?

(3) It would be interesting to know how does the extremal vector field Z^T_{ω}, which we introduced as an invariant of a maximal torus T in $\mathrm{Ham}(M, \omega)$, characterize this torus up to congugacy in $\mathrm{Symp}(M, \omega)$ or $\mathrm{Ham}(M, \omega)$; elaborating on the theory appearing in recent works [2, 30] would be an appealing direction of further investigation.

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