UNIVERSITÉ DU QUÉBEC À MONTRÉAL

ÉTUDE DES ORDRES ET TREILLIS

THÈSE

PRÉSENTÉE

COMME EXIGENCE PARTIELLE

DU DOCTORAT EN MATHÉMATIQUES

PAR

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JUIN 2026

REMERCIEMENTS

Tout d'abord, je tiens à remercier très chaleureusement mes deux directeurs de thèse, Samuele Giraudo et Hugh Thomas.

Après un stage de master en 2020 à l'université de Strasbourg sous la direction de Frédéric Chapoton, en pleine pandémie mondiale de Covid, j'ai décidé d'apprendre la combinatoire algébrique à l'université Gustave Eiffel. J'y ai croisé Samuele qui fut, avec Jean-Christophe Novelli, le professeur m'ayant initié au domaine. J'ai découvert lors de cette année la gentillesse de Samuele et ses qualités pédagogiques. Le hasard nous a réunis au Québec après ma première année de doctorat, et je lui suis très reconnaissant d'être devenu à ce moment mon co-directeur de thèse.

J'ai beaucoup d'admiration pour Hugh. J'ai découvert un mathématicien dévoué à sa passion, très actif et très reconnu dans la communauté. Merci pour la patience dont tu as fait preuve à mon égard, et tout le temps que tu m'as consacré. Merci aussi de m'avoir permis de venir faire un doctorat au Québec. Je t'en suis très reconnaissant.

Je tiens aussi à remercier les autres membres de mon jury Patricia Hersh, Nathan Reading et Franco Saliola. Merci pour le temps que vous avez pris pour lire ma thèse et pour vos commentaires qui m'encouragent à poursuivre le travail commencé dans cette thèse.

Je me dois aussi de remercier l'Institut des sciences mathématiques pour les différentes bourses que j'ai pu obtenir au cours de mon doctorat (bourse d'excellence, de thésard étoile et bourse de voyage). Ce fut un soutien très apprécié.

J'ai commencé mon doctorat en 2022 au sein du LACIM, le laboratoire d'algèbre, de combinatoire et d'informatique mathématiques de l'UQAM. C'est un lieu reconnu mondialement dans le domaine de la combinatoire algébrique, et tant avant moi ont également passé une partie de leur parcours mathématique dans ce laboratoire. J'y ai passé 4 très belles années, principalement grâce aux professeurs, étudiants et post-doctorants. De nature réservé, je me suis senti à l'aise grâce à l'atmosphère qui y règne où l'on sent moins, comparé à ce que j'ai ressenti en France, la distance entre étudiants et professeurs. Je remercie particulièrement les professeurs François Bergeron, Srečko Brlek, Christophe Hohlweg, Alejandro Morales, Christophe

Reutenauer et Franco Saliola. J'y ai aussi côtoyé des post-doctorants très sympathiques dont fait notamment partie José Bastidas, Jonathan Boretsky, Monica Garcia, Gayee Park et Luis Scoccola. Évidemment, j'y ai aussi rencontré des camarades étudiants sans qui ces années n'auraient pas été aussi belles. Sans être exhaustif, merci à Antoine, Félix, Yan, Daria, Louis, Nicolas, Benjamin, Karimatou, Alexis, Ismaël, Philippe, Yakob, Nicolas, Sooa, ... Merci à tous ceux qui font vivre ce laboratoire, j'en garderai un super souvenir !

Merci en particulier à Antoine Abram, Félix Gélinas, José Bastidas et Benjamin Dequêne qui sont devenus des bons amis. Un article de cette thèse est d'ailleurs coécrit avec Antoine, merci pour cette collaboration courte mais qui fut très intense !

Ayant quitté la France pour faire mon doctorat, je veux remercier mes amis qui malgré la distance restent toujours présents dès que je rentre au pays. Merci Pierre-Louis, Alban, Julien, Benoît, Medhi, Marc-Antoine, Arthur, Martin, Alann, Manon, ...

Merci aussi à mes colocs de Montréal qui m'ont supporté au début de mon doctorat : Victor, Lola, Étienne, Céphas, Bodler et Damien.

Je souhaite aussi remercier plusieurs professeurs qui ont pu me soutenir ou m'inspirer durant mon long parcours scolaire. Au collège, je pense à monsieur Loi et monsieur Varet. Au lycée, à madame Créhange et monsieur Rey. En classe préparatoire, je pense à monsieur Rauch. Enfin à l'université, je pense à Dominique Hulin, David Harari et Jean-Christophe Novelli.

Et bien sûr, je ne peux que terminer en mentionnant mes proches. Je remercie celle qui partage ma vie, ma plus belle rencontre à Montréal, mon bichon Charlie. Sans toi je n'aurais jamais pu obtenir tous les résultats présentés dans cette thèse. Merci à Sandrine et Serge pour vos encouragements. Je remercie mes trois frères adorés Benjamin, Alexandre et Sydney. Et enfin mes parents Sandrine et Philippe qui nous ont toujours encouragés, mes frères et moi, à poursuivre nos études à leur terme. Benjamin ayant déjà terminé sa thèse de médecine, et Alexandre son diplôme d'ingénieur, il ne reste que toi Sydney. Bon courage pour ta thèse de mathématiques ! Comme tu es bien meilleur que moi, ça ne sera qu'une formalité ☺.

TABLE DES MATIÈRES

TABLE DES FIGURES

RÉSUMÉ

Dans cette thèse, nous démontrons plusieurs résultats au sujet des ordres et des treillis.

Dans un premier temps, nous démontrons que tout ordre unicyclique, c'est-à-dire un ordre dont le graphe des couvertures a exactement un cycle, est de dimension au plus 3. Il s'agit d'un travail en collaboration (Abram, Segovia, 2025) qui résout une conjecture formulée en 1997 (Bollobás, Brightwell, 1997, Conjecture 1.1). Cette conjecture est motivée par le fait que ces ordres apparaissent naturellement comme ordres aléatoires. Nous donnons une preuve constructive, fournissant des triplets d'extensions linéaires réalisant ces ordres. Pour y parvenir, nous donnons en particulier une nouvelle preuve du fait que les ordres dont le graphe des couvertures est un arbre sont de dimension au plus 3.

Dans le reste de la thèse, on s'intéresse aux treillis. Les contributions sont dans deux directions : à la théorie générale des treillis dans un premier temps, puis à l'étude précise d'un treillis venant de la théorie des représentations des algèbres dans un second temps.

Pour la théorie générale des treillis, nous donnons une nouvelle définition équivalente des treillis modulaires à gauche à l'aide de certains étiquetages naturels des relations de couverture du treillis. Nous démontrons un critère sur les doublements d'un treillis congruence uniforme qui caractérise la modularité à gauche de ces treillis. Nous établissons également qu'un treillis congruence uniforme est shellable si et seulement s'il est extrémal. On donne également le premier exemple d'un complexe sup canonique d'un treillis semi-distributif qui contient un sous-complexe induit qui n'est pas un complexe sup canonique d'un treillis semi-distributif. Ceci répond à une question de Barnard (Barnard, 2017, Question 2.5.5). Une autre contribution à la théorie des treillis est une formule pour la dimension des treillis semi-distributifs extrémaux, qui est utilisée pour donner la dimension de plusieurs familles de treillis. Ceci inclut l'obtention de la dimension des treillis des classes de torsion des arbres aimables, qui sont de dimension n pour les arbres ayant n sommets.

Enfin, nous étudions en détail le treillis des classes de torsion supérieures des algèbres d'Auslander supérieures de type $\mathbf{A}$. En particulier, nous démontrons que ces treillis sont sup-congruence uniformes, modulaires à gauche et sup-extrémaux. Nous démontrons également que les treillis des classes de torsion supérieures des algèbres de Nakayama supérieures de type $\mathbf{A}$ sont des quotients de treillis de ceux d'Auslander supérieures. Ces différents résultats sont obtenus comme cas particulier d'une nouvelle construction de treillis que nous appelons les treillis (P, ϕ)-Tamari où P est un poset et ϕ est une chaîne de P.

INTRODUCTION

Cette thèse est une thèse par articles. Les trois articles présentés ont été rendus publics sur le site internet arXiv. Le premier de ces articles, qui fait l'objet du chapitre 2, écrit en collaboration avec Antoine Abram, a été soumis à une revue spécialisée. Le deuxième, qui fait l'objet du chapitre 3, a été rédigé seul et sera prochainement soumis à une revue spécialisée. Le troisième, qui fait l'objet du chapitre 4, a été rédigé seul et une version courte a été acceptée pour présentation à la conférence FPSAC 2025 à Sapporo et publiée dans un volume du Séminaire Lotharingien de Combinatoire [95]. L'article correspondant retranscrit dans le chapitre 4 est la version longue qui contient les preuves des résultats annoncés dans la version courte citée ci-dessus.

Cette thèse se situe dans le domaine de la combinatoire algébrique. Plus particulièrement, nous nous intéressons aux ensembles partiellement ordonnés finis, qui sont des objets centraux d'étude dans ce domaine. On les appelle aussi ordres ou posets (pour l'anglais partially ordered sets). Un ordre est un ensemble muni d'une relation binaire réflexive, antisymétrique et transitive.

Une sous-classe importante des ordres est celle des treillis. Ils sont apparus la première fois dans des travaux de Schröder [93] et Dedekind [31] à la fin du XIX$^{\text{e}}$ siècle, et la version moderne du domaine a été présentée en premier lieu dans un livre de Birkhoff en 1940 [11]. Par définition, il s'agit des ordres pour lesquels toute paire d'éléments possède une borne supérieure et une borne inférieure. Ceci nous donne deux opérations appelées respectivement sup et inf. Les treillis sont omniprésents en combinatoire algébrique, du fait que nos ordres proviennent souvent de structures algébriques. C'est d'ailleurs dans ce contexte qu'ils ont été découverts.

La recherche sur les ordres et les treillis se fait souvent dans deux directions, qui communiquent entre elles. La première est une étude générale, il s'agit de la théorie des ordres et treillis. Dans celle-ci, on étudie de façon abstraite ces structures et on déduit des résultats généraux pour des sous-classes d'ordres ou de treillis. La seconde est une étude spécifique d'un exemple ou une famille d'exemples. On étudie minutieusement des ordres et treillis particuliers, cherchant par là à en apprendre plus sur les éléments de ces structures particulières. Dans cette thèse, nous donnons des contributions dans ces deux directions.

Nous commencerons par introduire les différentes notions utiles à la compréhension des résultats de cette thèse dans le chapitre 1. Ce chapitre traitera donc des ordres et treillis, mais donnera également une courte introduction à la théorie des représentations des algèbres.

Une mesure bien connue et très étudiée de la complexité des ordres est celle de dimension d'un ordre, introduite en 1941 par Dushnik et Miller [38]. La dimension d'un ordre est difficile à obtenir en général. Par exemple, c'est un problème NP-complet de déterminer si un ordre donné est de dimension n pour un entier $n \geq 3$ [109]. Des ordres aléatoires peuvent être produits avec différents modèles, et pour l'un de ces modèles particuliers il est naturel de se poser la question de la dimension des ordres unicycliques, c'est-à-dire les ordres dont le graphe des couvertures a un unique cycle. Bollobás et Brightwell ont conjecturé que ces ordres sont toujours de dimension au plus 3. Cette thèse contient au chapitre 2 une preuve de cette conjecture. Il s'agit d'un travail en collaboration avec Abram [1].

Nous donnons ensuite, dans le chapitre 3, des résultats liés à la théorie des treillis qui se trouvent dans un article écrit par l'auteur [94]. Ces résultats ont essentiellement trait à l'extrémalité, la modularité à gauche et la dimension des treillis congruence uniformes et semi-distributifs. Nous résumons maintenant les contributions de ce chapitre.

Il a été démontré par Liu [67] que les treillis modulaires à gauche possèdent un certain étiquetage des relations de couverture très pratique pour établir des résultats de nature topologique sur ces treillis, et que ces étiquetages ont plusieurs définitions équivalentes. Nous donnons une nouvelle preuve du résultat de Liu et établissons la réciproque, au sens où nous montrons que si pour un treillis ces différentes définitions donnent le même étiquetage, alors le treillis est modulaire à gauche. Nous utilisons ce résultat pour donner une nouvelle preuve du fait que tout treillis semi-distributif extrémal est modulaire à gauche. Ce dernier résultat montre en particulier que si un treillis est congruence uniforme, l'extrémalité implique la shellabilité. Nous démontrons la réciproque, ce qui établit qu'un treillis congruence uniforme est extrémal si et seulement s'il est shellable. Aussi, nous caractérisons, à travers leurs doublements, les treillis congruence uniformes qui sont modulaires à gauche. On donne également le premier exemple d'un complexe sup canonique d'un treillis semi-distributif qui contient un sous-complexe induit qui n'est pas un complexe sup canonique d'un treillis semi-distributif. Ceci répond à une question de Barnard [7, Question 2.5.5]. Nous démontrons ensuite que la dimension d'un treillis semi-distributif extrémal est le nombre chromatique du complément de son graphe de Galois. Nous terminons le chapitre en donnant plusieurs applications de ce résultat. En particulier, on montre que le treillis des classes de torsion d'un arbre aimable à n sommets est de dimension n.

Dans le chapitre 4 se trouve une version longue de l'article [95] définissant les treillis (P, ϕ)-Tamari, dans lequel on trouve une étude détaillée des treillis des classes de torsion supérieures des algèbres supérieures d'Auslander et de Nakayama de type $\mathbf{A}$. Pour chaque ordre P et chaîne ϕ de P, on définit un treillis appelé (P, ϕ)-Tamari. Nous y démontrons, parmi d'autres résultats, que les treillis (P, ϕ)-Tamari sont sup-congruence uniformes, modulaires à gauche et sup-extrémaux. Deux exemples importants de cette nouvelle construction sont les suivants. Lorsque $P = \phi$ est une chaîne, le treillis obtenu est le treillis de Tamari. Pour P qui est l'ordre produit sur les d-uplets croissants dont les éléments sont dans $\{0, 1, \ldots, n-1\}$, et ϕ est la chaîne formée des d-uplets constants, le treillis obtenu est celui des d-classes de torsion de l'algèbre $(d-1)$-Auslander de type $\mathbf{A}_n$. De plus, nous démontrons que les treillis des d-classes de torsion des algèbres $(d-1)$-Nakayama de type $\mathbf{A}$ sont des quotients de treillis de ce dernier.

Finalement, nous terminons avec une conclusion donnant quelques questions ouvertes et axes de recherche pour continuer le travail présenté dans cette thèse.

CHAPITRE 1
ORDRES, TREILLIS ET THÉORIE DES REPRÉSENTATIONS DES ALGÈBRES

1.1 Introduction

Dans ce chapitre introductif, on définit les notions utiles pour comprendre les trois prochains chapitres. Tous les objets introduits dans ce chapitre, notamment les ensembles, sont toujours supposés finis. Si ce n'est pas le cas, nous y ferons explicitement mention.

Chaque article retranscrit dans les chapitres suivants a toujours une première partie donnant les notations et définitions utiles à sa compréhension. L'intérêt principal de ce chapitre introductif est de rassembler toutes ces notations et notions utiles au même endroit, et donner plus de détails pour certaines notions. Nous allons aborder de nombreux concepts différents, souvent en surface, car nos résultats dans cette thèse visent surtout à faire des liens entre différentes notions plutôt qu'une étude très approfondie d'une unique notion mathématique. Cependant, comme les treillis occupent une part prépondérante de nos résultats, une partie significative de ce chapitre traitera de ceux-ci.

Après avoir donné les notations usuelles des mathématiques que nous utiliserons, nous discuterons dans cet ordre des graphes, des ordres, de topologie des ordres, des treillis (en particulier distributifs, extrémaux, modulaires à gauche, semi-distributifs et congruence normaux) pour finir par la théorie des représentations des algèbres.

1.2 Notations usuelles

Pour un ensemble E sa cardinalité est notée $|E|$. Pour un entier naturel n, on note $[n] := \{1, 2, \dots, n\}$. Les coefficients binomiaux sont notés $\binom{n}{k}$, ce qui correspond au nombre de sous-ensembles de $[n]$ ayant k éléments. On écrit $\binom{E}{k}$ pour l'ensemble des sous-ensembles de E ayant k éléments. L'ensemble des sous-ensembles de E est noté $\mathcal{P}(E)$. Un multiensemble est une généralisation d'un ensemble dans lequel un même élément peut apparaître plusieurs fois.

Une relation binaire R sur un ensemble E est un sous-ensemble de $E \times E$. Si $(x, y) \in R$, on écrit $x\,R\,y$. On rappelle qu'une relation binaire R sur E est réflexive si pour tout $x \in E$, on a $x\,R\,x$. Elle est symétrique si pour tout $(x, y) \in E^2$, on a $x\,R\,y$ si et seulement si $y\,R\,x$. Elle est antisymétrique si pour tout $(x, y) \in E^2$,

$x \, R \, y$ et $y \, R \, x$ ensemble impliquent $x = y$. Enfin, elle est transitive si pour tout $(x, y, z) \in E^3$, $x \, R \, y$ et $y \, R \, z$ ensemble impliquent $x \, R \, z$. Une relation binaire est une relation d'équivalence si elle est réflexive, symétrique et transitive. Si R est une relation d'équivalence sur E, ses classes d'équivalence sont les ensembles $\mathcal{C}(x) = \{y \in E \mid x \, R \, y\}$ pour $x \in E$.

Sans donner de détails, on rappelle que la classe NP (pour non déterministe polynomial) regroupe les problèmes décisionnels pour lesquels on peut vérifier si une solution candidate est effectivement une solution avec une machine de Turing déterministe en temps polynomial par rapport à la taille de l'entrée. Un problème décisionnel est dit NP-complet s'il appartient à la classe NP et si tout problème de la classe NP se ramène à celui-ci par une réduction polynomiale. Lorsqu'un problème de décision est NP-complet nous dirons qu'il s'agit d'un *problème difficile*.

1.3 Graphes

Pour cette partie sur la théorie des graphes, nous recommandons deux livres de Bollobás [19, 18].

Un graphe (ou graphe simple) est un couple $G = (V, E)$ constitué d'un ensemble V et d'un ensemble $E \subseteq \binom{V}{2}$. Les éléments de V sont appelés sommets et les éléments de E sont appelés arêtes. Si G est un graphe, son ensemble de sommets est $V = V(G)$ et son ensemble d'arêtes $E = E(G)$. Les deux sommets qui forment une arête sont appelés les extrémités de l'arête. Si l'ensemble E est remplacé par un multiensemble, autorisant de plus les arêtes joignant un sommet à lui-même, représentées dans E par un singleton et appelée boucle, on dit que G est un multigraphe. Si les arêtes d'un graphe sont remplacées par des couples, choisissant donc un ordre sur les extrémités de chaque arête, alors G est appelé un graphe orienté et souvent noté $\vec{G}$. Dans ce cas une arête (x, y) est aussi appelée une flèche de x vers y. Si E est un multiensemble constitué de paires $(x, y) \in V \times V$, alors G est ce qu'on appelle un multigraphe orienté, et souvent noté $\vec{G}$. Dans ce cas les boucles correspondent aux paires $(x, x) \in E$. Voir figure 1.1 pour la manière dont nous représentons les graphes, les graphes orientés et multigraphes orientés.

Un chemin dans un graphe G est une suite de sommets $x_1, x_2, \ldots, x_k$ tel que $\{x_i, x_{i+1}\} \in E(G)$ pour tout $i \in [k-1]$. Il s'agit d'un chemin de x_1 à x_k et le chemin est dit non trivial si $k > 1$. Un cycle d'un graphe est un chemin non trivial de la forme $x_1, x_2, \ldots, x_k = x_1$ qui est sans répétitions avant d'y ajouter x_k. Dans ce cas on dit qu'il s'agit d'un k-cycle. La relation binaire R_G dans un graphe G définie par $x \, R_G \, y$ s'il existe un chemin de x à y est une relation d'équivalence. Les classes d'équivalence de R_G sont appelées les composantes connexes de G. Si G possède une unique composante connexe, on dit que G est connexe.

Un chemin orienté dans un graphe orienté ou multigraphe orienté est une suite de sommets $x_1, x_2, \ldots, x_k$ tel que pour tout $i \in [k-1]$ on a $x_i \neq x_{i+1}$ et il existe une flèche de x_i vers x_{i+1}. Ainsi on ne peut pas emprunter une boucle dans un chemin orienté. Le chemin orienté est dit non trivial si $k > 1$. Un graphe orienté est acyclique s'il ne possède pas de cycles orientés (aussi appelés circuits), c'est-à-dire de chemins orientés non triviaux $x_1, x_2, \ldots, x_k = x_1$. Dans un multigraphe orienté, un 2-cycle orienté désigne un cycle orienté x_1, x_2, x_1, c'est-à-dire nous avons une flèche de x_1 vers x_2 et de x_2 vers x_1, où $x_1 \neq x_2$. La fermeture transitive d'un graphe orienté $\vec{G}$ est le graphe orienté obtenu à partir de $\vec{G}$ en ajoutant pour tout $x, y \in V(\vec{G})$ une flèche de x vers y si une telle flèche n'existe pas déjà et qu'il existe un chemin orienté non trivial $x = x_1, x_2, \ldots, x_k = y$. Il est immédiat que la fermeture transitive d'un graphe orienté acyclique est encore acyclique. Un graphe orienté est transitivement réduit si dès qu'il existe un chemin orienté $x = x_1, x_2, \ldots, x_k = y$ de x à y avec $k > 2$, alors il n'existe pas de flèche de x vers y.

Exemple 1.1 *Dans la figure 1.1, le multigraphe orienté $\vec{H}$ possède un 2-cycle. Le graphe orienté $\vec{G}$ est acyclique, mais n'est pas transitivement réduit. On construit sa fermeture transitive en ajoutant une flèche de* 1 *vers* 5 *et une de* 4 *vers* 2.

On dit que $G' = (V', E')$ est un sous-graphe de $G = (V, E)$ si $V' \subseteq V$ et $E' \subseteq E$. Si G' contient toutes les arêtes de G qui joignent deux sommets de V', alors G' est un sous-graphe induit et noté $G[V']$. Si $\vec{G} = (V, E)$ est un graphe orienté, on appelle sous-graphe non orienté sous-jacent de $\vec{G}$ le graphe simple ayant pour sommets V et pour arêtes les $\{x, y\}$ où $(x, y) \in E$. De façon similaire est défini le sous-multigraphe non orienté sous-jacent d'un multigraphe orienté. Le complément d'un graphe G, noté $\overline{G}$, est le graphe ayant les mêmes sommets que G et une arête $\{x, y\}$ si et seulement si $\{x, y\} \notin E$. Similairement, le complément d'un multigraphe G est le graphe $\overline{G}$ ayant les même sommets que G et dont les arêtes sont les $\{x, y\}$ tels que $x \neq y$ et $\{x, y\} \notin E$. Les compléments des graphes orientés et multigraphes orientés sont définis comme les compléments de leurs sous-graphes et sous-multigraphes non orientés sous-jacents respectifs.

Exemple 1.2 *Dans la figure 1.1, le graphe G est le sous-graphe non orienté sous-jacent de $\vec{G}$, et on a représenté son complément $\overline{G}$ (qui correspond aussi au complément de $\vec{G}$). Le complément de $\vec{H}$ est le graphe de sommets* $[4]$ *avec une unique arête* $\{1, 4\}$.

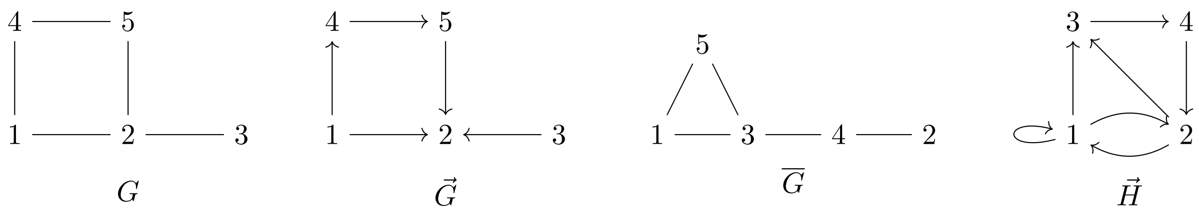


FIGURE 1.1 : De la gauche vers la droite nous avons un graphe G, un graphe orienté $\vec{G}$ dont le sous-graphe non orienté sous-jacent est G, le complément de G, et un multigraphe orienté $\vec{H}$.

Le graphe K_n, appelé graphe complet à n sommets, est le graphe $K_n = ([n], \binom{V}{2})$. Un graphe vide est un graphe n'ayant pas d'arêtes, mais il peut avoir un nombre arbitraire de sommets. Les arbres sont les graphes acycliques connexes. Un graphe unicyclique est un graphe connexe ayant exactement un cycle.

Les voisins d'un sommet $x \in V(G)$ sont les sommets qui sont reliés par une arête à x. Le degré maximal du graphe G, noté $d(G)$, est le nombre maximal de voisins d'un sommet de G. Une clique d'un graphe G est un sous-ensemble S des sommets de G tel que $G[S]$ est un graphe complet. Le nombre clique de G, noté $\omega(G)$, est la plus grande taille d'une clique de G. Un ensemble indépendant d'un graphe G est un sous-ensemble I des sommets de G tel que $G[I]$ n'a pas d'arêtes, c'est-à-dire est un graphe vide. Un sous-ensemble source d'un multigraphe orienté $\vec{G}$ est un sous-ensemble non vide $X \subsetneq V(\vec{G})$ tel qu'il n'y a pas de flèche allant d'un sommet de $V(G) \setminus X$ à un sommet de X.

Une coloration à n couleurs du graphe G est une application $\chi : V(G) \to [n]$. La couleur assignée au sommet $v \in V(G)$ est $\chi(v)$. Il s'agit d'une coloration propre si pour toute arête les couleurs assignées aux deux extrémités de l'arête sont différentes. Le nombre chromatique du graphe G, noté $\chi(G)$, est le nombre minimal de couleurs n tel qu'il existe une coloration propre de G à n couleurs. Le résultat suivant est immédiat :

Lemme 1.3 *Pour tout graphe G, on a* $\omega(G) \leq \chi(G) \leq d(G) + 1$.

Exemple 1.4 *On utilise les graphes représentés dans la figure 1.1 pour illustrer certaines notions que nous venons de définir. Les graphes G et $\overline{G}$ sont unicycliques. Les voisins du sommet* 2 *dans G sont* 1, 3 *et* 5. *On a* $d(G) = d(\overline{G}) = 3$, $\omega(G) = 2$ *et* $\omega(\overline{G}) = 3$. *L'ensemble* $\{1, 3, 5\}$ *est un ensemble indépendant de G. Dans* $\vec{G}$, $\{1, 4\}$ *et* $\{3\}$ *sont deux sous-ensembles sources disjoints. On a* $\chi(G) = 2$ *et* $\chi(\overline{G}) = 3$.

On termine cette partie sur les graphes en rappelant un modèle de génération de graphes aléatoires orientés. Soit un entier $n \geq 1$. Soit $p \in]0,1[$. On appelle $\vec{G}(n,p)$ le graphe aléatoire orienté dont les sommets sont $[n]$ et tel qu'on a une flèche de i vers j, avec $i < j$, avec probabilité p indépendamment de i et j. Son sous-graphe non orienté sous-jacent est noté $G(n,p)$. La probabilité p peut dépendre de n, on l'écrit donc p_n pour rappeler cette dépendance. Le résultat suivant est dû à Erdős et Rényi [39]. Dans leur formulation, au lieu de parler de la probabilité p_n, ils utilisent le nombre d'arêtes $N(n)$ du graphe $G(n,p)$. Mais cette reformulation s'en déduit.

Proposition 1.5 ([39, Theorem 5e]) *Supposons qu'il existe $c \in [0,1[$ tel que $\lim_{n\to\infty}(p_n\, n) = c$. Alors presque sûrement, quand n tend vers l'infini, $G(n,p)$ est une union disjointe d'arbres et de graphes unicycliques.*

Mentionnons, pour avoir un autre point de vue sur ces graphes, que la proposition précédente utilise notamment dans sa démonstration le fait qu'un graphe connexe G est un arbre si et seulement $|E(G)| = |V(G)|-1$, et est unicyclique si et seulement si $|E(G)| = |V(G)|$.

1.4 Ordres

Pour les généralités sur ordres, une référence que nous recommandons est le chapitre sur les ensembles partiellement ordonnés du livre de Stanley [98]. Pour la dimension d'un ordre notre référence est un livre de Trotter [105].

Un ordre est un couple $(P, \leq_P)$ formé d'un ensemble P et d'une relation binaire $\leq_P$ sur P qui est réflexive, antisymétrique et transitive. Un ordre est aussi communément appelé un ensemble partiellement ordonné (en anglais on utilise le terme poset pour *partially ordered set*). Un ordre est souvent désigné simplement par P, son ensemble sous-jacent. De plus la relation binaire $\leq_P$ est souvent désignée par $\leq$. La notation $x \geq y$ veut dire $y \leq x$. La notation $x < y$ veut dire $x \leq y$ et $x \neq y$. On définit similairement $>$. Deux éléments x et y dans P sont comparables si $x \leq y$ ou $y \leq x$. Si ce n'est pas le cas x et y sont dits incomparables. Si tous les éléments de P sont comparables deux à deux, on dit que P est totalement ordonné ou est une chaîne. Lorsque $x \leq y$, on dit que x est inférieur ou égal à y, ou y est supérieur ou égal à x. Si la relation est stricte, c'est-à-dire $<$ ou $>$, on utilisera les termes inférieur ou supérieur. Si un ordre P a un élément maximum, il sera toujours noté $\hat{1}$, et $\hat{0}$ pour un élément minimum. Étant donné que nos ordres sont finis, P a toujours des éléments maximaux et minimaux, mais l'existence d'un élément maximum ou minimum n'est pas toujours assurée.

Une relation de couverture d'un ordre P, notée $x \lessdot y$, est un couple $(x, y) \in P^2$ vérifiant $x < y$ et il n'existe pas de $z \in P$ tel que $x < z < y$. Si $x \lessdot y$, on dit que x est couvert par y, ou que y couvre x. L'ensemble des relations de couvertures de P est noté $E(P)$. Le diagramme de Hasse H d'un ordre P est le graphe orienté sur l'ensemble de sommets P et dont les arêtes sont les relations de couvertures $x \lessdot y$, orientées de x vers y. C'est un graphe orienté acyclique transitivement réduit. Inversement, tout graphe orienté acyclique transitivement réduit est le diagramme de Hasse d'un ordre. Le graphe sous-jacent non orienté de H est appelé le graphe des couvertures. Un étiquetage des relations de couvertures de P est une application $\gamma : E(P) \to Q$ où Q est un ordre. Souvent Q est l'ensemble des entiers naturels avec son ordre usuel.

Il est facile de montrer que si un ordre est fini, il est totalement déterminé par ses relations de couverture, et deux ordres sont isomorphes si et seulement s'ils ont des diagrammes de Hasse isomorphes. On va donc souvent définir un ordre en donnant son diagramme de Hasse. Très souvent, on représente un diagramme de Hasse sans mettre les flèches, car nous prenons la convention que si on a une flèche de x vers y, alors on la représente par une arête joignant x et y où y a une ordonnée plus grande (on suppose qu'on a un repère cartésien usuel où l'axe des ordonnées est vertical et pointe vers le haut). Un ordre est planaire s'il existe une manière de représenter son diagramme de Hasse dans le plan comme nous venons de le mentionner sans qu'il y ait de croisements des arêtes.

L'ordre pentagone est l'ordre noté N_5 dont le diagramme de Hasse est un pentagone, il est représenté sur la figure 1.2. L'ordre hexagone est l'ordre noté H_6 dont le diagramme de Hasse est un hexagone (il est obtenu à partir de N_5 en ajoutant un élément d entre a et $\hat{1}$, voir figure 1.2). L'ordre diamant est l'ordre noté M_3 ayant cinq éléments avec un minimum, un maximum, et trois éléments incomparables entre eux. L'ordre booléen d'ordre n, noté B_n, est l'ordre sur les sous-ensembles de $[n]$ ordonnés par inclusion. Un ordre arbre est un ordre dont le graphe des couvertures est un arbre. Un ordre est unicyclique si son graphe des couvertures est unicyclique. Un ordre cycle est un ordre dont le graphe des couvertures est un cycle. On a un ordre couronne pour chaque entier $n \geq 1$. L'ordre couronne $\mathscr{C}_1$, aussi appelé ordre carré, est l'ordre ayant 4 éléments x, a, b, z où x est le minimum, z le maximum et a et b sont incomparables. L'ordre couronne $\mathscr{C}_n$ pour $n \geq 2$ est l'ordre dont les éléments sont $x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n$ et les seules relations de

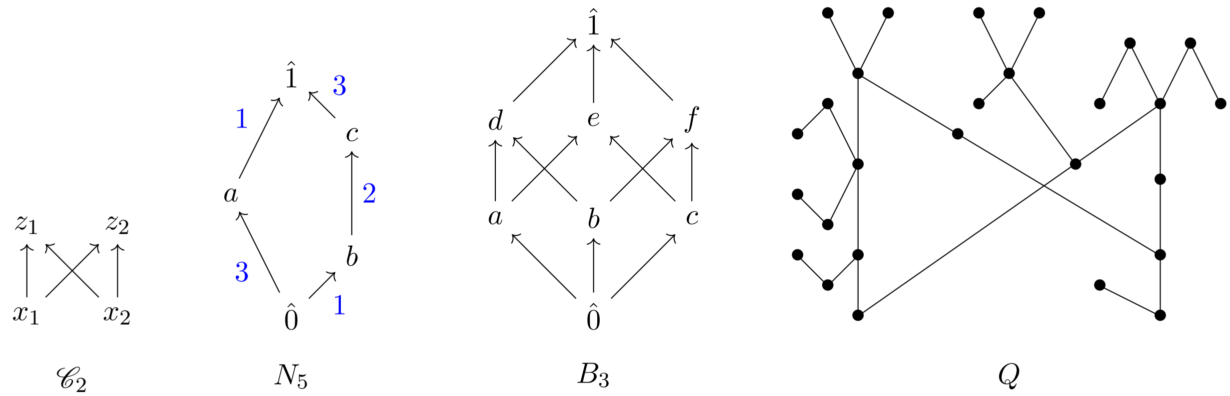


FIGURE 1.2 : Quatre ordres différents représentés par leur diagramme de Hasse.

couvertures sont $x_i \lessdot z_i$ et $x_i \lessdot z_{i-1}$ pour tout $i \in [n]$ où les indices sont considérés cycliquement.

Exemple 1.6 *On s'intéresse aux ordres de la figure 1.2, représentés à l'aide de leur diagramme de Hasse. Il s'agit de l'ordre couronne $\mathscr{C}_2$, l'ordre pentagone N_5, l'ordre booléen B_3 et un ordre unicyclique Q. Grâce à notre convention pour représenter ces diagrammes, il n'y a pas d'ambiguïté pour l'ordre Q et cela nous arrivera de représenter ou non les flèches. Le graphe des couvertures de $\mathscr{C}_2$ est un 4-cycle. Contrairement à ce que l'on pourrait croire, seul B_3 n'est pas planaire parmi ces 4 ordres. Nous avons représenté en bleu un étiquetage des relations de couvertures de N_5, les étiquettes faisant partie de l'ordre usuel sur* $[3]$.

Le dual de l'ordre P, noté P^{op}, est l'ordre $(P, \geq_P)$. C'est-à-dire $x \leq_{P^{op}} y$ si et seulement si $x \geq_P y$. Deux ordres $(P, \leq_P)$ et $(Q, \leq_Q)$ sont isomorphes s'il existe une application $f : P \to Q$ bijective telle que pour tout $x, y \in P$, on ait $x \leq_P y$ si et seulement si $f(x) \leq_Q f(y)$. Dans ce cas on écrit $P \cong Q$. L'union disjointe de deux ordres P et Q, notée $P \sqcup Q$, est l'ordre sur l'union disjointe des ensembles P et Q défini par $x \leq y$ si $x, y \in P$ et $x \leq_P y$, ou si $x, y \in Q$ et $x \leq_Q y$. Le produit direct de deux ordres P et Q, noté $P \times Q$, est l'ordre sur le produit cartésien $P \times Q$ défini par $(x, y) \leq (x', y')$ si $x \leq_P x'$ et $y \leq_Q y'$. Un sous-ordre Q de P est un sous-ensemble des éléments de P muni de l'ordre induit par l'ordre P.

Un ordre linéaire de P, où $|P| = n$, est une bijection $\mathcal{L} : P \to [n]$. Si cette bijection vérifie pour tout $x, y \in P$, $x \leq_P y$ implique $\mathcal{L}(x) \leq \mathcal{L}(y)$ où l'ordre sur $[n]$ est l'ordre usuel des entiers, on dit que $\mathcal{L}$ est une extension linéaire de P. On la représente souvent par le mot $\mathcal{L}^{-1}(1)\,\mathcal{L}^{-1}(2)\cdots\mathcal{L}^{-1}(n)$. Un réalisateur de taille d de P est un ensemble de d extensions linéaires $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_d\}$ de P tel que $x \leq y$ dans P si et seulement si $\mathcal{L}_i(x) \leq \mathcal{L}_i(y)$ pour tout $i \in [d]$. Pour cette dernière propriété on dit aussi que *l'intersection* des extensions linéaires est P. La dimension (d'ordre) de P, notée $\dim(P)$, est le plus petit entier naturel n tel que P soit isomorphe à un sous-ordre de l'ordre produit sur $\mathbb{R}^n$ (où $\mathbb{R}$ est muni de son ordre usuel). De façon équivalente, c'est la taille minimale d'un réalisateur de P, ce qui veut dire le nombre minimal d'extensions linéaires dont l'intersection est P. La dimension a été introduite en 1941 par Dushnik et Miller [38], et a depuis été très étudiée (voir le livre de Trotter [105] qui est entièrement consacré à cette notion). C'est connu que l'ensemble de toutes les extensions linéaires de P forme un réalisateur de P, donc la dimension d'un ordre existe toujours. Cependant, déterminer la dimension d'un ordre est un problème difficile. En effet, soit un entier $n \geq 3$. C'est un problème NP-complet de déterminer si un ordre donné est de dimension n [109]. Plus de résultats et références seront données au sujet de la dimension dans les chapitres 2 et 3.

Exemple 1.7 *Dans la figure 1.2, les ordres $\mathscr{C}_2$, N_5 et B_3 sont isomorphes à leur dual respectif, mais pas Q. Le sous-ordre de B_3 sur les éléments a, b, c, d, e, f est isomorphe à $\mathscr{C}_3$. Il est clair que $\mathscr{C}_2$ est un sous-ordre de Q. Les extensions linéaires de $\mathscr{C}_2$ sont $x_1\, x_2\, z_1\, z_2$, $x_1\, x_2\, z_2\, z_1$, $x_2\, x_1\, z_1\, z_2$ et $x_2\, x_1\, z_2\, z_1$. On a $\dim(\mathscr{C}_2) = \dim(N_5) = 2$ et $\dim(B_3) = \dim(Q) = 3$. Un réalisateur de taille 2 de $\mathscr{C}_2$ est donné par les deux extensions linéaires $x_1\, x_2\, z_1\, z_2$ et $x_2\, x_1\, z_2\, z_1$.*

L'idéal d'ordre de P engendré par le sous-ensemble $C \subseteq P$, noté $I_P(C)$ ou simplement $I(C)$, est $\{x \in P \mid \exists c \in C,\ x \leq c\}$. Il s'agit donc des sous-ordres formés par tous les éléments de P qui sont inférieurs ou égaux à certains éléments fixés. Si $C = \{c\}$ a un unique élément, l'idéal d'ordre engendré par C est simplement noté $I_P(c)$ et s'appelle l'idéal d'ordre principal engendré par c. On définit dualement les filtres d'ordres. Pour $C \subseteq P$, le filtre d'ordre de P engendré par C est $F_P(C) := \{x \in P \mid \exists c \in C,\ x \geq c\}$. L'ordre sur l'ensemble des ordres d'idéaux de P ordonnés par inclusion est noté $J(P)$. D'autres sous-ordres importants sont les intervalles (fermés) et les intervalles ouverts. Un intervalle est un sous-ordre $[x, y] := \{z \in P \mid x \leq z \leq y\}$ où $x \leq y$ dans P. Un intervalle ouvert est un sous-ordre $]x, y[:= \{z \in P \mid x < z < y\}$ où $x \leq y$ dans P. Un pseudo-intervalle inférieur est une union d'intervalles ayant tous le même minimum, et un pseudo-intervalle supérieur est une union d'intervalles ayant tous le même maximum. Un sous-ensemble $C \subseteq P$ est convexe si $z \in C$ dès que $x \leq z \leq y$ et $x, y \in C$. Les intervalles et pseudo-intervalles sont des exemples d'ensembles convexes.

Une antichaîne de P est un sous-ensemble dont les éléments sont deux à deux incomparables. La largeur de P, notée $\operatorname{width}(P)$, est la taille maximale d'une antichaîne de P. Les chaînes de P sont ses sous-ensembles totalement ordonnés. On appelle chaînes maximales les chaînes de P auxquelles on ne peut pas ajouter d'éléments de P pour former une chaîne ayant plus d'éléments. Si une chaîne F de P a $n+1$ éléments, sa longueur est $\ell(F) := n$. La longueur d'un ordre P, notée $\ell(P)$, est la longueur maximale d'une chaîne de P. Les chaînes de longueur maximale sont celles de longueur $\ell(P)$. La colonne vertébrale de P est l'ensemble des éléments $x \in P$ tel qu'il existe une chaîne de longueur maximale de P contenant x. Si toutes les chaînes maximales ont la même longueur, l'ordre est gradué. Si P a un minimum $\hat{0}$, la hauteur de $x \in P$ est la longueur maximale d'une chaîne de $\hat{0}$ à x

Exemple 1.8 *On s'intéresse à nouveau aux ordres de la figure 1.2. Dans B_3, on a $I_{B_3}(d) = [\hat{0}, d] \cong \mathscr{C}_1$ et le filtre d'ordre principal engendré par $c \in B_3$ est aussi isomorphe à $\mathscr{C}_1$. On a $\mathrm{width}(\mathscr{C}_2) = \mathrm{width}(N_5) = 2$, $\mathrm{width}(B_3) = 3$ et $\mathrm{width}(Q) = 10$. L'idéal d'ordre $I_{B_3}(\{d, f\})$ est un pseudo-intervalle inférieur de B_3 qui n'est pas un intervalle. Si R est l'ordre à trois éléments tous incomparables, alors $B_3 = J(R)$ (voir figure 1.4 pour un autre exemple). Les sous-ensembles $\{\hat{0}, c\}$ et $\{\hat{0}, a, \hat{1}\}$ sont deux chaînes de N_5 de longueur respective 1 et 2, mais seulement la seconde est une chaîne maximale (mais ce n'est pas une chaîne de longueur maximale). Ces deux sous-ensembles ne sont pas convexes. On a $\ell(\mathscr{C}_2) = 1$, $\ell(N_5) = \ell(B_3) = 3$ et $\ell(Q) = 4$. Seulement $\mathscr{C}_2$ et B_3 sont gradués.*

On finit cette partie en rappelant le théorème de Dilworth, paru en 1950, de décomposition d'un ordre.

Théorème 1.9 ([36, Theorem 1.1]) *La taille maximale d'une antichaîne d'un ordre P est égale au nombre minimal de chaînes de P dont la réunion est P.*

1.5 Topologie des ordres

Pour cette partie, on recommande un cours de Wachs [108], le chapitre sur les ensembles partiellement ordonnés du livre de Stanley [98] et les articles de Björner et Wachs qui introduisent la shellabilité des complexes simpliciaux non purs [15, 16]. Plus de détails sur les complexes simpliciaux peuvent être trouvés dans un livre de Munkres [81].

Un complexe simplicial (abstrait) est un couple (V, K) où V est un ensemble dont les éléments sont appelés sommets, et $K \subseteq \mathcal{P}(V)$ est un ensemble de parties de V appelées faces, vérifiant $V = \bigcup_{\sigma \in K} \sigma$ et, si $\sigma \subseteq \tau$ et $\tau \in K$, alors $\sigma \in K$. Le complexe simplicial (V, K) est souvent simplement désigné par K. Une face est appelée une facette si elle est maximale au sens de l'inclusion dans K. La dimension d'une face σ est $\dim(\sigma) := |\sigma| - 1$, et la dimension du complexe K est la dimension maximale d'une face de K. On note que $\emptyset$ est toujours une face de K, qui est de dimension -1, sauf si $K = \emptyset$. On note f_i le nombre de faces de K de dimension i. La caractéristique réduite d'Euler de K est $\tilde{\chi}(K) := \sum_{i \geq -1} (-1)^i f_i$. Le complexe simplicial K est pur si toutes ses facettes ont la même dimension. Sinon il est dit non pur. Le 1-squelette du complexe simplicial K est le graphe dont les sommets sont V et les arêtes sont les faces de dimension 1.

Un simplexe géométrique d-dimensionnel dans $\mathbb{R}^n$ est défini comme étant l'enveloppe convexe de $d+1$ points affinement indépendants dans $\mathbb{R}^n$, appelés sommets. Une face de ce simplexe géométrique est l'enveloppe convexe d'un sous-ensemble de ses sommets. Un complexe géométrique simplicial X dans $\mathbb{R}^n$ est une collection non vide de simplexes géométriques dans $\mathbb{R}^n$ tel que chaque face d'un simplexe de X est dans X, et l'intersection de deux simplexes de X est une face de ces deux simplexes. Partant d'un complexe géométrique simplicial X, on obtient un complexe simplicial abstrait K dont les faces sont les ensembles des sommets des simplexes de X. En fait, tous les complexes simpliciaux abstraits sont obtenus de cette façon. Même si, à K fixé, il n'y a pas unicité du complexe géométrique simplicial X associé, il y a unicité à homéomorphisme près de l'espace topologique obtenu (avec la topologie usuelle de $\mathbb{R}^n$) en prenant la réunion des simplexes d'un tel complexe géométrique associé au complexe simplicial K. Cet espace est appelé la réalisation géométrique de K, et on le notera également K. Ainsi on se permet de désigner K à la fois comme complexe simplicial et comme sa réalisation géométrique.

Soit P un ordre. Le complexe d'ordre de P est le complexe simplicial, noté $\Delta(P)$, dont les sommets sont P et les faces sont les chaînes de P. Les facettes de $\Delta(P)$ sont les chaînes maximales de P. Ainsi $\Delta(P)$ est pur si et seulement si P est gradué.

Exemple 1.10 *Sur la figure 1.3 sont représentés trois complexes simpliciaux. Nous avons notamment représenté le complexe d'ordre de N_5. Les faces de dimensions 2 et supérieures sont représentées en bleu. Le complexe simplicial $\Delta(N_5)$ a deux facettes, un tétraèdre plein qui est de dimension 3 et un triangle plein de dimension 2, ces deux facettes partageant la face de dimension 1 qui correspond à l'arête $\{\hat{0}, \hat{1}\}$. Le complexe simplicial K_1 a lui trois facettes et est de dimension 2. Son 1-squelette est constitué de deux 3-cycles reliés par une arête. Le complexe simplicial K_2 est de dimension 1, il possède 9 facettes, que nous avons étiquetées, qui sont toutes de dimension 1. Donc K_2 est pur, contrairement à $\Delta(N_5)$ et K_1 qui sont non purs. On a $\tilde{\chi}(\Delta(N_5)) = \tilde{\chi}(K_1) = 0$ et $\tilde{\chi}(K_2) = -4$.*

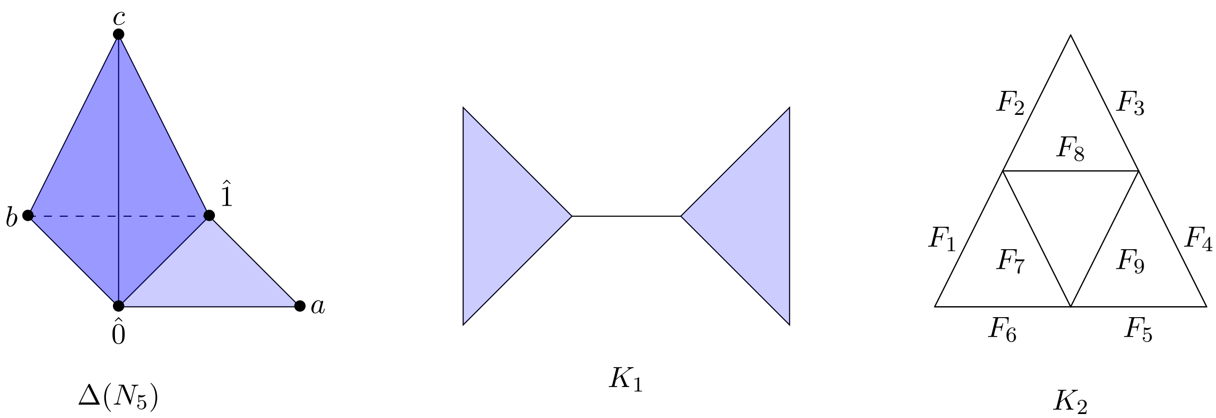


FIGURE 1.3 : Trois complexes simpliciaux.

Soit $G = (V, E)$ un graphe. Son complexe clique est le complexe simplicial de sommets V et dont les faces sont les cliques de G. Un complexe est dit clique si c'est le clique complexe de son 1-squelette, et il est donc entièrement déterminé par ce dernier. Un complexe simplicial (V, K) est dit drapeau si pour tout sous-ensemble $S \subseteq V$ tel que toute paire d'éléments de S est une face de K, alors S est une face de K. Il est facile de montrer que les complexes drapeaux sont les complexes cliques.

Soit P un ordre. Une paire critique de P est une paire $(a, b) \in P^2$ tel que a et b sont incomparables et si $x < a$, alors $x < b$, et si $y > b$, alors $y > a$. Le graphe orienté des paires critiques de P, noté $D(P)$, est le graphe orienté dont les sommets sont les paires critiques et tel qu'on ait une flèche de (a, b) vers (c, d) dès que $b \geq c$. Le complexe simplicial des paires critiques de P, noté $C(P)$, est le complexe simplicial dont les faces sont les sous-ensembles de paires critiques qui induisent un sous-graphe orienté acyclique de $D(P)$.

Exemple 1.11 *Pour l'ordre pentagone N_5 (voir figure 1.2) le graphe orienté des paires critiques $D(N_5)$ est constitué d'un 2-cycle orienté dont les sommets sont (a, c) et (b, a). Donc son complexe simplicial des paires critiques $C(N_5)$ est constitué de deux sommets isolés. C'est également le cas pour l'ordre hexagone H_6.*

Pour une face F d'un complexe simplicial K, on note $\langle F \rangle := \{G \mid G \subseteq F\}$ le sous-complexe engendré par F. Un complexe simplicial K est shellable s'il existe un ordre total $F_1, F_2, \ldots, F_k$ des facettes de K tel que pour tout $1 \leq j < k$, le complexe $\left(\bigcup_{i=1}^{j} \langle F_i \rangle\right) \bigcap \langle F_{j+1} \rangle$ est pur et de dimension $\dim(F_{j+1}) - 1$. Un tel ordre total sur les facettes est appelé un shelling. Cela est équivalent à demander que pour tout $1 \leq i < j \leq k$, il existe $l < j$ tel que $F_i \cap F_j \subseteq F_l \cap F_j$ et $|F_l \cap F_j| = |F_j| - 1$. Dans un shelling $F_1, F_2, \ldots, F_k$ une facette est appelée facette homologique si toute sa frontière est contenue dans l'union des facettes précédentes.

Un ordre P est dit shellable si son complexe d'ordre $\Delta(P)$ est shellable. La shellabilité a été initialement seulement définie et étudiée pour les complexes simpliciaux purs. On peut citer par exemple l'article [13] de Björner dans lequel se trouve plusieurs résultats de shellabilité d'ordres. La définition donnée ci-dessus est valable avec ou sans hypothèse que le complexe simplicial est pur. Cette généralisation est due à Björner et Wachs en 1996 [15, 16]. Déterminer si un complexe simplicial donné est shellable est un problème NP-complet, donc difficile [49]. Nous verrons plus tard, dans la partie 1.9, qu'il est possible d'obtenir la shellabilité d'ordres particuliers à l'aide d'un étiquetage des relations de couverture appelé un EL-étiquetage.

Exemple 1.12 *Sur la figure 1.3, $\Delta(N_5)$ est shellable, l'unique shelling étant de considérer comme première facette le tétraèdre plein puis le triangle plein $\hat{0}\,a\,\hat{1}$. Ce complexe simplicial n'a pas de facette homologique. Le complexe simplicial K_2 est également shellable. Un shelling est donné par $F_1, F_2, \ldots, F_9$. Pour ce shelling les facettes F_6, F_7, F_8 et F_9 sont des facettes homologiques. Par contre, K_1 n'est pas shellable puisqu'aucun des six ordres totaux sur ses facettes ne forme un shelling. Les deux complexes simpliciaux $C(N_5)$ et $C(H_6)$ décrits dans l'exemple 1.11 sont shellables.*

Un bouquet $\vee_{i=1}^{n} X_i$ de n espaces topologiques connexes mutuellement disjoints est l'espace obtenu en sélectionnant un point de base pour chaque X_i puis en identifiant ensemble tous ces points de base. La shellabilité d'un complexe simplicial n'est pas déterminée par la topologie de sa réalisation géométrique puisqu'il existe des complexes simpliciaux shellables qui sont homéomorphes à des complexes simpliciaux non shellables (voir par exemple [66]). Cependant, la shellabilité a quand même des belles conséquences topologiques comme l'illustre le résultat suivant.

Proposition 1.13 ([15]) *Un complexe simplicial shellable est homotopie équivalent à un bouquet de sphères, où pour tout i, le nombre de sphères de dimension i est le nombre de facettes homologiques de dimension i.*

La proposition précédente montre en particulier que le nombre de facettes homologiques d'une certaine dimension est le même quelque soit le shelling considéré.

Exemple 1.14 *La Proposition 1.13 montre que le complexe simplicial K_2 de la figure 1.3 est homotopie équivalent, comme attendu, au bouquet de quatre sphères de dimension* 1. *En effet, dans le shelling $F_1, F_2, \ldots, F_9$ les facettes homologiques sont F_6, F_7, F_8, F_9.*

La fonction de Möbius d'un ordre P, notée μ_P, est une application associant à tout intervalle $[x, y]$ de P un entier relatif $\mu_P(x, y)$. Elle est définie de façon unique par

$$\begin{cases} \forall x \in P,\ \mu_P(x,x) = 1, \\ \mu_P(x,y) = -\displaystyle\sum_{x \leq t < y} \mu_P(x,t) \text{ pour tout } x < y \text{ dans } P. \end{cases}$$

Nous avons décidé d'introduire la fonction de Möbius dans cette partie sur la topologie des ordres notamment pour le résultat suivant, attribué à Hall même s'il s'agit d'une reformulation (un autre résultat similaire dans le cas des treillis modulaires à gauche sera donné dans la partie 1.9). Pour un ordre P, on note $\hat{P}$ l'ordre P auquel on a ajouté un minimum $\hat{0}$ et un maximum $\hat{1}$.

Proposition 1.15 ([54]) *Soit P un ordre. Notons c_i le nombre de chaînes de longueur i contenant $\hat{0}$ et $\hat{1}$. Alors $\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P)) = \sum_{i \geq 0} (-1)^i c_i$.*

1.6 Généralités sur les treillis

Dans cette partie, nous donnons les généralités sur les treillis. Pour la théorie générale des treillis, nous recommandons les livres de Grätzer [53], Grätzer et Wehrung [52], Stanley [98], et de Freese, Ježek et Nation [44]. Ces références traitent des différentes classes de treillis que nous allons introduire, sauf les treillis extrémaux et modulaires à gauche. Pour les treillis extrémaux nous recommandons l'article de Markowsky [70] dans lequel il introduit la notion, ainsi que l'article de Thomas et Williams [103] qui contient le langage moderne que nous utiliserons pour ceux-ci. Pour les treillis modulaires à gauche nous recommandons l'article de Blass et Sagan qui introduit cette classe de treillis [17], la thèse de doctorat de Liu [67] (qui n'est pas facilement accessible) et un article de Liu et Sagan [68]. On trouve aussi des informations intéressantes dans l'article de Thomas [100] qui introduit les treillis trim, qui sont par définition les treillis à la fois modulaires à gauche et extrémaux.

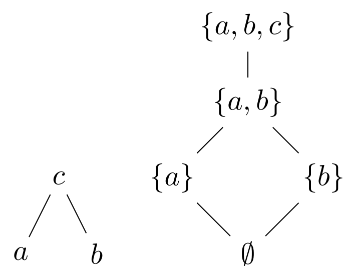
Un semi-treillis supérieur est un ordre P tel que pour tout $x, y \in P$, le sous-ensemble $\{x, y\}$ de P possède une borne supérieure. Quand elle existe, la borne supérieure est appelée le sup et notée $x \vee y$. Un semi-treillis inférieur est un ordre P tel que pour tout $x, y \in P$, le sous-ensemble $\{x, y\}$ de P possède une borne inférieure. Quand elle existe, la borne inférieure est appelée le inf et notée $x \wedge y$. Un treillis est un ordre L qui est à la fois un semi-treillis supérieur et un semi-treillis inférieur. Dans la suite, L désigne toujours un treillis. Si les bornes supérieures et inférieures existent pour tout sous-ensemble de L, on dit que L est un treillis complet. Un treillis fini est complet, mais il existe des treillis infinis qui ne sont pas complets. Comme on s'intéresse seulement aux treillis finis, nos treillis auront toujours un minimum $\hat{0}$ et maximum $\hat{1}$. Un isomorphisme de treillis de L vers K est une bijection f qui satisfait $f(x \vee y) = f(x) \vee f(y)$ et $f(x \wedge y) = f(x) \wedge f(y)$ pour tout $x, y \in L$. C'est équivalent de demander que f soit un isomorphisme d'ordres de L vers K. Un sous-treillis K de L est un sous-ordre K de L qui vérifie de plus $x \vee y \in K$ et $x \wedge y \in K$ pour tout $x, y \in K$ (le sup et l'inf sont ceux de L).

Exemple 1.16 *Parmi les ordres de la figure 1.2, seuls N_5 et B_3 sont des treillis. Même si on ajoute un minimum et un maximum à $\mathscr{C}_2$, ce ne sera quand même pas un treillis car $\{x_1, x_2\}$ n'a pas de borne supérieure.*

Un élément $x \in L$ est sup-irréductible s'il ne peut pas s'écrire $x = a \vee b$ avec $a < x$ et $b < x$. Un élément $x \in L$ est inf-irréductible s'il ne peut pas s'écrire $x = a \wedge b$ avec $a > x$ et $b > x$. Comme nos treillis sont finis, x est sup-irréductible est équivalent au fait que x couvre un unique élément, tandis que x est inf-irréductible s'il est couvert par un unique élément. L'ensemble de ces éléments sont respectivement notés $\mathrm{JIrr}(L)$ et $\mathrm{MIrr}(L)$ (en raison des termes anglais correspondant *join-irreducible* et *meet-irreducible*). On utilise la même notation pour désigner les sous-ordres de L sur ces deux ensembles, mais nous préciserons si nous considérons ces ensembles comme sous-ordres. On montre facilement, par récurrence sur la hauteur de $x \in L$, le résultat suivant :

Lemme 1.17 *Tout élément $x \in L$ est le sup des sup-irréductibles inférieurs ou égaux à x. Dualement, c'est aussi l'inf des inf-irréductibles supérieurs ou égaux à x.*

Pour des détails sur les congruences de treillis que nous discutons maintenant, nous recommandons [87] (il s'agit d'un chapitre écrit par Reading dans le livre [52]). Une congruence (de treillis) de L est une relation d'équivalence $\equiv$ sur L telle que pour tout $x_1, x_2, y_1, y_2 \in L$ vérifiant $x_1 \equiv x_2$ et $y_1 \equiv y_2$, on a $x_1 \wedge y_1 \equiv x_2 \wedge y_2$ et $x_1 \vee y_1 \equiv x_2 \vee y_2$. Les classes d'équivalence d'une telle congruence sont toujours des intervalles. De plus, une congruence sur L permet de définir un treillis quotient $\mathrm{Con}(L)$, qui est le treillis



P $J(P)$

FIGURE 1.4

dont les éléments sont les classes d'équivalence et qui est ordonné en considérant le sous-ordre induit de L sur les minimum des classes d'équivalence.

1.7 Treillis distributifs

Un treillis L est distributif si pour tout $x, y, z \in L$, on a $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Ou de façon équivalente, si pour tout $x, y, z \in L$, on a $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Il s'agit des treillis n'ayant aucun sous-treillis isomorphe à l'ordre pentagone N_5 et l'ordre diamant M_3. Ce sont les treillis les plus étudiés et les mieux compris en combinatoire algébrique. Le théorème suivant, dû à Birkhoff en 1937, est communément appelé le théorème fondamental des treillis distributifs finis (voir [98, Chapter 3, Section 3.4] pour plus de détails). Voir figure 1.4 pour un exemple.

Théorème 1.18 ([12]) *Soit L un treillis distributif. Alors il existe un unique ordre P tel que $L \cong J(P)$. Cet ordre P est en fait le sous-ordre* $\mathrm{JIrr}(L)$.

En particulier, un treillis distributif est gradué.

Exemple 1.19 *Des deux treillis N_5 et B_3 de la figure 1.2, seul B_3 est distributif. Plus généralement, l'ordre booléen B_n est un treillis distributif pour tout $n \geq 1$. L'ordre sur les diviseurs positifs d'un entier naturel n fixé, souvent noté D_n, est un treillis dont le sup est le ppcm et l'inf le pgcd. Il s'agit d'un treillis distributif.*

Dilworth déduit du théorème 1.9 le résultat suivant (il s'agit d'une reformulation, utilisant par exemple le théorème 1.18).

Théorème 1.20 ([36]) *Si L est un treillis distributif, alors* $\dim(L) = \mathrm{width}(\mathrm{JIrr}(L))$.

Ce résultat a été généralisé par Reading aux ordres dissectifs [89], et nous allons le généraliser dans une autre direction (voir chapitre 3 pour plus de détails).

1.8 Treillis extrémaux

Les treillis extrémaux ont été définis par Markowsky en 1992 [70]. Avant d'en donner la définition, on démontre un premier résultat vrai dans tout treillis.

Proposition 1.21 *On a* $\ell(L) \leq \min(\, |\mathrm{JIrr}(L)|\, , \, |\mathrm{MIrr}(L)|\,)$.

Preuve. Notons $n := \ell(L)$. On montre que $n \leq |\mathrm{JIrr}(L)|$, l'argument étant dual pour $n \leq |\mathrm{MIrr}(L)|$. Soit $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ une chaîne maximale de L de longueur n. Soit $i \in [n-1]$. D'après le lemme 1.17, comme $x_i \neq x_{i+1}$, il existe un sup-irréductible j_i qui est inférieur ou égal à x_{i+1} mais pas inférieur ou égal à x_i. Tous les $j_1, j_2, \ldots, j_n$ sont différents, donc $n \leq |\mathrm{JIrr}(L)|$. $\square$

Un treillis L est sup-extrémal s'il vérifie $\ell(L) = |\mathrm{JIrr}(L)|$, et inf-extrémal si $\ell(L) = |\mathrm{MIrr}(L)|$. Un treillis est extrémal s'il est sup-extrémal et inf-extrémal.

Exemple 1.22 *L'ordre hexagone H_6, qui est un treillis, n'est pas extrémal. En effet, il est de longueur* 3 *et a quatre sup-irréductibles. Par contre, les treillis N_5 et B_3 de la figure 1.2 sont extrémaux.*

Nous donnons maintenant deux résultats importants au sujet des treillis extrémaux. Le premier a été démontré avant que les treillis extrémaux aient été formellement définis [4, Theorem 4.6] (voir aussi [71, Theorem 3.1]).

Proposition 1.23 *Un treillis est distributif si et seulement s'il est extrémal et gradué.*

Proposition 1.24 ([70, Theorem 14]) *Soit L un treillis. Il existe un treillis K extrémal tel que L soit isomorphe à un intervalle de K.*

Nous terminons cette partie en donnant un théorème de caractérisation des treillis extrémaux dû à Markowsky (Théorème 1.26), que nous utiliserons à plusieurs reprises dans le chapitre 3. Nous utilisons la terminologie moderne donnée dans un article de Thomas et Williams [103, Section 2.3].

Pour un treillis extrémal L, le choix de n'importe quelle chaîne de longueur maximale $\phi :\ \hat{0} = x_0 \lessdot \cdots \lessdot x_n = \hat{1}$ fournit une numérotation des sup-irréductibles $j_1, j_2, \ldots, j_n$ et des inf-irréductibles $m_1, m_2, \ldots, m_n$ telle que $x_i = j_1 \vee \cdots \vee j_i = m_{i+1} \wedge \cdots \wedge m_n$ pour tout $i \in [n]$. On suppose qu'une telle numérotation est fixée pour le treillis extrémal L. On définit le graphe de Galois $G(L)$ comme étant le graphe orienté dont les sommets sont $\mathrm{JIrr}(L)$ et on a une flèche de j_i vers j_k si $i \neq k$ et $j_i \not\leq m_k$. Il est aussi souvent défini comme le graphe orienté sur $[n]$ avec une flèche de i vers k si $i \neq k$ et $j_i \not\leq m_k$. Nous utiliserons ces deux représentations équivalentes, en précisant à chaque fois laquelle nous considérerons.

Proposition 1.25 ([70, Theorem 11]) *Soit L un treillis extrémal. Alors $G(L)$ est un graphe orienté acyclique.*

Réciproquement, soit G un graphe orienté sur $[n]$ tel que si on a une flèche de i vers k, alors $i > k$. Une paire orthogonale maximale de G est une paire (X, Y) de deux sous-ensembles disjoints de $[n]$ telle qu'il n'y ait pas de flèches d'un sommet de X vers un sommet de Y, et qui est maximale pour ces conditions. Avec l'ordre $(X, Y) \leq (X', Y')$ si $X \subseteq X'$, les paires orthogonales maximales forment un treillis noté $L(G)$.

Théorème 1.26 ([70]) *Tout treillis extrémal L est isomorphe à $L(G(L))$. Réciproquement, si G est un graphe orienté sur $[n]$ tel qu'avoir une flèche de i vers k implique $i > k$, alors $L(G)$ est un treillis extrémal.*

D'après théorème 1.26, on peut définir un treillis extrémal en donnant son graphe de Galois (voir figure 1.5).

1.9 Treillis modulaires à gauche

Un treillis L est modulaire s'il vérifie pour tout $a, b, c \in L$, $b \leq c$ implique $(b \vee a) \wedge c = b \vee (a \wedge c)$. Pour a fixé, si cette dernière implication est vraie pour tout $b \leq c$, alors a est modulaire à gauche. Il est

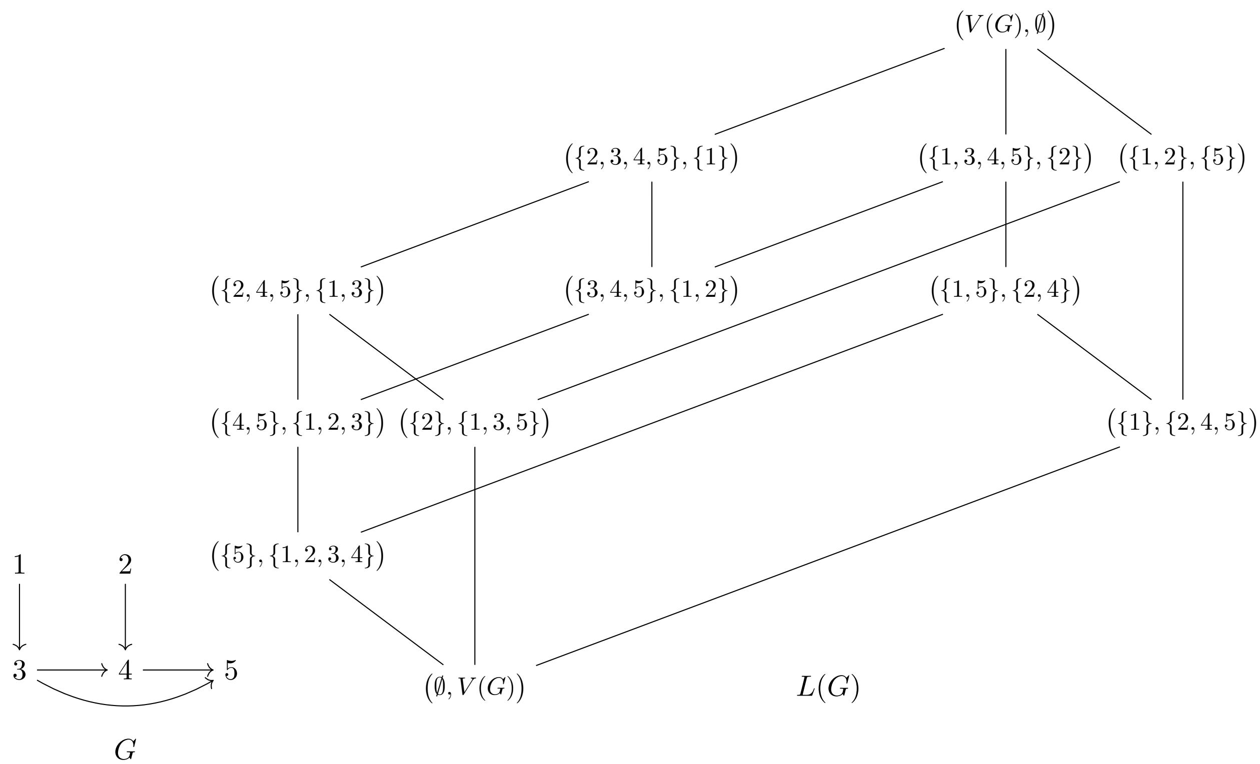


FIGURE 1.5 : Un graphe orienté G et son treillis extrémal associé $L(G)$ des paires orthogonales maximales.

habituel de dire que le couple (a, c) est modulaire lorsque l'égalité précédente est satisfaite pour tout b tel que $b \leq c$. Ceci explique le terme *à gauche* dans modulaire à gauche, car on demande que chaque paire avec a à gauche soit modulaire. Le fait que a soit modulaire à gauche est équivalent au fait qu'il n'existe pas de sous-treillis de L isomorphe à N_5 où a est le petit côté du pentagone (comme sur la figure 1.2). Un treillis modulaire se définit de façon équivalente comme un treillis dont tous les éléments sont modulaires à gauche, ou encore qui ne possède aucun sous-treillis isomorphe à N_5. Les treillis distributifs sont modulaires, donc les treillis modulaires généralisent les treillis distributifs. Les treillis modulaires sont sans doute, après les treillis distributifs, les treillis ayant reçu le plus d'attention.

Il existe de nombreux résultats sur les treillis modulaires, mais dans cette thèse nous nous intéresserons à une généralisation des treillis modulaires que sont les treillis modulaires à gauche. Ils ont été définis par Blass et Sagan en 1997 [17]. Un treillis L est modulaire à gauche s'il possède une chaîne maximale dont tous les éléments sont modulaires à gauche. Une telle chaîne maximale est appelée une chaîne modulaire à gauche. Les treillis modulaires sont modulaires à gauche.

Exemple 1.27 *L'ordre pentagone N_5 n'est pas modulaire puisque $a \in N_5$ n'est pas modulaire à gauche (voir figure 1.2). Mais N_5 est modulaire à gauche puisque $\hat{0} \lessdot b \lessdot c \lessdot \hat{1}$ est une chaîne modulaire à gauche. L'ordre diamant M_3, qui n'est pas distributif, est modulaire et donc modulaire à gauche. Mentionnons également que le treillis des partitions d'ensembles de $[n]$ où les partitions sont ordonnées par l'inclusion de leurs parts est modulaire non distributif lorsque $n \geq 2$, comme c'est le cas du treillis ordonné par inclusion de tous les sous-espaces vectoriels d'un espace vectoriel de dimension $n \geq 2$ sur le corps fini $\mathbb{F}_q$.*

D'autres définitions équivalentes des éléments modulaires à gauche, en plus de celle en terme d'évitement du sous-treillis N_5, sont résumées dans la proposition suivante.

Lemme 1.28 ([68]) *Soit $a \in L$ un élément d'un treillis. Les propriétés suivantes sont équivalentes :*

(1) *a est modulaire à gauche.*

(2) *Pour tout $b < c$, on a $a \vee b \neq a \vee c$ ou $a \wedge b \neq a \wedge c$.*

(3) *Pour tout $b \lessdot c$, on a $a \vee b \neq a \vee c$ ou $a \wedge b \neq a \wedge c$.*

La raison de notre intérêt pour les treillis modulaires à gauche réside notamment dans leurs propriétés topologiques. Commençons par introduire une nouvelle notion. Si on a un étiquetage des relations de couverture d'un treillis L, à chaque chaîne maximale de L est associé un mot obtenu en concaténant les étiquettes des relations de couverture que nous rencontrons en parcourant la chaîne de son minimum à son maximum. Dans ce contexte, une chaîne maximale est appelée chaîne strictement croissante si le mot associé est strictement croissant (similairement sont définies les chaînes strictement décroissantes). Un EL-étiquetage d'un treillis L est un étiquetage des relations de couverture tel que pour chaque intervalle de L, il existe une unique chaîne strictement croissante dans cet intervalle et le mot associé à cette chaîne est inférieur lexicographiquement aux mots associés aux autres chaînes maximales de cet intervalle. Notons $\overline{L} := L \setminus \{\hat{0}, \hat{1}\}$. Si L admet un EL-étiquetage, alors ordonner les chaînes maximales de $\overline{L}$ lexicographiquement donne un shelling de $\overline{L}$ (c'est un résultat de Björner [13]). Donc le complexe simplicial $\Delta(\overline{L})$ est shellable et homotopie équivalent à un bouquet de sphères par proposition 1.13. De plus, avoir un EL-étiquetage implique que pour tout $x < y$ dans L, $\mu_L(x, y)$ est égal à la différence entre le nombre de chaînes strictement décroissantes de longueurs paires de $[x, y]$ et le nombre de chaînes strictement décroissantes de longueurs impaires de $[x, y]$. Ainsi si

L a au plus une chaîne strictement décroissante dans chaque intervalle, alors $\mu(x, y) \in \{-1, 0, 1\}$ pour tout $x, y \in L$, et le complexe d'ordre de chaque intervalle ouvert non vide $]x, y[$ est homotopie équivalent à une sphère ou est contractile.

Le résultat suivant est un résultat de Liu [67].

Proposition 1.29 *Soit L un treillis modulaire à gauche ayant une chaîne modulaire à gauche $\phi : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = \hat{1}$. Soit $\gamma_\phi : E(L) \to \mathbb{N}^*$ défini par $\gamma_\phi(b \lessdot c) = \min\{i \mid b \vee x_i \geq c\}$. Alors γ_ϕ est un EL-étiquetage. Donc L est shellable.*

Au chapitre 3, nous donnerons une nouvelle preuve de ce résultat et introduirons d'autres étiquetages que nous montrerons être équivalents à γ_ϕ.

Thomas a étudié en 2005 les treillis qui sont à la fois extrémaux et modulaires à gauche [100]. Il a nommé ces treillis trim et ils possèdent de belles propriétés que nous verrons au chapitre 3. Nous mentionnons seulement que ces treillis sont fermés par intervalles et quotients, et ils inclus notamment les treillis de Tamari et certaines de ses généralisations.

1.10 Treillis semi-distributifs

Les treillis (sup) semi-distributifs ont été définis par Jónsson en 1961 [61], mais la terminologie est apparue plus tard. Ils forment une généralisation naturelle des treillis distributifs différente des treillis modulaires à gauche, et occupent aujourd'hui une place centrale dans l'étude des treillis en combinatoire algébrique. Par exemple, l'ordre sur les classes de torsion $\mathrm{Tors}(A)$ est un treillis complet semi-distributif qui a reçu beaucoup d'attention [35, 101] (voir partie 1.12). On peut aussi citer l'ordre des régions $\mathrm{Pos}(\mathcal{A}, B)$, Reading ayant montré qu'il est semi-distributif lorsque l'arrangement $\mathcal{A}$ est étroit par rapport à la région B [87].

Un treillis L est sup semi-distributif si pour tout $x, y, z \in L$, $x \vee y = x \vee z$ implique $x \vee (y \wedge z) = x \vee y$. Dualement, un treillis est inf semi-distributif si pour tout $x, y, z \in L$, $x \wedge y = x \wedge z$ implique $x \wedge (y \vee z) = x \wedge y$. Un treillis est semi-distributif s'il est sup semi-distributif et inf semi-distributif. La proposition suivante compile différents résultats intéressants sur ces treillis.

Proposition 1.30 ([44, Section 5]) *Un treillis L est semi-distributif si et seulement si L est sup semi-distributif et $|\mathrm{JIrr}(L)| = |\mathrm{MIrr}(L)|$. Un treillis L est sup semi-distributif si et seulement si pour tout $b \lessdot c$, l'ensemble $I_L(c) \setminus I_L(b)$ a un minimum, noté $\gamma_J(b \lessdot c)$. Il est inf semi-distributif si et seulement si pour tout $b \lessdot c$, l'ensemble $F_L(b) \setminus F_L(c)$ a un maximum, noté $\gamma_M(b \lessdot c)$. Dans ces cas-là, $\gamma_J(b \lessdot c) \in \mathrm{JIrr}(L)$ et $\gamma_M(b \lessdot c) \in \mathrm{MIrr}(L)$.*

D'après la proposition 1.30, si L est semi-distributif, on a donc deux étiquetages des relations de couverture γ_J et γ_M. De plus, l'application $\kappa : j \in \mathrm{JIrr}(L) \mapsto \gamma_M(j_* \lessdot j) \in \mathrm{MIrr}(L)$ est une bijection [44, Theorem 2.54]. Un des sens du théorème suivant (extrémal implique modulaire à gauche) a été démontré par Thomas et Williams [103, Theorem 1.4], tandis que l'autre sens est un résultat de Mühle [79, Theorem 3.2].

Théorème 1.31 *Soit L un treillis semi-distributif. Alors L est extrémal si et seulement s'il est modulaire à gauche.*

Soit L un treillis sup semi-distributif. Pour tout $x \in L$, notons $D(x) := \{\gamma_J(y \lessdot x) \mid y \in \ L, y \lessdot x\}$. Cet ensemble est l'ensemble des étiquettes inférieures de x. Dualement, on définit l'ensemble des étiquettes supérieures de x par $U(x) := \{\gamma_J(x \lessdot y) \mid y \in \ L, x \lessdot y\}$. Le complexe sup-canonique d'un treillis sup semi-distributif L, noté $\mathrm{CJC}(L)$ (pour l'anglais *canonical join complex*), est l'ensemble formé de tous les ensembles des étiquettes inférieures de L, c'est-à-dire $\mathrm{CJC}(L) := \{D(x) \mid x \in L\}$. Il a été défini par Reading en 2015 [88] puis étudié en détail par Barnard [6]. Souvent, il est défini à l'aide de ce qu'on appelle les représentations sup-canoniques des éléments de L, mais la définition donnée ci-dessus est équivalente. Son nom est justifié par le fait qu'il s'agit d'un complexe simplicial ([88, Proposition 2.2]). Il s'agit même d'un complexe simplicial drapeau si et seulement si L est semi-distributif ([6, Theorem 1]). Dans la suite de cette thèse, on s'intéressera toujours à $\mathrm{CJC}(L)$ lorsque L est semi-distributif, pas seulement sup semi-distributif. Dans ce cas $\mathrm{CJC}(L)$ est un complexe simplicial drapeau, donc est le clique complexe de son 1-squelette. Ce dernier, qui est un graphe noté $CJG(L)$, est appelé le graphe sup-canonique de L.

Supposons que L est semi-distributif et extrémal. Comme rappelé dans la partie 1.8, le choix d'une chaîne de longueur maximale de L fournit une numérotation particulière des sup-irréductibles $j_1, \ldots, j_n$ et des inf-irréductibles $m_1, \ldots, m_n$. On suppose qu'un tel choix de chaîne maximale a été fait, même si on ne le rappelle pas. En utilisant le fait que $\kappa\big(\gamma_J(b \lessdot c)\big) = \gamma_M(b \lessdot c)$ pour tout $b \lessdot c$ (voir [103, Proposition 6.2] ou [6]), on obtient le résultat suivant.

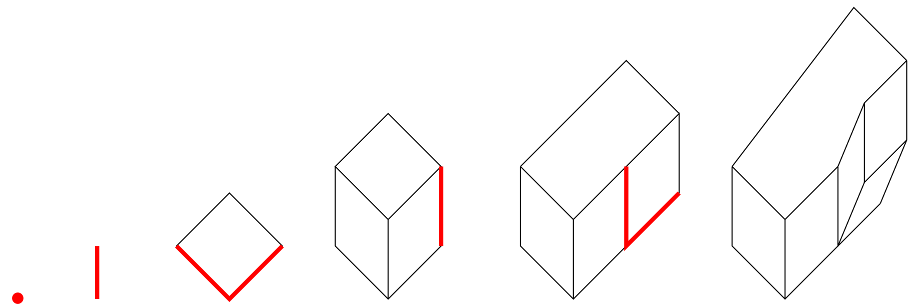


FIGURE 1.6 : Nous représentons cinq doublement consécutifs d'un treillis par des pseudo-intervalles inférieurs, lesquels sont représentés par des arêtes rouges.

Proposition 1.32 *Soit L un treillis semi-distributif et extrémal. On a $\kappa(j_i) = m_i$ pour tout $i \in [n]$. Donc le graphe de Galois $G(L)$ est le graphe orienté dont les sommets sont $\mathrm{JIrr}(L)$, et on a une flèche de i vers j si $i \neq j$ et $i \not\leq \kappa(j)$.*

Le résultat suivant nous permet, pour les treillis semi-distributifs extrémaux, de passer aisément de son graphe de Galois à son graphe sup-canonique.

Proposition 1.33 ([103, Corollary 6.7]) *Le graphe sup-canonique d'un treillis semi-distributif extrémal L est le complément de son graphe de Galois. C'est-à-dire $\overline{G(L)} = \mathrm{CJG}(L)$.*

Nous finissons cette partie en mentionnant qu'il existe un théorème fondamental des treillis semi-distributifs finis qui a été obtenu relativement récemment par Reading, Speyer et Thomas [91]. Ce résultat généralise le théorème 1.18 de Birkhoff sur les treillis distributifs. Nous rappellerons ce théorème dans le chapitre 3.

1.11 Treillis congruence normaux

Soit C un sous-ensemble convexe d'un treillis L. Soit $\{0, 1\}$ la chaîne à deux éléments où $0 < 1$. Nous définissons le doublement $L[C]$ comme le sous-ordre de $L \times \{0, 1\}$ sur le sous-ensemble

$$\Big(I_L(C) \times \{0\}\Big) \bigsqcup \Big[\Big(\big(L \setminus I_L(C)\big) \cup C\Big) \times \{1\}\Big].$$

En fait, $L[C]$ est un treillis dont le sup et l'inf sont donnés, pour $(x, \epsilon), (y, \epsilon') \in L[C]$, par

$$(x, \epsilon) \vee (y, \epsilon') = \begin{cases} (x \vee y, \max(\epsilon, \epsilon')) & \text{si } x \vee y \in I_L(C), \\ (x \vee y, 1) & \text{sinon,} \end{cases}$$

$$(x, \epsilon) \wedge (y, \epsilon') = \begin{cases} (x \wedge y, \min(\epsilon, \epsilon')) & \text{si } x \wedge y \in ((L \setminus I_L(C)) \cup C), \\ (x \wedge y, 0) & \text{sinon.} \end{cases}$$

Nous avons un morphisme surjectif d'ordres $L[C] \to L$ qui envoie (x, ϵ) sur x, ainsi qu'un morphisme injectif d'ordres $\psi : L \to L[C]$ qui envoie $a \in L$ sur $(a, 0)$ si $a \in I_L(C)$, ou sur $(a, 1)$ sinon. Voir la figure 1.6 pour des exemples de doublements.

On obtient aisément de la définition du doublement les deux résultats suivants.

Lemme 1.34 ([23, Lemma 1]) *Les sup-irréductibles de $L[C]$ sont de deux types. Chaque $j \in \mathrm{JIrr}(L)$ donne un élément $(j, \epsilon) \in \mathrm{JIrr}(L[C])$, avec $\epsilon = 0$ si $j \in I_L(C)$ et $\epsilon = 1$ sinon. On a également $(p, 1) \in \mathrm{JIrr}(L[C])$ pour tout élément minimal p de C.*

De même, les éléments inf-irréductibles de $L[C]$ sont de deux types. Chaque $m \in \mathrm{MIrr}(L)$ donne un élément $(m, \epsilon) \in \mathrm{MIrr}(L[C])$, avec $\epsilon = 0$ si $m \in I_L(C) \setminus C$ et $\epsilon = 1$ sinon. On a également $(q, 0) \in \mathrm{MIrr}(L[C])$ pour tout élément maximal q de C.

Lemme 1.35 *On a $\ell(L[C]) = \ell(L)$ si C ne contient pas d'éléments de la colonne vertébrale de L, et $\ell(L[C]) = \ell(L) + 1$ sinon.*

Un treillis L est congruence normale s'il est obtenu à partir du treillis à un élément par une suite de doublements de sous-ensembles convexes. Nous écrivons $L = E[C_1, C_2, \ldots, C_n]$ pour le treillis obtenu à partir du treillis à un élément par le doublement successif de $C_1, C_2, \ldots, C_n$, ce qui signifie que $E[C_1, \ldots, C_{i+1}] = E[C_1, \ldots, C_i][C_{i+1}]$ pour tout i, où C_{i+1} est un sous-ensemble convexe de $E[C_1, \ldots, C_i]$ et où E désigne le treillis à un élément. Si chaque C_i est un pseudo-intervalle inférieur, alors L est sup-congruence uniforme. Si tous les C_i sont des pseudo-intervalles supérieurs, alors L est inf-congruence uniforme. Si tous les C_i

sont des intervalles, alors L est congruence uniforme. Le résultat suivant de Day suit aussi de la définition du doublement.

Lemme 1.36 ([29]) *Un treillis sup-congruence uniforme est sup semi-distributif. Un treillis inf-congruence uniforme est inf semi-distributif. Donc un treillis congruence uniforme est semi-distributif.*

Le lemme 1.36 montre par exemple que les treillis congruence uniformes sont des cas particuliers de treillis semi-distributif. Le plus petit treillis semi-distributif non congruence uniforme a 14 éléments, il est représenté sur la figure 3.6a du chapitre 3. Nous verrons à plusieurs reprises ces treillis aux chapitres 3 et 4. Mentionnons que le treillis de Tamari et la plupart de ses généralisations sont congruence uniformes, ainsi que l'ordre faible. Aussi, tout treillis distributif est congruence uniforme. En effet, les treillis distributifs sont les treillis congruence uniformes $L = E[C_1, C_2, \ldots, C_n]$ tel que C_{i+1} contient le maximum de $E[C_1, \ldots, C_i]$ pour tout i ([10, Corollary 5.6]).

Le doublement d'intervalles a été défini par Day en 1970 [27]. Ainsi, pour citer les travaux les plus anciens, les treillis congruence uniformes et leurs généralisations que sont les treillis sup-congruence uniformes et inf-congruence uniformes ont été introduits dans les années 1970 [27, 28] (mais la terminologie moderne est apparue plus tard), puis leur généralisation que sont les treillis congruence normaux semblent avoir été formellement étudiée à la fin des années 1980 et début des années 1990 [30, 29, 48].

1.12 Théorie des représentations des algèbres

Dans cette partie, on rappelle du vocabulaire de la théorie des représentations des algèbres. Notre référence est le livre [2] d'Assem, Simson et Skowroński.

Soit $\mathbb{K}$ un corps algébriquement clos. Soit A une $\mathbb{K}$-algèbre de dimension finie. On note $\mathrm{mod} A$ la catégorie des A-modules à droite qui sont de dimension finie. Un module est indécomposable s'il n'est pas isomorphe à une somme directe de deux modules non nuls. L'algèbre A est dite de type de représentation fini si elle possède, à isomorphisme près, un nombre fini de modules indécomposables.

On suppose que les sous-catégories sont pleines et fermées par somme directe et facteur direct. Par conséquent, une sous-catégorie est entièrement déterminée par ses modules indécomposables. Cela nous arrivera d'identifier une sous-catégorie avec l'ensemble de ses modules indécomposables. Les classes de torsion sont les sous-catégories de $\mathrm{mod}A$ fermées par extensions et quotients. L'ordre obtenu (éventuellement infini) sur toutes les classes de torsion de $\mathrm{mod}A$, considérées à isomorphisme près, et ordonnées par inclusion est noté $\mathrm{Tors}(A)$ (voir figure 1.7 pour un exemple). L'ordre $\mathrm{Tors}(A)$ est un treillis complet semi-distributif qui a reçu beaucoup d'attention [35, 101].

Une brique B de $\mathrm{mod}A$ est un A-module tel que $\mathrm{End}(B) \cong \mathbb{K}$. Deux briques B et B' sont hom-orthogonales si $\mathrm{Hom}(B, B') = 0$ et $\mathrm{Hom}(B', B) = 0$. Une semi-brique est un ensemble $\{B_i\}_{i \in I}$ de briques deux à deux hom-orthogonales.

Un cycle dans la catégorie $\mathrm{mod}\, A$ est une suite de modules indécomposables $M_0, M_1, \ldots, M_r = M_0$ et de morphismes qui ne sont pas des isomorphismes $f_i : M_i \to M_{i+1}$ pour $0 \leq i \leq r-1$ où $r > 0$. Si A est de type de représentation fini et $\mathrm{mod}A$ n'a pas de cycles, alors ses modules indécomposables sont les briques [37].

Un carquois $Q = (Q_0, Q_1, s, t)$ est un graphe orienté dont les sommets forment l'ensemble Q_0, les flèches l'ensemble Q_1, et de plus s et t sont deux applications de Q_1 dans Q_0 tels que l'image d'une flèche de x à y est respectivement x, appelée source de la flèche, et y, appelée but de la flèche. En plus d'être fini, on suppose que Q est également connexe. Un chemin de longueur $m \geq 1$ est une suite finie de flèches $\gamma_1\gamma_2 \ldots \gamma_m$ avec $t(\gamma_i) = s(\gamma_{i+1})$ pour tout $i \in [m-1]$. On note $\mathbb{K}Q$ l'algèbre de chemins de Q, et on note R_Q son idéal engendré par les flèches. Une relation nulle de Q est donnée par un chemin $\gamma_1\gamma_2 \ldots \gamma_m$ avec $m \geq 2$. Un idéal bilatère I de $\mathbb{K}Q$ est un idéal admissible s'il existe $m \geq 2$ tel que $R_Q^m \subseteq I \subseteq R_Q^2$. Dans la suite, on suppose que I est un idéal admissible engendré par des relations nulles.

Une algèbre $A = \mathbb{K}Q/I$ est une algèbre aimable si elle satisfait les propriétés suivantes :

- L'idéal I est engendré par des relations nulles de longueur deux.
- Il y a au plus deux flèches entrantes et sortantes en chaque sommet de Q, et il y a au plus une flèche β et une flèche γ tel que $0 \neq \alpha\beta \in I$ et $0 \neq \gamma\alpha \in I$.
- Pour toute flèche α, il y a au plus une flèche β et une flèche γ tel que $\alpha\beta \notin I$ et $\gamma\alpha \notin I$.

Les arbres aimables sont les algèbres aimables dont le sous-graphe non orienté sous-jacent de leur carquois

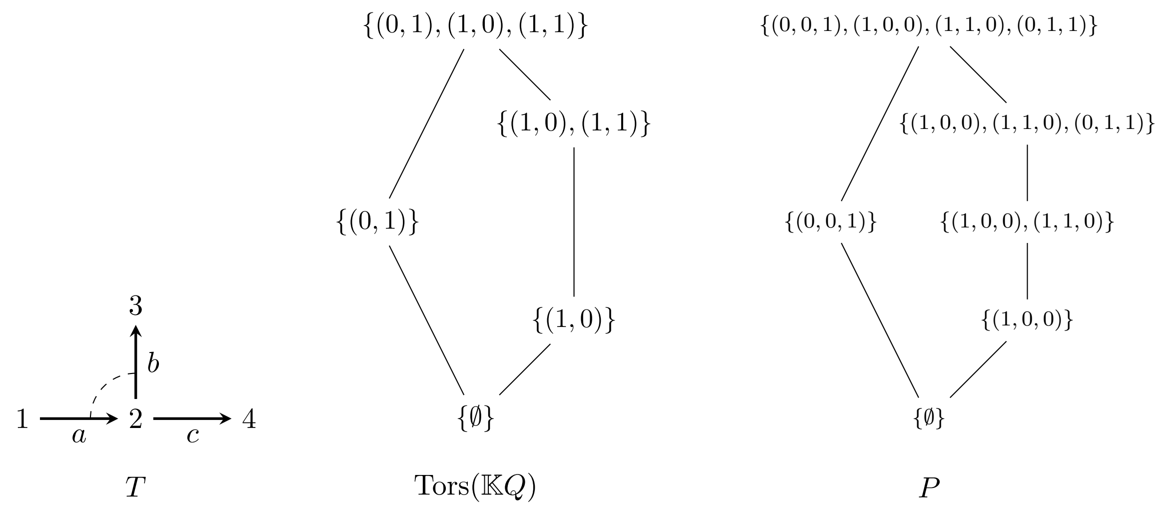


FIGURE 1.7 : Un arbre aimable T avec la relation $ab = 0$, l'ordre $\mathrm{Tors}(\mathbb{K}Q)$ pour $Q : 1 \to 2$, et l'ordre P sur les 2-classes de torsion de la catégorie $\mathcal{M}$ de l'exemple 1.37. Dans ces ordres, nous avons remplacé les classes de torsion et 2-classes de torsion par leur ensemble d'indécomposables écrits comme vecteurs dimensions.

est un arbre (voir figure 1.7). Il s'agit des seuls exemples d'algèbres aimables que nous verrons, au chapitre 3. Les arbres aimables sont de type de représentation fini et nous verrons que leurs indécomposables, appelés modules cordes, sont bien compris combinatoirement.

Une représentation M de Q est définie par les données suivantes : à chaque sommet $x \in Q_0$, on associe un $\mathbb{K}$-espace vectoriel M_x, et à chaque flèche $\alpha : x \to y$ de Q_1, on associe une application $\mathbb{K}$-linéaire $\varphi_\alpha : M_x \to M_y$. Si Q possède des relations nulles, on demande en outre que la composition des applications correspondant à une relation nulle soit l'application linéaire nulle. Une telle représentation est notée $M = (M_x, \varphi_\alpha)$. Son vecteur dimension est $\dim(M) := \big(\dim_{\mathbb{K}}(M_x)\big)_{x\in Q_0}$. Si (M_x, φ_α) et (M'_x, φ'_α) sont deux représentations de Q, un morphisme f entre ces deux représentations est une famille $f = (f_x)_{x\in Q_0}$ d'applications $\mathbb{K}$-linéaires vérifiant $\varphi'_\alpha f_x = f_y \varphi_\alpha$ pour toute flèche $\alpha : x \to y$ de Q_1. Les représentations de Q, munies de leurs morphismes, forment une catégorie abélienne équivalente à la catégorie $\mathrm{mod}\,\mathbb{K}Q$ (plus généralement $\mathrm{mod}\,\mathbb{K}Q/I$ s'il y a des relations). Nous nous autoriserons donc à passer d'un langage à l'autre.

Une théorie plus récente, apparue dans des articles d'Iyama [59, 57, 58], est celle de l'algèbre homologique supérieure. Les catégories considérées dans ce contexte ne sont en général plus abéliennes. Nous terminons cette partie en donnant deux définitions liées à cette théorie.

Soit un entier naturel $d \geq 1$. La définition suivante, qui est [65, Proposition 4.4], est équivalente à la définition originelle de Iyama (quand la catégorie ambiante est $\mathrm{mod}A$). Une sous-catégorie $\mathcal{M}$ de $\mathrm{mod}A$ est appelée d-amas basculante si $\mathrm{Ext}^i_A(\mathcal{M}, \mathcal{M}) = 0$ pour tout $0 < i < d$, et si pour tout $M \in \mathrm{mod}A$ il existe des suites exactes

$$0 \to X_d \to \cdots \to X_1 \to M \to 0 \qquad \text{et} \qquad 0 \to M \to X^1 \to \cdots \to X^d \to 0$$

avec $X_1, \ldots, X_d, X^1, \ldots, X^d \in \mathcal{M}$.

Exemple 1.37 *Soit le carquois* $Q : 1 \xrightarrow{a} 2 \xrightarrow{b} 3$. *On définit l'algèbre* A *comme étant l'algèbre de chemins de* Q *quotientée par l'idéal engendré par* ab. *Alors la sous-catégorie* $\mathcal{M}$ *dont les modules indécomposables sont ceux de vecteur dimension* $(0,0,1), (0,1,1), (1,1,0), (1,0,0)$ *est* 2*-amas basculante.*

Une sous-catégorie $\mathcal{U}$ d'une sous-catégorie $\mathcal{M}$ d-amas basculante est une d-classe de torsion si pour tout $M \in \mathcal{M}$ il existe une suite exacte $0 \to \mathcal{U}_M \to M \to X^1 \to \cdots \to X^d \to 0$ avec $X^1, \ldots, X^d \in \mathcal{M}$ et $\mathcal{U}_M \in \mathcal{U}$, tel que la suite suivante est exacte pour tout $U \in \mathcal{U}$:

$$0 \to \mathrm{Hom}_A(U, X^1) \to \cdots \to \mathrm{Hom}_A(U, X^d) \to 0.$$

Les d-classes de torsion sont aussi appelées classes de torsion supérieures et ont été définies par Jørgensen [62]. Elles généralisent les classes de torsion usuelles, puisque pour $d = 1$ on peut considérer $\mathcal{M} = \mathrm{mod}A$ et les 1-classes de torsion correspondent aux classes de torsion de $\mathrm{mod}A$. Voir figure 1.7 pour l'ordre d'inclusion P sur les 2-classes de torsion de la sous-catégorie 2-amas basculante de l'exemple 1.37. L'ordre d'inclusion sur les d-classes de torsion d'une sous-catégorie amas-basculante est un treillis complet qui n'est pas semi-distributif en général [3]. Nous nous intéresserons aux d-classes de torsion de certaines algèbres dans le chapitre 4. Nous verrons plus d'exemples dans ce chapitre.

CHAPITRE 2
LA DIMENSION DES ORDRES UNICYCLIQUES

2.1 Avant-Propos et Résumé

Comme nous l'avons rappelé dans la partie 1.4, déterminer la dimension d'un ordre est un problème difficile. Bollobás et Brightwell ont obtenu des résultats pour la dimension d'ordres aléatoires [20]. Leur modèle de génération d'ordres aléatoires est le suivant (nous l'avons rappelé à la fin de la partie 1.3).

Fixons un entier $n \geq 1$. Soit $p \in]0, 1[$. On crée un graphe orienté aléatoire $\vec{G}(n, p)$ dont les sommets sont $[n]$ et tel qu'on a une arête orientée de i vers j, où $i < j$, avec probabilité p indépendamment de i et j. La probabilité p peut dépendre de n, on l'écrit donc p_n pour rappeler cette dépendance. On en déduit un *ordre aléatoire* $\mathbf{P}(n, p)$ en prenant la clôture transitive de $\vec{G}(n, p)$.

Les résultats de dimension pour $\mathbf{P}(n, p)$ sont des résultats asymptotiques lorsque n tend vers l'infini, et dépendent de l'asymptotique de p_n. Pour p_n relativement petit, précisément s'il existe $c \in [0, 1[$ tel que $\lim(p_n\, n) = c$, il est connu en théorie des graphes aléatoires (voir proposition 1.5) que presque sûrement, quand n tend vers l'infini, $G(n, p)$ est une union disjointe d'arbres et de graphes unicycliques.

Bollobás et Brightwell ont conjecturé que les ordres unicycliques (ceux dont le graphe des couvertures est connexe avec exactement un cycle), sont de dimension au plus 3 [20, Conjecture 1.1]. Ceci impliquerait que la dimension des ordres aléatoires pour p_n relativement petit comme décrit ci-dessus serait presque sûrement inférieure ou égale à 3 quand n tend vers l'infini.

Dans l'article retranscrit dans ce chapitre, nous démontrons cette conjecture. Cet article de 20 pages intitulé « Dimension of unicycle posets », écrit en collaboration avec Abram, a été soumis le 26 mai 2025 à la revue Combinatorics, Probability and Computing. Il est disponible sur arXiv [1].

En premier lieu, nous établissons le résultat facile que les ordres unicycliques peuvent être obtenus à partir de certains ordres appelés ordres couronnes, par ajout d'éléments et greffes d'arbres.

On utilise alors la définition de la dimension à l'aide des réalisateurs, obtenant des réalisateurs de taille 3 des ordres couronnes et arbres. On fournit notamment une nouvelle preuve algorithmique que les ordres arbres sont de dimension 3 [107]. Surtout, cela nous donne une forme particulière de réalisateur qui est fondamental par la suite.

On obtient alors des réalisateurs de taille 3 pour les ordres couronnes avec des éléments ajoutés (appelés ordres cycles), et pour les ordres couronnes avec des arbres greffés. En utilisant ces réalisateurs, en quelque sorte en les entrelaçant, on obtient un réalisateur de taille 3 pour tout ordre unicyclique, ce qui démontre la conjecture citée plus haut.

2.2 Abstract

Motivated by the study of the dimension of random posets, it was conjectured by Bollobás and Brightwell in 1997 that if P is a finite poset whose cover graph contains at most one cycle then its order dimension is at most 3. In this paper we prove this conjecture by giving a constructive proof with explicit triplets of linear extensions realizing such posets.

2.3 Introduction

Let $(\mathbb{R}^n, \leq)$ be the poset defined by $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if $x_i \leq y_i$ for every $i \in [n]$. The (order) *dimension* of a poset **P** is the smallest integer n such that **P** is isomorphic to a subposet of $(\mathbb{R}^n, \leq)$. Equivalently, this is the minimum number of linear extensions whose intersection is **P**, where the intersection is the poset on the set **P** whose relations are $x \leq y$ if in each of the chosen linear extensions we have $x \leq y$. A list of linear extensions whose intersection is **P** is called a *realizer* of **P**. Thus $\dim(\mathbf{P})$ is the smallest size of a realizer of **P**. This notion was first introduced in [38] and since then was extensively studied (see [105] for a survey).

The *cover graph* of a poset is the graph formed by its undirected Hasse diagram. A poset is a *tree poset*, a *cycle poset*, or a *unicycle poset*, if their cover graphs are respectively a tree, a cycle, or is connected and contains a unique cycle.

The only posets of dimension 1 are the chains. The *planar* lattices, which correspond to the bounded planar posets, are of dimension 2 [5]. But there exist planar posets of any dimension [63]. It is known that the tree posets and all planar posets with a minimum element are of dimension at most 3 [107]. In general, knowing the dimension of a given poset is difficult; it is NP-complete to determine if a given poset **P** is of dimension

n, for any $n \geq 3$ [109], even if **P** is of height 2 [41].

Shortly after the death of Paul Erdős in 1996, two volumes were published with papers by experts on the mathematics Erdős was interested in [50, 51]. Our main result is a proof of a conjecture of Bollobás and Brightwell that appears in one of these papers.

Theorem 2.1 ([20, Conjecture 1.1]) *The dimension of a unicycle poset is at most* 3.

This conjecture is motivated by the study of the dimension of *random posets*. Let $[n] := \{1, \ldots, n\}$ with its usual order denoted by $\preccurlyeq$. A random simple graph $G(n, p)$ on vertices $[n]$ is generated by creating an edge $\{i, j\}$ with probability p_n, for any $i \neq j$ independently. In this model, the probability p_n depends on n. We orient the edges of $G(n, p)$: any edge $\{i, j\}$ with $i \preccurlyeq j$ is directed from i to j. A *random poset* $(\mathbf{P}(n, p), \leq)$ is obtained from the directed version of $G(n, p)$ by taking its transitive closure. It is well-known for random graphs [39, Theorem 5e], and pointed out in [20], that when there exists $0 \leq c < 1$ such that $\lim(p_n\, n) = c$, then as n goes to infinity $\mathbf{P}(n, p)$ is almost surely a disjoint union of tree posets and unicycle posets. Thus Theorem 2.1 implies that in this case we have almost surely $\dim(\mathbf{P}(n, p)) \leq 3$.

We now list a few results that gave upper bounds on the dimension of unicycle posets. First, from a special case of a result of Trotter [104], as mentioned in [20], the dimension of a unicycle poset is at most 5. In [43], it is proved that if a poset has an *outerplanar* cover graph, which means that its cover graph is planar and all its vertices belong to its unbounded face, then its dimension is at most 4. Since a unicycle poset has an outerplanar cover graph, it follows that it is of dimension at most 4. More recently, Gao and Kumar gave in [46] a short proof, provided by Brightwell via personal communication, of the fact that the dimension of a unicycle poset is at most 4.

We start with some basics in Section 2.4, and state in Section 2.5 that a unicycle poset is a cycle poset with *grafted trees* on each of its vertices (Proposition 2.9). We give in Section 2.6 a specific realizer of the *crowns*, which are the cycle posets with only minimal and maximal elements. In Section 2.7 we give a specific realizer for the tree posets. In particular, we recover the result that the tree posets are of dimension at most 3. From these, we deduce realizers for a cycle poset with grafted trees on its vertices that are neither minimal nor maximal elements (Section 2.8), and realizers for a crown with grafted trees on any of its vertices (Section 2.9). We finally combine the graftings in order to build a realizer of any unicycle poset in

Section 2.10. These realizers have size 3, proving Theorem 2.1[1].

Acknowledgments

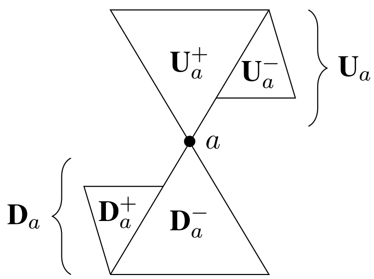
We are very thankful to Nathan Chapelier-Laget for numerous discussions on this project. We thank Samuele Giraudo and Hugh Thomas for their careful reading and proof-checking. We thank Clément Chenevière for earlier discussions. Finally, we want to thank Oliver Pechenik for presenting the conjecture at the BIRS workshop on Lattice theory at Banff in January 2025, and Jane Gao for providing the reference of the conjecture.

2.4 Basics

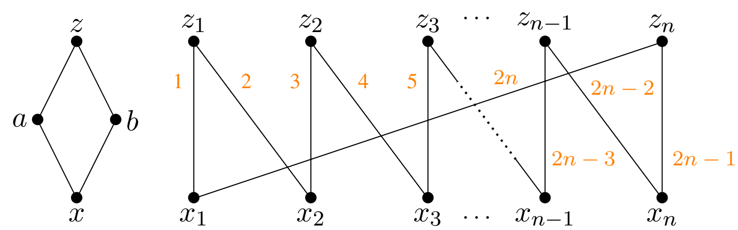
We denote $[n] := \{1, 2, \ldots, n\}$. We denote by $\mathbf{P}^{op}$ the dual poset of $\mathbf{P}$. A poset is *planar* if there exists a planar drawing without crossings of its Hasse diagram, where edges go upward from smaller to bigger elements. A *linear order* of a poset is any total order of the elements of the poset. A *linear extension* of a poset $(\mathbf{P}, \leq)$ is a linear order $\preccurlyeq$ on $\mathbf{P}$ such that for all relations $a \leq b$, we have $a \preccurlyeq b$. For $x, z \in \mathbf{P}$, a cover relation of $\mathbf{P}$ is denoted by $x \lessdot z$. A *saturated chain* is a chain $x_1 \lessdot \cdots \lessdot x_k$ made of cover relations. We mostly use lowercase letters for elements, capital letters for linear extensions and bold capital letters for posets or subsets of a poset (*i.e.* U is a linear extension of $\mathbf{U}$). To avoid using unnecessary symbols, we denote the linear orders and linear extensions as words; for example we write $a\,c\,b\,d$ for $a < c < b < d$. Thus we will use the terminology of words, for example $a\,b\,d$ is a subword of $a\,c\,b\,d$, and a is to the left of b in $a\,c\,b\,d$. We use the notations $\prod_{i=1}^{k} a_i := a_1 a_2 \cdots a_k$ and $\prod_{i=k}^{1} a_i := a_k a_{k-1} \cdots a_1$. For two elements, $a \sim b$ means $a \leq b$ or $a \geq b$, and if it is not the case we write $a \not\sim b$ and say that a and b are *incomparable*. We also use this notation for two sets $\mathbf{X}$ and $\mathbf{Y}$: by writing $\mathbf{X} \not\sim \mathbf{Y}$ we mean that every element of $\mathbf{X}$ is incomparable to every element of $\mathbf{Y}$.

Let $\mathbf{T}$ be a tree poset and $a \in \mathbf{T}$. See Figure 2.1a and Example 2.15 for the way we draw the trees and illustrations of the following definitions. For any $x \in \mathbf{T} \setminus \{a\}$ there exists a unique non-empty path in the Hasse diagram of $\mathbf{T}$ from a to x. Denote $\mathbf{U}_a$ the set of elements $x \in \mathbf{T} \setminus \{a\}$ such that the path from a to x starts with an up-step, meaning an element bigger than a, and similarly denote $\mathbf{D}_a$ the set of elements that can

[1] One can find SageMath code based on the results of this article here: `https://github.com/AntoineAbram/Dimension-of-Unicycle-Posets`



a A tree poset $\mathbf{T}$ with $a \in \mathbf{T}$.



b The crowns $\mathscr{C}_1$ and $\mathscr{C}_n$ for $n \geq 2$. The orange edge labelling is the one explained in Definition 2.6.

Figure 2.1

be reached from a by starting with a down-step. We have $\mathbf{T} \setminus \{a\} = \mathbf{U}_a \sqcup \mathbf{D}_a$. Denote by $\mathbf{U}_a^+ \subseteq \mathbf{U}_a$ the subset of the elements of $\mathbf{U}_a$ that are bigger than a, and $\mathbf{U}_a^- := \mathbf{U}_a \setminus (\mathbf{U}_a^+)$ the complement of $\mathbf{U}_a^+$ in $\mathbf{U}_a$. Similarly, denote by $\mathbf{D}_a^- \subseteq \mathbf{D}_a$ the subset of the elements of $\mathbf{D}_a$ that are smaller than a, and $\mathbf{D}_a^+ := \mathbf{D}_a \setminus (\mathbf{D}_a^-)$ the complement of $\mathbf{D}_a^-$ in $\mathbf{D}_a$. We have $\mathbf{U}_a = \mathbf{U}_a^+ \sqcup \mathbf{U}_a^-$ and $\mathbf{D}_a = \mathbf{D}_a^- \sqcup \mathbf{D}_a^+$. It will be useful to let $\mathbf{I}_a^+ := \mathbf{U}_a \sqcup \mathbf{D}_a^+$ and $\mathbf{I}_a^- := \mathbf{D}_a \sqcup \mathbf{U}_a^-$. Observe that in $\mathbf{T}$ all the elements incomparable to a are those in $\mathbf{U}_a^- \sqcup \mathbf{D}_a^+$. When no confusion is possible, we drop the subscript a from the notations that we just introduced.

For a linear extension L of $\mathbf{P}$ represented as a word, we denote by L^{rev} the reverse of L, meaning the word obtained from L by reading it from right to left. The following result is immediate:

Lemma 2.2 *Let $L_1, \ldots, L_k$ be a realizer of a poset $\boldsymbol{P}$. Then $L_1^{rev}, \ldots, L_k^{rev}$ is a realizer of $\boldsymbol{P}^{op}$.*

In most of the proofs we will make extensive use of the following easy result without reference to it:

Lemma 2.3 *Let $\boldsymbol{P} = \mathbf{X} \cup \mathbf{Y}$ and let $L_1, \ldots, L_k$ be linear extensions of $\boldsymbol{P}$ such that the restrictions of these linear extensions to $\mathbf{X}$, respectively $\mathbf{Y}$, give a realizer of $\mathbf{X}$, rescpectively $\mathbf{Y}$. If for all $(x, y) \in \mathbf{X} \times \mathbf{Y}$ such that $x \not\sim y$, there exist $i, j \in [k]$ such that x is to the left of y in L_i and to the right of y in L_j, then $L_1, \ldots, L_k$ is a realizer of $\boldsymbol{P}$.*

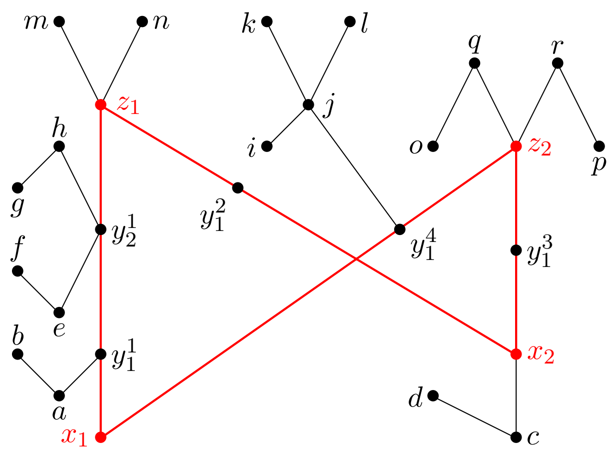


Figure 2.2: A unicycle poset $\mathbf{P}$ with an underlying cycle poset $\mathbf{C}_2$ on the elements x_1, y_1^1, y_2^1, z_1, y_1^2, x_2, y_1^3, z_2, y_1^4. The underlying crown $\mathscr{C}_2$ is depicted in red.

Example 2.4 *For the unicycle poset **P** of Figure 2.2, the following three linear orders form a realizer of **P** (they are obtained using Proposition 2.26):*

$$\mathcal{L}_1 = g\ e\ f\ a\ b\ x_1\ y_1^1\ y_2^1\ h\ c\ x_2\ y_1^2\ z_1\ m\ n\ y_1^3\ y_1^4\ z_2\ o\ q\ p\ r\ i\ j\ k\ l\ d$$

$$\mathcal{L}_2 = o\ p\ c\ d\ x_2\ y_1^3\ y_1^2\ i\ x_1\ y_1^4\ j\ k\ l\ z_2\ r\ q\ a\ y_1^1\ e\ y_2^1\ z_1\ n\ m\ g\ h\ f\ b$$

$$\mathcal{L}_3 = x_1\ y_1^4\ i\ j\ l\ k\ c\ d\ x_2\ a\ b\ y_1^1\ e\ f\ y_2^1\ g\ h\ y_1^3\ z_2\ p\ r\ o\ q\ y_1^2\ z_1\ n\ m$$

For example, the only elements below y_1^2 are c and x_2. This can be seen by intersecting the three linear orders $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$. Indeed, c and x_2 are to the left of y_1^2 in each of the linear orders, whereas the only other elements to the left of y_1^2 in $\mathcal{L}_2$ are o, p, d, y_1^3, but these elements are to the right of y_1^2 in $\mathcal{L}_1$.

Remark 2.5 *We want to point out that after finding a specific realizer of the crowns in Lemma 2.10 and of the tree posets in Proposition 2.13, all the remaining results of the paper starting from Corollary 2.14 will have always the same kind of proofs. We will be given three linear orders of a poset **P** and the fact that these are linear extensions is straightforward to prove (the pictures help). Then we will observe that realizers of some relevant subposets are obtained using previous results as subwords of the three given linear extensions. We will say that we* recover *these subposets. Then we will finish the proofs using Lemma 2.3, by proving the incomparabilities that are not part of the subposets described previously.*

2.5 Unicycle posets

A *cycle poset* is a poset **P** whose cover graph is a cycle. A cycle poset such that all its elements are minimal or maximal elements is called a *fundamental cycle poset*. An immediate result is that a cycle poset has all its vertices of degree 2 and the same number of minimal and maximal elements. It follows that the fundamental cycle posets are the crowns $\mathscr{C}_n$ for $n \geq 2$, whose definition is the following (see Figure 2.1b):

Definition 2.6 *For $n \geq 2$, the* crown $\mathscr{C}_n$ *is the poset having $2n$ elements $x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n$ where for all i the x_i's are minimal elements and the z_i's are maximal elements and the relations are x_{i+1} is less than z_i and z_{i+1} for indices taken cyclically. We also define a numbering of all covers of the crown $\mathscr{C}_n$ for $n \geq 2$; for all $i \in [n]$, the number $2i-1$ is assigned to the cover $x_i \lessdot z_i$, and $2i$ is assigned to $x_{i+1} \lessdot z_i$. It means that for all $p \in [2n]$ the cover numbered by p is $x_{\lfloor \frac{p}{2} \rfloor + 1} \lessdot z_{\lceil \frac{p}{2} \rceil}$. We define the crown $\mathscr{C}_1$ in a different way as the square poset on four elements, with minimum x, maximum z and incomparable elements a, b.*

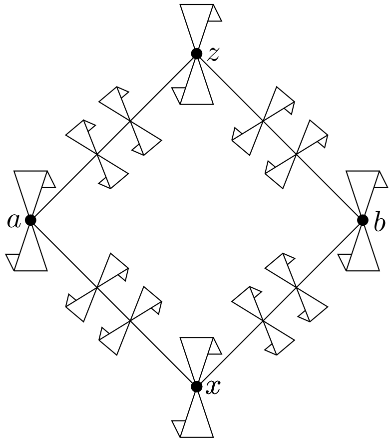
In the sequel, the indices within the crowns will always be taken cyclically.

Lemma 2.7 *A cycle poset **C** is a crown with a certain number of added vertices (possibly zero) on each cover of the crown.*

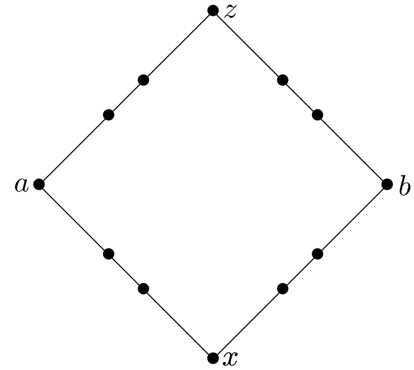
Proof. If **C** has a minimum element (thus a maximum), it is $\mathscr{C}_1$ with possibly extra vertices added on its covers since all its vertices have degree 2. If **C** does not have a minimum element, consider the subposet of **C** defined by keeping only its minimal and maximal elements. It is a fundamental cycle, thus a crown $\mathscr{C}_n$ for $n \geq 2$. All other elements of **C** are of degree 2 and since they are not minimal or maximal elements, they form chains between two such elements. The result follows. □

We denote by $\mathbf{C}_n$ a cycle poset that has $\mathscr{C}_n$ as a subposet; we say that $\mathscr{C}_n$ is the *underlying crown of* $\mathbf{C}_n$. A *rooted tree poset* is a pair $(\mathbf{T}, r)$ of a tree poset **T** and a distinguished element r of **T**.

Definition 2.8 *Let **P** be a poset. Let $\mathcal{T}(\boldsymbol{P}) := \{(\boldsymbol{T}_x, r_x) \mid x \in \boldsymbol{P}\}$ be a family of rooted tree posets indexed by **P**. The* grafting of $\mathcal{T}$ on **P** *is the poset on $\boldsymbol{R} := \{(x,t) \mid x \in \boldsymbol{P},\, t \in \boldsymbol{T}_x\}$ such that $(a,b) \leq (c,d)$ if and only if*



a A unicycle poset whose underlying cycle poset is the poset in Figure 2.3b.



b A cycle poset with underlying crown $\mathscr{C}_1$.

Figure 2.3

(i) $a = c$ *and* $b \leq d$ *in* $\boldsymbol{T}_a$*; or*

(ii) $a < c$ *in* $\boldsymbol{P}$*,* $b \leq r_a$ *in* $\boldsymbol{T}_a$ *and* $r_c \leq d$ *in* $\boldsymbol{T}_c$*.*

The grafting is a special case of a more general construction called *amalgamation* [104]. Intuitively, the Hasse diagram of the grafting of $\mathcal{T}(\mathbf{P})$ on $\mathbf{P}$ corresponds to the directed graph obtained by identifying each $x \in \mathbf{P}$ with r_x. For example, the result of the grafting of rooted trees on the poset of Figure 2.3b is given in Figure 2.3a.

Since a unicycle poset is a cycle poset with tree posets grafted to any of its vertices, it follows:

Proposition 2.9 *Let* $\boldsymbol{P}$ *be a unicycle poset. There exist* $n \geq 1$ *and* $\mathcal{T}(\boldsymbol{C}_n)$ *such that the grafting of* $\mathcal{T}$ *on* $\boldsymbol{C}_n$ *is isomorphic to* $\boldsymbol{P}$*.*

For $\mathbf{P}$ a unicycle poset, the cycle poset $\mathbf{C}_n$ from which $\mathbf{P}$ is constructed is called its *underlying cycle poset*. For example see Figure 2.3a for a unicycle poset whose underlying cycle poset is the cycle poset in Figure 2.3b. See Figure 2.9 for an example of a unicycle poset whose underlying cycle poset is $\mathbf{C}_n$ with $n \geq 2$. Using Proposition 2.9, a unicycle poset $\mathbf{P}$ can be describe using a pair $(\mathbf{C}_n, \mathcal{T}(\mathbf{C}_n))$.

2.6 Crowns

The fact that the crowns $\mathscr{C}_1$ and $\mathscr{C}_2$ are of dimension 2 and the crown $\mathscr{C}_n$ for $n \geq 3$ is of dimension 3 is a well-known result [106, Theorem 4.3]. But we need to work with a specific realizer for later results. After a thorough study of the realizers of crowns, we found that the following realizer is well suited for our construction.

Lemma 2.10 *For all $n \geq 2$, the following three linear orders of $\mathscr{C}_n$ form a realizer:*

$$x_1\ x_2\ z_1 \prod_{i=2}^{n-1} \left(x_{i+1}\ z_i\right)\ z_n, \tag{2.1}$$

$$x_2 \prod_{i=2}^{n-1} \left(x_{i+1}\ z_i\right)\ x_1\ z_n\ z_1, \tag{2.2}$$

$$x_1 \prod_{i=n}^{2} \left(x_i\ z_i\right)\ z_1. \tag{2.3}$$

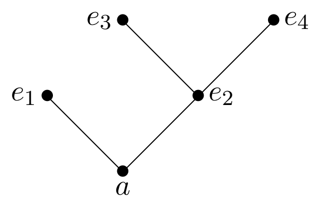
Proof. A linear order is a linear extension of $\mathscr{C}_n$ if and only if for any i, z_i is to the right of x_i and x_{i+1}. It follows that the linear orders (2.1), (2.2) and (2.3) of $\mathscr{C}_n$ are linear extensions of $\mathscr{C}_n$. It remains to prove that we get all the incomparabilities by intersecting the linear extensions (2.1), (2.2) and (2.3); the only comparabilities being $x_i < z_{i-1}, z_i$. For all i the z_i's are mutually incomparable by looking at (2.1) and (2.3). For x_1, with (2.1) and (2.2) we have $x_1 < z_1, z_n$ and no other relations. Since we treated x_1, we can look at the linear extensions restricted to all elements except x_1. Let $i \neq 1$. We look at x_i. By (2.1) and (2.3) we have only the relations $x_i < z_{i-1}, z_i$. This concludes the proof. □

Lemma 2.11 *A realizer of the crown $\mathscr{C}_1$ is given by the two linear extensions $x\,a\,b\,z$ and $x\,b\,a\,z$.*

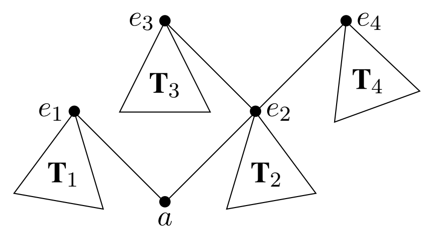
Even if $\mathscr{C}_2$ is of dimension 2, we still use the realizer of size three given by Lemma 2.10. Similarly, for $\mathscr{C}_1$ we consider the realizer of size 3 obtained by repeating its first linear extension $x\,a\,b\,z$.

2.7 Tree posets

We know that tree posets are of dimension at most 3 [107, Theorem 2] but, as for crowns, an important step in the proof of our main result is to obtain a specific realizer of the tree posets (see Corollary 2.14). Recall



a A tree poset **P** with a minimum a. The left cover of a is e_1 and its right cover is e_2. A realizer of size 2 is given by $a\, e_1\, e_2\, e_3\, e_4$ and $a\, e_2\, e_4\, e_3\, e_1$.



b Illustration for the proof of Proposition 2.13.

Figure 2.4

the notations from Section 2.4 related to the tree posets. If a tree poset **T** has a minimum a, choose a planar representation of **T**, which means choosing an order on the covers of any vertex. The *prefix left-to-right reading word* of **T** is the word obtained by the following procedure; starting at a, each time one visits a vertex v, one then has to recursively visit the cover elements of v starting from the leftmost one. The *prefix right-to-left reading word* of **T** is exactly the same procedure but visiting the cover elements from right to left instead. See Figure 2.4a for an example of the following result:

Lemma 2.12 *Let **P** be a tree poset. If **P** has a minimum or a maximum, then its dimension is at most* 2*.*

Proof. First suppose that **P** has a minimum. It is easy to see that a realizer of size 2 of **P** is given by the linear extensions obtained from the prefix left-to-right and prefix right-to-left reading words of the tree. If **P** has a maximum, the result follows by Lemma 2.2 as $\mathbf{P}^{op}$ has a minimum. □

Proposition 2.13 *Let **P** be a tree poset. If a is a minimal element of **P**, then there exist linear extensions U_a^- and U_a^+ of respectively $\boldsymbol{U}_a^-$ and $\boldsymbol{U}_a^+$, and linear extensions $U_{a,1}$ and $U_{a,2}$ of $\boldsymbol{U}_a$ such that the three following linear orders form a realizer of **P**:*

$$U_a^- \; a \; U_a^+, \quad a \, U_{a,1}, \quad \textit{and} \quad a \, U_{a,2}.$$

*If a is a maximal element of **P**, then there exist linear extensions D_a^- and D_a^+ of respectively $\boldsymbol{D}_a^-$ and $\boldsymbol{D}_a^+$, and linear extensions $D_{a,1}$ and $D_{a,2}$ of $\boldsymbol{D}_a$ such that the three following linear orders form a realizer of **P**:*

$$D_a^- \; a \; D_a^+, \quad D_{a,1} \, a, \quad \textit{and} \quad D_{a,2} \, a.$$

Proof. We make a proof by induction on the number of elements of **P**, our induction hypothesis containing both cases together. For $n = 1$ it is true for both cases (a is a minimal and maximal element of **P**) using the three linear extensions a, a, and a.

Assume that the results are true for any tree posets with n or fewer elements. Suppose that **P** is a tree poset having $n + 1$ elements. Without loss of generality we can assume that a is a minimal element (otherwise we consider the dual poset and reverse the linear extensions, see Lemma 2.2). Denote by $e_1, e_2, \ldots, e_r$ the elements of $\mathbf{U}_a^+$ taken in prefix left-to-right order, and by $e_{s_1}, e_{s_2}, \ldots, e_{s_r}$ these elements taken in prefix right-to-left order (looking at the tree $\mathbf{U}_a^+ \cup \{a\}$ but without writing the root a). By Lemma 2.12, a realizer of $\mathbf{U}_a^+$ is given by $e_1 e_2 \cdots e_r$ and $e_{s_1} e_{s_2} \cdots e_{s_r}$. See Figure 2.4b for an illustration of the following.

For all $e_i \in \mathbf{U}_a^+$, let $\mathbf{T}_i$ be the biggest tree such that $\mathbf{T}_i \cap \mathbf{U}_a^+ = \{e_i\}$. Then $\mathbf{U}_a^- = \bigsqcup_{i=1}^r \big(\mathbf{T}_i \setminus \{e_i\}\big)$. The trees $\mathbf{T}_i$ are all disjoint tree posets and contain e_i as a maximal element. The notations $\mathbf{D}_{e_i}$, $\mathbf{D}_{e_i}^-$, $\mathbf{D}_{e_i}^+$ will be used for the tree $\mathbf{T}_i$. In particular, $\mathbf{D}_{e_i} = \mathbf{T}_i \setminus \{e_i\}$. Since $|\mathbf{T}_i| \leq n$ because $a \notin \mathbf{T}_i$, by the induction hypothesis we have a realizer of $\mathbf{T}_i$ for all $i \in [r]$ of the form

$$D_i^- \; e_i \; D_i^+, \quad D_{i,1} \; e_i, \quad D_{i,2} \; e_i,$$

where D_i^- is a linear extension of $\mathbf{D}_{e_i}^-$, D_i^+ is a linear extension of $\mathbf{D}_{e_i}^+$, and $D_{i,1}$ and $D_{i,2}$ are linear extensions of $\mathbf{D}_{e_i} = \mathbf{T}_i \setminus \{e_i\}$. Let us consider the three following linear orders of **P**:

$$\prod_{i=r}^{1} \big(D_{i,1}\big) \; a \; \prod_{i=1}^{r} e_i, \tag{2.4}$$

$$a \; \prod_{i=1}^{r} \big(D_{i,2} \; e_i\big), \tag{2.5}$$

$$a \; \prod_{i=1}^{r} \big(D_{s_i}^- \; e_{s_i}\big) \; \prod_{i=r}^{1} D_i^+. \tag{2.6}$$

Let us prove that (2.4), (2.5) and (2.6) form a realizer of **P**. The fact that these are three linear extensions is a straightforward verification. We recover the trees $\mathbf{T}_i$ for all $i \in [r]$ since $D_i^- \; e_i \; D_i^+$ is a subword of (2.6), we have $D_{i,1} \; e_i$ is a subword of (2.4) and $D_{i,2} \; e_i$ is a subword of (2.5). Moreover, we recover the poset $\mathbf{U}_a^+$ since $e_1 \cdots e_r$ is a subword of (2.4) and (2.5) and $e_{s_1} \cdots e_{s_r}$ is a subword of (2.6). In order to be a realizer of **P**, we need the following incompatibilities:

$$\mathbf{T}_i \setminus \{e_i\} \not\sim a, \; \forall i, \qquad \mathbf{T}_i \setminus \{e_i\} \not\sim \mathbf{T}_j \setminus \{e_j\}, \; \forall i \neq j,$$

$$\mathbf{D}_{e_i}^+ \not\sim e_j,\ \forall i \neq j, \qquad \mathbf{D}_{e_i}^- \not\sim \mathbf{T}_j,\ \text{ for } e_i \not< e_j.$$

By looking at (2.4) and (2.5) we obtain the incomparabilities $\mathbf{T}_i \setminus \{e_i\} \not\sim a$ for all $i \in [r]$, and also $\mathbf{T}_i \setminus \{e_i\} \not\sim \mathbf{T}_j \setminus \{e_j\}$ for all $i \neq j$. By looking at (2.4) and (2.6), we obtain $\mathbf{D}_{e_i}^+ \not\sim e_j$ for all $i \neq j$. The remaining incomparabilities $\mathbf{D}_{e_i}^- \not\sim \mathbf{T}_j$ for $e_j \not< e_j$ follow from (2.5) and (2.6).

Finally, these three linear extensions are of the form we wanted; $\prod_{i=r}^{1} D_{i,1}$ is a linear extension of $\mathbf{U}_a^-$, $e_1 \cdots e_r$ is a linear extension of $\mathbf{U}_a^+$, and $\prod_{i=1}^{r} D_{i,2}\, e_i$ and $\prod_{i=1}^{r} \left(D_{s_i}^-\, e_{s_i}\right)\ \prod_{i=r}^{1} D_i^+$ are linear extensions of $\mathbf{U}_a$. □

The following result implies the well-known result that all tree posets are of dimension at most 3. This is needed in order to extract a particular realizer of a tree poset.

Corollary 2.14 *Let* $\boldsymbol{P}$ *be a tree poset and* $a \in \boldsymbol{P}$*. The three following linear orders form a realizer of* $\boldsymbol{P}$*:*

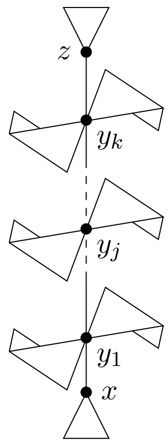
$$U_a^-\ D_{a,1}\ a\ U_a^+, \tag{2.7}$$

$$D_a^-\ a\ U_{a,1}\ D_a^+, \tag{2.8}$$

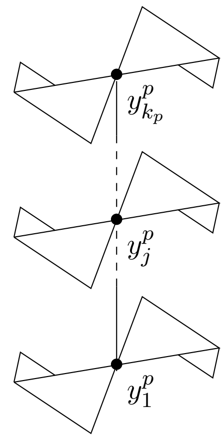
$$D_{a,2}\ a\ U_{a,2}, \tag{2.9}$$

where U_a^-*,* U_a^+*,* $U_{a,1}$*, and* $U_{a,2}$ *are the linear extensions obtained in Proposition 2.13 for the tree* $\boldsymbol{U}_a \cup \{a\}$*, and* D_a^-*,* D_a^+*,* $D_{a,1}$*, and* $D_{a,2}$ *are the linear extensions obtained in Proposition 2.13 for the tree* $\boldsymbol{D}_a \cup \{a\}$*.*

Proof. The fact that these three linear orders are linear extensions is straightforward (see Figure 2.1a). Since these three linear extensions contain respectively the subwords $U_a^-\, a\, U_a^+$, $a\, U_{a,1}$ and $a\, U_{a,2}$, we know by Proposition 2.13 that their restrictions to the elements of $\mathbf{U}_a \cup \{a\}$ form a realizer of $\mathbf{U}_a \cup \{a\}$. The same goes for $\mathbf{D}_a \cup \{a\}$. It only remains to verify that $\mathbf{D}_a^+ \not\sim \mathbf{U}_a \cup \{a\}$, which follows from (2.8) and (2.9), and $\mathbf{U}_a^- \not\sim \mathbf{D}_a \cup \{a\}$, which follows from (2.7) and (2.9). □



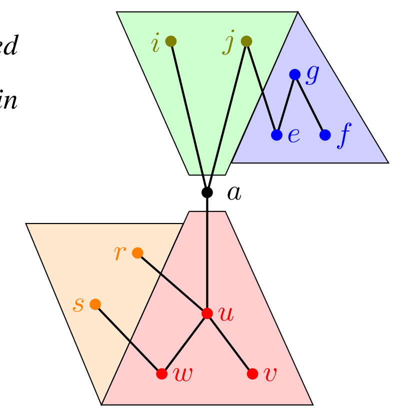
a The tree poset $\mathbf{Q}$ of Corollary 2.16.



b The tree $\mathbf{T}_p$.

Figure 2.5

Example 2.15 *For the tree on the right, the root is a and, $\boldsymbol{D}^+_a$ is coloured orange, $\boldsymbol{D}^-_a$ red, $\boldsymbol{U}^-_a$ blue and $\boldsymbol{U}^+_a$ green. Using Corollary 2.14, we obtain the following realizer:*



$$f \; e \; g \; w \; s \; v \; u \; r \; a \; i \; j,$$

$$w \; v \; u \; a \; i \; e \; f \; g \; j \; r \; s,$$

$$v \; w \; s \; u \; r \; a \; e \; j \; i \; f \; g.$$

Let $\mathbf{Q}$ be a tree poset with $x, z \in \mathbf{Q}$ such that $[x, z] = \{x \lessdot y_1 \lessdot \cdots \lessdot y_k \lessdot z\}$ has at least two elements and such that $\mathbf{D}^+_x = \emptyset$, x is only covered by y_1, $\mathbf{U}^-_z = \emptyset$ and z covers only y_k (see Figure 2.5a). We now define some subtrees of $\mathbf{Q}$. Denote by $\mathbf{T}_x$ the tree formed by x and the elements below it, and by $\mathbf{T}_z$ the tree formed by z and the elements above it. Let $\mathbf{T} := \bigsqcup_{j=1}^{k} \mathbf{T}_{y_j}$ where $\mathbf{T}_{y_j}$ is the tree grafted to y_i as in Figure 2.5a. Let $\mathbf{U} := \bigsqcup_{j=1}^{k} \mathbf{U}_{y_j}$ where $\mathbf{U}_{y_j}$ is defined as before in the tree $\mathbf{T}_{y_j}$, and similarly for $\mathbf{U}^+, \mathbf{U}^-, \mathbf{D}, \mathbf{D}^-, \mathbf{D}^+, \mathbf{I}^+, \mathbf{I}^-$.

Corollary 2.16 *Let $\mathbf{Q}$ be as above and keep the notations from Corollary 2.14. The three following linear* *orders form a realizer of* $\mathbf{Q}$*:*

$$\prod_{j=k}^{1}\left(U_{y_j}^{-}\ D_{y_j,1}\right)\ T_{x,1}\ x\ \prod_{j=1}^{k}\left(y_j\ U_{y_j}^{+}\right)\ z\ T_{z,1}, \tag{2.10}$$

$$T_{x,2}\ x\ \prod_{j=1}^{k}\left(D_{y_j}^{-}\ y_j\right)\ z\ T_{z,2}\ \prod_{j=k}^{1}\left(U_{y_j,1}\ D_{y_j}^{+}\right), \tag{2.11}$$

$$T_{x,3}\ x\ \prod_{j=1}^{k}\left(D_{y_j,2}\ y_j\ U_{y_j,2}\right)\ z\ T_{z,3}, \tag{2.12}$$

where $T_{x,1}\,x$*,* $T_{x,2}\,x$*,* $T_{x,3}\,x$ *is any realizer of* $\boldsymbol{T}_x$*, and* $z\,T_{z,1}$*,* $z\,T_{z,2}$*,* $z\,T_{z,3}$ *is any realizer of* $\boldsymbol{T}_z$*.*

Proof. It is straightforward to check that these three linear orders are linear extensions. For all $j \in [k]$, the realizer formed by $U_{y_j}^{-}\,D_{y_j,1}\,y_j\,U_{y_j}^{+}$, and $D_{y_j}^{-}\,y_j\,U_{y_j,1}\,D_{y_j}^{+}$ and $D_{y_j,2}\,y_j\,U_{y_j,2}$ of Corollary 2.14 for $\mathbf{T}_{y_j}$ is obtained as subwords of (2.10), (2.11) and (2.12). Thus we know that for all $j \in [k]$ we recover the subposet $\mathbf{T}_{y_j}$. It is also clear that we recover the subposets $\mathbf{T}_x$ and $\mathbf{T}_z$. It remains to prove the incomparabilities:

$$\mathbf{T}_x \not\sim \mathbf{D} \sqcup \mathbf{U}^-, \qquad \mathbf{T}_z \not\sim \mathbf{U} \sqcup \mathbf{D}^+, \qquad \mathbf{D}_{y_i}^{+}, \mathbf{U}_{y_i}^{-} \not\sim y_j \text{ for all } i \neq j, \tag{2.13}$$

$$\mathbf{U}_{y_i} \not\sim y_j \text{ for all } i < j, \qquad \mathbf{U}_{y_i} \not\sim \mathbf{U}_{y_j} \sqcup \mathbf{D}_{y_j}^{+} \text{ for all } i \neq j, \tag{2.14}$$

$$\mathbf{D}_{y_i}^{-} \not\sim \mathbf{U}_{y_j}^{+} \text{ and } \mathbf{D}_{y_i} \not\sim y_j \text{ for all } i > j, \qquad \mathbf{D}_{y_i} \not\sim \mathbf{D}_{y_j} \sqcup \mathbf{U}_{y_j}^{-} \text{ for all } i \neq j. \tag{2.15}$$

To verify the incomparabilities of (2.13) it suffices to look at (2.10) and (2.11). For (2.14), it suffices to look at (2.11) and (2.12). Finally, for the incomparabilities of (2.15) it suffices to look at (2.10) and (2.12). □

2.8 Non-extremal grafting on cycle posets

By non-extremal we mean the vertices that are neither minimal nor maximal elements. Assume $n \geq 2$ and consider a cycle poset $\mathbf{C}_n$ such that $x_{\lfloor \frac{p}{2} \rfloor+1} \lessdot y_1^p \lessdot \cdots \lessdot y_{k_p}^p \lessdot z_{\lceil \frac{p}{2} \rceil}$ is a saturated chain, for every $p \in [2n]$. These p correspond to the edge labelling defined in Definition 2.6 and represented in Figure 2.1b (see also Figure 2.2). We denote by $\mathcal{T}(\mathbf{C}_n)$ a family of rooted trees indexed by the elements of $\mathbf{C}_n$ such that $\mathbf{T}_{x_i} = \{r_{x_i}\}$ and $\mathbf{T}_{z_i} = \{r_{z_i}\}$, for every $i \in [n]$. Thus, the grafting of $\mathcal{T}(\mathbf{C}_n)$ on $\mathbf{C}_n$, which defines a unicycle poset $\mathbf{P}_n$, only grafts nontrivial trees on the non-extremal elements of $\mathbf{C}_n$ (see Figure 2.6). We denote by $\mathbf{T}_p := \bigsqcup_{j=1}^{k_p} \mathbf{T}_{y_j^p}$ the tree obtained by grafting $\{(\mathbf{T}_{y_j^p}, y_j^p) \mid 1 \leq j \leq k_p\}$ on the saturated chain $y_1^p \lessdot \cdots \lessdot y_{k_p}^p$ of $\mathbf{C}_n$ for every $p \in [2n]$ (see Figure 2.5b). See the paragraph before Corollary 2.16, where we defined the $\mathbf{U}, \mathbf{D}, \mathbf{I}$ related to this tree. For $n = 1$, we only have two such trees: $\mathbf{T}_1$ on the maximal chain containing a, and $\mathbf{T}_2$ on the one containing b. This gives us a unicycle poset $\mathbf{P}_1$ (see Figure 2.7). Using these notations, for any $p \in [2n]$ and any word X denote:

$$\begin{aligned} A_p(X) &:= \prod_{j=k_p}^{1} \left(U^-_{y^p_j} \; D_{y^p_j,1} \right) \; X \; \prod_{j=1}^{k_p} \left(y^p_j \; U^+_{y^p_j} \right), \\ B_p(Z) &:= \prod_{j=1}^{k_p} \left(D^-_{y^p_j} \; y^p_j \right) \; Z \; \prod_{j=k_p}^{1} \left(U_{y^p_j,1} \; D^+_{y^p_j} \right), \\ C_p &:= \prod_{j=1}^{k_p} \left(D_{y^p_j,2} \; y^p_j \; U_{y^p_j,2} \right). \end{aligned}$$

Remark 2.17 *In the last section of this paper, we will need the following observation. We have that $A_p(X)$ is composed of three parts; it starts with a linear extension of $\boldsymbol{I}^-_p$, then X, and then a linear extension of $\{y^p_1, \ldots, y^p_{k_p}\} \sqcup \boldsymbol{U}^+_p$. Similarly for $B_p(Z)$, the first part is a linear extension of $\{y^p_1, \ldots, y^p_{k_p}\} \sqcup \boldsymbol{D}^-_p$, then Z, and then a linear extension of $\boldsymbol{I}^+_p$.*

Recall that we always take the indices related to the crowns cyclically. We can rewrite the equations of Corollary 2.16 where $x = x_{\lfloor \frac{p}{2} \rfloor + 1}$, $z = z_{\lceil \frac{p}{2} \rceil}$ and the y_j's replaced by the y^p_j's, respectively for $p = 2i-1$ and $p = 2i$, in the following way where the realizer on the left if one for $\mathbf{T}_{2i-1} \sqcup \{x_i, z_i\}$, and on the right one for $\mathbf{T}_{2i} \sqcup \{x_{i+1}, z_i\}$:

$$\begin{aligned} &A_{2i-1}(x_i)\, z_i, && A_{2i}(x_{i+1})\, z_i, \\ &x_i \; B_{2i-1}(z_i), && x_{i+1} \; B_{2i}(z_i), \\ &x_i \; C_{2i-1} \; z_i, && x_{i+1} \; C_{2i} \; z_i. \end{aligned}$$

Let us set $A^i_j(X) := A_i(A_j(X))$ and $B^i_j(X) := B_i(B_j(X))$.

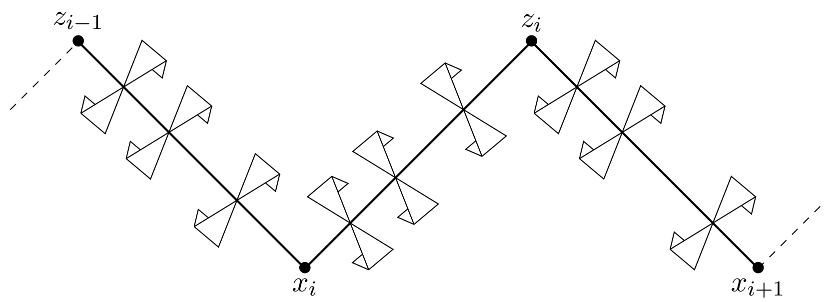


Figure 2.6: A part of the crown illustrating Proposition 2.18 with added trees on its covers from left to right $\mathbf{T}_{2i-2}$, $\mathbf{T}_{2i-1}$ and $\mathbf{T}_{2i}$.

Proposition 2.18 *Let $n \geq 4$. The following three linear orders form a realizer of $\boldsymbol{P}_n$:*

$$A_1(x_1)\ x_2\ B_2(z_1)\ C_{2n} \prod_{i=2}^{n-1} \Big(C_{2i-1}\ A_{2i}(x_{i+1})\ z_i\Big)\ B_{2n-1}(z_n), \tag{2.16}$$

$$A_3(x_2)\ \big(x_3\ B_4(z_2)\big)\ C_2 \prod_{i=3}^{n-1} \Big(x_{i+1}\ B_{2i}^{2i-1}(z_i)\Big)\ C_{2n-1}\ A_{2n}(x_1)\ z_n\ B_1(z_1), \tag{2.17}$$

$$x_1\ A_{2n-1}(x_n)\ B_{2n}(z_n)\ C_1 \prod_{i=n-1}^{3} \Big(C_{2i}\ A_{2i-1}(x_i)\ z_i\Big)\ C_4\ A_2(x_2)\ B_3(z_2)\ z_1. \tag{2.18}$$

Proof. It is straightforward to check that these are three linear extensions of $\mathbf{P}$ (see Figure 2.6). By forgetting about the A, B and C, meaning looking at the subwords on the elements x_i and z_i, we recover using Lemma 2.10 the crown $\mathscr{C}_n$. Looking at the subwords of these linear extensions on the elements of $\mathbf{T}_{2i-1} \sqcup \{x_i, z_i\}$, we recover the realizer $A_{2i-1}(x_i)\, z_i$, $x_i\, B_{2i-1}(z_i)$ and $x_i\ C_{2i-1}\ z_i$ of this subposet, the same being true for $\mathbf{T}_{2i} \sqcup \{x_{i+1}, z_i\}$ (see above Proposition 2.18). The remaining incomparabilities to prove are the following:

$$\mathbf{T}_i \not\sim \mathbf{T}_j,\ \forall i \neq j, \tag{2.19}$$

$$\mathbf{T}_{2i-1} \not\sim x_j, z_j,\ \forall j \neq i, \tag{2.20}$$

$$\mathbf{T}_{2i} \not\sim x_j, z_k,\ \forall j \neq i+1,\ \forall k \neq i. \tag{2.21}$$

Let $i \neq j$. All the elements of $\mathbf{T}_i$ are in $A_i(X)$, but also in $B_i(Z)$ and in C_i, regardless of X and Z. Thus proving that $\mathbf{T}_i \not\sim \mathbf{T}_j$ is equivalent to showing that in a linear extension one of the symbol A, B or C with a subscript i appears before one of those symbols with subscript j, which gives $\mathbf{T}_i < \mathbf{T}_j$, and, in another linear extension they appear in the opposite order, which gives $\mathbf{T}_j < \mathbf{T}_i$. Thus we just look at the sequences of subscripts of the symbols A, B and C obtained from (2.16), (2.17) and (2.18):

$$1, 2, 2n, \prod_{i=2}^{n-1}(2i-1, 2i), 2n-1,$$

$$3, 4, 2, \prod_{i=3}^{n-1} {}^{2i-1}_{2i}, 2n-1, 2n, 1,$$

$$2n-1, 2n, 1, \prod_{i=n-1}^{3}(2i, 2i-1), 4, 2, 3,$$

where, in the second linear extension, because of $B_{2i}^{2i-1}(z_i)$, we can conclude neither $\mathbf{T}_{2i-1} < \mathbf{T}_{2i}$ nor $\mathbf{T}_{2i} < \mathbf{T}_{2i-1}$. It is thus straightforward to verify the incomparabilities (2.19) for any pair $i \neq j$.

For (2.20) and (2.21), let $i \in [n]$. First remark that, any letter that comes before x_i in (2.1), comes before $\mathbf{T}_{2i-1}$ in (2.16). Similarly, any letter that comes after z_i in (2.1), comes after $\mathbf{T}_{2i-1}$ in (2.16). The same goes for the pairs (2.2), (2.17) and, (2.3), (2.18). Because $x_i < z_i$, an element incomparable to both x_i and z_i appears before x_i in one of the linear extensions and after z_i in another. Therefore, for any $j, k \in [n]$, having both $x_i \not\sim x_j, z_k$ and $z_i \not\sim x_j, z_k$ imply $\mathbf{T}_{2i-1} \not\sim x_j, z_k$. We proceed similarly for $\mathbf{T}_{2i}$. Thus we have (see Figure 2.6):

$$\mathbf{T}_{2i-1} \not\sim x_j, z_k, \quad \text{for all } j \neq i, i+1 \text{ and } k \neq i, i-1, \tag{2.22}$$

$$\mathbf{T}_{2i} \not\sim x_j, z_k, \quad \text{for all } j \neq i, i+1 \text{ and } k \neq i, i+1. \tag{2.23}$$

To finish the proof, we need to look at the linear extension containing C_{2i-1}. In this linear extension we have $z_{i-1} < \mathbf{T}_{2i-1} < x_{i+1}$. Looking at the two others, one can easily verify that, at least one of the two has $x_{i+1} < \mathbf{T}_{2i-1} < z_{i-1}$ and (2.20) is satisfied. Similarly, in the linear extension containing C_{2i}, we have $z_{i+1} < \mathbf{T}_{2i} < x_{i-1}$. In at least one of the others, we have $x_{i-1} < \mathbf{T}_{2i} < z_{i+1}$ and (2.21) is satisfied. This finishes the proof. $\square$

For $n = 3$, it is exactly the same formula as in Proposition 2.18 but without the middle products of (2.17) and (2.18). The proof is exactly the same but we state this particular case separately as we will need to treat $\mathbf{P}_3$ separately in the last section.

Proposition 2.19 *The following three linear orders form a realizer of* $\boldsymbol{P}_3$*:*

$$A_1(x_1)\ x_2\ B_2(z_1)\ C_6\ C_3\ A_4(x_3)\ z_2\ B_5(z_3), \tag{2.24}$$

$$A_3(x_2)\ x_3\ B_4(z_2)\ C_2\ C_5\ A_6(x_1)\ z_3\ B_1(z_1), \tag{2.25}$$

$$x_1\ A_5(x_3)\ B_6(z_3)\ C_1\ C_4\ A_2(x_2)\ B_3(z_2)\ z_1. \tag{2.26}$$

Proposition 2.20 *The following three linear orders form a realizer of* $\boldsymbol{P}_2$*:*

$$A_1(x_1)\ x_2\ B_2(z_1)\ B_4^3(z_2), \tag{2.27}$$

$$A_3^2(x_2)\ A_4(x_1)\ z_2\ B_1(z_1), \tag{2.28}$$

$$x_1\ C_4\ x_2\ C_1\ C_3\ z_2\ C_2\ z_1. \tag{2.29}$$

Proof. It is straightforward to check that these are three linear extensions of $\mathbf{P}$. By forgetting about the A, B and C, meaning looking at the subwords on the elements x_i and z_i, we recover the crown $\mathscr{C}_2$. Looking at the subwords of these linear extensions on the elements of $\mathbf{T}_{2i-1}\sqcup\{x_i, z_i\}$, we recover the realizer $A_{2i-1}(x_i)\ z_i$, $x_i\ B_{2i-1}(z_i)$ and $x_i\ C_{2i-1}\ z_i$ of this subposet, the same being true for $\mathbf{T}_{2i}\sqcup\{x_{i+1}, z_i\}$ (see above Proposition 2.18). The remaining incomparabilities to prove are the following:

$$\mathbf{T}_i \not\sim \mathbf{T}_j,\ \forall i \neq j, \qquad \mathbf{T}_1 \not\sim x_2, z_2, \qquad \mathbf{T}_3 \not\sim x_1, z_1,$$
$$\mathbf{T}_2 \not\sim x_1, z_2, \qquad \mathbf{T}_4 \not\sim x_2, z_1.$$

Let $i \neq j$. In a similar manner as in Proposition 2.18, to prove that $\mathbf{T}_i \not\sim \mathbf{T}_j$, we look at the sequences of subscripts of the symbols A, B and C in (2.27), (2.28) and (2.29). This gives $1, 2, \frac{3}{4}$, and $\frac{2}{3}, 4, 1$ and $4, 1, 3, 2$. Then $\mathbf{T}_i \not\sim \mathbf{T}_j$ follows by the fact that in one of the sequence i appears before j and in another one j appears before i. We have that $\mathbf{T}_1 \not\sim x_2, z_2$ and $\mathbf{T}_3 \not\sim x_1, z_1$ follow from (2.27) and (2.28) just by looking at A_1, B_1 and A_3, B_3 respectively. For $\mathbf{T}_2 \not\sim x_1, z_2$ we look at A_2, C_2 in (2.28) and (2.29). For $\mathbf{T}_4 \not\sim x_2, z_1$ we look at B_4, C_4 in (2.27) and (2.29). $\square$

Proposition 2.21 *The following three linear orders form a realizer of* $\boldsymbol{P}_1$*:*

$$A_1(x)\ B_2(z), \tag{2.30}$$

$$A_2(x)\ B_1(z), \tag{2.31}$$

$$x\ C_1\ C_2\ z. \tag{2.32}$$

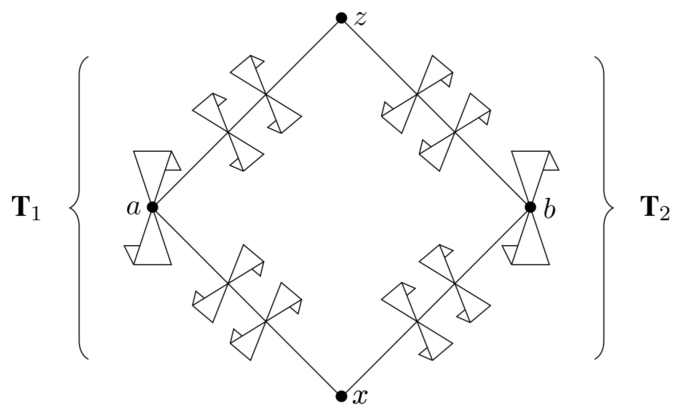


Figure 2.7: A cycle poset $\mathbf{C}_1$ with grafted trees from Proposition 2.21.

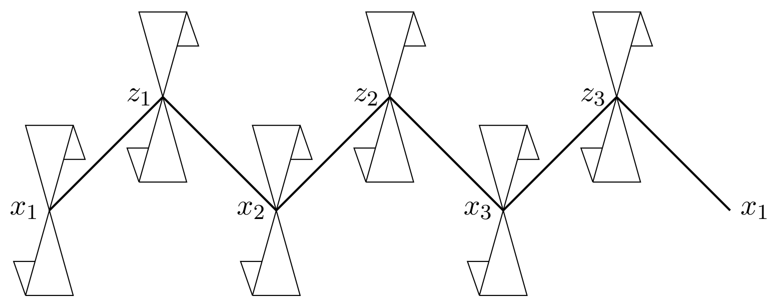


Figure 2.8: The crown $\mathscr{C}_3$ with grafted trees to each of its vertices. The last edge on the right is supposed to go back to the first vertex x_1.

Proof. It is straightforward to check that these are three linear extensions of $\mathbf{P}$ (see Figure 2.7). Also we recover the crown $\mathscr{C}_1$ as a subposet since we have respectively in these linear extensions the subwords $x\,a\,b\,z$, $x\,b\,a\,z$ and $x\,a\,b\,z$. For $i = 1, 2$, looking at the subwords of these linear extensions on the elements of $\mathbf{T}_i \sqcup \{x, z\}$, we recover the realizer $A_i(x)\,z$, $x\,B_i(z)$ and $x\ C_i\ z$ of this subposet. The only incompatibility to check is $\mathbf{T}_1 \not\sim \mathbf{T}_2$, which follows from $A_1 < B_2$ in (2.30) and $A_2 < B_1$ in (2.31). $\square$

2.9 Grafting on the crowns

We will describe realizers for posets obtained by grafting trees on the crowns. Recall that in Corollary 2.14 we proved that the following is a realizer of $\mathbf{T}_w$:

$$(U_w^-\ D_{w,1})\ (w\ U_w^+), \qquad (D_w^-\ w)\ (U_{w,1}\ D_w^+), \qquad D_{w,2}\ w\ U_{w,2}.$$

We denote $I_w^- := U_w^- \, D_{w,1}$, $I_w^+ := U_{w,1} \, D_w^+$, $W^+ := w \, U_w^+$, $W^- := D_w^- \, w$, and $W^\bullet \;=\; D_{w,2} \; w \; U_{w,2}$. Thus a realizer of $\mathbf{T}_w$ is given by:

$$I_w^- \; W^+, \tag{2.33}$$

$$W^- \; I_w^+, \tag{2.34}$$

$$W^\bullet. \tag{2.35}$$

We will replace w by x_i or z_i, thus we just defined $I_{x_i}^-, I_{x_i}^+$, X_i^-, X_i^+, $X_i^\bullet$ and $I_{z_i}^-, I_{z_i}^+$, Z_i^-, Z_i^+, $Z_i^\bullet$.

For the remainder of this section, for any $n > 1$, let $\mathcal{T}(\mathscr{C}_n)$ be a family of rooted tree posets indexed by the elements of $\mathscr{C}_n$, and let $\mathbf{P}_n$ be the unicycle poset obtained from the grafting of $\mathcal{T}(\mathscr{C}_n)$ on $\mathscr{C}_n$ (see Figure 2.8).

Proposition 2.22 *Let $n \geq 4$. The following three linear orders form a realizer of $\boldsymbol{P}_n$:*

$$I_{z_1}^- \; I_{x_2}^- \; X_1^- \; X_2^+ \; Z_1^+ \; (X_3^- \; Z_2^- \; I_{z_2}^+) \prod_{i=3}^{n-1} \Big(X_{i+1}^- \; Z_i^- \; I_{z_i}^+ \; I_{x_i}^+ \Big) \; Z_n^\bullet \; I_{x_n}^+ \; I_{x_1}^+, \tag{2.36}$$

$$I_{z_2}^- \; X_2^\bullet \; \prod_{i=2}^{n-2} \Big(I_{z_{i+1}}^- \; I_{x_{i+1}}^- \; X_{i+1}^+ \; Z_i^+ \Big) \; (I_{x_n}^- \; X_n^+ \; Z_{n-1}^+) \; X_1^\bullet \; Z_n^- \; Z_1^\bullet \; I_{z_n}^+, \tag{2.37}$$

$$I_{z_n}^- \; I_{x_1}^- \; X_1^+ \; X_n^\bullet \; Z_n^+ \; \prod_{i=n-1}^{3} \Big(X_i^\bullet \; Z_i^\bullet \Big) \; X_2^- \; Z_2^\bullet \; Z_1^- \; I_{z_1}^+ \; I_{x_2}^+. \tag{2.38}$$

Proof. It is straightforward to check that these three linear orders are linear extensions of $\mathbf{P}$ (see Figure 2.8). Moreover, the restrictions of the three linear extensions to the set $\mathbf{T}_w$ form a realizer of $\mathbf{T}_w$ for any $i \in [n]$ and $w \in \{x_i, z_i\}$ (see above Proposition 2.22). The same goes for the crown $\mathscr{C}_n$, thus for $\mathscr{C}_n$ with $\mathbf{X}_i^-$ grafted to each x_i and $\mathbf{Z}_i^+$ grafted to each z_i. We need to check the following incomparabilities for all $i \in [n]$:

$$\mathbf{T}_{x_i} \not\sim \mathbf{T}_{x_j}, \forall j \neq i, \tag{2.39}$$

$$\mathbf{T}_{x_i} \not\sim \mathbf{T}_{z_j}, \forall j \neq i, i-1, \tag{2.40}$$

$$\mathbf{I}_{x_i}^+ \not\sim \mathbf{T}_{z_j}, \text{ for } j = i, i-1, \tag{2.41}$$

$$\mathbf{T}_{x_i} \not\sim \mathbf{I}_{z_j}^-, \text{ for } j = i, i-1, \tag{2.42}$$

$$\mathbf{T}_{z_i} \not\sim \mathbf{T}_{z_j}, \forall j \neq i. \tag{2.43}$$

Note that for example in (2.36) the elements of $\mathbf{T}_{x_1}$ are in X_1^- and $I_{x_1}^+$. Keep in mind (2.33), (2.34) and (2.35) to understand the following. We first verify the incomparabilities (2.39) to (2.42) for each i and finish with the incomparabilities of (2.43). We begin with $i = 1$. In (2.38), $\mathbf{T}_{x_1}$ is smaller than everything except $\mathbf{I}_{z_n}^-$ and in (2.37), it is bigger than everything except $\mathbf{T}_{z_1}$ and $\mathbf{T}_{z_n}$. This verifies that we have the incompatibilities of (2.39) and (2.40) and $\mathbf{T}_{x_i} \not< \mathbf{I}_{z_n}^-$. In (2.36), $\mathbf{I}_{z_1}^- < \mathbf{T}_{x_1}$, which completes the verification of (2.42), and $\mathbf{T}_{z_1}, \mathbf{T}_{z_n} < \mathbf{I}_{x_1}^+$, which completes the verification of (2.41).

For $i = 2$, we already have all incomparabilities involving $\mathbf{T}_{x_1}$, thus we restrict the linear extensions to the subset $\mathbf{P} \setminus \mathbf{T}_{x_1}$. In (2.36), $\mathbf{T}_{x_2}$ is smaller than every element except $\mathbf{I}_{z_1}^-$. In (2.37), $\mathbf{T}_{x_2}$ is smaller than every element except $\mathbf{I}_{z_2}^-$. In particular we have $\mathbf{T}_{x_2} \not< \mathbf{I}_{z_1}^-, \mathbf{I}_{z_2}^-$, which gives (2.42). In (2.38), $\mathbf{T}_{x_2}$ is bigger than everything except $\mathbf{T}_{z_1}$ and $\mathbf{T}_{z_2}$, thus verifying the incomparabilities (2.39) and (2.40). We also have $\mathbf{T}_{z_1}, \mathbf{T}_{z_2} < \mathbf{I}_{x_2}^+$, which proves (2.41).

For $i = n$, similarly, we can restrict to the subset $\mathbf{P} \setminus (\mathbf{T}_{x_1} \cup \mathbf{T}_{x_2})$. In (2.37), $\mathbf{T}_{x_n}$ is bigger than everything except $\mathbf{Z}_{n-1}^+$ and $\mathbf{T}_{z_n}$. In (2.38), $\mathbf{T}_{x_n}$ is smaller than everything except $\mathbf{I}_{z_n}^-$. Thus we have the incompatibilities of (2.39), (2.40) and (2.42). For (2.41), we need to remark that $\mathbf{I}_{x_n}^+$ is bigger than $\mathbf{T}_{z_{n-1}}, \mathbf{T}_{z_n}$ in (2.37), and we have $\mathbf{T}_{x_n} < \mathbf{T}_{z_{n-1}}, \mathbf{T}_{z_n}$ in (2.36).

For $3 \leq i \leq n-1$, we can restrict to the subset $\mathbf{P} \setminus (\mathbf{T}_{x_1} \cup \mathbf{T}_{x_2} \cup \mathbf{T}_{x_n})$. We have in (2.37) and (2.38) respectively

$$\mathbf{T}_{z_j}, \mathbf{T}_{x_k} < \mathbf{T}_{x_i} < \mathbf{T}_{z_l}, \mathbf{T}_{x_m}, \quad \text{for } j < i-1,\ k < i,\ i < l,\ i < m, \tag{2.44}$$

$$\mathbf{T}_{z_j}, \mathbf{T}_{x_k} < \mathbf{T}_{x_i} < \mathbf{T}_{z_l}, \mathbf{T}_{x_m}, \quad \text{for } i < j,\ i < k,\ l \leq i,\ m < i, \tag{2.45}$$

verifying the incompatibilities (2.39) and (2.40). We finally need to remark that, in (2.36), we have $\mathbf{T}_{z_i}, \mathbf{T}_{z_{i-1}} < \mathbf{I}_{x_i}^+$ and, in (2.37), we have $\mathbf{I}_{z_i}^-, \mathbf{I}_{z_{i-1}}^- < \mathbf{T}_{x_i}$. Both paired with (2.45) verify, respectively, (2.41) and (2.42).

The only missing incomparabilities are the ones of the type (2.43), which are satisfied because for all i, in (2.36) we have $\mathbf{T}_{z_i} < \mathbf{T}_{z_{i+1}}$ and in (2.38) we have $\mathbf{T}_{z_{i+1}} < \mathbf{T}_{z_i}$. $\square$

For the crown $\mathscr{C}_3$, it is the same formula stated in Proposition 2.22 but without the middle products of (2.36), (2.37) and (2.38). The proof is exactly the same but we state this particular case separately as we will need to treat $\mathscr{C}_3$ separately in the last section.

Proposition 2.23 *The following three linear orders form a realizer of* $\boldsymbol{P}_3$*:*

$$I_{z_1}^- \; I_{x_2}^- \; X_1^- \; X_2^+ \; Z_1^+ \; (X_3^- \; Z_2^- \; I_{z_2}^+) \; Z_3^\bullet \; I_{x_3}^+ \; I_{x_1}^+, \tag{2.46}$$

$$I_{z_2}^- \; X_2^\bullet \; (I_{x_3}^- \; X_3^+ \; Z_2^+) \; X_1^\bullet \; Z_3^- \; Z_1^\bullet \; I_{z_3}^+, \tag{2.47}$$

$$I_{z_3}^- \; I_{x_1}^- \; X_1^+ \; X_3^\bullet \; Z_3^+ \; X_2^- \; Z_2^\bullet \; Z_1^- \; I_{z_1}^+ \; I_{x_2}^+. \tag{2.48}$$

Proposition 2.24 *The following three linear orders form a realizer of* $\boldsymbol{P}_2$*:*

$$X_1^- \; X_2^- \; Z_1^\bullet \; Z_2^\bullet \; I_{x_1}^+ \; I_{x_2}^+, \tag{2.49}$$

$$I_{z_2}^- \; I_{z_1}^- \; X_2^\bullet \; X_1^\bullet \; Z_2^+ \; Z_1^+, \tag{2.50}$$

$$I_{x_1}^- \; X_1^+ \; I_{x_2}^- \; X_2^+ \; Z_2^- \; I_{z_2}^+ \; Z_1^- \; I_{z_1}^+. \tag{2.51}$$

Proof. It is straightforward to check that each linear order is a linear extension of $\mathbf{P}$. Moreover, the restrictions of the three linear extensions to the set $\mathbf{T}_w$ form a realizer of $\mathbf{T}_w$ for any $i \in [n]$ and $w \in \{x_i, z_i\}$ (see above Proposition 2.22). The same goes for the crown $\mathscr{C}_2$, thus for $\mathscr{C}_2$ with $\mathbf{X}_i^-$ grafted to each x_i and $\mathbf{Z}_i^+$ grafted to each z_i. We only need to check the following incomparabilities

$$\mathbf{T}_{x_1} \not\sim \mathbf{T}_{x_2}, \qquad \mathbf{T}_{z_1} \not\sim \mathbf{T}_{z_2}, \qquad \mathbf{I}_{x_i}^+ \not\sim \mathbf{P} \setminus \mathbf{T}_{x_i} \text{ for } i = 1, 2, \qquad \mathbf{I}_{z_i}^- \not\sim \mathbf{P} \setminus \mathbf{T}_{z_i} \text{ for } i = 1, 2.$$

We have $\mathbf{T}_{x_1} < \mathbf{T}_{x_2}$ in (2.51) and $\mathbf{T}_{x_2} < \mathbf{T}_{x_1}$ in (2.50). Similarly, $\mathbf{T}_{z_1} < \mathbf{T}_{z_2}$ in (2.49) and $\mathbf{T}_{z_2} < \mathbf{T}_{z_1}$ in (2.51). Finally, we finish the proof of the proposition with:

$$\begin{aligned} &\mathbf{T}_{z_1}, \mathbf{T}_{z_2} < \mathbf{I}_{x_1}^+, \mathbf{I}_{x_2}^+ \quad \text{in (2.49)}, \\ &\mathbf{I}_{z_1}^-, \mathbf{I}_{z_2}^- < \mathbf{T}_{x_1}, \mathbf{T}_{x_2} \quad \text{in (2.50)}, \\ &\mathbf{T}_{x_1}, \mathbf{T}_{x_2} < \mathbf{T}_{z_1}, \mathbf{T}_{z_2} \quad \text{in (2.51)}. \end{aligned}$$

□

Proposition 2.25 *Let* $\mathcal{T} = \{(\boldsymbol{T}_x, r_x), (\boldsymbol{T}_z, r_z), (\{r_a\}, r_a), (\{r_b\}, r_b)\}$ *be a family of rooted tree posets indexed by the elements of* $\mathscr{C}_1$*, and* $\boldsymbol{P}$ *be the unicycle poset obtained from the vertex-grafting of* $\mathcal{T}$ *on* $\mathscr{C}_1$*. Then*

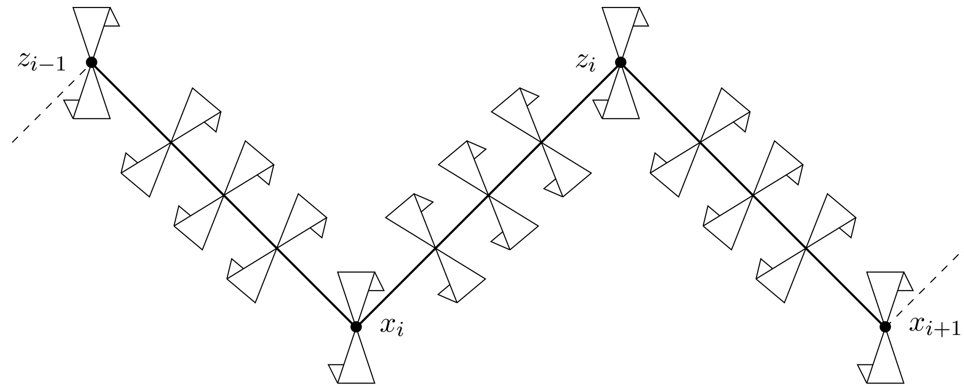


Figure 2.9: A part of a general unicycle poset whose underlying crown is $\mathscr{C}_n$ for $n \geq 2$.

the following three linear orders form a realizer of $\boldsymbol{P}$*:*

$$X^- \; a \; b \; Z^\bullet \; I_x^+, \tag{2.52}$$

$$I_z^- \; X^\bullet \; b \; a \; Z^+, \tag{2.53}$$

$$I_x^- \; X^+ \; a \; b \; Z^- \; I_z^+. \tag{2.54}$$

Proof. It is straightforward to check that these linear orders are linear extensions of $\mathbf{P}$ with $\mathscr{C}_1$ as a subposet. The restrictions of the three linear extensions to the set $\mathbf{T}_w$ form a realizer of $\mathbf{T}_w$ for any $w = x, z$ (see above Proposition 2.22). Moreover, by Corollary 2.16, we know that the restrictions of these linear extensions form a realizer for both $\mathbf{X}^- \sqcup \mathbf{Z}^+ \sqcup \{a, x, z\}$ and $\mathbf{X}^- \sqcup \mathbf{Z}^+ \sqcup \{b, x, z\}$. We only need to check that $\mathbf{I}_x^+ \not\sim \mathbf{P} \setminus \mathbf{T}_x$ and $\mathbf{I}_z^- \not\sim \mathbf{P} \setminus \mathbf{T}_z$, which is easy to see. $\square$

2.10 Grafting on cycle posets

Joining the work done in the previous sections, we now build, for any unicycle poset $\mathbf{P}$, a realizer of $\mathbf{P}$ of size 3. As proved in Section 2.5, a unicycle poset $\mathbf{P}$ can be obtained by grafting rooted trees $\mathcal{T}(\mathbf{C}_n)$ on a cycle poset $\mathbf{C}_n$. Moreover, $\mathbf{C}_n$ can be itself built by adding elements between two elements that form a cover of its underlying crown $\mathscr{C}_n$ (Lemma 2.7). We merge the realizers described in Section 2.8 with the ones described in Section 2.9. Note that in the next result, the first two linear orders are split in two lines.

Proposition 2.26 *Let* $n \geq 4$*. Let* $\boldsymbol{C}_n$ *be a cycle poset,* $\mathcal{T}(\boldsymbol{C}_n)$ *be a family of rooted trees indexed by* $\boldsymbol{C}_n$*, and* $\boldsymbol{P}$ *be the unicycle poset obtained from the grafting of* $\mathcal{T}(\boldsymbol{C}_n)$ *on* $\boldsymbol{C}_n$*. The following three linear orders form* *a realizer of **P**:*

$$I^-_{z_1}\ I^-_{x_2}\ A_1(X^-_1)\ X^+_2\ B_2(Z^+_1)\ C_{2n}\ \left(C_3\ A_4(X^-_3)\ Z^-_2\ I^+_{z_2}\right)\prod_{i=3}^{n-1}\Big(C_{2i-1}\ A_{2i}(X^-_{i+1})\ Z^-_i\ I^+_{z_i}\ I^+_{x_i}\Big) \qquad B_{2n-1}(Z^\bullet_n)\ I^+_{x_n}\ I^+_{x_1}, \tag{2.55}$$

$$I^-_{z_2}\ A_3(X^\bullet_2)\ \left(I^-_{z_3}\ I^-_{x_3}\ X^+_3\ B_4(Z^+_2)\right)\ C_2\ \prod_{i=3}^{n-2}\Big(I^-_{z_{i+1}}\ I^-_{x_{i+1}}\ X^+_{i+1}\ B^{2i-1}_{2i}(Z^+_i)\Big) \qquad \left(I^-_{x_n}\ X^+_n\ B^{2n-3}_{2n-2}(Z^+_{n-1})\right)\ C_{2n-1}\ A_{2n}(X^\bullet_1)\ Z^-_n\ B_1(Z^\bullet_1)\ I^+_{z_n}, \tag{2.56}$$

$$I^-_{z_n}\ I^-_{x_1}\ X^+_1\ A_{2n-1}(X^\bullet_n)\ B_{2n}(Z^+_n)\ C_1\ \prod_{i=n-1}^{3}\Big(C_{2i}\ A_{2i-1}(X^\bullet_i)\ Z^\bullet_i\Big)\ C_4\ A_2(X^-_2)\ B_3(Z^\bullet_2)\ Z^-_1\ I^+_{z_1}\ I^+_{x_2}. \tag{2.57}$$

Proof. It is straightforward to check that these linear orders are linear extensions of **P** (see Figure 2.9). If we restrict these linear extensions respectively to the subposets considered in Propositions 2.18 and 2.22, we recover the realizers of these subposets. Let $p \in [2n]$. It suffices to prove the following incomparabilities

$$\mathbf{T}_p \not\sim \mathbf{T}_{x_j}, \mathbf{T}_{z_k}, \text{ for all } j \neq \lfloor p/2\rfloor + 1, k \neq \lceil p/2\rceil, \tag{2.58}$$

$$\mathbf{T}_p \not\sim \mathbf{I}^+_{x_{\lfloor p/2\rfloor+1}}, \mathbf{I}^-_{z_{\lceil p/2\rceil}}, \tag{2.59}$$

$$\mathbf{I}^-_p \not\sim \mathbf{T}_{x_{\lfloor p/2\rfloor+1}}, \tag{2.60}$$

$$\mathbf{I}^+_p \not\sim \mathbf{T}_{z_{\lceil p/2\rceil}}. \tag{2.61}$$

Recall that A_p, B_p and C_p are linear extensions of $\mathbf{T}_p$. They appear in the linear extensions (2.55), (2.56) and (2.57), all in different ones.

We designed our realizers such that $*$ in any $A_p(*)$ is either a $X^\bullet_{\lfloor p/2\rfloor+1}$ or $X^-_{\lfloor p/2\rfloor+1}$ but never $X^+_{\lfloor p/2\rfloor+1}$, and $*$ in any $B_p(*)$ is either a $Z^\bullet_{\lceil p/2\rceil}$ or $Z^+_{\lceil p/2\rceil}$ but never $Z^-_{\lceil p/2\rceil}$. It follows, by Remark 2.17, that in the linear extension where A_p appears, $\mathbf{I}^-_p < \mathbf{T}_{x_{\lfloor p/2\rfloor+1}}$, and in the one where B_p appears, $\mathbf{T}_{z_{\lceil p/2\rceil}} < \mathbf{I}^+_p$. One can also note that in the linear extension where C_p appears, $\mathbf{T}_{x_{\lfloor p/2\rfloor+1}} < \mathbf{T}_p < \mathbf{T}_{z_{\lceil p/2\rceil}}$. This proves (2.60) and (2.61).

We now prove (2.59). In the linear extension where $X^-_{\lfloor p/2\rfloor+1}$ appears, $\mathbf{T}_p < \mathbf{I}^+_{x_{\lfloor p/2\rfloor+1}}$. Indeed, for $i \neq 2$, X^-_i is in (2.55), and either $i = 1, n$ for which the assertion is trivial, or we have $\mathbf{T}_{2i-2} < \mathbf{T}_{2i-1} < \mathbf{I}^+_{x_i}$. Also X^-_2 is in (2.57) for which the assertion is trivial. In the one where $X^+_{\lfloor p/2\rfloor+1}$ appears, $\mathbf{T}_{x_{\lfloor p/2\rfloor+1}} < \mathbf{T}_p$. Indeed, for $i \neq 1, 2$, X^+_i is in (2.56) and $X^+_i < \mathbf{T}_{2i-2} < \mathbf{T}_{2i-1}$. We also have X^+_1 in (2.57) in which $\mathbf{T}_{x_1} < \mathbf{T}_{2n} < \mathbf{T}_1$, and $\mathbf{X}^+_2$ in (2.55) in which $\mathbf{T}_{x_2} < \mathbf{T}_2 < \mathbf{T}_3$. This proves the part $\mathbf{T}_p \not\sim \mathbf{I}^+_{x_{\lceil p/2\rceil+1}}$ of (2.59).

Similarly, in the linear extension where $Z^+_{\lceil p/2 \rceil}$ appears, $\mathbf{I}^-_{z_{\lceil p/2 \rceil}} < \mathbf{T}_p$. Indeed, for $i \neq 1, n$, Z_i^+ is in (2.56), and either $i = 2$ for which the assertion is trivial, or we have $\mathbf{I}^-_{z_i} < \mathbf{T}_{2i}, \mathbf{T}_{2i-1}$. We have that Z_1^+ and Z_n^+ appear respectively in (2.55) and (2.57) for which the assertion is trivial for both. In the one where $Z^-_{\lceil p/2 \rceil}$ appears, $\mathbf{T}_p < \mathbf{T}_{z_{\lceil p/2 \rceil}}$. Indeed, for $i \neq 1, n$, we have Z_i^- is in (2.55) in which $\mathbf{T}_{2i-1} < \mathbf{T}_{2i} < Z_i^-$. We also have Z_1^- in (2.57) in which $\mathbf{T}_{2n} < \mathbf{T}_1 < \mathbf{T}_{z_1}$ and Z_n^- in (2.55) in which $\mathbf{T}_{2n} < \mathbf{T}_{2n-1} < \mathbf{T}_{z_n}$. Thus proving the part $\mathbf{T}_p \not< \mathbf{I}^-_{z_{\lceil p/2 \rceil}}$ of (2.59).

It remains to prove (2.58).

For $p = 1$, we want $\mathbf{T}_1 \not< \mathbf{T}_{x_j}, \mathbf{T}_{z_k}$ for $j, k \neq 1$. In (2.56) and (2.57), we have respectively

$$\mathbf{T}_{x_j}, \mathbf{T}_{z_k} < \mathbf{T}_1, \qquad j \neq 1 \text{ and } k \neq 1, n.$$

$$\mathbf{T}_{x_n}, \mathbf{T}_{z_n} < \mathbf{T}_1 < \mathbf{T}_{x_j}, \mathbf{T}_{z_k}, \qquad j \neq 1, n \text{ and } k \neq 1, n.$$

The only incomparabilities that we still need to prove are $\mathbf{T}_1 \not< \mathbf{T}_{x_n}, \mathbf{T}_{z_n}$, which follow from (2.55) where $\mathbf{T}_1 < \mathbf{T}_{x_n}, \mathbf{T}_{z_n}$.

For $p = 2$, we want $\mathbf{T}_2 \not< \mathbf{T}_{x_j}, \mathbf{T}_{z_k}$ for $j \neq 2$ and $k \neq 1$. In (2.56) and (2.57), we have respectively

$$\mathbf{T}_{x_3}, \mathbf{T}_{z_2} < \mathbf{T}_2 < \mathbf{T}_{x_j}, \mathbf{T}_{z_k}, \qquad j \neq 2, 3 \text{ and } k \neq 1, 3.$$

$$\mathbf{T}_{x_j}, \mathbf{T}_{z_k} < \mathbf{T}_2 < \mathbf{T}_{z_2}, \qquad j \neq 2 \text{ and } k \neq 1, 2.$$

The only incomparabilities that we still need to prove are $\mathbf{T}_2 \not< \mathbf{T}_{x_3}, \mathbf{T}_{z_3}$, which follow from (2.55) where $\mathbf{T}_2 < \mathbf{T}_{x_3}, \mathbf{T}_{z_3}$.

For $p = 2n$, we want $\mathbf{T}_{2n} \not< \mathbf{T}_{x_j}, \mathbf{T}_{z_k}$ for $j \neq 1$ and $k \neq n$. In (2.55) and (2.56), we have respectively

$$\mathbf{T}_{x_2}, \mathbf{T}_{z_1} < \mathbf{T}_{2n} < \mathbf{T}_{x_j}, \mathbf{T}_{z_k}, \qquad j \neq 1, 2 \text{ and } k \neq 1.$$

$$\mathbf{T}_{x_j}, \mathbf{T}_{z_k} < \mathbf{T}_{2n} < \mathbf{T}_{z_1}, \qquad j \neq 1 \text{ and } k \neq 1, n.$$

The only incomparability that we still need to prove is $\mathbf{T}_{2n} \not< \mathbf{T}_{x_2}$, which follows from (2.57) where $\mathbf{T}_{2n} < \mathbf{T}_{x_2}$.

For $p = 2n-1$, we want $\mathbf{T}_{2n-1} \not< \mathbf{T}_{x_j}, \mathbf{T}_{z_k}$ for $j \neq n$ and $k \neq n$. In (2.56) and (2.57), we have respectively

$$\mathbf{T}_{x_j}, \mathbf{T}_{z_k} < \mathbf{T}_{2n-1} < \mathbf{T}_{x_1}, \mathbf{T}_{z_1}, \qquad j \neq 1 \text{ and } k \neq 1, n.$$

$$\mathbf{T}_{x_1} < \mathbf{T}_{2n-1} < \mathbf{T}_{x_j}, \mathbf{T}_{z_k}, \qquad j \neq 1, n \text{ and } k \neq n.$$

The only incomparability that we still need to prove is $\mathbf{T}_{2n-1} \not\sim \mathbf{T}_{z_1}$, which follows from (2.55) where $\mathbf{T}_{z_1} < \mathbf{T}_{2n-1}$.

For $p = 2i$ with $2 \leq i \leq n-1$, we want $\mathbf{T}_{2i} \not\sim \mathbf{T}_{x_j}, \mathbf{T}_{z_k}$ for $j \neq i+1$ and $k \neq i$. In (2.56) and (2.57), we have respectively

$$\mathbf{T}_{x_j} < \mathbf{T}_{2i} < \mathbf{T}_{x_1}, \mathbf{T}_{x_l}, \qquad 1 < j \leq i+1 \text{ and } i+1 < l.$$

$$\mathbf{T}_{x_1}, \mathbf{T}_{x_j}, \mathbf{T}_{z_k} < \mathbf{T}_{2i} < \mathbf{T}_{x_l}, \mathbf{T}_{z_m}, \qquad i < j;\ i < k;\ l \leq i;\ m < i+1.$$

The only incomparabilities that we still need to prove are $\mathbf{T}_{2i} \not\sim \mathbf{T}_{z_k}$ for $k \neq i$, which follow from (2.55) where $\mathbf{T}_{z_k} < \mathbf{T}_{2i} < \mathbf{T}_{z_m}$ for $k < i$ and $i \leq m$.

For $p = 2i-1$ with $2 \leq i \leq n-1$, we want $\mathbf{T}_{2i-1} \not\sim \mathbf{T}_{x_j}, \mathbf{T}_{z_k}$ for $j, k \neq i$. In (2.55), (2.56) and (2.57), we have respectively

$$\mathbf{T}_{x_j}, \mathbf{T}_{z_k} < \mathbf{T}_{2i-1} < \mathbf{T}_{x_l}, \mathbf{T}_{z_m}, \qquad j < i;\ k < i;\ i < l;\ i \leq m,$$

$$\mathbf{T}_{x_j} < \mathbf{T}_{2i-1} < \mathbf{T}_{x_1}, \mathbf{T}_{x_l}, \qquad 1 < j \leq i+1 \text{ and } i+1 < l,$$

$$\mathbf{T}_{z_k} < \mathbf{T}_{2i-1} < \mathbf{T}_{z_m}, \qquad 1 < k < i \text{ and } m < i.$$

Thus, we have $\mathbf{T}_{2i-1} \not\sim \mathbf{T}_{x_j}$, for all $j \neq i$, and $\mathbf{T}_{2i-1} \not\sim \mathbf{T}_{z_k}$, for all $k \neq i$.

This proves (2.58), which finishes the proof. □

The realizer for a unicycle poset that has $\mathbf{C}_3$ as an underlying cycle poset is very similar. We will have the same linear extensions (2.55) and (2.57) but without the middle product, whereas (2.56) is slightly different; we are missing the middle product, as well as $I^-_{z_3}$ from the first parenthesis and we do not have the parenthesis after the product. But the proof is very similar to that of Proposition 2.26, thus we do not rewrite it.

Proposition 2.27 *Let $\boldsymbol{C}_3$ be a cycle poset, $\mathcal{T}(\boldsymbol{C}_3)$ be a family of rooted trees indexed by $\boldsymbol{C}_3$, and $\boldsymbol{P}$ be the unicycle poset obtained from the grafting of $\mathcal{T}(\boldsymbol{C}_3)$ on $\boldsymbol{C}_3$. The following three linear orders form a realizer of $\boldsymbol{P}$:*

$$I^-_{z_1}\ I^-_{x_2}\ A_1(X^-_1)\ X^+_2\ B_2(Z^+_1)\ C_6\ C_3\ A_4(X^-_3)\ Z^-_2\ I^+_{z_2}\ B_5(Z^\bullet_3)\ I^+_{x_3}\ I^+_{x_1}, \tag{2.62}$$

$$I^-_{z_2}\ A_3(X^\bullet_2)\ I^-_{x_3}\ X^+_3\ B_4(Z^+_2)\ C_2\ C_5\ A_6(X^\bullet_1)\ Z^-_3\ B_1(Z^\bullet_1)\ I^+_{z_3}, \tag{2.63}$$

$$I^-_{z_3}\ I^-_{x_1}\ X^+_1\ A_5(X^\bullet_3)\ B_6(Z^+_3)\ C_1\ C_4\ A_2(X^-_2)\ B_3(Z^\bullet_2)\ Z^-_1\ I^+_{z_1}\ I^+_{x_2}. \tag{2.64}$$

Proposition 2.28 *Let $\boldsymbol{C}_2$ be a cycle poset, $\mathcal{T}(\boldsymbol{C}_2)$ be a family of rooted trees indexed by $\boldsymbol{C}_2$, and $\boldsymbol{P}$ the unicycle poset obtained from the grafting of $\mathcal{T}(\boldsymbol{C}_2)$ on $\boldsymbol{C}_2$. The following three linear orders form a realizer of $\boldsymbol{P}$:*

$$A_1(X_1^-)\ X_2^-\ B_2(Z_1^\bullet)\ B_4^3(Z_2^\bullet)\ I_{x_1}^+\ I_{x_2}^+, \tag{2.65}$$

$$I_{z_2}^-\ I_{z_1}^-\ A_3^2(X_2^\bullet)\ A_4(X_1^\bullet)\ Z_2^+\ B_1(Z_1^+), \tag{2.66}$$

$$I_{x_1}^-\ X_1^+\ C_4\ I_{x_2}^-\ X_2^+\ C_1\ C_3\ Z_2^-\ I_{z_2}^+\ C_2\ Z_1^-\ I_{z_1}^+. \tag{2.67}$$

Proof. It is straightforward to check that these linear orders are linear extensions of **P**. If we restrict these linear extensions respectively to the subposets considered in Propositions 2.20 and 2.24, we recover the realizers of these subposets. Let $p \in [4]$. It suffices now to prove the following incomparabilities:

$$\mathbf{I}_p^- \not< \mathbf{T}_{x_{\lfloor p/2 \rfloor+1}}, \tag{2.68}$$

$$\mathbf{I}_p^+ \not< \mathbf{T}_{z_{\lceil p/2 \rceil}}, \tag{2.69}$$

$$\mathbf{T}_p \not< \mathbf{T}_{z_{\lceil p/2 \rceil+1}}, \mathbf{I}_{x_{\lfloor p/2 \rfloor+1}}^+, \tag{2.70}$$

$$\mathbf{T}_p \not< \mathbf{T}_{x_{\lfloor p/2 \rfloor}}, \mathbf{I}_{z_{\lceil p/2 \rceil}}^+. \tag{2.71}$$

Recall that A_p, B_p and C_p each appears only once in (2.65), (2.66) and (2.67), all in different ones. One can note that, in the linear extension where A_p appears, $\mathbf{I}_p^- < \mathbf{T}_{x_{\lfloor p/2 \rfloor+1}}$, in the one where B_p appears, $\mathbf{T}_{z_{\lceil p/2 \rceil}} < \mathbf{I}_p^+$, and, in the one where C_p appears, $\mathbf{T}_{x_{\lfloor p/2 \rfloor+1}} < \mathbf{T}_p < \mathbf{T}_{z_{\lceil p/2 \rceil}}$. This proves (2.68) and (2.69).

For (2.70) and (2.71), in the case $p = 1$, one only has to look at (2.65) and (2.66). For $p \neq 1$, for the incomparabilities of (2.70), one has to look at (2.65) and (2.67). For the incomparabilities of (2.71), one has to look at (2.66) and (2.67). □

Proposition 2.29 *Let $\boldsymbol{C}_1$ be a cycle poset, $\mathcal{T}(\boldsymbol{C}_1)$ be a family of rooted trees indexed by $\boldsymbol{C}_1$, and $\boldsymbol{P}$ the unicycle poset obtained from the grafting of $\mathcal{T}(\boldsymbol{C}_1)$ on $\boldsymbol{C}_1$. The following three linear orders form a realizer of $\boldsymbol{P}$:*

$$A_1(X^-)\ B_2(Z^\bullet)\ I_x^+, \tag{2.72}$$

$$I_z^-\ A_2(X^\bullet)\ B_1(Z^+), \tag{2.73}$$

$$I_x^-\ X^+\ C_1\ C_2\ Z^-\ I_z^+. \tag{2.74}$$

Proof. It is straightforward to check that these linear orders are linear extensions of **P** (see Figure 2.3a). If we restrict these linear extensions respectively to the subposets considered in Propositions 2.21 and 2.25, we recover the realizers of these subposets. It suffices now to prove the following incomparabilities

$$\mathbf{I}_x^+, \mathbf{I}_z^- \not\sim \mathbf{T}_1, \mathbf{T}_2.$$

With (2.72) and (2.74) we obtain respectively $\mathbf{T}_1, \mathbf{T}_2 < \mathbf{I}_x^+$ and $\mathbf{T}_x < \mathbf{T}_1, \mathbf{T}_2$. Thus $\mathbf{I}_x^+ \not\sim \mathbf{T}_1, \mathbf{T}_2$. Similarly for $\mathbf{I}_z^- \not\sim \mathbf{T}_1, \mathbf{T}_2$ with (2.73) and (2.74). $\square$

CHAPITRE 3
L'EXTRÉMALITÉ DANS LES TREILLIS SEMI-DISTRIBUTIFS

3.1 Avant-Propos et Résumé

Comme nous l'avons vu dans le chapitre 1, il existe plusieurs classes importantes de treillis, ayant toutes des propriétés particulières. Pendant le doctorat, nous avons cherché à comprendre le lien entre ces différentes classes. Dans ce chapitre, nous présentons l'article compilant nos contributions à la théorie des treillis. C'est un article de 30 pages intitulé « Extremality in semidistributive lattices » qui a été déposé sur arXiv le 23 novembre 2025 [94]. Nous nous sommes particulièrement intéressés à l'extrémalité dans les treillis semi-distributifs et congruence uniformes, ce qui explique le titre de l'article.

Liu a démontré [67] qu'un treillis modulaire à gauche a un étiquetage des relations de couverture admettant plusieurs définitions équivalentes. Nous donnons une nouvelle démonstration et montrons la réciproque, à savoir que si pour un treillis les différentes définitions pour l'étiquetage sont équivalentes, alors le treillis est modulaire à gauche. Nous nous servons de ce résultat pour donner une nouvelle preuve du résultat de Thomas et Williams qu'un treillis semi-distributif extrémal est modulaire à gauche [103, Theorem 1.4]. Nous utiliserons également ce résultat pour établir au chapitre 4 que les treillis (P, ϕ)-Tamari sont modulaires à gauche.

Ensuite, nous étudions l'extrémalité et la modularité à gauche des treillis congruence normaux. Nous caractérisons ces propriétés par rapport aux doublements des sous-ensembles convexes qui permettent de construire ces treillis. Cela nous montre en particulier que lorsqu'on perd une de ces propriétés après un doublement, elle ne sera jamais récupérée par la suite quels que soient les nouveaux doublements opérés. Cela nous permet aussi de prouver qu'un treillis congruence normal modulaire à gauche est extrémal. De plus, lorsqu'on se restreint aux treillis congruence uniformes on obtient que l'extrémalité est équivalente à la modularité à gauche. Cela nous permet de comprendre l'équivalence de ces propriétés pour cette sous-classe des treillis semi-distributifs. En effet, la réciproque au résultat de Thomas et Williams cité plus haut, à savoir qu'un treillis semi-distributif modulaire à gauche est extrémal, a été établi par Henri Mühle [77, Theorem 3.2].

Ayant compris les doublements pour les treillis congruence uniformes qui mènent à l'extrémalité, nous démontrons ensuite que tout treillis congruence uniforme shellable est extrémal. Ceci permet de montrer, avec notamment le résultat de Thomas et Williams cité plus haut, que pour les treillis congruence uniformes il y a équivalence entre extrémalité, modularité à gauche, EL-shellabilité et shellabilité.

Nous nous intéressons ensuite au complexe sup-canonique défini par Barnard [6, 7]. Des questions structurelles demeurent ouvertes sur ces complexes, et nous fournissons une réponse négative à une de ces questions [7, Question 2.5.5]. En effet, nous donnons un exemple de treillis semi-distributif dont le complexe sup-canonique possède un sous-complexe induit qui n'est pas lui-même le complexe sup-canonique d'un treillis semi-distributif. Pour ce faire, nous utilisons le théorème fondamental des treillis semi-distributifs finis de Reading, Speyer et Thomas [91].

Notre dernière contribution à la théorie des treillis est une formule pour la dimension des treillis semi-distributifs extrémaux. Nous démontrons, en se servant d'un résultat de Reading établissant un lien entre la dimension et certaines décompositions du complexe simplicial des paires critiques [89], que la dimension d'un treillis semi-distributif extrémal est le nombre chromatique de son graphe sup-canonique. C'est aussi le nombre chromatique du complément de son graphe de Galois. Ceci généralise un résultat de Dilworth pour la dimension des treillis distributifs. Ceci nous permet ensuite de donner la dimension de plusieurs familles de treillis semi-distributifs extrémaux, incluant notamment les treillis bulles de McConville et Mühle [73, 74], treillis (m, n)-words de Pilaud et Poliakova [84], et treillis Tamari paraboliques de Mühle et Williams [80]. Nous terminons en montrant que le treillis des classes de torsion d'un arbre aimable à n sommets est de dimension n.

Finalement, nous terminerons en donnant des questions ouvertes.

3.2 Abstract

We establish several independent results concerning extremal, left modular, congruence uniform, and semidistributive lattices. An equivalent characterization of left modular lattices is obtained in terms of edge-labellings, together with necessary and sufficient conditions on the doubling steps in the construction of congruence normal lattices that ensure left modularity or extremality. We prove that a congruence uniform lattice is shellable if and only if it is extremal. We answer a question of Barnard by constructing a counterexample showing that an induced subcomplex of a canonical join complex need not itself be such a complex. Finally, we show that the order dimension of a semidistributive extremal lattice equals the chromatic number of the complement of its Galois graph, generalizing a theorem of Dilworth for distributive

lattices. As an application, we determine the dimensions of generalizations of the Hochschild lattice, of the parabolic Tamari lattice, and of some lattices of torsion classes.

3.3 Introduction

In this article all posets and lattices are assumed to be finite. The definitions will be given in this context. All the notions will be defined in Section 3.4.

A lattice is *extremal* if the maximum size of a chain in the lattice corresponds to the number of join-irreducible and to the number of meet-irreducible elements, which are the elements that respectively cover or are covered by a unique element. It is known that the distributive lattices are the extremal and graded lattices [70]. We will be mainly interested in the *semidistributive* lattices, which are a generalization of the distributive lattices, and to a subclass of them called *congruence uniform* lattices. These latter are the lattices that can be obtained from the one element lattice by doubling intervals using the doubling construction of A. Day [29]. By doubling convex subsets, we obtain the *congruence normal lattices*, which generalize the congruence uniform lattices but are not semidistributive, unless they are congruence uniform. Some examples of congruence uniform lattices are the distributive lattices, the Tamari lattice and most of its generalizations, and the weak order on a finite Coxeter group. With the exception of the weak order which is not an extremal lattice, the others are semidistributive and extremal and we will give more examples of such lattices in Section 3.6.

An element of a lattice is left modular if it cannot be obtained as the small side of a pentagonal sublattice. A lattice is *left modular* if it contains a maximal chain made of left modular elements [17, 68]. We call such a chain a left modular chain. As a distributive lattice does not have pentagonal sublattices, all its elements are left modular, in particular it is a left modular lattice. A poset is *shellable* when its order complex is a shellable simplicial complex, which is a nice topological property that is often difficult to establish. In fact, determining whether a given poset is shellable is NP-complete [49]. Proofs of shellability of lattices often rely on finding a particular edge-labelling called an EL-labelling [14]. Such a labelling can be obtained in a left modular lattice [67], thus a left modular lattice is shellable.

Let L be a lattice. Choosing a chain $\psi : \hat{0} = x_0 < \cdots < x_k = \hat{1}$ that contains the minimum $\hat{0}$ and maximum $\hat{1}$ of L, we get a numbering of its join-irreducible and meet-irreducible elements, that enables us to define four edge-labellings $\gamma_{1,\psi}, \gamma_{1',\psi}, \gamma_{2,\psi}, \gamma_{2',\psi}$ of L (see Definition 3.27). We will prove:

Theorem 3.1 *For any lattice L, $\gamma_{1,\psi} = \gamma_{1',\psi} = \gamma_{2,\psi} = \gamma_{2',\psi}$ if and only if for all i, x_i is left modular.*

This gives us a way to prove that lattices are left modular using edge-labellings. A first application, not present in this paper, is the proof of the left modularity of the (P, ϕ)-Tamari lattices defined recently by the author [95]. Another application is a new short proof (Theorem 3.29) of the fact that every semidistributive extremal lattice is left modular [103, Theorem 1.4]. The converse statement, that any semidistributive left modular lattice is extremal, was proved in [77, Theorem 3.2]. Thus for semidistributive lattices, extremality and left modularity are equivalent properties. We were interested in understanding this equivalence in the case of the congruence uniform lattices. For this, we obtain necessary and sufficient conditions on the doubling steps in the construction of a congruence normal lattice that ensure extremality or left modularity. To state these results we need the following two definitions. The *spine* of a lattice L is the subset of elements that lie on some maximum length chain. We define the *heart* of a subset C of L to be the subset of elements that are less than all the maximal elements of C and bigger than all its minimal elements.

Theorem 3.2 *A congruence uniform lattice is extremal if and only if at each doubling step in its construction, the interval to be doubled intersects the spine of the lattice we are doubling. A congruence normal lattice is left modular if and only if at each doubling step in its construction, the heart of the convex subset to be doubled intersects a left modular chain of the lattice we are doubling.*

From Theorem 3.2 we recover the equivalence between extremality and left modularity for congruence uniform lattices (Corollary 3.48), but we will be more precise and prove more general results in Section 3.5.2.

As any lattice can be seen as an interval in an extremal lattice [70, Theorem 14] and shellability is preserved under taking intervals [13, Proposition 4.2], being extremal is far from being equivalent to being shellable for general lattices. However, building on the above result regarding the extremality of a congruence uniform lattice, we obtain:

Theorem 3.3 *A congruence uniform lattice is shellable if and only if it is extremal.*

We said that establishing shellability is often difficult, but Theorem 3.3 proves that for congruence uniform lattices this is easy as extremality can be obtained in polynomial time for any lattice. Furthermore, with known results explained earlier, it follows from Theorem 3.3:

Corollary 3.4 *Let L be a congruence uniform lattice. The following properties are equivalent:*

(i) *L is extremal.*

(ii) *L is left modular.*

(iii) *L is EL-shellable.*

(iv) *L is shellable.*

Part of what was known, that an extremal semidistributive lattice is left modular, has led to recent results on shellability of lattices [40, 32, 69]. What we add with Theorem 3.3 is a way to prove that certain lattices are not shellable, as now we know that any congruence uniform lattice that is not extremal is not shellable.

In a semidistributive lattice, each element can be obtained canonically as the join of some join-irreducible elements. These subsets of join-irreducible elements are called the canonical join representations, and the set of all of them form a simplicial complex called the *canonical join complex* [6]. Different structural properties of these simplicial complexes were proved, but open questions remain. In Section 3.5.4, we give a counterexample to [7, Question 2.5.5], by exhibiting a join canonical complex that has an induced subcomplex which is not itself a canonical join complex. This counterexample is produced using the fundamental theorem of finite semidistributive lattices [91].

Finally, we are interested in the (order) dimension of semidistributive extremal lattices. Let $(\mathbb{R}^n, \leq)$ be the poset defined by $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if $x_i \leq y_i$ for every $i \in [n]$. The *dimension* of a poset P is the smallest integer n such that P is isomorphic to a subposet of $(\mathbb{R}^n, \leq)$ ([38]). This notion was extensively studied over the years (see the book [105]), but it is difficult to obtain the dimension in general, as it is even an NP-complete problem to determine if a given poset is of dimension n, for any $n \geq 3$. In the past, this problem was linked to the coloring of graphs and hypergraphs [109, 45, 42, 90], but often to obtain bounds on the dimension.

The *Galois graph* of an extremal lattice is a directed graph on its join-irreducible elements. It is acyclic and from it we recover the lattice as the poset of inclusion on its maximal orthogonal pairs [70, 91, 103, 102].

Building on results of [89] which relate the dimension to cover sets of a simplicial complex called the critical pairs simplicial complex, and results about the Galois graphs of extremal lattices, we obtain the following:

Theorem 3.5 *The dimension of a semidistributive extremal lattice is the chromatic number of the complement of its Galois graph.*

This generalizes a result of Dilworth that the dimension of a distributive lattice is the maximum size of an antichain of its subposet of join-irreducible elements (see Corollary 3.79).

The complement of the Galois graph is often explicitly described for semidistributive lattices, as it corresponds to the *canonical join graph*, which is the 1-skeleton of its canonical join complex [6, 103]. Thus, using Theorem 3.5 we are able to give the dimension of different families of semidistributive extremal lattices, and we expect that this result could be applied effectively to obtain the dimension of other families of such lattices. We will focus on some recent families of semidistributive extremal lattices appearing in algebraic combinatorics, mainly generalizations of the Hochschild lattices and of the Tamari lattice. For the latter, this includes results about the dimension of some lattices of torsion classes. In particular, we prove that the lattice of torsion classes of a gentle tree on n vertices has dimension n (Proposition 3.97).

We start with background on the relevant poset, lattice and representation theory in Section 3.4. Then Section 3.5 gives our main results. In Section 3.5.1 we prove Theorem 3.1. In Section 3.5.2, we prove Theorem 3.2 and more results relating the doubling operation to extremality and left modularity. In Section 3.5.3, we focus on shellability and prove Theorem 3.3. Then in Section 3.5.4 we give a counterexample to [7, Question 2.5.5]. We then focus on the dimension of semidistributive extremal lattices in Section 3.5.5, proving Theorem 3.5. Finally, we give applications of Theorem 3.5 in Section 3.6, and finish by giving a list of open questions in Section 3.7.

Some of the results from Sections 3.5.1 and 3.5.2 appeared, without proofs, in an extended abstract accepted to FPSAC 2025 [95].

Acknowledgments

I am very grateful to my advisors Samuele Giraudo and Hugh Thomas. I also want to thank Antoine Abram for helpful discussions, Félix Reutenauer who used a supercomputer to run my program to obtain Proposition 3.93, and Clément Chenevière for pointing out some known results on the Cambrian lattices. I was supported by NSERC Discovery Grants RGPIN-2022-03960 and RGPIN-2024-04465.

3.4 Background

3.4.1 Vocabulary on posets and lattices

We denote by $|E|$ the cardinality of a set E. For a positive integer n, we denote $\{1, 2, \dots, n\}$ by $[n]$. The terms *smaller*, *bigger*, *below* and *above* will always refer to weak relations.

If a partially ordered set (poset) $(P, \leq)$ has a minimum, it will always be denoted $\hat{0}$, and $\hat{1}$ for the maximum. The cover relations of P are denoted $x \lessdot y$ and we say that y *covers* x or x *is covered by* y. They form the set $E(P)$ of the edges of its Hasse diagram, which is drawn with smaller elements at the bottom. The interval of P between x and y is denoted by $[x, y]$. A subset C of P is *convex* if for all x and y in C, we have $[x, y] \subseteq C$. An *order ideal* of a poset P is a subset I such that for all $x, y \in P$, if $y \leq x$ and $x \in I$, then $y \in I$. The order ideal generated by a subset C is denoted by $I_P(C) := \{y \in P \mid \exists x \in C,\, y \leq x\}$. If $C = \{x\}$, we write $I_P(x)$. Dually, an *order filter* is a subset F such that for all $x, y \in P$, if $y \geq x$ and $x \in F$, then $y \in F$. The order filter generated by a subset C is denoted by $F_P(C) := \{y \in P \mid \exists x \in C,\, y \geq x\}$. If $C = \{x\}$, we write $F_P(x)$. The *chains* are the totally ordered subsets of P.

A chain F of $n + 1$ elements is said to have length n, which is denoted by $\ell(F)$. If we cannot add elements to a chain, it is called a *maximal chain*. The *length* of a poset, denoted $\ell(P)$, is the maximum length of a chain in P. The chains of length $\ell(P)$ are called *longest chains*. The *spine* of a poset is the subset of the elements that lie on some longest chain. If P and Q are two posets, then the *direct product* $P \times Q$ is the poset on the Cartesian product $P \times Q$ defined by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

A *lattice* L is a poset such that any pair of elements $\{x, y\}$ admits a least upper bound, called the *join* and written $x \vee y$, and a greatest lower bound, called the *meet* and written $x \wedge y$. A *join-irreducible* $j \in L$ is an element that covers a unique element, denoted j_*, and a *meet-irreducible* $m \in L$ is one that is covered by a unique element, denoted m^*. The sets of these elements are respectively denoted $\mathrm{JIrr}(L)$ and $\mathrm{MIrr}(L)$. An *edge-labelling* of L is a map $\gamma : E(L) \to P$ where P is a poset. We will say that $x \in L$ is *incomparable* to a subset $Y \subseteq L$ if x is incomparable to any element of Y.

3.4.2 Extremal lattices

It is well known that in a lattice, an element is always the join of the join-irreducibles below it.

Lemma 3.6 *For any lattice L we have $\ell(L) \leq \min\big(|\mathrm{JIrr}(L)|, |\mathrm{MIrr}(L)|\big)$.*

Proof. Denote $k = \ell(L)$. Let $\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_k = \hat{1}$ be a maximal chain of length k. For any $i \in [k-1]$, there is always a join-irreducible that is below c_{i+1} but not c_i. If it was not the case, then all the join-irreducibles below c_{i+1} would be below c_i, thus c_{i+1} which is the join of them would be below c_i, which is absurd. It follows that $k \leq |\mathrm{JIrr}(L)|$. A dual argument gives $k \leq |\mathrm{MIrr}(L)|$. $\square$

The previous result motivated the following definition by G. Markowsky [70]:

Definition 3.7 *A lattice L is* join-extremal *if $\ell(L) = |\mathrm{JIrr}(L)|$,* meet-extremal *if $\ell(L) = |\mathrm{MIrr}(L)|$ and* extremal *if it is both join-extremal and meet-extremal.*

Two notable results on these lattices are that the graded extremal lattices are the distributive lattices [70, Theorem 17], and any lattice is isomorphic to an interval of an extremal lattice [70, Theorem 14]. G. Markowsky also proved a representation theorem of extremal lattices, that we now explain with recent terminology (see [103, Section 2.3]). For an extremal lattice L, the choice of any longest chain $\psi : \hat{0} = x_0 \lessdot \cdots \lessdot x_n = \hat{1}$ gives a numbering of the join and meet-irreducibles $j_1, j_2, \ldots, j_n$ and $m_1, m_2, \ldots, m_n$ such that $x_i = j_1 \vee \cdots \vee j_i = m_{i+1} \wedge \cdots \wedge m_n$.

Definition 3.8 ([103]) *Let L be an extremal lattice. We define a directed graph $G(L)$, called the* Galois graph*, whose vertices are $\mathrm{JIrr}(L)$ and with an edge $j_i \to j_k$ if $i \neq k$ and $j_i \not\leq m_k$.*

We defined the Galois graph $G(L)$ on the set $\mathrm{JIrr}(L)$ as that will be the way we see it in Section 3.5.5, but with our numbering of join and meet-irreducibles, it can be equivalently seen as the graph on $[n]$ with directed edges $i \to k$ if $i \neq k$ and $j_i \not\leq m_k$.

Proposition 3.9 ([70, Theorem 11]) *For any extremal lattice L, the Galois graph $G(L)$ is acyclic.*

Let G be a directed graph on $[n]$ without multiple edges and such that whenever $i \to k$ we have $i > k$. A *maximal orthogonal pair* of G is a pair (X, Y) of disjoint subsets of $[n]$ with no directed edges from X to Y, and which is maximal for this condition. With the partial order $(X, Y) \leq (X', Y')$ if $X \subseteq X'$, the maximal orthogonal pairs form a lattice denoted by $L(G)$ (see Figure 3.9).

Theorem 3.10 ([70]) *Every finite extremal lattice L is isomorphic to $L(G(L))$. Conversely, if G is a directed graph on $[n]$ without multiple edges and such that whenever $i \to k$ we have $i > k$, then $L(G)$ is an extremal lattice.*

This result will enable us in Section 3.6 to define quickly an extremal lattice by giving its Galois graph.

3.4.3 Semidistributive lattices

Definition 3.11 *A lattice L is* join-semidistributive *if for all $x, y, z \in L$, we have $x \vee y = x \vee z \Longrightarrow x \vee (y \wedge z) = x \vee y$. It is* meet-semidistributive *if for all $x, y, z \in L$, we have $x \wedge y = x \wedge z \Longrightarrow x \wedge (y \vee z) = x \wedge y$. It is* semidistributive *if it is both join-semidistributive and meet-semidistributive.*

The following is well-known:

Lemma 3.12 ([44, Section 5]) *A lattice L is join-semidistributive if and only if for all covers $b \lessdot c$, the set $I_L(c) \setminus I_L(b)$ has a minimum element, denoted $\gamma_J(b \lessdot c)$. It is meet-semidistributive if and only if for all covers $b \lessdot c$, the set $F_L(b) \setminus F_L(c)$ has a maximum element, denoted $\gamma_M(b \lessdot c)$. In these cases, the minimum is join-irreducible and the maximum meet-irreducible.*

Thus by Lemma 3.12, if L is semidistributive we have two edge-labellings γ_J and γ_M. In this case, the map $\kappa : j \in \mathrm{JIrr}(L) \mapsto \gamma_M(j_* \lessdot j) \in \mathrm{MIrr}(L)$ is a bijection [44, Theorem 2.54].

Proposition 3.13 ([6] and [103, Proposition 6.2]) *For all $b \lessdot c$ in a semidistributive lattice L, we have $\kappa\big(\gamma_J(b \lessdot c)\big) = \gamma_M(b \lessdot c)$.*

Recall as explained earlier that if L is extremal, choosing a longest chain of L gives a numbering of the join-irreducible elements $j_1, \ldots, j_n$ and the meet-irreducible elements $m_1, \ldots, m_n$. The next statement follows from Proposition 3.13:

Proposition 3.14 *Let L be a semidistributive extremal lattice. We have $\kappa(j_i) = m_i$ for all $i \in [n]$. Thus the Galois graph $G(L)$ is the directed graph whose vertices are $\mathrm{JIrr}(L)$ and there is an edge $i \to j$ if $i \neq j$ and $i \not\leq \kappa(j)$ (we use the same convention as explained after Definition 3.8; the join-irreducible j_i being replaced by the integer i for all $i \in [n]$).*

Let L be a semidistributive lattice. For any $x \in L$, we denote $D(x) := \{\gamma_J(y \lessdot x) \mid y \in\ L,\ y \lessdot x\}$ and call it the *downward label set* of x. The canonical join complex is often defined using the notion of canonical join representations, but the following definition is equivalent:

Definition 3.15 ([6]) *The* canonical join complex *of a semidistributive lattice is the set of downward label sets.*

It is more generally defined for join-semidistributive lattices, but in this article we will always assume that the canonical join complexes come from a semidistributive lattice. The name is justified by the fact that it is a simplicial complex ([88, Proposition 2.2]). The *clique complex* of a graph G is the simplicial complex whose faces are the cliques of G. The canonical join complex is flag ([6, Theorem 1]), which by definition means that it is the clique complex of its 1-skeleton. This 1-skeleton is called the *canonical join graph* of the lattice.

The *complement* of a (directed) graph H, denoted by $\overline{H}$, is the simple undirected graph on the same vertices where we have an edge between two vertices if there are no (directed) edges between the two in H. Thus, the cliques of $\overline{H}$ are the independence sets of H.

Lemma 3.16 ([103, Corollary 6.7]) *The canonical join graph of a semidistributive extremal lattice is the complement of its Galois graph.*

3.4.4 Left modular lattices

The left modular lattices were first defined and studied by A. Blass and B. Sagan [17].

Definition 3.17 *An element $a \in L$ is* left modular *if for all $b < c$ in L, we have $(b \vee a) \wedge c = b \vee (a \wedge c)$ (we gave an equivalent definition in the introduction, which is part of the next result). A maximal chain made of left modular elements is called a* left modular chain. *The lattice L is called left modular if there exists a left modular chain.*

Proposition 3.18 ([68]) *Let $a \in L$ be an element of a lattice L. The following statements are equivalent :*

(1) *The element a is left modular.*

(2) *For all $b < c$, we have $a \vee b \neq a \vee c$ or $a \wedge b \neq a \wedge c$.*

(3) *For all $b \lessdot c$, we have $a \vee b \neq a \vee c$ or $a \wedge b \neq a \wedge c$.*

(4) *There is no pentagonal sublattice N_5 of L such that a is the middle element on the smaller maximal chain of N_5.*

If $b \lessdot c$ is such that (3) of Proposition 3.18 is not satisfied for a, we will say that $b \lessdot c$ is a *counterexample to the left modularity* of a.

H. Thomas defined a lattice to be *trim* if it is extremal and left modular [100]. Trim lattices form a class of lattices, closed under taking intervals and quotients, that include in particular the Tamari lattices and more generally the finite Cambrian lattices. With N. Williams, they obtained additional results on them, including the fact that extremal semidistributive lattices are trim [103, Theorem 1.4]. We will give a new proof of this result in Section 3.5.1.

3.4.5 Congruence normal lattices

We will use the following doubling construction of A. Day [29]:

Definition 3.19 *Let C be a convex subset of a lattice L. Let $\{0,1\}$ be the chain on two elements with order $0 < 1$. We define the* doubling *$L[C]$ to be the subposet of $L \times \{0,1\}$ on the subset*

$$\Big(I_L(C) \times \{0\}\Big) \bigsqcup \Big[\Big((L \setminus I_L(C)) \cup C\Big) \times \{1\}\Big].$$

In fact $L[C]$ is a lattice with meet and join given for $(x,\epsilon),(y,\epsilon') \in L[C]$ by

$$(x,\epsilon) \vee (y,\epsilon') = \begin{cases} (x \vee y, \max(\epsilon,\epsilon')) & \textit{if } x \vee y \in I_L(C) \\ (x \vee y, 1) & \textit{otherwise,} \end{cases}$$

$$(x,\epsilon) \wedge (y,\epsilon') = \begin{cases} (x \wedge y, \min(\epsilon,\epsilon')) & \textit{if } x \wedge y \in ((L \setminus I_L(C)) \bigcup C) \\ (x \wedge y, 0) & \textit{otherwise.} \end{cases}$$

We have a surjective poset morphism $L[C] \to L$ that sends (x,ϵ) to x, and an injective poset morphism $\psi : L \to L[C]$ that sends $a \in L$ to $(a,0)$ if $a \in I_L(C)$ or to $(a,1)$ otherwise. If F is a maximal chain of $L[C]$, we say that F *goes through the doubling* of C if there exists $c \in C$ such that $\{(c,0),(c,1)\} \subseteq F$. See Figure 3.2 for examples of the doubling construction.

From the definition of the doubling construction, the next two results follow.

Lemma 3.20 ([23, Lemma 1]) *The join-irreducible elements of $L[C]$ are of two types; each $j \in \mathrm{JIrr}(L)$ gives $(j,\epsilon) \in \mathrm{JIrr}(L[C])$ with $\epsilon = 0$ if $j \in I_L(C)$ and $\epsilon = 1$ otherwise, and we also get $(p,1) \in \mathrm{JIrr}(L[C])$ for any minimal element p of C.*

Similarly the meet-irreducible elements of $L[C]$ are of two types; each $m \in \mathrm{MIrr}(L)$ gives $(m,\epsilon) \in \mathrm{MIrr}(L[C])$ with $\epsilon = 0$ if $m \in I_L(C) \setminus C$ and $\epsilon = 1$ otherwise, and we also get $(q,0) \in \mathrm{MIrr}(L[C])$ for any maximal element q of C.

Lemma 3.21 *We have $\ell(L[C]) = \ell(L)$ if C does not intersect the spine of L, or $\ell(L[C]) = \ell(L) + 1$ otherwise.*

A subset C is a *lower pseudo-interval* if C is a union of intervals sharing the same minimum element. It is an *upper pseudo-interval* if it is a union of intervals sharing the same maximum element. A lattice L is *congruence normal* if it is obtained from the one element lattice by successive doublings of convex subsets. We write $L = E[C_1, C_2, \ldots, C_n]$ for the lattice obtained from the one element lattice E by doubling successively $C_1, C_2, \ldots, C_n$, which means $E[C_1, \ldots, C_{i+1}] = E[C_1, \ldots, C_i][C_{i+1}]$ for all i where C_{i+1} is a convex subset of $E[C_1, \ldots, C_i]$. We will always assume that $C_i \neq \emptyset$ for all i. If C_i is a lower pseudo-interval for all i then L is *join-congruence uniform*. If all the C_i are upper pseudo-intervals then L is *meet-congruence uniform*. If all the C_i are intervals, L is called *congruence uniform*[1].

Lemma 3.22 ([29]) *A join-congruence uniform lattice is join-semidistributive. A meet-congruence uniform lattice is meet-semidistributive. A congruence uniform lattice is semidistributive.*

3.4.6 Shellability

We refer the reader to the papers of A. Björner and M. Wachs [14, 15, 16] for details related to the following topological notions. The *order complex* $\Delta(P)$ of a poset P is the simplicial complex on vertex set P whose faces are the chains of P. Thus the facets of $\Delta(L)$ are the maximal chains of L. The *dimension* of a face F is $\dim(F) := |F| - 1$. For any face F of a simplicial complex, we denote by $\langle F \rangle := \{G \mid G \subseteq F\}$ the subcomplex generated by F. A simplicial complex is *shellable* if there exists a linear order of its facets $F_1, F_2, \ldots, F_k$ such that for any $1 \leq j < k$, the complex

$$\Big(\bigcup_{i=1}^{j} \langle F_i \rangle\Big) \bigcap \langle F_{j+1} \rangle \tag{3.1}$$

is pure of dimension $\dim(F_{j+1}) - 1$. We call such a linear order on the facets a *shelling* of $\Delta(L)$. Equivalently, $F_1, F_2, \ldots, F_k$ is a shelling if and only if for all $1 \leq i < j \leq k$, there exists $l < j$ such that $F_i \cap F_j \subseteq F_l \cap F_j$ and $|F_l \cap F_j| = |F_j| - 1$. We will say that L is shellable if $\Delta(L)$ is shellable. It is an easy fact that the first simplex of a shelling is always of longest length. If we have an edge-labelling, a

[1] *Join-congruence uniform* and *meet-congruence uniform* lattices are often called respectively *lower-bounded* and *upper-bounded* in the literature, but it conflicts with the term *bounded* poset which usually refers to a poset with a minimum and a maximum.

maximal chain gives a word by concatenating the labels of the edges we see by following the chain from bottom to top. In this context we call a maximal chain *increasing* if the associated word is increasing. An EL*-labelling* of a lattice L is an edge-labelling such that in any interval, when reading the labels following the chains from bottom to top, there is a unique maximal increasing chain and the label word of the increasing chain lexicographically precedes the label word of any other maximal chains. Denote $\overline{L} := L \setminus \{\hat{0}, \hat{1}\}$. If L admits an EL-labelling, then the order complex $\Delta(\overline{L})$ is shellable and homotopy equivalent to a wedge of spheres. Any left modular lattice admits an EL-labelling (see Remark 3.32). For any left modular chain, there exists a shelling of $\Delta(L)$ that starts with this chain, thus the left modular chains are always of longest length. It follows from the fact that all semidistributive extremal lattice are trim, thus left modular, that $(i) \implies (ii) \implies (iii) \implies (iv)$ of Theorem 3.4 is true for semidistributive lattices. By Lemma 3.22, these implications are true in particular for congruence uniform lattices.

3.4.7 Order dimension

We now recall what is the (order) dimension of a poset P. The book [105] of W.T. Trotter is a good reference.

Definition 3.23 ([38]) *Let* $(\mathbb{R}^n, \leq)$ *be the poset defined by* $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ *if* $x_i \leq y_i$ *for every* $i \in [n]$*. The* dimension *of a poset* P *is the smallest integer* n *such that* P *is isomorphic to a subposet of* $(\mathbb{R}^n, \leq)$.

Equivalently, this is the minimum number of linear extensions whose intersection is P, where the intersection is the poset on the set P whose relations are $x \leq y$ if in each of the chosen linear extensions we have $x \leq y$. The posets of dimension smaller than 3 include, for example, the planar posets with a minimum [107], the interval orders [85] and the posets whose Hasse diagrams contain at most one cycle [1]. It seems that for families of lattices the dimension is easier to determine. For example, a lattice is planar if and only if its dimension is smaller than 2 [5]. This is false for posets since there exist planar posets of any dimension [63]. For some families of semidistributive extremal lattices, we will be able to find their dimension in Section 3.6.

We finish by recalling a general bound on the dimension of a poset P. The *width* of a poset P, denoted by $\mathrm{width}(P)$, is the biggest size of an antichain of P. Denote by $\mathrm{Dis}(P)$ the subposet of P on its dissectors, where a *dissector* is an element x such that $P \setminus [x, \hat{1}]$ has a maximum element.

Proposition 3.24 ([89, Theorem 6]) *For any poset P, we have*

$$\mathrm{width}(\mathrm{Dis}(P)) \leq \dim(P) \leq \mathrm{width}(\mathrm{JIrr}(P)).$$

N. Reading gives two proofs of his Proposition 3.24. One of these proofs use results that we will recall in Section 3.5.5. This implies the following result, which is proved similarly in [73, Lemma 4.22].

Lemma 3.25 *Let L be a semidistributive lattice such that the maximum number of elements that cover an element of L is n. Then $\dim(L) \geq n$.*

Proof. Let $x \in L$ be an element covered by n elements. Let $L' = [x, \hat{1}]$. Then L' is a semidistributive lattice. Any atom of L' is a dissector. Indeed, let y be an atom of L'. We have $F_{L'}(x) \setminus F_{L'}(y) = L' \setminus [y, \hat{1}]$. Since L' is semidistributive and $x \lessdot y$ is a cover relation, it follows from Lemma 3.12 that $F_{L'}(x) \setminus F_{L'}(y)$ has a maximum element. Thus $L' \setminus [y, \hat{1}]$ has a maximum, which means that y is a dissector. Since each of the n atoms of L' is a dissector, then $\dim(L) \geq \dim(L') \geq \mathrm{width}(\mathrm{Dis}(L')) \geq n$, which finishes the proof. □

3.4.8 Representation theory of algebras

We finish this background by recalling some vocabulary of the representation theory of associative algebras, which will be only used in Section 3.6.4. A good reference is [2], and for gentle algebras we recommend [22, 83].

Let A be a finite-dimensional $\mathbb{K}$-algebra, where $\mathbb{K}$ is an algebraically closed field. We denote by $\mathrm{mod}A$ the category of finite-dimensional right A-modules. Subcategories are assumed to be full and closed under finite direct sums and summands. The *torsion classes* are the subcategories of $\mathrm{mod}A$ closed by extensions and quotients. They form a complete lattice with meet given by intersection, denoted by $\mathrm{Tors}(A)$, that was extensively studied [35, 101].

A module is called *indecomposable* if it is not isomorphic to a direct sum of two non-zero modules. A *brick* B of $\mathrm{mod}A$ is a A-module such that $\mathrm{End}(B) \cong \mathbb{K}$. Two bricks B and B' are *hom-orthogonal* if $\mathrm{Hom}(B, B') = 0$ and $\mathrm{Hom}(B', B) = 0$. A *semibrick* is a set $\{B_i\}_{i \in I}$ of pairwise hom-orthogonal bricks. The algebra A is *representation finite* if it has a finite number of non-isomorphic indecomposable modules.

A *cycle* in the category $\operatorname{mod} A$ is a sequence of indecomposable modules $M_0, M_1, \ldots, M_r = M_0$ and non-isomorphisms $f_i : M_i \to M_{i+1}$ for $0 \le i \le r-1$ where $r > 0$. It is known that if A is representation finite and has no cycles, then its indecomposable modules are the bricks [37]. Moreover, in this case $\operatorname{Tors}(A)$ is finite [34].

A *quiver* $Q = (Q_0, Q_1, s, t)$ is a directed graph whose vertices are Q_0 and its edges, called arrows, are Q_1. Moreover, s and t are two maps from Q_1 to Q_0 that respectively specify what is the source and target of an arrow. We will assume that Q is finite and connected. A *path* of length $m \ge 1$ is a finite sequence of arrows $\gamma_{1,\psi}\gamma_2 \ldots \gamma_m$ with $t(\gamma_i) = s(\gamma_{i+1})$ for all $i \in [m-1]$. We denote by $\mathbb{K}Q$ the path algebra of Q and by R_Q its ideal generated by arrows. A zero relation in Q is given by a path $\gamma_{1,\psi}\gamma_2 \ldots \gamma_m$ with $m \ge 2$. A two sided ideal I of $\mathbb{K}Q$ is an *admissible ideal* if there exists $m \ge 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$. In the sequel I is always assumed to be an admissible ideal generated by zero relations. For background on gentle algebras see [22, 83].

Definition 3.26 *An algebra $A = \mathbb{K}Q/I$ is a* gentle algebra *if all the following properties are satisfied:*

- *The ideal I is generated by zero relations of length two.*
- *There are at most two incoming and two outgoing arrows at every vertex of Q, and there is at most one arrow β_ψ and one arrow γ such that $0 \neq \alpha\beta \in I$ and $0 \neq \gamma\alpha \in I$.*
- *For every arrow α, there is at most one arrow β_ψ and one arrow γ such that $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.*

The only examples of gentle algebras that we will see are the *gentle trees*, which are the gentle algebras whose underlying undirected graph is a tree (see Figure 3.14a). The gentle trees are representation finite and their indecomposables are certain modules called string modules (see Section 3.6.4).

A *representation* M of Q is defined by the following data: to each vertex $x \in Q_0$ we associate a $\mathbb{K}$-vector space M_x, and to each arrow $\alpha : x \to y$ in Q_1 we associate a $\mathbb{K}$-linear map $\varphi_\alpha : M_x \to M_y$. Such a representation is denoted by (M_x, φ_α). If Q has zero relations, we ask additionally that the composition of maps corresponding to a zero relation gives the zero linear map. If (M_x, φ_α) and (M'_x, φ'_α) are two representations of Q, a morphism f between these two representations is a family $f = (f_x)_{x \in Q_0}$ of $\mathbb{K}$-linear maps that satisfy $\varphi'_\alpha f_x = f_y \varphi_\alpha$ for all $\alpha : x \to y$ in Q_1. The representations of Q with their morphisms form an abelian category which is equivalent to the category $\operatorname{mod} \mathbb{K}Q$. Thus we will allow ourselves to switch between the two languages.

3.5 The main results

3.5.1 Edge-labellings and left modularity

In this section we give a result that characterizes left modularity as the equality of some edge-labellings.

Definition 3.27 *Let L be a lattice. Denote by $\psi : \hat{0} = x_0 < \cdots < x_k = \hat{1}$ a chain containing $\hat{0}$ and $\hat{1}$. Using ψ, we have four labellings $\gamma_{1,\psi}$, $\gamma_{1',\psi}$, $\gamma_{2,\psi}$ and $\gamma_{2',\psi}$ of the cover relations of L (the edges of its Hasse diagram). For $j \in \mathrm{JIrr}(L)$, denote $\delta_\psi(j) := \min\{i \mid j \le x_i\}$. For $m \in \mathrm{MIrr}(L)$, denote $\beta_\psi(m) := \max\{i \mid m \ge x_{i-1}\}$. Then, for a cover relation $b \lessdot c$ we define*

$$\gamma_{1,\psi}(b \lessdot c) := \min\{\delta_\psi(j) \mid j \text{ join-irreducible}, \ j \le c, \ j \not\le b\},$$

$$\gamma_{1',\psi}(b \lessdot c) := \max\{i \mid c \wedge x_{i-1} \le b\},$$

$$\gamma_{2,\psi}(b \lessdot c) := \max\{\beta_\psi(m) \mid m \text{ meet-irreducible}, \ m \ge b, \ m \not\ge c\},$$

$$\gamma_{2',\psi}(b \lessdot c) := \min\{i \mid b \vee x_i \ge c\}.$$

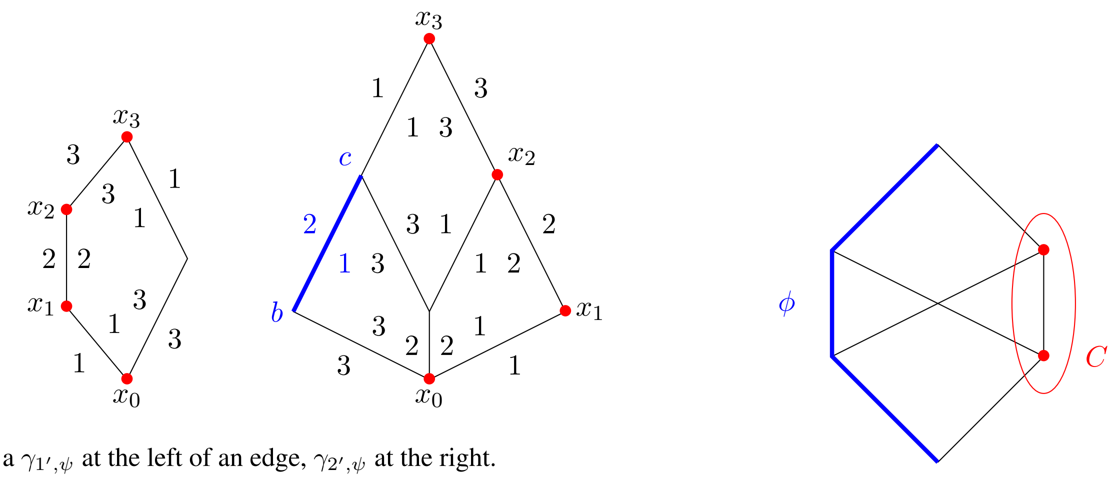
Remark 3.28 *Recall the edge-labellings γ_J and γ_M from Section 3.4. If L is semidistributive, then $\gamma_J(b \lessdot c) = j \Longrightarrow \gamma_{1,\psi}(b \lessdot c) = \delta_\psi(j)$ and $\gamma_M(b \lessdot c) = m \Longrightarrow \gamma_{2,\psi}(b \lessdot c) = \beta_\psi(m)$.*

The following was partly stated as Theorem 3.1 in the introduction:

Theorem 3.29 *Using the notations from Definition 3.27, for any lattice L, we have $\gamma_{2,\psi} = \gamma_{2',\psi} \le \gamma_{1,\psi} = \gamma_{1',\psi}$. Moreover $\gamma_{1',\psi} = \gamma_{2',\psi}$ if and only if for all i, x_i is left modular.*

Before giving the proof of Theorem 3.29, we give two lemmas.

Lemma 3.30 *Using the same notations from Definition 3.27, we have $\gamma_{1,\psi} = \gamma_{1',\psi}$ and $\gamma_{2,\psi} = \gamma_{2',\psi}$.*



a $\gamma_{1',\psi}$ at the left of an edge, $\gamma_{2',\psi}$ at the right. On the right x_1 is not left modular because of $b \lessdot c$.

b The elements of the blue maximal chain ϕ are all comparable to at least one element in C.

Figure 3.1

Proof. Let $b \lessdot c$ be a cover relation. We only prove $\gamma_{2,\psi} = \gamma_{2',\psi}$ as the argument for $\gamma_{1,\psi} = \gamma_{1',\psi}$ is dual.

Denote $i_0 := \gamma_{2',\psi}(b \lessdot c) = \min\{i \,|\, b \vee x_i \geq c\}$. It means that $b \vee x_{i_0} \geq c$ but $b \vee x_{i_0-1} \ngeq c$. Let us prove that $i_0 = \gamma_{2,\psi}(b \lessdot c)$.

First, let us prove that $\gamma_{2,\psi}(b \lessdot c) \leq i_0$. We just have to prove that for any m meet-irreducible such that $m \geq b$ and $m \ngeq c$, we have $\beta_\psi(m) \leq i_0$. Seeking a contradiction, suppose $\beta_\psi(m) > i_0$. Then by definition of β_ψ we have $m \geq x_{i_0}$ since the x_i's form a chain. Since $m \geq b$ and $m \geq x_{i_0}$, then $m \geq b \vee x_{i_0} \geq c$. It is absurd because $m \ngeq c$, which proves that $\gamma_{2,\psi}(b \lessdot c) \leq i_0$.

Now, let us prove that $i_0 \leq \gamma_{2,\psi}(b \lessdot c)$. By definition of $\gamma_{2,\psi}$, we want to find a meet-irreducible m such that $m \geq b$, $m \ngeq c$ and such that $i_0 \leq \beta_\psi(m)$. The latter condition is equivalent to $m \geq x_{i_0-1}$. So we want to find a meet-irreducible m such that $m \geq b \vee x_{i_0-1}$ and $m \ngeq c$. Such a meet-irreducible exists because $b \vee x_{i_0-1} \ngeq c$, as it was noted in the beginning. If such a meet-irreducible does not exist, then all the meet-irreducibles above $b \vee x_{i_0-1}$ would also be above c, and since an element is the meet of the meet-irreducibles above it, then $b \vee x_{i_0-1} \geq c$, which is absurd. Thus $i_0 \leq \gamma_{2,\psi}(b \lessdot c)$. This finishes the proof of $\gamma_{2,\psi} = \gamma_{2',\psi}$. $\square$

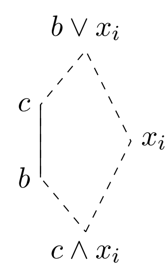
Lemma 3.31 *Using the same notation as in Definition 3.27, we have* $\gamma_{2',\psi} \leq \gamma_{1',\psi}$.

Proof. Let $b \lessdot c$ be a cover relation. Denote $i := \gamma_{2',\psi}(b \lessdot c)$. It means that $b \vee x_i \geq c$ and $b \vee x_{i-1} \ngeq c$. We want to prove that $\gamma_{1',\psi}(b \lessdot c) \geq i$, which is equivalent to $c \wedge x_{i-1} \leq b$. Seeking a contradiction, suppose that $c \wedge x_{i-1} \nleq b$. Since $b < c$ and $c \wedge x_{i-1} \leq c$, we have $b \leq b \vee (c \wedge x_{i-1}) \leq c$. Since $c \wedge x_{i-1} \nleq b$, then $b \vee (c \wedge x_{i-1}) \neq b$. Thus $b \vee (c \wedge x_{i-1}) = c$. But $b \vee x_{i-1} \geq b \vee (c \wedge x_{i-1}) = c$, which is absurd. □

Proof. [Proof of Theorem 3.29] Lemma 3.30 and Lemma 3.31 give the first part of Theorem 3.29. For the rest, using Lemma 3.31 we have

$$\gamma_{1',\psi} \neq \gamma_{2',\psi} \iff \exists i,\, \exists\, b \lessdot c, \begin{cases} \gamma_{2',\psi}(b \lessdot c) \leq i \\ \gamma_{1',\psi}(b \lessdot c) \geq i+1. \end{cases}$$

By definition of $\gamma_{1',\psi}$ and $\gamma_{2',\psi}$, it is also equivalent to the fact that there exist an i and a cover $b \lessdot c$ such that the following subposet of L is a sublattice:



Using (4) of Proposition 3.18, we deduce that $\gamma_{1',\psi} \neq \gamma_{2',\psi}$ is equivalent to the fact that there exists i such that x_i is not left modular. This finishes the proof. □

Remark 3.32 *S.-C. Liu proved in [67], for the case of a longest chain ψ, that $\gamma_{2',\psi} \leq \gamma_{1,\psi}$ and that if for all i, x_i is left modular, then $\gamma_{2',\psi} = \gamma_{1,\psi}$. He also proved that the equal labellings that we obtain starting from a left modular chain are EL-labellings. What we add to the story, other than an easier proof (by considering $\gamma_{1',\psi}$ we also add some symmetry), is a way to use these labellings to prove that lattices are left modular.*

P. McNamara and H. Thomas [75] proved that the above equal labellings of a left modular lattice are interpolating labellings (an EL-labelling with an additional property), and also proved that if a finite lattice has an interpolating labelling, then it is left modular. Our Theorem 3.29 is then of the same flavor, helping us prove left modularity, but without mentioning EL-labelling.

Recall the definition of the bijection κ from Section 3.4. We can use Theorem 3.29 to give a short proof of the following result:

Corollary 3.33 ([103, Theorem 1.4]) *Semidistributive extremal lattices are left modular.*

Proof. Since L is extremal, the choice of any longest chain $\psi :\ \hat{0} = x_0 \lessdot \cdots \lessdot x_n = \hat{1}$ gives a numbering of the join and meet-irreducibles $j_1, j_2, \ldots, j_n$ and $m_1, m_2, \ldots, m_n$ such that $x_i = j_1 \vee \cdots \vee j_i = m_{i+1} \wedge \cdots \wedge m_n$. We have $\kappa(j_i) = m_i$ for all $i \in [n]$ by Proposition 3.14. Moreover $\delta_\psi(j_i) = i$ and $\beta_\psi(m_i) = i$. Since L is semidistributive, using Proposition 3.13 we have that $\gamma_J(b \lessdot c) = j_i$ if and only if $\gamma_M(b \lessdot c) = m_i$, for any $b \lessdot c$. Thus with Remark 3.28, we have $\gamma_{1,\psi} = \gamma_{2,\psi}$. Then Theorem 3.29 gives that L is left modular. $\square$

Remark 3.34 *Theorem 3.29 was proved and first used in order to obtain the left modularity of the (P, ϕ)-Tamari lattices (defined by the author in [95]).*

3.5.2 Extremality and left modularity of congruence normal lattices

In this section, we characterize the congruence normal lattices that are extremal or left modular by necessary and sufficient conditions on each doubling step. In the sequel, unless stated otherwise, C is a convex subset of a lattice L.

Definition 3.35 *We call the* heart *of C, written $H(C)$, all the elements of C that are less than all the maximal elements of C and bigger than all its minimal elements.*

Lemma 3.36 *For any convex subset C of L,*

(i) *$H(C) \neq \emptyset$ if and only if the meet of the maximal elements is bigger than the join of the minimal elements of C. In this case, if these elements are respectively given by M_i for $i \in I$ and m_j for $j \in J$, we have $H(C) = [\vee_{i \in I} m_i\, ,\, \wedge_{j \in J} M_j]$.*

(ii) *$H(C)$ is convex.*

(iii) $H(C) = C$ *if and only if* C *is an interval.*

(iv) $H(C) = \{a \in C \,|\, \forall b \in C,\, a \vee b \in C$ *and* $a \wedge b \in C\}$.

Proof. The statement (i) is easy to prove and implies (ii) and (iii). Let us prove (iv).

Let $a \in H(C)$. Let $b \in C$. There exists a minimal element $m \in C$ and a maximal element $M \in C$ such that $m \leq b \leq M$. Since $a \in H(C)$, we have $m \leq a \leq M$. Then $a \leq a \vee b \leq M$, thus by convexity of C we have $a \vee b \in C$. Also $m \leq a \wedge b \leq a$ implies by convexity that $a \wedge b \in C$. Thus $a \in \{a \in C \,|\, \forall b \in C,\, a \vee b \in C$ and $a \wedge b \in C\}$.

Now let $a \in C$ such that for all $b \in C$, we have $a \vee b \in C$ and $a \wedge b \in C$. Let M be a maximal element and m a minimal element of C. We have by hypothesis that $a \vee M \in C$ and $a \wedge m \in C$. But since M is maximal and m is minimal in C, this happens if and only if $a \leq M$ and $a \geq m$. Then a is smaller than all the maximal elements of C and bigger than all the minimal elements of C. Thus $a \in H(C)$. $\square$

The next two results were partly stated as Theorem 3.2 in the introduction:

Proposition 3.37 *Let* $L = E[C_1, \ldots, C_n]$ *be a congruence normal lattice. Then*

(1) L *is join-extremal if and only if it is join-congruence uniform and* C_i *intersects the spine of* $E[C_1, \ldots, C_{i-1}]$*, for all* i*.*

(2) *We have the dual result of* (1) *for meet-extremality by replacing* join *by* meet.

(3) L *is extremal if and only if it is congruence uniform and* C_i *intersects the spine of* $E[C_1, \ldots, C_{i-1}]$*, for all* i*.*

Proof. Since L is congruence normal we construct it by successive doublings of convex subsets starting from the one element lattice, which is extremal. By Lemma 3.20, at each doubling of the convex subset C_i, we add exactly as many join-irreducibles as the number of minimal elements of C_i, and we add as many meet-irreducibles as the number of maximal elements of C_i. By Lemma 3.21 the doubling of C_i will add

0 or 1 to the length of L; 0 if C_i does not contain an element of the spine and 1 otherwise. Thus, we keep the initial join-extremality through the doubling procedure if and only if at each step C_i is a lower-pseudo interval that contains an element of the spine. Indeed, these are the subsets that have a minimum element. Similarly for meet-extremality with upper pseudo-intervals. Finally, for extremality we need C_i to have a minimum and a maximum, and since it is convex this means that it has to be an interval. This finishes the proof. □

Theorem 3.38 *Let $L = E[C_1, \dots, C_n]$ be a congruence normal lattice. Then L is left modular if and only if $H(C_i)$ has an element that lies on a left modular chain of $E[C_1, \dots, C_{i-1}]$, for all i.*

To prove Theorem 3.38, we will need some lemmas. The next two results follow from Definition 3.19.

Lemma 3.39 *Let $c \in C$ and $(a, \epsilon) \in L[C]$. We have*

$$(a, \epsilon) \vee (c, 0) \neq (a, \epsilon) \vee (c, 1) \iff (a \vee c \in C \text{ and } \epsilon = 0)$$
$$(a, \epsilon) \wedge (c, 0) \neq (a, \epsilon) \wedge (c, 1) \iff (a \wedge c \in C \text{ and } \epsilon = 1)$$

Lemma 3.40 *In $L[C]$ we have a cover $(b, 0) \lessdot (c, 1)$ if and only if $b = c \in C$, or $b \in I_L(C) \setminus C$, $c \notin I_L(C)$ and $b \lessdot c$ is a cover in L.*

Lemma 3.41 *The element $(a, \epsilon) \in L[C]$ does not have a cover $(b, \alpha) \lessdot (c, \alpha')$ with $b \neq c$ that is a counterexample to its left modularity if and only if $a \in L$ is left modular.*

Proof. Suppose that $a \in L$ is not left modular because of $b \lessdot c$. By (3) of Proposition 3.18 this means $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$. We use the injection $\psi : L \to L[C]$ following Definition 3.19 and define α and α' by $(b, \alpha) = \psi(b)$ and $(c, \alpha') = \psi(c)$, unless $b \in C$ and $c \notin C$, in which case we define $\alpha = \alpha' = 1$. These choices guarantee that $(b, \alpha) \lessdot (c, \alpha')$ is a cover of $L[C]$. We have:

$$(a, \epsilon) \vee (b, \alpha) \neq (a, \epsilon) \vee (c, \alpha') \iff (a \vee c \in I_L(C),\ \epsilon = 0,\ \alpha = 0 \text{ and } \alpha' = 1) \tag{3.2}$$

$$(a, \epsilon) \wedge (b, \alpha) \neq (a, \epsilon) \wedge (c, \alpha') \iff (a \wedge b \in \big(L \setminus I_L(C)\big) \cup C,\ \epsilon = 1,\ \alpha = 0 \text{ and } \alpha' = 1) \tag{3.3}$$

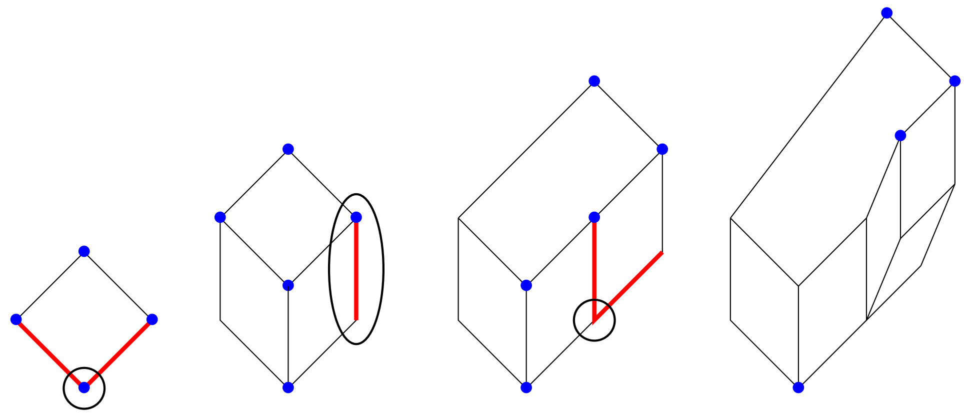


Figure 3.2: We represent 3 successive doublings. The left modular elements are the blue dots. The thick red edges form the convex subsets C that we double and we circled the elements of $H(C)$.

We want to conclude that $(a,\epsilon) \vee (b,\alpha) = (a,\epsilon) \vee (c,\alpha')$ and $(a,\epsilon) \vee (b,\alpha) = (a,\epsilon) \vee (c,\alpha')$, so it suffices to show that one of the conditions on the RHS of (3.2) and (3.3) is false. Suppose that $\alpha = 0$ and $\alpha' = 1$. Then by Lemma 3.40, since $b \neq c$ we have $b \in I_L(C) \setminus C$ and $c \notin I_L(C)$. Thus $a \vee c \notin I_L(C)$ and $a \wedge b \notin \big(L \setminus I_L(C)\big) \cup C$, which concludes.

It follows from (3) of Proposition 3.18 that $(b,\alpha) \lessdot (c,\alpha')$ with $b \neq c$ is a counterexample to the left modularity of (a,ϵ) in $L[C]$.

Conversely, assume that $a \in L$ is left modular. Let $(b,\alpha) \lessdot (c,\alpha')$ be a cover in $L[C]$ such that $b \neq c$. Since $b \neq c$, we have by Lemma 3.40 that $b \lessdot c$ is a cover in L. Since a is left modular it implies by (3) of Proposition 3.18 that $a \vee b \neq a \vee c$ or $a \wedge b \neq a \wedge c$. By definition of the join in $L[C]$, it is clear that $(a,\epsilon) \vee (b,\alpha) \neq (a,\epsilon) \vee (c,\alpha')$ or $(a,\epsilon) \wedge (b,\alpha) \neq (a,\epsilon) \wedge (c,\alpha')$, which again using (3) of Proposition 3.18 finishes the proof. $\square$

We can now prove a characterization of the left modular elements of the doubling $L[C]$.

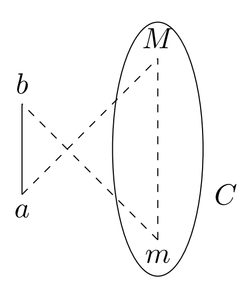
Proposition 3.42 *We have $(a,0) \in L[C]$ is left modular if and only if $a \in L$ is left modular and a is smaller than all the maximal elements of C. We have $(a,1) \in L[C]$ is left modular if and only if $a \in L$ is left modular and a is bigger than all the minimal elements of C. Thus $(a,0) \lessdot (a,1)$ with $a \in C$ are two left modular elements if and only if $a \in L$ is left modular and $a \in H(C)$.*

Proof. Lemma 3.41 tells us that the left modularity of $(a, \epsilon) \in L[C]$ is seen, if $a \in L$ is left modular, by looking only at the covers of the doubling $(c, 0) \lessdot (c, 1)$ with $c \in C$. We described what happens with these covers in Lemma 3.39. Thus by (3) of Proposition 3.18 we obtain that $(a, 0) \in L[C]$ is left modular if and only if $a \in L$ is left modular and $a \vee c \in C$ for all $c \in C$. This latter condition is equivalent to the fact that a is smaller than all the maximal elements of C. We deduce similarly the case of $(a, 1)$. The last part with $(a, 0) \lessdot (a, 1)$ is a direct consequence of the first part. □

Now that we understand the left modularity of the elements of $L[C]$, we still need to look at the left modular chains. The next result is true for any lattice. A particular case gives that any maximal chain that does not intersect an interval has an element incomparable to that interval, which is not true in a general poset (see Figure 3.1b).

Lemma 3.43 *Let C be a convex subset of L. Any maximal chain of L that does not intersect C has an element that is neither smaller than all the maximal elements of C nor bigger than all its minimal elements.*

Proof. Let Δ be a maximal chain of L that does not intersect C. The elements of this chain are split between 3 disjoint totally ordered sets : the set A of these that are less than all the maximal elements of C, the set I of these that are not less than all the maximal elements nor bigger than all the minimal elements of C, and the set B of these that are bigger than all the minimal elements of C. By convexity of C, we have $A \cap B = \emptyset$. Since Δ is maximal it contains $\hat{0}$ and $\hat{1}$, thus $A \neq \emptyset$ and $B \neq \emptyset$. We want to prove that $I \neq \emptyset$. Seeking a contradiction, suppose that $I = \emptyset$. Then Δ is partitioned between A, and B which consists of the bigger elements. Since Δ is maximal we have a cover relation $a \lessdot b$ with $a \in A$ and $b \in B$, giving us the following situation where m and M are minimal and maximal elements of C respectively such that $m \leq M$, $a < M$ and $m < b$:



If $a < m$, then $a < m < b$, which is absurd since $a \lessdot b$ is a cover relation. Thus $a \not< m$. If $m < a$, then by convexity of C we have $m < a < M$ implies $a \in C$, which is absurd. Thus $m \not< a$. Thus, a and m are incomparable. Then $a \vee m = b$. Since M is bigger than a and m, it implies $M \geq a \vee m = b$. Thus $m < b < M$, which by convexity of C implies $b \in C$. This is absurd, which proves that $I \neq \emptyset$. $\square$

Lemma 3.44 *Let L be a lattice and C be a convex subset that does not contain an element of any left modular chain of L. Then $L[C]$ is not left modular.*

Proof. We need to prove that $L[C]$ does not have a left modular chain.

To begin with, no such chain goes through the doubling, meaning no such chain contains an edge $(c, 0) \lessdot (c, 1)$ with $c \in C$. Indeed, if it was the case, take the projection of this chain to L, meaning we replace the elements (a, ϵ) of this chain by a. By forgetting the repetition of c coming from $(c, 0) \lessdot (c, 1)$, by Lemma 3.41 this would give a left modular chain of L. It is absurd because C would contain the element c of this chain but C does not contain any element of a left modular chain of L by hypothesis.

It remains to prove that we do not have a left modular chain of $L[C]$ that does not go through the doubling. By Lemma 3.43 for $C \times \{0\}$, if such a chain exists we would have an element (a, ϵ) of this chain that is neither smaller than all the maximal elements of $C \times \{0\}$ nor bigger than all its minimal elements. It means that a is neither smaller than all the maximal elements of C nor bigger than all its minimal elements. By Lemma 3.42 the element (a, ϵ) would not be left modular in $L[C]$. Then such a left modular chain does not exist. $\square$

Now we know that if $L[C]$ is left modular it is necessary that there exists a left modular chain Δ in $L[C]$ that goes through the doubling, meaning it comes from a left modular chain Δ in L that intersects C and by Lemma 3.42 this intersection contains an element of $H(C)$. The following lemma proves that this is also a sufficient condition.

Lemma 3.45 *Let L be a lattice and C a convex subset of L. Suppose that there exists $a \in H(C)$ that lies on a left modular chain $\Delta :\ x_0 \lessdot x_1 \lessdot \cdots \lessdot x_l = a \lessdot \cdots \lessdot x_n$ of L. Then the maximal chain*

$$(x_0, 0) \lessdot \cdots \lessdot (x_{l-1}, 0) \lessdot (a, 0) \lessdot (a, 1) \lessdot (x_{l+1}, 1) \lessdot \cdots \lessdot (x_n, 1) \tag{3.4}$$

of $L[C]$ is a left modular chain. Thus $L[C]$ is left modular.

Proof. Since $a \in L$ is left modular and $a \in H(C)$, we know using Proposition 3.42 that $(a, 0) \lessdot (a, 1)$ is a cover relation made of two left modular elements of $L[C]$. Since (3.4) is a maximal chain, it just remains to prove that any of its elements other than $(a, 0)$ and $(a, 1)$ are left modular in $L[C]$. It follows from Lemma 3.42 since for any other element (x_i, ϵ) of the chain (3.4) we have $x_i \in L$ is left modular and either smaller or bigger than $a \in H(C)$. □

We can now prove Theorem 3.38.

Proof of Theorem 3.38. By Lemma 3.45 we know that if for all i we have that $H(C_i)$ has an element that lies on a left modular chain of $E[C_1, \ldots, C_{i-1}]$, then L is left modular.

Conversely, suppose that $E[C_1, \ldots, C_n]$ is left modular. Let $i \leq n$. We want to prove that $H(C_i)$ has an element that lies on a left modular chain of $E[C_1, \ldots, C_{i-1}]$. By Lemma 3.44, it is required that C_i contains an element of a left modular chain of $E[C_1, \ldots, C_{i-1}]$. Indeed, if at some point no such chain exists in our lattice then it is too late; we will never recover a left modular lattice by doublings. As recalled at the end of Section 3.4, all left modular chains are of longest length. With the previous requirement, we know that at each doubling step we add one to the length of the lattice. Then to have a left modular chain after the doubling, since it is a longest chain, we need it to go through the doubling. Thus by Lemma 3.42 it is forced that $H(C_i)$ contains an element that lies on a left modular chain of $E[C_1, \ldots, C_{i-1}]$. □

Before giving some corollaries of our results, we give a lemma.

Lemma 3.46 *Let $L = E[C_1, \ldots, C_n]$ be an extremal congruence uniform lattice. Then all the elements of the spine of L are left modular elements.*

Proof. We prove this by induction on the number of doublings. For the one element lattice, the spine is the lattice itself and the only element is left modular. By induction, assume that all the elements of the spine of $E[C_1, \ldots, C_{i-1}]$ are left modular. We want to prove that all the elements of the spine of $E[C_1, \ldots, C_i] = E[C_1, \ldots, C_{i-1}][C_i]$ are left modular. We have the equivalence between the fact that all the elements of the spine are left modular and the fact that any longest chain is a left modular chain. Thus the induction hypothesis gives us that the longest chains of $E[C_1, \ldots, C_{i-1}]$ are left modular chains. By (3) of Proposition 3.37, we have that C_i intersects the spine of $E[C_1, \ldots, C_{i-1}]$. It follows by Lemma 3.21 that the longest chains of $E[C_1, \ldots, C_i]$ go through the doubling; these correspond to the longest chains of $E[C_1, \ldots, C_{i-1}]$ intersecting C_i. Precisely, these are the chains (3.4) of Lemma 3.45 for a longest chain $\Delta : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_l := a \lessdot \cdots \lessdot x_r$ of $E[C_1, \ldots, C_{i-1}]$, where $a \in C_i$. Since C_i is an interval, by (iii) of Lemma 3.36 we have $H(C_i) = C_i$. Thus $a \in H(C_i)$. As Δ is a left modular chain by hypothesis, by Lemma 3.45 the longest chains (3.4) are also left modular chains, which finishes the proof. □

Corollary 3.47 *If $L = E[C_1, \ldots, C_n]$ is congruence uniform, then L is left modular if and only if C_i contains an element of the spine of $E[C_1, \ldots, C_{i-1}]$, for all i.*

Proof. By (iii) of Lemma 3.36, we have $H(C_i) = C_i$, for all i. Thus Theorem 3.38 tells us that L is left modular if and only if C_i has an element that lies on a left modular chain of $E[C_1, \ldots, C_{i-1}]$, for all i. Then using the fact that the left modular chains are of longest length gives one direction of Corollary 3.47, the other direction is obtained using additionally Lemma 3.46. □

Corollary 3.48 *Join-congruence uniform left modular lattices are join-extremal. Similarly meet-congruence uniform left modular lattices are meet-extremal. For congruence uniform lattices, extremality and left modularity are equivalent.*

Proof. The first statements on join and meet-congruence uniform lattices are a direct consequence of Theorem 3.38 with the fact that left modular chains are of longest length. The statement on congruence uniform lattices is the comparison of (2) of Proposition 3.37 and Corollary 3.47. □

Remark 3.49 *Lemma 3.46 is a special case of the combination of [100, Lemma 6] and of [103, Theorem 1.4], but is proved in our context by a simple induction.*

The first statements of Corollary 3.48 relative to join and meet-congruence uniform lattices are a special case of [77, Theorem 3.2], as these lattices are respectively join and meet-semidistributive as recalled in Lemma

3.22. Combining this result with [103, Theorem 1.4] gives, as was observed in [77], a generalization of the last statement of Corollary 3.48 to all semidistributive lattices.

3.5.3 Shellability of congruence normal lattices

Recall the definitions related to shellability from Section 3.4.6. In the sequel C is always a non-empty convex subset of L. We give a necessary condition for a congruence normal lattice to be shellable.

Theorem 3.50 *Let $L = E[C_1, \ldots, C_n]$ be a congruence normal lattice. If L is shellable, then C_i intersects the spine of $E[C_1, \ldots, C_{i-1}]$, for all i.*

We will prove Theorem 3.50 in the rest of this section, but we start with two corollaries:

Corollary 3.51 *If a join-congruence uniform lattice is shellable, then it is join-extremal. If a meet-congruence uniform lattice is shellable, then it is meet-extremal.*

Proof. This follows from Proposition 3.37 and Theorem 3.50. □

The following was stated as Theorem 3.4 in the introduction, and contains Theorem 3.3:

Theorem 3.52 *Let L be a congruence uniform lattice. The following properties are equivalent:*

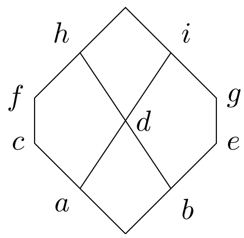
(i) *L is extremal.*

(ii) *L is left modular.*

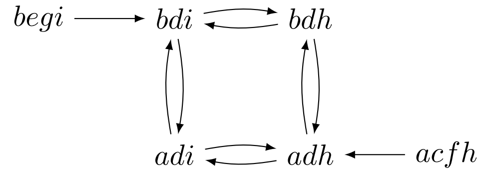
(iii) *L is EL-shellable.*

(iv) *L is shellable.*

Proof. Corollary 3.48 gives $(i) \iff (ii)$. As we recalled in Section 3.4, we have that $(ii) \implies (iii) \implies (iv)$ is true for any lattice. It remains to prove $(iv) \implies (i)$, which follows from Corollary 3.51 since a congruence uniform lattice is both join-congruence uniform and meet-congruence uniform. □



a A non-extremal congruence uniform lattice L.



b The graph $FA(L)$ from Figure 3.3a. The sets $\{begi\}$ and $\{acfh\}$ are two disjoint source sets.

Figure 3.3

We are now interested in proving Theorem 3.50.

Definition 3.53 *Let F be a chain of L. If $F \cap C = \emptyset$, then F corresponds to a chain in $L[C]$ that we denote by $(F, 0)$. Otherwise, let $F \cap C = \{c_1, c_2, \ldots, c_k\}$ and we denote by (F, i) the chain*

$$\big(F \setminus (F \cap C)\big) \cup \{(c_1, 0), (c_2, 0), \ldots, (c_i, 0)\} \cup \{(c_i, 1), (c_{i+1}, 1), \ldots, (c_k, 1)\}.$$

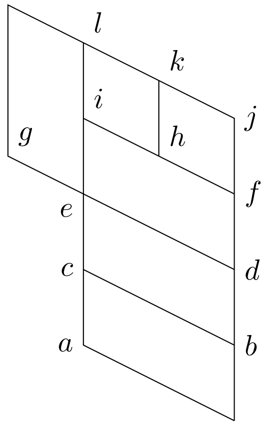
Although it is in conflict with our convention that for $n \in \mathbb{Z}_{\geq 0}$, we have $[n] = \{1, \ldots, n\}$, it turns out to be convenient to define $[0] = \{0\}$.

The following lemma follows from the definition of the doubling construction.

Lemma 3.54 *The facets of $\Delta(L[C])$ are all the (F, i) for any maximal chain F of L and $i \in [\,|F \cap C|\,]$.*

For any finite lattice L, we define a directed multigraph $FA(L)$, called the *facet adjacency graph* of L, whose vertices are the maximal chains of L, and we have an edge $F \to G$ between two such chains F and G if $|F \cap G| = |G| - 1$. A *source set* of a directed multigraph G is a non-empty subset of the vertices $X \subsetneq V(G)$ such that there are no $y \to x$ with $y \notin X$ and $x \in X$. This means that no edges starting outside X are pointing to a vertex of X. Two source sets X and Y are disjoint if $X \cap Y = \emptyset$. See Figure 3.3 for an example of a non-extremal congruence uniform lattice, and Figure 3.4 for the case of an extremal congruence uniform lattice. In these figures we write the chains as words without including $\hat{0}$ and $\hat{1}$.

The following is immediate from the definition of shellability:



$A := a\,c\,e\,i\,l$
$B := a\,c\,e\,g$
$C := b\,c\,e\,i\,l$
$D := b\,c\,e\,g$
$E := b\,d\,e\,i\,l$
$F := b\,d\,e\,g$
$G := b\,d\,f\,h\,i\,l$
$H := b\,d\,f\,h\,k\,l$
$I := b\,d\,f\,j\,k\,l$

a An extremal congruence uniform lattice L with a labeling of the maximal chains, omitting the extremum elements.

$$\begin{tikzcd}
I \arrow[r, shift left] & H \arrow[l, shift left] \arrow[r, shift left] & G \arrow[l, shift left] \arrow[r] & E \arrow[r, shift left] \arrow[d] & C \arrow[l, shift left] \arrow[r, shift left] \arrow[d] & A \arrow[l, shift left] \arrow[d] \\
 & & & F \arrow[r, shift left] & D \arrow[l, shift left] \arrow[r, shift left] & B \arrow[l, shift left]
\end{tikzcd}$$

b The graph $FA(L)$ from Figure 3.4a, whose source sets are $\{I, H, G\}$ and $\{I, H, G, E, C, A\}$.

Figure 3.4

Lemma 3.55 *If $F_1, F_2, \ldots, F_k$ is a shelling of $\Delta(L)$, then for all $2 \leq j \leq k$, there exists $i \in \{1, \ldots, j-1\}$ such that $F_i \to F_j$.*

We deduce a necessary condition for shellability of L on its facet adjacency graph.

Lemma 3.56 *If L is shellable, then $FA(L)$ cannot have two disjoint source sets.*

Proof. Assume that $FA(L)$ has two disjoint source sets S and T. Seeking a contradiction, suppose that L is shellable. Thus we have a shelling $F_1, F_2, \ldots, F_k$ of $\Delta(L)$. Let F_l and F_j be the left-most maximal chains of respectively S and T in our shelling. Since $F_l \neq F_j$ because the source sets S and T are disjoint, without loss of generality we can assume that $j \geq 2$. Using Lemma 3.55, there exists $i \in [j-1]$ such that $F_i \to F_j$. Since F_j is the left-most maximal chain of T in our shelling, it implies that $F_i \notin T$. Since T is a source set, this is absurd. $\square$

We can obtain $FA(L[C])$ from $FA(L)$:

Proposition 3.57 *The vertices of $FA(L[C])$ are the (F, i) for any maximal chain F of L and $i \in [\,|F \cap C|\,]$. We give the edges of $FA(L[C])$ knowing these of $FA(L)$:*

- *We have* $(F,i) \to (G,0)$ *whenever* $F \to G$ *in* $FA(L)$ *with* $G \cap C = \emptyset$*, where* $i \in [\,|F \cap C|\,]$.

- *For* $F \cap C \neq \emptyset$ *and* $G \cap C \neq \emptyset$*, we have* $(F,i) \to (G,j)$ *whenever* $F \to G$ *in* $FA(L)$ *and* $|(F,i) \cap (G,j)| = |G|$*, for any* $i \in [\,|F \cap C|\,]$ *and* $j \in [\,|G \cap C|\,]$.

- *We have* $(F,i) \to (F,i+1)$ *and* $(F,i+1) \to (F,i)$ *whenever* $|F \cap C| \geq 2$*, for any* $i \in [\,|F \cap C| - 1]$.

Proof. We have $|(F,0)| = |F|$ and $|(F,i)| = |F|+1$ if $i \neq 0$. Then the result follows from Lemma 3.54 and the definition of the doubling construction. □

Lemma 3.58 *If* $FA(L)$ *has at least two disjoint source sets, it is also the case for* $FA(L[C])$.

Proof. Let S and T be two disjoint source sets of $FA(L)$. We define two subsets $S_C := \{(F,i) \mid F \in S, i \in [\,|F \cap C|\,]\}$ and $T_C := \{(G,i) \mid G \in T, i \in [\,|G \cap C|\,]\}$ of $FA(L[C])$. We conclude by proving that the sets S_C and T_C are disjoint source sets of $FA(L[C])$. The fact that they are disjoint nonempty proper subsets of $FA(L[C])$ is clear. By Proposition 3.57, if we do not have $F \to G$ in $FA(L)$ with $F \neq G$, then we do not have $(F,i) \to (G,j)$ in $FA(L[C])$. It follows, from this observation and the fact that S and T are source sets, that S_C and T_C are source sets. □

Lemma 3.59 *Suppose that* C *does not intersect the spine of* L*. Then the two sets*

$$S := \{(F,i) \mid F \text{ is a maximal chain of } L,\ F \cap C \neq \emptyset,\ i \in [\,|F \cap C|\,]\}$$

and

$$T := \{(G,0) \mid G \text{ is a longest chain of } L\}$$

are two disjoint source sets of $FA(L[C])$.

Proof. Since for any $(F,i) \in S$ there exists $c \in C$ such that $\{(c,0),(c,1)\} \subseteq (F,i)$, and no maximal chain of $L[C]$ outside of S contains such elements, we know that no edges from a vertex outside S is pointing to (F,i) in $FA(L[C])$. This proves that S is a source set of $FA(L[C])$.

Seeking a contradiction, suppose that there exist $(F, i) \notin T$ and $(G, 0) \in T$ such that $(F, i) \to (G, 0)$. Since $(F, i) \notin T$, we have $F \cap C \neq \emptyset$, meaning $i \neq 0$. We have $|(G, 0)| = |G|$ and $|(F, i)| = |F| + 1$ since $i \neq 0$. In general, if $F \to F'$ in $FA(L)$, it follows that $|F| \geq |F'|$. Then $(F, i) \to (G, 0) \implies |F| + 1 \geq |G|$. Since C does not intersect the spine of L, by Lemma 3.21 we have $\ell(L[C]) = \ell(L)$. Thus $|G| \leq |F| + 1 = |(F, i)| \leq \ell(L[C]) = \ell(L) = |G|$. Then $|(F, i)| = |G| = \ell(L[C])$, thus (F, i) is a longest chain of $L[C]$. Since $(F, i) \notin T$, which means F is not a longest chain of L, it implies that $(F, i) \in S$ since the chains of longest length of $L[C]$ are either longest chains of L or go through the doubling of C. Then, as observed in the first part of the proof, (F, i) has at least two elements $(c, 0)$ and $(c, 1)$ not in $(G, 0)$, which with $|(F, i)| = |(G, 0)|$ makes it absurd to have $|(F, i) \cap (G, 0)| = |(G, 0)| - 1$, which is implied by $(F, i) \to (G, 0)$. It follows that T is a source set.

With $S \cap T = \emptyset$ since C does not intersect the spine of L, this finishes the proof. □

Now we can prove Theorem 3.50:

Proof. [Proof of Theorem 3.50] Let $L = E[C_1, \ldots, C_n]$ be a congruence normal lattice. Suppose that there exists i such that C_i does not intersect the spine of $E[C_1, \ldots, C_{i-1}]$. Then by Lemma 3.59 the graph $FA(E[C_1, \ldots, C_i])$ has at least two disjoint source sets. By Lemma 3.58, since $L = E[C_1, \ldots, C_n]$ is obtained from $E[C_1, \ldots, C_i]$ by doublings, we have that L has also at least two disjoint source sets, which by Lemma 3.56 implies that L is not shellable. □

3.5.4 Induced subgraphs of a canonical join graph

In this section we give, using the fundamental theorem of finite semidistributive lattices [91], a counterexample to the following:

Question 3.60 ([7, Question 2.5.5]) *Is an induced subcomplex of a canonical join complex still a canonical join complex?*

As explained in Section 3.4, for us a canonical join complex is always one coming from a semidistributive lattice (not just a join-semidistributive lattice). Question 3.60 was asked in this context. The canonical join complex being the clique complex of its 1-skeleton, which is the canonical join graph, this question is

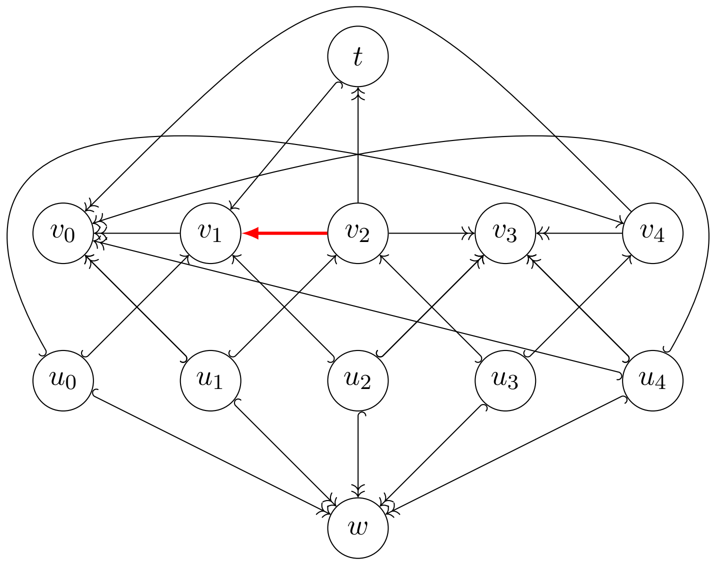


Figure 3.5: The Galois graph of a semidistributive lattice having 167 elements. The arrows are of the three types $\rightarrow$, $\hookrightarrow$ and $\twoheadrightarrow$ as in [91] and they form a two-acyclic factorization system.

equivalent to asking whether an induced subgraph of a canonical join graph is still such a graph. Indeed, an induced subcomplex on vertices X of a flag simplicial complex Δ is still flag, and its 1-skeleton is the induced subgraph of the 1-skeleton of Δ on vertices X.

From now on in this section, all directed graphs and multigraphs are assumed to have a loop at every vertex, but they are not represented on the figures. An illustration of the following definitions is given in Figure 3.5.

Definition 3.61 *Let $(G, \rightarrow)$ be a directed multigraph. We deduce from it another directed multigraph G_A by replacing the arrows $\rightarrow$ with up to three kinds of arrows $\twoheadrightarrow, \hookrightarrow, \hookrightarrow\!\!\!\!\rightarrow$, where some arrows $\rightarrow$ can remain. An arrow $x \rightarrow y$ is replaced by $x \hookrightarrow y$ if for every $z \in G$ such that $z \rightarrow x$, we have an arrow $z \rightarrow y$. An arrow $x \rightarrow y$ is replaced by $x \twoheadrightarrow y$ if for every $z \in G$ such that $y \rightarrow z$, we have an arrow $x \rightarrow z$. If an arrow $x \rightarrow y$ can be replaced by both the arrows $\hookrightarrow$ and $\twoheadrightarrow$, then we replace it by $x \hookrightarrow\!\!\!\!\rightarrow y$. We understand this latter arrow to be both an $\hookrightarrow$-arrow and an $\twoheadrightarrow$-arrow.*

Definition 3.62 ([91]) *Let $(G, \rightarrow)$ be a directed multigraph. We say that G forms a* two-acyclic factorization system *if G_A satisfies the following conditions:*

(i) *(order condition) We do not have $x \twoheadrightarrow y \twoheadrightarrow x$ or $x \hookrightarrow y \hookrightarrow x$ with $x \neq y$.*

(ii) *(brick condition) We do not have $x \twoheadrightarrow y \hookrightarrow x$ with $x \neq y$.*

(iii) *(multiplication) For each $x \to z$ there exists y such that $x \twoheadrightarrow y \hookrightarrow z$.*

If G is a multigraph, we say that it admits a *two-acyclic factorization system* if there exists an orientation of its edges such that the resulting directed multigraph $(G, \to)$ forms a two-acyclic factorization system. The following is a reformulation of results of [91]:

Proposition 3.63 *The canonical join graphs are the complements of the multigraphs that admit a two-acyclic factorization system.*

Proof. By [91, Theorem 1.2], a multigraph G admits a two-acyclic factorization system if and only if G can be oriented such that the resulting directed multigraph $(G, \to)$ gives a semidistributive lattice by taking the associated lattice of maximal orthogonal pairs. By [91, Theorem 5.13], this is equivalent to $\overline{G}$ being a canonical join graph. This concludes. $\square$

In Figure 3.5, we give an example of the Galois graph $G(L)$ of a semidistributive extremal lattice. Indeed, conditions (i) and (ii) of Definition 3.62 are satisfied since there are no cycles, and for condition (iii) we have only one regular arrow $v_2 \to v_1$ in $G(L)_A$; thus, this condition is satisfied since $v_2 \twoheadrightarrow t \hookrightarrow v_1$. Let $H := \overline{G(L)}$. By Proposition 3.63, H is a canonical join graph.

For a graph $G = (V, E)$ and $X \subseteq V$, we denote by $G[X]$ the induced subgraph of G on the elements of X. Let $X := V(H) \setminus \{t\}$, where $V(H)$ is the vertex-set of H. Our goal is now to prove that $H[X]$ is not a canonical join graph. Thus it will be a counterexample to Question 3.60.

The following result is easy:

Lemma 3.64 *Let $G = (V, E)$ be a multigraph and $X \subseteq V$. We have $\overline{G[X]} = \overline{G}[X]$.*

By definition, $\overline{H} = G(L)$. Thus by Lemma 3.64, we have $\overline{H[X]} = \overline{H}[X] = G(L)[X]$. By Proposition 3.63, $H[X]$ is a canonical join graph if and only if the underlying undirected graph of $G(L)[X]$, or any multigraph obtained from it by adding some edges between two elements of $G(L)[X]$ already joined by an edge, is a two-acyclic factorization system. The underlying undirected graph of $G(L)[X]$ is called the *Grötzsch graph*; it is a triangle-free graph of chromatic number 4. These two properties will play an important role.

Lemma 3.65 *Let G be a simple graph. There exists an orientation of the edges of G such that the resulting directed graph has no directed walk of length 3 if and only if $\chi(G) \leq 3$.*

Proof. Suppose that $\chi(G) \leq 3$. Let us choose a proper coloring $C : V \to \{0, 1, 2\}$ of G. We deduce an orientation of the edges of G by orienting an edge $x \to y$ if $C(x) < C(y)$. This is an orientation of G without directed walks of length 3. Conversely, suppose that there exists an orientation of the edges of G such that the resulting directed graph $(G, \to)$ has no directed walk of length 3. Then $(G, \to)$ does not have cycles. This proves that the following coloring is well-defined; color any vertex x with the longest length of a directed walk ending at x. This gives a coloring $C : V \to \{0, 1, 2\}$, which is proper since for each $x \to y$, we have $C(x) < C(y)$. $\square$

Lemma 3.66 *A triangle-free simple directed graph G forms a two-acyclic factorization system if and only if G does not have a directed walk of length 3.*

Proof. Since G is simple, conditions (i) and (ii) of Definition 3.62 are satisfied. On the contrary, condition (iii) cannot be satisfied if there are any $\to$ arrows since G is triangle-free. Thus G forms a two-acyclic factorization system if and only if G_A does not have regular arrows $\to$. By the definitions of $\hookrightarrow$ and $\twoheadrightarrow$ and the fact that G is triangle-free, we have regular arrows $\to$ in G_A if and only if G does not have a directed walk of length 3. $\square$

The following result is a consequence of Lemmas 3.65 and 3.66:

Proposition 3.67 *A triangle-free simple graph G admits a two-acyclic factorization system if and only if $\chi(G) \leq 3$.*

Proposition 3.68 *Let G be a triangle-free multigraph. Let G' be its underlying simple graph, which is obtained by replacing multiple edges by a single edge. If G admits a two-acyclic factorization system, then it is also the case of G'.*

Proof. Let $(G, \to)$ be an orientation of G that forms a two-acyclic factorization system. From it we deduce an orientation $(G', \to')$ of G'; for multiple edges between x and y in G, choose arbitrarily one of the arrows between x and y in $(G, \to)$. Let us prove that $(G', \to')$ forms a two-acyclic factorization system. As seen in the proof of Lemma 3.66, since $(G', \to')$ does not have 2-cycles we only need to look at condition (iii) of Definition 3.62. As G and G' are triangle-free, condition (iii) cannot be satisfied if there are any $\to$ arrows. This means that G_A, which is obtained from $(G, \to)$, does not have regular arrows $\to$. By definitions of $\hookrightarrow$ and $\twoheadrightarrow$, it is also the case of G'_A. This proves that G' admits a two-acyclic factorization system. $\square$

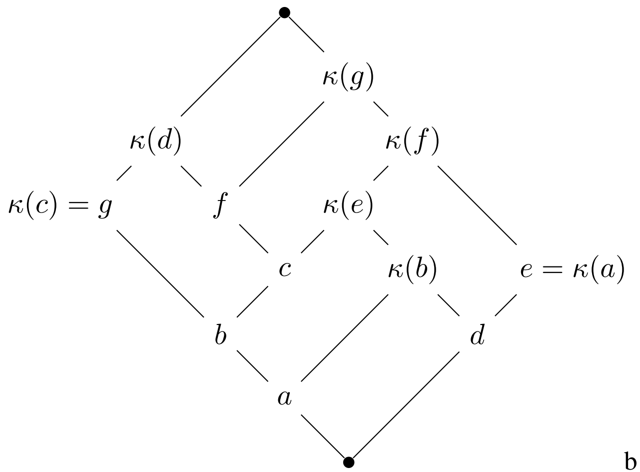
Then the underlying undirected graph of $G(L)[X]$, which is triangle-free with a chromatic number equal to 4, does not admit a two-acyclic factorization system by Proposition 3.67. By Proposition 3.68, any multigraph obtained from the underlying undirected graph of $G(L)[X]$ by adding some edges between two elements of $G(L)[X]$ already joined by an edge, does not admit either a two-acyclic factorization system. This proves that $H[X]$ is not a canonical join graph.

3.5.5 Dimension of semidistributive extremal lattices

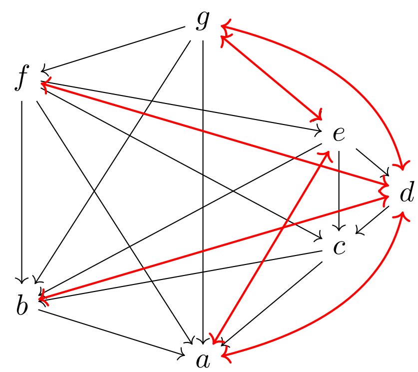
In this section we prove Theorem 3.5 from the introduction, that gives a way to compute the dimension of a semidistributive extremal lattice.

Definition 3.69 *Let P be a poset. The* critical pairs *of P are the pairs $(a, b) \in P \times P$ such that a and b are incomparable, for any $x < a$ we have $x < b$, and for any $y > b$ we have $y > a$.*

Definition 3.70 ([89]) *Let P be a poset. The* graph of critical pairs *$D(P)$ associated to P is the directed graph whose vertices are the critical pairs and edges are $(a, b) \to (c, d)$ whenever $b \geq c$. The* critical pairs simplicial complex *$C(P)$ associated to P is the simplicial complex whose faces correspond to the subsets of critical pairs that induce an acyclic subgraph of $D(P)$ (we mean that we need to avoid directed cycles).*



a A semidistributive extremal lattice L



b In black the Galois graph $G(L)$ of the lattice from Figure 3.6a. With the extra red 2-cycles it is $D(L)$.

Figure 3.6

A *cover set* of a simplicial complex Δ is a subset S of the faces of Δ such that every vertex of Δ is contained in a face of S. The next result follows from [105, Chapter 4, Proposition 2.1], and we restate it as it is done in [89, Proposition 14 and the discussion that follows].

Theorem 3.71 *For any poset P, the dimension of P is equal to the minimum size of a cover set of $C(P)$.*

The following is easy to prove.

Lemma 3.72 *Let L be a semidistributive lattice. Then the critical pairs are exactly the pairs $(j, \kappa(j))$ for $j \in \mathrm{JIrr}(L)$ with $j_* \notin \mathrm{MIrr}(L)$, which means $\kappa(j) \neq j_*$.*

Thus, for L a semidistributive lattice, we identify a critical pair with the join-irreducible element given by its first coordinate. Then the graph $D(L)$ is the graph on vertices the join-irreducible elements j such that $\kappa(j) \neq j_*$, with an edge $i \to j$ whenever $\kappa(i) \geq j$. Looking at Proposition 3.14, when L is a semidistributive extremal lattice, we see that $D(L)$ and the Galois graph $G(L)$ are similar. We make this observation precise in the next result.

A 2-cycle on two vertices x and y in a directed graph will be represented by $x \longleftrightarrow y$. See Figure 3.6, which

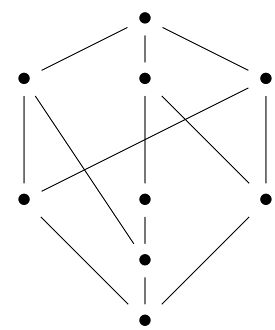


Figure 3.7: An extremal lattice L that is not semidistributive.

is partly taken from [91, Figures 1 and 2], for an illustration of the following result in a case where there are no j such that $\kappa(j) = j_*$.

Lemma 3.73 *Let L be a semidistributive extremal lattice. The graph of critical pairs $D(L)$ is obtained from the Galois graph $G(L)$ by adding a 2-cycle between all pairs of join-irreducible elements i and j that form an independent set of $G(L)$, and then removing the vertices j such that $\kappa(j) = j_*$.*

Proof. Let us prove that if $i, j \in \mathrm{JIrr}(L)$ such that $\kappa(i) \neq i_*$ and $\kappa(j) \neq j_*$, then we have $i \to j$ in $D(L)$ if and only if $j \to i$ is not in $G(L)$. This follows from the definitions of the arrows in each graph; $i \to j$ is in $D(L)$ means $\kappa(i) \geq j$, whereas by Proposition 3.14, $j \to i$ is not in $G(L)$ means $j \leq \kappa(i)$. Thus if we have 2-cycles in $G(L)$ they disappear in $D(L)$, but since $G(L)$ is acyclic by Proposition 3.9, $G(L)$ does not have 2-cycles. The Lemma follows. $\square$

Lemma 3.74 *Let L be a semidistributive extremal lattice. The isolated vertices of $\overline{G(L)}$ are the join-irreducible elements $i \in \mathrm{JIrr}(L)$ such that $\kappa(i) = i_*$.*

Proof. Suppose that $i \in \mathrm{JIrr}(L)$ such that $\kappa(i) = i_*$. Let us prove that i is an isolated vertex of $\overline{G(L)}$, which means that for every $j \in \mathrm{JIrr}(L)$, $\{i, j\}$ is not an independent set. Seeking a contradiction, assume that $\{i, j\}$ is an independent set of $G(L)$, which means $i \leq \kappa(j)$ and $j \leq \kappa(i) = i_*$. This gives $j < \kappa(j)$, which is absurd.

Suppose that $i \in \mathrm{JIrr}(L)$ such that $\kappa(i) \neq i_*$. We want to prove that i is not an isolated vertex of $\overline{G(L)}$. Since L is semidistributive and extremal, by Lemma 3.16 the downward label sets of L are the independence sets of its Galois graph $G(L)$. In a semidistributive lattice, the set of all downward label sets is the same as that of all upward label sets, which are denoted by $U(x)$ for an element $x \in L$ [103, Corollary 6.4]. By definition of γ_J, we have $i \in U(i_*)$. Since $\kappa(i) \neq i_*$, we have $|U(i_*)| \geq 2$. Thus i is part of an independence set of size bigger than two, which proves that it is not an isolated vertex of $\overline{G(L)}$. $\square$

For L a semidistributive extremal lattice, we denote by $K(L)$ the undirected graph formed by the subgraph of $D(L)$ obtained by keeping only the 2-cycles $x \longleftrightarrow y$ of $D(L)$ and replacing these by an edge joining x and y. The *null graph* is the graph that does not have vertices. An *empty graph* is a graph without edges (but it can have vertices).

Proposition 3.75 *Let L be a semidistributive extremal lattice. Then $\overline{G(L)}$ is the disjoint union of the graph $K(L)$ with the empty graph whose vertices are the $i \in \mathrm{JIrr}(L)$ such that $\kappa(i) = i_*$.*

Proof. By Lemma 3.74, we know that the isolated vertices of $\overline{G(L)}$ are the join-irreducible elements $i \in \mathrm{JIrr}(L)$ such that $\kappa(i) = i_*$. Then the proposition follows from Lemma 3.73. $\square$

Proposition 3.76 *Let L be a semidistributive extremal lattice. The cover sets of $C(L)$ are the partitions of the vertices of $K(L)$ into independent sets.*

Proof. By definition the vertices of $C(L)$ and $K(L)$ are the same.

Let S be a cover set of $C(L)$. We know that S is a partition of the vertices of $K(L)$, and we want to prove that it is a partition into independent sets. Let $F \in S$. By hypothesis F is a face of $C(L)$, which means that the induced subgraph of $D(L)$ on F does not have a directed cycle. In particular it does not contain 2-cycles, which proves that F is an independent set of $K(L)$.

Conversely, let S be a partition of $K(L)$ into independent sets. We know that it is a partition of the vertices of $C(L)$. Let $F \in S$. By hypothesis F does not contain 2-cycles. But by Lemma 3.73 and the fact that

$G(L)$ does not have a directed cycle (Proposition 3.9), it follows that the only directed cycles in $D(L)$ are 2-cycles. Thus the induced subgraph of $D(L)$ on F does not contain directed cycles, which proves that F is a face of $C(L)$. Thus S is a cover set of $C(L)$. $\square$

We can now prove the main result of this section (which is Theorem 3.5 from the introduction):

Theorem 3.77 *Let L be a semidistributive extremal lattice with $|L| > 1$, then* $\dim(L) = \chi(\overline{G(L)})$.

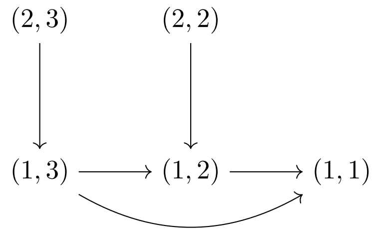
Proof. If L is a chain with at least two elements, then the underlying undirected graph of $G(L)$ is the complete graph on $|L|-1$ vertices. Thus $\overline{G(L)}$ is an empty graph with at least one vertex, and the result follows. If L is not a chain, then the underlying undirected graph of $G(L)$ is not the complete graph. Thus, by Proposition 3.75, $K(L)$ is not the null graph and $\chi(K(L)) = \chi(\overline{G(L)})$. Then we conclude using Theorem 3.71 and Proposition 3.76 which together give $\chi(K(L)) = \dim(L)$. $\square$

Remark 3.78 *We defined the Galois graph for any extremal lattice L, without the assumption that L is semidistributive. The lattice L from Figure 3.7 shows that it is necessary to have the semidistributivity in Theorem 3.77. Indeed, L is extremal but not semidistributive,* $\dim(L) = 3$ *and the complement of its Galois graph is a path on* 4 *vertices, thus* $\chi(\overline{G(L)}) = 2$.

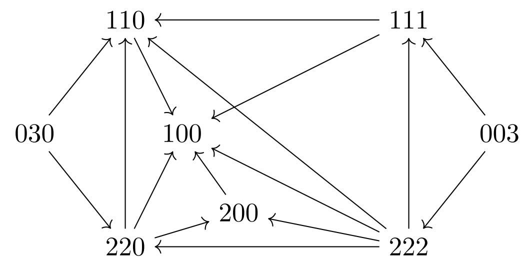
Corollary 3.79 ([36]) *Let L be a distributive lattice. Then* $\dim(L) = \mathrm{width}(\mathrm{JIrr}(L))$.

Proof. By the fundamental theorem of finite distributive lattices, the Galois graph $G(L)$ is isomorphic to the Hasse diagram of the subposet $\mathrm{JIrr}(L)$. Indeed, we have an edge $i \to j$ in $G(L)$ if and only if $i \geq j$ in the subposet $\mathrm{JIrr}(L)$. Thus $\overline{G(L)}$ is isomorphic to the incomparability graph of $\mathrm{JIrr}(L)$. By Dilworth's decomposition theorem, the chromatic number of this latter graph equals the maximum size of an antichain, which is $\mathrm{width}(\mathrm{JIrr}(L))$. We conclude using Theorem 3.77. $\square$

For a finite simple graph G, we denote by $\omega(G)$ its clique number and by $d(G)$ its maximum degree. The following is well-known and easy:



a The Galois graph $G(\mathrm{Hoch}(3))$.



b The Galois graph $G(W(2,3))$.

Figure 3.8: Two Galois graphs.

Lemma 3.80 *For any simple graph G, we have* $\omega(G) \leq \chi(G) \leq d(G) + 1$.

The number of independent sets of $G(L)$ is the number of elements of L ([103, Corollary 5.6]). This corresponds to the number of cliques of $\overline{G(L)}$. By Lemma 3.16, a clique of $\overline{G(L)}$ corresponds to the downward label set, or upward label set using [103, Corollary 6.4], of some element of L. Thus $\omega(\overline{G(L)})$ is the maximum number of elements that cover, or are covered, by some vertex of L. This is a lower bound for the dimension of L by Theorem 3.77 and Lemma 3.80. We obtained differently this result in Proposition 3.25, proving it is true for any semidistributive lattice.

3.6 Applications of our dimension result

In this section, we use Theorem 3.77 to obtain the dimension of some semidistributive extremal lattices that appeared recently in algebraic combinatorics.

3.6.1 Hochschild and bubble lattices

In this subsection we determine the dimension of the Hochschild lattice [25, 78] and of one of its generalizations called the bubble lattices [73, 74]. The Hochschild lattice is obtained as an orientation of the 1-skeleton of the Hochschild polytopes, which appeared in algebraic topology [92].

Recall the discussion on extremal lattices from Section 3.4.2, in particular Theorem 3.10, which enables us to define an extremal lattice by giving its Galois graph.

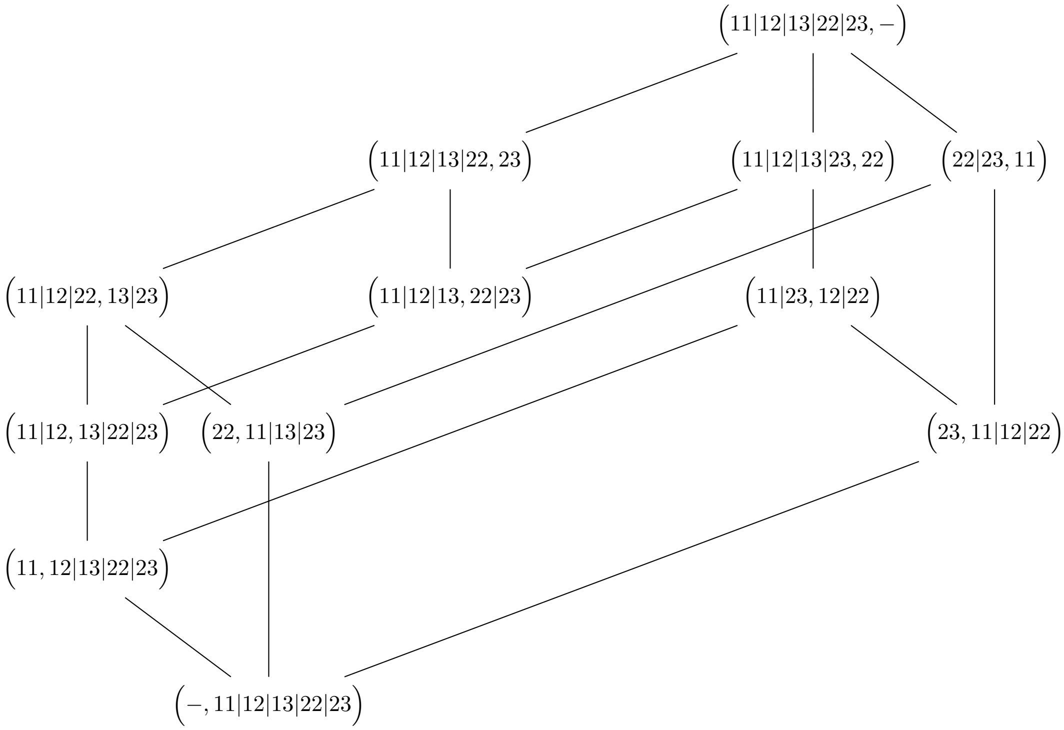


Figure 3.9: The lattice of maximal orthogonal pairs of $G(\mathrm{Hoch}(3))$ from Figure 3.8a, where pairs (i, j) are abbreviated by the words ij and we omitted set brackets and replaced commas by vertical bars.

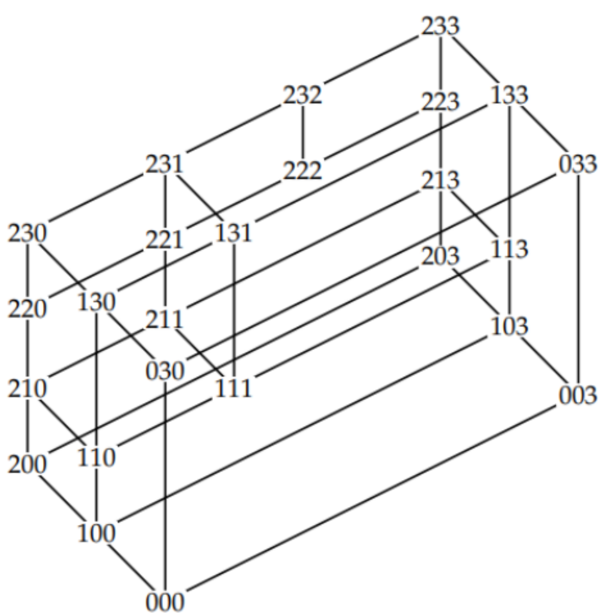


Figure 3.10: The lattice $W(2, 3)$.

Definition 3.81 ([78, Theorem 1.1]) *For any $n > 0$, the* Hochschild lattice $\mathrm{Hoch}(n)$ *is an extremal lattice whose Galois graph is isomorphic to the directed graph whose vertices are $\{(i, j) \mid i \in \{1, 2\},\ i \leq j \leq n\}$ and the edges are $(i, j) \to (i', j')$ if $(i, j) \neq (i', j')$ and either we have $i = 2$, $i' = 1$ and $j = j'$, or we have $i = i' = 1$ and $j > j'$.*

We give in Figure 3.8a the Galois graph of $\mathrm{Hoch}(3)$, and we give this lattice in Figure 3.9 (this figure is taken from [78]).

Definition 3.82 ([73, Proposition 4.27]) *Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be two disjoint linearly ordered alphabets. For any $m, n \geq 0$, the* bubble lattice $\mathrm{Bub}(m, n)$ *is an extremal lattice whose Galois graph is the directed graph whose vertices are $X \sqcup Y \sqcup (X \times Y)$ and the edges are $l_1 \to l_2$ if $l_1 \neq l_2$ and either:*

- *$l_1 = (x_s, y_t)$ and $l_2 = x_s$, or*
- *$l_1 = y_t$ and $l_2 = (x_s, y_t)$, or*
- *$l_1 = (x_s, y_t)$ and $l_2 = (x_{s'}, y_{t'})$ with $s \geq s'$ and $t \leq t'$.*

It is proved in [73, Proposition 4.19] that $\mathrm{Bub}(n-1, 1)$ is isomorphic to $\mathrm{Hoch}(n)$. Moreover, these lattices are semidistributive and extremal [73, Theorem 1.1].

Now we use Theorem 3.77 to obtain the dimension of $\mathrm{Hoch}(n)$.

Proposition 3.83 *We have $\dim(\mathrm{Hoch}(n)) = n$ for all $n > 0$.*

Proof. From the Galois graph of $L := \mathrm{Hoch}(n)$ given in Definition 3.81, we can easily deduce $\overline{G(L)}$. Let $A := \{(1, i) \mid i \in [n]\}$ and $B := \{(2, i) \mid 2 \leq i \leq n\}$. Then $\overline{G(L)}$ is the graph whose vertices are $A \sqcup B$, and it can be described by the fact that the induced subgraph on A is an empty graph, and for every $i \geq 2$ the induced subgraph on $\{(1, i)\} \sqcup B$ is the complete graph on n vertices. From the latter we obtain $\chi(\overline{G(L)}) \geq n$. Moreover, by coloring the vertices of $\{(1, 1)\} \sqcup B$ with n different colors, and coloring the vertices of A all with the same color as $(1,1)$, this gives a proper coloring with n colors. Thus $\chi(\overline{G(L)}) = n$, which implies the proposition, by Theorem 3.77. □

In fact, we can directly obtain the dimension of the bubble lattices (which is [73, Conjecture 4.24]) using the bounds on dimension from Proposition 3.24, which also implies Proposition 3.83.

Proposition 3.84 *We have* $\dim(\mathrm{Bub}(m,n)) = m+n$ *for all* $m,n \geq 0$.

Proof. In [73, Corollary 4.10], it is proved that the subposet $\mathrm{JIrr}(\mathrm{Bub}(m,n))$ on the join-irreducible elements of $\mathrm{Bub}(m,n)$ is the disjoint union of an m-element antichain with n chains. This proves that $\mathrm{Bub}(m,n)$ has $m+n$ atoms and $\mathrm{width}\big(\mathrm{JIrr}(\mathrm{Bub}(m,n))\big) = m+n$, which is an upper-bound of $\dim(\mathrm{Bub}(m,n))$ by Proposition 3.24. Since $\mathrm{Bub}(m,n)$ is a semidistributive lattice with $m+n$ atoms, using Proposition 3.25 we have $\dim(\mathrm{Bub}(m,n)) \geq m+n$. This implies the proposition. □

3.6.2 (m,n)-word lattices

We focus in this section on another generalization of the Hochschild lattice, called (m,n)-word lattices ([84, Definition 77],[76]).

Definition 3.85 ([84, Definition 77]) *Let* $m,n \geq 0$. *The* (m,n)*-words are the words* $w = w_1 \cdots w_n$ *of length* n *on the alphabet* $\{0,1,\ldots,m+1\}$ *such that* $w_1 \neq m+1$ *and for* $s \in [m]$, $w_i = s$ *implies* $w_j \geq s$ *for all* $j < i$. *The* (m,n)-word poset $W(m,n)$ *is the poset on* (m,n)*-words with the product order, which means* $w \leq w'$ *if* $w_i \leq w'_i$ *for all* $i \in [n]$.

The poset $W(m,n)$ is a lattice [84, Corollary 84] and is semidistributive and extremal [76, Theorem 1.1]. Similarly to the bubble lattice $\mathrm{Bub}(m,n)$, it is isomorphic to $\mathrm{Hoch}(n)$ when $m=1$ [84, Example 78]. But these are different generalizations of the Hochschild lattice. See Figure 3.8b for the Galois graph of $W(2,3)$, and Figure 3.10 for the lattice $W(2,3)$.

Lemma 3.86 ([76, Lemma 12]) *The join-irreducible elements of* $W(m,n)$ *are the* (m,n)*-words* $a^{(i,j)}$ *for*

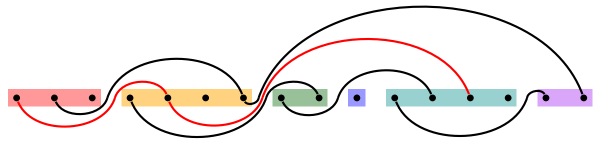


Figure 3.11: A noncrossing α-partition.

$i \in [n]$ and $j \in [m]$, and $b^{(i)}$ for $i \in \{2, \ldots, n\}$, where $a^{(i,j)}$ is the (m,n)-word with the first i letters equal to j and other letters are 0, and $b^{(i)}$ is the (m,n)-word whose i-th letter is $m+1$ and all other letters are 0.

Proposition 3.87 ([76, Proposition 22]) *The downward label set of $w \in W(m,n)$ is*

$$\{a^{(i,j)} \mid i \text{ is the position of the rightmost letter } j \text{ in } w, \text{ where } j \in [m]\} \sqcup \{b^{(i)} \mid w_i = m+1\}.$$

Proposition 3.88 *We have $\dim(W(m,n)) = n$ for all $m, n \geq 0$.*

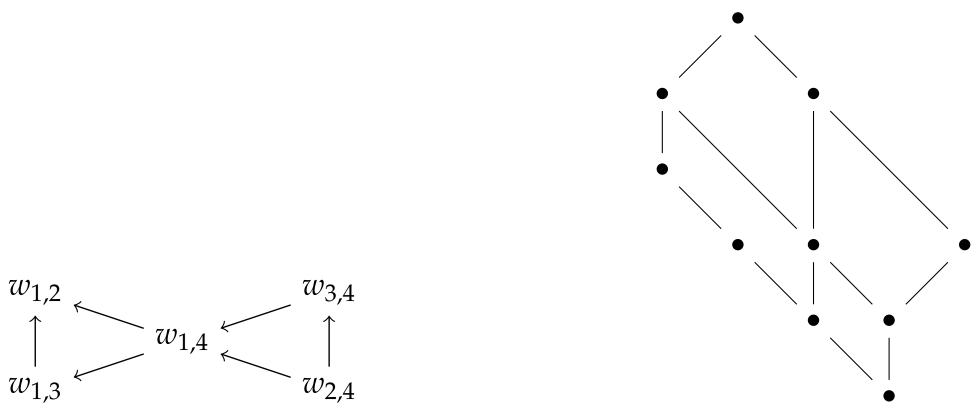
Proof. Let $m, n \geq 0$ and let $L := W(m,n)$. By Theorem 3.77, we have $\dim(L) = \chi(\overline{G(L)})$, which is the chromatic number of the canonical join graph of L. Color a join-irreducible element of L by the position of its rightmost non-zero letter. By Proposition 3.87, this is a proper coloring with n colors of $\overline{G(L)}$, thus $\dim(L) \leq n$. As the downward label set of the maximum element $m(m+1)\ldots(m+1)$ of L has n elements, thus $\chi(\overline{G(L)}) \geq n$, and the proposition follows. $\square$

Remark 3.89 *We saw that the bounds from Proposition 3.24 were enough to find the dimension of the bubble lattices. For $W(m,n)$ it is not enough as [76, Lemma 24] gives its subposet on its join-irreducible elements and it follows that $\operatorname{width}\big(\operatorname{JIrr}(W(m,n))\big) = n+1$, which is strictly bigger than $\dim(W(m,n)) = n$.*

3.6.3 Parabolic Tamari lattices

In this section, we are interested in the parabolic Tamari lattice [80, 79], which is a generalization of the Tamari lattice.

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be an integer composition of $n > 0$. We draw n horizontally aligned nodes labeled $1, \ldots, n$ and put vertical bars separating the node $\alpha_1 + \cdots + \alpha_i$ from the node $\alpha_1 + \cdots + \alpha_i + 1$ for all



a The Galois graph $G(\mathrm{Tam}(1,2,1))$. b The lattice $\mathrm{Tam}(1,2,1)$.

Figure 3.12: The parabolic Tamari lattice $\mathrm{Tam}(1,2,1)$.

$i \in [k]$. The sets of nodes between two such separating bars are the α-regions, they are naturally labeled $1,\ldots,k$ from left to right.

An α-arc is a pair (a,b), where $1 \leq a < b \leq n$ and a,b are in different α-regions. We represent graphically an α-arc (a,b) by drawing a curve leaving the bottom of node a, staying below the α-region containing a, moving up and above the following α-regions until it reaches the top of node b. We will also denote an α-arc (a,b) by $w_{a,b}$. Two α-arcs (a_1,b_1) and (a_2,b_2) are *compatible* if $a_1 \neq a_2$, $b_1 \neq b_2$, and if $a_1 < a_2 < b_1 < b_2$ then either a_1,a_2 or a_2,b_1 are in the same α-region, and if $a_1 < a_2 < b_2 < b_1$ then a_1,a_2 are in the same α-region. A set of pairwise compatible α-arcs are called *noncrossing α-partitions*. The set of all noncrossing α-partitions is denoted by $\mathrm{Nonc}(\alpha)$. See Figure 3.11 (which is taken from [79]) for an illustration of a noncrossing α-partition, where α-regions are depicted with different colors. In this figure $\alpha = (3,4,2,1,4,2)$ and the two red α-arcs are $(1,5)$ and $(5,13)$.

Definition 3.90 ([79, Theorem 1.3]) *Let α be an integer composition of $n > 0$. The* parabolic Tamari lattice $\mathrm{Tam}(\alpha)$ *is an extremal lattice whose Galois graph is isomorphic to the directed graph whose vertices are the α-arcs $w_{a,b}$ and the edges are between different α-arcs $w_{a_1,b_1} \to w_{a_2,b_2}$ such that either a_1 and a_2 are in the same α-region and $a_1 \leq a_2 < b_2 \leq b_1$, or a_1 and a_2 are in different α-regions and $a_2 < a_1 < b_2 \leq b_1$, where a_1 and b_2 are also in different α-regions.*

See Figure 3.12 for an example. This lattice was defined in [80]. The fact that they are semidistributive and extremal is [79, Theorem 1.1]. If $\alpha = (1, 1, \dots, 1)$ then $\mathrm{Tam}(\alpha)$ is the Tamari lattice [80, Proposition 8].

Using [79, Proposition 3.5], it follows that the canonical join graph of $\mathrm{Tam}(\alpha)$, which is $\overline{G(\mathrm{Tam}(\alpha))}$, is the graph whose vertices are the α-arcs, with an edge between two compatible such arcs. Thus its cliques are the noncrossing α-partitions.

Proposition 3.91 *Let* $\alpha = (\alpha_1, \dots, \alpha_k)$ *be an integer composition of* $n > 0$*. Denote by* $\max(\alpha)$ *the biggest integer appearing in* α*. We have* $\dim(\mathrm{Tam}(\alpha)) = n - \max(\alpha)$*.*

Proof. Let i such that $\alpha_i = \max(\alpha)$. A proper coloring of $\overline{G(\mathrm{Tam}(\alpha))}$ with $n - \max(\alpha)$ colors can be obtained by coloring an α-arc (a, b) by a if a is in an α-region labeled by $k < i$, and the other α-arcs (a', b') which have not yet been assigned a color are colored by b'. Thus the colors are labels of nodes that are not in the i-th α-region, thus this uses at most (in fact exactly) $n - \max(\alpha)$ colors. This is a proper coloring since two α-arcs having the same color share the same starting node, or ending node, thus they are not compatible. This proves that $\chi(\overline{G(\mathrm{Tam}(\alpha))}) \leq n - \max(\alpha)$.

We still need to prove that $\chi(\overline{G(\mathrm{Tam}(\alpha))}) \geq n - \max(\alpha)$. This comes from the fact that we can find a noncrossing α-partition with this number of α-arcs; for every $r \in [k]$ draw an α-arc between the j^{th} node of the r-th α-region and the j^{th} node of the $(r+1)$-st α-region whenever these nodes exist. This forms a noncrossing α-partition with $n - \max(\alpha)$ total number of α-arcs. □

For example, the planar lattice $\mathrm{Tam}(1, 2, 1)$ represented in Figure 3.12 is of dimension 2.

3.6.4 On some lattices of torsion classes

In this section we will mainly focus on the lattices of torsion classes of a gentle tree, but we also include a short discussion about the lattices of torsion classes of the simply laced Dynkin quivers $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{E}$.

Proposition 3.92 ([103, Theorem 2.10]) *If* A *is representation finite and* $\mathrm{mod}\, A$ *has no cycles, then* $\mathrm{Tors}(A)$ *is a finite semidistributive extremal lattice.*

This proposition applies to the simply laced Dynkin quivers $\mathbf{A}$, $\mathbf{D}$, $\mathbf{E}$. Other examples, which contain type

$\mathbf{A}$, are the gentle trees (see Section 3.4.8). Let us first focus on the former.

Denote by $\mathrm{Camb}(\vec{Q})$ the Cambrian lattice associated to an orientation of a simply laced Dynkin diagram Q ([86]). It is proved in [56, Section 4.2] that if $\vec{Q}$ is a simply laced Dynkin quiver, then $\mathrm{Tors}(\mathbb{K}\vec{Q}) \cong \mathrm{Camb}(\vec{Q})$. For example, it is known that when the path quiver $\mathbf{A}$ is linearly oriented, we obtain the Tamari lattice. See also Figure 3.14b for another example, which is a Cambrian lattice of type $\mathbf{A}_3$.

If $\vec{Q}$ has n vertices, then $\dim(\mathrm{Camb}(\vec{Q})) \geq n$ using Proposition 3.25. Also, by definition, $\mathrm{Camb}(\vec{Q})$ is a subposet of the weak order of the Coxeter group associated to Q. In [90], it is proved that the dimension of the weak order of type $\mathbf{A}_n$ and $\mathbf{D}_n$ is n, and bounds are given for the dimension of the weak orders on the exceptional types $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$. It follows that for all orientations of the corresponding Coxeter diagrams, $\dim\left(\mathrm{Camb}(\overrightarrow{\mathbf{A}_n})\right) = n$ and $\dim\left(\mathrm{Camb}(\overrightarrow{\mathbf{D}_n})\right) = n$. For the exceptional types, with the help of the computer, we obtain:

Proposition 3.93 *For all orientations of the corresponding Coxeter diagrams, we have* $\dim\left(\mathrm{Camb}(\overrightarrow{\mathbf{E}_6})\right) = 6$, $\dim\left(\mathrm{Camb}(\overrightarrow{\mathbf{E}_7})\right) = 7$ *and* $\dim\left(\mathrm{Camb}(\overrightarrow{\mathbf{E}_8})\right) = 8$.

Proof (using a computer). The computations were performed on the Nibi supercomputer, operated by Calcul Québec and the Digital Research Alliance of Canada (Alliance). The Cambrian lattices of any orientation of types $\mathbf{E}_6$, $\mathbf{E}_7$, and $\mathbf{E}_8$ have respectively 833, 4160, and 25080 elements. The number of vertices of their respective Galois graphs are 36, 63 and 120. With SageMath, using Theorem 3.77, we were able to exhaustively check that the Cambrian lattices of the 32 different orientations of $\mathbf{E}_6$ have all dimension 6, the Cambrian lattices of the 64 different orientations of $\mathbf{E}_7$ have all dimension 7, and the Cambrian lattices of the 128 different orientations of $\mathbf{E}_8$ have all dimension 8. The latter required roughly one week of computation. □

Remark 3.94 *For the Cambrian lattices of type* $\mathbf{A}$*, it is also possible to use a similar combinatorial model to the above discussion about the parabolic Tamari lattice in terms of noncrossing partitions ([88, Example 4.9], [9, section 8.3]). In this model, by coloring an arc with the label of its starting node, we prove analogously to Proposition 3.91 that* $\dim\left(\mathrm{Camb}(\overrightarrow{\mathbf{A}_n})\right) = n$ *for any orientation of* $\mathbf{A}_n$.

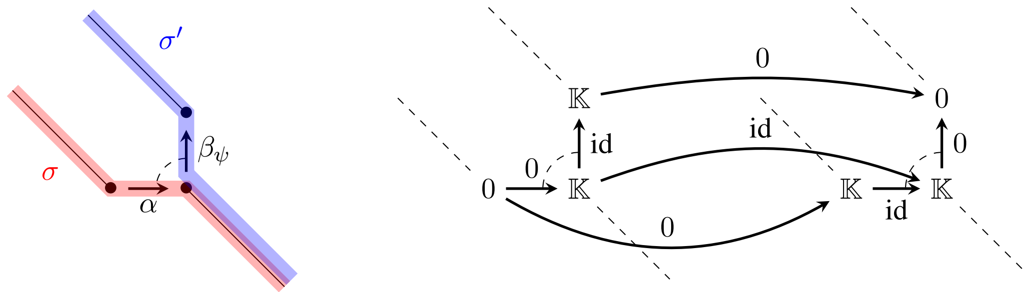
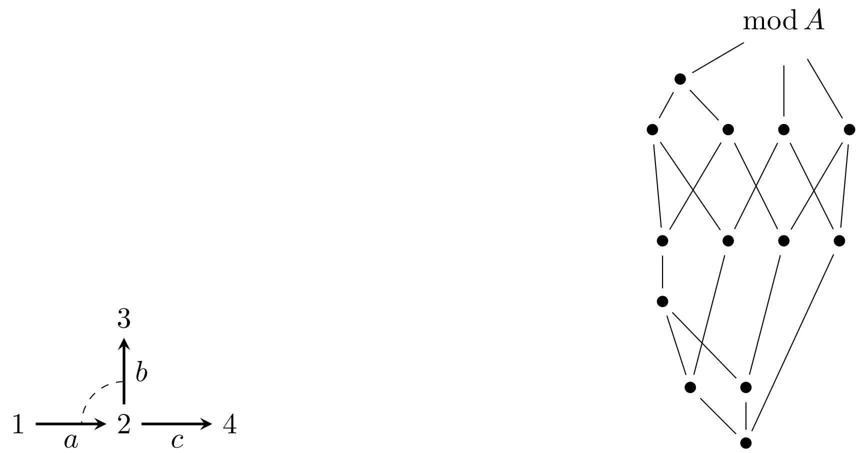


Figure 3.13: Illustration of the proof of Lemma 3.95, showing on the right a part of the non-zero morphism from $C_{\sigma'}$ to C_σ.

Now we focus on the torsion classes of a gentle tree. To see more details about the combinatorics of gentle algebras, see [22, 83]. Given a gentle tree T, there is an (almost) canonical way to draw it so that arrows always go horizontally to the right or vertically upwards, and such that the relations correspond to a change of direction. (It is almost canonical because for a given arrow, we can choose whether to draw it horizontally or vertically; this forces the direction of all other arrows). From now on, we assume that our gentle trees are drawn in this way (having fixed an orientation of the edges). The horizontal arrows can be thought of as east steps and the vertical ones as north steps. Thus, the relations correspond to taking an east step followed by a north step, or a north step followed by an east step. See Figure 3.14a for an example of a gentle tree drawn with our conventions.

The indecomposable modules of T correspond to the walks in the tree which take the unique shortest path in the underlying graph between their two endpoints without containing two steps that form a relation. Those are called the *string modules* and we write C_σ for the indecomposable module corresponding to the walk σ. As a representation this corresponds to putting $\mathbb{K}$ at every vertex visited by σ, 0 at every other vertex and the identity map id between any two adjacent copies of $\mathbb{K}$. With the orientation of the gentle tree fixed as above, there is a canonical way to read a walk σ: from top left to bottom right. Each walk therefore has a well-defined top left endpoint, and a well-defined bottom right endpoint. We refer to them simply as the *left endpoint* and the *right endpoint*. Then, any walk σ can be uniquely written as the word recording the arrows α or opposite arrows α^{-1} met on the way from its left endpoint to its right endpoint.

Lemma 3.95 *Let σ and σ' be two different walks on a gentle tree T that share the same right endpoint. Then C_σ and $C_{\sigma'}$ are not hom-orthogonal.*



a A gentle tree T with relation $ab = 0$.

b The lattice of torsion classes of $1 \to 2 \leftarrow 3$.

Figure 3.14

Proof. The walks σ, σ', sharing the same right endpoint, can be written $\sigma = \sigma_1 \alpha \gamma$ and $\sigma' = \sigma'_1 \beta^{-1} \gamma$ where $\alpha \neq \beta_\psi$. Moreover since the underlying undirected graph of T is a tree, we know that the walks σ and σ' have disjoint supports. Without loss of generality, we can assume that α is an east step and β_ψ a north step. Then we have a non-zero morphism from $C_{\sigma'}$ to C_σ, which is the identity map on the vertices of γ and the zero map on every other vertex. See Figure 3.13, with on the right the crucial part of this non-zero morphism. □

Proposition 3.96 ([8, Theorem 1.8]) *If A is representation finite, the faces of the canonical join complex are the semibricks.*

Thus for A representation finite, $\overline{G(\mathrm{Tors}(A))}$ is the graph whose vertices are the bricks and we have an edge between B and B' if they are hom-orthogonal.

Proposition 3.97 *Let T be a gentle tree having n vertices. Then $\dim(\mathrm{Tors}(T)) = n$.*

Proof. Since a gentle tree is representation finite, Proposition 3.96 applies. The category $\operatorname{mod} A$ has n simples and they form a semibrick as their supports are disjoint. Thus $\dim(\mathrm{Tors}(T)) \geq n$. To conclude, let us prove that $\dim(\mathrm{Tors}(T)) \leq n$. For this we find a proper coloring of $\overline{G(\mathrm{Tors}(T))}$ with n colors. Since T is a gentle tree, every brick of T is a string representation C_σ, thus it has a right endpoint. Color each brick with the label of its right endpoint. This gives a coloring with n colors of $\overline{G(\mathrm{Tors}(T))}$, which is proper by Lemma 3.95. This proves the proposition. □

Corollary 3.98 *We have* $\dim\left(\mathrm{Camb}(\overrightarrow{\mathbf{A_n}})\right) = n$ *for any orientation of* $\mathbf{A}_n$.

Proof. This follows from Proposition 3.97 since $\mathrm{Camb}(\overrightarrow{\mathbf{A_n}})$ is the lattice of torsion classes of an orientation of the path quiver $\mathbf{A}_n$, which is a particular gentle tree having n vertices. □

3.7 Open questions

We finish by giving some open questions.

In [6, Figure 9] an example is given of an extremal congruence uniform lattice of dimension 3 obtained from the doubling of a chain, thus of dimension 1, of an extremal congruence uniform lattice of dimension 2. This is a counterexample to the observation that often $\dim(L[C]) = \dim(L)$ when the interval C satisfies $\dim(C) < \dim(L)$.

Question 3.99 *Let L be a lattice. Under which condition on an interval C of L do we have* $\dim(L[C]) = \dim(L)$*?*

In Theorem 3.52, we proved the equivalence between extremality and shellability in the case of congruence uniform lattices. For the more general case of semidistributive lattices, we know that extremality implies shellability (Corollary 3.33). What about the converse?

Question 3.100 *Is it true that any semidistributive shellable lattice is extremal?*

Another direction to generalize Theorem 3.52 is to look at join-congruence uniform and meet-congruence uniform lattices. We proved in Corollary 3.51 that any join-congruence uniform lattice that is shellable is join-extremal. What about the converse:

Question 3.101 *Is it true that any join-congruence uniform lattice that is join-extremal is shellable?*

In Section 3.5.4 we proved that an induced subgraph of a canonical join graph is not always a canonical join graph, even if the initial canonical join graph can be obtained from a semidistributive lattice that is extremal. For similar questions, it would be nice to have an affirmative answer to the following question, as it would restrict the class of lattices we have to look at:

Question 3.102 *Let G be the canonical join graph of a semidistributive lattice. Is it true that G can also be obtained as the canonical join graph of an extremal semidistributive lattice?*

Different lattices that we considered in this paper can be obtained as the orientation of the 1-skeleton of a polytope by choosing a generic cost vector (see the precise definition in [55]). In our examples, all the ones coming from a n-dimensional polytope were of order dimension n. Since in these examples those lattices are all semidistributive we know that we have the lower bound n for their order dimension by Proposition 3.25.

Question 3.103 *If a lattice L is obtained as an orientation of the 1-skeleton of a n-dimensional polytope, is it true that $\dim(L) = n$? If it is false, does it hold with the extra hypothesis that L is semidistributive and/or extremal?*

Note that it seems that the face lattice associated to a polytope behaves very differently since for any $n \geq 4$ there exist n-dimensional polytopes whose face lattice has arbitrarily large order dimension, and for $n = 3$ the face lattice has order dimension less than or equal to 4 [21].

The lattices of torsion classes are *Hasse-regular* lattices, meaning any element x in their Hasse diagram has the same degree, which is the sum of the number of elements x covers with the number of elements that cover x. A lattice is called *r-Hasse-regular* if the degree of any element is r.

Question 3.104 *Is it true that any semidistributive r-Hasse-regular lattice is of dimension r? If not, is it true if the lattice is additionally congruence uniform?*

CHAPITRE 4
COMBINATOIRE DES TREILLIS DE CLASSES DE TORSIONS SUPÉRIEURES

4.1 Avant-Propos et Résumé

Dans ce dernier chapitre, on présente une nouvelle construction de treillis appelée (P, ϕ)-Tamari. Sa définition est motivée par le fait que dans un cas très particulier ils donnent des treillis apparus récemment en théorie des représentations des algèbres, à savoir les treillis des classes de torsion supérieures des algèbres supérieures d'Auslander et Nakayama de type $\mathbf{A}$ [3]. L'étude détaillée des treillis (P, ϕ)-Tamari nous permet donc d'obtenir de nombreuses propriétés combinatoires de ces treillis.

Il s'agit d'un article de 35 pages, intitulé « (P, ϕ)-Tamari and higher torsion lattices of type $\mathbf{A}$ », qui sera prochainement déposé sur arXiv. Une version courte de 12 pages a été acceptée pour une présentation à la conférence FPSAC 2025 et publiée dans un volume du Séminaire Lotharingien de Combinatoire [95].

Soit P un ordre dont les idéaux d'ordre principaux sont finis, et $\phi : \phi(0) < \phi(1) < \cdots$ une chaîne de P. On rappelle que $I_P(\phi(k))$ est l'idéal d'ordre principal de P engendré par $\phi(k)$. On définit l'ordre $\mathcal{C}_P^\phi = \bigsqcup_{k\geq 0} I_P(\phi(k)) \times \{k\}$. Alors l'ordre (P, ϕ)-Tamari est l'ordre d'inclusion sur les filtres d'ordre T de $\mathcal{C}_P^\phi$ vérifiant pour tout $0 \leq i < j$, si $(x, i) \in T$ et $(\phi(i+1), j) \in T$, alors $(x, j) \in T$.

L'ordre (P, ϕ)-Tamari est toujours un treillis, noté $\mathrm{Tam}(P, \phi)$. Nous montrerons que lorsque $P = \phi$ est la chaîne à n éléments, alors $\mathrm{Tam}(P, \phi)$ est le treillis de Tamari de taille $n + 1$. De plus, lorsque P est l'ordre produit sur les d-uplets croissants dont les éléments sont dans $\{0, 1, \ldots, n-1\}$, et ϕ est la chaîne formée des d-uplets constants, le treillis obtenu est celui des d-classes de torsion de l'algèbre $(d-1)$-Auslander de type $\mathbf{A}$.

Nous montrerons qu'en général $\mathrm{Tam}(P, \phi)$ est sup-congruence uniforme, modulaire à gauche et sup-extrémal. Nous montrerons également que ses intervalles non vides sont contractiles ou homotopie équivalentes à une sphère. Lorsque P est un sup-semitreillis, alors $\mathrm{Tam}(P, \phi)$ a certaines congruences de treillis que nous appelons de types Kupisch. Utilisant ce dernier résultat, nous montrons que le treillis des d-classes de torsion de l'algèbre $(d-1)$-Nakayama associée à une série de Kupisch est un quotient du treillis des d-classes de torsion de l'algèbre $(d-1)$-Auslander de type $\mathbf{A}$. En particulier ceci permet de montrer que ces treillis sont également sup semi-distributifs.

Finalement, nous terminerons en donnant des questions ouvertes.

4.2 Abstract

The goal of this work is to study the combinatorics of the lattices of higher torsion classes of the higher Auslander and Nakayama algebras of type $\mathbf{A}$. Combinatorial descriptions of these higher torsion classes were recently obtained by August *et al.* (2025), and it was observed that the lattices that they form are not semidistributive. We study in some depth these lattices, proving in particular that the lattices of higher torsion classes of the higher Auslander algebras of type $\mathbf{A}$ are join-semidistributive, join-extremal and left modular. We also prove that the lattices of higher torsion classes of the higher Nakayama algebras of type $\mathbf{A}$ are lattice quotients of them. In order to prove these results, we define a general construction that produces a lattice, which we call (P, ϕ)-Tamari, for any choice of poset P and chain ϕ in P. We prove the lattice results for this general construction. When $P = \phi$ is a chain, we recover the Tamari lattice, whereas the lattices of higher torsion classes that we study are obtained for another very particular choice.

4.3 Introduction

The Tamari lattice is familiar to many in combinatorics. It is perhaps most simply described as the lattice of planar binary trees ordered by tree rotation. For our purposes, it is also important to point out that it arises as the lattice of torsion classes for the type $\mathbf{A}_n$ linearly oriented path algebra. Our goal is to present a certain combinatorial generalization of the Tamari lattice which we call the (P, ϕ)-Tamari lattice, where P is a poset and ϕ is a chain in P. We establish certain properties for these lattices, including join-semidistributivity, join-extremality, and left modularity.

The reason that we were inspired to formulate the definition of the (P, ϕ)-Tamari lattices is that they include, as a very special case, certain lattices which recently appeared in representation theory. For any finite-dimensional algebra, its torsion classes ordered by inclusion form a lattice. These have attracted considerable interest [35, 101]. One property in particular of interest is that they are all semidistributive. There is a generalization due to Jørgensen of torsion classes [62] in the setting of higher homological algebra [59, 57, 58], known as d-torsion classes (or higher torsion classes). Recently the authors of [3] showed that Jørgensen's definition is equivalent to being closed under d-extensions and d-quotients, thus yielding a lattice ordered by inclusion on these d-torsion classes. They obtained a combinatorial description of the d-torsion classes for the higher Auslander and Nakayama algebras of type $\mathbf{A}$, which are the two main examples where we are able to compute in higher homological algebra. They noticed that the lattices of d-torsion classes of these algebras are not semidistributive in general, but we prove that they are a special case of our construction. Thus they inherit the properties of the (P, ϕ)-Tamari lattices.

In Section 4.4 we give background on the relevant lattice theory (Section 4.4.1) and introduce the lattices of higher torsion classes (Section 4.4.2). In Section 4.5 we define and study the (P, ϕ)-Tamari lattices; the definition is given in Section 4.5.1, among other results the join-semidistributivity and join-extremality are proved in Section 4.5.2, we briefly talk about enumeration in Section 4.5.3 and results related to labellings, topology and congruences are proved in Section 4.5.4. In Section 4.6 we look at particular examples of our construction; the case of a poset P that is a chain is treated in Section 4.6.1, whereas the particular cases of the lattices of higher torsion classes of the higher Auslander and Nakayama algebras are respectively studied in Sections 4.6.2 and 4.6.3. We finish by giving some open questions in Section 4.7.

This paper is the full-length version of an extended abstract that was accepted to FPSAC 2025 [95].

Acknowledgments

I am very grateful to Hugh Thomas for proposing this project and for his support. I also want to thank my other advisor Samuele Giraudo, and Baptiste Rognerud for interesting discussions. I was supported by NSERC Discovery Grants RGPIN-2022-03960 and RGPIN-2024-04465.

4.4 Background

4.4.1 Posets and lattices

We denote by $|E|$ the cardinality of a set E. For a positive integer n, denote $[n] := \{1, 2, \ldots, n\}$. By $k < n$ we will mean $k \in \{0, 1, \ldots, n-1\}$.

If a partially ordered set (abbreviated as *poset*) $(P, \leq)$ has a minimum, it is denoted $\hat{0}$, and $\hat{1}$ for the maximum. If a subset $X \subseteq P$ has a minimum element, it is denoted by $\min(X)$. If it has a maximum element, it is denoted by $\max(X)$. The dual of P is denoted P^{op}. Two elements $x, y \in P$ are *comparable* if $x \leq y$ or

$y \leq x$. If this is not the case, then x and y are *incomparable* and we denote it by $x \not\sim y$. A subset $X \subseteq P$ is an *antichain* if the elements of X are pairwise incomparable. The cover relations of P are denoted $x \lessdot y$ and we say that y *covers* x or x *is covered by* y. We denote by $\mathrm{Cov}^{\uparrow}_P(x)$ the set of elements that cover x, and by $\mathrm{Cov}^{\downarrow}_P(x)$ the set of elements that are covered by x. They form the set $E(P)$ of the edges of its Hasse diagram, which is drawn with smaller elements at the bottom. If P has a minimum $\hat{0}$, the *atoms* are the elements of P that cover $\hat{0}$. If P has a maximum $\hat{1}$, the *coatoms* are the elements of P that are covered by $\hat{1}$. The *degree* of an element $x \in P$ is $\deg(x) = |\mathrm{Cov}^{\uparrow}_P(x)| + |\mathrm{Cov}^{\downarrow}_P(x)|$. The poset P is *Hasse-regular* if all its elements have the same degree. The interval of P between x and y is denoted by $[x, y]$. By a *subposet* we always mean an induced subposet. Let $X \subseteq P$. If $P \setminus X$ is viewed as a poset, it corresponds to the subposet of P on the elements $P \setminus X$. The subposet of P on the elements $[x, y] \setminus \{y\}$ is denoted by $[x, y[$. Its Möbius function is written $\mu(x, y)$. A subset C of P is *convex* if for all x and y in C, we have $[x, y] \subseteq C$. An *order ideal* of a poset P is a subset I such that for all $x, y \in P$, if $y \leq x$ and $x \in I$, then $y \in I$. Dually, an *order filter* is a subset F such that for all $x, y \in P$, if $y \geq x$ and $x \in F$, then $y \in F$. The order ideal generated by a subset C is denoted by $I_P(C) := \{y \in P \mid \exists x \in C,\, y \leq x\}$. If $C = \{x\}$, $I_P(\{x\})$ is a *principal order ideal* and is denoted by $I_P(x)$. Dually the *principal order filter* generated by $x \in P$ is denoted $F_P(x)$. The poset ordered by inclusion on the order ideals of a poset P is denoted $J(P)$. Then the number of order ideals of P, which is also its number of order filters, is $|J(P)|$. We denote by C_n the chain $0 < 1 < \cdots < n-1$. We also call *chains* the totally ordered subsets of P. Equivalently, they are the image of an injective order preserving map $\phi : \{0, 1, \ldots, n\} \to P$, and we will use $\phi : \phi(0) < \phi(1) < \cdots < \phi(n)$ to denote them. In this case, the length of this chain, denoted by $|\phi|$, is n. We will also briefly consider the case where ϕ is infinite, meaning ϕ is an injective order preserving map from the nonnegative integers to P. If we cannot add elements to a chain, it is called a *maximal chain*. The *length* of a poset, denoted $\ell(P)$, is the maximum length of a chain in P. The chains of length $\ell(P)$ are called *longest chains*. A finite poset is *graded* if all its maximal chains have the same length. The *spine* of a poset is the subset of the elements that lie on some longest chain. A *linear extension* of a poset $(P, \leq)$ is a total order $\preceq$ on P such that $x \leq y$ implies $x \preceq y$. The *disjoint union* of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$, denoted by $P \bigsqcup Q$, is the poset on $P \sqcup Q$ defined by $x \leq y$ if $x \leq_P y$ and $x, y \in P$, or $x \leq_Q y$ and $x, y \in Q$. If P and Q are two posets, then the *direct product* $P \times Q$ is the poset on the cartesian product $P \times Q$ defined by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. We will refer to this latter partial order as the *product order*, sometimes denoted by $\leq_{prod}$.

A *lattice* L is a poset such that any pair of elements $\{x, y\}$ admits a least upper bound, called the *join* and written $x \vee y$, and a greatest lower bound, called the *meet* and written $x \wedge y$. If only the join exists for any pair of elements, then L is a *join-semilattice*. A *complete lattice* is a poset in which any subset of the elements admits a least upper bound and greatest lower bound. Any finite lattice is a complete lattice but the converse is not true in general. We will denote Tam_n the Tamari lattice of size n which has cardinality $\frac{1}{n+1}\binom{2n}{n}$.

The following definitions of join-irreducibles and meet-irreducibles are given for a finite lattice. A *join-irreducible* $j \in L$ is an element that covers a unique element, denoted j_*, and a *meet-irreducible* $m \in L$ is one that is covered by a unique element. The sets of these elements are respectively denoted by $\mathrm{JIrr}(L)$ and $\mathrm{MIrr}(L)$. An *edge-labelling* of L is a map $\gamma : E(L) \to P$ where P is a poset. It is well-known that in a finite lattice, an element is always the join of the join-irreducibles below it. It follows:

Lemma 4.1 *For any finite lattice L, we have* $\ell(L) \leq \min\big(|\mathrm{JIrr}(L)|, |\mathrm{MIrr}(L)|\big)$.

Proof. We prove $\ell(L) \leq |\mathrm{JIrr}(L)|$ as for the meet-irreducibles it is dual. Let $\phi : \phi(0) \lessdot \cdots \lessdot \phi(n)$ be a maximal chain of length n. For all $i < n$, there is always a join-irreducible that is below $\phi(i+1)$ but not $\phi(i)$. If it was not the case, then $\phi(i+1)$, which is the join of the join-irreducibles below it, would be below $\phi(i)$, which is absurd. This proves that we have at least n join-irreducibles. □

Definition 4.2 ([70]) *A lattice L is* join-extremal *if* $\ell(L) = |\mathrm{JIrr}(L)|$, meet-extremal *if* $\ell(L) = |\mathrm{MIrr}(L)|$ *and* extremal *if it is both join-extremal and meet-extremal.*

Definition 4.3 *A lattice L is* complemented *if any element $x \in L$ has a* complement *$y \in L$, which is an element such that $x \vee y = \hat{1}$ and $x \wedge y = \hat{0}$. A lattice L is* meet-pseudocomplemented *if for any element $x \in L$, the set $\{y \in L \mid x \wedge y = \hat{0}\}$ has a maximum element. The* join-pseudocomplemented *lattices are defined dually, and a lattice is* pseudocomplemented *if it is both meet-pseudocomplemented and join-pseudocomplemented.*

Definition 4.4 *A lattice L is* join-semidistributive *if for all $x, y, z \in L$, $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$.* Meet-semidistributive *lattices are defined dually, and it is* semidistributive *if it is both join-semidistributive and meet-semidistributive.*

The following two results can be found in [44, Section 5].

Lemma 4.5 *A lattice L is semidistributive if and only if L is join-semidistributive and* $|\mathrm{JIrr}(L)| = |\mathrm{MIrr}(L)|$.

Lemma 4.6 *A lattice L is join-semidistributive if and only if for all covers $b \lessdot c$, the set $I_L(c) \setminus I_L(b)$ has a minimum element. In this case, the minimum is join-irreducible, and $\gamma : E(L) \to \mathrm{JIrr}(L)$ that sends a cover $b \lessdot c$ to $\min(I_L(c) \setminus I_L(b))$ is a well-defined edge-labelling.*

Definition 4.7 *An element $a \in L$ is* left modular *if for all $b < c$ in L, we have $(b \vee a) \wedge c = b \vee (a \wedge c)$. A maximal chain made of left modular elements is called a maximal left modular chain. The lattice L is left modular if there exists a maximal left modular chain.*

We now recall a result that enables us to prove that a lattice is left modular using some edge-labellings. The names for the labellings in the following definition are chosen to be consistent with [94].

Definition 4.8 *Let L be a lattice. Denote by $\psi :\ \hat{0} = x_0 < \cdots < x_k = \hat{1}$ a chain containing $\hat{0}$ and $\hat{1}$. Using ψ, we define two edge-labellings $\gamma_{1,\psi}$ and $\gamma_{2',\psi}$. For $j \in \mathrm{JIrr}(L)$, denote $\delta_\psi(j) := \min\{i \,|\, j \leq x_i\}$. Then, for a cover relation $b \lessdot c$, we define*

$$\gamma_{1,\psi}(b \lessdot c) := \min\{\delta(j) \,|\, j \textit{ join-irreducible},\ j \leq c,\ j \not\leq b\},$$
$$\gamma_{2',\psi}(b \lessdot c) := \min\{i \,|\, b \vee x_i \geq c\}.$$

Proposition 4.9 ([94, Theorem 3.3]) *Using the notations from Definition 4.8, for any lattice L, we have $\gamma_{1,\psi} = \gamma_{2',\psi}$ if and only if for all i, x_i is left modular.*

We refer the reader to the papers of A. Björner and M. Wachs [14, 15, 16] for details related to the following topological notions. Denote $\overline{L} := L \setminus \{\hat{0}, \hat{1}\}$. The *order complex* $\Delta(P)$ of a poset P is the simplicial complex of vertex set P whose faces are the chains of P. An *EL-labelling* of a lattice L is an edge-labelling such that in any interval, when reading the labels following the maximal chains from bottom to top, there is a unique maximal increasing chain and the label word of the increasing chain lexicographically precedes the label word of any other maximal chain. If L admits an EL-labelling, then its order complex $\Delta(\overline{L})$ is

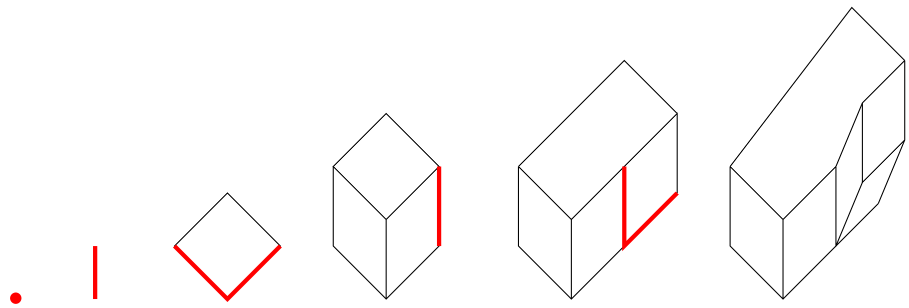


Figure 4.1: We represent five successive doublings of a lattice by lower pseudo-intervals, which are represented with thick red edges.

shellable and homotopy equivalent to a wedge of spheres. Moreover, for all $x < y$ in L we have that $\mu(x, y)$ is given by the difference between the number of even length maximal decreasing chains and the number of odd length maximal decreasing chains of $[x, y]$. Thus if L has at most one maximal decreasing chain in any interval, then $\mu(x, y) \in \{-1, 0, 1\}$ for all $x, y \in L$. Also, the order complex of each nonempty open interval $]x, y[$ has the homotopy type of either a sphere or a point.

Proposition 4.10 ([67]) *If ψ is a left modular chain, then $\gamma_{1,\psi} = \gamma_{2',\psi}$ and this is an EL-labelling. Thus a left modular lattice admits an EL-labelling.*

Definition 4.11 ([28]) *Let C be a convex subset of L. The* doubling *$L[C]$ is the subposet of $L \times C_2$ on the subset $\big(I_L(C) \times \{0\}\big) \bigsqcup \big[\big((L \setminus I_L(C)) \cup C\big) \times \{1\}\big]$. It is in fact a lattice.*

A subset C is a *lower pseudo-interval* if C is a union of intervals sharing the same minimum element. A lattice L is *congruence normal* if it is obtained from the one-element lattice by successive doublings of convex subsets. If at each step we double a lower pseudo-interval then L is *join-congruence uniform* (sometimes called lower-bounded in the literature). If we use only doublings of intervals, L is called *congruence uniform*. See Figure 4.1 for an example of the construction of a join-congruence uniform lattice.

A (lattice) *congruence* on L is an equivalence relation $\equiv$ on L such that for all x_1, x_2, y_1, y_2 in L, we have that $x_1 \equiv x_2$ and $y_1 \equiv y_2$ imply both $x_1 \wedge y_1 \equiv x_2 \wedge y_2$ and $x_1 \vee y_1 \equiv x_2 \vee y_2$. We say that a congruence $\equiv$ *contracts* a join-irreducible j if $j_* \equiv j$. Two different congruences always contract different sets of join-irreducibles. Thus we will identify the congruences with the set of join-irreducibles they contract. Let D, called the *join dependency relation*, be the binary relation on $\mathrm{JIrr}(L)$ defined by aDb if $a \neq b$ and there exists $p \in L$ such that $a \leq b \vee p$ and $a \not\leq b_* \vee p$. Note that aDb implies $a \not\leq b$. A *D-cycle* is a sequence of elements $a_1,\ a_2,\ \ldots, a_k$ with $k \geq 2$ such that $a_1 D a_2 D \cdots D a_k D a_1$.

Proposition 4.12 ([28, 44]) *The lattice L is join-congruence uniform if and only if it contains no D-cycles. In this case, the congruences of L correspond to the subsets $S \subseteq \mathrm{JIrr}(L)$ such that if aDb and $b \in S$, then $a \in S$.*

A lattice congruence on a lattice L gives rise to a *lattice quotient*, which is the lattice on the equivalence classes ordered by the induced order of L on the minimum elements of these classes (which are in fact intervals).

4.4.2 Higher torsion classes

The goal of this section is to introduce the lattices of higher torsion classes and to recall the combinatorial descriptions of these higher torsion classes for the higher Auslander and Nakayama algebras of type $\mathbf{A}$.

We first recall some vocabulary of the representation theory of associative algebras (see for example [2]). Let A be a finite-dimensional $\mathbb{K}$-algebra, where $\mathbb{K}$ is an algebraically closed field. We denote by $\mathrm{mod}A$ the category of finite-dimensional right A-modules. A module is called *indecomposable* if it is not isomorphic to a direct sum of two non-zero modules. Subcategories are assumed to be full and closed under finite direct sums and summands. It follows that a subcategory is the additive hull of its indecomposable modules. Thus we will identify a subcategory with the set of its indecomposable modules. The *torsion classes* are the subcategories of $\mathrm{mod}A$ closed under extensions and quotients. They form a complete lattice with meet given by intersection that was extensively studied [35, 101].

Higher homological algebra is a subject whose study was initiated by O. Iyama [59, 57, 58]. We are interested in a generalization of the torsion classes in the context of higher homological algebra. Let d be a positive integer. The following is an equivalent definition of the d-cluster tilting subcategories (we take it as the definition to avoid introducing more notations):

Definition 4.13 ([65, Proposition 4.4]) *A subcategory $\mathcal{M}$ of $\mathrm{mod}A$ is called d-*cluster tilting* if*

$\mathrm{Ext}^i_A(\mathcal{M},\mathcal{M}) = 0$ *for all* $0 < i < d$*, and if for all* $M \in \mathrm{mod}A$ *there exist exact sequences*

$$0 \to X_d \to \cdots \to X_1 \to M \to 0 \qquad \textit{and} \qquad 0 \to M \to X^1 \to \cdots \to X^d \to 0$$

with $X_1, \ldots, X_d, X^1, \ldots, X^d \in \mathcal{M}$.

In the sequel, $\mathcal{M}$ is a d-cluster tilting subcategory of $\mathrm{mod}A$.

Definition 4.14 ([62]) *A subcategory* $\mathcal{U}$ *of* $\mathcal{M}$ *is a* d-torsion class *if for all* $M \in \mathcal{M}$ *there exists an exact sequence* $0 \to \mathcal{U}_M \to M \to X^1 \to \cdots \to X^d \to 0$ *with* $X^1, \ldots, X^d \in \mathcal{M}$ *and* $\mathcal{U}_M \in \mathcal{U}$*, such that the following sequence is exact for all* $U \in \mathcal{U}$*:*

$$0 \to \mathrm{Hom}_A(U, X^1) \to \cdots \to \mathrm{Hom}_A(U, X^d) \to 0.$$

The d-torsion classes are also called *higher torsion classes*. When $d = 1$ we have $\mathcal{M} = \mathrm{mod}A$ and the 1-torsion classes correspond to the usual torsion classes. Thus the d-torsion classes are a generalization of the usual torsion classes in higher homological algebra. What makes this generalization maybe more challenging to work with is that the d-cluster tilting subcategories are not abelian in general, so one needs to be careful when using standard homological algebra techniques.

Definition 4.15 *A subcategory* $\mathcal{U}$ *of* $\mathcal{M}$ *is closed under*

(1) d-extensions *if for all exact sequences* $0 \to X \to X_1 \to \cdots \to X_d \to Y \to 0$ *of* $\mathcal{M}$ *with* $X, Y \in \mathcal{U}$*, there exists an equivalent* d*-extension where all the objects are in* $\mathcal{U}$*.*

(2) d-quotients *if for all* $f : M \to U$ *from* $M \in \mathcal{M}$ *to* $U \in \mathcal{U}$*, there exists an exact sequence* $M \to U \to X_1 \to \cdots \to X_d \to 0$ *with* $X_1, \ldots, X_d \in \mathcal{U}$*.*

The main result of [3] is the following equivalent definition of the d-torsion classes:

Theorem 4.16 ([3, Theorem 1.1]) *A subcategory* $\mathcal{U}$ *of* $\mathcal{M}$ *is a* d*-torsion class if and only if it is closed under* d*-extensions and* d*-quotients.*

It follows from Theorem 4.16 that an arbitrary intersection of d-torsion classes is a d-torsion class, which proves the following:

Theorem 4.17 ([3, Theorem 1.2]) *The poset of d-torsion classes in $\mathcal{M}$ ordered by inclusion is a complete lattice with meet given by intersection.*

In higher homological algebra, the two main examples where we are able to compute and have a combinatorial understanding are the $(d-1)$-Auslander and the $(d-1)$-Nakayama algebras of type $\mathbf{A}$ (introduced respectively in [58] and [60]). These algebras are also called respectively *higher* Auslander and Nakayama algebras of type $\mathbf{A}$. We do not recall their definitions, but the interested reader can look at [3, Section 5 and 6], where the definitions and essential properties are given. The only thing we need to know is that the module categories of these algebras always contain a unique d-cluster tilting subcategory, whose indecomposable modules are respectively indexed by os_n^{d+1} and $os_{\underline{l}}^{d+1}$, two posets that we will shortly define. When we will talk about the d-torsion classes of one of these algebras, we mean the d-torsion classes of its unique d-cluster tilting subcategory. We finish this background by recalling the combinatorial descriptions of the higher torsion classes of these algebras obtained in [3]. First we need to introduce some notations.

Let n be a positive integer. Let os_n^d be the set of non-decreasing sequences $x = (x_1, \dots, x_d)$ of size d of elements of $\{0, 1, \dots, n-1\}$. Let $\leq$ be the product order on os_n^d (see Figure 4.11a for an example). On os_n^d we also define a binary relation $x \rightsquigarrow y$ if $x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_d \leq y_d$. On $\mathbb{Z}^d$ we define the map $\tau_d(x_1, \dots, x_d) = (x_1 - 1, x_2 - 1, \dots, x_d - 1)$.

Theorem 4.18 ([3, Theorem 5.13]) *The d-torsion classes of the $(d-1)$-Auslander algebra of type $\mathbf{A}_n$ are identified with the subsets $T \subseteq os_n^{d+1}$ that satisfy the following two conditions:*

1. *For all $i < n$, $\{x \in T \mid x_{d+1} = i\}$ is an order filter of the poset $\{x \in os_n^{d+1} \mid x_{d+1} = i\}$ ordered with the product order.*
2. *For all $x, z \in T$, if $x \rightsquigarrow \tau_d(z)$, then any $y \in os_n^{d+1}$ with $y_i \in \{x_i, z_i\}$ for each i must be in T.*

Denote by L_n^d the lattice of the d-torsion classes of the $(d-1)$-Auslander algebra of type $\mathbf{A}_n$. The only combinatorial properties that were discussed in [3] about the lattices L_n^d are that they are not semidistributive nor Hasse-regular in general. Indeed, L_3^3 is neither semidistributive nor Hasse-regular ([3, Example 5.21]). We will give different combinatorial properties satisfied by L_n^d in Section 4.6.2.

Remark 4.19 *Condition (2) of Theorem 4.18 is harder to manipulate than condition (1), but in Section 4.6.2 we will give a reformulation of Theorem 4.18, which will greatly simplify the study of these d-torsion classes.*

A *Kupisch series of type $\boldsymbol{A}_n$* is a sequence $\underline{l} = (l_0, l_1, \ldots, l_{n-1})$ of positive integers satisfying $l_0 = 1$ and $\forall i \geq 1,\ 2 \leq l_i \leq l_{i-1} + 1$. Let $os_{\underline{l}}^{d+1} := \{y \in os_n^{d+1} \mid y_1 \geq y_{d+1} - l_{y_{d+1}} + 1\} \subseteq os_n^{d+1}$.

Theorem 4.20 ([3, Theorem 6.1]) *The d-torsion classes of the $(d-1)$-Nakayama algebra associated to the Kupisch series $\underline{l}$ are identified with the subsets $T \subseteq os_{\underline{l}}^{d+1}$ that satisfy the following two conditions:*

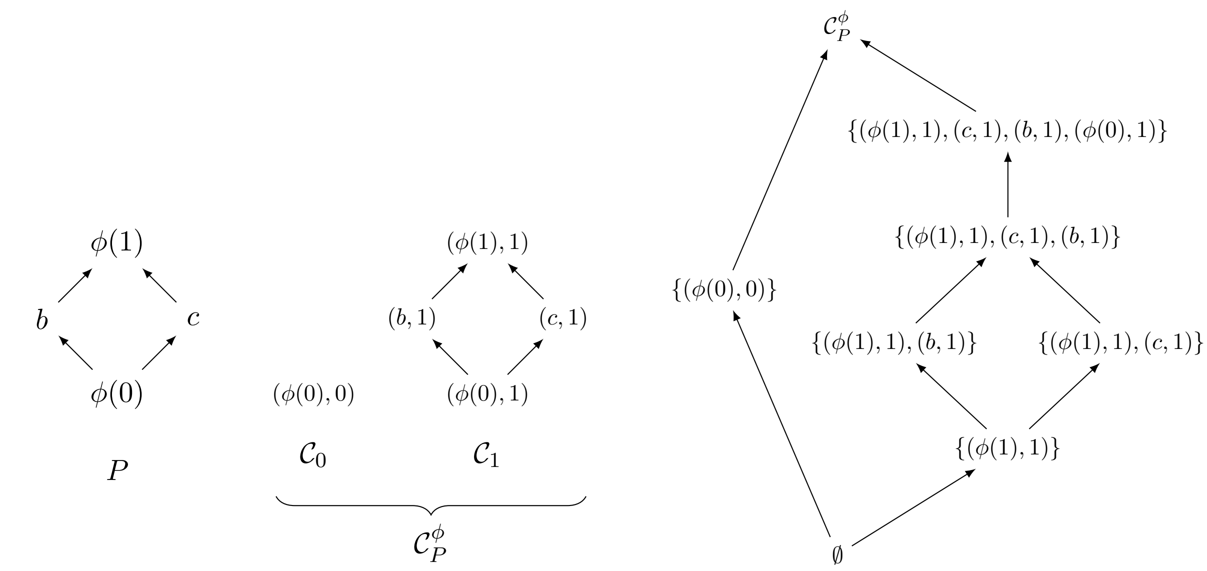
1. *For all $i < n$, $\{x \in T \mid x_{d+1} = i\}$ is an order filter of the poset $\{x \in os_{\underline{l}}^{d+1} \mid x_{d+1} = i\}$ ordered with the product order.*
2. *For all $x, z \in T$, if $x \rightsquigarrow \tau_d(z)$, then any $y \in os_{\underline{l}}^{d+1}$ with $y_i \in \{x_i, z_i\}$ for each i must be in T.*

Denote by $L_{\underline{l}}^d$ the lattice of the d-torsion classes of the $(d-1)$-Nakayama algebra of type $\mathbf{A}$ associated to $\underline{l}$. This is a generalization of the case of the higher Auslander algebras, as this particular case is obtained by taking $\underline{l} = (1, 2, \ldots, n)$. We will give different combinatorial properties satisfied by $L_{\underline{l}}^d$ in Section 4.6.3.

4.5 (P, ϕ)-Tamari lattices

4.5.1 Definition of the (P, ϕ)-Tamari lattices

Let P be a poset such that all its principal order ideals are finite. Let $\phi : \phi(0) < \phi(1) < \cdots$ be any chain of P. For any $i \geq 0$, denote $\mathcal{C}_i := I_P(\phi(i)) \times \{i\}$ where $\times$ is the direct product. The finite poset $\mathcal{C}_i$ is called the *i-th component* of the poset $\mathcal{C}_P^\phi := \bigsqcup_{i \geq 0} \mathcal{C}_i$. Very particular elements of $\mathcal{C}_P^\phi$ that will play an important role are the $(\phi(k), i)$ for all $0 \leq k \leq i$. In $\mathcal{C}_i$, these elements form a chain $(\phi(i), i) > (\phi(i-1), i) > \cdots > (\phi(0), i)$ and $(\phi(i), i)$ is the maximum element of $\mathcal{C}_i$. If T is an order filter of $\mathcal{C}_P^\phi$, then for any i, we denote $T_i := T \cap \mathcal{C}_i$ and $T_{<i} := \sqcup_{k<i} T_k$ (and similarly for $T_{\leq i}$, $T_{>i}$ and $T_{\geq i}$). See Figures 4.2 and 4.3 for examples of the following definition.



a A poset P with a chain ϕ, and the poset $\mathcal{C}_P^\phi$.

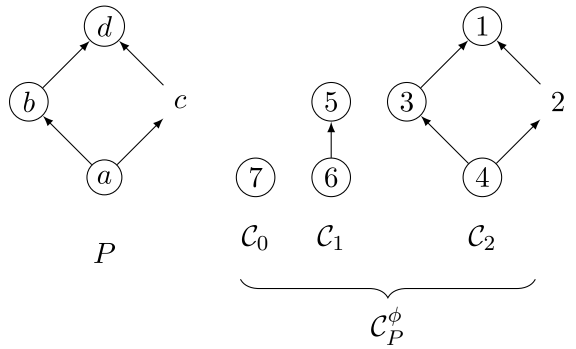
b $\mathrm{Tam}(P, \phi)$ from Figure 4.2a.

Figure 4.2

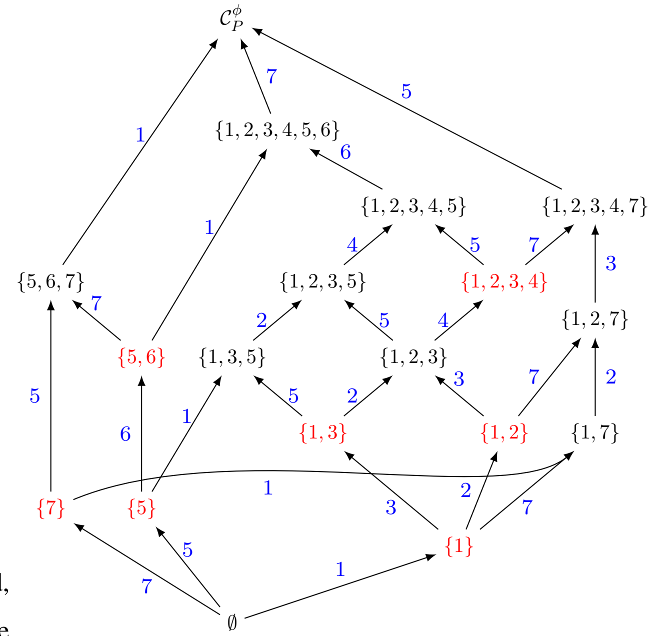
Definition 4.21 *An order filter T of $\mathcal{C}_P^\phi$ is* torclosed *if and only if for all $i < j$, having both $(x, i) \in T$ and $(\phi(i+1), j) \in T$ implies $(x, j) \in T$. We call (P, ϕ)-*Tamari *the poset on the torclosed sets of $\mathcal{C}_P^\phi$ ordered by inclusion. Since the intersection of torclosed sets is torclosed and this poset has a maximum $\mathcal{C}_P^\phi$, it is a complete lattice with meet given by intersection, that we denote by $\mathrm{Tam}(P, \phi)$. For T an order filter of $\mathcal{C}_P^\phi$, we define the* completion *of T to be the minimal torclosed set containing T.*

Except in this section, in this article we will restrict our study to the finite (P, ϕ)-Tamari lattices. Moreover, when we will look at applications of this construction in Section 4.6, all our examples will be with posets P having a minimum $\hat{0}$. To have more concise statements, we will often restrict to this setting (but more general statements can sometimes be derived).

Remark 4.22 *Definition 4.21 is motivated by Theorem 4.18. We will see in Section 4.6.2 that for a particular choice of poset P and chain ϕ, the torclosed sets correspond to d-torsion classes and $\mathrm{Tam}(P, \phi)$ to the lattice of d-torsion classes ordered by inclusion. This explains the name* torclosed *for* torsion closed*. We will prove in Section 4.6.1 that $\mathrm{Tam}(C_n, C_n) \cong \mathrm{Tam}_{n+1}$ (Proposition 4.57), which explains the name* Tamari *for these lattices, as well as the fact that any $\mathrm{Tam}(P, \phi)$ with $|\phi| = n$ contains a sublattice isomorphic to Tam_{n+1} (Proposition 4.58).*



a A poset P with a chain ϕ whose elements are circled, and the poset $\mathcal{C}_P^\phi$ whose elements $(\phi(i), j)$ are circled. The numbering is the one from Section 4.5.4



b $\mathrm{Tam}(P, \phi)$ from Figure 4.3a.

Figure 4.3

The following four results are immediate:

Lemma 4.23 $\mathrm{Tam}(P, \phi)$ *is finite if and only if* ϕ *is finite.*

Lemma 4.24 *In* $\mathrm{Tam}(P, \phi)$*, we have* $I_{\mathrm{Tam}(P,\phi)}\left(\bigcup_{i<k} \mathcal{C}_i\right) = \mathrm{Tam}(P, \phi_{|\{0,1,\dots,k-1\}})$ *for every* k*.*

Lemma 4.25 *Let* $T, T' \in \mathrm{Tam}(P, \phi)$*. Then* $(T \bigvee T')_{<k} = T_{<k} \bigvee T'_{<k}$*, where on the right this is the join in the finite lattice* $\mathrm{Tam}(P, \phi_{|\{0,1,\dots,k-1\}})$.

Lemma 4.26 *Let* k *be a nonnegative integer. Let* $T \in \mathrm{Tam}(P, \phi_{|\{0,1,\dots,k-1\}})$ *and* F *be an order filter of* $\mathcal{C}_k$*. The completion of* $T \cup F$ *only differs from* $T \cup F$ *in the component* $\mathcal{C}_k$*. It can be obtained by first computing the completion of* $T_{k-1} \cup F$*, whose elements in* $\mathcal{C}_k$ *form an order filter denoted by* F_1*. Then computing the completion of* $T_{k-2} \cup F_1$*, whose elements in* $\mathcal{C}_k$ *form an order filter denoted by* F_2*. And so on until obtaining* F_k*. Then the completion of* $T \cup F$ *is* $T \cup F_k$*.*

Proposition 4.27 $\mathrm{Tam}(P, \phi)$ *has a minimum element* $\emptyset$ *and a maximum element* $\mathcal{C}_P^\phi$. *Its atoms are* $\{(\phi(i), i)\}$ *for all* $i \geq 0$. *Let* $k \geq 0$. *If* m *is a minimal element of* $I_P(\phi(k))$ *that is not in* $I_P(\phi(k-1))$, *then* $\mathcal{C}_P^\phi \setminus \{m\}$ *is a coatom. If no such elements exist then* $\mathcal{C}_P^\phi \setminus \mathcal{C}_k$ *is a coatom. All coatoms are obtained in this way.*

Proof. We only prove the statement related to the coatoms, as the rest is straightforward. Any element of the form $\mathcal{C}_P^\phi \setminus \{m\}$, without assumptions on m, is a coatom. Let $k \geq 0$. The element $\mathcal{C}_P^\phi \setminus \mathcal{C}_k$ is torclosed, and any element that covers $\mathcal{C}_P^\phi \setminus \mathcal{C}_k$ contains $(\phi(k), k)$, which is the maximum of $\mathcal{C}_k$. Suppose that every minimal element of $I_P(\phi(k))$ is an element in $I_P(\phi(k-1))$. In this case, the minimal torclosed set that contains both the elements of $\mathcal{C}_{k-1}$ and the element $(\phi(k), k)$ is $\mathcal{C}_{k-1} \sqcup \mathcal{C}_k$. This proves that $\mathcal{C}_P^\phi \setminus \mathcal{C}_k$ is a coatom.

Conversely, suppose that T is a coatom. If there exist two elements $i < j$ such that $T_i \neq \mathcal{C}_i$ and $T_j \neq \mathcal{C}_j$, then $T < T \vee \mathcal{C}_j \neq \mathcal{C}_P^\phi$ since $T \vee \mathcal{C}_j$ does not contain $\mathcal{C}_i$, which proves that T is not a coatom. Thus there exists a unique k such that $T_k \neq \mathcal{C}_k$. Minimal elements of $I_P(\phi(k))$ that are not in $I_P(\phi(k-1))$ cannot be obtained through a completion. Thus we must have $T = \mathcal{C}_P^\phi \setminus \{m\}$ with m such an element if these minimal elements exist. But if there are no such elements, we saw that we must have $T = \mathcal{C}_P^\phi \setminus \mathcal{C}_k$ since the minimal torclosed set containing $T \cup \{(\phi(k), k)\}$ is the maximum $\mathcal{C}_P^\phi$. $\square$

4.5.2 Lattice properties of $\mathrm{Tam}(P, \phi)$

In the remainder of this article, we always assume that $\mathrm{Tam}(P, \phi)$ is finite, and that the chain ϕ in P is given by $\phi(0) < \phi(1) < \cdots < \phi(n-1)$.

Lemma 4.28 *Let* $T \lessdot T'$ *be a cover relation of* $\mathrm{Tam}(P, \phi)$. *Then there exists* $k < n$ *such that* $T' \setminus T \subseteq \mathcal{C}_k$ *and* $T' \setminus T$ *has a maximum element.*

Proof. Recall that $T < T'$ means $T \subsetneq T'$. Let k be the biggest nonnegative integer such that $T'_k \setminus T_k \neq \emptyset$. Let R be the subset of T' obtained from T by replacing T_k with T'_k. Then R is torclosed, and $T < R \leq T'$. Since $T \lessdot T'$, we have $T' = R$. This proves that there exists $k < n$ such that $T' \setminus T \subseteq \mathcal{C}_k$. Now let us prove that $T' \setminus T$ has a maximum. We know that $T' \setminus T \subseteq \mathcal{C}_k$. Seeking a contradiction, suppose that m_1 and m_2

are two maximal elements of $T' \setminus T$. It could be that one of them is of the form $(\phi(i), k)$ with $i \leq k$. But since these latter elements form a chain in $\mathcal{C}_k$, and m_1 and m_2 are incomparable, without loss of generality we can assume that m_1 is not of the form $(\phi(i), k)$ with $i \leq k$. Thus $T \cup \{m_1\}$ is torclosed, but it is strictly between T and T', which is a contradiction with the fact that $T \lessdot T'$ is a cover relation. □

We now characterize the join-irreducible elements of $\mathrm{Tam}(P, \phi)$ (for an example see Figure 4.3b where these elements are colored in red).

Proposition 4.29 *The join-irreducible elements of* $\mathrm{Tam}(P, \phi)$ *are the principal order filters of* $\mathcal{C}_P^\phi$*. Thus the subposet of* $\mathrm{Tam}(P, \phi)$ *formed from its join-irreducible elements is the dual of the poset* $\mathcal{C}_P^\phi$*. Then* $|\,\mathrm{JIrr}(\mathrm{Tam}\,(P, \phi))\,| = |\mathcal{C}_P^\phi|$.

Proof. It is immediate that these order filters are join-irreducible elements. We need to prove that these are the only join-irreducible elements. Seeking a contradiction, suppose that $T \in \mathrm{JIrr}(\mathrm{Tam}(P, \phi))$ is not of the previous form. Let $k \geq 0$ be minimal such that $T_k \neq \emptyset$. If $T_k \subseteq \mathcal{C}_k$ has two minimal elements a and b, then T covers both $T \setminus \{a\}$ and $T \setminus \{b\}$, which is absurd since T is a join-irreducible element. Thus there exists a such that T_k is the principal order filter of $\mathcal{C}_k$ generated by a. Then the only torclosed set that T covers is $T \setminus \{a\}$. By hypothesis $T \neq T_k$, thus there exists $j > k$ such that $T_j \neq \emptyset$. The subset $T \setminus T_j$ is torclosed, and $(T \setminus T_j) < T$. But $(T \setminus T_j) \not\leq T \setminus \{a\}$, which is absurd since the only element that T covers is $T \setminus \{a\}$.

The other statements follow immediately from what we just proved. □

We denote $J_{(x,i)} = F_{\mathcal{C}_P^\phi}((x, i))$, which is the minimal torclosed set containing (x, i). Then Proposition 4.29 tells us that the join-irreducibles of $\mathrm{Tam}(P, \phi)$ are the $J_{(x,i)}$ for $(x, i) \in \mathcal{C}_P^\phi$.

For any $x \leq \phi(n-1)$, if $x \not\geq \phi(0)$, let $l(x) = 0$, otherwise let $l(x) = \max\{i < n \mid x \geq \phi(i)\}$. In order to be able to analyze the meet-irreducible elements, we add the hypothesis that P has a minimum, which is not very restrictive as all the examples that we will see have this property.

Proposition 4.30 *Suppose that* P *has a minimum element* $\hat{0}$*. The meet-irreducible elements of* $\mathrm{Tam}(P, \phi)$ *are of two kinds, which are the following. For any* $(x, k) \in \mathcal{C}_P^\phi$*, the following element is the* first kind *of* *meet-irreducible:*

$$M_{(x,k)} = \Big(\bigsqcup_{0 \le i < l(x)} \mathcal{C}_i \Big) \bigsqcup \Big(\bigsqcup_{l(x) \le i \le k} \{(y,i) \in \mathcal{C}_P^\phi \mid y \not\le x\} \Big) \bigsqcup \Big(\bigsqcup_{k < i < n} \mathcal{C}_i \Big).$$

For any $(x,k) \in \mathcal{C}_P^\phi$ *and any* $(\phi(j),k)$ *with* $1 \le j \le k$ *such that* $x \not\to \phi(j)$*, the following element is the* second kind *of meet-irreducible:*

$$M_{(x,k),(\phi(j),k)} = \Big(\bigsqcup_{0 \le i < j} \mathcal{C}_i \Big) \bigsqcup \Big(\bigsqcup_{j \le i \le k} \{(y,i) \in \mathcal{C}_P^\phi \mid y \not\le x,\, y \not\le \phi(j)\} \Big) \bigsqcup \Big(\bigsqcup_{k < i < n} \mathcal{C}_i \Big).$$

Proof. We start by proving that any meet-irreducible is of one of the two kinds. Let T be a meet-irreducible, covered by T^*. Let $k < n$ be the biggest nonnegative integer such that $T_k \neq \mathcal{C}_k$. Since $T < T \cup \mathcal{C}_k$, then $T^* \setminus T \subseteq \mathcal{C}_k$. Moreover, by Lemma 4.28, $T^* \setminus T$ has a maximum element (x,k). In particular, (x,k) is a maximal element of $\mathcal{C}_k \setminus T_k$. In fact, for any maximal element m of $\mathcal{C}_k \setminus T_k$ such that $m \neq (\phi(i),k)$ for every $i \le k$, we have $T \lessdot T \cup \{m\}$. Since T is meet-irreducible and the elements $(\phi(i),k)$ for $i \le k$ form a chain, then (x,k) is the maximum of $\mathcal{C}_k \setminus T_k$, or there exists $j \le k$ such that the maximal elements of $\mathcal{C}_k \setminus T_k$ are (x,k) and $(\phi(j),k)$. Let us prove that in the former case $T = M_{(x,k)}$, whereas in the latter case $T = M_{(x,k),(\phi(j),k)}$. In the remainder of this proof, for these two different cases we will refer to the *former case* and to the *latter case*.

Let $A_i = \{(y,i) \in \mathcal{C}_P^\phi \mid y \not\le x\}$ and $B_i = \{(y,i) \in \mathcal{C}_P^\phi \mid y \not\le x,\, y \not\le \phi(j)\}$. In the former case we have $(\phi(l(x)+1),k) \in T$ but $(x,k) \notin T$, and in the latter case we have $(\phi(j+1),k) \in T$ but $(x,k) \notin T$ and $(\phi(j),k) \notin T$. Since T is torclosed, then in the former case for all $i \in \{l(x), l(x)+1, \dots, k\}$, we have $T_i \subseteq A_i$, and in the latter case for all $i \in \{j, j+1, \dots, k\}$, we have $T_i \subseteq B_i$.

Then this forces in the former case to have $T_{<l(x)} = \bigsqcup_{0 \le i < l(x)} \mathcal{C}_i$. Indeed, since $T_i \subseteq A_i$ for all $i \in \{l(x), l(x)+1, \dots, k\}$, then $(\phi(p),i) \notin T$ for all $p \le l(x)$ and $i \in \{l(x), l(x)+1, \dots, k\}$. Moreover, $T_q = \mathcal{C}_q$ for all $k < q < n$, which proves that the addition of elements of $\sqcup_{i<l(x)} \mathcal{C}_i$ to T does not affect the others components of T. Thus $T \le T \vee \big(\sqcup_{i<l(x)} \mathcal{C}_i \big) = T \cup \big(\sqcup_{i<l(x)} \mathcal{C}_i \big)$. If the inequality is strict, then $T^* \setminus T \subseteq \sqcup_{i<l(x)} \mathcal{C}_i$, which is absurd. Thus, we have $T = T \cup \big(\sqcup_{i<l(x)} \mathcal{C}_i \big)$. This proves that $T_{<l(x)} = \bigsqcup_{0 \le i < l(x)} \mathcal{C}_i$. We would prove very similarly, by replacing $l(x)$ by j, that in the latter case $T_{<j} = \bigsqcup_{0 \le i < j} \mathcal{C}_i$.

Moreover, it is immediate that $\sqcup_{l(x)\leq i\leq k} A_i$ and $\sqcup_{j\leq i\leq k} B_i$ are torclosed sets, and with the same argument as above we have that $T_{<l(x)} \sqcup \big(\sqcup_{l(x)\leq i\leq k} \ A_i\big) \sqcup T_{>k}$ and $T_{<j} \sqcup \big(\sqcup_{j\leq i\leq k} \ B_i\big) \sqcup T_{>k}$ are torclosed. But both of these torclosed sets contain T, and since they can only differ in components that are not the k-th component, then in fact they are equal to T in the respective cases.

It remains to prove that the two torclosed sets $M_{(x,k)}$ and $M_{(x,k),(\phi(j),k)}$ are meet-irreducibles. Adding to these torclosed sets respectively an element $(y,i) \notin A_i$, for $i \in \{l(x), l(x)+1, \dots, k\}$, or $(y,i) \notin B_i$, for $i \in \{j, j+1, \dots, k\}$, will by completion add (y,k) to each of these torclosed sets. Then any torclosed set that covers one of these two torclosed sets add only elements of the k-th component. Thus any element that covers $M_{(x,k)}$ contains (x,k), then $M_{(x,k)}$ is meet-irreducible and its only cover is the minimal torclosed set containing $M_{(x,k)} \cup \{(x,k)\}$. For $M_{(x,k),(\phi(j),k)}$, the minimal torclosed set containing $M_{(x,k),(\phi(j),k)} \cup \{(\phi(j),k)\}$ contains also (x,k). Indeed, $(\hat{0}, j-1) \in M_{(x,k),(\phi(j),k)}$ thus by completion $(\hat{0},k)$ is in this minimal torclosed set, then (x,k) also. Thus $M_{(x,k),(\phi(j),k)}$ has only one cover which is $M_{(x,k),(\phi(j),k)} \cup \{(x,k)\}$. □

Proposition 4.31 *Let $L = \mathrm{Tam}\,(P,\phi)$. The lattice L is join-semidistributive. Suppose additionally that P has a minimum element. Then L is semidistributive if and only if each element of $I_P(\phi(n-1))$ is comparable to all the elements of the chain $\phi(1) < \phi(2) < \cdots < \phi(n-1)$.*

Proof. We use Lemma 4.6. Let $T \lessdot T'$ in L. By Lemma 4.28, there exists $k < n$ such that $T' \setminus T \subseteq \mathcal{C}_k$ and it has a maximum element m. The torclosed set generated by m, which is J_m, is the minimum element of $I_L(T') \setminus I_L(T)$. Using Lemma 4.6, this proves that L is join-semidistributive.

The second statement follows from Lemma 4.5, with the number of join-irreducibles given in Proposition 4.29 which is equal to the number of meet-irreducibles if and only if we do not have meet-irreducibles of the second kind as described in Proposition 4.30. □

Example 4.32 *See Figure 4.3 for the smallest counter-example to semidistributivity. The number of join-irreducibles in this figure is 7, whereas the number of meet-irreducibles is 8. This is because we have the meet-irreducible of the second kind $M_{2,3} = \{1,7\}$.*

The following result proves that the distributive lattices that are (P, ϕ)-Tamari lattices are the ones that have exactly one atom. Thus by forgetting the minimum $\emptyset$ of these (P, ϕ)-Tamari lattices, we recover all distributive lattices.

Proposition 4.33 *If ϕ has only one element, then $\mathrm{Tam}(P, \phi) \cong J(I_P(\phi(0))^{op})$. These correspond to all the distributive lattices that have exactly one atom. If $|\phi| \geq 2$, then $\mathrm{Tam}(P, \phi)$ is not graded, and thus is not distributive.*

Proof. If ϕ has one element, then the torclosed sets are the order filters of $I_P(\phi(0))$. Then the result follows from the fundamental theorem of finite distributive lattices. Suppose that $|\phi| \geq 2$. By Proposition 4.35, $\ell(\mathrm{Tam}(P, \phi)) = |\mathcal{C}_P^\phi|$. But all maximal chains in the order filter of $\mathrm{Tam}(P, \phi)$ generated by the atom $\{(\phi(0), 0)\}$ have length at most $|\mathcal{C}_P^\phi| - 2 = \ell(\mathrm{Tam}(P, \phi)) - 2$. Indeed, the moment we will add $(\phi(1), 1)$, the completion will also add with it at least another element $(\phi(0), 1)$. Thus the maximal chains of $\mathrm{Tam}(P, \phi)$ do not have all the same length, which means that $\mathrm{Tam}(P, \phi)$ is not graded. $\square$

Proposition 4.34 *Suppose that P has a minimum $\hat{0}$ and $\phi(0) = \hat{0}$. Let $L = \mathrm{Tam}(P, \phi)$. Then L is complemented and pseudocomplemented.*

Proof. Let $T \in L$ be a torclosed set that is neither $\emptyset$ nor $\mathcal{C}_P^\phi$. Let $I = \{i < n \mid T_i \neq \emptyset\}$. Since $\phi(0) = \hat{0}$ and $T \neq \mathcal{C}_P^\phi$, then $J := \{0, 1, \ldots, n-1\} \setminus I$ is not empty. Indeed, if $J = \emptyset$ then $(\phi(i), i) \in T$ for all $i < n$, and successive completions would give $(\hat{0}, i) \in T$ for all $i < n$, but then $T = \mathcal{C}_P^\phi$. Let $T' := \bigsqcup_{i \in J} \mathcal{C}_i$. Then T' is torclosed and $T \wedge T' = \emptyset$. Also, we have $T \vee T' = \mathcal{C}_P^\phi$ for the same reason as why $J \neq \emptyset$ using $\phi(0) = \hat{0}$. This proves that T' is a complement of T. Thus L is complemented. Moreover, it is immediate that $T' = \max\{R \in L \mid T \wedge R = \emptyset\}$, which proves that L is meet-pseudocomplemented. We conclude that L is pseudocomplemented using for example [24, Théorème 4.3], which proves that a complemented lattice is pseudocomplemented if and only if it is meet-pseudocomplemented. $\square$

We define on $\mathcal{C}_P^\phi$ another partial order $\leq_{prod}$, called the *product order*, defined by $(x, i) \leq_{prod} (y, j)$ if and only if $x \leq y$ and $i \leq j$. See Figure 4.4 for an example of this poset.

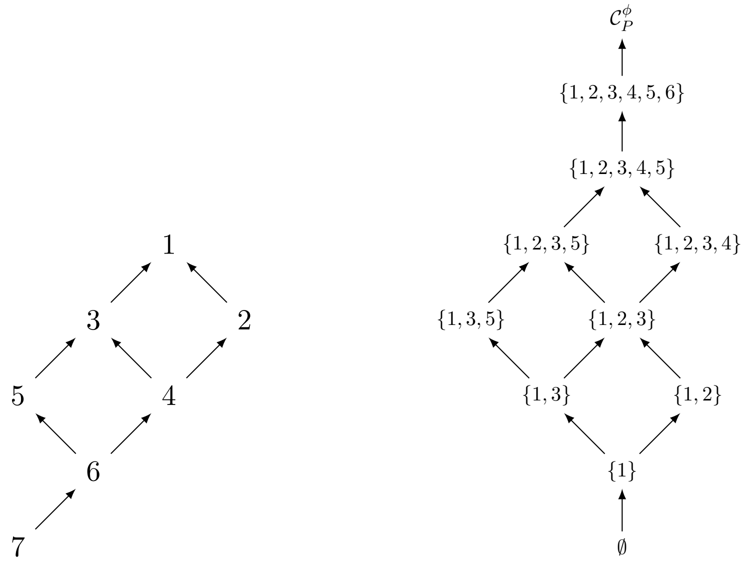


Figure 4.4: On the left we have $(\mathcal{C}_P^\phi, \leq_{prod})$ from Figure 4.3, and on the right $J\big((\mathcal{C}_P^\phi, \leq_{prod})^{op} \big)$.

Proposition 4.35 *The linear extensions $e_1 \prec e_2 \prec \cdots \prec e_m$ of the dual of $(\mathcal{C}_P^\phi, \leq_{prod})$ correspond to the longest chains $\emptyset \subsetneq \{e_1\} \subsetneq \{e_1, e_2\} \subsetneq \cdots \subsetneq \{e_1, e_2, \ldots, e_m\} = \mathcal{C}_P^\phi$ of $\mathrm{Tam}(P, \phi)$. Thus $\ell(\mathrm{Tam}\,(P, \phi)) = |\mathcal{C}_P^\phi|$.*

Proof. By definition, a linear extension of the dual of $(\mathcal{C}_P^\phi, \leq_{prod})$ is one that takes (y, j) before taking (x, i), for all $j \geq i$ and $y \geq x$ in P, where $x \leq \phi(i)$ and $y \leq \phi(j)$. It follows that if $e_1 \prec e_2 \prec \cdots \prec e_m$ is a linear extension of the dual of $(\mathcal{C}_P^\phi, \leq_{prod})$, then $\{e_1, e_2, \ldots, e_k\}$ is torclosed for any $k \in [m]$. Thus $\emptyset \subsetneq \{e_1\} \subsetneq \{e_1, e_2\} \subsetneq \cdots \subsetneq \{e_1, e_2, \ldots, e_m\} = \mathcal{C}_P^\phi$ is a maximal chain of $\mathrm{Tam}(P, \phi)$, and is obviously of longest length as we only add one new element at each step. Thus the longest chains are of length $m = |\mathcal{C}_P^\phi|$, which is the length of $\mathrm{Tam}(P, \phi)$. Conversely, let $\emptyset \subsetneq \{e_1\} \subsetneq \{e_1, e_2\} \subsetneq \cdots \subsetneq \{e_1, e_2, \ldots, e_m\} = \mathcal{C}_P^\phi$ be a longest chain of $\mathrm{Tam}(P, \phi)$ (now we know that they have this form). Let $T(i) = \{e_1, e_2, \ldots, e_i\}$. Then the longest chain is $\emptyset \subsetneq T(1) \subsetneq T(2) \subsetneq \cdots \subsetneq T(m)$. Let us prove that the sequence $e_1, e_2, \ldots, e_m$ defines a linear extension of the dual of $(\mathcal{C}_P^\phi, \leq_{prod})$. Seeking a contradiction, suppose that there exists $i < j$ such that $e_i \leq_{prod} e_j$. Note that since $i < j$, then $e_j \notin T(i)$. Let $e_i = (x, k)$ and $e_j = (y, p)$. Then $e_i \leq_{prod} e_j$ means $x \leq y$ in P and $k \leq p$. If $p = k$, since $e_i = (x, k) \leq (y, k) = e_j$ and $e_i \in T(i)$ which is torclosed, then $e_j \in T(i)$. This is absurd, thus $p > k$.

Let $e_l = (\phi(k+1), p)$. If $e_l \in T(i)$, then by completion we have $(x, p) \in T(i)$. But $e_j = (y, p) \geq (x, p)$, thus since $T(i)$ is torclosed we have $e_j \in T(i)$. This is absurd, thus $e_l \notin T(i)$. Then $i < l$. Let us explain why this is impossible. Since $(x,p) < (\phi(k+1),p) = e_l$ then a torclosed set that contains (x,p) also contains e_l. Since both these elements are not in $T(i)$ and following the chain $\emptyset \subsetneq T(1) \subsetneq T(2) \subsetneq \cdots \subsetneq T(m)$ we add only a unique element at each step, we will add to a torclosed set that contains $T(i)$ the element e_l before adding later the element (x,p). But we proved that as soon as a torclosed set contains both $T(i)$ and the element e_l, then it contains (x,p). This is absurd. This finishes the proof of the proposition. $\square$

Corollary 4.36 *The lattice* $\mathrm{Tam}(P,\phi)$ *is join-extremal.*

Proof. This follows from Proposition 4.29 and 4.35. $\square$

Proposition 4.37 *The elements of the spine of* $\mathrm{Tam}(P,\phi)$ *correspond to the order filters of* $(\mathcal{C}_P^\phi, \leq_{prod})$.

Proof. This follows from Proposition 4.35 where we proved that the torclosed sets on a longest chain of $\mathrm{Tam}(P,\phi)$ are the order ideals of the linear extensions of the dual of $(\mathcal{C}_P^\phi, \leq_{prod})$. $\square$

Example 4.38 *The spine of the lattice from Figure 4.3b is drawn on the right of Figure 4.4.*

4.5.3 Enumeration of $\mathrm{Tam}(P,\phi)$

In this short section, we study the number of elements of $\mathrm{Tam}(P,\phi)$.

Let $m_P^\phi = |\operatorname{Tam}(P,\phi)|$. Let $I_i = I_P(\phi(i))$ for any $i < n$. Then $|J(I_i)|$ is the number of order filters of I_i, which also corresponds to the number of order ideals or the number of antichains of I_i. We will later prove in Proposition 4.58 that $\mathrm{Tam}(P,\phi)$ always contains a sublattice isomorphic to the Tamari lattice of size $n+1$, which gives $m_P^\phi \geq \frac{1}{n+2}\binom{2n+2}{n+1}$. Counting torclosed sets whose elements lie in only one component gives another lower bound $\sum_{i<n} |J(I_i)| - (n-1)$. Counting the order filters gives the upper bound $|J(I_0)| \times |J(I_1)| \times \cdots \times |J(I_{n-1})|$.

Proposition 4.39 *If* ϕ *has one element, then* $m_P^\phi = |J(I_0)|$. *If* ϕ *has two elements, then* $m_P^\phi = |J(I_0)| + |J(\mathcal{C}_P^\phi, \leq_{prod})| - 1$.

Proof. The case $|\phi| = 1$ is immediate. If ϕ has two elements, then a torclosed set may or may not contain $(\phi(1), 1)$. The number of torclosed sets that do not contain $(\phi(1), 1)$ is $|J(I_0)|$ since they correspond to the torclosed sets T such that $T_1 = \emptyset$. On the other hand, the number of torclosed sets containing $(\phi(1), 1)$ is $|J(\mathcal{C}_P^\phi, \leq_{prod})| - 1$. Indeed, in this case an order filter T of $\mathcal{C}_P^\phi$ is torclosed if and only if $(x, 0) \in T$ implies $(x, 1) \in T$. $\square$

Proposition 4.40 *Suppose that P has a minimum $\hat{0}$ and $\phi(0) = \hat{0}$. If ϕ has three elements, then $m_P^\phi = 1 + |J(I_1)| + \left|J(\mathcal{C}_P^\phi, \leq_{prod})\right| + |J(I_2 \setminus I_1)|$.*

Proof. We first count the torclosed sets that do not contain $(\phi(2), 2)$. Among these, the number of torclosed sets that do not contain $(\phi(1), 1)$ is 2, whereas we have $|J(I_1)|$ such torclosed sets that contain $(\phi(1), 1)$. Now, let us count the torclosed sets that contain $(\phi(2), 2)$. Among these, the number of torclosed sets that do not contain $(\phi(0), 0)$ is $|J(\mathcal{C}_P^\phi, \leq_{prod})| - 2$, whereas we have $|J(I_2 \setminus I_1)| + 1$ such torclosed sets that contain $(\phi(0), 0)$. The result follows by summing the cardinalities of these disjoint families of torclosed sets. $\square$

Example 4.41 *For Figure 4.3b, Proposition 4.40 gives $m_P^\phi = 1 + 3 + 11 + 3 = 18$ elements.*

We will go slightly further, allowing $|\phi| = 4$, in the particular example of the higher torsion classes (see Proposition 4.77). We will also give a formula for the number of torclosed sets in the particular case where P is a chain (see Section 4.6.1). However, no general counting formula is known.

4.5.4 Edge-labellings and congruences

In this section, we obtain further properties of $\mathrm{Tam}(P, \phi)$, mainly results about its topology and congruences. We still assume that $\mathrm{Tam}(P, \phi)$ is finite, the chain being $\phi : \phi(0) < \phi(1) < \cdots < \phi(n-1)$.

By Proposition 4.35, the choice of any linear extension $\mathcal{L} : e_1 \prec e_2 \prec \cdots \prec e_m$ of the dual of $(\mathcal{C}_P^\phi, \leq_{prod})$ yields a longest chain $\psi_{\mathcal{L}} : \emptyset = T(0) \lessdot T(1) \lessdot T(2) \lessdot \cdots \lessdot T(m) = \mathcal{C}_P^\phi$ of $\mathrm{Tam}(P, \phi)$, where $T(i) := \{e_1, e_2, \ldots, e_i\}$. Moreover, $\mathcal{L}$ induces a numbering $\alpha_{\mathcal{L}} : \mathcal{C}_P^\phi \to [m]$ defined by $\alpha_{\mathcal{L}}(e_i) = i$. For $T \subseteq \mathcal{C}_P^\phi$, we also define $\alpha_{\mathcal{L}}(T) := \{\alpha_{\mathcal{L}}(e_i) \mid e_i \in T\}$. This enables us to identify a torclosed set as a set of integers. We define an edge-labelling $\omega_{\mathcal{L}}$ of $\mathrm{Tam}(P, \phi)$, which is well-defined by Lemma 4.28, which maps $T \lessdot T'$

to the integer $\alpha_{\mathcal{L}}(\max(T' \setminus T))$. This latter integer is easily seen to be equal to $\min\big(\alpha_{\mathcal{L}}(T') \setminus \alpha_{\mathcal{L}}(T)\big)$. This latter formulation shows that the edge-labelling $\omega_{\mathcal{L}}$ is particularly convenient if the torclosed sets are already represented as set of integers.

For the remainder of this section, let $\mathcal{L}$ be a linear extension of the dual of $(\mathcal{C}_P^{\phi}, \leq_{prod})$ such that:

- $(y, j) \prec (x, i)$ whenever $j > i$, with $(x, i) \in \mathcal{C}_i$ and $(y, j) \in \mathcal{C}_j$,
- if $(x, k) \not\leq (\phi(i), k)$, then $(x, k) \prec (\phi(i), k)$, for all $(x, k) \in \mathcal{C}_k$ and all $i \leq k$.

Such linear extensions exist since the elements $(\phi(i), k)$ for $0 \leq i \leq k$ form a chain in $\mathcal{C}_k$. They correspond to the linear extensions that add elements in decreasing order of component number, and avoid adding each element of the form $(\phi(i), k)$ as long as possible. See Figure 4.3a for an example of the numbering $\alpha_{\mathcal{L}}$ of $\mathcal{C}_P^{\phi}$ and Figure 4.3b for the associated (blue) edge-labelling $\omega_{\mathcal{L}}$.

Proposition 4.42 *In Definition 4.8, if we set ψ to be the chain $\psi_{\mathcal{L}} : T(0) \lessdot \cdots \lessdot T(m)$, we have $\gamma_{1,\psi_{\mathcal{L}}} = \omega_{\mathcal{L}}$ and $\gamma_{2',\psi_{\mathcal{L}}} = \omega_{\mathcal{L}}$.*

Proof. The first equality $\gamma_{1,\psi_{\mathcal{L}}} = \omega_{\mathcal{L}}$ follows easily from the definitions of the labellings, since with our conventions $\delta_{\psi_{\mathcal{L}}}(J_{e_i}) = i$ for all $i \in [m]$. The second equality requires our particular choice of linear extension $\mathcal{L}$. Recall that $\gamma_{2',\psi_{\mathcal{L}}}(T \lessdot T') = \min\{i \mid T \vee T(i) \geq T'\}$. Since $T \lessdot T'$ and $\max(T' \setminus T) = e_{\omega_{\mathcal{L}}(T \lessdot T')}$, we have $T \vee T(i) \geq T'$ if and only if $e_{\omega_{\mathcal{L}}(T \lessdot T')} \in T \vee T(i)$. Thus $\gamma_{2',\psi_{\mathcal{L}}}(T \lessdot T') \leq \omega_{\mathcal{L}}(T \lessdot T')$. Seeking a contradiction, suppose that there exists $j < \omega_{\mathcal{L}}(T \lessdot T')$ such that $\gamma_{2',\psi_{\mathcal{L}}}(T \lessdot T') = j$. Then $T \vee T(j) \geq T'$, thus $e_{\omega_{\mathcal{L}}(T \lessdot T')} \in T \vee T(j)$. By definition $e_{\omega_{\mathcal{L}}(T \lessdot T')} \notin T$, and since $j < \omega_{\mathcal{L}}(T \lessdot T')$, we have $e_{\omega_{\mathcal{L}}(T \lessdot T')} \notin T(j)$. Then $e_{\omega_{\mathcal{L}}(T \lessdot T')} \notin T \cup T(j)$. But $e_{\omega_{\mathcal{L}}(T \lessdot T')} \in T \vee T(j)$, which proves that $e_{\omega_{\mathcal{L}}(T \lessdot T')}$ is obtained with a completion. By minimality of j, we have $T \vee T(j-1) \not\geq T'$, thus the completion that makes $e_{\omega_{\mathcal{L}}(T \lessdot T')} \in T \vee T(j)$ comes from the addition of e_j to $T(j-1)$. Since by hypothesis $j \neq \omega_{\mathcal{L}}(T \lessdot T')$, thus there exists $p \leq k$ such that $e_j = (\phi(p), k)$, where $T' \setminus T \subseteq \mathcal{C}_k$. Moreover, e_j and $e_{\omega_{\mathcal{L}}(T \lessdot T')}$ are incomparable. By definition of our particular choice of linear extension $\mathcal{L}$, then $e_{\omega_{\mathcal{L}}(T \lessdot T')} \prec e_j$, which means $\omega_{\mathcal{L}}(T \lessdot T') < j$. This is absurd. $\square$

Remark 4.43 *Our particular choice of linear extension $\mathcal{L}$ is important. For example, in Figure 4.3 let us take the linear extension $\mathcal{L}'$ that switches the numbers 2 and 3 in our numbering of $\mathcal{C}_P^{\phi}$ in Figure 4.3a. Then let $T = \{1, 7\}$ and $T' = \{1, 3, 7\}$. We have $\omega_{\mathcal{L}'}(T \lessdot T') = \gamma_{1,\psi_{\mathcal{L}'}}(T \lessdot T') = 3$, but $\gamma_{2',\psi_{\mathcal{L}'}}(T \lessdot T') = 2$.*

Theorem 4.44 $\mathrm{Tam}(P, \phi)$ *is left modular. The edge-labelling $\omega_{\mathcal{L}}$ is an EL-labelling.*

Proof. From Propositions 4.9 and 4.42 follows the fact that $\mathrm{Tam}(P, \phi)$ is left modular. Then $\omega_{\mathcal{L}}$ is an EL-labelling by Proposition 4.10. □

Using $\omega_{\mathcal{L}}$ we obtain the following topological result about $\mathrm{Tam}(P, \phi)$.

Proposition 4.45 *The EL-labelling $\omega_{\mathcal{L}}$ is such that in any interval $[T, T']$ of $\mathrm{Tam}(P, \phi)$ there is at most one maximal decreasing chain. Thus $\mu(T, T') \in \{-1, 0, 1\}$ and if $]T, T'[$ is nonempty, then its order complex has the homotopy type of either a sphere or a point.*

Proof. Let $[T, T']$ be an interval of $\mathrm{Tam}(P, \phi)$, and let $I = \{i_1, i_2, \ldots, i_k\}$ be the set of indices $0 \leq i_1 < \cdots < i_k < n$ of the nonempty components of $T' \setminus T$. Let us prove that we have at most one maximal decreasing chain in $[T, T']$ with respect to $\omega_{\mathcal{L}}$. By our choice of linear extension $\mathcal{L}$, if $e_k \in \mathcal{C}_i$ and $e_l \in \mathcal{C}_j$ with $i < j$, thus $l < k$. Then, since by Lemma 4.28 the cover relations happen between two torclosed sets that differ in only one component, any maximal decreasing chain of $[T, T']$ will contain the subchain $T < T \cup T'_{i_1} < T \cup T'_{i_1} \cup T'_{i_2} < \cdots < T \cup T'_{i_1} \cup \cdots \cup T'_{i_k} = T'$. Seeking a contradiction, suppose that we have two different maximal decreasing chains C_1 and C_2 of $[T, T']$. As both chains contain the subchain described above, and two consecutive elements of this subchain differ in only one component, we look to the component $\mathcal{C}_{i_j}$ with the least index where C_1 and C_2 differ. Suppose that the first time these chains differ, following them from bottom to top, is witnessed by $R \lessdot S$ in C_1 and $R \lessdot U$ in C_2. Since $T \cup T'_{i_1} \cup \cdots \cup T'_{i_j}$ is torclosed and greater than both R and S, the elements in components $\mathcal{C}_p$ for $p > i$ have no effect on which subsets of elements from $\mathcal{C}_i$ can be added to R and S. This just means that we add all the elements of $T'_{i_j} \setminus R$ in C_1 and C_2 as if we did not have elements in components with bigger indices. By Lemma 4.28, if $V \lessdot W$ is a cover relation in $\mathrm{Tam}(P, \phi)$, then $W \setminus V$ has a maximum element e_l, and $\omega_{\mathcal{L}}(V \lessdot W) = l$ by definition of $\omega_{\mathcal{L}}$. Without loss of generality suppose that $\omega_{\mathcal{L}}(R \lessdot S) < \omega_{\mathcal{L}}(R \lessdot U)$.

Let $V \lessdot W$ be the cover relation of C_1 such that $e_{\omega_{\mathcal{L}}(R \lessdot U)} \in W \setminus V$. Since $e_{\omega_{\mathcal{L}}(R \lessdot U)}$ is a maximal element of $T'_{i_j} \setminus R$, then $\omega_{\mathcal{L}}(V \lessdot W) = \omega_{\mathcal{L}}(R \lessdot U)$. Thus the chain C_1 is not decreasing. This is absurd.

Then the last sentence of the proposition follows by results recalled in Section 4.4.1 in the paragraph about poset topology. □

We now turn our attention to the congruences of $\mathrm{Tam}(P, \phi)$. The next two results characterize the join dependency relation D. Recall that we denote by $J_{(x,i)}$ the minimal torclosed set containing (x, i), and that Proposition 4.29 tells us that the join-irreducibles of $\mathrm{Tam}(P, \phi)$ are the $J_{(x,i)}$ for $(x, i) \in \mathcal{C}_P^{\phi}$.

Lemma 4.46 *Let $i < n$. If $J_{(x,i)}\, D\, J_{(y,i)}$, then there exists $k \le i$ such that $y = \phi(k)$ and $y \not\le x$. If P has a minimum $\hat{0}$ and $\phi(0) = \hat{0}$, then the last statement is an equivalence.*

Proof. We suppose $J_{(x,i)}\, D\, J_{(y,i)}$, which means that $x \neq y$ and there exists $p \in \mathrm{Tam}(P, \phi)$ such that $J_{(x,i)} \le J_{(y,i)} \vee p$ and $J_{(x,i)} \not\le (J_{(y,i)})_* \vee p$. Then $J_{(x,i)} \not\le J_{(y,i)}$, as noted at the end of Section 4.4.1. This means $(y, i) \not\le (x, i)$, thus $y \not\le x$. It remains to prove that there exists $k \le i$ such that $y = \phi(k)$. Seeking a contradiction, suppose that it is not the case. Then $J_{(y,i)}$ and $(J_{(y,i)})_* = J_{(y,i)} \setminus \{(y, i)\}$ contain the same elements of the form $(\phi(k), i)$ for $k \le i$. Thus $(\,(J_{(y,i)})_* \vee p)_i = (\,J_{(y,i)} \vee p)_i$ or $(\,(J_{(y,i)})_* \vee p)_i = (J_{(y,i)} \vee p)_i \setminus \{(y, i)\}$. As $J_{(x,i)} = (J_{(x,i)})_i$, we have $(J_{(x,i)})_i \le (J_{(y,i)} \vee p)_i$ and $(J_{(x,i)})_i \not\le (\,(J_{(y,i)})_* \vee p)_i$. In particular, $(\,(J_{(y,i)})_* \vee p)_i \neq (J_{(y,i)} \vee p)_i$, thus $(\,(J_{(y,i)})_* \vee p)_i = (J_{(y,i)} \vee p)_i \setminus \{(y, i)\}$. Then, we have $J_{(x,i)} \le (J_{(y,i)} \vee p)_i$ and $J_{(x,i)} \not\le (\,(J_{(y,i)})_* \vee p)_i = (J_{(y,i)} \vee p)_i \setminus \{(y, i)\}$. This means that $(x, i) = (y, i)$, which is absurd since $y \not\le x$.

Conversely, suppose that P has a minimum $\hat{0}$, that $\phi(0) = \hat{0}$, and let (x, i) and (y, i) be in $\mathcal{C}_i$ such that there exists $k \le i$ such that $y = \phi(k)$ and $y \not\le x$. Since $y \not\le x$, then $y \neq \phi(0) = \hat{0}$. This proves that $k \ge 1$. Let $p = \mathcal{C}_{k-1} \in \mathrm{Tam}(P, \phi)$. We have $J_{(x,i)} \subseteq \mathcal{C}_i \cup \mathcal{C}_{k-1} = J_{(\phi(k),i)} \vee p$ since this latter torclosed set contains $(\hat{0}, k - 1)$ and $(\phi(k), i)$, thus by completion contains $(\hat{0}, i) = \min(\mathcal{C}_i)$. But $(J_{(\phi(k),i)})_* \vee p = (J_{(\phi(k),i)})_* \cup \mathcal{C}_{k-1}$ since $(\phi(k), i) \notin (J_{(\phi(k),i)})_*$. Then $J_{(x,i)} \not\le (J_{(\phi(k),i)})_* \vee p$. This proves that $J_{(x,i)}\, D\, J_{(y,i)}$. □

Lemma 4.47 *For all $i \neq j$, we have $J_{(x,j)}\, D\, J_{(y,i)}$ if and only if $i < j$, $y \le x$, $\phi(i + 1) \not\le x$ and there is no $y' \le \phi(i)$ such that $y < y' \le x$.*

Proof. Let (x, j) and (y, i) be in $\mathcal{C}_P^\phi$ such that $i < j$, $y \leq x$, $\phi(i+1) \not\leq x$ and there is no $y' \leq \phi(i)$ such that $y < y' \leq x$. Let $p = J_{(\phi(i+1),j)}$. We have both (y, i) and $(\phi(i+1), j)$ in the torclosed set $J_{(y,i)} \vee p$, thus by completion $(y, j) \in J_{(y,i)} \vee p$. Since $(x, j) \geq (y, j)$ and they are in the same component $\mathcal{C}_j$, then $(x, j) \in J_{(y,i)} \vee p$. Thus $J_{(x,j)} \leq J_{(y,i)} \vee p$.

If $y = \phi(i)$, then $(J_{(y,i)})_* = \emptyset$. And since $\phi(i+1) \not\leq x$, then $(x, j) \notin J_{(\phi(i+1),j)} = (\, J_{(y,i)})_* \vee J_{(\phi(i+1),j)}$. This proves $J_{(x,j)} \, D \, J_{(y,i)}$. Otherwise $y \neq \phi(i)$. Then $(\, (J_{(y,i)})_* \vee J_{(\phi(i+1),j)})_j = \{(z, j) \mid \exists y' \in P,\, y < y' \leq \phi(i)$ and $y' \leq z \leq \phi(j)\}$. Then (x, j) is not in this latter set since there is no $y' \leq \phi(i)$ such that $y < y' \leq x$. Thus $J_{(x,j)} \not\leq (\, (J_{(y,i)})_* \vee J_{(\phi(i+1),j)})_j$, which proves that $J_{(x,j)} \, D \, J_{(y,i)}$.

Conversely, suppose that we have $J_{(x,j)} \, D \, J_{(y,i)}$, which means that there exists $p \in \mathrm{Tam}(P, \phi)$ such that $J_{(x,j)} \leq J_{(y,i)} \vee p$ and $J_{(x,j)} \not\leq (J_{(y,i)})_* \vee p$. If $j < i$, then $J_{(y,i)}$ does not contribute in $J_{(x,j)} \leq J_{(y,i)} \vee p$, which means that $J_{(x,j)} \leq p$. Then $J_{(x,j)} \leq (J_{(y,i)})_* \vee p$, which is absurd. This proves that $i < j$. As observed, we do not have $J_{(x,j)} \leq p$, thus $x \notin p$. Since $J_{(x,j)} \leq J_{(y,i)} \vee p$, this means that $J_{(y,i)}$ is essential when computing the join to contain $J_{(x,j)}$, even though it is not in the same component. Thus we have completions involving $J_{(y,i)}$, which proves both $(\phi(i+1), j) \in p$ and $(y, j) \leq (x, i)$. The latter gives $y \leq x$, while the former gives $\phi(i+1) \not\leq x$, because if it was not the case then $(x, j) \in p$ since p is torclosed, which is absurd. It remains to prove that there is no $y' \leq \phi(i)$ such that $y < y' \leq x$. Seeking a contradiction, suppose that such a y' exists. Then $(y', i) \in (J_{(y,i)})_*$, which with $(\phi(i+1), j) \in p$ gives by completion $(y', j) \in (J_{(y,i)})_* \vee p$. Since $y' \leq x$, this gives $(x, i) \in (J_{(y,i)})_* \vee p$. Thus $J_{(x,j)} \leq (J_{(y,i)})_* \vee p$, which is absurd. $\square$

Example 4.48 *In Figure 4.3a, the D relations between join-irreducibles generated by elements in the same component are $J_6 \, D \, J_5$, $J_4 \, D \, J_3$, $J_4 \, D \, J_1$, $J_3 \, D \, J_1$, $J_2 \, D \, J_3$, $J_2 \, D \, J_1$ and the other relations are $J_6 \, D \, J_7$, $J_4 \, D \, J_7$, $J_2 \, D \, J_7$, $J_4 \, D \, J_6$, $J_3 \, D \, J_5$, $J_2 \, D \, J_5$.*

The following theorem gives another proof that $\mathrm{Tam}(P, \phi)$ is join-semidistributive, since a join-congruence uniform lattice is join-semidistributive.

Theorem 4.49 $\mathrm{Tam}(P, \phi)$ *is join-congruence uniform.*

Proof. By Proposition 4.12, we need to prove that the D relation is acyclic. This follows from Lemmas 4.46 and 4.47. $\square$

Remark 4.50 *The lattice* $\mathrm{Tam}(P, \phi)$ *is not polygonal in general (see Figure 4.2).*

Definition 4.51 *Let* $K := (K_0, K_1, \ldots, K_n)$ *be a non-decreasing sequence of non-negative integers such that* $K_i \leq i$ *for all* $i \leq n$*. We call such a sequence a* Kupisch-like series*. Denote* $R_K := \{(x, i) \in \mathcal{C}_P^\phi \mid x \geq \phi(K_i)\}$*. On* $\mathrm{Tam}(P, \phi)$ *we define the* Kupisch equivalence relation $T \equiv_K T'$ *if and only if* $T \cap R_K = T' \cap R_K$*.*

Proposition 4.52 *Suppose that* P *is a join-semilattice. Then a Kupisch equivalence relation is a lattice congruence of* $\mathrm{Tam}(P, \phi)$*.*

Proof. Let K be a Kupisch-like series. Since $\mathrm{Tam}(P, \phi)$ is join-congruence uniform (Theorem 4.49), by Proposition 4.12 it suffices to prove that the set S of join-irreducibles that $\equiv_K$ contracts satisfies that $J_{(x,j)} D J_{(y,i)}$ and $J_{(y,i)} \in S$ together imply $J_{(x,j)} \in S$. We have

$$J_{(y,i)} \in S \iff J_{(y,i)} \equiv_K (J_{(y,i)})_* \iff (y, i) \notin R_K \iff y \ngeq \phi(K_i).$$

Suppose that $J_{(x,j)} D J_{(y,i)}$ and $y \ngeq \phi(K_i)$. We need to prove that $x \ngeq \phi(K_j)$. We consider two cases; first $j = i$, then $j \neq i$.

Suppose $j = i$. By Lemma 4.46, $J_{(x,i)} D J_{(y,i)}$ implies that there exists $k \leq i$ such that $y = \phi(k)$ and $x \ngeq y$. Since $\phi(k) = y \ngeq \phi(K_i)$, and $\phi(k)$ is comparable to $\phi(K_i)$, then $\phi(K_i) > \phi(k) = y$. This proves that $x \ngeq \phi(K_j)$, because if it was not the case then $x \geq \phi(K_j) > y$, which is absurd since $x \ngeq y$.

Suppose $j \neq i$. By Lemma 4.47, $J_{(x,j)} D J_{(y,i)}$ implies that $i < j$, $y \leq x$, $\phi(i+1) \nleq x$ and there is no $y' \leq \phi(i)$ such that $y < y' \leq x$. Seeking a contradiction, assume that $x \geq \phi(K_j)$. Since $x \ngeq \phi(i+1)$ and $\phi(i+1)$ is comparable to $\phi(K_j)$, then $\phi(K_j) < \phi(i+1)$. Thus $\phi(K_j) \leq \phi(i)$. Since $y \ngeq \phi(K_i)$, either $y < \phi(K_i)$, or $\phi(K_i)$ and y are incomparable.

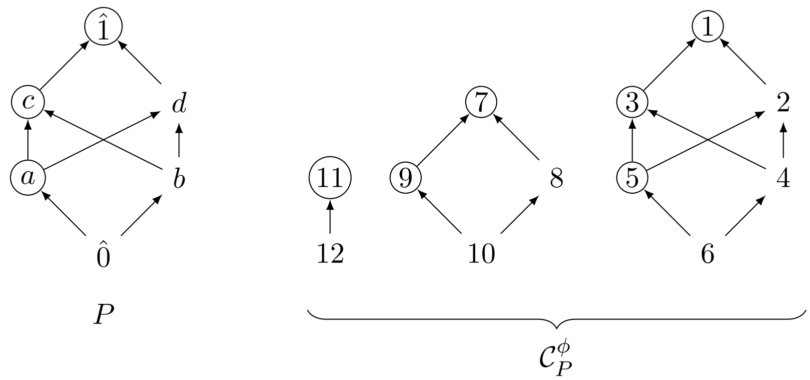


Figure 4.5: On the left is a poset P with a chain ϕ whose elements are circled. On the right is $\mathcal{C}_P^\phi$, with the numbering from Section 4.5.4.

If $y < \phi(K_i)$, then $y < \phi(K_i) \leq \phi(K_j)$ since K is non-decreasing. Thus $y < \phi(K_j) \leq x$, and we proved above that $\phi(K_j) \leq \phi(i)$. This is absurd since we know that no such element $y' := \phi(K_j)$ exists.

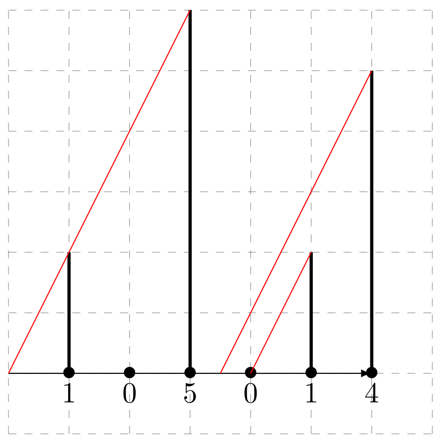
Otherwise $\phi(K_i)$ and y are incomparable. Since P is a join-semilattice, we can define in P the element $y' := \phi(K_j) \vee y$. We proved that $\phi(K_j) \leq \phi(i)$, and by definition $y \leq \phi(i)$. Then $y' \leq \phi(i)$. By hypothesis, we have $\phi(K_j) \leq x$ and $y \leq x$, thus $y' \leq x$. To have something absurd, it suffices to prove that $y < y'$. By definition of y', we have $y \leq y'$. If $y = y'$, then $\phi(K_j) \leq y$, and since $\phi(K_i) \leq \phi(K_j)$ because K is non-decreasing, it would give $\phi(K_i) \leq y$, which is absurd. Thus $y < y'$. This finishes the proof of the proposition. $\square$

Remark 4.53 *The hypothesis that P is a join-semilattice in Proposition 4.52 is important. For example, the choice of P and ϕ in Figure 4.5 produces a lattice $\mathrm{Tam}(P, \phi)$ of 50 elements in which there are Kupisch equivalence relations that are not lattice congruences. Indeed, let $K = (0, 0, 0)$ and $\equiv_K$ be the associated Kupisch equivalence relation on $\mathrm{Tam}(P, \phi)$. Then $J_2 \; D \; J_8$ and $J_8 \equiv_K (J_8)_*$, but $J_2 \not\equiv_K (J_2)_*$.*

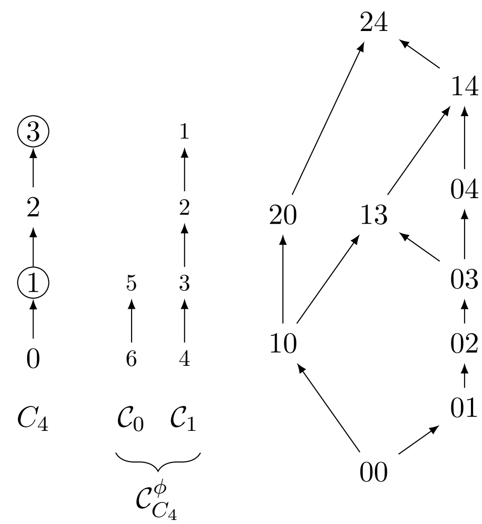
4.6 Main examples

4.6.1 The case of the chains

In this section, we study the particular case where P is a chain. We suppose that $P = C_m$ where $C_m : 0 < 1 < \cdots < m-1$. Since ϕ is a subchain of C_m, it can be viewed as a nonempty subset of $\{0, 1, \ldots, m-1\}$. If $\phi = \{0, 1, \ldots, m-1\}$, we will write $\phi = C_m$. The components $\mathcal{C}_i$ for $i < n$



a The word $u = 105014$ which represents a torclosed set of $\mathrm{Tam}(C_{12}, \phi_2)$. The lines of slope 2 from the top of each segment are represented in red.



b $\mathrm{Tam}(C_4, \phi_2)$ with elements the words u.

Figure 4.6

are themselves chains, isomorphic to $C_{\phi(i)}$, and having an order filter of $\mathcal{C}^{\phi}_{C_m}$ just amounts to knowing how many elements we have in each component $\mathcal{C}_i$ for $i < n$. Thus, these order filters T are identified with words on nonnegative integers $u = u_0 u_1 \dots u_{n-1}$ where for all $i < n$, $u_i := |\mathcal{C}_i \cap T|$. The *length* of a word is its number of letters.

Proposition 4.54 *$\mathrm{Tam}(C_m, \phi)$ is a congruence uniform, extremal and left modular lattice.*

Proof. The left modular property follows from Theorem 4.44. Using Proposition 4.31, it follows that $\mathrm{Tam}(C_m, \phi)$ is semidistributive, and $\mathrm{Tam}(C_m, \phi)$ is join-congruence uniform by Theorem 4.49. Thus $\mathrm{Tam}(C_m, \phi)$ is congruence uniform since a semidistributive join-congruence uniform lattice is congruence uniform (this is an easy consequence of [23, Lemma 1]). By Corollary 4.36, $\mathrm{Tam}(C_m, \phi)$ is join-extremal, but since it is semidistributive then it is extremal. □

Proposition 4.55 *The word u corresponds to an order filter of $\mathcal{C}^{\phi}_{C_m}$ if and only if for all $i \in [n]$, $u_i \leq \phi(i) + 1$. Moreover u corresponds to a torclosed set if and only if for all $i < n$ and $1 \leq k < n - i$, having both $u_{i+k} > \phi(i+k) - \phi(i+1)$ and $u_i \neq 0$ implies $u_{i+k} \geq u_i + \phi(i+k) - \phi(i)$.*

Proof. The word u corresponds to an order filter of $\mathcal{C}^{\phi}_{C_m}$ if and only if for all $i \in [n]$, $u_i \leq |\mathcal{C}_i| = \phi(i) + 1$. Denote T the order filter associated to u. It remains to understand on the word u the completion operation. Let $i < n$ and $1 \leq k < n - i$. We look at the completions between components $\mathcal{C}_i$ and $\mathcal{C}_{i+k}$. The completions between these two components happen if and only if $T_i \neq \emptyset$ and $(\phi(i+1), i+k) \in T_{i+k}$, and in this case we need $(x, i) \in T_i$ implies $(x, i+k) \in T_{i+k}$. We have $T_i \neq \emptyset \iff u_i \neq 0$ and $(\phi(i+1), i+k) \in T_{i+k} \iff u_{i+k} > \phi(i+k) - \phi(i+1)$. Moreover, $(x, i) \in T_i \Longrightarrow (x, i+k) \in T_{i+k}$ means $u_{i+k} - u_i \geq \phi(i+k) - \phi(i)$. This finishes the proof of the proposition. □

Let p be a positive integer. Before giving a general recursive formula to compute $|\operatorname{Tam}(C_m, \phi)|$ (Proposition 4.66), we explore in some details the case $\operatorname{Tam}(C_{np}, \phi_p)$ where by definition $\phi_p(i) = (i+1)\,p - 1$ for all $i < n$. For every $i < n$ such that $u_i \neq 0$, draw a segment from $(i, 0)$ to $(i, u_i + p - 1)$ in the Cartesian plane, where these points are respectively called the base and the top of the segment. Then Lemma 4.55 says that one can draw lines of slope p passing through the x-axis and the top of each segment without crossing any segment nor touching the base of a segment (it can touch the top of a segment). These drawings are called *diagrams*. See Figure 4.6a for an example of a diagram, and see Figure 4.6b for the lattice $\operatorname{Tam}(C_4, \phi_2)$ whose elements are represented as their associated words u. When $u_i = 0$, we have what we call an empty segment, which is represented as a point corresponding to the base of this empty segment. We call a word u which represents a torclosed set of $\operatorname{Tam}(C_n, C_n)$ a *Tamari word* of length m. For a Tamari word u of length n, let $B_p(u)$ be the word of length m whose i-th letter is $p\,u_i - (p-1)$ for all $i < n$, and let $E_p(u)$ be the word of length n whose i-th letter is $p\,u_i$ for all $i < n$.

Lemma 4.56 *Let u be a Tamari word of length n. Then both $B_p(u)$ and $E_p(u)$ correspond to elements of* $\operatorname{Tam}(C_{np}, \phi_p)$*. Moreover, the intervals $I_p(u) := [B_p(u), E_p(u)]$ for all Tamari words u of length n form a partition of* $\operatorname{Tam}(C_{np}, \phi_p)$.

Proof. The fact that $B_p(u)$ and $E_p(u)$ are both elements of $\operatorname{Tam}(C_{np}, \phi_p)$ follows easily from the diagrams using the fact that $u_j - u_i \geq j - i$ for all $i < j$ since u is a Tamari word. This can also be seen directly on the chains forming $\mathcal{C}^{\phi}_{C_{np}}$ (see the right of Figure 4.9). From this latter point of view follows the result about the partition of $\operatorname{Tam}(C_{np}, \phi_p)$, since for a torclosed set T of $\operatorname{Tam}(C_{np}, \phi_p)$ the unique u such that $T \in I_p(u)$ is the word whose i-th letter is the number of elements of $(\phi(j), i)$ in T for $j \leq i$. □

A binary tree T is full if each node of T has either 0 or 2 children. In the remainder of this paper, all binary

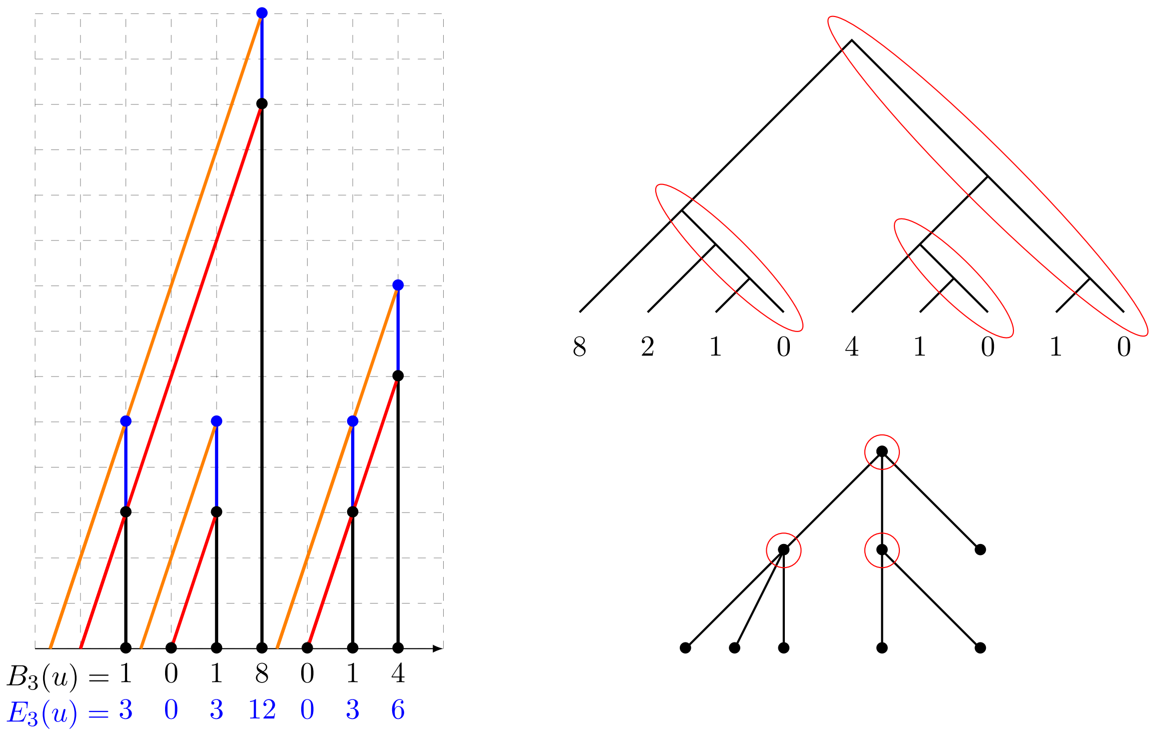


Figure 4.7: On the top right is represented the tree $\varphi(u)$ for the Tamari word $u = 1014012$. On the left are represented $B_3(u)$ and $E_3(u)$. On the bottom right is the tree T_u of the interval $[B_3(u), E_3(u)]$.

trees and subtrees are assumed to be full. The leaves of a binary tree T are numbered by a reverse preorder traversal of T, which means visit the root, then traverse its subtrees from right to left. Given a binary tree T, each leaf is the leftmost leaf of some binary subtrees of T. For the leaf numbered i, let w_i be the number of leaves minus one of the largest binary subtree of T whose leftmost leaf is the one numbered i. We call w_i the weight of the leaf i, and the sequence $(w_1, w_2, \ldots, w_k)$ of all weights is called the weight sequence of the binary tree. Let φ be the map that sends a Tamari word u of length n to the binary tree T whose weight sequence is $(0, u_0, u_1, \ldots, u_{n-1}, n+1)$. For an example, see the top right of Figure 4.7 which represents $\varphi(1014012)$. The extra $n+1$ can be seen on the diagram with the segments by adding a segment of height $n+1$ at the x-coordinate n. The map φ is in fact a bijection (see [82]). It follows (this corresponds to the case $p = 1$):

Proposition 4.57 ([82]) $\mathrm{Tam}(C_n, C_n) \cong \mathrm{Tam}_{n+1}$.

Thus the Tamari lattice is a very special case of a (P, ϕ)-Tamari lattice. In addition to being a special case of a (P, ϕ)-Tamari lattice, we can recover the Tamari lattice in any (P, ϕ)-Tamari lattice as a sublattice:

Proposition 4.58 *Let P be any finite poset and $\phi : \phi(0) < \phi(1) < \cdots < \phi(n-1)$ be any chain in P. The induced subposet of $\mathrm{Tam}(P, \phi)$ on the torclosed sets that are order filters generated only by elements of the form $(\phi(i), j)$ for $i \leq j < n$ is a sublattice and is isomorphic to Tam_{n+1}.*

Proof. By definition of this subposet, denoted by P_ϕ, and of the completions, it is immediate that P_ϕ is a sublattice of $\mathrm{Tam}(P, \phi)$. Since, for each $j < n$, the elements $(\phi(i), j)$ for $i \leq j$ form a chain in $\mathcal{C}_j$, an element T of P_ϕ can be identified as the word $u = u_0 u_1 \dots u_{n-1}$ where u_j is the number of elements of the chain $(\phi(j), j) > (\phi(j-1), j) > \cdots > (\phi(0), j)$ in T. Then $P_\phi \cong \mathrm{Tam}_{n+1}$ follows from Proposition 4.57. $\square$

Example 4.59 *In Figure 4.3b, forgetting the torclosed sets $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 7\}$ and $\{1, 2, 3, 5\}$ gives the sublattice Tam_4.*

We now want to study $\mathrm{Tam}(C_{np}, \phi_p)$ when $p \geq 2$. We define a map from binary trees to planar rooted trees, called the *gluing* map. It maps a binary tree to the planar rooted tree obtained by identifying together all the vertices on a right-descendant path starting at any internal node. For example, the binary tree on the top right of Figure 4.7 is mapped by the gluing to the planar rooted tree on the bottom right of the same figure (we circled in red the elements that are glued together). Let ψ be the bijection that maps a Tamari word u of length n to the gluing of the tree $\varphi(u)$. For example, the planar rooted tree on the bottom right of Figure 4.7 is $\psi(1014012)$. See also on the right of Figure 4.8. The following result follows from the definition of φ. For an illustration, see the left of Figure 4.8 with its caption, and compare it to the top right of Figure 4.7.

Lemma 4.60 *The bijection φ is equivalently defined to be the map which sends a Tamari word u of length n to the binary tree obtained by taking the mirror image (reflection with respect to the y-axis) of the diagram of u drawn with an extra empty segment indexed by 0 at the beginning, an extra segment of height $n+1$ at the end, and lines of slope 1 from the empty segments (0 letters) until reaching the last segment of height $n+1$ or before crossing another segment.*

It follows from Lemma 4.60 (the lines of slope -1 correspond to the red lines in Figure 4.8):

Proposition 4.61 *The bijection ψ is equivalently defined to be the map which sends a Tamari word u of length n to the planar rooted tree obtained by identifying together all the nodes that are on the same line of slope 1 of $\varphi(u)$ as constructed in Lemma 4.60 (or lines of slope -1 after the mirror image).*

Proposition 4.62 *Let u be a Tamari word of length n. We have*

$$|I_p(u)| = \prod_{x \text{ an internal node of } \varphi(u)} \binom{(\text{number of children of } x) - 1 + (p-1)}{p-1} \tag{4.1}$$

Proof. We consider the diagrams of $B_p(u)$ and $E_p(u)$ drawn together as is done on the left of Figure 4.7. Passing from $B_p(u)$ to $E_p(u)$ corresponds to adding $p-1$ to the height of all the nonempty segments (this corresponds to adding the blue segments in Figure 4.7). The elements of $I_p(u)$ correspond to the diagrams with segments whose heights are between these of $B_p(u)$ and these of $E_p(u)$. We can increase freely the height of a given segment of x-coordinate i independently of the others before reaching the maximum height $p\,u_i + (p-1)$, unless the top of this segment in u is touched by the line of slope p from the top of another segment (in Figure 4.7 the segments which we cannot freely increase the height independently from the other segments are the first and fourth segments).

If we have $k \geq 1$ segments in u of x-coordinate $x_1, x_2, \ldots, x_k$ whose tops are on the same line of slope p (and there are no other segments whose top is on that line), then just looking at these segments without changing the others we have as many elements in $I_p(u)$ as the number of sequences $i_1, i_2, \ldots, i_k$ satisfying $0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq p-1$. Here i_j corresponds to the number we add to the height of the segment of x-coordinate x_j. The number of such sequences is $\binom{p-1+k}{k} = \binom{k+(p-1)}{p-1}$. The line of slope p we consider reaches a point on the x-axis whose x-coordinate is an integer. This point corresponds to a 0 (empty segment) of the weight sequence of the binary tree $\varphi(u)$ (see the left of Figure 4.8). By Proposition 4.61, in $\psi(u)$ the top of the segments of x-coordinates $x_1, x_2, \ldots, x_k$ are identified, thus the resulting internal node that it creates has $k+1$ children. Thus there are as many elements of $I_p(u)$ as number of trees $\psi(u)$ where each internal node is labeled by a positive integer less than or equal to $\binom{(\text{number of children of x})-1+(p-1)}{p-1}$. This finishes the proof of the proposition. □

Let $d_{0,p} = d_{1,p} = 1$ and $d_{n,p} = |\operatorname{Tam}(C_{(n-1)p}, \phi_p)|$ for all $n > 1$.

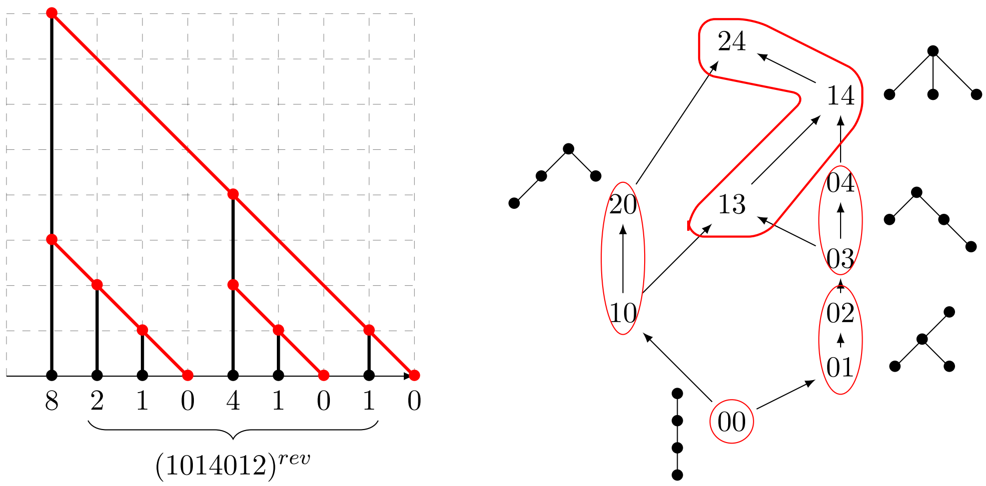


Figure 4.8: On the left this is the mirror image of the diagram of $u = 1014012$ with an extra initial 0 and extra final 8 and lines of slope 1 drawn in red starting at each 0. The segments with the red lines form the binary tree $\varphi(u)$, and $\psi(u)$ is obtained by identifying the red points that are on the same red lines. On the right we show the partition of $\mathrm{Tam}(C_4, \phi_2)$ with the intervals $I_2(u)$ in red, and we represented the associated tree $\psi(u)$ for each interval.

Theorem 4.63 *The generating function $D_p(x)$ of $(d_{n,p})_{n\geq 0}$ satisfies $D_p(x) = 1 + \frac{x\,D_p(x)}{(1-x\,D_p(x))^p}$.*

Proof. By Lemma 4.56, we have $d_{n,p} = \sum_{u \text{ a Tamari word of length } n} |I_p(u)|$. With Proposition 4.62, it follows

$$d_{n,p} = \sum_{u \text{ a Tamari word of length } n} \quad \prod_{x \text{ an internal node of } \varphi(u)} \binom{(\text{number of children of x}) - 1 + (p-1)}{p-1}.$$

Thus $d_{n,p}$ is the number of planar rooted trees on $n+1$ vertices where each internal node is labeled by a positive integer less than or equal to $\binom{(\text{number of children of x}) - 1 + (p-1)}{p-1}$. It follows:

$$D_p(x) = 1 + \sum_{k\geq 1} \binom{k+p-2}{p-1} \big(x\,D_p(x)\big)^k = 1 + x\,D_p(x) \sum_{k\geq 0} \binom{k+p-1}{p-1} \big(x\,D_p(x)\big)^k.$$

Using $\frac{1}{(1-X)^p} = \sum_{k\geq 0} \binom{k+p-1}{p-1} X^k$, we have $D_p(x) = 1 + \frac{x\,D_p(x)}{(1-x\,D_p(x))^p}$. □

Corollary 4.64 *For all $n \geq 0$ and $p \geq 1$ we have* $d_{n,p} = \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} \binom{n+1}{k} \binom{(p-1)\,k + n - 1}{p\,k - 1}$.

Proof. This follows from Theorem 4.63 by using the Lagrange inversion formula. Let $R(x) = x\,D_p(x)$. The formula of Theorem 4.63 can be written $R(x) = x\left(1 + \frac{R(x)}{(1-R(x))^p}\right)$. Let $\Phi(u) = 1 + \frac{u}{(1-u)^p}$. Thus we have

$R(x) = x\,\Phi(R(x))$ with $R(0) = 0$. By the Lagrange inversion formula, we have

$$d_{n,p} = [x^{n+1}]\,R(x) = \frac{1}{n+1}[u^n]\Phi(u)^{n+1} = \frac{1}{n+1}[u^n]\Big(1+\frac{u}{(1-u)^p}\Big)^{n+1}.$$

By the binomial theorem and the expansion of $\frac{1}{(1-X)^p}$ recalled in the proof of Theorem 4.63, we have

$$\Big(1+\frac{u}{(1-u)^p}\Big)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{u^k}{(1-u)^{p\,k}} = \sum_{k=0}^{n+1}\binom{n+1}{k}u^k\sum_{m\geq 0}\binom{m+p\,k-1}{p\,k-1}u^m.$$

This finishes the proof of the corollary by taking $m = n-k$ and using the above formula for $d_{n,p}$. $\square$

Note that when $p = 1$, we recover as expected by Proposition 4.57 a formula giving the Catalan numbers, the terms in the sum being the Narayana numbers. The sequence $(d_{n,2})_{n\geq 0}$ corresponds to the sequence $A109081$ of the OEIS, whose first terms are $1, 1, 3, 10, 37, 146, 602, 2563$. When $p \in \{3, 4, 5\}$ the associated sequences $(d_{n,p})_{n\geq 0}$ are respectively the sequences $A161797$, $A321798$ and $A321799$, but for $p \geq 6$ there are no corresponding entries in the OEIS.

Remark 4.65 *Combe and Giraudo [26] also obtained similar, but different, generalizations of the Tamari lattice called δ-canyon lattices. In our generalization $\mathrm{Tam}(C_{np}, \phi_p)$ we have lines of slope p, whereas they keep the lines of slopes 1 that give the Tamari lattice but they relax the conditions on the height of the segments.*

Now we are interested in the general case $\mathrm{Tam}(C_n, \phi)$. Denote by $c_{k_0,k_1,\ldots,k_{n-1}}$ the number of elements of $\mathrm{Tam}(C_n, \phi)$ where $k_0 = \phi(0)+1$ and $k_i = \phi(i) - \phi(i-1)$ for all $1 \leq i < n$ (see the left-hand side of Figure 4.9). By convention, let $c_\emptyset = 1$. For example, the case $\phi = C_n$, which is the Tamari lattice Tam_{n+1} by Proposition 4.57, gives $|Tam_{n+1}| = c_{1,1,\ldots,1}$ with n ones in the index.

Proposition 4.66 *We have $c_{k_0} = k_0 + 1$ and $c_{k_0,k_1} = 1 + k_1 + \dfrac{k_0(k_0+5)}{2}$. For any $n \geq 3$, we have*

$$\begin{aligned} c_{k_0,k_1,\ldots,k_{n-1}} = {} & c_{k_0,k_1,\ldots,k_{n-2}} + c_{k_0,k_1,\ldots,k_{n-3}} \times k_{n-1} + c_{k_0,k_1,\ldots,k_{n-4}}\Big(\frac{(k_{n-2}+2)(k_{n-2}+1)}{2} - 1\Big) \\ & + \sum_{i=0}^{n-3} c_{k_0,k_1,\ldots,k_{i-2}}\sum_{j=1}^{k_i} c_{j,k_{i+1},k_{i+2},\ldots,k_{n-2}}. \end{aligned}$$

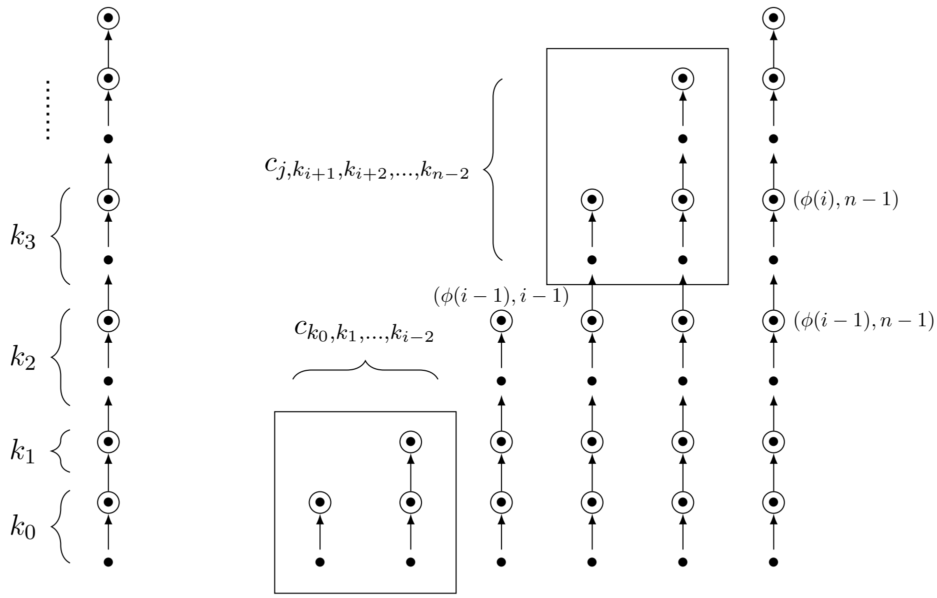


Figure 4.9: On the left we have C_n with ϕ whose elements are circled. On the right we have an illustration of $S(i,j)$ in the proof of Proposition 4.66 where the elements $(\phi(i),j) \in \mathcal{C}_P^{\phi}$ for $i \leq j < n$ are circled.

Proof. The first two formulas follow from Proposition 4.39. Now we suppose that $n \geq 3$. We consider the lattice $\mathrm{Tam}(C_n,\phi)$ and denote $k_0 = \phi(0)+1$ and $k_q = \phi(q) - \phi(q-1)$ for all $q \in [n-1]$. Let $i < n$ and $j \in [k_i]$. Let $S(i,j)$ be the set of the torclosed sets of $\mathrm{Tam}(C_n,\phi)$ that contain $(\phi(i)-j+1, n-1)$ but not $(\phi(i)-j, n-1)$, and $S(n)$ be the set of torclosed sets that do not contain $(\phi(n-1), n-1)$. Since all the sets that we just introduced form a partition of the elements of $\mathrm{Tam}(C_n,\phi)$, then

$$c_{k_0,k_1,\ldots,k_{n-1}} = |S(n)| + \sum_{i=0}^{n-1}\sum_{j=1}^{k_i} |S(i,j)|.$$

We have $|S(n)| = c_{k_0,k_1,\ldots,k_{n-2}}$. Now we want to know $|S(i,j)|$. A torclosed set in $S(i,j)$ is the disjoint union of a torclosed set of $\mathrm{Tam}(C_n, \phi_{|\{0,1,\ldots,i-2\}})$ and of a specific torclosed set T of $\mathrm{Tam}(C_n,\phi)$ with elements only in the components $\mathcal{C}_i$ through $\mathcal{C}_{n-2}$, where T_i has at most j elements. More precisely, these latter torclosed sets in $S(i,j)$ correspond bijectively to the torclosed sets of $\mathrm{Tam}(C_{n-\phi(i-1)}, \phi')$ where $\phi'(0) = j$ and $\phi'(p) - \phi'(p-1) = k_{i+p}$ for all $p \in [n-i-2]$. See Figure 4.9 for an illustration of $S(i,j)$. This gives for all $i < n-2$, $|S(i,j)| = c_{k_0,k_1,\ldots,k_{i-2}} \times c_{j,k_{i+1},k_{i+2},\ldots,k_{n-2}}$, and $|S(n-2,j)| = c_{k_0,k_1,\ldots,k_{n-4}} \times c_j$, and $|S(n-1,j)| = c_{k_0,k_1,\ldots,k_{n-3}}$. The formula of the proposition for $n \geq 3$ then follows from the above formula for $c_{k_0,k_1,\ldots,k_{n-1}}$ by extracting the terms of the sum corresponding to $i = n-1$ and $i = n-2$. $\square$

By letting $\mathrm{Cat}_0 = \mathrm{Cat}_1 = 1$ and, for any $n > 1$, $\mathrm{Cat}_n = c_{1,1,\ldots,1}$ where the number of 1s in the index is $n-1$, we recover from Proposition 4.66 the formula $\mathrm{Cat}_{n+1} = \mathrm{Cat}_n + \mathrm{Cat}_{n-1} + \mathrm{Cat}_{n-2} \times 2 + \sum_{i=0}^{n-3} \mathrm{Cat}_i \mathrm{Cat}_{n-i} = \sum_{i=0}^{n} \mathrm{Cat}_i \mathrm{Cat}_{n-i}$, which is a well-known recursive formula of the Catalan numbers.

4.6.2 Lattices of d-torsion classes of the $(d-1)$-Auslander algebras of type A

Let d and n be positive integers. Recall the notations and results from Section 4.4.2. The poset os_n^{d+1} is ordered with the product order $\leq$ (see Figure 4.11a for an example). By Theorem 4.18, a subset $T \subseteq os_n^{d+1}$ is an element of L_n^d if and only if it satisfies the following two conditions:

(1) For all $i < n$, $\{x \in T \mid x_{d+1} = i\}$ is an order filter of the subposet $\{x \in os_n^{d+1} \mid x_{d+1} = i\}$.

(2) For all $x, z \in T$, if $x \rightsquigarrow \tau_d(z)$, then any $y \in os_n^{d+1}$ with $y_i \in \{x_i, z_i\}$ for each i must be in T.

We now prove that we can replace condition (2) by a simpler one (in a sense that will be explained later). We first prove a lemma.

Lemma 4.67 *Let $T \subseteq os_n^{d+1}$. Then $T \in L_n^d$ if and only if it satisfies* (1) *above and*

$$\begin{gathered}\text{For all } i < j < n, \text{ if } (x_1, \dots, x_d, i) \in T \text{ and } (x_2+1, x_3+1, \dots, x_d+1, i+1, j) \in T, \\ \text{then } (x_1, \dots, x_d, j) \in T.\end{gathered} \tag{$2'$}$$

Proof. Let $T \subseteq os_n^{d+1}$. We prove that T satisfies both (1) and (2′) if and only if it satisfies both (1) and (2). We suppose that T satisfies (1).

Suppose that T satisfies (2). Let $i < j < n$. Suppose that $(x_1, \dots, x_d, i) \in T$ and $(x_2+1, x_3+1, \dots, x_d+1, i+1, j) \in T$. Since $(x_1, \dots, x_d, i) \rightsquigarrow \tau_d(x_2+1, x_3+1, \dots, x_d+1, i+1, j)$, by (2) we have $(x_1, \dots, x_d, j) \in T$. This proves that T satisfies (2′).

Conversely, suppose that T satisfies (2′). Let $x, z \in T$. Suppose $x \rightsquigarrow \tau_d(z)$, meaning $x_1 \leq z_1 - 1 \leq x_2 \leq z_2 - 1 \leq \cdots \leq x_{d+1} \leq z_{d+1} - 1$. We have that any $y \in os_n^{d+1}$ with $y_i \in \{x_i, z_i\}$ for each i is above $(x_1, \dots, x_d, z_{d+1})$ for the product order. Thus, using (1), it suffices to prove $(x_1, \dots, x_d, z_{d+1}) \in T$. Since $x = (x_1, \dots, x_d, x_{d+1}) \in T$, it suffices to prove that $(x_2+1, x_3+1, \dots, x_d+1, x_{d+1}+1, z_{d+1}) \in T$, as

(2′) would then imply that $(x_1, \dots, x_d, z_{d+1}) \in T$. Since we have $x_1 \leq z_1 - 1 \leq x_2 \leq z_2 - 1 \leq \cdots \leq x_{d+1} \leq z_{d+1} - 1$, we know that $z \leq (x_2 + 1, x_3 + 1, \dots, x_{d+1} + 1, z_{d+1})$. Since $z \in T$ and T satisfies (1), we have $(x_2 + 1, x_3 + 1, \dots, x_{d+1} + 1, z_{d+1}) \in T$. This finishes the proof of the lemma. □

Proposition 4.68 *Let $T \subseteq os_n^{d+1}$. Then $T \in L_n^d$ if and only if it satisfies* (1) *above and*

$$\begin{aligned}&\textit{For all } i < j < n, \textit{ if } (x_1, \dots, x_d, i) \in T \textit{ and } (i+1, \dots, i+1, j) \in T,\\ &\textit{then } (x_1, \dots, x_d, j) \in T.\end{aligned} \tag{2′′}$$

Proof. Let $T \subseteq os_n^{d+1}$. We prove that T satisfies both (1) and (2′′) if and only if it satisfies both (1) and (2′). Using Lemma 4.67, this will prove the proposition. We suppose that T satisfies (1).

Suppose that T satisfies (2′′). Let $i < j < n$. Suppose that $(x_1, \dots, x_d, i) \in T$ and $(x_2 + 1, x_3 + 1, \dots, x_d + 1, i + 1, j) \in T$. We need to prove that $(x_1, \dots, x_d, j) \in T$. Since $(x_1, \dots, x_d, i) \in os_n^{d+1}$, then $x_p \leq i$ for all $p \in [d]$. Thus $(i + 1, \dots, i + 1, j) \geq (x_2 + 1, x_3 + 1, \dots, x_d + 1, i + 1, j) \in T$. Since T satisfies (1), then $(i + 1, \dots, i + 1, j) \in T$. Since T satisfies (2′′), then $(x_1, \dots, x_d, j) \in T$. Thus T satisfies (2′).

Conversely, suppose that T satisfies (2′). Let $i < j < n$. Suppose that $(x_1, \dots, x_d, i) \in T$ and $(i+1, \dots, i+1, j) \in T$. We need to prove that $(x_1, \dots, x_d, j) \in T$. For $k \in \{0, 1, \dots, i\}$, let

$$\begin{aligned}
X_k &:= (\max(x_1, i - k), \max(x_2, i - k), \dots, \max(x_d, i - k), i),\\
L_k &:= (\max(x_2, i - k) + 1, \max(x_3, i - k) + 1, \dots, \max(x_d, i - k) + 1, i + 1, j),\\
Y_k &:= (\max(x_1, i - k), \max(x_2, i - k), \dots, \max(x_d, i - k), j).
\end{aligned}$$

See Figure 4.10 for an illustration of these elements and of the following arguments. Since T satisfies (2′), having both $X_k \in T$ and $L_k \in T$ implies $Y_k \in T$. Let P_k be the property $(X_k, L_k) \in T^2$. Let us prove by induction that P_k is true for all $k \in \{0, 1, \dots, i\}$.

Since $(x_1, \dots, x_d, i) \in os_n^{d+1}$, then $x_p \leq i$ for all $p \in [d]$. Thus $X_0 = (i, i, \dots, i) \geq (x_1, \dots, x_d, i) \in T$. Since T satisfies (1), then $X_0 \in T$. We already know that $L_0 = (i + 1, \dots, i + 1, j) \in T$. Thus P_0 is true.

Let $k \in \{0, 1, \dots, i - 1\}$ and assume that P_k is true. It means that $(X_k, L_k) \in T^2$, which implies $Y_k \in T$ since T satisfies (2′). We have $X_{k+1} \geq (x_1, x_2, \dots, x_d, i) \in T$, thus $X_{k+1} \in T$ since T satisfies (1). Let

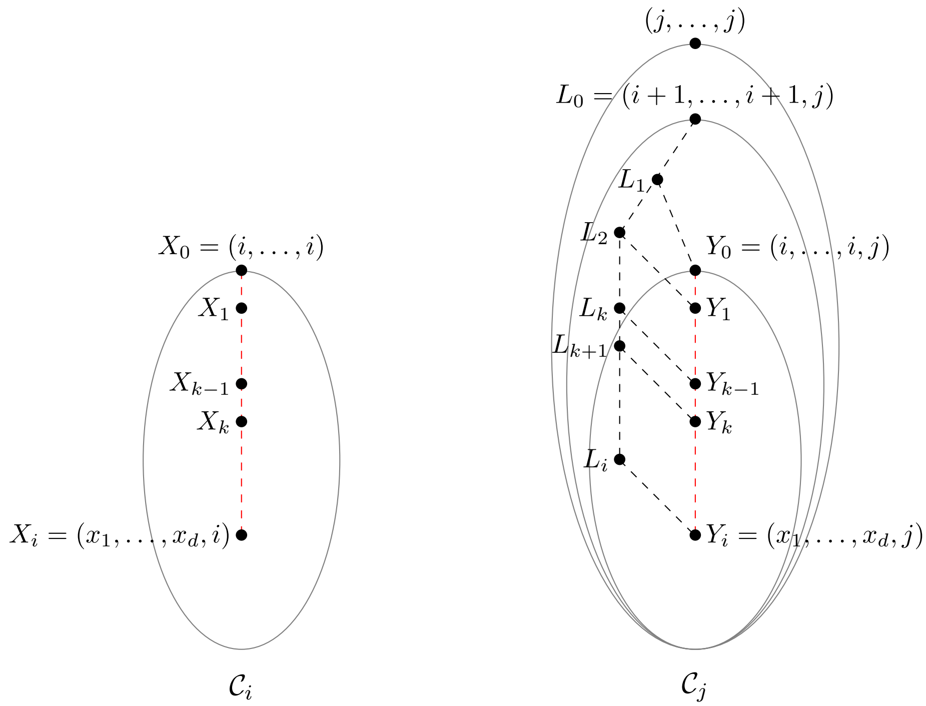


Figure 4.10: Illustration of the proof of Proposition 4.68

$l \in [d-1]$. Since $x_{l+1} \geq x_l$, we have $\max(x_{l+1}, i-k-1)+1 \geq \max(x_l, i-k)$. This proves $L_{k+1} \geq Y_k$, and since $Y_k \in T$, then $L_{k+1} \in T$ since T satisfies (1). Then $(X_{k+1}, L_{k+1}) \in T^2$, meaning P_{k+1} is true.

By induction, this proves in particular that P_i is true. Thus $(X_i, L_i) \in T^2$. This implies that $Y_i \in T$ since T satisfies (2′). Thus $Y_i = (x_1, \dots, x_d, j) \in T$. This finishes the proof of the proposition. □

Note that Proposition 4.68 is a nice improvement on Theorem 4.18. Indeed, we replaced (2) with (2″), which still says that $(x_1, \dots, x_d, j) \in T$ as soon as both $(x_1, \dots, x_d, i)$ and another element are in T. But the other element no longer depends on $x_1, \dots, x_d$, only on i.

The next theorem follows from the previous result and motivated the definition of the (P, ϕ)-Tamari lattices.

Theorem 4.69 *$L_n^d = \mathrm{Tam}\big((os_n^d, \leq), \phi\big)$ with ϕ defined by $\phi(i) = (i, \dots, i)$ for all $i < n$.*

Proof. Both L_n^d and $\mathrm{Tam}\big((os_n^d, \leq), \phi\big)$ are lattices ordered by inclusion, thus it suffices to prove that the d-torsion classes $T \in L_n^d$ are the torclosed sets of $\mathrm{Tam}\big((os_n^d, \leq), \phi\big)$. Recall the definition of $\mathcal{C}_P^\phi$ in Section

4.5.1. By definition, $T \in L_n^d$ satisfies (1) if and only if T is an order filter of $\mathcal{C}_P^{\phi}$. Since $(\phi(i+1), j) = (i+1, \ldots, i+1, j)$, then $T \in L_n^d$ satisfies $(2'')$ if and only if T is a torclosed set of $\mathrm{Tam}\big((os_n^d, \leq), \phi\big)$. This proves the theorem using Proposition 4.68. □

Note that L_n^d is a (P, ϕ)-Tamari lattice that enjoys nice properties as os_n^d is a lattice and the associated ϕ satisfies $\phi(0) = (0, \ldots, 0) = \hat{0}_{os_n^d}$. Recall that the formula $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$, for all integers $n \geq r$, is called the *hockey-stick identity*. From Theorem 4.69 and the results of Section 4.5 follow:

Corollary 4.70 *The lattice L_n^d inherits all properties of the (P, ϕ)-Tamari lattices. In particular, it is a join-semidistributive lattice, and it is semidistributive if and only if $n \leq 2$ or $d = 1$. The spine of L_n^d corresponds to $J((os_n^{d+1})^{op})$. We have $\ell(L_n^d) = |\mathrm{JIrr}(L_n^d)| = \binom{n+d}{d+1}$ and $|\mathrm{MIrr}(L_n^d)| = (d-1)\binom{n+d}{d+2} + \frac{n(n+1)}{2}$.*

Proof. It is immediate that $|os_n^d| = \binom{n+d-1}{d}$. We only need to prove the formula for the number of meet-irreducibles. We use Proposition 4.30. The number of the first kind of meet-irreducibles is $\binom{n+d}{d+1}$. Counting the second kind of meet-irreducibles amounts to count, for any $i < n$ and any $j \leq i$, the number of elements of $\mathcal{C}_i$ incomparable to $(\phi(j), i)$. Since $\phi(0) = \hat{0}_{os_n^d}$, no such elements exist if $i \in \{0, 1\}$, nor if $j = \{0, i\}$ since $(\phi(0), i)$ and $(\phi(i), i)$ are respectively the minimum and maximum elements of $\mathcal{C}_i$. Then

$$|\mathrm{MIrr}(L_n^d)| = \binom{n+d}{d+1} + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} |\{(x, i) \in \mathcal{C}_i \mid (x, i) \not\sim (\phi(j), i)\}|$$

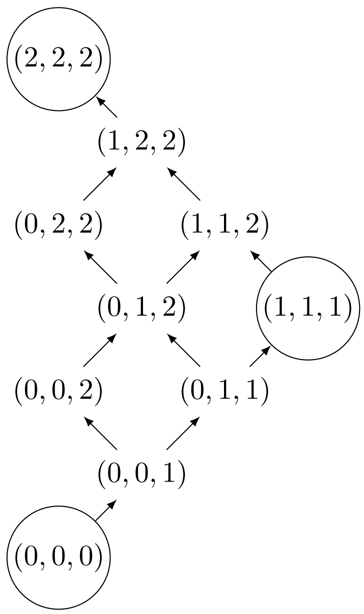
Let i and j such that $2 \leq i < n$ and $0 < j < i$. The integer in the sum above is also the number of elements of os_n^d less than $\phi(i) = (i, \ldots, i)$ that are incomparable to $\phi(j) = (j, \ldots, j)$. Such an element needs to start with a $k < j$, and finish with a $l \in \{j+1, j+2, \ldots, i\}$, and between the two it should be non-decreasing. Thus when k and l are fixed, the number of such elements is $|os_{l-k+1}^{d-2}| = \binom{l-k+d-2}{d-2}$. This proves that

$$|\mathrm{MIrr}(L_n^d)| = \binom{n+d}{d+1} + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} \sum_{k=0}^{j-1} \sum_{l=j+1}^{i} \binom{l-k+d-2}{d-2}.$$

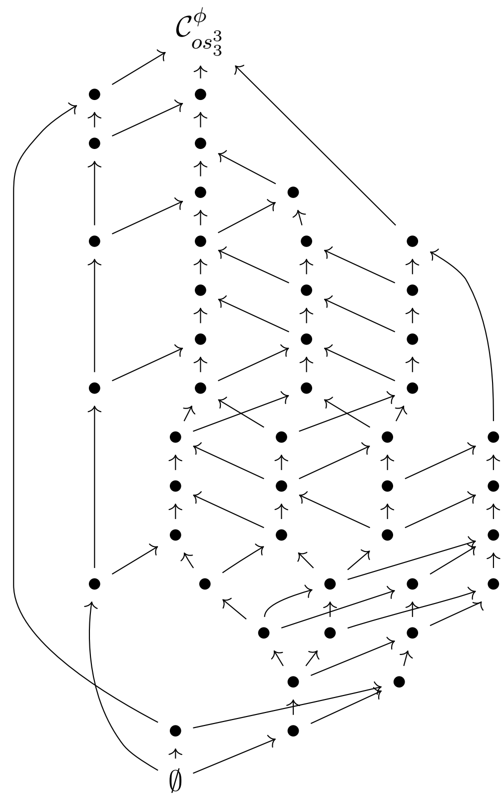
By successive use of the hockey-stick identity, this reduces to

$$|\mathrm{MIrr}(L_n^d)| = \binom{n+d}{d+1} + \sum_{i=2}^{n-1} (i-1)\binom{i+d}{d} - 2\binom{n+d}{d+2} + \frac{(n-1)(n+2)}{2}$$

Using the identity $i\binom{i+d}{d} = (d+1)\binom{i+d}{d+1}$ and again the hockey-stick identity, we obtain the desired formula. □



a The poset os_3^3 with the chain $\phi : i \mapsto (i, \dots, i)$ whose elements are circled.



b The lattice L_3^3.

Figure 4.11

Example 4.71 *The lattice L_3^3 is given in Figure 4.11b (which is taken from [3]). As proved in Corollary 4.70 it is not semidistributive (as observed in [3, Example 5.21]), and we can check that the number of meet-irreducibles is* 18*, as predicted by our formula.*

Definition 4.72 *The* totally symmetric d-dimensional partitions *are the order ideals I of $(\mathbb{N}^d, \leq_{prod})$ such that if $x = (x_1, x_2, \dots, x_d) \in I$, then $x_\sigma := (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(d)}) \in I$ for all permutations $\sigma \in \mathfrak{S}_d$, where $\mathfrak{S}_d$ is the set of all permutations of $[d]$.*

Lemma 4.73 *Let I be a totally symmetric d-dimensional partition and $x \in I$ be a maximal element of I. Then x_σ is also a maximal element of I for all $\sigma \in \mathfrak{S}_d$.*

Proof. Seeking a contradiction, suppose that there exists $\sigma \in \mathfrak{S}_d$ such that x_σ is not a maximal element of I. Then there exists $y \in I$ such that $x_\sigma <_{prod} y$. By applying σ^{-1} to both sides, which keeps the product order inequality, we obtain $x <_{prod} y_{\sigma^{-1}}$. But $y_{\sigma^{-1}} \in I$ and x is a maximal element of I. This is absurd. $\square$

We recall the following result of J.R. Stembridge:

Proposition 4.74 ([99]) *The number of totally symmetric* 3*-dimensional partitions, also called* totally symmetric plane partitions*, which fit in a* 3*-dimensional box whose sides all have length* n *is given by the sequence* $(a_n)_{n\geq 0}$ *(*$A005157$ *of the* OEIS *[97]) defined for all* $n \geq 0$ *by* $a_n := \displaystyle\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n} \frac{i+j+k-1}{i+j+k-2}$.

The first ten values of the sequence $(a_n)_{n\geq 0}$ are $1, 2, 5, 16, 66, 352, 2431, 21760, 252586, 3803648$.

Proposition 4.75 *We have an isomorphism of posets between* os_n^d *and* os_{d+1}^{n-1}*, thus* $|J(os_n^d)| = |J(os_{d+1}^{n-1})|$. *We have that* $|J(os_n^d)|$ *is the number of totally symmetric* d*-dimensional partitions which fit in a* d*-dimensional box whose sides all have length* n*. We deduce that* $|J(os_3^d)| = 2^{d+1}$ *and* $|J(os_4^d)| = a_{d+1}$ *where* $(a_d)_{d\geq 0}$ *is defined in Proposition 4.74.*

Proof. The function which maps $(x_1, x_2, \ldots, x_d) \in os_n^d$ to the non-decreasing sequence of size $n-1$ with $n-1-x_d$ zeros, $x_d - x_{d-1}$ ones, and so on where the number of i's is $x_{d-i+1} - x_{d-i}$ for all $i \in [d]$, is an isomorphism of posets from os_n^d to os_{d+1}^{n-1}. Indeed this bijection comes from reading row by row or column by column the Ferrers diagram of $(x_1, x_2, \ldots, x_d) \in os_n^d$. The fact that it is an isomorphism of posets is easily seen with this representation.

We now prove the combinatorial interpretation for $|J(os_n^d)|$. By definition, we can identify a totally symmetric d-dimensional partition with its set of maximal elements. By Lemma 4.73, we in fact just need to specify the maximal elements that are non-decreasing. This shows that the totally symmetric d-dimensional partitions which fit in a d-dimensional box whose sides all have length n correspond to the antichains of $(os_n^d, \leq)$. These latter are counted by $|J(os_n^d)|$. This proves what we wanted.

Finally, from this combinatorial interpretation the formulas for $|J(os_3^d)|$ and $|J(os_4^d)|$ follow. $\square$

Unfortunately, no formulas are known for the number of totally symmetric d-dimensional partitions which fit in a d-dimensional box whose sides all have length $n \geq 5$. Building on the above results, we are able to obtain formulas for the number of elements of L_n^d when $n \leq 4$ (no formulas were known for $n \in \{3, 4\}$). We start with a lemma.

Lemma 4.76 *Let P be a finite poset and $x \in P$. Let I be the subposet of P on the elements that are incomparable to x. Then the number of order filters of P where x is a generator (which means x is a minimal element) is equal to $|J(I)|$. Moreover $|J(P)| = |J(P \setminus \{x\})| + |J(I)|$.*

Proof. These order filters are identified with their minimal elements, which are the antichains of P that contain x. Thus the number of these order filters is the number of antichains of P formed by elements that are all incomparable to x. The results follow. $\square$

Proposition 4.77 *Let $d \geq 1$. We have $|L_1^d| = 2$, $|L_2^d| = d + 4$, and $|L_3^d| = d + 3 + 5 \times 2^d$. Let $K_1 := [(0, \ldots, 0, 2, 2), (1, 3, \ldots, 3)]$, $K_2 := [(0, \ldots, 0, 2, 3), (1, 3, \ldots, 3)]$ and $K_3 := \{x \in os_4^{d+1} \mid x \not\geq 1 \cdots 13\}$ be three subposets of os_4^{d+1}. Then*

$$|L_4^d| = 8 + d + 3 \times 2^{d+1} + (d + 4)(a_d - 1) + \sum_{i=2}^{d} a_i + 2\left(|J(K_1)| + |J(K_2)|\right) + |J(K_3)|, \qquad (*)$$

where the sequence $(a_d)_{d \geq 0}$ is defined in Proposition 4.74.

Proof. The formulas for $|L_n^d|$ with $n \leq 3$ follow from Theorem 4.69 and Proposition 4.40, using for $|L_3^d|$ the formula $|J(os_3^d)| = 2^{d+1}$ obtained in Proposition 4.75. For $|L_4^d|$ it requires more work. By Theorem 4.69, $L_4^d = \mathrm{Tam}(os_4^d, \phi)$ with ϕ defined by $\phi(i) = (i, \ldots, i)$ for all $i < n$. Let:

- $R_1 := \{T \in L_4^d \mid (\phi(3), 3) \notin T\}$
- $R_2 := \{T \in L_4^d \mid (\phi(3), 3) \in T,\ (\phi(2), 3) \notin T\}$
- $R_3 := \{T \in L_4^d \mid (\phi(2), 3) \in T,\ (\phi(1), 3) \notin T\}$
- $R_4 := \{T \in L_4^d \mid (\phi(1), 3) \in T,\ (\phi(0), 3) \notin T,\ (\phi(2), 2) \notin T\}$

- $R_5 := \{T \in L_4^d \mid (\phi(1), 3) \in T,\ (\phi(0), 3) \notin T,\ (\phi(2), 2) \in T\}$

- $R_6 := \{T \in L_4^d \mid (\phi(0), 3) \in T\}$

These sets form a partition of all the torclosed sets of L_4^d, thus $|L_4^d| = \sum_{i=1}^{6} |R_i|$. It is immediate that $|R_1| = |R_6| = |L_3^d| = d + 3 + 5 \times 2^d$.

A torclosed set of R_2 is a torclosed set of $\mathrm{Tam}(os_4^d, \phi_{|\{0,1\}})$ together with an order filter of $\big(\mathcal{C}_3 \setminus \{(\phi(3), 3)\}\big) \setminus I_{\mathcal{C}_3}(\phi(2), 3)$. The elements in this latter poset are those elements of os_4^{d+1} that end with two letters 3, but are not $(3, 3, \ldots, 3)$. It follows that $|R_2| = (d + 4)\,(|J(os_4^{d-1})| - 1)$. Thus, by Proposition 4.75, $|R_2| = (d + 4)(a_d - 1)$.

Let $Q = (\mathcal{C}_2 \cup \mathcal{C}_3, \leq_{prod})$. A torclosed set of R_3 may or may not contain $(\phi(0), 0)$, does not contain elements of $\mathcal{C}_1$, and on the other elements it should be an order filter of $Q \setminus \big(I_Q(\phi(1), 3) \cup F_Q(\phi(2), 3)\big)$. The elements in this latter poset are those of os_4^{d+1} that are in $[(0, \ldots, 0, 2, 2), (1, 3, \ldots, 3)] \cup \{(2, \ldots, 2)\} = K_1 \cup \{(2, \ldots, 2)\}$. Then by Lemma 4.76, $|R_3| = 2 \times (|J(K_1)| + |J(I)|)$ where I is the subposet of $K_1 \cup \{(2, \ldots, 2)\}$ on the incomparable elements of $(2, \ldots, 2)$. We have $I = [(0, \ldots, 0, 2, 3), (1, 3, \ldots, 3)] = K_2$. Thus $|R_3| = 2\,(|J(K_1)| + |J(K_2)|)$.

The torclosed sets of R_4 are the order filters of the subposet of os_4^{d+1} on the elements $[(0, \ldots, 0, 1, 1), (1, \ldots, 1)] \cup [(0, \ldots, 0, 1, 3), (0, 3, \ldots, 3)]$. Denote by V this subposet. We have $|R_4| = |J(V)|$. It will be helpful to add $(0, \ldots, 0, 3)$ to V (the adding forms a new subposet of os_4^{d+1}). To do this we use Lemma 4.76, which gives $|R_4| = |J(V \cup \{(0, \ldots, 0, 3)\})| - |J(I')|$ where I' is the subposet of $V \cup \{(0, \ldots, 0, 3)\}$ on the elements that are incomparable to $(0, \ldots, 0, 3)$. Then $I' = [(0, \ldots, 0, 1, 1), (1, \ldots, 1)]$, thus $|R_4| = |J(V \cup \{(0, \ldots, 0, 3)\})| - (d + 1)$. In $V \cup \{(0, \ldots, 0, 3)\}$ we give the subposet formed by the incomparable elements to each of the element of the chain $[(0, \ldots, 0, 1, 1), (1, \ldots, 1)]$; there is only one element incomparable to $(0, \ldots, 0, 1, 1)$ which is $(0, \ldots, 0, 3)$, for $(1, \ldots, 1)$ the incomparable elements form the subposet $[(0, \ldots, 0, 3), (0, 3, \ldots, 3)] \cong os_4^{d-1}$, and for any $(0, \ldots, 0, 1, \ldots, 1)$ with the number of zeros equal to $i \in [d - 2]$ the incomparable elements are the elements starting with $i + 1$ zeros and ending with a 3, which form a subposet isomorphic to os_4^{d-i-1}. Remember that $|J(os_4^j)| = a_{j+1}$ (Proposition 4.75). Then applying Lemma 4.76 successively for all the elements of the chain $[(0,\ldots,0,1,1),(1,\ldots,1)]$ gives at the end

$$|J(V\cup\{(0,\ldots,0,3)\})| = |J([(0,\ldots,0,3),(0,3,\ldots,3)])| + 2 + a_d + \sum_{i=1}^{d-2} a_{d-i} = 2 + 2\,a_d + \sum_{i=1}^{d-2} a_{d-i}.$$

Thus $|R_4| = 2 + 2\,a_d + \sum_{i=1}^{d-2} a_{d-i} - (d+1) = 1 + a_d - d + \sum_{i=2}^{d} a_i$.

The torclosed sets of R_5 are the order filters of the subposet of $os_4^{d+1} \setminus \{(2,\ldots,2)\}$ on the elements that are neither bigger than $(1,\ldots,1,3)$ nor smaller than $(0,\ldots,0,3)$. Denote by W this subposet. We have $K_3 = W \cup \{(0,\ldots,0,1),(0,\ldots,0),(0,\ldots,0,2),(0,\ldots,0,3),(2,\ldots,2)\}$. We now use five times Lemma 4.76 to add the five above elements to W to obtain K_3. As $(0,\ldots,0,1)$ is smaller than all the elements of $W \cup \{(0,\ldots,0,1)\}$, we have $|J(W \cup \{(0,\ldots,0,1)\})| = |J(W)| + 1$. Then as $(0,\ldots,0,0)$ is smaller than all the elements of $W \cup \{(0,\ldots,0,1),(0,\ldots,0,0)\}$, we have $|J(W \cup \{(0,\ldots,0,1),(0,\ldots,0)\})| = |J(W)| + 2$. The subposet of $W \cup \{(0,\ldots,0,1),(0,\ldots,0),(0,\ldots,0,2)\}$ on the incomparable elements of $(0,\ldots,0,2)$ is $[(0,\ldots,0,1,1),(1,\ldots,1)]$, thus $|J(W\cup\{(0,\ldots,0,1),(0,\ldots,0),(0,\ldots,0,2)\})| = |J(W)|+2+(d+1) = |J(W)| + 3 + d$. The subposet of $W \cup \{(0,\ldots,0,1),(0,\ldots,0),(0,\ldots,0,2),(0,\ldots,0,3)\}$ on the incomparable elements of $(0,\ldots,0,3)$ is $[(0,\ldots,0,1,1),(2,\ldots,2)[$. Since

$$[(0,\ldots,0,1,1),(2,\ldots,2)[\cup\{(2,\ldots,2)\}\cup[(0,\ldots,0),(0,\ldots,0,2)] = [(0,\ldots,0),(2,\ldots,2)] \cong os_3^{d+1},$$

by using Lemma 4.76 we can obtain $|J([(0,\ldots,0,1,1),(2,\ldots,2)[)| = |J(os_3^{d+1})| - d - 4 = 2^{d+2} - d - 4$. Thus

$$|J(W\cup\{(0,\ldots,0,1),(0,\ldots,0),(0,\ldots,0,2),(0,\ldots,0,3)\})| = |J(W)| + 2^{d+2} - 1.$$

The incomparable elements of $(2,\ldots,2)$ in K_3 are the ones that start with a 0 and end with a 3, thus they form a subposet of K_3 isomorphic to os_4^{d-1}. Then

$$|J(K_3)| = (|J(W)| + 2^{d+2} - 1) + |J(os_4^{d-1})| = |J(W)| + 2^{d+2} - 1 + a_d.$$

Thus $|R_5| = |J(W)| = |J(K_3)| - 2^{d+2} - a_d + 1$.

Putting all together using $|L_4^d| = \sum_{i=1}^{6} |R_i|$ finishes the proof of the proposition. □

Remark 4.78 *With Proposition 4.74 and* $(*)$*, we are able to compute* $|L_4^7| = 6543848$ *and* $|L_4^8| = 130286256$*. This completes the column* $n = 4$ *of Table* 2 *of [3].*

4.6.3 Lattices of d-torsion classes of the $(d-1)$-Nakayama algebras of type A

Let d and n be positive integers. Recall the notations and results from the end of Section 4.4.2, where we introduced the lattice $L^d_{\underline{l}}$ of the d-torsion classes of the $(d-1)$-Nakayama algebra associated to the Kupisch series $\underline{l}$. We defined $os^{d+1}_{\underline{l}} = \{y \in os^{d+1}_n \mid y_1 \geq y_{d+1} - l_{y_{d+1}} + 1\} \subseteq os^{d+1}_n$. We have that $\mathcal{C}^{\phi}_{os^d_n} \cap os^{d+1}_{\underline{l}}$ is an order filter of $\mathcal{C}^{\phi}_{os^d_n}$. Thus any element of $L^d_{\underline{l}}$ is an order filter of $\mathcal{C}^{\phi}_{os^d_n}$. Recall also that by Theorem 4.69, $L^d_n = \mathrm{Tam}((os^d_n, \leq), \phi)$ with ϕ defined by $\phi(i) = (i, \ldots, i)$ for all $i < n$.

Lemma 4.79 *Let $R \in L^d_{\underline{l}}$. Let $i < j < n$. Suppose that $(x, i) \in R$ and $(\phi(i+1), j) \in R$. Then $(y, j) \in R$ for any $(y, j) \in \mathcal{C}_j \cap os^{d+1}_{\underline{l}}$ such that $(y, j) \geq (x, j)$.*

Proof. Let $(y, j) \in \mathcal{C}_j \cap os^{d+1}_{\underline{l}}$ such that $(y, j) \geq (x, j)$. Let us prove that $(y, j) \in R$. We have $(y, i) \geq (x, i) \in R$, thus $(y, i) \in R$ since R is an order filter of $\mathcal{C}^{\phi}_{os^d_n}$. For $k \in \{0, 1, \ldots, i\}$, let

$$X_k := (\max(y_1, i-k), \max(y_2, i-k), \ldots, \max(y_d, i-k), i),$$
$$L_k := (\max(y_2, i-k)+1, \max(y_3, i-k)+1, \ldots, \max(y_d, i-k)+1, i+1, j),$$
$$Y_k := (\max(y_1, i-k), \max(y_2, i-k), \ldots, \max(y_d, i-k), j).$$

Note that these elements were already introduced in the proof of Proposition 4.68 and were illustrated in Figure 4.10 (where $x_1, \ldots, x_d$ are replaced by $y_1, \ldots, y_d$). Since $X_k \geq X_i = (y, i) \in R$ for all $k \leq i$, then $X_k \in R$ for all $k \leq i$ since R is an order filter of $\mathcal{C}^{\phi}_{os^d_n}$. Since $Y_k \geq Y_i = (y, j) \in os^{d+1}_{\underline{l}}$, then $Y_k \in os^{d+1}_{\underline{l}}$ for all $k \leq i$. Let us prove by induction that $Y_k \in R$ for all $k \leq i$. This will finish the proof of the lemma since $Y_i = (y, j)$. We know that $X_0 = (i, \ldots, i) \in R$, and $L_0 = (i+1, \ldots, i+1, j) = (\phi(i+1), j) \in R$ by hyphothesis. Moreover $X_0 \rightsquigarrow \tau_d(L_0)$. Thus by condition 2 of Theorem 4.20, since $(i, \ldots, i, j) = Y_0 \in os^{d+1}_{\underline{l}}$, we have $Y_0 \in R$. Let $k \in [i]$. Suppose that for all $p < k$, $Y_p \in R$. Let us prove that $Y_k \in R$. We know that $X_k \in R$. Since $L_k \geq Y_{k-1}$ and $Y_{k-1} \in R$ by hypothesis, then $L_k \in R$. Moreover $X_k \rightsquigarrow \tau_d(L_k)$. Thus by condition 2 of Theorem 4.20, since $Y_k \in os^{d+1}_{\underline{l}}$ and the l-th coordinate of this element is either the l-th coordinate of X_k or L_k for all $l \in [d+1]$, we have $Y_k \in R$. This finishes the proof of the induction, thus of the lemma. □

Proposition 4.80 *The lattice $L^d_{\underline{l}}$ is the lattice, ordered by inclusion, on the sets $T \cap os^{d+1}_{\underline{l}}$ for all torclosed sets $T \in L^d_n$.*

Proof. If $T \in L^d_n$, then $T \cap os^{d+1}_{\underline{l}} \in L^d_{\underline{l}}$. Conversely, let $R \in L^d_{\underline{l}}$. Then $R \subseteq \mathcal{C}^{\phi}_{os^d_n}$. Let T be the smallest torclosed set of L^d_n containing R. Let us prove that $R = T \cap os^{d+1}_{\underline{l}}$. We prove by induction that $T_k \cap os^{d+1}_{\underline{l}} = R_k$ for all $k < n$. It is immediate for $k = 0$. Let $k \in [n-1]$. We suppose that $T_i \cap os^{d+1}_{\underline{l}} = R_i$ for all $i < k$. We want to prove that $T_k \cap os^{d+1}_{\underline{l}} = R_k$.

By definition of T, we know that $R_k \subseteq T_k \cap os^{d+1}_{\underline{l}}$. Thus we need to prove that $T_k \cap os^{d+1}_{\underline{l}} \subseteq R_k$. If $R_k = \emptyset$ this is immediate, thus we suppose that $R_k \neq \emptyset$. By Lemma 4.26, we can compute T_k by first looking at the completion of $T_{k-1} \cup R_k$, which in $\mathcal{C}_k$ forms the order filter that we denote F_1. Then at the completion of $T_{k-2} \cup F_1$, which in $\mathcal{C}_k$ forms the order filter that we denote F_2, and so on. At the end we will have $F_k = T_k$. Let us prove by induction that $F_j \cap os^{d+1}_{\underline{l}} \subseteq R_k$ for all $j \in [k]$. This will finish the proof of the proposition since for $j = k$ this will give $T_k \cap os^{d+1}_{\underline{l}} \subseteq R_k$.

Let us prove that $F_1 \cap os^{d+1}_{\underline{l}} \subseteq R_k$. Let $(y,k) \in F_1 \cap os^{d+1}_{\underline{l}}$. If (y,k) is not obtained through the completion of $T_{k-1} \cup R_k$, then $(y,k) \in R_k$ and there is nothing to prove. Thus we assume that there exists $(z, k-1) \in T_{k-1}$ such that $(y,k) \geq (z,k)$. In particular $(\phi(k), k) \in R_k$. Let $t = (\min(y_1, k-1), \min(y_2, k-1), \dots, \min(y_d, k-1)) \in os^d_n$. We have $(y,k) \geq (t,k)$. Let us prove that $(t, k-1) \in R_{k-1}$.

Since $(y,k) \in \mathcal{C}_k \cap os^{d+1}_{\underline{l}} = \{(x,k) \in \mathcal{C}_k \mid x \geq \phi(k - l_k + 1)\}$, then $y_1 \geq k - l_k + 1$. Since $k \geq 1$, we have $l_k \geq 2$, which gives $k - 1 \geq k - l_k + 1$. Thus $\min(y_1, k-1) \geq k - l_k + 1$. Then $(t,k) \in os^{d+1}_{\underline{l}}$. Since $(t,k) \leq (k-1, \dots, k-1, k)$, we have $(t, k-1) \in \mathcal{C}_{k-1}$. Moreover, $(t, k-1) \in os^{d+1}_{\underline{l}}$. Indeed, since $l_k - l_{k-1} \leq 1$, then $(k-1) - l_{k-1} + 1 \leq k - l_k + 1$. Thus if $(t, k-1) \notin os^{d+1}_{\underline{l}}$, which means $t \ngeq k-1-l_{k-1}+1$, then $t \ngeq k - l_k + 1$, which is absurd since $(t,k) \in os^{d+1}_{\underline{l}}$. Thus $(t, k-1) \in \mathcal{C}_{k-1} \cap os^{d+1}_{\underline{l}}$. Since $(z, k-1) \in \mathcal{C}_{k-1}$, we have $z_i \leq k-1$ for all $i \in [d]$. Moreover $(z,k) \leq (y,k)$ gives $z_i \leq y_i$ for all $i \in [d]$. Then $(z, k-1) \leq (t, k-1)$. Since $(z, k-1) \in T_{k-1}$, then $(t, k-1) \in T_{k-1}$. But we proved that $(t, k-1) \in \mathcal{C}_{k-1} \cap os^{d+1}_{\underline{l}}$, thus $(t, k-1) \in T_{k-1} \cap os^{d+1}_{\underline{l}} = R_{k-1}$ by induction hypothesis.

Then we use Lemma 4.79 with $(t, k-1) \in R$ and $(\phi(k), k) \in R$. This gives $(y,k) \in R_k$ since $(y,k) \in \mathcal{C}_k \cap os^{d+1}_{\underline{l}}$ and $(y,k) \geq (t,k)$. This is what we wanted to prove. Now if we let $j \in [k]$ and we suppose that $F_i \cap os^{d+1}_{\underline{l}} \subseteq R_k$ for all $i < j$, we need to prove that $F_j \cap os^{d+1}_{\underline{l}} \subseteq R_k$. The proof below uses the same structure as in the proof of $F_1 \cap os^{d+1}_{\underline{l}} \subseteq R_k$ above. To help the reader follow these similar arguments, as no confusion is possible since we do not use previously introduced elements, we will reuse some variable names.

Let $(y,k) \in F_j \cap os_{\underline{l}}^{d+1}$. Let us prove that $(y,k) \in R_k$. If (y,k) is not obtained through the completion of $T_{k-j} \cup F_{j-1}$, then $(y,k) \in F_{j-1} \cap os_{\underline{l}}^{d+1} \subseteq R_k$ by induction hypothesis, and there is nothing to prove. Thus we assume that there exists $(z, k-j) \in T_{k-j}$ such that $(y,k) \geq (z,k)$. In particular $(\phi(k-j+1),k) \in F_{j-1}$. If $(\phi(k-j+1),k) \notin os_{\underline{l}}^{d+1}$, then $(\phi(k-j+1),k) \leq (\phi(k-l_k+1),k)$. Since $(\phi(k-j+1),k) \in F_{j-1}$ then $(\phi(k-l_k+1),k) \in F_{j-1} \cap os_{\underline{l}}^{d+1} \subseteq R_k$ by induction hypothesis, and this proves that $F_j \cap os_{\underline{l}}^{d+1} \subseteq \mathcal{C}_k \cap os_{\underline{l}}^{d+1} = R_k$ since $\mathcal{C}_k \cap os_{\underline{l}}^{d+1} = \{(x,k) \in \mathcal{C}_k \mid x \geq \phi(k-l_k+1)\}$. Thus we assume that $(\phi(k-j+1),k) \in os_{\underline{l}}^{d+1}$, which gives us $l_k \geq j$. Then $(\phi(k-j+1),k) \in F_{j-1} \cap os_{\underline{l}}^{d+1} \subseteq R_k$ by induction hypothesis, then $(\phi(k-j+1),k) \in R_k$. To have something to prove we thus further assume that $\phi(k-l_k+1) < \phi(k-j+1)$, which gives $l_k > j$. Let $t = (\min(y_1, k-j), \ldots, \min(y_d, k-j)) \in os_n^d$. We have $(y,k) \geq (t,k)$. Let us prove that $(t, k-j) \in R_{k-j}$.

Since $(y,k) \in \mathcal{C}_k \cap os_{\underline{l}}^{d+1} = \{(x,k) \in \mathcal{C}_k \mid x \geq \phi(k-l_k+1)\}$, then $y_1 \geq k-l_k+1$. We also proved that $l_k > j$, which gives $k-j \geq k-l_k+1$. Thus $\min(y_1, k-j) \geq k-l_k+1$. Then $(t,k) \in os_{\underline{l}}^{d+1}$. Since $(t,k) \leq (k-j, \ldots, k-j, k)$, we have $(t,k-j) \in \mathcal{C}_{k-j}$. Moreover, $(t,k-j) \in os_{\underline{l}}^{d+1}$. Indeed, since $l_k - l_{k-j} \leq j$, then $(k-j) - l_{k-j} + 1 \leq k - l_k + 1$. Thus if $(t,k-j) \notin os_{\underline{l}}^{d+1}$, which means $t \ngeq k-j-l_{k-j}+1$, then $t \ngeq k-l_k+1$, which is absurd since $(t,k) \in os_{\underline{l}}^{d+1}$. Thus $(t,k-j) \in \mathcal{C}_{k-j} \cap os_{\underline{l}}^{d+1}$. Since $(z,k-j) \in \mathcal{C}_{k-j}$, we have $z_i \leq k-j$ for all $i \in [d]$. Moreover $(z,k) \leq (y,k)$ gives $z_i \leq y_i$ for all $i \in [d]$. Then $(z,k-j) \leq (t,k-j)$. Since $(z,k-j) \in T_{k-j}$, then $(t,k-j) \in T_{k-j}$. But we proved that $(t,k-j) \in \mathcal{C}_{k-j} \cap os_{\underline{l}}^{d+1}$, thus $(t,k-j) \in T_{k-j} \cap os_{\underline{l}}^{d+1} = R_{k-j}$ by induction hypothesis.

Then we use Lemma 4.79 with $(t,k-j) \in R$ and $(\phi(k-j+1),k) \in R$. This gives $(y,k) \in R_k$ since $(y,k) \in \mathcal{C}_k \cap os_{\underline{l}}^{d+1}$ and $(y,k) \geq (t,k)$. This proves that $F_j \cap os_{\underline{l}}^{d+1} \subseteq R_k$. As explained, this finishes the proof of the proposition. □

Theorem 4.81 *The lattice $L_{\underline{l}}^d$ is a lattice quotient of L_n^d. Thus it inherits the properties of the (P,ϕ)-Tamari lattices that are preserved by lattice quotient. In particular it is a join-semidistributive lattice.*

Proof. Let $\equiv$ be the equivalence relation on L_n^d defined by $T \equiv T'$ if and only if $T \cap os_{\underline{l}}^{d+1} = T' \cap os_{\underline{l}}^{d+1}$. By Proposition 4.80, the elements of $L_{\underline{l}}^d$ are identified with the equivalence classes of the relation $\equiv$. Let $K = (K_0, K_1, \ldots, K_{n-1})$ be the Kupisch-like series defined by $K_i := \max(0, i - l_i + 1)$ for all $i < n$.

Then the associated Kupisch equivalence relation $\equiv_K$ on L_n^d corresponds to $\equiv$. By Theorem 4.69, $L_n^d = \mathrm{Tam}\left((os_n^d, \leq), \phi\right)$ with ϕ defined by $\phi(i) = (i, \ldots, i)$ for all $i < n$. Thus the proposition follows from Proposition 4.52 since $(os_n^d, \leq)$ is a lattice. □

4.7 Open questions

We finish with some open questions.

Question 4.82 *Can we find a general formula for the number of elements of* $\mathrm{Tam}(P, \phi)$*?*

Maybe a general closed formula is too much to ask as counting the number of order ideals is a problem known to be hard, but we could have one for special cases or depending on the number of ideals of (subposets of) P. For example, in Section 4.6.1 we looked at the special case where $P = C_n$ is a chain. In that case we obtained a recursive formula (Proposition 4.66) to count the number of elements of $\mathrm{Tam}(C_n, \phi)$. For the case of $\mathrm{Tam}(C_{np}, \phi_p)$ we were able to obtain a closed-form expression (Corollary 4.64) using a formula satisfied by the generating function of $|\mathrm{Tam}(C_{np}, \phi_p)|$ (Theorem 4.63).

Question 4.83 *Can we obtain a closed-form expression for the number of elements of* $\mathrm{Tam}(C_n, \phi)$*?*

In the other special case $P = os_n^d$ that we studied in Section 4.6.2, we were able to give a formula for the number of meet-irreducibles of L_n^d (Corollary 4.70).

Question 4.84 *Can we find a simple bijective explanation for the number of meet-irreducibles of L_n^d obtained in Corollary 4.70?*

We have made some progress on the enumeration of the higher torsion classes in that case by giving formulas when $n \leq 4$ is fixed, and d is any positive integer (Proposition 4.77). The following asymptotic behavior seems to hold, which says that when the number of components n is fixed, in some ways the lattice L_n^d converges to a distributive lattice when d goes to infinity as the number of elements of L_n^d is comparable to the number of elements of its spine (which is $|J(os_n^{d+1})|$ by Corollary 4.70). For instance, as d goes to infinity, the quotient $\dfrac{|L_n^d|}{|J(os_n^{d+1})|}$ converges to 1 when $n = 2$ and to $\frac{5}{4}$ when $n = 3$.

Question 4.85 *For n fixed, $|L_n^d| = \mathcal{O}_{d\to\infty}\big(|J(os_n^{d+1})|\big)$?*

We finish with two more algebraically oriented questions.

Question 4.86 *Are the higher Nakayama algebras of type $\mathbf{A}_\infty^\infty$ also examples of (P,ϕ)-Tamari lattices?*

Question 4.87 *Do the higher torsion classes of a d-cluster tilting subcategory always form a join-semidistributive lattice?*

CONCLUSION

Dans cette thèse nous avons apporté des contributions à la théorie des ordres et des treillis, mais de nombreuses questions ouvertes demeurent. Nous en avons listé quelques-unes à la fin des chapitres 3 et 4. Nous citons à nouveau certaines de ces questions, mais discutons aussi d'autres perspectives.

Dans le chapitre 2, nous avons démontré que tous les ordres unicycliques sont de dimension au plus 3. Pour ce faire, nous avons exhibé des triplets d'extensions linéaires particuliers pour les ordres arbres. Nous pensons que ces triplets pourraient être utilisés pour démontrer d'autres résultats de dimensions faisant intervenir ces ordres. Il serait donc intéressant d'explorer la littérature sur la dimension des ordres pour y trouver d'autres questions faisant intervenir les ordres arbres. Ces triplets pourraient aussi être utilisés pour l'étude de processus aléatoires sur les ordres arbres, liés ou non à la dimension.

Le chapitre 3 contient nos contributions à la théorie des treillis. Il est naturel de se poser la question de la généralité de nos résultats. Par exemple, nous avons montré que pour un treillis congruence uniforme, l'extrémalité et la shellabilité sont des propriétés équivalentes. Est-ce vrai plus généralement pour les treillis semi-distributifs ? Nous ne connaissons pas de contre-exemples. Dans ce même chapitre, nous avons utilisé le théorème fondamental des treillis semi-distributifs finis obtenu relativement récemment par Reading, Speyer et Thomas [91]. Nous pensons qu'il serait intéressant de l'utiliser pour résoudre d'autres problèmes sur les treillis semi-distributifs. De plus, il pourrait être intéressant d'avoir un théorème fondamental au niveau plus général des treillis sup semi-distributifs. Ayant montré au chapitre 4 que certains treillis de classes de torsion supérieure sont sup semi-distributifs, et des idées venant des treillis des classes de torsion usuelles ayant servi à établir le théorème fondamental pour les treillis semi-distributifs, nous croyons qu'un théorème fondamental pour les treillis sup semi-distributifs pourrait être inspiré de ces classes de torsion supérieures. Bien sûr cela est à ce stade purement hypothétique.

Nous avons également dans ce chapitre obtenu une formule pour la dimension des treillis semi-distributifs extrémaux. Cette formule est utile en pratique et nous a permis d'établir différents résultats de dimensions. Nous pensons que ce résultat pourrait être utilisé pour établir la dimension de nombreuses autres familles de treillis semi-distributifs extrémaux. De manière plus générale, les treillis, surtout en combinatoire algébrique, ont souvent des propriétés intéressantes qu'un ordre aléatoire ne possède pas. Cela nous amène à penser que la dimension peut sans doute être obtenue plus facilement, mais la théorie des treillis manque aujourd'hui de techniques spécifiques dans ce sens.

Enfin, au chapitre 4 nous nous sommes intéressés à des treillis que nous avons introduits et décidé d'appeler (P, ϕ)-Tamari. Leur dénombrement demeure ouvert en général, et comme nous l'avons observé, même dans des cas très particuliers nous sommes confrontés à des problèmes de dénombrements compliqués (comptage de partitions d-dimensionnelles totalement symétriques). Ces nouveaux treillis ont été introduits car ils contiennent comme cas particulier le treillis des d-classes de torsion des algèbres $(d-1)$-Auslander de type $\mathbf{A}$. Nous ne savons pas lesquelles des propriétés combinatoires obtenues pour ces treillis sont vraies plus généralement pour tous les treillis des classes de torsion supérieure. Il serait très intéressant de les étudier en toute généralité, ce qui demandera une étude plus algébrique, pas seulement combinatoire.

Par souci d'unité, nous n'avons pas voulu inclure d'autres travaux effectués pendant le doctorat ([47, 96, 33]). Cependant, nous profitons de cette conclusion pour donner un court résumé de chacun de ces travaux.

Dans l'article [47], écrit en collaboration avec Gélinas et Thomas, nous démontrons une conjecture de Matherne, Morales, et Selover [72]. Ce que nous démontrons est que deux bijections différentes des ordres d'intervalles unités vers les chemins de Dyck peuvent être reliées à l'aide de la fonction zêta définie par Haglund sur les chemins de Dyck.

Dans l'article [96], écrit seul, nous définissons deux actions du groupe symétrique sur le monoïde des mots tels que les invariants sous ces actions donnent respectivement les algèbres de Hopf $\mathbf{FQSym}$ et $\mathbf{PQSym}$. Il s'agit d'algèbres de Hopf respectivement sur les permutations et fonctions de parking. Une étude détaillée est menée pour l'action donnant $\mathbf{PQSym}$, des généralisations de cette action nous permettant de définir une suite infinie imbriquée de sous-algèbres de Hopf de $\mathbf{PQSym}$.

Enfin, dans l'article [33], écrit en collaboration avec Defant, Jiang, Marczinzik, Speyer, Thomas et Williams, nous étudions une généralisation du rowmotion définie pour n'importe quel ordre, que nous appelons échelonmotion. L'échelonmotion d'un ordre dépend du choix d'une extension linéaire de l'ordre. Nous étendons un résultat de Klász, Marczinzik et Thomas [64] en démontrant que pour un treillis l'échelonmotion est indépendante du choix de l'extension linéaire si et seulement si le treillis est semi-distributif. De plus, nous démontrons que pour un treillis semi-distributif l'échelonmotion correspond au rowmotion. Nous y démontrons également, parmi d'autres résultats, qu'il existe un choix canonique d'extension linéaire pour chaque treillis trim tel que l'échelonmotion correspond au rowmotion.

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