Université du Québec à Montréal

Variétés graphiques SU(2)-abéliennes avec un seul tore JSJ

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Giacomo Bascapè

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SU(2)-abelian graph manifolds with a single JSJ torus

Thesis

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Giacomo Bascapè

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In memory of Giulio Regeni,
Italian martyr for the freedom of research

and to the people who taught me to be free,
my parents.

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Résumé

Français

On dit qu'une 3-variété est $SU(2)$-*abélienne* si toute les $SU(2)$-représentations de son groupe fondamental ont une image abélienne. Dans cette travail, nous classifions toutes les variétés graphe $SU(2)$-abéliennes qui sont des sphères d'homologie rationnelle de dimension 3 avec un seul tore JSJ. Pour une 3-variété Y à bord torique, nous définissons l'invariant $T(Y, \partial Y)$ qui décrit $\mathrm{Hom}(\pi_1(Y), SU(2))$ jusqu'à la conjugaison. En particulier, l'invariant $T(Y, \partial Y)$ est un sous-espace d'un tore. Étant donné une variété fermée $Y_1 \cup_\Sigma Y_2$, nous déterminons si elle est $SU(2)$-abélienne en étudiant l'intersection de $T(Y_1, \partial Y_1)$ et $T(Y_2, \partial Y_2)$. Enfin, nous démontrons que si une variété de graphe qui est une sphère d'homologie de dimension 3 est $SU(2)$-abélienne, alors c'est un L-espace au sens de l'homologie de Heegaard Floer.

Mots-clés: 3-variétés, représentations de $SU(2)$, L-espaces.

Anglais

We say that a 3-manifold is $SU(2)$-*abelian* if every $SU(2)$-representation of its fundamental group has abelian image. In this work we classify all $SU(2)$-abelian graph manifold rational homology 3-spheres with a single JSJ torus. For a 3-manifold Y with torus boundary, we define the invariant $T(Y, \partial Y)$ that describes $\mathrm{Hom}(\pi_1(Y), SU(2))$ up to conjugation. In particular, the invariant $T(Y, \partial Y)$ is a subspace of a torus. For a generic closed manifold $Y_1 \cup_\Sigma Y_2$, we determine the $SU(2)$-abelian status of $Y_1 \cup_\Sigma Y_2$ by studying the intersection of $T(Y_1, \partial Y_1)$ with $T(Y_2, \partial Y_2)$. Finally, we prove that if a graph manifold rational homology 3-sphere with a single JSJ torus is $SU(2)$-abelian, then it is an L-space for Heegaard Floer homology.

Keywords: 3-manifolds, $SU(2)$-representations, L-spaces.

Introduction

We take the opportunity to begin this production by recalling the Greek origin of the word "Topology". This comes from the combination of the Greek word $\tau\acute{o}\pi o\varsigma$, which means "place" or "space" and the word $\lambda\acute{o}\gamma o\varsigma$, a word very popular in philosophy that can be translated as "study" but also as "word" or "reason". In fact, as one does on the first day of a topology course, we say that this is the branch of mathematics that studies the global properties of spaces that do not change under reasonable modifications. This work is about Low Dimensional Topology, and indeed it examines those spaces that contain a small number of dimensions, seen as different directions. In particular, this studies three dimensional compact manifolds.

A fascinating problem that arises in this branch of topology can be simply stated as follows: given two n-manifolds is it possible to decide whether they are homeomorphic? Moreover if the latter has a positive answer, which are the steps we need to follow to reach such a result?

One of the milestones of three manifold theory is provided by Perelman's proof of Thurston's geometrization conjecture ([Per02],[Per03a],[Per03b]), which implies that closed, orientable prime 3-manifolds are determined, up to orientation, via fundamental groups, with the exception of lens spaces. In particular, Perelman proved that the fundamental group of a compact 3-manifold is trivial if and only if the manifold is S^3.

Let Y be a closed, prime 3-manifold. If one wants to determine whether Y is homeomorphic to S^3, one is naturally led to ask whether $\pi_1(Y)$ is trivial or not. Unfortunately, this is not a winning strategy since, as a consequence of the Adian-Rabin theorem, deciding if a group is trivial is hard. Therefore, a more promising way to understand the non-triviality of Y can be found in the following conjecture:

Conjecture 1 ([Kir95, Problem 3.105(A)]). *If Y is a closed 3-manifold other than S^3, then $\pi_1(Y)$ admits a non-trivial representation $\pi_1(Y) \to SU(2)$.*

The 3-manifolds which are the simplest from the $SU(2)$-representation perspective are the $SU(2)$-abelian 3-manifolds. A 3-manifold is $SU(2)$-*abelian* if every $SU(2)$-representation of its fundamental group has abelian image. We say that a rational homology 3-sphere Y is an *L-space* if it has minimal Heegaard Floer homology, i.e. $\operatorname{rank}\widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. The following conjecture, which links these two notions, somehow justifies why one wonders about $SU(2)$-abelian 3-manifolds.

Conjecture 2 ([Zha20, Conjecture 4]). *Let Y be a rational homology 3-sphere. If Y is $SU(2)$-abelian, then Y is a Heegaard Floer L-space.*

In general, the theory of $SU(2)$-representations provides a fruitful technique for studying the fundamental group of closed 3-manifolds. In particular, $SU(2)$-abelian rational homology spheres are interesting from the instanton Floer homology perspective. For instance, in [BS18] Baldwin and Sivek proved that an $SU(2)$-abelian rational homology sphere Y is an instanton Floer L-space if every $SU(2)$-representation of $\pi_1(Y)$ is non-degenerate in a Morse-Bott sense. Furthermore, it has been conjectured in [KM10, Conjecture 7.24] that the instanton Floer homology group of a rational homology 3-sphere Y is isomorphic, as a $\mathbb{C}$-vector space, to the $\mathbb{C}$-tensored Heegaard Floer homology group $\widehat{HF}(Y;\mathbb{C})$. Therefore, Conjecture 2 with [KM10, Conjecture 7.24] suggests that if a rational homology sphere is $SU(2)$-abelian, then Y is an instanton Floer L-space. This is schematically summed up in Figure 0.1.

Conjecture 3. *Let Y be a rational homology 3-sphere. If Y is $SU(2)$-abelian, then Y is an instanton L-space*

In this work we focus on graph manifold rational homology 3-spheres and the classification of those that are $SU(2)$-abelian. The first step in this classification was taken by Sivek and Zentner in [SZ21] when they classified $SU(2)$-abelian Seifert fibred spaces. Consequently, we study non-Seifert fibred graph manifold rational homology 3-spheres and we shall classify all $SU(2)$-abelian graph manifold rational homology 3-spheres that have only one JSJ torus. The first examples of $SU(2)$-abelian non-Seifert fibred graph manifolds are contained in [Mot88], where it is proven that a specific gluing of torus knot complements is $SU(2)$-abelian.

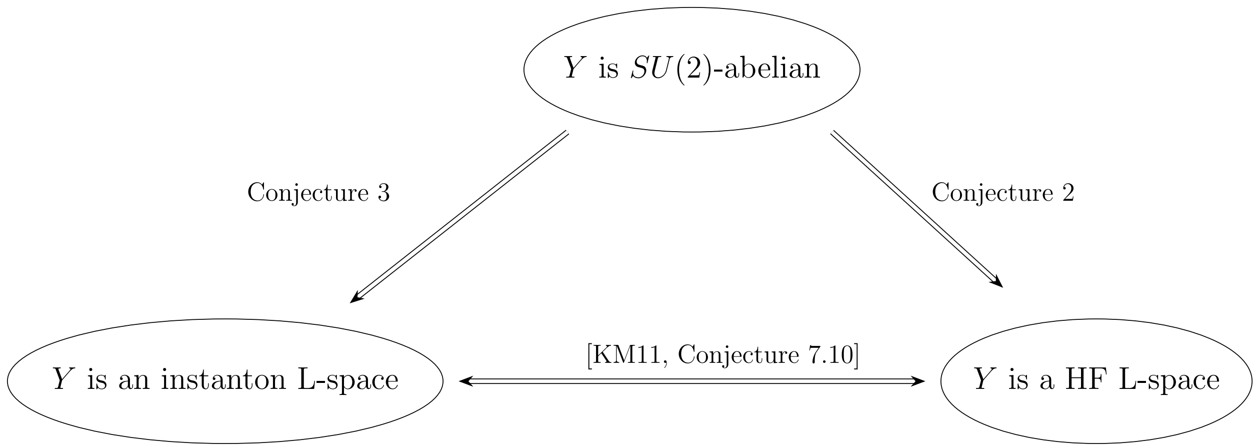


FIGURE 0.1. A triangle of Conjectures. The manifold Y is assumed to be a rational homology 3-sphere.

Let Y_1 and Y_2 be two 3-manifolds with torus boundary. We denote by $Y = Y_1 \cup_\Sigma Y_2$ a closed 3-manifold with an embedded torus $\Sigma \subset Y$ such that

$$\overline{Y \setminus \Sigma} = Y_1 \cup Y_2.$$

We denote by $\Delta(\gamma_1, \gamma_2)$ the geometric intersection number between the curves γ_1 and γ_2 on Σ. The first step of the classification is the following:

Theorem 5.1.6. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a graph manifold rational homology 3-sphere such that $\Sigma \subset Y$ is the only JSJ torus of Y. If Y is $SU(2)$-abelian, then all the following hold:*

- *both Y_1 and Y_2 admit a Seifert fibration with disk base space;*
- *up to swapping the two JSJ pieces, Y_1 has exactly two singular fibres;*
- *if $h_1, h_2 \subset \Sigma = \partial Y_1 = \partial Y_2$ are regular fibres of Y_1 and Y_2, then $\Delta(h_1, h_2) = 1$.*

For $p_i \geq 2$, then the writing $\mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_n}/_{q_n})$ denotes a Seifert space fibred over a disk with n cone points and singular fibres of orders $(p_1, \cdots, p_n)$. In particular, $\mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_n}/_{q_n})$ has torus boundary. Let $Y = Y_1 \cup_\Sigma Y_2$ be a 3-manifold as in Theorem 5.1.6. In Theorem 5.1.14 we give necessary and sufficient conditions for the manifold Y to be $SU(2)$-abelian. For $i \in \{1, 2\}$, we denote by t_i the order of the torsion subgroup of $H_1(Y_i; \mathbb{Z})$ and by o_i the order of the rational longitude λ_i of Y_i. As an application of Theorem 5.1.6, we can assume Y_1 and Y_2 to be fibred over a disk, therefore we denote by h_1 and h_2 be the regular fibres of Y_1 and Y_2 in $\partial Y_1 = \partial Y_2 = \Sigma \subset Y$.

Theorem 5.1.14. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a graph manifold rational homology 3-sphere such that $\Sigma \subset Y$ is the unique JSJ torus. The manifold Y is $SU(2)$-abelian if and only if Y is contained in one of the classes of Table 1 and Table 2.*

A more algorithmic, but less clean version of the Theorem 5.1.14 can be found in Theorem 5.1.8. If Y_1 and Y_2 are as in Table 1 or in Table 2, then Proposition 5.1.12 and Proposition 5.1.13 provide a diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$ such that the graph manifold $Y_1 \cup_\Sigma Y_2 = Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian. Thus, each class in Table 1 and in Table 2 is nonempty. Such a gluing is in general not unique, as shown in Example 5.2.3.

Let Y be the gluing of torus knot exteriors defined in [Mot88], where Motegi proved that Y is $SU(2)$-abelian.

#	Y_1	Y_2	Additional Requirements	(Δ_1, Δ_2)
1)	$\mathbb{D}^2\left(\frac{2}{1}, \frac{p_2}{q_2}\right)$	$\mathbb{D}^2\left(\frac{r_1}{s_1}, \frac{r_1}{r_1-s_1}\right)$	$r_1 \equiv_2 1$ $2q_2 + p_2 \equiv_{2p_2} \pm o_1 t_1$	$(0,1)$
2)	$\mathbb{D}^2\left(\frac{2}{1}, \frac{p_2}{q_2}\right)$	$\mathbb{D}^2\left(\frac{3}{1}, \frac{3}{2}\right)$	$o_1 = 1$ $2q_2 + p_2 \equiv_{2p_2} \pm 3t_1$	$(0,3)$
3)	$\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}\right)$	$\mathbb{D}^2\left(\frac{r_1}{s_1}, \frac{r_2}{s_2}\right)$	$o_2 = 1$ $r_1 s_2 + r_2 s_1 \equiv_{r_1 r_2} \pm 4 g_2$	$(4,1)$
4)	$\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}\right)$	$\mathbb{D}^2\left(\frac{3}{s_1}, \frac{3}{s_1}\right)$	$\Delta(\lambda_1, \lambda_2) \neq 0$	$(4,3)$
5)	$\mathbb{D}^2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$	$\mathbb{D}^2\left(\frac{3}{s_1}, \frac{3}{s_1}\right)$	$p_1 p_2 \equiv_2 1, t_1 = 1$ $p_1 q_2 + p_2 q_1 \equiv_{p_1 p_2} \pm 3$	$(1,3)$
6)	$\mathbb{D}^2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$	$\mathbb{D}^2\left(\frac{4}{s_1}, \frac{4}{s_1}\right)$	$\gcd(p_1 p_2, p_1 q_2 + p_2 q_1) \leq 2$ $p_1 q_2 + p_2 q_1 \equiv_{p_1 p_2} \pm 2 t_1$	$(1,2)$
7)	$\mathbb{D}^2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$	$\mathbb{D}^2\left(\frac{r_1}{s_1}, \frac{r_2}{s_2}\right)$	$o_1 t_1 \leq 2 \vee o_2 t_2 \leq 2$ $o_1 \leq 2,\ o_2 \leq 2$ $(p_1 q_2 + p_2 q_1) \equiv_{p_1 p_2} \pm o_1 t_1$ $(r_1 s_2 + r_2 s_1) \equiv_{r_1 r_2} \pm o_2 g_2$	$(1,1)$

TABLE 1. Classes of $SU(2)$-abelian graph manifolds rational homology spheres of the form $Y_1 \cup_\Sigma Y_2$. We use a normalization for which $0 < q_i < p_i$ and $0 < s_j < r_j$. We always suppose that $\Delta(h_1, h_2) = 1$, $\Delta(\lambda_1, \lambda_2) \neq 0$, and $t_1 \leq t_2$. We write Δ_1 for $\Delta(\lambda_2, h_1)$ and Δ_2 for $\Delta(\lambda_1, h_2)$.

Corollary 5.2.2. *For $i \in \{1,2\}$, let $E(T_i)$ be the exterior of a open tubular neighborhood of the torus knot $T_i \subset S^3$. We denote by λ_i and μ_i the null-homologous longitude and the meridian of T_i. The manifold $E(T_1) \cup_\Sigma E(T_2)$ is $SU(2)$-abelian if and only if $\Delta(\lambda_1, \mu_2) = 0$ and $\Delta(\lambda_2, \mu_1) = 0$.*

Corollary 5.2.2 implies that if Y is an $SU(2)$-abelian graph manifold whose JSJ pieces are two torus knot exteriors, then Y is one of the gluing of torus knot exteriors as in [Mot88]. Since Y has two JSJ pieces which are two torus knot exteriors, Y belongs to class (7) of Table 1.

By means of an explicit calculation of the L-space interval of Y_1 and Y_2, and by applying the "gluing theorem" for graph L-space manifolds (i.e. [Ras17, Proposition 1.5]) we prove the following:

Theorem 6.1.19. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a 3-manifold as in Theorem 5.1.14. If Y is an $SU(2)$-abelian rational homology sphere, then Y is an L-space.*

#	Y_1	Y_2	Additional Requirements	(Δ_1, Δ_2)
8)	$\mathbb{D}^2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$	$\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{1}, \frac{4}{3}\right)$	$\gcd(p_1p_2, p_1q_2 + p_2q_1) \leq 2$ $(p_1q_2 + p_2q_1) \equiv_{p_1p_2} \pm o_1t_1$	$(1,1)$
9)	$\mathbb{D}^2\left(\frac{2}{1}, \frac{p_2}{q_2}\right)$	$\mathbb{D}^2\left(\frac{3}{r_1}, \frac{3}{r_1}, \frac{3}{r_3}\right)$	$\gcd(p_1p_2, p_1q_2 + p_2q_1) \leq 2$, $(p_1q_2 + p_2q_1) \equiv_{p_1p_2} \pm o_1t_1$ $r_1 \neq r_3$, $\Delta(\lambda_1, \lambda_2) = 2n$, $n \neq 0$	$(1,1)$
10)	$\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}\right)$	$\mathbb{D}^2\left(\frac{3}{r_1}, \frac{3}{r_1}, \frac{3}{r_3}\right)$	$r_1 \neq r_3$, $\Delta(\lambda_1, \lambda_2) = 2n$, $n \neq 0$	$(4,1)$
11)	$\mathbb{D}^2\left(\frac{p_1}{q_1}, \frac{p_1}{p_1 - q_1}\right)$	$\mathbb{D}^2\left(\frac{2}{1}, \cdots, \frac{2}{1}, \frac{r_m}{s_m}\right)$	$p_1 \equiv_2 1$, $o_2 \leq 2$, $(m-1)r_m + 2s_m \equiv_{2r_m} \pm 1$	$(1,0)$
12)	$\mathbb{D}^2\left(\frac{3}{1}, \frac{3}{2}\right)$	$\mathbb{D}^2\left(\frac{2}{1}, \cdots, \frac{2}{1}, \frac{r_m}{s_m}\right)$	$o_2 = 3$, $(m-1)r_m + 2s_m \equiv_{2r_m} \pm 1$	$(1,0)$
13)	$\mathbb{D}^2\left(\frac{3}{1}, \frac{3}{2}\right)$	$\mathbb{D}^2\left(\frac{2}{1}, \cdots, \frac{2}{1}, \frac{r_m}{s_m}\right)$	$o_2 = 1$, $\Delta(\lambda_1, \lambda_2) = 2n$, $n \neq 0$ $(m-1)r_m + 2s_m \equiv_{2r_m} \pm 3$	$(3,0)$

TABLE 2. Classes of $SU(2)$-abelian graph manifolds rational homology spheres of the form $Y_1 \cup_\Sigma Y_2$. We use a normalization for which $0 < q_i < p_i$ and $0 < s_j < r_j$. We always suppose that $\Delta(h_1, h_2) = 1$ and $\Delta(\lambda_1, \lambda_2) \neq 0$. We write Δ_1 for $\Delta(\lambda_2, h_1)$ and Δ_2 for $\Delta(\lambda_1, h_2)$.

This gives further evidence to the conjecture that if a rational homology sphere is $SU(2)$-abelian, then it is a Heegaard Floer L-space (see Conjecture 2):

Corollary 6.1.20. *Conjecture 2 holds for graph manifolds with at most one JSJ torus.*

We denote by $U(1)$ the subgroup of diagonal matrices of $SU(2)$. For a manifold with torus boundary Y, we define $T(Y, \partial Y) \subset \mathrm{Hom}(\pi_1(\partial Y), U(1))$ as the set of representations $\pi_1(\partial Y) \to U(1)$ that extend to a representation $\pi_1(Y) \to SU(2)$. The object $T(Y, \partial Y)$ can be seen as the $SU(2)$ version of the A-polynomial, defined in [Coo+94], which uses $SL_2(\mathbb{C})$-representations or the translation extension locus, defined in [Dun19], which uses $SL_2(\mathbb{R})$-representations. The set $T(Y, \partial Y)$ has a natural decomposition as $H(Y) \cup A(Y) \cup P(Y)$, where $H(Y)$ (resp. $A(Y)$ resp. $P(Y)$) is the set of representation $\pi_1(\partial Y) \to U(1)$ that extend to an irreducible (resp. abelian resp. abelian and non-central) representation $\pi_1(Y) \to SU(2)$. Let $Y = Y_1 \cup_\Sigma Y_2$, since $\partial Y_1 = \partial Y_2 = \Sigma$, we have that $T(Y_1, \partial Y_1)$ and $T(Y_2, \partial Y_2)$ are both subsets of $\mathrm{Hom}(\pi_1(\Sigma), U(1))$. Therefore, the intersection $T(Y_1, \partial Y_1) \cap T(Y_2, \partial Y_2)$ is well defined and we can determine the $SU(2)$-abelian status of Y via the following:

Theorem 2.1.8. *Let Y_1 and Y_2 be two 3-manifolds with torus boundary. The manifold $Y = Y_1 \cup_\Sigma Y_2$ is $SU(2)$-abelian if and only if $H(Y_1) \cap H(Y_2)$, $H(Y_1) \cap A(Y_2)$, $A(Y_1) \cap H(Y_2)$, and $P(Y_1) \cap P(Y_2)$ are empty.*

Let $Y = Y_1 \cup_\varphi Y_2$ be the manifold defined as before. We also determine an obstruction to the existence of an irreducible $SU(2)$-representation of $\pi_1(Y)$, whose analogue in the Heegaard Floer world is obtained as a consequence of [HRW23, Theorem 1.14].

Corollary 2.1.14. *If the manifold $Y = Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian, then the manifolds*

$$Y_1(\lambda_{Y_2}) \quad \text{and} \quad Y_2(\lambda_{Y_1})$$

are $SU(2)$-abelian. Here λ_{Y_1} and λ_{Y_2} are the rational longitudes of Y_1 and Y_2.

Stepping aside from the preceding discussion, this work naturally leads to the following conjecture. We refer the discussion of the latter to the final part of Chapter 6.

Conjecture 4. *Every toroidal manifold such that $|H_1(Y; \mathbb{Z})| \leq 6$ is not $SU(2)$-abelian.*

Chapter 1

Background and Notation

1.1. Our favourite group: SU(2)

As we mentioned in the Introduction, this work focuses on understanding representations $\pi_1(Y) \to G$ for a fixed group G. Therefore, the first step is to choose the group G wisely.

The Geometrization theorem, implies that three-manifolds have residually finite fundamental groups [Hem87], meaning that for every non-trivial element of the group there exists a homomeomorphism to a finite group that maps this element in a non-trivial element. Therefore, the first attempt could be to take the group G as a *finite* group. This is not a wise choice since, for every finite group G, it is possible to find a closed 3-manifold Y such that

$$\mathrm{Hom}(\pi_1(Y), G) = \mathrm{Hom}(\pi_1(S^3), G) = \{1\}.$$

For instance, it suffices to take Y as a lens space whose fundamental group is of coprime order with the order of G.

Another attempt could be to take $G = SL_2(\mathbb{C})$. This choice may be motivated, for example, by a Thurston's famous result that claims that if Y is a *hyperbolic* manifold, then there exists a discrete faithful representation

$$\pi_1(Y) \to SL_2(\mathbb{C})$$

given by the hyperbolic metric of Y (see [Thu77, page 98]). Indeed, the theory of $SL_2(\mathbb{C})$-representations has been largely developed in the last two decades. For instance, the following is the result the author finds particularly within the $SL_2(\mathbb{C})$-representation theory of 3-manifold:

Theorem 1.1.1 ([Zen16, Theorem 1.1]). *Every integer homology 3-sphere Y admits a non-trivial representation $\pi_1(Y) \to SL_2(\mathbb{C})$.*

The logic of Theorem 1.1.1 can be summed up in the following way: if Y is a hyperbolic 3-manifold, then it has a "for free" non-trivial representation $\pi_1(Y) \to SL_2(\mathbb{C})$ by Thurston. If Y is geometric but not hyperbolic, then a non-trivial $SL_2(\mathbb{C})$-representation can be constructed by hand. If Y is prime and not geometric, meaning that it has a non-trivial JSJ decomposition, then Zentner proved that there exists a surjective homeomorphism

$$\pi_1(Y) \twoheadrightarrow \pi_1(Y_0),$$

where Y_0 is an integer homology sphere obtained by gluing together two knot exteriors. In particular, extending [KM10, Theorem 1] he proved that the group $\pi_1(Y_0)$ admits an irreducible, and hence non-trivial, $SU(2)$-representation. We recall that $SU(2)$ is a subgroup of $SL_2(\mathbb{C})$, therefore an $SU(2)$-representation is an $SL_2(\mathbb{C})$-representation.

The importance of the group $SU(2)$ is, in the author's opinion, bifold. On one hand $SU(2)$ is the universal cover of $SO(3)$, and these two groups are the two compact non-abelian Lie groups of smallest dimension. On the other, the group $SU(2)$ is deeply connected with the theory of instanton Floer homology. In a nutshell, under specific hypotheses, the instanton Floer chain complex of a closed 3-manifold Y is generated by $SU(2)$-representations of $\pi_1(Y)$. We recommend [BS18] and [LPZ23] as references.

We now take a brief moment to define the $SU(2)$ group comprehensively and satisfactorily, we give [Sav12] as a reference. The Lie group $SU(2)$ consists of all complex 2×2 matrices A such that

$$A\overline{A}^\top = \mathbb{1} \quad \text{and} \quad \det A = 1.$$

Since A is invertible, we obtain that $\overline{A}^\top = A^{-1}$. More explicitly,

$$\overline{A}^\top = \overline{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^\top = \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix}^\top = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix} = A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This is equivalent of asking that $\overline{a} = d$ and $\overline{c} = -b$. Thus, any matrix $A \in SU(2)$ is of the form

$$A = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}, \quad \text{where} \quad a, b \in \mathbb{C} \quad \text{and} \quad \det A = |a|^2 + |b|^2 = 1.$$

This provides the identification

$$SU(2) = \left\{(a,b) \in \mathbb{C}^2 \big| |a|^2 + |b|^2 = 1\right\} = S^3.$$

The group $U(1)$ is an abelian subgroup of $SU(2)$ and it can be identified as

$$U(1) = \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \middle| \theta \in [0, 2\pi] \right\}.$$

In this work an abuse of notation is in use: when not confusing, we write $e^{i\theta} \in SU(2)$ for the matrix

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

The center of $SU(2)$ is

$$\mathcal{Z}(SU(2)) := \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \{\pm 1\}.$$

Remark 1.1.2 ([Sav12, Theorem 13.2]). Two $SU(2)$-matrices A and B are conjugate, meaning that there exists a third matrix $X \in SU(2)$ such that $XAX^{-1} = B$, if and only if $\operatorname{Tr} A = \operatorname{Tr} B$.

We now define a very important subgroup of $SU(2)$, which, with very little imagination, has been called a *maximal torus*. A *torus* of a Lie group is a compact closed abelian Lie subgroup. It can be proven that a torus is diffeomorphic to $(S^1)^n$, so at least the lack of imagination is justified in being an actual torus. A *maximal torus* of a Lie group is a torus that is maximal with respect to inclusion.

Let z be a non-central element of $SU(2)$. We denote by $\Lambda_z \subset SU(2)$ the centralizer subgroup of the element z. The subgroups Λ_z and $U(1)$ are known to be conjugate in $SU(2)$, details can be found in [Sav12, Lecture 13]. The following is well known.

Fact 1.1.3. *Let x, y be two non-central elements of $SU(2)$. The elements x and y commute if and only if the two centralizer subgroups Λ_x and Λ_y coincide.*

Lemma 1.1.4. *The centralizers Λ_z, with $z \in SU(2)\backslash\mathcal{Z}(SU(2))$, are precisely the maximal tori of $SU(2)$.*

PROOF. It is known that $U(1)$ is a maximal abelian (Lie) subgroup of $SU(2)$. Therefore, $U(1)$ is a maximal torus of $SU(2)$. Furthermore, it is easy to see that $U(1)$ is the centralizer of $e^{i\theta} \in SU(2)$, for $\theta \in (0, \pi)$. By [Hal03, Theorem 11.9] two maximal tori of a compact Lie group are conjugate. Therefore, every maximal torus of $SU(2)$ is of the form

$$xU(1)x^{-1} = x\Lambda_{e^{i\theta}}x^{-1}, \quad \text{for} \quad x \in SU(2) \quad \text{and} \quad \theta \in (0, \pi).$$

A basic computation of group theory shows that

$$x\Lambda_y x^{-1} = \Lambda_{xyx^{-1}}.$$

Therefore, all maximal tori of $SU(2)$ are of the form

$$\Lambda_{xe^{i\theta}x^{-1}} \quad \text{with} \quad \theta \in (0, \pi).$$

Since, two matrices in $SU(2)$ are conjugate if and only if they share the same trace, we obtain the conclusion. □

We are ready now to give probably the most important definition of this work.

Definition 1.1.5. A group G is said to be $SU(2)$-*abelian* (resp. $SU(2)$-*central*) if every representation $G \to SU(2)$ has abelian image (resp. has image contained in the center $\mathcal{Z}(SU(2)) = \{\pm 1\}$). A 3-manifold is said to be $SU(2)$-*abelian* if its fundamental group is $SU(2)$-abelian.

A representation $G \to SU(2)$ is said to be *abelian* (resp. *central*) if its image is abelian (resp. contained in the center $\mathcal{Z}(SU(2))$). A non-abelian representation is also called *irreducible.*

1.2. Our favourite walks: the slopes

Let M be a manifold with torus boundary. A (rational) slope on the boundary M is an element $[\alpha]$ of the projective space of $H_1(\partial M; \mathbb{Q})$, where $\alpha \in H_1(\partial M; \mathbb{Q}) \setminus \{0\}$. Slopes can be identified as either

- a $\pm$-pair of primitive elements of $H_1(\partial M; \mathbb{Z}) \equiv \pi_1(\partial M)$;
- a ∂M-isotopy class of essential simple closed curves on ∂M.

We denote by $\mathscr{S}(\partial M)$ the set of slopes in ∂M. For further details, see [BC24, Subsection 4.2].

Let Σ be a torus, and $\{\mu, \lambda\}$ a basis of $H_1(\Sigma; \mathbb{Z})$. We use the convention, depending on the choice of the basis $\{\mu, \lambda\}$, such that an element $p/q \in \mathbb{Q} \cup \{1/0\}$ corresponds to the slope $p\mu + q\lambda \in H_1(\Sigma; \mathbb{Z})$.

Definition 1.2.1. Let $\{\mu, \lambda\}$ be a basis of $H_1(\Sigma; \mathbb{Z})$, where Σ is a 2-torus. Let γ_1 and γ_2 be two slopes on a 2-torus Σ such that $[\gamma_1] = p_1\mu + q_1\lambda$ and $[\gamma_2] = p_2\mu + q_2\lambda$ in $H_1(\Sigma; \mathbb{Z})$. The *geometric intersection number* between γ_1 and γ_2 is $\Delta(\gamma_1, \gamma_2) \coloneqq |p_1q_2 - q_2p_1|$.

Let Y be a 3-manifold such that ∂Y contains a torus Σ. Let $\gamma \subset \Sigma$ be a slope. We call *Dehn filling of Y along γ* the 3-manifold

$$Y(\Sigma; \gamma) = Y \cup_f S^1 \times \mathbb{D}^2,$$

where $f \colon \partial(S^1 \times \mathbb{D}^2) \to \Sigma$ is an orientation reversing diffeomorphism such that

$$f(\{*\} \times \partial\mathbb{D}^2) = \gamma \quad \text{with} \quad * \in S^1. \tag{1.2.1}$$

We chose f the be orientation *reversing* as if Y is oriented, then its orientation can be extended to an orientation of $S^1 \times \mathbb{D}^2$. Therefore, if Y is oriented and f is orientation reversing diffeomorphism, then $Y(\gamma)$ is orientable with an orientation compatible with Y.

It is straightforward to see that if f and f' are two orientation reversing diffeomorphisms $\partial(S^1 \times \mathbb{D}^2) \to \Sigma$ both satisfying the relation (1.2.1), then

$$Y \cup_f S^1 \times \mathbb{D}^2 \cong_{\text{homeo}} Y \cup_{f'} S^1 \times \mathbb{D}^2.$$

Therefore, $Y(\Sigma; \gamma)$ is indepent of the choice of f, as long as this satisfies the relation (1.2.1). Everyone's favorites theorem, meaning Seifert–Van Kampen one, implies that

$$\pi_1(Y(\gamma)) = \frac{\pi_1(Y)}{\langle\langle \gamma \rangle\rangle},$$

where $\langle\langle \gamma \rangle\rangle$ denotes the smallest normal subgroup containing the element $\gamma \in \pi_1(Y)$ [1].

[1]To be more precise, there is a natural inclusion $\iota \colon \Sigma \to Y$ that indices a homomorphism $\iota_* \colon \pi_1(\Sigma) \to \pi_1(Y)$. Since γ is a slope, it is an element of $\pi_1(\Sigma)$. Therefore, when considered as an element of $\pi_1(Y)$, we call $\gamma \in \pi_1(Y)$ the image $\iota_*(\gamma) \in \pi_1(Y)$

If Y is a compact 3-manifold with toroidal boundary components $\Sigma_1, \cdots, \Sigma_n$ with fixed bases $\{\mu_i, \lambda_i\}$ for each $H_1(\Sigma_i; \mathbb{Z})$, then

$$Y(\Sigma_1, \cdots, \Sigma_n; {}^{r_1}/_{s_n}, \cdots {}^{r_n}/_{s_n})$$

denotes the closed 3-manifold obtained by performing Dehn fillings along a simple closed curve representing ${}^{r_i}/_{s_i}$ on Σ_i for each $i = 1, \cdots, n$. If Y has a single torus boundary and $\gamma \subset \partial Y$ is a slope, we make the notation lighter by writing

$$Y(\gamma) \coloneqq Y(\partial Y; \gamma).$$

Let Y be a 3-manifold with torus boundary and let $\iota \colon \partial Y \to Y$ be the natural inclusion. According to the "half lives and half dies" Theorem [Mar16, Corollary 9.1.5], we have that

$$\dim\Big(\ker\Big(H_1(\partial Y; \mathbb{Q}) \xrightarrow{\iota_*} H_1(Y; \mathbb{Q})\Big)\Big) = 1.$$

We define the *rational longitude* as the slope $\lambda_Y \subset \partial Y$ such that its class in homology generates $\ker \iota_* \subseteq H_1(\partial Y; \mathbb{Q})$. Equivalently, the rational longitude of Y is a simple and closed curve $\lambda_Y \subset \partial Y$ such that its class is a torsion element of $H_1(Y; \mathbb{Z})$, which is unique up to isotopy.

In this work an abuse of notation is in use: for a given simple closed curve γ in the torus Σ, when we refer to its homotopy class $[\gamma] \in \pi_1(\Sigma)$ we omit the brackets. Consequently, γ indicates both a curve in Σ and its homotopy class $\gamma \in \pi_1(\Sigma)$.

1.3. Our favourite bricks: Seifert fibred spaces

The concept of *Seifert fibred space* was originally studied by Herbert Seifert in [Sei33]. Since the article just quoted is in German and the prerequisites for this work do not include knowledge of that language, we will give a definition of a Seifert fibred spaces.

A compact space Y is said to be a *Seifert fibred space* if there exists a collection of pairwise disjoint circles $f_\alpha \subset Y$, which are called *fibres*, such that

$$Y = \bigcup_\alpha f_\alpha,$$

and every fibre f_α admits a tubular neighbourhood in Y that is union of fibres. We call such a tubular neighbourhood a *fibred neighbourhood*.

Every fibre f_α is associated with an invariant called *order* that counts how many times any other close by fibre wraps around f_α. A fibre of order one is called *regular fibre* and the ones with order 2 or greater are called *singular fibres*. A compact Seifert fibred space admits finite many singular fibres. A regular fibre is sometimes called *general fibre*.

Definition 1.3.1. Let Y be a Seifert fibred space and let $S \subset Y$ be an embedded or immersed surface. We say that S is *vertical* if it is union of fibres of Y.

Let Y be a Seifert fibred space. It can be proven that

$$Y/\sim \quad \text{with} \quad x \sim y \text{ if and only if } x, y \in f_\alpha,$$

is a 2-dimensional *orbifold* $\mathcal{B}$ with a number of cone points equal to the number of the singular fibres, each of order equal to the order of the corresponding singular fibre. We give [Sco83] and [Car19] as references for the concept of orbifold. Let B be underlying surface of the orbifold $\mathcal{B}$ and $\{p_1, \cdots, p_n\}$ the orders of its cone points. The orbifold $\mathcal{B}$ is sometimes denoted by

$$B(p_1, \cdots, p_n).$$

We say that Y is fibred over the orbifold $\mathcal{B} = B(p_1, \cdots, p_n)$ and, when the Seifert fibration of Y is fixed, that B is the *base space* of Y. To be even more accurate, when we say that Y has base space B we mean that there exists a Seifert fibration of Y whose base space is B.

Let $n \geq 1$. A Seifert fibred manifold with n singular fibres can be described from the surface B and a set of rational numbers $\{p_1/q_n, \cdots, p_n/q_n\}$, where p_i is the order of the i^{th} singular fibre. We call the fraction p_i/q_i a *Seifert coefficient* of Y. We can therefore write

$$Y = B\left(\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n}\right).$$

In this case, Y is fibred over the orbifold $B(p_1, \cdots, p_n)$ and has base space B. Details, can be found in [Mar16, Section 10.3.2].

Let us now go into the details of two types of Seifert fibred spaces to fix the notation once and for all. Let us suppose that $p_i \geq 2$. Let $\{\mathbb{D}^2_i\}_{i\in\{1,\cdots,n\}}$ be a system of n open disjoint disks in S^2. We define $\hat{Y}$ as

$$\hat{Y} = S^1 \times \left(S^2 \setminus \coprod_{i=1}^{n} \mathbb{D}^2_i\right) = S^1 \times S^2 \setminus \left(\coprod_{i=1}^{n} S^1 \times \mathbb{D}^2_i\right).$$

We define m_i and l_i on the i^{th}-torus boundary component of $\hat{Y}$ as

$$m_i = \{*\} \times \partial\mathbb{D}^2_i \quad \text{and} \quad l_i = S^1 \times \{*\}.$$

The closed manifold $Y = S^2(p_1/q_1, \cdots p_n/q_n)$ is the result of performing a Dehn filling of $\hat{Y}$ along the curve $p_i m_i + q_i l_i$ for every i. This construction gives the following presentation for the fundamental group of Y:

$$\pi_1(Y) = \pi_1\big(S^2(p_1/q_1, \cdots p_n/q_n)\big) = \langle x_1, \cdots, x_n, h \,|\, x_i^{p_i} h^{q_i}, [x_i, h], x_1 x_2 \cdots x_n \rangle.$$

Here $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$. Let h be a regular fibre of the Seifert fibred space $Y = S^2(p_1/q_1, \cdots p_n/q_n)$, we define the space $\mathbb{D}^2(p_1/q_1, \cdots p_n/q_n)$ as

$$\mathbb{D}^2(p_1/q_1, \cdots p_n/q_n) = Y \setminus \nu(h),$$

where $\nu(h)$ is a small fibred open neighborhood of the fibre h. The manifold $\mathbb{D}^2(p_1/q_1, \cdots p_n/q_n)$ is a Seifert space fibred over a disk. If $M = \mathbb{D}^2(p_1/q_1, \cdots p_n/q_n)$, then its fundamental group is

$$\pi_1(M) = \pi_1\big(\mathbb{D}^2(p_1/q_1, \cdots p_n/q_n)\big) = \langle x_1, \cdots, x_n, h \,|\, x_i^{p_i} h^{q_i}, [x_i, h] \rangle. \tag{1.3.1}$$

Definition 1.3.2. Let $M = \mathbb{D}^2(p_1/q_1, \cdots p_n/q_n)$. Let us suppose that $\pi_1(M)$ is presented as in (1.3.1). We define the *fibration meridian* of this presentation as the, unique up to isotopy, simple closed curve $\mu \subset \partial M$, such that

$$[\mu] = x_1 x_2 \cdots x_{n-1} x_n \in \pi_1\big(\mathbb{D}^2(p_1/q_1, \cdots, p_n/q_n)\big).$$

Similarly, let us suppose that $p_i \geq 2$. Let $\{\mathbb{D}^2_i\}_{i \in \{1, \cdots, n\}}$ be a system of n open disjoint disks in $\mathbb{RP}^2$. We define $\hat{Y}$ as

$$\hat{Y} = S^1 \widetilde{\times} \left(\mathbb{RP}^2 \setminus \coprod_{i=1}^{n} \mathbb{D}^2_i \right),$$

here $\widetilde{\times}$ denotes the twisted product (see [MS74, Chapter 2]). We define m_i and l_i on the i^{th}-torus boundary component of $\hat{Y}$ as

$$m_i = \{*\} \times \partial\mathbb{D}^2_i \quad \text{and} \quad l_i = S^1 \times \{*\}.$$

As before, the closed manifold $Y = \mathbb{RP}^2(p_1/q_1, \cdots p_n/q_n)$ is the result of performing a Dehn filling of $\hat{Y}$ along the curve $p_i m_i + q_i l_i$ for every i. This construction gives the following presentation for the fundamental group of Y:

$$\pi_1(Y) = \pi_1\big(\mathbb{RP}^2(p_1/q_1, \cdots p_n/q_n)\big) = \big\langle x_1, \cdots, x_n, z, h \,\big|\, x_i^{p_i} h^{q_i}, [x_i, h], zhz^{-1}h, x_1 x_2 \cdots x_n z^2 \big\rangle.$$

Let h be a regular fibre of the Seifert fibred space $Y = \mathbb{RP}^2(p_1/q_1, \cdots p_n/q_n)$, we define the space $\mathcal{M}(p_1/q_1, \cdots p_n/q_n)$ as

$$\mathcal{M}(p_1/q_1, \cdots p_n/q_n) = Y \setminus \nu(h),$$

where $\nu(h)$ is a small fibred open neighborhood of the fibre h. It is easy to see that $\mathcal{M}^2(p_1/q_1, \cdots p_n/q_n)$ is a Seifert space fibred over the Möbius band, that is a punctured $\mathbb{RP}^2$. The fundamental group of $\mathcal{M}^2(p_1/q_1, \cdots p_n/q_n)$ is

$$\pi_1\left(\mathcal{M}\left(\frac{p_1}{q_1}, \cdots \frac{p_n}{q_n}\right)\right) \;=\; \big\langle x_1, \cdots, x_n, y, z, h \,\big|\, x_i^{p_i} h^{q_i}, [y, h], [x_i, h] zhz^{-1}h, yx_1 x_2 \cdots x_n z^2 \big\rangle. \tag{1.3.2}$$

Definition 1.3.3. Let $M = \mathcal{M}(p_1/q_1, \cdots, p_n/q_n)$. Let us suppose that $\pi_1(M)$ is presented as in (1.3.2). We define the *fibration meridian* of this presentation as the, unique up to isotopy, simple closed curve $\mu \subset \partial M$, such that

$$[\mu] = y \in \pi_1(\mathcal{M}(p_1/q_1, \cdots, p_n/q_n)).$$

The classification of Seifert fibrations [Mar16, Proposition 10.3.11] states that there exists a fibre-preserving diffeomorphism

$$\mathbb{D}^2\left(\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n}\right) \longrightarrow \mathbb{D}^2\left(\frac{p_1}{q_1}, \cdots, \frac{p_i}{kp_i + q_i}, \cdots, \frac{p_n}{q_n}\right) \tag{1.3.3}$$

for every $k \in \mathbb{Z}$ and $i \in \{1, \cdots, n\}$. In particular, we can suppose q_i to be odd for every i.

Let $M = \mathbb{D}^2(p_1/q_1, \cdots p_n/q_n)$ with $n \geq 2$ and $p_i \geq 2$. The Seifert fibration on M is unique up to isotopy unless $M \cong \mathbb{D}^2(2/1, 2/1)$; see [Mar16, Proposition 10.4.16]. The latter admits exactly two isotopy classes of Seifert fibration. One has base orbifold the Möbius band and the other has base orbifold the disk with two cone points.

Let Y be a Seifert manifold with a given fibration which has a non-trivial singular fibre, then we denote by $\mathcal{O}(Y)$ the vector whose entries are the orders of the singular fibres in ascending order. For instance, if $Y = \mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_n}/_{q_n})$ with $2 \leq p_1 \leq \cdots \leq p_n$, then $\mathcal{O}(Y) = (p_1, \cdots, p_n)$.

Lemma 1.3.4. *Let Y be a Seifert space fibred over a Möbius band. The rational longitude of Y coincides with the regular fibre $h \subset \partial Y$.*

Proof. This presentation in (1.3.2) shows that h has finite order in $H_1(Y;\mathbb{Z})$ and, indeed, it is the rational longitude of Y. □

Now we need to define a *graph manifold*, in simple terms we can say that graph manifolds are 3-manifolds resulting from the union of Seifert fibred spaces.

Definition 1.3.5. Let M be a closed and orientable 3-manifold. A *JSJ decomposition* on M is a minimal collection of disjointly embedded incompressible tori $\mathcal{T} \subset M$ such that each connected component of $M \setminus \mathcal{T}$ is either atoroidal or Seifert fibreed. A torus in $\mathcal{T}$ is called *JSJ torus.*

In [JS79], it is proven that every closed, orientable, irreducible 3-manifold admits a JSJ decomposition. Moreover, they proved that a JSJ decomposition is unique up to isotopy.

Definition 1.3.6. Let M be a closed 3-manifold and $\mathcal{T} \subset M$ its JSJ decomposition. We say that M is a *graph manifold* if every connected component of $\overline{M \setminus \mathcal{T}}$ is a Seifert fibred space.

Graph manifolds are named after the fact that they can be described by a graph; see [Neu81] for further details

Definition 1.3.7. A closed 3-manifold is said to be *toroidal* if contains an embedded *incompressible* torus.

For the meaning of 'incompressible' we give [Hem76, Chapter 6] as a reference. We emphasize [Hem76, Corollary 6.2], which states that 2-sided and incompressible surfaces are π_1-injective. This in particular implies that if a 3-manifold has a non-trivial JSJ decomposition, then it is toroidal.

1.4. The Pillowcase and the image of the character variety

We give [HHK13] as a reference for this section. For a given group G we define

$$\chi(G) \coloneqq \mathrm{Hom}(G, SU(2))/_{\sim},$$

where $\sim$ is conjugation in $SU(2)$. We call $\chi(G)$ the $SU(2)$-*character variety of* G. It is easy to see that

$$\chi\colon \mathbb{G}roup \longrightarrow \mathbb{T}op$$

is a contravariant functor from the category of groups to the category of topological spaces. This means that if for a group homomorphism $f\colon G \to H$ we have a map

$$\chi(f)\colon \chi(H) \to \chi(G)$$

If M is a manifold and $G = \pi_1(M)$, we define the $SU(2)$-*character variety of* M, by $\chi(M)$ denoted, as $\chi(\pi_1(M))$.

Let Σ be a 2-torus. Since $\pi_1(\Sigma)$ is abelian, every representation $\pi_1(\Sigma) \to SU(2)$ has image in a maximal torus of $SU(2)$. Since the maximal tori are conjugate, every $SU(2)$-representation of $\pi_1(\Sigma)$ is conjugate to one whose image is contained in $U(1)$.

Definition 1.4.1. We define the *pillowcase* as the quotient of the rectangle $[0,\pi]\times[0,2\pi]$ with the following identifications along its boundary

$$(x,0) \sim (x,2\pi), \quad (0,y) \sim (0, 2\pi - y), \quad \text{and} \quad (\pi, y) \sim (\pi, 2\pi - y).$$

We denote this as $\mathbb{P}$.

Figure 1.1 shows the pillowcase and, in particular, it is homeomorphic to the 2-sphere. Figure 1.1 is taken from [HHK13]. In [HHK13] it is proven that the pillowcase $\mathbb{P}$ is homeomorphic to the orbifold $S^2(2,2,2,2)$. We call the four corner points $\{0,\pi\}^2 \subset \mathbb{P}$ of the pillowcase *cone points*.

Lemma 1.4.2. *Let Σ be a 2-torus. The $SU(2)$ character variety $\chi(\Sigma)$ is homeomorphic to the pillowcase $\mathbb{P}$.*

Proof. Let $\{x_1, x_2\}$ be an ordered basis of $\pi_1(\Sigma)$. Let $(\theta,\psi) \in \mathbb{P} = ([0,\pi]\times[0,2\pi])/\sim$, and map this point to the conjugacy class $[\eta] \in \chi(\Sigma)$, where η is the representation $\eta\colon \pi_1(\Sigma) \to$

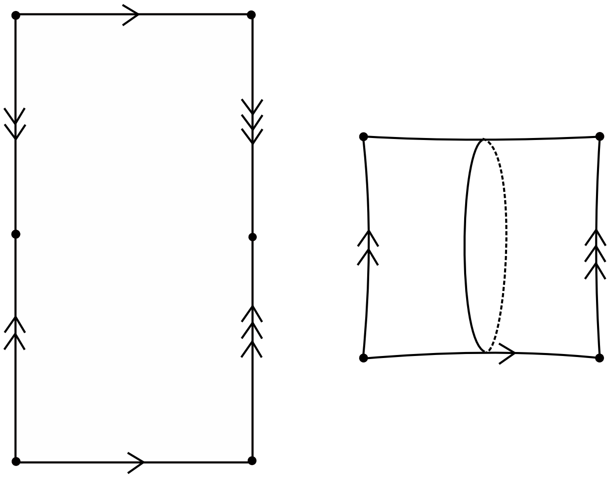


FIGURE 1.1. The pillowcase from [HHK13]

$U(1)$ such that

$$x_1 \mapsto \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \quad \text{and} \quad x_2 \mapsto \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}. \tag{1.4.1}$$

This defines a map $F\colon \mathbb{P} \to \chi(\Sigma)$. The map F is clearly well-defined. We shall prove that it is surjective and injective.

Let $\eta\colon \pi_1(\Sigma) \to U(1)$ be a representation as in (1.4.1) such that $\theta \in [0, \pi]$ and $\psi \in [0, 2\pi]$, then $[\eta] \in \operatorname{Im} F$.

Let $\eta\colon \pi_1(\Sigma) \to U(1)$ be a representation as in (1.4.1) such that $\theta \in (\pi, 2\pi)$. The representation η stays in the same conjugation class of $\eta' = X\eta X^{-1}$, where

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We notice that

$$X\eta(x_1)X^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} e^{i(2\pi-\theta)} & 0 \\ 0 & e^{-i(2\pi-\theta)} \end{bmatrix}.$$

Therefore, $[\eta'] = [\eta] \in \operatorname{Im} F$. This implies that the map F is surjective.

Let us suppose that the points $(\theta_1, \psi_1) \in \mathbb{P}$ and $(\theta_2, \psi_2) \in \mathbb{P}$ are such that

$$F(\theta_1, \psi_1) = F(\theta_2, \psi_2) = [\eta],$$

where $\eta\colon \pi_1(\Sigma) \to U(1)$. This implies that there exists a matrix $Y \in SU(2)$ such that

$$\begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{bmatrix} = Y \begin{bmatrix} e^{i\theta_2} & 0 \\ 0 & e^{-i\theta_2} \end{bmatrix} Y^{-1} \quad \text{and} \quad \begin{bmatrix} e^{i\psi_1} & 0 \\ 0 & e^{-i\psi_1} \end{bmatrix} = Y \begin{bmatrix} e^{i\psi_2} & 0 \\ 0 & e^{-i\psi_2} \end{bmatrix} Y^{-1}.$$

Therefore, as a consequence of Remark 1.1.2, we obtain that their traces coincide. Hence, either $(\theta_1, \psi_1) = (\theta_2, \psi_2)$ or $(\theta_1, \psi_1) = (2\pi - \theta_2, 2\pi - \psi_2)$. In both cases, $(\theta_1, \psi_1) = (\theta_2, \psi_2)$ as points of $\mathbb{P}$. □

Definition 1.4.3. Let Y be a 3-manifold with torus boundary. Let $\iota\colon \partial Y \hookrightarrow Y$ be the natural inclusion with induced map $\chi(\iota)\colon \chi(Y) \to \chi(\partial Y)$. We call $I(Y)$ the image of $\chi(Y)$ through the map $\chi(\iota)$.

We will not make use of the orbifold structure of $\mathbb{P} = \chi(\Sigma)$, but we are interested in its cone points. Specifically, we refer to the four points $\{0, \pi\}^2 \subset \mathbb{P}$ as the cone points of $\chi(\Sigma)$; these correspond to the characters of the four central $SU(2)$-representations

$$\pi_1(\Sigma) \to \mathcal{Z}(SU(2)) = \{\pm 1\}.$$

Definition 1.4.4. Let G be a group and H a linear group [2]. For a given matrix M, we denote by $M^\top$ the transpose of a M. Let $\rho\colon G \to H$ be a representation. We define the *jewelled representation* $\rho^\dagger : G \to H$ as

$$\rho^\dagger(x) = \left(\rho(x)^{-1}\right)^\top = \left(\rho(x)^\top\right)^{-1}.$$

As we mentioned in the proof of Lemma 1.4.2, if Σ is a torus and $\eta\colon \pi_1(\Sigma) \to U(1)$ is a representation, then η is conjugate, via the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

to the representation $\eta^\dagger$.

[2] A linear group is a group of invertible matrices over a given field and with the operation of matrix multiplication.

Chapter 2

Topological Set Up

2.1. General results

Definition 2.1.1. Let Σ be a 2-torus. We define $\mathcal{R}_{U(1)}(\Sigma)$ as $\mathrm{Hom}(\pi_1(\Sigma), U(1))$.

Let Y be an n-manifold, we denote by $\mathcal{R}(Y)$ the set $\mathrm{Hom}(\pi_1(Y), SU(2))$. There exists a natural inclusion

$$\mathcal{R}_{U(1)}(\Sigma) \hookrightarrow \mathcal{R}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), SU(2)).$$

Let $\{x_1, x_2\} \subseteq \pi_1(\Sigma)$ be a basis for the fundamental group of Σ. The space $\mathcal{R}_{U(1)}(\Sigma)$ is homeomorphic to the torus $[0, 2\pi]^2/_\sim$ in the following way: the point

$$(\theta, \psi) \in [0, 2\pi]^2/_\sim = S^1 \times S^1$$

is associated to the unique representation

$$\pi_1(\partial Y_1) \to U(1), \quad \text{with} \quad x_1 \mapsto \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \quad \text{and} \quad x_2 \mapsto \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}. \tag{2.1.1}$$

Definition 2.1.2. Let Y be a 3-manifold and Σ an embedded torus via the map $\iota\colon \Sigma \to Y$. We define the *$U(1)$-representation space of Σ relative to Y* as the set of representations $\pi_1(\Sigma) \to U(1)$ that extend to a representation $\pi_1(Y) \to SU(2)$. We denote this by $T(Y, \Sigma) \subseteq \mathcal{R}_{U(1)}(\Sigma)$.

Explicitly, we have

$$T(Y, \Sigma) \coloneqq \left\{ \eta \in \mathcal{R}_{U(1)}(\Sigma) \,\middle|\, \exists \rho \in \mathcal{R}(Y) \text{ such that } \rho|_{\iota_* \pi_1(\Sigma)} \equiv \eta \right\}.$$

We can thus see $T(Y, \partial Y)$ as the $SU(2)$-version of the zero set of the A-polynomial of Y, a formal definition of this polynomial can be found in [Coo+94]. Similarly, $T(Y, \partial Y)$ can be seen as the $SU(2)$ version of the translation extension locus, see [Khô03] and [CD16], which uses $SL_2(\mathbb{R})$-representations. The set $T(Y, \partial Y)$ was previously defined in [Lin13, Definition 2.10] in terms of the flat connections of a trivialized rank 2 unitary bundle over Y.

We now emphasize that the object $T(Y, \partial Y)$ is very close to the *pillowcase image* $I(Y)$. This latter is defined in Section 1.4. As an application of Lemma 1.4.2, $\chi(\partial Y)$ is homeomorphic to the pillowcase $\mathbb{P} = [0, \pi] \times [0, 2\pi]/\sim$. Let

$$\pi\colon \mathcal{R}_{U(1)}(\partial Y) \to \chi(\partial Y), \quad \text{with} \qquad \eta \mapsto [\eta],$$

be the quotient map. Let $\{x_1, x_2\}$ be a basis for $\pi_1(\Sigma)$ and we denote by $\eta_{(\theta,\psi)}\colon \pi_1(\partial Y) \to U(1)$ the representation corresponding to the point $(\theta, \psi) \in \mathcal{R}_{U(1)}(\partial Y)$ as in (2.1.1). Therefore,

$$\pi(\eta_{(\theta,\psi)}) = [(\theta, \psi)]/\sim \in \mathbb{P}.$$

It can be proven that

$$[\eta_{(\theta_1,\psi_1)}] = [\eta_{(\theta_2,\psi_2)}] \in \chi(\partial Y)$$

if and only if $(\theta_1, \psi_1) = (2\pi - \theta_2, 2\pi - \psi_2)$. Therefore, if $c \in \chi(\partial Y)$, then

$$\pi^{-1}(c) = \left\{\eta_{(\theta,\psi)}, \eta^{\dagger}_{(\theta,\psi)}\right\},$$

for some $(\theta, \psi) \in S^1 \times S^1$. Here $\eta^{\dagger}$ denotes the jewelled representation as in Definition 1.4.4.

This implies that the map π is the double cover of the 2-sphere branched over the four cone points $\{0, \pi\}^2 \subset \mathbb{P}$. We prove now that preimage $\pi^{-1}I(Y) \subseteq \mathcal{R}_{U(1)}(\partial Y)$ is exactly $T(Y, \partial Y)$.

Lemma 2.1.3. *Let Y be a 3-manifold with torus boundary. Let $\pi\colon \mathcal{R}_{U(1)}(\partial Y) \to \chi(\partial Y)$ be as above, then $\pi^{-1}I(Y) = T(Y, \partial Y)$.*

Proof. Let $\iota\colon \partial Y \hookrightarrow Y$. If $\eta \in T(Y, \partial Y)$, then $\pi(\eta) \in I(Y)$ by definition. Thus, $T(Y, \partial Y) \subseteq \pi^{-1}I(Y)$. Conversely, if $c \in I(Y)$, then by definition there exists a representation $\rho_c\colon \pi_1(Y) \to SU(2)$ such that

$$\left[\rho_c|_{\iota_*\pi_1(\partial Y)}\right] = \pi(\rho_c) = c.$$

It is easy to see that if $c \in I(Y)$ one of branched point $\{0,\pi\}^2 \subset \chi(\partial Y)$, then the preimage $\pi^{-1}(c)$ contains only one representation $\eta\colon \pi_1(\partial Y) \to \mathcal{Z}(SU(2))$. Therefore, $\pi^{-1}(c) \subseteq T(Y, \partial Y)$ by definition. If $c \in I(Y)$ is not a branched point of $\chi(\partial Y)$, then $\pi^{-1}(c) = \{\eta_c, \eta_c^\dagger\}$, where $\eta_c^\dagger$ is the jellewed representation of η_c. By definition, we know that there exists a representation $\rho_c\colon \pi_1(Y) \to SU(2)$ and a matrix $X \in SU(2)$ such that

$$X^{-1}\rho_c\big|_{\iota_*\pi_1(\partial Y)}X = \eta_c.$$

Up to conjugation, we can suppose that $X = 1$. It is easy to see that

$$\left(\rho_c\big|_{\iota_*\pi_1(\partial Y)}\right)^\dagger = \rho_c^\dagger\big|_{\iota_*\pi_1(\partial Y)} = \eta_c^\dagger.$$

Therefore, the representation $\eta_c^\dagger$ extends to $\pi_1(Y)$ too. We conclude that $\pi^{-1}I(Y) \subseteq T(Y, \partial Y)$.

□

The author prefers working with $\mathcal{R}_{U(1)}(\partial Y)$ rather than the pillowcase $\chi(\partial Y)$ as they find the former more natural when working with objects such as slopes, Dehn surgeries, and coordinates. It should be noted, however, that as a consequence of Lemma 2.1.3, $T(Y, \partial Y)$ and $I(Y)$ are almost equivalent. Consequently, given a certain result concerning the structure of $T(Y, \partial Y)$ in $\mathcal{R}_{U(1)}(\partial Y)$ one can find a corresponding to $I(Y)$ in the pillowcase $\chi(\partial Y)$, and vice versa.

Let Y_1 and Y_2 be a pair of 3-manifolds with torus boundary and $\varphi\colon \partial Y_1 \to \partial Y_2$ a diffeomorphism. We denote by $Y = Y_1 \cup_\varphi Y_2$ be the manifold obtained by gluing Y_1 and Y_2 along the map φ. Let Σ be the torus $\partial Y_1 = \partial Y_2$ in Y and $\iota\colon \Sigma \to Y$ the inclusion. Then, the spaces $T(Y_1, \Sigma)$ and $T(Y_2, \Sigma)$ are both contained in $\mathcal{R}_{U(1)}(\Sigma)$. Therefore, the intersection $T(Y_1, \Sigma) \cap T(Y_2, \Sigma)$ is well defined in $\mathcal{R}_{U(1)}(\Sigma)$.

If $\eta\colon \pi_1(\Sigma) \to U(1)$ is a representation in $T(Y_1, \Sigma) \cap T(Y_2, \Sigma)$, then there exist $\rho_1 \in \mathcal{R}(Y_1)$ and $\rho_2 \in \mathcal{R}(Y_2,)$ such that

$$\rho_1\big|_{\iota_*\pi_1(\Sigma)} \equiv \eta \equiv \rho_2\big|_{\iota_*\pi_1(\Sigma)}.$$

Let $\rho\colon \pi_1(Y) \to SU(2)$ obtained by gluing the representations ρ_1 and ρ_2 along η. Thus, the representation ρ is such that

$$\rho\big|_{\pi_1(Y_1)} \equiv \rho_1, \quad \rho\big|_{\pi_1(Y_2)} \equiv \rho_2, \quad \text{and} \quad \rho\big|_{\iota_*\pi_1(\Sigma)} \equiv \eta.$$

This means that every point in $T(Y_1,\Sigma)\cap T(Y_2,\Sigma)$ corresponds to an $SU(2)$-representation of $\pi_1(Y)$. Conversely, up to conjugation, a representation $\rho\colon \pi_1(Y)\to SU(2)$ satisfies

$$\operatorname{Im}\rho|_{\iota_*\pi_1(\Sigma)}\subseteq U(1).$$

This implies that $\rho|_{\iota_*\pi_1(\Sigma)}\in T(Y_1,\Sigma)\cap T(Y_2,\Sigma)$.

Proposition 2.1.4. *Let $Y=Y_1\cup_\varphi Y_2$ be the 3-manifold defined above. The manifold Y is $SU(2)$-abelian if and only if every $SU(2)$-representation of $\pi_1(Y)$ that restricts to a representation in $T(Y_1,\Sigma)\cap T(Y_2,\Sigma)$ is $SU(2)$-abelian.*

Proof. If Y is $SU(2)$-abelian, the conclusion is trivial. If Y is not $SU(2)$-abelian, there exists a representation ρ whose image is not abelian. Up to conjugation, we can suppose that $\rho(\iota_*\pi_1(\Sigma))\subseteq U(1)$. The restriction $\rho|_{\iota_*\pi_1(\Sigma)}$ extends to both $\pi_1(Y_1)$ and $\pi_1(Y_2)$. Hence, $\rho|_{\iota_*\pi_1(\Sigma)}\in T(Y_1,\Sigma)\cap T(Y_2,\Sigma)$. □

Definition 2.1.5. Let Y be a 3-manifold with torus boundary, let $\iota\colon \partial Y\to Y$ be the natural inclusion. We define the sets $A(Y), H(Y)$, and $P(Y)$ as:

$$\begin{aligned}
A(Y) &:= \left\{\eta\in\mathcal{R}_{U(1)}(\partial Y)\,\middle|\,\exists\rho\in\mathcal{R}(Y)\text{ such that }\rho|_{\iota_*\pi_1(\partial Y)}\equiv\eta\text{ and }\rho\text{ is abelian}\right\},\\
H(Y) &:= \left\{\eta\in\mathcal{R}_{U(1)}(\partial Y)\,\middle|\,\exists\rho\in\mathcal{R}(Y)\text{ such that }\rho|_{\iota_*\pi_1(\partial Y)}\equiv\eta\text{ and }\rho\text{ is irreducible}\right\},\\
P(Y) &:= \Big\{\eta\in\mathcal{R}_{U(1)}(\partial Y)\,\Big|\,\exists\rho\in\mathcal{R}(Y)\text{ such that }\rho|_{\iota_*\pi_1(\partial Y)}\equiv\eta,\\
&\qquad\qquad\eta\text{ is central, and }\rho\text{ is abelian and non-central}\Big\}.
\end{aligned}$$

Equivalently, a representation $\pi_1(\partial Y)\to U(1)$ is in $A(Y)$ (resp. $H(Y)$) if and only if it extends to an abelian (resp. an irreducible) representation $\pi_1(Y)\to SU(2)$. Similarly, a representation $\pi_1(\partial Y)\to\mathcal{Z}(SU(2))$ is in $P(Y)$ if and only if it extends to an abelian representation $\pi_1(Y)\to SU(2)$ whose image is not in $\mathcal{Z}(SU(2))$. Notice that $P(Y)\subset A(Y)$ and $A(Y)\cup H(Y)=T(Y,\partial Y)\subseteq\mathcal{R}_{U(1)}(\partial Y)$.

Let $Y=Y_1\cup_\varphi Y_2$ and $\rho\in\mathcal{R}(Y)$. We write ρ_1 for $\rho|_{\pi_1(Y_1)}$ and ρ_2 for $\rho|_{\pi_1(Y_2)}$.

Proposition 2.1.6. *Let $Y=Y_1\cup_\varphi Y_2$ and $\Sigma\subset Y$ the torus corresponding to $\partial Y_1=\partial Y_2$. If $\rho\in\mathcal{R}(Y)$ is an irreducible representation such that ρ_1 and ρ_2 are both abelian, then $\rho(\iota_*\pi_1(\Sigma))\subseteq\mathcal{Z}(SU(2))$.*

PROOF. Since ρ is irreducible, neither $\operatorname{Im}\rho_1$ nor $\operatorname{Im}\rho_2$ is central. Let us suppose, by contradiction, that there exists $g \in \iota_*\pi_1(\Sigma)$ such that $\rho_1(g)$ is not in the center. Since $g = \varphi_*(g)$ in $\pi_1(Y)$, we have that $\rho_1(g) = \rho_2(\varphi_*(g))$. In particular $\operatorname{Im}\rho_1 \subseteq \Lambda_{\rho_1(g)}$ and $\operatorname{Im}\rho_2 \subseteq \Lambda_{\rho_2(\varphi_*(g))}$, where $\Lambda_z \subset SU(2)$ is the centralizer subgroup of the non-central element $z \in SU(2)$. Fact 1.1.3 implies that $\Lambda_{\rho_1(g)} = \Lambda_{\rho_2(\varphi_*(g))}$. This implies that $\operatorname{Im}\rho \subseteq \Lambda_{\rho_1(g)} = \Lambda_{\rho_2(\varphi_*(g))}$, which contradicts the irreducibility of the representation ρ. □

Proposition 2.1.7. *Let $Y = Y_1 \cup_\varphi Y_2$ be as above. If there exists $\rho \in \mathcal{R}(Y)$ such that $\rho(\iota_*\pi_1(\Sigma)) \subseteq \mathcal{Z}(SU(2))$ and neither ρ_1 nor ρ_2 is central, then the manifold Y is not $SU(2)$-abelian.*

PROOF. If ρ is irreducible we get our conclusion. Let us suppose that ρ is reducible and therefore both ρ_1 and ρ_2 are reducible as well. Without loss of generality, we assume that the images $\operatorname{Im}\rho_1$ and $\operatorname{Im}\rho_2$ are both in $U(1)$. Let $z \in SU(2) \setminus U(1)$. If $a \in zU(1)z^{-1}$ and a is not in the center $\mathcal{Z}(SU(2)) = \{\pm 1\}$, then the matrix a does commute with any element of $U(1) \setminus \mathcal{Z}(SU(2))$. Let $\gamma\colon \pi_1(Y) \to SU(2)$ be defined as

$$\gamma(x) = \begin{cases} \rho_1(x) & \text{if } x \in \pi_1(Y_1) \\ z\rho_2(x)z^{-1} & \text{if } x \in \pi_1(Y_2) \end{cases}.$$

As $\rho_1(x) = \rho_2(x) \in \mathcal{Z}(SU(2))$ for every $x \in \pi_1(\Sigma)$, then

$$z\rho_2(y)z^{-1} = \rho_2(y) = \rho_1(y)$$

for every $y \in \pi_1(\Sigma)$. Therefore the representation γ is a well defined representation. Since neither ρ_1 nor ρ_2 is central, the representation γ has non-abelian image. This implies that γ is irreducible. □

Let Y_1 and Y_2 be two 3-manifolds with torus boundary and let $Y = Y_1 \cup_\varphi Y_2$. Let $\Sigma \subset Y$ be the torus corresponding to $\partial Y_1 = \partial Y_2$, let $\iota\colon \Sigma \to Y$ be inclusion. In order to make the notation a little lighter, the sets $A(Y_i)$, $H(Y_i)$, and $P(Y_i)$ of Definition 2.1.5 will be denoted by A_i, H_i, and P_i, where $i \in \{1,2\}$, where there is no risk of confusion. Proposition 2.1.6 and Proposition 2.1.7 imply that there exists an irreducible representation $\rho \in \mathcal{R}(Y)$ such that $\rho|_{\iota_*\pi_1(\Sigma)} \in A_1 \cap A_2$ if and only if $\rho|_{\iota_*\pi_1(\Sigma)} \in P_1 \cap P_2$.

Theorem 2.1.8. *Let Y_1 and Y_2 be two 3-manifolds with torus boundary. The manifold $Y = Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian if and only if $H_1 \cap H_2$, $H_1 \cap A_2$, $A_1 \cap H_2$, and $P_1 \cap P_2$ are empty.*

Proof. If Y is $SU(2)$-abelian, then, for every $\rho \in \mathcal{R}(Y)$, the restrictions $\rho_1 = \rho|_{\iota_*\pi_1(Y_1)}$ and $\rho_2 = \rho|_{\iota_*\pi_1(Y_2)}$ are abelian. This implies that $H_1 \cap H_2 = \emptyset$, $A_1 \cap H_2 = \emptyset$, and $H_1 \cap A_2 = \emptyset$.

If $\eta \in P_1 \cap P_2$, then there exists a representation $\rho \in \mathcal{R}(Y)$ such that ρ_1 and ρ_2 are non-central. Proposition 2.1.7 implies that Y is not $SU(2)$-abelian, which is a contradiction. Thus, $P_1 \cap P_2 = \emptyset$.

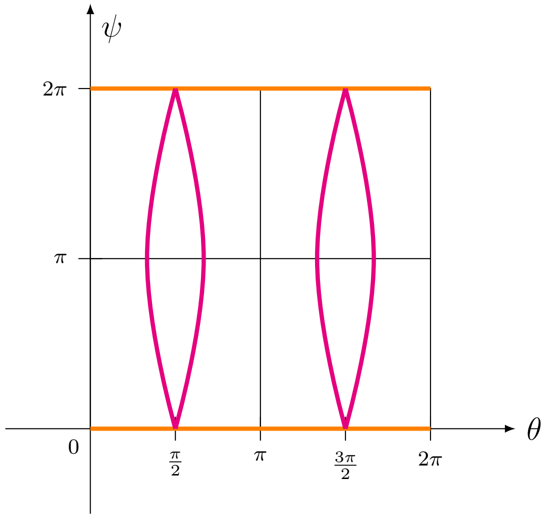
Conversely, let $\rho \in \mathcal{R}(Y)$ be an irreducible representation. In particular, Y is not $SU(2)$-abelian. Up to conjugation, we can suppose that $\rho(\iota_*\pi_1(\Sigma)) \subseteq U(1)$. If $\rho|_{\iota_*\pi_1(\Sigma)}$ is either in $H_1 \cap H_2$, $A_1 \cap H_2$, or in $H_1 \cap A_2$, then we get our conclusion. If $\rho|_{\iota_*\pi_1(\Sigma)}$ is in neither $H_1 \cap H_2$, $A_1 \cap H_2$, nor $H_1 \cap A_2$, then ρ_1 and ρ_2 are both abelian. Proposition 2.1.6 implies that $\rho(\iota_*\pi_1(\Sigma)) \subseteq \mathcal{Z}(SU(2))$. Since ρ is irreducible and the restrictions ρ_1 and ρ_2 are $SU(2)$-abelian, neither ρ_1 nor ρ_2 is $SU(2)$-central. This implies that $\rho|_{\iota_*\pi_1(\Sigma)}$ is in $P_1 \cap P_2$. Therefore, $P_1 \cap P_2$ is nonempty. □

Example 2.1.9. We give now an application of Theorem 2.1.8. Let $Y_1 = Y_2$ be the exterior of the figure eight knot. For $i \in \{1, 2\}$, we denote by $\mu_i, \lambda_i \subseteq \partial Y_i$ the knot meridian and the nullhomologous longitude. Let $\varphi\colon \partial Y_1 \to \partial Y_2$ be an orientation reversing diffeomorphism defined as

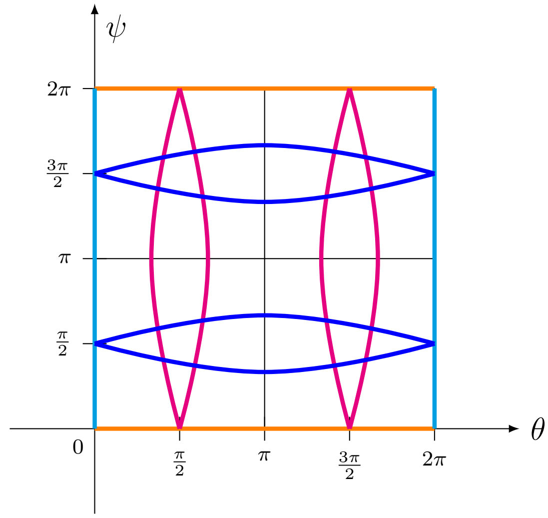
$$\varphi(\mu_1) = \lambda_2 \quad \text{and} \quad \varphi(\lambda_1) = \mu_2. \tag{2.1.2}$$

We want to prove that $Y = Y_1 \cup_\varphi Y_2$ is not $SU(2)$-abelian by applying Theorem 2.1.8. Note that $\mathcal{R}_{U(1)}(\partial Y_i)$ comes with coordinates (θ_i, ψ_i) according to the ordered base $\{\mu_i, \lambda_i\}$ as in (2.1.1). Figure 2.1a shows the space $T(Y_i, \partial Y_i) \subseteq \mathcal{R}_{U(1)}(\partial Y_i)$ with the chosen coordinates. Details of Figure 2.1a can be found in [KK90].

Since $\partial Y_1 = \partial Y_2$ in Y, (θ_1, ψ_1) and (θ_2, ψ_2) are two parameterization of the same space. In particular, the (2.1.2) implies that $(\theta_1, \psi_1) = (\psi_2, \theta_2)$. Figure 2.1b represents the sets $T(Y_1, \partial Y_1)$ and $T(Y_2, \partial Y_2)$ in $\mathcal{R}_{U(1)}(\partial Y_1)$ with the coordinates (θ_1, ψ_1). Figure 2.1b shows that $H(Y_1) \cap H(Y_2)$ is nonempty. Therefore, as a consequence of Theorem 2.1.8, the manifold Y is not $SU(2)$-abelian.



(A) $T(Y_i, \partial Y_i) \subseteq \mathcal{R}_{U(1)}(\partial Y_i)$ with coordinates (θ, ψ) corresponding to $\{\mu_i, \lambda_i\}$. The set $A(Y_i)$ (resp. $H(Y_i)$) is in orange (resp. magenta).



(B) $T(Y_1, \partial Y_1) \cap T(Y_2, \partial Y_2)$ as a subset of $\mathcal{R}_{U(1)}(\partial Y_1)$. The set $A(Y_1)$ (resp. $A(Y_2)$, $H(Y_1)$, $H(Y_2)$) is in orange (resp. cyan, magenta, blue).

FIGURE 2.1. Example 2.1.9

Let Y be as in Example 2.1.9, since $\varphi(\mu_1) = \lambda_2$, then $\Delta(\lambda_1, \lambda_2) = 1$. It can be proven, for example as a consequence of Remark 2.2.2 below, that this implies that Y is an integer homology 3-sphere. Clearly, Y is toroidal. Therefore, Y was known to be non $SU(2)$-abelian as a consequence of [LPZ23, Theorem 1.1].

Now we focus on the set $A(Y)$ of Definition 2.1.5. Let Y be a compact orientable 3-manifold with torus boundary, we identify the group $H_1(\partial Y; \mathbb{Z}) \cong \mathbb{Z}^2$ with the group $\pi_1(\partial Y) \cong \mathbb{Z}^2$ in the natural way. With an abuse of notation, we consider the group

$$\ker(\iota_* : H_1(\partial Y; \mathbb{Z}) \to H_1(Y; \mathbb{Z})) \leq H_1(\partial Y; \mathbb{Z}),$$

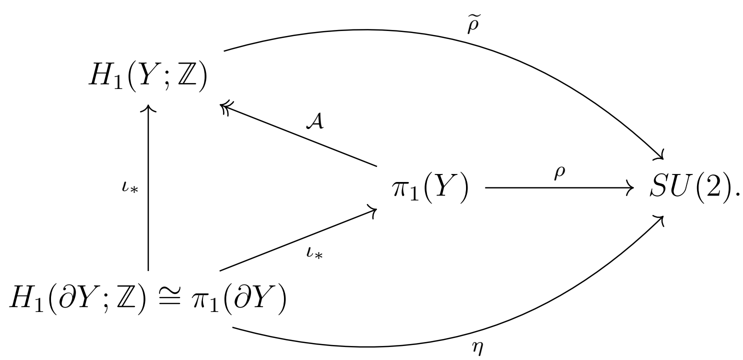
where $\iota \colon \partial Y \to Y$ is the inclusion, as a subgroup of $\pi_1(\partial Y)$.

Before moving to next proposition, we recall a definition:

Definition 2.1.10. A group G is said to be *divisible* if for every $x \in G$ and every $n \in \mathbb{N}_{\geq 1}$, there exists a $y \in G$ such that $x = y^n$.

Proposition 2.1.11. *Let Y be a 3-manifold with torus boundary and $\iota \colon \partial Y \to Y$ the natural inclusion. A representation $\eta \colon \pi_1(\partial Y) \to SU(2)$ extends to an abelian representation $\rho : \pi_1(Y) \to SU(2)$ if and only if $\eta|_{\ker \iota_*} \equiv 1$.*

Proof. The abelianization map $\pi_1(Y) \twoheadrightarrow H_1(Y;\mathbb{Z})$ is denoted by $\mathcal{A}$. Let us suppose that the representation $\eta\colon \pi_1(\partial Y) \to SU(2)$ extends to an abelian representation $\rho : \pi_1(Y) \to SU(2)$. Then, $\eta = \rho \circ \iota_*$. Since ρ is abelian, there exists a representation $\widetilde{\rho}\colon H_1(Y;\mathbb{Z}) \to SU(2)$, such that $\rho = \widetilde{\rho} \circ \mathcal{A}$. This implies that all the triangles in diagram (2.1.3) commute.



(2.1.3)

Thus, $\eta \equiv \widetilde{\rho} \circ \iota_*$. Therefore, $\eta(\ker \iota_*) = \widetilde{\rho} \circ \iota_*(\ker \iota_*) = 1$. This concludes one direction.

Conversely, let $\eta\colon \pi_1(\partial Y) \to SU(2)$ be a representation that is trivial on $\ker \iota_*$. Up to conjugating, we can suppose that η has image in $U(1) \subset SU(2)$. Therefore, we define the (abelian) representation $\gamma\colon \operatorname{Im} \iota_* \to U(1)$ as $\eta(x) = \gamma(x)$. Since the group $U(1)$ is divisible, [Lam99, Proposition 3.19] implies that there exists a representation $\widetilde{\rho}\colon H_1(Y;\mathbb{Z}) \to U(1)$ that extends γ. The representation ρ is given by pre-composing $\widetilde{\rho}$ with the abelianization homomorphism $\mathcal{A}$. □

We recall that an abuse of notation in use: for a given simple closed curve γ in the torus Σ, when we refer to its homotopy class $[\gamma] \in \pi_1(\Sigma)$ we omit the brackets. Consequently, γ indicates both a curve in Σ and its homotopy class $\gamma \in \pi_1(\Sigma)$.

The following is a more operational formulation of Proposition 2.1.11.

Corollary 2.1.12. *Let Y be a 3-manifold with torus boundary and let λ_Y be its rational longitude. Let n be the order of λ_Y in $H_1(Y;\mathbb{Z})$. A representation $\eta\colon \pi_1(\partial Y) \to SU(2)$ extends to an abelian representation $\pi_1(Y) \to SU(2)$ if and only if $\eta(\lambda_Y)^n = 1$.*

Proof. The subgroup $\ker \iota_* \leq H_1(\partial Y;\mathbb{Z})$ is generated by the element $n \cdot \lambda_Y \in H_1(\partial Y;\mathbb{Z})$. Hence, $\eta(\ker \iota_*) = 1$ if and only if $\eta(\lambda_Y)^n = 1$. The conclusion holds by Proposition 2.1.11 □

In what follows λ_1 and λ_2 are the rational longitudes of Y_1 and Y_2. Furthermore, o_1 and o_2 are the orders of the corresponding rational longitudes in $H_1(Y_1;\mathbb{Z})$ and $H_1(Y_2;\mathbb{Z})$.

Proposition 2.1.13. *Let Y_1 and Y_2 be two 3-manifolds with torus boundary and let $\varphi\colon \partial Y_1 \to \partial Y_2$ be a diffeomorphism. If the manifold $Y = Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian, then the groups*

$$\frac{\pi_1(Y_1)}{\langle\langle {\lambda_2}^{o_2} \rangle\rangle} \quad \textit{and} \quad \frac{\pi_1(Y_2)}{\langle\langle {\lambda_1}^{o_1} \rangle\rangle}$$

are $SU(2)$-abelian. Here, $\langle\langle x \rangle\rangle \leq \pi_1(Y_1)$ denotes the smallest normal subgroup of $\pi_1(Y_1)$ containing $x \in \pi_1(Y_1)$.

Proof. Let us suppose that the group $\pi_1(Y_1)/\langle\langle {\lambda_2}^{o_2} \rangle\rangle$ admits an irreducible $SU(2)$-representation. Hence, there exists an irreducible representation $\rho_1\colon \pi_1(Y_1) \to SU(2)$ such that $\rho_1({\lambda_2}^{o_2}) = 1$. Let Σ be the embedded torus in Y corresponding to $\partial Y_1 = \partial Y_2$ and let $\iota\colon \Sigma \to Y$ be the natural inclusion. Let η be the restriction $\rho_1|_{\iota_* \pi_1(\Sigma)}$. Since $\eta(\lambda_2)^{o_2} = \rho_1(\lambda_2)^{o_2} = 1$, Corollary 2.1.12 implies that the representation η extends to an abelian representation $\rho_2\colon \pi_1(Y_2) \to SU(2)$. Thus, there exists a representation $\rho\colon \pi_1(Y) \to SU(2)$ such that $\rho|_{\pi_1(Y_1)} \equiv \rho_1$ and $\rho|_{\pi_1(Y_2)} \equiv \rho_2$. The representation ρ is irreducible. The conclusion for $\pi_1(Y_1)/\langle\langle {\lambda_2}^{o_2} \rangle\rangle$ holds similarly. $\square$

Corollary 2.1.14. *If Y is $SU(2)$-abelian, then the manifolds $Y_1(\lambda_2)$ and $Y_2(\lambda_1)$ are $SU(2)$-abelian as well.*

Proof. The conclusion holds by Proposition 2.1.13 and the existence of a surjective homomorphism

$$\frac{\pi_1(Y_1)}{\langle\langle {\lambda_2}^{o_2} \rangle\rangle} \twoheadrightarrow \pi_1(Y_1(\lambda_2)) = \frac{\pi_1(Y_1)}{\langle\langle \lambda_2 \rangle\rangle} = \frac{\pi_1(Y_1)}{\langle\langle {\lambda_2}^{o_2} \rangle\rangle} \Big/ \langle\langle \lambda_2 \rangle\rangle.$$

$\square$

As shown in [Mot88], there exists a way to glue together two copies of the exterior of the trefoil to get an $SU(2)$-abelian manifold. Corollary 2.1.14 combined with Theorem 2.1.15 above has an interesting application: we cannot do the same for the figure-eight knot.

Theorem 2.1.15 ([BH07, Theorem 1.1]). *Let Y be the exterior of a nontrivial $2-$bridge knot in S^3 which is not a torus knot, and let α be any non-meridional slope in ∂Y. Then there exists an irreducible representation $\pi_1(Y(\alpha)) \to SU(2)$.*

Corollary 2.1.16. *For $i \in \{1, 2\}$, let $K_i \subset S^3$ a non-trivial 2-bridge knot which is not a torus knot and $Y_i = S^3 \setminus \nu(K_i)$. For every diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$, the manifold $Y_1 \cup_\varphi Y_2$ is not $SU(2)$-abelian.*

PROOF. Since K_i is a non-trivial 2-bridge knot, Y_i cannot be a solid torus. Therefore for every diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$, the manifold $Y_1 \cup_\varphi Y_2$ is toroidal. Let $\{\mu_i, \lambda_i\} \subseteq \partial Y$ be a system of the knot meridian and null homologous longitude. If $\varphi(\mu_1) = \lambda_2$ and $\varphi(\lambda_1) = \mu_2$, then manifold Y is an integer homology sphere. In this case the conclusion holds by [LPZ23, Theorem 1.1].

Let us suppose that the diffeomorphism φ is not as above, therefore either $\varphi(\lambda_1) \neq \mu_2$ or $\varphi^{-1}(\lambda_2) \neq \mu_1$. Without loss of generality, we suppose that $\varphi(\lambda_1) \neq \mu_2$. The manifold $Y_2(\varphi(\lambda_1))$ is known to be not $SU(2)$-abelian by Theorem 2.1.15. The conclusion holds by Corollary 2.1.14. □

Since the figure eight knot is a 2-bridge knot, the following is a trivial consequence of Corollary 2.1.16.

Corollary 2.1.17. *Let Y_1 and Y_2 be two copies of the exterior of the figure eight knot. Let $\varphi\colon \partial Y_1 \to \partial Y_2$ be a diffeomorphism. The manifold $Y = Y_1 \cup_\varphi Y_2$ is not $SU(2)$-abelian.*

2.2. Graph manifold rational homology 3-spheres

In this section we shall prove that graph manifold rational homology spheres decompose into Seifert pieces whose base spaces are relatively simple. More explicitly, we are going to prove that every JSJ piece of a graph manifold rational homology sphere, which is a Seifert fibred space, has a base space which is either a punctured 2-sphere S^2 or a punctured $\mathbb{RP}^2$. For details see [BC17, Section 2.2].

Lemma 2.2.1. *Let Y be a graph manifold rational homology sphere with a nontrivial JSJ decomposition and let $Y_0 \subseteq Y$ be a JSJ piece. The Seifert space Y_0 admits a Seifert fibration over either a punctured 2-sphere S^2 or a punctured $\mathbb{RP}^2$.*

PROOF. Let us fix a Seifert fibration of Y_0 and let $\mathcal{B}$ be the base orbifold of this Seifert fibration. Let B be its underlying surface, we shall prove that this is either a punctured S^2 or a punctured $\mathbb{RP}^2$. If B admits a non-separating circle, then Y_0 admits a vertical embedded orientable non-separating surface. This implies that Y admits an embedded orientable non-separating surface. We recall that if a closed 3-manifold admits an embedded orientable non-separating surface, then it is not a rational homology sphere. Therefore, the base surface Σ

does not admit a non-separating orientation-preserving circle. Thus, Σ is either a punctured 2-sphere or a punctured $\mathbb{RP}^2$. □

We give now an useful consequence of Lemma 2.2.1. Before going through this we need to mention some facts.

Remark 2.2.2. Let Σ be a closed surface and Y a Seifert fibred manifold with boundary, n singular fibres, and a punctured Σ as a base surface. Let us suppose that Y is not a solid torus. Let γ be a slope on ∂Y. If γ is a regular fibre for a Seifert fibration of Y, then $Y(\gamma)$ is a reducible manifold. If γ is not a regular fibre, then $Y(\gamma)$ is a closed Seifert space fibred over Σ with n or $n+1$ singular fibres. More precisely, $Y(\gamma)$ has n singular fibres with the same orders as Y and one additional singular fibre of order $\Delta(h, \gamma)$, where $h \subset \partial Y$ is a regular fibre.

Remark 2.2.3. Let Y_1 and Y_2 be two rational homology solid tori and $Y = Y_1 \cup_\Sigma Y_2$ a closed three manifold. Let $\sigma \in \mathbb{N}$ be defined as

$$\sigma := \Delta(\lambda_1, \lambda_2) o_1 t_1 o_2 t_2,$$

where o_i is the order of the rational longitude λ_i and t_i is the order of the torsion subgroup of $H_1(Y_i; \mathbb{Z})$. A standard homology computation shows that if Y is a rational homology sphere, then

$$\sigma = |H_1(Y; \mathbb{Z})|.$$

In particular, if $\Delta(\lambda_1, \lambda_2) = 0$, then Y is not a rational homology sphere. Details about this can be found in [BGH21, Section 10.1].

Proposition 2.2.4. *[SZ21, Proposition 3.5] Let Y be a Seifert fibre space with base space $\mathbb{RP}^2$ and any number of singular fibres. Then Y is $SU(2)$-abelian if and only if Y is a lens space or $\mathbb{RP}^3 \# \mathbb{RP}^3$.*

We remind the reader that $\mathbb{RP}^3 \# \mathbb{RP}^3$ is a Seifert manifold fibred over $\mathbb{RP}^2$ and with no singular fibres. Furthermore, a Seifert manifold with base space $\mathbb{RP}^2$ is a lens space if and only if it has exactly one singular fibre and this has an integer Seifert coefficient. For the notation used here see [SZ21, Section 3].

Corollary 2.2.5. *Let Y be an $SU(2)$-abelian graph manifold rational homology sphere with one JSJ torus. The two JSJ pieces of Y admit either a Seifert fibration disk base space or a Seifert fibration over a Möbius band with exactly one cone point.*

Proof. Let $\Sigma \subset Y$ be the JSJ torus. We call Y_1 and Y_2 the two connected components of $\overline{Y \setminus \Sigma}$. Lemma 2.2.1 implies that Y_1 and Y_2 have base spaces that are either a disk or a Möbius band. Without loss of generality, we suppose that Y_1 has base space a Möbius band and that it has $n \geq 0$ singular fibres. As an application of Corollary 2.1.14, since Y is $SU(2)$-abelian, the manifold $Y_1(\lambda_2)$ is $SU(2)$-abelian. According to Remark 2.2.2 we have two possibilities:

- $\lambda_2 \subseteq \partial Y_1$ is a regular fibre of Y_1 and $Y_1(\lambda_2)$ is reducible;
- $\lambda_2 \subseteq \partial Y_1$ is not regular fibre of Y_1 and $Y_1(\lambda_2)$ has base space $\mathbb{RP}^2$.

According to Lemma 1.3.4, the regular fibre of Y_1 coincides, as a slope of ∂Y_1, with the rational longitude of Y_1. In other words, we have that $h_1 = \lambda_1$. This implies that, if $\Delta(h_1, \lambda_2) = 0$, then $\Delta(\lambda_1, \lambda_2) = 0$ and therefore Y is not a rational homology sphere by Remark 2.2.3. This is not possible since Y is a rational homology sphere by hypothesis.

Let us suppose that $\Delta(h_1, \lambda_2) \neq 0$. As before, Remark 2.2.2 implies that $Y_1(\lambda_2)$ is a Seifert space fibred with base space $\mathbb{RP}^2$. By Corollary 2.1.14, the manifold $Y_1(\lambda_2)$ is $SU(2)$-abelian. By Proposition 2.2.4 and the remarks above, the manifold $Y_1(\lambda_2)$ has at most one non-trivial fibre.

If $Y_1(\lambda_2)$ does not have any non-trivial fibre, then Y_1 is the result of removing a vertical tubular neighborhood of a regular fibre. This implies that Y_1 is a Seifert space fibred over a Möbius band with no cone points. According to [Mar16, Proposition 10.4.16], Y_1 admits a fibration over a disk with two cone points, both of order 2.

If $Y_1(\lambda_2)$ has one singular fibre, then Y_1 is fibred over a Möbius band and this at most one non-trivial singular fibre. If Y_1 has does not have any non-trivial singular fibres, then the conclusion holds as in the case proven previously. □

To prove the next result, we need to fix the notation. In particular, we adopt the one introduced in [Bas25a]. Let C_2 be the twisted I-bundle over an once-punctured Möbius band, this is a a Seifert fibred manifold whose fundamental group is

$$\pi_1(C_2) = \langle x_1, x_2, z, h | zhz^{-1}h = 1, x_1x_2z^2 = 1 \rangle. \tag{2.2.1}$$

The boundary ∂C_2 consists of two tori. As in the disk case, the element $h \in \pi_1(C_2)$ in the presentation (2.2.1) is the homotopy class of the regular fibre. We name Σ_1 and Σ_2 the two tori boundary of C_2. For $i \in \{1,2\}$, if h is considered as a slope in $\pi_1(\Sigma_i)$, then we call it h_i. With this notation, the group

$$\pi_1(\Sigma_i) \leq \pi_1(C_2)$$

is generated by the ordered basis $\{x_i, h_i\}$. We consider the space $\mathcal{R}_{U(1)}(\Sigma_i)$ with coordinates (θ_i, ψ_i) according to this ordered basis as in (2.1.1).

Theorem 2.2.6. *[Bas25a, Theorem 7.6] Let $Y = Y_1 \cup_{\Sigma_1} C_2$. Let the set $T(Y_1, \partial Y_1) = A(Y_1) \cup H(Y_1)$ be considered as a subset of $\mathcal{R}_{U(1)}(\Sigma_1)$. As a subset of $\mathcal{R}_{U(1)}(\Sigma_2)$ the set $T(Y, \partial Y) = A(Y) \cup H(Y)$ where*

$$A(Y) = \begin{cases} \{\psi_2 = 0\} & \text{if } A(Y_1) \cap \{\psi_1 = \pi\} = \emptyset \\ \{\psi_2 = 0\} \cup \{\psi_2 = \pi\} & \text{otherwise.} \end{cases} \tag{2.2.2}$$

and $H_{Y,t} \cup H_{Y,0} \cup H_{Y,\pi} \subset H(Y)$, where for $\varepsilon \in \{0, \pi\}$,

$$H_{Y,t} \coloneqq \left\{(-\theta_1 + \pi, \psi_1) \in \mathcal{R}_{U(1)}(\Sigma_2) \middle| (\theta_1, \psi_1) \in T(Y_1, \partial Y_1), \psi_1 \notin \pi\mathbb{Z}\right\},$$

$$H_{Y,\varepsilon} \coloneqq \begin{cases} \{\psi_2 = \varepsilon\} \subset \mathcal{R}_{U(1)}(\Sigma_2) & \text{if } H(Y_1) \cap \{\psi_1 = \varepsilon\} \neq \emptyset \\ \{\psi_2 = \varepsilon\} \setminus \{0, \pi\}^2 \subset \mathcal{R}_{U(1)}(\Sigma_2) & \text{if } T(Y_1, \partial Y_1) \cap \left(\{\psi_1 = \varepsilon\} \setminus \{0, \pi\}^2\right) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Proposition 2.2.7. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a graph manifold rational homology sphere with a single JSJ torus. If Y_1 is a Seifert space fibred over a Möbius band and with one singular fibre, then Y is not $SU(2)$-abelian.*

Proof. Pet $p \in \mathbb{N}$ be the order of the singular fibre of Y_1. Since Y_1 has base space a Möbius band, there exists a vertical torus $\Sigma_1 \subset Y_1$ such that

$$\overline{Y_1 \setminus \Sigma_1} = C_2 \cup St,$$

where $St = S^1 \times \mathbb{D}^2$. With the notation above, $\Sigma_2 = \Sigma \subset \partial C_2$ has the JSJ torus of Y. We call $\lambda_{St} = pt \times \partial\mathbb{D}^2 \subset \Sigma_1$ the rational longitude of St. Since Y_1 as a nontrivial singular fibre, then $\Delta(\lambda_{St}, h_1) = p \geq 2$ by Remark 2.2.2. According to Proposition 2.2.4, the space

$$C_2(\Sigma_1, \Sigma_2; \lambda_{St}, \lambda_2),$$

is $SU(2)$-abelian if and only if it is a lens space. This happens if and only if

$$\lambda_{St} = nx_1 + mh_1 \in \pi_1(\Sigma_1) \quad \text{and} \quad \lambda_2 = x_2 \in \pi_1(\Sigma_2), \tag{2.2.3}$$

where $|n| = \Delta(\lambda_{St}, h_1) = p$ and $|m| = 1$. We assume the two identities in (2.2.3). For $i \in \{1, 2\}$, we consider the space $\mathcal{R}_{U(1)}(\Sigma_i)$ with coordinates (θ_i, ψ_i) as above. According to (2.2.3) and Corollary 2.1.12,

$$T(St, \partial St) = A(St) = \{n\theta_1 + m\psi_1 = 0\} \subseteq \mathcal{R}_{U(1)}(\Sigma_1).$$

Figure 2.2b shows part of $T(Y_1, \partial Y_1)$, as a subset of $\mathcal{R}_{U(1)}(\Sigma_2)$ is the case $n = -2$. Figure 2.2b is obtained by applying Theorem 2.2.6 to $A(St)$ for $n = -2$. Therefore, by Theorem 2.2.6, we obtain that for every $|n| \geq 2$

$$A(Y_1) = \{\psi_2 = 0\} \cup \{\psi_2 = \pi\} \quad \text{and} \quad \{\psi_2 = \pi\} \setminus \{(0, \pi), (\pi, \pi)\} \subseteq H(Y_1) \tag{2.2.4}$$

Claim 1. Every central representation $\eta\colon \pi_1(\Sigma_2) \to \mathcal{Z}(SU(2))$ is in $P(Y_1)$.

Proof. We consider $\pi_1(Y_1)$ presented as

$$\pi_1(Y_1) = \langle x, y, z, h | [x, h], [y, h], y^p h^m, zhz^{-1}h, xyz^2\rangle.$$

As before, the element $h \in \pi_1(Y_1)$ is the homotopy class of the regular fibre of Y_1. We recall that $\pi_1(\partial Y_1)$ is generated by the set $\{x, h\} \subset \pi_1(Y_1)$. Let $\eta \in \mathcal{R}_{U(1)}(\Sigma_2)$ be a representation of coordinates $(\pi\varepsilon_1, \pi\varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. Clearly, $\operatorname{Im}\eta \subset \mathcal{Z}(SU(2))$. We define $\rho\colon \pi_1(Y_1) \to SU(2)$ as

$$\rho(x) = e^{i\varepsilon_1\pi}, \quad \rho(h) = e^{i\varepsilon_2\pi}, \quad \rho(y) = e^{-\frac{i\varepsilon_2 m\pi}{p} + \frac{i2\pi}{p}}, \quad \text{and} \quad \rho(z) = e^{-i\frac{\varepsilon_1\pi}{2} + i\frac{\varepsilon_2 m\pi}{2p} - i\frac{\pi}{p}}.$$

We prove now that ρ is a representation by checking that the relators map to the identity. We start by noticing that $\rho(x), \rho(y), \rho(h), \rho(z) \in U(1)$, that is an abelian group. Therefore $\rho[y, h] = \rho[x, h] = 1$. Similarly, $\rho(z)\rho(h)\rho(z)^{-1}\rho(h) = \rho(h)^2 = 1$. Moreover,

$$\rho(y)^p\rho(h)^m = e^{-i\varepsilon_2 m\pi + i2\pi}e^{i\varepsilon_2 m\pi} = e^{i2\pi} = 1.$$

Finally,

$$\rho(x)\rho(y)\rho(z)^2 = e^{i\varepsilon_1\pi}e^{-\frac{i\varepsilon_2 m\pi}{p}+\frac{i2\pi}{p}}e^{-i\varepsilon_1\pi}e^{i\frac{\varepsilon_2 m\pi}{p}-\frac{i2\pi}{p}} = 1.$$

Therefore, ρ is a well-defined representation.

It is clear that the representation ρ is abelian and that

$$\rho|_{\pi_1(\Sigma_2)} \equiv \eta.$$

If $p \geq 3$ then $\rho(y) \notin \mathcal{Z}(SU(2))$ and hence ρ is non-central. Therefore $\eta \in P(Y_1)$.

Let us suppose that $p = 2$. If $\eta(h) = -1$, then $\varepsilon_2 = 1$. This implies that $\rho(y) \notin \mathcal{Z}(SU(2))$ and again ρ is non-central. If $\eta(h) = 1$, then $\varepsilon_2 = 0$. If $\eta(x) = -1$, then $\varepsilon_1 = -1$. In this case $\rho(z) \notin \mathcal{Z}(SU(2))$. This implies that ρ is non-central. It $\eta\colon \pi_1(\Sigma_2) \to \mathcal{Z}(SU(2))$ is trivial, then $\varepsilon_1 = \varepsilon_2 = 0$. This implies that $\rho(xy) = -1$. Thus $\rho(z)^2 = -1$. This implies that ρ is non-central. This concludes the claim. □

The identity (2.2.3) implies that, as a subset of $\mathcal{R}_{U(1)}(\Sigma_2)$,

$$A(Y_2) = \left\{\psi_2 = \frac{2\pi i k}{o_2}\right\}_{k\in\{1,\cdots,o_2\}}. \tag{2.2.5}$$

The (2.2.4) and (2.2.5) imply that if $o_2 \geq 3$, then $H(Y_1) \cap A(Y_2) \neq \emptyset$. Therefore, Y is not $SU(2)$-abelian as an application of Theorem 2.1.8.

Let us suppose that $o_2 = 1$. Thus, the (2.2.5) implies that $A(Y_2) = \{\psi_2 = 0\} \subset \mathcal{R}_{U(1)}(\Sigma_2)$. We remind the reader that, as proven in [Bas25a], $H(Y_2) \subset \mathcal{R}_{U(1)}(\Sigma_2)$ contains a non-trivial path $\gamma\colon (-1,1) \to H(Y_2)$ contained in a straight line of $\mathcal{R}_{U(1)}(\partial Y_1)$ such that

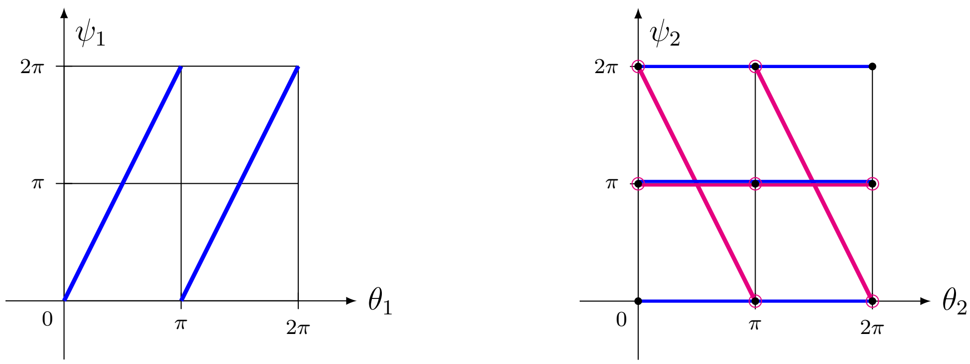
$$\lim_{x\mapsto\pm1} \gamma(x) \in A(Y_2).$$

Let $\Gamma = \operatorname{Im}\gamma$ be the image of this path. Thus, $\Gamma \subseteq H(Y_2)$. Figure 2.2b implies that, if $o_2 = 1$, then Γ intersects either $A(Y_1)$ or $H(Y_1)$. Therefore,

$$H(Y_2) \cap (A(Y_1) \cup H(Y_1)) \neq \emptyset.$$

Theorem 2.1.8 implies the conclusion. Thus, we can suppose that $o_1 = 2$.

By [Bas25a, Corollary 9.6], if a point $\eta \in \{0,\pi\}^2$ is an end point of $H(Y_2)$, i.e. it is contained in $\overline{H(Y_2)} \setminus H(Y_2)$, then $\eta \in P(Y_2)$. This consideration, Figure 2.2b, and Claim 1



(A) $T(St, \partial St) \subset \mathcal{R}_{U(1)}(\Sigma_1)$ for $(n, m) = (2, -1)$.

(B) A subset of $T(Y_1, \partial Y_1) \subset \mathcal{R}_{U(1)}(\Sigma_2)$.

FIGURE 2.2. Part of the proof of Proposition 2.2.7. The sets $A(St)$ and $A(Y_1)$ are in blue. The set $H(Y_1)$ is in red and $P(Y_1)$ is represented by the black dots.

imply that if $o_1 = 2$, then either

$$\Gamma \cap (A(Y_1) \cup H(Y_1)) \neq \emptyset \quad \text{or} \quad P(Y_1) \cap P(Y_2) \neq \emptyset.$$

In both cases, the conclusion is given by Theorem 2.1.8. □

We recall that if Y is a Seifert manifold with a given Seifert fibration which has a non-trivial singular fibre, then we denoted by $\mathcal{O}(Y)$ the vector whose entries are the orders of the singular fibres in ascending order. For instance, if $Y = \mathbb{D}^2(p_1/q_1, \cdots, p_n/q_n)$ with $2 \leq p_1 \leq \cdots \leq p_n$, then $\mathcal{O}(Y) = (p_1, \cdots, p_n)$.

Theorem 2.2.8. *[SZ21, Theorem 1.2] Let Y be a closed Seifert manifold fibred with base space either S^2 or $\mathbb{RP}^2$. Then Y is $SU(2)$-abelian if and only if one of the following holds:*

- *Y is either S^3, a lens space, $S^1 \times S^2$, or $\mathbb{RP}^3 \# \mathbb{RP}^3$;*
- *Y has base space S^2 and $\mathcal{O}(Y) = (2, 4, 4)$,*
- *Y has base space S^2, $\mathcal{O}(Y) = (3, 3, 3)$, and either $|H_1(Y; \mathbb{Z})| = \infty$ or $|H_1(Y; \mathbb{Z})| \equiv_2 0$.*

We define the set S as

$$S \coloneqq \{((2,4,4),1),((3,3,3),1),((4,4),2),((3,3),3),((2,4),4)\} \cup \\ \cup \{((2,\cdots,2,n),0),((n,m),1)\}_{n,m \in \mathbb{N}_{\geq 2}}.$$

Corollary 2.2.9. *Let $Y = Y_1 \cup_\Sigma Y_2$ be an $SU(2)$-abelian graph manifold rational homology sphere with one JSJ torus. The two JSJ pieces of Y admit a fibration whose base space is a* *disk. Furthermore, if Y_2 has at least three singular fibres, then*

$$(\mathcal{O}(Y_2), \Delta(\lambda_1, h_2)) \in S,$$

where $h_2 \subset \Sigma$ is the regular fibre of Y_2.

Proof. The manifolds Y_1 and Y_2 a disk base space as an application of Corollary 2.2.5 and Proposition 2.2.7.

If Y is $SU(2)$-abelian, then $Y_1(\lambda_2)$ and $Y_2(\lambda_1)$ are $SU(2)$-abelian by Corollary 2.1.14. Let us suppose that Y_2 has at least three singular fibres. If $\Delta(\lambda_2, h_1) \geq 1$, then $Y_1(\lambda_2)$ is a Seifert space fibred over S^2. The conclusion holds by Theorem 2.2.8.

If $\Delta(\lambda_2, h_1) = 0$, then

$$\pi_1(Y_1(\lambda_2)) = \frac{\mathbb{Z}}{p_1\mathbb{Z}} * \cdots * \frac{\mathbb{Z}}{p_n\mathbb{Z}}, \tag{2.2.6}$$

where $(p_1, \cdots, p_n) = \mathcal{O}(Y_1)$. We conclude the proof by proving that $\pi_1(Y_2(\lambda_1))$ is $SU(2)$-abelian if and only if $p_1 = \cdots = p_{n-1} = 2$.

Let ρ be an $SU(2)$-representation of $\pi_1(Y_1(\lambda_2))$. If $p_1 = \cdots = p_{n_1} = 2$, then the restriction of ρ to the first $n-1$ components of (2.2.6) has image in $\mathcal{Z}(SU(2))$. Therefore ρ is $SU(2)$-abelian. Conversely, let us suppose that $p_{n-1} \geq 3$ and $p_n \geq 3$. Let

$$X = e^{i\frac{2\pi}{p_{n-1}}} \quad Y = je^{i\frac{2\pi}{p_n}}j^{-1}.$$

The matrices X and Y do not commute. The group in (2.2.6) admits the representation that sends the generator of the $(n-1)^{th}$ component in X and the generator of the n^{th} component in Y. This representation is not $SU(2)$-abelian. ☐

2.3. The graph manifold $Y_1 \cup_\Sigma Y_2$

Let Y be a graph manifold rational homology sphere with a single JSJ torus $\Sigma \subset Y$. Let Y_1 and Y_2 be the two JSJ pieces of Y, in other words Y_1 and Y_2 are the closures in Y of the two components of $Y \setminus \Sigma$. Thus,

$$\overline{Y \setminus \Sigma} = Y_1 \cup Y_2.$$

As a consequence of Corollary 2.2.9, if either Y_1 or Y_2 does not admit a fibration with disk base space, then Y is not $SU(2)$-abelian. Therefore, from now on we assume that the two

JSJ pieces Y_1 and Y_2 are given with a fibation whose base space is the disk $\mathbb{D}^2$. We call $h_1 \subset \partial Y_1$ (resp. $h_2 \subset Y_2$) the regular fibre of Y_1 (resp. Y_2).

The orientation of Y gives an orientation to the manifolds Y_1 and Y_2. Let $\varphi\colon \partial Y_1 \to \partial Y_2$ be a diffeomorphism such that

$$Y = Y_1 \cup_\Sigma Y_2 = Y_1 \cup_\varphi Y_2. \tag{2.3.1}$$

Since Y is an oriented manifold, the diffeomorphism φ is an orientation reversing diffeomorphism. The goal of this thesis is to determine which of these manifolds are $SU(2)$-abelian.

Let us suppose that the manifolds Y_1 and Y_2 are parameterized as

$$Y_1 = \mathbb{D}^2\left(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}\right), \quad Y_2 = \mathbb{D}^2\left(\frac{r_1}{s_1}, \cdots, \frac{r_m}{r_m}\right); \tag{2.3.2}$$

with $\gcd(p_i, q_i) = \gcd(r_i, s_i) = 1$, $p_i \geq 2$, $r_i \geq 2$, $n \geq 2$, and $m \geq 2$. We remind the reader that we can suppose the manifold Y_1 and Y_2 admit a Seifert fibration over the disk as a consequence of Corollary 2.2.9. We further suppose that the orders of the singular fibres are given in ascending order: we assume that

$$p_1 \leq \cdots \leq p_n \quad \text{and} \quad r_1 \leq \cdots \leq r_m.$$

Without loss of generality, we assume $n \leq m$. Therefore the JSJ piece Y_1 is assumed to have a number of singular fibres less than or equal to the number of singular fibres of Y_2.

Let Y_1 be presented as in (2.3.2). As explained in section 1.3, the Seifert coefficients of Y_1 are not unique. In fact, for every $k \in \mathbb{Z}$ and for every $j \in \{1, \cdots, n\}$, there exists a fibre preserving diffeomorphism

$$\mathbb{D}^2\left(\frac{p_1}{q_1}, \cdots, \frac{p_i}{q_i}, \cdots \frac{p_n}{q_n}\right) \longrightarrow \mathbb{D}^2\left(\frac{p_1}{q_1}, \cdots, \frac{p_i}{kp_i + q_i}, \cdots \frac{p_n}{q_n}\right).$$

Therefore, q_i can be chosen to be either in $\{1, \cdots p_i\}$ or positive and odd. As we shall see later, it is algebraically favourable to consider q_i odd. Similar conclusions follow for Y_2 and thus for the coefficients s_j.

The parameterizations in (2.3.2) give the following presentations for the fundamental groups of Y_1 and Y_2:

$$\begin{aligned} \pi_1(Y_1) &= \langle x_1, \cdots, x_n, h_1 \,|\, x_i^{p_i} h_1^{q_i}, [h_1, x_i] \rangle \quad \text{and} \\ \pi_1(Y_2) &= \langle y_2, \cdots, y_m, h_2 \,|\, y_j^{r_j} h_2^{s_j}, [h_2, y_j] \rangle; \end{aligned} \tag{2.3.3}$$

where $[a, b] \coloneqq aba^{-1}b^{-1}$.

Let γ be a simple closed curve in Σ. Recall that we are abusing notation by considering γ as either the curve in Σ or its homotopy class in $\pi_1(\Sigma)$. Therefore, h_1 indicates either a regular fibre of Y_1 in ∂Y_1, its homotopy class in $\pi_1(Y_1)$, or its homotopy class in the fundamental group of the boundary $\pi_1(\partial Y_1)$. More schematically,

$$h_1 \subset \partial Y_1, \quad h_1 \in \pi_1(\partial Y_1), \quad \text{and} \quad h_1 \in \pi_1(Y_1).$$

The same holds for h_2.

Given these presentations, the curves $\mu_1 \subset \partial Y_1$ and $\mu_2 \subset \partial Y_2$ will represent the fibration meridians of Y_1 and Y_2 as in Definition 1.3.2, respectively. We recall that, when μ_1 and μ_2 are considered as an elements of $\pi_1(Y_1)$ and $\pi_1(Y_2)$ respectively, then

$$\mu_1 = x_1 \cdots x_n \quad \text{and} \quad \mu_2 = y_1 \cdots y_m.$$

The groups $\pi_1(\partial Y_1)$ and $\pi_1(\partial Y_2)$ admit the following presentations:

$$\pi_1(\partial Y_1) = \langle \mu_1, h_1 \,|\, [h_1, \mu_1] \rangle, \quad \text{and} \quad \pi_1(\partial Y_2) = \langle \mu_2, h_2 \,|\, [h_2, \mu_2] \rangle.$$

We set the convention that the vectors

$$\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in \mathbb{Z}^2 \cong \pi_1(\partial Y_1)$$

correspond to μ_1 and h_1 respectively. Similarly, the vectors

$$\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in \mathbb{Z}^2 \cong \pi_1(\partial Y_2)$$

correspond to μ_2 and h_2. We will use the integer matrix

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{with} \quad \alpha\delta - \beta\gamma = -1 \tag{2.3.4}$$

to represent the map $\varphi_*\colon \pi_1(\partial Y_1) \to \pi_1(\partial Y_2)$ with respect to these ordered bases. In particular, we have that $\Delta(h_1, h_2) = |\beta|$. This gives an explicit presentation for the fundamental group of $Y = Y_1 \cup_\varphi Y_2$:

$$\pi_1(Y) = \frac{\pi_1(Y_1) * \pi_1(Y_2)}{\langle\langle \mu_1 = \mu_2^\alpha h_2^\gamma, h_1 = \mu_2^\beta h_2^\delta \rangle\rangle} \tag{2.3.5}$$

$$\begin{aligned} &= \langle x_1, x_n, y_1, b_m, h_1, h_2 \,|\, x_i^{p_i} h_1^{q_i}, [h_1, x_i], i \in \{1, \cdots, n\}, \\ &\qquad y_j^{r_j} h_2^{s_j}, [h_2, y_j], j \in \{1, \cdots, m\}, \\ &\qquad x_1 \cdots x_n = (y_1 \cdots y_n)^\alpha h_2^\gamma, h_1 = (y_1 \cdots y_n)^\beta h_2^\delta \rangle. \end{aligned} \tag{2.3.6}$$

If the diffeomorphism $\varphi : \partial Y_1 \to \partial Y_2$ is such that $\Delta(h_1, h_2) = \beta = 0$, then the matrix φ_* is lower triangular. Thus, we have that $\varphi(h_1) = h_2$. This means the fibration of Y_1 can be extended over Y_2. Therefore, the manifold $Y = Y_1 \cup_\varphi Y_2$ is a closed Seifert fibred manifold. In particular, Y is fibred over S^2 and it has at least four singular fibres. Therefore, Y is not $SU(2)$-abelian by Theorem 2.2.8. Thus, we assume that $|\beta| \geq 1$.

The presentations in (2.3.3) implies that the first homology groups $H_1(Y_1; \mathbb{Z})$ and $H_1(Y_2; \mathbb{Z})$ are presented as $\mathbb{Z}$-module as

$$H_1(Y_1; \mathbb{Z}) = \text{coker} \begin{bmatrix} p_1 & 0 & 0 & \cdots & 0 \\ 0 & p_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & p_m \\ q_1 & q_2 & \cdots & q_{m-1} & q_m \end{bmatrix}$$

and

$$H_1(Y_2; \mathbb{Z}) = \text{coker} \begin{bmatrix} r_1 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & r_m \\ s_1 & s_2 & \cdots & s_{m-1} & s_m \end{bmatrix}$$

We define t_1 as cardinality of torsion subgroup of $H_1(Y_1;\mathbb{Z})$. We define t_2 similarly. It is easy to see if $n=2$, then $t_1=\gcd(p_1,p_2)$. Similarly, if $m=2$, then $t_2=\gcd(r_1,r_2)$.

Notation. *Let us summarize here the objects we have introduced and the notation used:*

- *Y is a graph manifold rational homology 3-sphere with a single JSJ torus Σ;*
- *Y_1 and Y_2 are the two JSJ pieces of Y;*
- *For $i\in\{1,2\}$, the manifold Y_i admits a fibration over a disk with at least two cone points. They are presented as in 2.3.2;*
- *Let Y_1 and Y_2 be presented as in 2.3.2, the q_i and s_j can be chosen either positive and smaller than p_i and r_i respectively or positive and odd;*
- *Y_1 has n singular fibres and Y_2 has m singular fibres. We assume $2\le n\le m$;*
- *$\lambda_1\subset\partial Y_1$ and $\lambda_2\subset\partial Y_2$ are the rational longitudes of Y_1 and Y_2 respectively;*
- *Similarly, $h_1\subset\partial Y_1$ and $h_2\subset\partial Y_2$ are the regular fibres of Y_1 and Y_2;*
- *$o_1\in\mathbb{N}$ and $o_2\in\mathbb{N}$ are the orders of the rational longitudes λ_1 and λ_2 in corresponding first homology groups $H_1(Y_1;\mathbb{Z})$ and $H_1(Y_2;\mathbb{Z})$;*
- *$\mu_1\subset\partial Y_1$ and $\mu_2\subset Y_2$ are the fibration meridians as in Definition 1.3.2;*
- *for $i\in\{1,2\}$, we consider the ordered basis $\{\mu_i,h_i\}$ for the group $\pi_1(\partial Y_i)$ and we parameterize the space $\mathcal{R}_{U(1)}(\partial Y_i)$ according to this basis as in (2.3.7);*
- *$t_1\in\mathbb{N}$ and $t_2\in\mathbb{N}$ are the orders of the torsion subgroups of $H_1(Y_1;\mathbb{Z})$ and $H_1(Y_2;\mathbb{Z})$ respectively;*
- *If $n=m=2$, then we assume that $t_1=\gcd(p_1,p_2)\le t_2=\gcd(r_1,r_2)$.*

We recall that, for the non-central element $z\in SU(2)$, we call Λ_z the centralizer subgroup of z in $SU(2)$.

Lemma 2.3.1. *For Y as above, if $\pi_1(Y)$ admits an irreducible $SU(2)$-representation ρ, then either $\rho(h_1)\in\mathcal{Z}(SU(2))$ or $\rho(h_2)\in\mathcal{Z}(SU(2))$.*

PROOF. We recall that $\Sigma=\partial Y_1=\partial Y_2$. Since the torus Σ contains the regular fibres h_1 and h_2, their homotopy classes commute in $\pi_1(M)$.

Let us suppose by contradiction that $\rho(h_1)\notin\mathcal{Z}(SU(2))$ and $\rho(h_2)\notin\mathcal{Z}(SU(2))$. According to Fact 1.1.3, the centralizers $\Lambda_{\rho(h_1)}$ and $\Lambda_{\rho(h_2)}$ coincide. According to the presentation (2.3.6) we have $[x_i,h_1]=[y_j,h_2]=1$ for $i\in\{1,\cdots,n\}$ and $j\in\{1,\cdots,m\}$, we conclude that $\rho(\pi_1(Y_1)) \subset \Lambda_{\rho(h_1)}$ and $\rho(\pi_1(Y_2)) \subset \Lambda_{\rho(h_2)}$. Then,

$$\rho(x_i), \rho(y_j), \rho(h_1), \rho(h_2) \in \Lambda_{\rho(h_1)} = \Lambda_{\rho(h_2)}.$$

This implies that ρ is abelian, which is a contradiction. □

We recall that $\mathcal{R}_{U(1)}(\partial Y_1)$ is a torus, as shown in (2.1.1). We choose $\{\mu_1, h_1\}$ as an ordered basis for $\pi_1(\partial Y_1)$ as we give the space $\mathcal{R}_{U(1)}(\partial Y_1)$ coordinates (θ_1, ψ_1) according to this ordered basis. Explicitly, the point $(\theta_1, \psi_1) \in [0, 2\pi]^2/_\sim$ is associated to the unique representation

$$\pi_1(\partial Y_1) \to U(1), \quad \text{with} \quad \mu_1 \mapsto e^{i\theta_1} = \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{bmatrix} \quad \text{and} \quad h_1 \mapsto e^{\psi_1} = \begin{bmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_1} \end{bmatrix}. \tag{2.3.7}$$

From now on, the sets $H(Y_i)$, $A(Y_i)$, and $P(Y_i)$, for $i \in \{1, 2\}$ of Definition 2.1.5 are denoted by H_i, A_i, and P_i. Figure 2.3 shows the spaces $T(Y_1, \partial Y_1)$ for Y_1 homeomorphic to

$$\mathbb{D}^2(2/1, 2/1), \quad \mathbb{D}^2(2/1, 4/1), \quad \mathbb{D}^2(3/1, 3/1), \quad \mathbb{D}^2(4/1, 4/1),$$
$$\mathbb{D}^2(2/1, 4/1, 4/1), \quad \text{or} \quad \mathbb{D}^2(3/1, 3/1, 3/2).$$

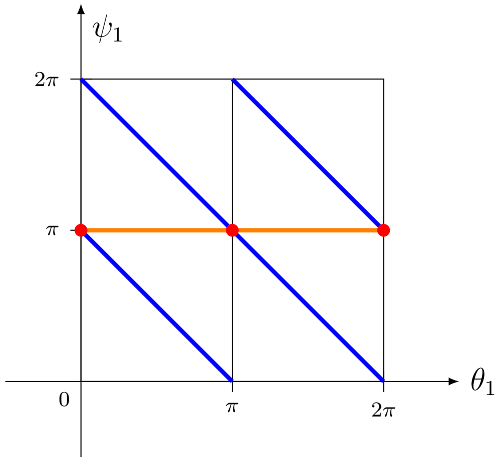
In Figure 2.3 the tori $\mathcal{R}_{U(1)}(\partial Y_1)$ are parameterized with coordinates (θ_1, ψ_1) as above. Since P_1 consists of central representations, it is contained in $\{0, \pi\}^2 \subset \mathcal{R}_{U(1)}(\partial Y_1)$. According to Fact 1.1.3, if the representation $\rho \in \mathcal{R}(Y)$ is such that $\rho|_{\pi_1(\partial Y_1)} \in H_1$, then $\rho(h_1) \in \mathcal{Z}(SU(2))$. This implies that H_1 is contained in the "horizontal" lines $\{\psi_1 \equiv_\pi 0\} \subset \mathcal{R}_{U(1)}(\partial Y_1)$. That is why we chose the notation as $H(Y)$. For explicit descriptions of the set A_1 and H_1, we refer the reader to the next chapter.

Here is an example of how Theorem 2.1.8 can be applied to determinate the $SU(2)$-abelian status of a 3-manifold.

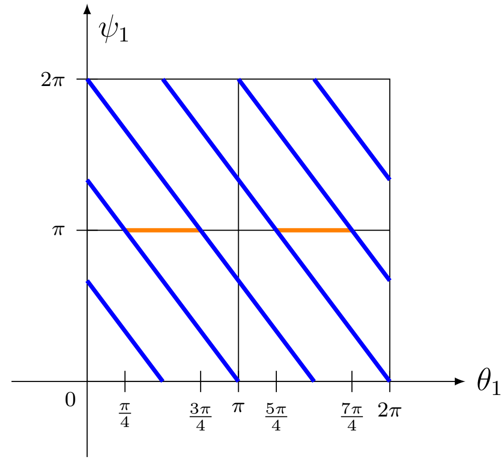
Example 2.3.2. Let $Y_1 = Y_2 = S^3 \backslash \nu(T_{2,3})$, where $T_{2,3}$ is the trefoil in S^3 and $\nu(T_{2,3}) \subset S^3$ is an open tubular neighborhood of the knot $T_{2,3}$. It is known that

$$Y_1 = Y_2 = \mathbb{D}^2\left(\frac{2}{-1}, \frac{3}{1}\right).$$

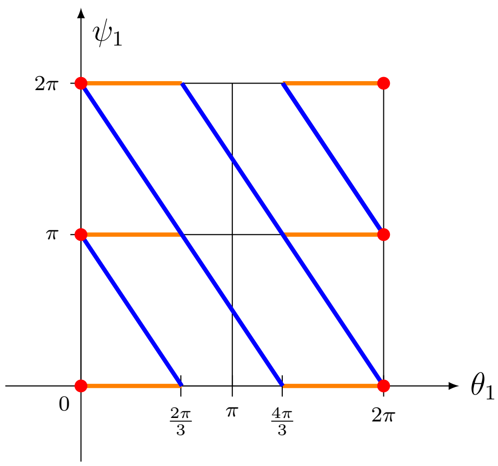
Let Y be the manifold obtained by gluing Y_1 and Y_2 along a diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$ that sends a knot meridian of Y_1 into a regular fibre of Y_2 and a regular fibre of Y_1 into a



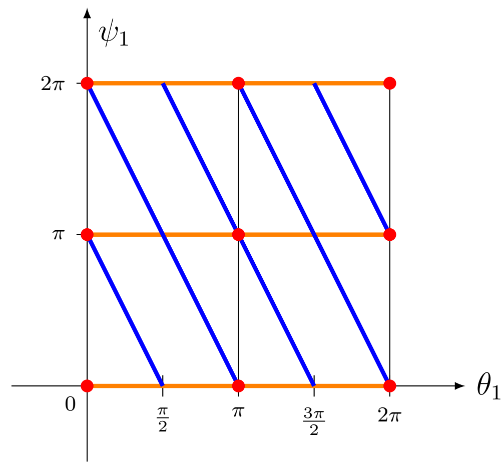
(A) $Y_1 = \mathbb{D}^2(2/1, 2/1)$



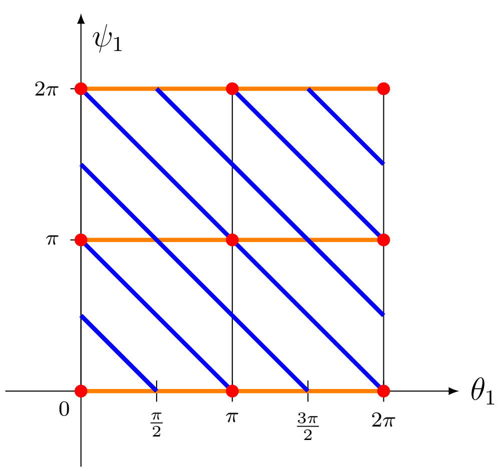
(B) $Y_1 = \mathbb{D}^2(2/1, 4/1)$



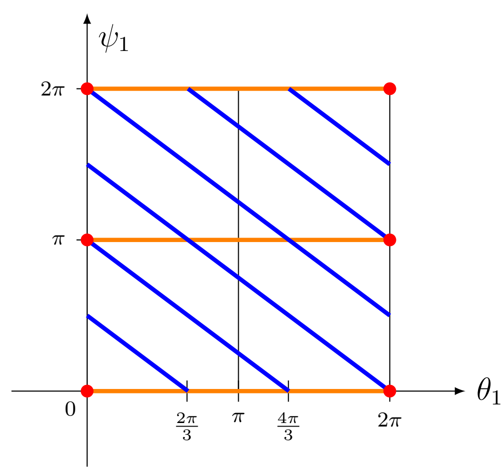
(C) $Y_1 = \mathbb{D}^2(3/1, 3/1)$



(D) $Y_1 = \mathbb{D}^2(4/1, 4/1)$



(E) $Y_1 = \mathbb{D}^2(2/1, 4/1, 4/1)$



(F) $Y_1 = \mathbb{D}^2(3/1, 3/1, 3/2)$

FIGURE 2.3. Six examples of $T(Y_1, \partial Y_1) \subset \mathcal{R}_{U(1)}(\partial Y_1)$. The set H_1, A_1, and P_1 are respectively in orange, blue, and red.

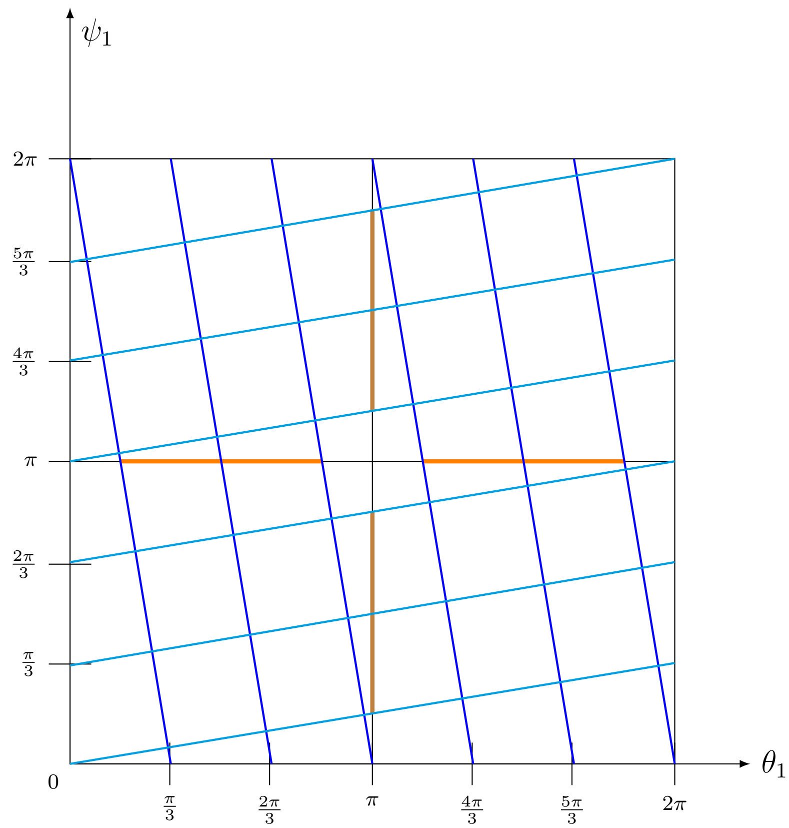


FIGURE 2.4. The intersection $T(Y_1, \partial Y_1) \cap T(M_2, \partial M_2)$ as in Example 2.3.2.

knot meridian of Y_2. With respect to our bases,

$$\varphi_* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let Σ be the torus in Y corresponding to $\partial Y_1 = \partial Y_2$. We remind the reader that the rational longitude of a knot exterior is the null-homologous longitude of the knot. This implies that the rational longitudes of Y_1 and Y_2 both have order 1. Hence $o_1 = o_2 = 1$. A famous construction in [Rol03, p. 327] shows that, as slopes in ∂Y_1 we have that $6\mu_1 - \lambda_1 = h_1$. Hence $\lambda_1 = 6\mu_1 - h_1$. Similarly, $\lambda_2 = 6\mu_2 - h_2$. In particular

$$\rho(\lambda_1) = \rho(\mu_1)^6 \rho(h_1)^{-1} = e^{i(6\theta_1 - \psi_1)}$$

As a consequence of Corollary 2.1.12

$$A_1 = \left\{\eta \in \mathcal{R}_{U(1)}(\Sigma) \middle| \eta(\lambda_1) = 1\right\} = \left\{(\theta_1, \psi_1) \in \mathcal{R}_{U(1)}(\Sigma) \middle| 6\theta_1 - \psi_1 = 0\right\}$$

Similarly,

$$\rho(\lambda_2) = \rho(\mu_2)^6 \rho(h_2)^{-1} = \rho(h_1)^6 \rho(\mu_1)^{-1} = e^{i(6\psi_1 - \theta_1)}$$

This implies that

$$A_2 = \left\{\eta \in \mathcal{R}_{U(1)}(\Sigma) \middle| \eta(\lambda_2) = 1\right\} = \left\{(\theta_1, \psi_1) \in \mathcal{R}_{U(1)}(\Sigma) \middle| \theta_1 - 6\psi_1 = 0\right\}$$

Figure 2.4 exhibits the sets $T(Y_1, \Sigma)$ and $T(Y_2, \Sigma)$ as a subsets of $\mathcal{R}_{U(1)}(\Sigma)$. This latter comes with coordinates (θ_1, ψ_1) as in (2.3.7). In particular, H_1 and H_2 are in orange and brown, A_1 and A_2 are in blue and light blue.

For explicit descriptions of the set H_1 and H_2, we refer the reader to the next chapter. Let $\rho_1 \colon \pi_1(Y_1) \to SU(2)$ be a representation such that $\rho_1(\pi_1(\Sigma)) \subset \mathcal{Z}(SU(2))$. This implies that $\rho_1(\mu) = \pm 1$, where $\mu \in \pi_1(Y_1)$ is the homotopy class of the knot meridian. By the Wirtinger presentation, see [Rol03, Section 3.D], we known that the fundamental group of $\pi_1(Y_1) = \pi_1(S^3 \setminus T_{2,3})$ is normally generated by the meridian: the group is generated by the set

$$\{g\mu g^{-1}\}_{g \in \pi_1(Y_1)}.$$

Hence, if $\rho_1(\mu) = \pm 1$, then the image of ρ_1 is contained in the center $\mathcal{Z}(SU(2))$. This implies that $P_1 = \emptyset$. Therefore, $P_1 \cap P_2 = \emptyset$. Figure 2.4 implies that

$$T(Y_1, \Sigma) \cap T(Y_2, \Sigma) = (A_1 \cap A_2) \subset \mathcal{R}_{U(1)}(\Sigma).$$

In particular, we obtain that $H_1 \cap H_2$, $A_1 \cap H_2$, $H_1 \cap A_2$, and $P_1 \cap P_2$ are empty. Theorem 2.1.8 implies that Y is $SU(2)$-abelian. This manifold was known to be $SU(2)$-abelian after [Mot88].

Chapter 3

The Ingredients

In this chapter we shall describe the sets A_i, H_i, and P_i of Definition 2.1.5. This will enable us to compute the intersections

$$A_1 \cap H_2, \quad H_1 \cap A_2, \quad H_1 \cap H_2, \quad \text{and} \quad P_1 \cap P_2,$$

as required by Theorem 2.1.8.

Corollary 2.2.9 implies that we have to study A_i, H_i, and P_i only for the following classes of manifolds:

$$\mathbb{D}^2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right), \quad \mathbb{D}^2\left(\frac{3}{q_1}, \frac{3}{q_2}, \frac{3}{q_3}\right), \quad \mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}, \frac{4}{q_3}\right), \quad \text{and} \quad \mathbb{D}^2\left(\frac{2}{1}, \cdots, \frac{2}{1}, \frac{p_n}{q_n}\right)$$

The sets of Definition 2.1.5 for the fist class of manifolds are exhaustively studied in [Bas25b]. We remind the reader that the space $\mathcal{R}_{U(1)}(\partial Y_1)$ is taken with coordinates (θ_1, ψ_1) according to the ordered basis $\{\mu_1, h_1\}$ as in (2.3.7).

3.1. Abelian representations and the set A_1.

Corollary 2.1.12 implies that the set $A_1 \subset T(Y_1, \partial Y_1)$ is equal to

$$A_1 = \left\{\eta \in \mathcal{R}_{U(1)}(\partial Y_1) \,\middle|\, \eta(\lambda_1)^{o_1} = 1\right\}. \tag{3.1.1}$$

Hence, to describe the set A_1 with coordinates (θ_1, ψ_1) it suffices to write the rational longitude $\lambda_1 \in \pi_1(\partial Y_1)$ in terms of the chosen basis $\{\mu_1, h_1\}$ and to determine its order.

Lemma 3.1.1. *Let $Y_1 = \mathbb{D}^2({}^{p_1}\!/_{q_1}, {}^{p_2}\!/_{q_2})$ and let us suppose that $\pi_1(Y_1)$ is presented as in* (2.3.3)*. Let $\mu_1 \subset \partial Y_1$ be the fibration meridian as in Definition 1.3.2 and $h_1 \subset \partial Y_1$ a regular* *fibre. Let* $t_1 = \gcd(p_1, p_2)$ *and* o_1 *be the order of the rational longitude of* Y_1 *in* $H_1(Y_1;\mathbb{Z})$. *Then*

$$\lambda_1 \coloneqq \left(\frac{p_1p_2}{o_1t_1}\right)\mu_1 + \left(\frac{p_1q_2+p_2q_1}{o_1t_1}\right)h_1 \subset \partial Y_1$$

is the rational longitude of Y_1. *Furthermore,* $o_1 = \gcd\left(\frac{p_1p_2}{t_1}, \frac{p_1q_2+p_2q_1}{t_1}\right)$ *and* o_1 *divides* t_1.

Proof. As we mentioned in the previous chapter, the $\mathbb{Z}$-module $H_1(Y_1;\mathbb{Z})$ is presented as

$$H_1(Y_1;\mathbb{Z}) = \operatorname{coker}\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \\ q_1 & q_2 \end{bmatrix}.$$

Therefore, the $\mathbb{Z}$-module $H_1(Y_1;\mathbb{Z})$ is generated by the set $\{x_1, x_2, h_1\}$ and related by the equations

$$p_1x_1 + q_2h_1 = 0 \quad \text{and} \quad p_2x_2 + q_2h_1 = 0. \tag{3.1.2}$$

This presentation implies that $H_1(Y_1;\mathbb{Z}) = \mathbb{Z}\times{}^{\mathbb{Z}}/_{t_1\mathbb{Z}}$. Since the element λ_1 is a torsion element of $H_1(Y_1;\mathbb{Z})$ by definition, its order o_1 divides t_1.

We remind the reader that $\mu_1 = x_1x_2 \in \pi_1(\partial Y_1)$. Thus, $[\mu_1] = x_1 + x_2 \in H_1(Y_1;\mathbb{Z})$. In particular, the additivite group $H_1(\partial Y_1;\mathbb{Z})$ is generated by $\{x_1+x_2, h_1\} \subseteq H_1(\partial Y_1;\mathbb{Z})$. Let $\alpha \in H_1(\partial Y_1;\mathbb{Z})$ be

$$\alpha = \frac{p_1p_2}{t_1}(x_1+x_2) + \left(\frac{p_1q_2+p_2q_1}{t_1}\right)h_1 \in H_1(\partial Y_1;\mathbb{Z}).$$

The relations (3.1.2) imply that the element α is null-homologous in $H_1(Y_1;\mathbb{Z})$. Since

$$\gcd\left(\frac{p_1p_2}{t_1k}, \frac{p_1q_2+p_2q_1}{t_1k}\right) = 1 \quad \text{where} \quad k = \gcd\left(\frac{p_1p_2}{t_1}, \frac{p_1q_2+p_2q_1}{t_1}\right)$$

then the following is a simple closed curve on ∂Y_1:

$$\lambda_1 \coloneqq \left(\frac{p_1p_2}{t_1k}\right)\mu_1 + \left(\frac{p_1q_2+p_2q_1}{t_1k}\right)h_1.$$

Clearly $\alpha = k\cdot\lambda_1$. Therefore, $k\cdot\lambda_1$ is null-homologous in $H_1(Y_1;\mathbb{Z})$. Since, as we have just proven, the slope λ_1 is a torsion element of $H_1(Y_1;\mathbb{Z})$, this is the rational longitude of Y_1.

We prove now the expression for the order o_1. Since $o_1 \cdot \lambda_1$ is trivial in $H_1(Y_1;\mathbb{Z})$, there exist two integers n and m such that the following identities hold in $H_1(Y_1;\mathbb{Z})$:

$$o_1\left(\frac{p_1p_2}{t_1k}\right)(x_1+x_2)+o_1\left(\frac{p_1q_2+p_2q_1}{t_1k}\right)h_1 = o_1\cdot\lambda_1 = n(p_1x_1+q_1h_1)+m(p_2x_2+q_2h_1). \quad (3.1.3)$$

Thus, we obtain that

$$o_1\left(\frac{p_1p_2}{t_1k}\right) = np_1 = mp_2.$$

Therefore, there exists a $j \in \mathbb{Z}_{\neq 0}$ such that

$$j\,\mathrm{lcm}(p_1,p_2) = o_1\left(\frac{p_1p_2}{t_1k}\right).$$

Since $\mathrm{lcm}(p_1,p_2) = \frac{p_1p_2}{t_1}$, we obtain that

$$j\left(\frac{p_1p_2}{t_1}\right) = o_1\left(\frac{p_1p_2}{t_1k}\right) = np_1 = mp_2.$$

This latter implies that $n = \frac{jp_2}{t_1}$ and $m = \frac{jp_1}{t_1}$. Hence, the (3.1.3) becomes

$$o_1\cdot\lambda_1 = \frac{jp_2}{t_1}(p_1x_1+q_1h_1)+\frac{jp_1}{t_1}(p_2a_2+q_2h_1) = j\left(\frac{p_2}{t_1}(p_1x_1+q_1h_1)+\frac{p_1}{t_1}(p_2a_2+q_2h_1)\right).$$

The presentation of the first homology implies that the element

$$\frac{p_2}{t_1}(p_1x_1+q_1h_1)+\frac{p_1}{t_1}(p_2a_2+q_2h_1) \in H_1(Y_1;\mathbb{Z})$$

is trivial. The minimality of o_1 implies $j=\pm 1$. Hence,

$$\frac{p_1p_2}{t_1} = o_1\left(\frac{p_1p_2}{t_1k}\right).$$

This implies that $o_1 = k = \gcd\left(\frac{p_1p_2}{t_1}, \frac{p_1q_2+p_2q_1}{t_1}\right)$. □

Lemma 3.1.2. *Let $Y_1 = \mathbb{D}^2(3/q_1, 3/q_2, 3/q_3)$ and let us suppose that $\pi_1(Y_1)$ is presented as in (2.3.3). Let $\mu_1 \subset \partial Y_1$ be the fibration meridian as in Definition 1.3.2 and $h_1 \subset \partial Y_1$ a regular fibre. Let o_1 be the order of the rational longitude λ_1 of Y_1. Then*

$$o_1 = \gcd(3, q_1+q_2+q_3) \quad \textit{and} \quad \lambda_1 = \frac{3}{o_1}\mu_1 + \frac{q_1+q_2+q_3}{o_1}h_1.$$

PROOF. The proof follows with the same strategy of Lemma 3.1.1. As in Lemma 3.1.1, the $\mathbb{Z}$-module $H_1(Y_1;\mathbb{Z})$ is presented as

$$H_1(Y_1;\mathbb{Z}) = \operatorname{coker}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \\ q_1 & q_2 & q_3 \end{bmatrix}.$$

The module above is generated by $\{x_1, x_2, x_3, h_1\}$. The module $H_1(\partial Y_1;\mathbb{Z})$ is generated by the basis $\{x_1 + x_2 + x_3, h_1\}$. Let $\alpha \in H_1(Y_1;\mathbb{Z})$ be

$$\alpha = 3(x_1 + x_2 + x_3) + (q_1 + q_2 + q_3)h_1.$$

Clearly, the element α is nullhomologous in $H_1(Y_1;\mathbb{Z})$. As before, the following is a slope in $H_1(\partial Y_1;\mathbb{Z})$

$$\lambda_1 = \frac{3}{k}\mu_1 + \frac{q_1 + q_2 + q_3}{k}h_1 \subset \partial Y_1 \quad \text{with} \quad k = \gcd(3, q_1 + q_2 + q_3).$$

We recall that $\mu_1 = x_1 + x_2 + x_3$ in $H_1(Y_1;\mathbb{Z})$. Since $k \cdot \lambda_1 = \alpha$ is trivial, the slope λ_1 is torsion in $H_1(Y_1;\mathbb{Z})$. This implies that λ_1 as above is the rational longitude of Y_1.

We prove now that $o_1 = k$. Since k divides 3, we get that $k \in \{1, 3\}$. If $k = \gcd(3, q_1+q_2+q_3) = 1$, then λ_1 is nullhomologous in $H_1(Y_1;\mathbb{Z})$. Therefore, $o_1 = 1 = k$. If $\gcd(3, q_1+q_2+q_3) = 3$, then

$$\lambda_1 = \mu_1 + \frac{q_1 + q_2 + q_3}{3}h_1$$

is not nullhomologous. However $3 \cdot \lambda_1 = \alpha$ is trivial in $H_1(Y_1;\mathbb{Z})$. Therefore o_1 divides 3 and cannot be 1. We conclude that $o_1 = 3 = k$. $\square$

The following two lemmata follow exactly as Lemma 3.1.1 and Lemma 3.1.2. Therefore, we give the statements, but the proofs will be left to the willing reader as an exercise.

Lemma 3.1.3. *Let $Y_1 = \mathbb{D}^2(2/1, 4/q_2, 4/q_3)$ and let us suppose that $\pi_1(Y_1)$ is presented as in* (2.3.3). *Let $\mu_1 \subset \partial Y_1$ be the fibration meridian as in Definition 1.3.2 and $h_1 \subset \partial Y_1$ a regular fibre. Let o_1 be the order of the rational longitude λ_1 of Y_1. Then*

$$o_1 = \gcd(4, 2 + q_2 + q_3) \quad \textit{and} \quad \lambda_1 = \frac{4}{o_1}\mu_1 + \frac{2 + q_2 + q_3}{o_1}h_1.$$

Lemma 3.1.4. *Let $Y_1 = \mathbb{D}^2(2/1, 2/1, \cdots, p_n/q_n)$ and let us suppose that $\pi_1(Y_1)$ is presented as in (2.3.3). Let $\mu_1 \subset \partial Y_1$ be the fibration meridian as in Definition 1.3.2 and $h_1 \subset \partial Y_1$ a regular fibre. Let o_1 be the order of the rational longitude λ_1 of Y_1. Let $g = \gcd(2, p_n)$, then*

$$o_1 = \gcd\left(\frac{2p_n}{g}, \frac{(n-1)p_n + 2q_n}{g}\right) \quad and \quad \lambda_1 = \frac{2p_n}{o_1 g}\mu_1 + \frac{(n-1)p_n + 2q_n}{o_1 g}h_1.$$

3.2. Central representations and the set P_1

In this section we give a description of the set $P_1 \subset \mathcal{R}_{U(1)}(\partial Y_1)$. Let us assume that $Y_1 = \mathbb{D}^2(p_1/q_1, \cdots, p_n/q_n)$. The diffeomorphism in (1.3.3) implies that all q_i can be chosen to be odd. In this section we consider $\pi_1(Y_1)$ to be presented as in (2.3.3) with all q_i odd.

We consider $\pi_1(\partial Y_1)$ generated by the usual basis $\{\mu_1, h_1\}$ where $\mu_1 = x_1 \cdots x_n$ when it is considered as an element of $\pi_1(Y_1)$.

In the first part of the section we study the problem for $n = 2$, and thus for $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$. In the second part we extend the results to the manifolds

$$\mathbb{D}^2\left(\frac{3}{q_1}, \frac{3}{q_2}, \frac{3}{q_3}\right), \quad \mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}, \frac{4}{q_3}\right), \quad \text{and} \quad \mathbb{D}^2\left(\frac{2}{1}, \cdots, \frac{2}{1}, \frac{p_n}{q_n}\right).$$

Let us suppose that $n = 2$, therefore $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$. We remind the reader that

$$\mathcal{R}(Y_1) \coloneqq \mathrm{Hom}(\pi_1(Y_1); SU(2)).$$

If the representation $\rho \in \mathcal{R}(Y_1)$ is such that the restriction $\rho|_{\pi_1(\partial Y_1)}$ is central, then

$$\rho(\mu_1) = \rho(x_1 x_2) = \pm 1.$$

The latter implies that $\rho(x_1)$ commutes with $\rho(x_2)$, and hence, since x_1 and x_2 commute with h_1 in $\pi_1(Y_1)$, therefore ρ is an abelian representation by Fact 1.1.3. We remind the reader that t_1 is the order of the torsion subgroup of $H_1(Y_1; \mathbb{Z})$. If $n = 2$, then $t_1 = gcd(p_1, p_2)$.

Lemma 3.2.1. *Let $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$. Let $\eta \colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ be a representation such that $\eta(h_1) = -1$ and let $t_1 \equiv_2 0$. If η extends to a representation $\pi_1(Y_1) \to SU(2)$, then every such extension is non-central.*

Proof. Suppose that the representation η extends to a central representation $\rho \colon \pi_1(Y_1) \to SU(2)$. This implies that $\rho(x_1)$ and $\rho(x_2)$ are both in $\mathcal{Z}(SU(2))$ and $\rho(h_1) = \eta(h_1) = -1$.

Since p_1 is even and q_1 is odd, we obtain that

$$1 = \rho(x_1)^{p_1}\rho(h_1)^{q_1} = (\pm 1)^{p_1}(-1)^{q_1} = -1.$$

This is a contradiction. ☐

Lemma 3.2.2. *Let $Y_1 = \mathbb{D}^2({p_1}/{q_1}, {p_2}/{q_2})$. If $t_1 \geq 3$, then the trivial representation $\eta\colon \pi_1(\partial Y_1) \to 1$ extends to a non-central representation $\pi_1(Y_1) \to SU(2)$.*

Proof. Let $\mathcal{A}$ be the abelianization homomorphism and G the abelian group

$$G = \mathcal{A}\left(\frac{\pi_1(Y_1)}{\langle\langle x_1x_2, h_1\rangle\rangle}\right).$$

We recall that $\langle\langle S\rangle\rangle$ denotes the smallest normal subgroup containing the set $S \subset \pi_1(Y_1)$. The presentation (2.3.3) of $\pi_1(Y_1)$ implies that G is isomorphic to ${\mathbb{Z}}/{t_1\mathbb{Z}}$. Let $q : \pi_1(Y_1) \twoheadrightarrow G$ be the quotient map. Since $t_1 \geq 3$, the group G admits a non-central $SU(2)$-representation. Let $\gamma\colon G \to SU(2)$ be a non-central representation, then $\gamma \circ q\colon \pi_1(Y_1) \to SU(2)$ is a non-central $SU(2)$-representation of $\pi_1(Y_1)$. Furthermore, $(\gamma \circ q)|_{\pi_1(\partial Y_1)}$ is the trivial representation η. This implies that the representation η admits a non-central extension. ☐

Lemma 3.2.3. *Let $Y_1 = \mathbb{D}^2({p_1}/{q_1}, {p_2}/{q_2})$. Let $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ be a representation and let $t_1 \geq 3$. If η extends to a representation $\pi_1(Y_1) \to SU(2)$, then it admits a non-central extension.*

Proof. We split the proof in the cases $t_1 \equiv_2 0$ and $t_1 \equiv_2 1$.

Let us suppose that $t_1 \equiv_2 0$. This implies that $t_1 \geq 4$. If $\eta(h_1) = -1$, then Lemma 3.2.1 gives the conclusion. If η is the trivial representation, the conclusion holds by Lemma 3.2.2. Therefore, we prove the remaining case: we suppose that $\eta(x_1x_2) = -1$ and $\eta(h_1) = 1$. Since $p_1 \equiv_2 p_2 \equiv_2 0$, we define a representation $\rho\colon \pi_1(Y_1) \to SU(2)$ as

$$\rho(x_1) = \begin{bmatrix} e^{\frac{2\pi i}{t_1}} & 0 \\ 0 & e^{-\frac{2\pi i}{t_1}} \end{bmatrix}, \quad \rho(x_2) = -\begin{bmatrix} e^{-\frac{2\pi i}{t_1}} & 0 \\ 0 & e^{\frac{2\pi i}{t_1}} \end{bmatrix}, \quad \text{and} \quad \rho(h_1) = 1.$$

Since $t_1 \geq 4$, such a representation ρ is non-central and it restricts to η.

Let us suppose that $t_1 \equiv_2 1$ and $t_1 \geq 3$. As a consequence of Lemma 3.1.1, the order o_1 is odd. Since t_1, q_1, and q_2 are odd, we obtain that

$$\frac{p_1 p_2}{t_1} \equiv_2 p_1 p_2 \quad \text{and} \quad \frac{p_1 q_2 + p_2 q_1}{t_1} \equiv_2 p_1 + p_2.$$

Let $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ be a representation that extends to $\pi_1(Y_1)$. As we said before, every such an extension is abelian. By Corollary 2.1.12, we obtain that $\eta(\lambda_1)^{o_1} = 1$. According to Lemma 3.1.1, we obtain that

$$1 = \eta(\lambda_1)^{o_1} = \eta(x_1 x_2)^{\frac{p_1 p_2}{t_1}} \eta(h_1)^{\frac{p_1 q_2 + p_2 q_1}{t_1}} = \eta(x_1 x_2)^{p_1 p_2} \eta(h_1)^{p_1 + p_2}. \tag{3.2.1}$$

If $p_1 p_2 \equiv_2 1$, then p_1 and p_2 are both odd and $p_1 + p_2 \equiv_2 0$. Equation (3.2.1) implies that $\eta(x_1 x_2) = 1$. If $\eta(h_1) = 1$, then the conclusion holds by Lemma 3.2.2. Without loss of generality, we can suppose that $\eta(h_1) = -1$. The representation $\rho\colon \pi_1(Y_1) \to SU(2)$ with

$$\rho(x_1) = \begin{bmatrix} e^{i\frac{\pi}{t_1}} & 0 \\ 0 & e^{-i\frac{\pi}{t_1}} \end{bmatrix}, \quad \rho(x_2) = \begin{bmatrix} e^{-i\frac{\pi}{t_1}} & 0 \\ 0 & e^{i\frac{\pi}{t_1}} \end{bmatrix} \quad \text{and} \quad \rho(h_1) = -1,$$

restricts to η. Since $t_1 \geq 3$, the representation ρ is non-central.

If $p_1 p_2 \equiv_2 0$, then $p_1 + p_2 \equiv_2 1$. Equation (3.2.1) implies that $\eta(h_1) = 1$. Again, if $\eta(x_1 x_2) = 1$, then the conclusion is implied by Lemma 3.2.2. Without loss of generality, we can suppose $\eta(x_1 x_2) = -1$. Since o_1 divides t_1 by Lemma 3.1.1, the quantity o_1 is odd. Since $t_1 \equiv_2 1$ and $p_1 p_2 \equiv_2 0$, we can also assume that $p_1 \equiv_2 0$ and $p_2 \equiv_2 1$. Let $t_1 = 2n + 1$ with $n \in \mathbb{N}$. The representation η extends to the representation $\rho\colon \pi_1(Y_1) \to SU(2)$ with

$$\rho(x_1) = \begin{bmatrix} e^{i\frac{\pi}{t_1}} & 0 \\ 0 & e^{-i\frac{\pi}{t_1}} \end{bmatrix}, \quad \rho(x_2) = \begin{bmatrix} e^{i\frac{2\pi n}{t_1}} & 0 \\ 0 & e^{-i\frac{2\pi n}{t_1}} \end{bmatrix}, \quad \text{and} \quad \rho(h_1) = 1.$$

If $t_1 \geq 3$, then ρ has a non-central image. $\square$

Remark 3.2.4 ([Wik25]). Let A be a nonzero $m \times n$ matrix over $\mathbb{Z}$. There exist invertible $m \times m$ and $n \times n$ integer matrices S, T such that the product SAT is

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & \alpha_r & & & \\ 0 & & \cdots & & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \cdots & & 0 & \cdots & 0 \end{pmatrix}.$$

where α_i divides α_{i+1} for all $1 \leq i < r$. The elements α_i are unique up to multiplication by ± 1. They can be computed (up to multiplication by ± 1) as:

$$\alpha_i = \frac{d_i}{d_{i-1}},$$

where d_i equals the greatest common divisor of the determinants of all $i \times i$ minor of the matrix A and $d_0 \coloneqq 1$. In particular,

$$\alpha_0 \alpha_1 \cdots \alpha_n = \frac{\cancel{d_1}}{d_0} \frac{\cancel{d_2}}{\cancel{d_1}} \cdots \frac{\cancel{d_{n-1}}}{\cancel{d_{n-2}}} \frac{d_n}{\cancel{d_{n-1}}} = \frac{d_n}{d_0} = d_n.$$

Lemma 3.2.5. *Let $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$. Let $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ be a representation, then η extends to a representation $\rho\colon \pi_1(Y_1) \to SU(2)$ if and only if $\eta(\lambda_1)^{o_1} = 1$. If this happens, the following hold:*

- *If $t_1 = 1$, then η extends only to central representations;*
- *If $t_1 = 2$, then η extends to a non-central representation if and only if $o_1 = 2$ and $\eta(h_1) = -1$;*
- *If $t_1 \geq 3$, then η extends to a non-central representation.*

Proof. As we stated before, if $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ extends to the representation $\rho\colon \pi_1(Y_1) \to SU(2)$, then ρ has abelian image. Thus, the first part of the statement is a consequence of Corollary 2.1.12. Moreover, if $t_1 \geq 3$, then the conclusion holds by Lemma 3.2.3.

Let $\varepsilon_1, \varepsilon_2 \in \{\pm 1\} = \mathcal{Z}(SU(2))$ and let $\tau_1, \tau_2 \in \{1, 2\}$ be their orders in $SU(2)$. Let $\rho\colon \pi_1(Y_1) \to SU(2)$ be such that $\rho(h_1) = \varepsilon_1$ and $\rho(x_1 x_2) = \varepsilon_2$. Let η be the restriction of ρ

to $\pi_1(\partial Y_1)$. We define the group $F_{\varepsilon_1,\varepsilon_2}$ as the the abelianization of

$$\frac{\pi_1(Y_1)}{\langle\!\langle h_1^{\tau_1}, (x_1x_2)^{\tau_2}\rangle\!\rangle},$$

for the corresponding τ_1 and τ_2. Consequently, the following diagram commutes:

$$\begin{array}{ccccc} & \overset{\eta}{\nearrow} & SU(2) & \overset{\bar{\rho}}{\longleftarrow} & \\ & & \uparrow \rho & & \\ \pi_1(\partial Y_1) & \xrightarrow{\iota_*} & \pi_1(Y_1) & \xrightarrow{\mathcal{F}} & \mathcal{A}\left(\frac{\pi_1(Y_1)}{\langle\!\langle h_1^{\tau_1},(x_1x_2)^{\tau_2}\rangle\!\rangle}\right) =: F_{\varepsilon_1,\varepsilon_2}. \end{array} \tag{3.2.2}$$

Here $\mathcal{F}$ is the quotient map and ι_* is the map induced by the inclusion $\iota : \partial Y_1 \to Y_1$.

Let us suppose that $t_1 = 1$. If η extends to a representation ρ. Diagram 3.2.2 implies that ρ factors through the group

$$F_{\varepsilon_1,\varepsilon_2} = \operatorname{coker}\begin{bmatrix} p_1 & 0 & 0 & \tau_2 \\ 0 & p_2 & 0 & \tau_2 \\ q_1 & q_2 & \tau_1 & 0 \end{bmatrix}.$$

We recall that $d_3(F_{\varepsilon_1,\varepsilon_2})$ is the greater common divisor of the determinant of all 3×3 minor of the matrix $F_{\varepsilon_1,\varepsilon_2}$. We recall that $t_1 = \gcd(p_1, p_2) = 1$. Explicitly,

$$\begin{aligned} d_3(F_{\varepsilon_1,\varepsilon_2}) &= \gcd(\tau_1 p_1 p_2, \tau_2(p_1q_2 + p_2q_1), p_1\tau_1\tau_2, p_2\tau_1\tau_2) \\ &= \gcd(\tau_1 p_1 p_2, \tau_2 \gcd(p_1q_2 + p_2q_1, \tau_1)). \end{aligned}$$

We show now that $d_3(F_{\varepsilon_1,\varepsilon_2}) \in \{1, 2\}$. If $\gcd(p_1q_2 + p_2q_1, \tau_1) = 1$, then $d_3(F_{\varepsilon_1,\varepsilon_2}) = \gcd(\tau_1 p_1 p_2, \tau_2)$. Since $\tau_2 \in \{1, 2\}$, the $d_3(F_{\varepsilon_1,\varepsilon_2}) \in \{1, 2\}$. If $\gcd(p_1q_2 + p_2q_1, \tau_1) = 2$, then $\tau_1 = 2$, $p_1 \equiv_2 1$, and $p_2 \equiv_2 1$. Thus, $d_3(F_{\varepsilon_1,\varepsilon_2}) = \gcd(2p_1p_2, 2\tau_2) = 2\gcd(p_1p_2, \tau_2)$. Since p_1p_2 is odd and $\tau_2 \in \{1, 2, \}$, then $\gcd(p_1p_2, \tau_2) = 1$ and $d_3(F_{\varepsilon_1,\varepsilon_2}) = 2$.

Remark 3.2.4 implies that $F_{\varepsilon_1,\varepsilon_2} \in \{\{1\}, \mathbb{Z}/2\mathbb{Z}\}$. Thus, the representation ρ factors through the center $\mathcal{Z}(SU(2))$. This implies the conclusion for the case $t_1 = 1$.

Let us suppose that $t_1 = 2$. We are going to prove that η extends to a non-central representation if and only if $\eta(h_1) = -1$ and $o_1 = 2$. Let us suppose that $o_1 = 2$. Let $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ be a representation such that $\eta(h_1) = -1$. Since η is a central representation, we obtain that $\eta(\lambda_1)^{o_1} = 1$. According to Corollary 2.1.12, the representation

η extends to $\pi_1(Y_1)$. Lemma 3.2.1 implies that η extends to a non-central representation. This concludes one direction of the case $t_1 = 2$.

Conversely, suppose that $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ extends to a non-central representation $\pi_1(Y_1) \to SU(2)$. We need to show that this implies that $o_1 = 2$ and $\eta(h_1) = -1$. For this purpose, we first prove that if the representation $\rho\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ extends to $\pi_1(Y_1)$ and $\rho(h_1) = 1$, then every extension of ρ is central.

Let $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ be a representation that extends to $\pi_1(Y_1)$ such that $\eta(h_1) = 1$. Let $\rho\colon \pi_1(Y_1) \to SU(2)$ be an extension of η. According to diagram 3.2.2, the representation ρ factors through the group

$$F_{1,\varepsilon_2} = \operatorname{coker}\begin{bmatrix} p_1 & 0 & 0 & \tau_2 \\ 0 & p_2 & 0 & \tau_2 \\ q_1 & q_2 & 1 & 0 \end{bmatrix} = \operatorname{coker}\begin{bmatrix} p_1 & 0 & 0 & \tau_2 \\ 0 & p_2 & 0 & \tau_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For $i \in \{1,2,3\}$, we define $d_i := d_i(F_{1,\varepsilon_2})$. We recall that $t_1 = \gcd(p_1,p_2) = 2$ It is straightforward to see that

$$d_1 = 1, \quad d_2 \in \{1, \gcd(p_1,p_2)\} = \{1,2\}, \quad \text{and} \quad d_3 = \gcd(p_1p_2, \tau_2p_1, \tau_2p_2) = 2d_2.$$

Remark 3.2.4 implies that $F_{1,\varepsilon_2} \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$. Hence, ρ is central. This implies that if the representation η extends to a non-central representation, then $\eta(h_1) = -1$.

Let $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ be a representation that extends to an $SU(2)$-representation of $\pi_1(Y_1)$ whose image is non-central. We proved before that $\eta(h_1) = -1$. We are going to prove that $o_1 = 2$. Since $t_1 = 2$, the integer $\frac{p_1p_2}{2}$ is even. Since η extends to $\pi_1(Y_1)$ by hypothesis, Corollary 2.1.12 and Lemma 3.1.1 imply that

$$1 = \eta(\lambda_1)^{o_1} = \left(\cancel{\eta(x_1x_2)^{\frac{p_1p_2}{2}}}\eta(h_1)^{\frac{p_1q_2+p_2q_1}{2}}\right)^{o_1} = (-1)^{o_1\left(\frac{p_1q_2+p_2q_1}{2}\right)}. \tag{3.2.3}$$

In particular, equation (3.2.3) implies that $o_1\left(\frac{p_1q_2+p_2q_1}{2}\right)$ is even. Lemma 3.1.1 implies that o_1 is even if and only if the integer $\frac{p_1q_2+p_2q_1}{2}$ is even. Hence, if o_1 is odd, then $o_1\left(\frac{p_1q_2+p_2q_1}{2}\right)$ is also odd, which contradicts (3.2.3). This implies that o_1 has to be even. Since, by Lemma 3.1.1, o_1 divides $t_1 = 2$, we obtain that $o_1 = 2$. $\square$

Lemma 3.2.5 describes completely $P(Y_1)$ in the case that $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$. We focus now on the manifolds

$$\mathbb{D}^2\left(\frac{3}{q_1}, \frac{3}{q_2}, \frac{3}{q_3}\right), \quad \mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}, \frac{4}{q_3}\right), \quad \text{and} \quad \mathbb{D}^2\left(\frac{2}{1}, \cdots, \frac{2}{1}, \frac{p_n}{q_n}\right).$$

Lemma 3.2.6. *Let $Y_1 = \mathbb{D}^2(3/q_1, 3/q_2, 3/q_3)$ with q_i odd. The representation $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ is in $P(Y_1)$ if and only if $\eta(\mu_1) = \eta(h_1)$.*

Proof. Let $\rho\colon \pi_1(Y_1) \to SU(2)$ be the abelian non-central representation defined by

$$\rho(x_1) = \begin{bmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{bmatrix}, \quad \rho(x_2) = \begin{bmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{bmatrix}, \quad \rho(x_3) = \begin{bmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{bmatrix}, \quad \text{and} \quad \rho(h_1) = 1.$$

Thus, $\eta \coloneqq \rho|_{\pi_1(\partial Y_1)}$ is such that

$$\eta(h_1) = \eta(\mu_1) = \eta(x_1 x_2 x_3) = 1$$

and $\eta \in P(Y_1)$.

Let $\rho\colon \pi_1(Y_1) \to SU(2)$ be the abelian non-central representation defined by

$$\rho(x_1) = \begin{bmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{bmatrix}, \quad \rho(x_2) = \begin{bmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{bmatrix}, \quad \rho(x_3) = \begin{bmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{bmatrix}, \quad \text{and} \quad \rho(h_1) = -1.$$

Again, $\eta \coloneqq \rho|_{\pi_1(\partial Y_1)}$ is such that

$$\eta(h_1) = \eta(\mu_1) = \eta(x_1 x_2 x_3) = -1$$

and $\eta \in P(Y_1)$. This concludes one direction: we have just proven that if $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ is such that $\eta(\mu_1) = \eta(h_1)$, then $\eta \in P(Y_1)$.

Conversely, let $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ be a representation such that

$$\eta(\mu_1) \neq \eta(h_1).$$

We write $\eta(\mu_1) = \pm 1$ and $\eta(h_1) = \mp 1$. Lemma 3.1.2 implies that

$$\eta(\lambda_1)^{o_1} = \eta(\mu_1)^3 \eta(h_1)^{q_1+q_2+q_3} = (\pm 1)^3 (\mp 1)^{q_1+q_2+q_3} = 1. \tag{3.2.4}$$

We recall that q_i are odd, therefore the sum $q_1 + q_2 + q_3$ is odd as well. The (3.2.4) implies that $\eta(\lambda_1)^{o_1} \neq 1$, therefore η cannot extend to an abelian representation $\pi_1(Y_1) \to SU(2)$ by

Corollary 2.1.12. This implies that if $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ is as above, then $\eta \notin P(Y_1)$. This concludes the proof. □

Lemma 3.2.7. *Let $Y_1 = \mathbb{D}^2(2/1, 4/q_2, 4/q_3)$. Every representation $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ extends to an abelian non-central representation $\pi_1(Y_1) \to SU(2)$.*

Proof. Let $\varepsilon \in \{1, 2\}$. Let $\rho\colon \pi_1(Y_1) \to SU(2)$ be defined as

$$\rho(x_1) = (-1)^{\varepsilon} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho(x_2) = \rho(x_3) = \begin{bmatrix} e^{\frac{\pi i}{2}} & 0 \\ 0 & e^{-\frac{\pi i}{2}} \end{bmatrix}, \quad \text{and} \quad \rho(h_1) = 1.$$

The representation ρ is abelian and, as $\rho(x_2) \notin \mathcal{Z}(SU(2))$, it is non-central. Moreover,

$$\rho|_{\pi_1(\partial Y_1)}(\mu_1) = \rho(x_1 x_2 x_3) = (-1)^{\varepsilon} \quad \text{and} \quad \rho(h_1) = 1.$$

This implies that if $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ is such that $\eta(h_1) = 1$, then $\eta \in P(Y_1)$

Similarly, let $\rho\colon \pi_1(Y_1) \to SU(2)$ be defined as

$$\rho(x_1) = (-1)^{\varepsilon} \begin{bmatrix} e^{\frac{\pi i}{2}} & 0 \\ 0 & e^{-\frac{\pi i}{2}} \end{bmatrix}, \quad \rho(x_2) = \rho(x_3) = \begin{bmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{bmatrix}, \quad \text{and} \quad \rho(h_1) = -1.$$

Clearly, ρ is abelian and non-central. Moreover,

$$\rho|_{\pi_1(\partial Y_1)}(\mu_1) = \rho(x_1 x_2 x_3) = (-1)^{\varepsilon+1} \quad \text{and} \quad \rho(h_1) = -1.$$

This implies that if $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ is such that $\eta(h_1) = -1$, then $\eta \in P(Y_1)$.

This concludes the proof that every representation $\pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ extends to an abelian non-central representation $\pi_1(Y_1) \to SU(2)$, and therefore every representation $\pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ is in $P(Y_1)$. □

Lemma 3.2.8. *Let $Y_1 = \mathbb{D}^2(2/1, \cdots, 2/1, p_n/q_n)$. If $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ is a representation such that $\eta(h_1) = 1$, then $\eta \notin P(Y_1)$.*

Proof. If $n = 2$, then the conclusion holds by Lemma 3.2.5. Therefore, we suppose that $n \geq 3$.

If ρ is an abelian representation of $\pi_1(Y_1)$ such that $\rho(h_1) = 1$, then for every $i \in \{1, \cdots, n-1\}$,

$$\rho(x_i)^2 = \rho(h_1) = 1, \quad \text{and therefore,} \quad \rho(x_i) \in \mathcal{Z}(SU(2)).$$

This implies that if $\rho(\mu_1) = \rho(x_1 \cdots x_{n-1}x_n) = \pm 1$, then

$$\rho(x_n) = \pm\rho(x_1 \cdots x_{n-1}) \in \mathcal{Z}(SU(2)).$$

Thus, ρ is central. Therefore, if $\eta\colon \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ is a representation that extends to an abelian representation of $\pi_1(Y_1)$ and such that $\eta(h_1) = 1$, then this extension must be central. Thus $\eta \notin P(Y_1)$. $\square$

Note that in Lemma 3.2.8 we only consider two representations out of the four central representation $\pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$, as we do not mention the representations $\eta : \pi_1(\partial Y_1) \to \mathcal{Z}(SU(2))$ such that $\eta(h_1) = -1$. In fact, we will prove in the next section that η extends to an irreducible representation $\pi_1(Y_1) \to SU(2)$.

3.3. The irreducible representations and the set H_1

Let $Y_1 = \mathbb{D}^2({}^{p_1}\!/_{q_1}, \cdots, {}^{p_n}\!/_{q_n})$. Let $\rho\colon \pi_1(Y_1) \to SU(2)$ be such that $\rho|_{\pi_1(\partial Y_1)} \coloneqq \eta \in H_1 = H(Y_1)$, Fact 1.1.3 implies that $\rho(h_1) = \eta(h_1) \in \mathcal{Z}(SU(2))$. Hence, we divide H_1 into the sets $H_{1,0}$ and $H_{1,\pi}$ where

$$H_{1,0} = \{\eta \in H_1 \,|\, \eta(h_1) = 1\} \quad \text{and} \quad H_{1,\pi} = \{\eta \in H_1 \,|\, \eta(h_1) = -1\}.$$

Clearly $H_1 = H_{1,0} \cup H_{1,\pi}$. We use the coordinates (θ_1, ψ_1) for the space $\mathcal{R}_{U(1)}(\partial Y_1)$ with respect to the basis ordered $\{\mu_1, h_1\}$ as in (2.3.7). With this parameterization of $\mathcal{R}_{U(1)}(\partial Y_1)$, we have that $H_{1,0} \subset \{\psi_1 = 0\} \subset \mathcal{R}_{U(1)}(\partial Y_1)$ and $H_{1,\pi} \subset \{\psi_1 = \pi\} \subset \mathcal{R}_{U(1)}(\partial Y_1)$.

Lemma 3.3.1. *Let $n = 2$, thus $Y_1 = \mathbb{D}^2({}^{p_1}\!/_{q_1}, {}^{p_2}\!/_{q_2})$. Let $\pi_1(Y_1)$ be presented as in* (2.3.3)*. If the representation $\eta\colon \pi_1(\partial Y_1) \to SU(2)$ is such that $\operatorname{Tr}\eta(x_1x_2) = \pm 2$, then η is not in H_1.*

Proof. Let us suppose that $\rho : \pi_1(Y_1) \to SU(2)$ is an extension of η. Since $\operatorname{Tr}\eta(x_1x_2) = \operatorname{Tr}\rho(x_1x_2) = \pm 2$, then $\rho(x_1x_2) = \pm 1$. Thus, $[\rho(x_1), \rho(x_2)] = 1$. This implies that ρ is an abelian representation and hence $\rho|_{\pi_1(\partial Y_1)} = \eta \notin H_1$. $\square$

Lemma 3.3.1 implies that if Y_1 admits $n = 2$ nontrivial singular fibres, then the set H_1 has no intersection with the lines

$$\{\theta_1 \equiv_{2\pi} 0\} \cup \{\theta_1 \equiv_{2\pi} \pi\} \subset \mathcal{R}_{U(1)}(\partial Y_1).$$

The next proposition will be central in this chapter.

Proposition 3.3.2. *Given $a, b, c \in (-2, 2)$, there exist two matrices $A, B \in SU(2)$ with $\operatorname{Tr} A = a$, $\operatorname{Tr} B = b$, $\operatorname{Tr} AB = c$, and $AB \neq BA$ if and only if*

$$\frac{1}{2}\Big(ab - \sqrt{(4-a^2)(4-b^2)}\Big) < c < \frac{1}{2}\Big(ab + \sqrt{(4-a^2)(4-b^2)}\Big).$$

Proof. In this proof we consider S^1 as the subset of $\mathbb{C}$ in the usual way. Let z be a complex number, we denote by $\Re(z)$ and $\Im(z)$ its real and imaginary part. Let us suppose that there exist two matrices $A, B \in SU(2)$ such that $\operatorname{Tr} A = a$, $\operatorname{Tr} B = b$, $\operatorname{Tr} AB = c$, and $AB \neq BA$. Up to conjugation, we can suppose that

$$A = \begin{bmatrix} u & 0 \\ 0 & \overline{u} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix},$$

where $u \in S^1$ and $\alpha, \beta \in \mathbb{C}$. Since A and B are assumed not to commute, B is not diagonal. This implies that $\beta \neq 0$ and $|\alpha|^2 < 1$. Thus, there exist $v \in S^1$ and $t \in (0, 1)$ such that

$$\alpha = (1-t)v + t\overline{v}.$$

By hypothesis, the complex numbers α, u, and v are such that

$$2\Re(u) = \operatorname{Tr} A = a, \quad 2\Re(\alpha) = 2\Re(v) = \operatorname{Tr} B = b, \quad \text{and} \quad \operatorname{Tr} AB = c.$$

Let us consider the following:

$$\begin{aligned} c = \operatorname{Tr} AB &= u\alpha + \overline{u\alpha} = 2\Re(u\alpha) = 2((1-t)\Re(uv) + t\Re(u\overline{v})) \\ &= 2\Re(u)\Re(v) + 2(2t-1)\Im(u)\Im(v) = \frac{ab}{2} + 2(2t-1)\sqrt{1-\frac{a^2}{4}}\sqrt{1-\frac{b^2}{4}} \qquad (3.3.1) \\ &= \frac{1}{2}\Big(ab + (2t-1)\sqrt{(4-a^2)(4-b^2)}\Big). \end{aligned}$$

Since t is neither 0 nor 1, we conclude that

$$\frac{1}{2}\Big(ab - \sqrt{(4-a^2)(4-b^2)}\Big) < c < \frac{1}{2}\Big(ab + \sqrt{(4-a^2)(4-b^2)}\Big).$$

Conversely, let $u, v \in S^1$ with

$$u = \frac{a}{2} + i\sqrt{1-\frac{a^2}{4}} \quad \text{and} \quad v = \frac{b}{2} + i\sqrt{1-\frac{b^2}{4}}.$$

For $t \in (0,1)$ we set $\alpha(t) = (1-t)v + t\overline{v}$ and define $A, B(t) \in SU(2)$ by

$$A = \begin{bmatrix} u & 0 \\ 0 & \overline{u} \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} \alpha(t) & \sqrt{1-|\alpha(t)|^2} \\ -\sqrt{1-|\alpha(t)|^2} & \overline{\alpha}(t) \end{bmatrix}.$$

It is easy to see that $\operatorname{Tr} A = a$ and $\operatorname{Tr} B(t) = b$. Since t is neither 0 nor 1, the matrix B is not diagonal and hence $AB(t) \neq B(t)A$ for every $t \in (0,1)$. The computation in (3.3.1) implies that the trace of the multiplication $AB(t)$ is

$$\operatorname{Tr} AB(t) = u\alpha(t) + \overline{u\alpha}(t) = \frac{1}{2}\Big(ab + (2t-1)\sqrt{(4-a^2)(4-b^2)}\Big).$$

Therefore, there is some $t \in (0,1)$ for which $\operatorname{Tr} AB(t) = c$. □

Definition 3.3.3. For $a, b \in [-2,2] \subset \mathbb{R}$ we define $I(a,b)$ as the open interval

$$\left(\frac{1}{2}\Big(ab - \sqrt{(4-a^2)(4-b^2)}\Big), \frac{1}{2}\Big(ab + \sqrt{(4-a^2)(4-b^2)}\Big)\right) \subseteq [-2,2].$$

Let $a, b \in [-2,2]$. Let $A, B \in SU(2)$ be two matrices such that

$$\operatorname{Tr} A = a, \quad \text{and} \quad \operatorname{Tr} B = b.$$

A consequence of Proposition 3.3.2 is that if $c \in \partial I(a,b)$ then, there exist two matrices $A, B \in SU(2)$ such that

$$\operatorname{Tr} A = a, \quad \operatorname{Tr} B = b, \quad \text{and} \quad [A,B] = 1.$$

Thus, we can say that the points in the interior of $I(a,b)$ correspond to irreducible representations and the two endpoints $\partial I(a,b)$ to abelian representations. We summarize this in the following Corollary.

Corollary 3.3.4. *Let $a, b, c \in [-2,2]$, there exist two matrices $A, B \in SU(2)$ with $\operatorname{Tr} A = a$, $\operatorname{Tr} B = b$, $\operatorname{Tr} AB = c$ and $AB = BA$ if and only if $c \in \partial I(a,b)$.*

PROOF. Let $\alpha, \beta \in [0, 2\pi]$ be two angles such that $a = 2\cos\alpha$ and $2\cos\beta$. A direct computation on $I(2\cos\alpha, 2\cos\beta)$ shows that

$$\begin{aligned}\partial I(a,b) = \partial I(2\cos\alpha, 2\cos\beta) =& \left\{2\cos\alpha\cos\beta \pm 2\sqrt{\sin^2\alpha\sin^2\beta}\right\} \\ & \{2\cos\alpha\cos\beta \pm 2\sin\alpha\sin\beta\} \\ & \{2\cos(\alpha\pm\beta)\}.\end{aligned}$$

If $AB = BA$, then up to conjugation A and B are both in $U(1)$. This implies that

$$A = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}, \quad B = \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{i\beta} \end{bmatrix}, \quad \text{and} \quad AB = \begin{bmatrix} e^{i(\alpha+\beta)} & 0 \\ 0 & e^{-i(\alpha+\beta)} \end{bmatrix}.$$

Therefore, $\operatorname{Tr} AB = 2\cos(\alpha+\beta)$ and $\operatorname{Tr} AB \in \partial I(a,b)$.

Conversely, let us suppose $c \in \partial I(2\cos\alpha, 2\cos\beta) = \{2\cos(\alpha\pm\beta)\}$. Let A and B defined as

$$A = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} e^{\pm i\beta} & 0 \\ 0 & e^{\mp i\beta} \end{bmatrix}.$$

These two matrices give the conclusion: they commute, $\operatorname{Tr} A = 2\cos\alpha = a$, $\operatorname{Tr} B = 2\cos(\pm\beta) = b$, and $\operatorname{Tr} AB = 2\cos(\alpha\pm\beta) = c$. ☐

In the following definitions an abuse of notation is in use. Let $x \in \mathbb{R}$ be a real number, then the open interval (x, x) is empty. However, we set $\partial(x, x) = \{x\}$.

Definition 3.3.5. Let $2 \leq p_1 \leq p_2$ be two natural numbers, we define the interval $J_0(p_1, p_2)$ as

$$J_0(p_1,p_2) = \bigcup_{k_1,k_2\in\mathbb{Z}} I\left(2\cos\left(\frac{2\pi k_1}{p_1}\right), 2\cos\left(\frac{2\pi k_2}{p_2}\right)\right) \subseteq [-2,2].$$

Definition 3.3.6. Let $2 \leq p_1 \leq \cdots \leq p_n$ be $n \geq 3$ natural numbers. We inductively define the interval $J_0(p_1, \cdots, p_n)$ as

$$J_0(p_1,\cdots,p_n) := \bigcup_{k_n\in\mathbb{Z}} \left(\bigcup_{c\in J_{n-1}} \overline{I}\left(2\cos\left(\frac{2\pi k_n}{p_n}\right), c\right) \cup \bigcup_{c\in\partial J_{n-1}} I\left(2\cos\left(\frac{2\pi k_n}{p_n}\right), c\right)\right) \subseteq [-2,2],$$

where J_{n-1} is $J_0(p_1, \cdots, p_{n-1})$ and $\overline{I}$ denotes the closed interval defined by the same formulae as in Definition 3.3.3.

For instance, $I(2,2) = \emptyset$ by Definition 3.3.3, however $\overline{I}(2,2) = [2,2]$ is the singleton $\{2\}$. Therefore $\partial\overline{I}(a,b)$, with $a,b \in [-2,2]$ is well defined and never empty. For example $\partial\overline{I}(2,2) = \partial\{2\} = \{2\}$.

Definition 3.3.7. Let $2 \leq p_1 \leq p_2$ be two natural numbers, we define the $J_\pi(p_1,p_2)$ as

$$J_\pi(p_1,p_2) = \bigcup_{\substack{k_1,k_2\in\mathbb{Z}\\ k_i \text{ odd}}} I\left(2\cos\left(\frac{\pi k_1}{p_1}\right), 2\cos\left(\frac{\pi k_2}{p_2}\right)\right) \subseteq [-2,2].$$

Definition 3.3.8. Let $2 \leq p_1 \leq \cdots \leq p_n$ be $n \geq 3$ natural numbers. We inductively define the interval $J_0(p_1,\cdots,p_n)$.

$$J_\pi(p_1,\cdots,p_n) \coloneqq \bigcup_{\substack{k_n\in\mathbb{Z}\\ k_n \text{ odd}}} \left(\bigcup_{c\in J_{n-1}} \overline{I}\left(2\cos\left(\frac{\pi k_n}{p_n}\right), c\right) \cup \bigcup_{c\in\partial J_{n-1}} I\left(2\cos\left(\frac{\pi k_n}{p_n}\right), c\right)\right) \subseteq [-2,2],$$

where J_{n-1} is $J_\pi(p_1,\cdots,p_{n-1})$ and $\overline{I}$ denotes the closed interval defined by the same formulae in Definition 3.3.3.

In order to make the notation lighter, the interval $I\left(2\cos\left(\frac{2\pi k_1}{p_1}\right), 2\cos\left(\frac{2\pi k_2}{p_2}\right)\right)$ as in Definition 3.3.5 will be denoted as $I(k_1/p_1, k_2/p_2)$. Thus, $J_\pi(p_1,p_2)$ is the union of $I(k_1/2p_1, k_2/2p_2)$ with k_1 and k_2 odd. The length of $I(k_1/p_1, k_2/p_2)$ can be computed from Definition 3.3.3 and it equals

$$m\left(I\left(\frac{k_1}{p_1}, \frac{k_2}{p_2}\right)\right) = 4\left|\sin\left(\frac{2\pi k_1}{p_1}\right)\sin\left(\frac{2\pi k_2}{p_2}\right)\right|. \tag{3.3.2}$$

Similarly, the length of $I(k_1/2p_1, k_2/2p_2)$ is computed from (3.3.2).

Lemma 3.3.9. *Let $2 \leq p_1 \leq p_2$. The set $J_0(p_1,p_2)$ is empty if and only if $p_1 = 2$. Furthermore, the set $J_\pi(p_1,p_2)$ is not empty.*

Proof. The identity (3.3.2) implies that $I(k_1/p_1, k_2/p_2)$ has positive measure if and only if $\sin(2\pi k_1/p_1)$ and $\sin(2\pi k_2/p_2)$ are both non-zero. This implies that $J_0(2,p_2)$ is empty. Conversely, if $3 \leq p_1 \leq p_2$, then there exists $(k_1,k_2) \in \mathbb{Z}^2$ so that $\sin(2\pi k_1/p_1) \neq 0$ and $\sin(2\pi k_2/p_2) \neq 0$. This implies that $I(k_1/p_1, k_2/p_2)$ is nonempty, and therefore that $J_0(p_1,p_2)$ is nonempty as well.

The set $J_\pi(p_1,p_2)$ contains the interval $I(1/2p_1, 1/2p_2)$ that is not empty by the (3.3.2). Hence, the set $J_\pi(p_1,p_2)$ is not empty. $\square$

Lemma 3.3.10. *Let $n \geq 3$, the set $J_0(2,\cdots,2,p_n)$ is empty and $J_\pi(2,\cdots,2,p_n) = [-2,2]$.*

Proof. Let us focus on $J_0(2,\cdots,2,p_n)$. We remind the reader that we are using the notation for which $\partial(x,x)=\{x\}$. Definition 3.3.5 implies that $I(2,2)=(2,2)$ is empty and $\partial I(2,2)=\{2\}$. By induction, Definition 3.3.6 implies that

$$J_0(2,\cdots,2)=\bigcup_{c,d\in\{2\}} I(c,d).$$

Therefore, $J_0(2,\cdots,2)=(2,2)$ is empty and $\partial J_0(2,\cdots,2)=\{2\}$. According to Definition 3.3.6, the set $J_0(2,\cdots,2,p_n)$ is equal to

$$\bigcup_{\substack{k\in\mathbb{Z}\\ c\in\partial J_0(2,\cdots,2)}} I\left(2\cos\left(\frac{2\pi k}{p_n},c\right)\right).$$

According, to Definition 3.3.3, $J_0(2,\cdots,2,p_n)$ is the union of empty interval and therefore, it is empty as well.

Let us focus on $J_\pi(2,\cdots,2,p_n)$. Definition 3.3.7 shows that $J_\pi(2,2)=(-2,2)$. We recall that, by Definition 3.3.3, for every $x\in[-2,2]$ the interval $I(x,\pm 2)$ is empty. Therefore, Definition 3.3.8 implies that

$$J_\pi(2,2,p_3)=\bigcup_{\substack{k_3\in\mathbb{Z}\\ k_3\ \text{odd}\\ c\in(-2,2)}} \overline{I}\left(2\cos\left(\frac{\pi k_3}{p_3}\right),c\right)$$

We notice that $2\cos(\pi k/p_3)\in(-2,2)$ if and only if $k\notin p_3\mathbb{Z}$. Thus, the set $J_\pi(2,2,p_3)$ contains

$$\bigcup_{k=1}^{p_3-1}\overline{I}\left(2\cos\left(\frac{\pi}{p_3}\right),2\cos\left(\frac{\pi k}{p_3}\right)\right). \tag{3.3.3}$$

This latter is the union of the closed intervals whose endpoints are

$$2\cos\left(\pi\frac{k+1}{p_3}\right)\quad\text{and}\quad 2\cos\left(\pi\frac{k-1}{p_3}\right).$$

Thus, the (3.3.3) implies that $J_\pi(2,2,p_3)=[-2,2]$. By induction, we obtain the conclusion for $J_\pi(2,\cdots,2,p_3)$. ☐

Remark 3.3.11. Let $n \geq 1$ be a natural number. Let $A \in SU(2)$. The matrix A is such that $A^n = 1$ if and only if $\operatorname{Tr} A = 2\cos\left(\frac{2\pi k}{n}\right)$ for a $k \in \mathbb{Z}$. Moreover, $A^n = -1$ if and only if $\operatorname{Tr} A = 2\cos\left(\frac{\pi k}{n}\right)$ for an odd $k \in \mathbb{Z}$.

The next lemma and Proposition 3.3.2 are, the two key results for constructing the set $H_1 \subset T(Y_1, \partial Y_1)$ for the case $Y_1 = \mathbb{D}^2({}^{p_1}/_{q_1}, {}^{p_2}/_{q_2})$.

Lemma 3.3.12. *Let $n = 2$, thus $Y_1 = \mathbb{D}^2({}^{p_1}/_{q_1}, {}^{p_2}/_{q_2})$. Let $\pi_1(Y_1)$ be presented as in* (2.3.3) *with q_1 and q_2 odd. Then the maps*

$$\begin{aligned} f_0 \colon H_{1,0} &\to J_0(p_1, p_2) & \quad \textit{and} \quad f_\pi \colon H_{1,\pi} &\to J_\pi(p_1, p_2) \\ \eta &\mapsto \operatorname{Tr} \eta(x_1 x_2) & \eta &\mapsto \operatorname{Tr} \eta(x_1 x_2) \end{aligned}$$

are well-defined and surjective.

Proof. Let us suppose that $p_1 = 2$. The set $J_0(2, p_2)$ is empty by Lemma 3.3.9. Since $p_1 = 2$, every representation $\rho \in \mathcal{R}(Y_1)$ with $\rho(h_1) = 1$ is such that $\rho(x_1) \in \mathcal{Z}(SU(2))$. Hence, the representation ρ has abelian image. This implies that $H_{1,0}$ is empty.

Let us suppose that $3 \leq p_1 \leq p_2$. If $\eta \in H_{1,0}$, then there exists an irreducible representation $\rho \in \mathcal{R}(Y_1)$ such that $\rho|_{\pi_1(\partial Y_1)} \equiv \eta$ and $\rho(h_1) = 1$. In particular, we have that

$$\rho(x_1)^{p_1} \rho(h_1)^{q_1} = \rho(x_1)^{p_1} = 1 \quad \text{and} \quad \rho(x_2)^{p_2} \rho(h_1)^{q_2} = \rho(x_2)^{p_2} = 1.$$

Since ρ is irreducible, we obtain that $\rho(x_1)\rho(x_2) \neq \rho(x_2)\rho(x_1)$. According to Remark 3.3.11, there exist two integers k_1 and k_2 such that $\operatorname{Tr} \rho(x_1) = 2\cos\left(\frac{2\pi k_1}{p_1}\right)$ and $\operatorname{Tr} \rho(x_2) = 2\cos\left(\frac{2\pi k_2}{p_2}\right)$. Proposition 3.3.2 implies that

$$\operatorname{Tr} \eta(x_1 x_2) = \operatorname{Tr} \rho(x_1 x_2) \in I\left(\frac{k_1}{p_1}, \frac{k_2}{p_2}\right) \subseteq J_0(p_1, p_2).$$

This implies that the map f_0 is well defined.

Let $z \in J_0(p_1, p_2)$. Thus, there exist two integers k_1 and k_2 such that $z \in I({}^{k_1}/_{p_1}, {}^{k_2}/_{p_2}) \subseteq J_0(p_1, p_2)$. According to Proposition 3.3.2, there exist two matrices A and B of $SU(2)$ such that

$$\operatorname{Tr} A = 2\cos\left(\frac{2\pi k_1}{p_1}\right), \quad \operatorname{Tr} B = 2\cos\left(\frac{2\pi k_2}{p_2}\right), \quad \operatorname{Tr} AB = z, \quad \text{and} \quad AB \neq BA.$$

Remark 3.3.11 implies that $A^{p_1} = 1$ and $B^{p_2} = 1$. Let $\rho \in \mathcal{R}(Y_1)$ be the representation defined as

$$\rho(x_1) = A, \quad \rho(x_2) = B, \quad \text{and} \quad \rho(h_1) = 1.$$

Since A and B do not commute, ρ is irreducible. Up to conjugation, we can suppose that $\rho|_{\pi_1(\partial Y_1)} \in \mathcal{R}_{U(1)}(\partial Y_1)$. In particular this implies that $\rho|_{\pi_1(\partial Y_1)} \in H_{1,0}$. Thus, $f_0(\rho|_{\pi_1(\partial Y_1)}) = \operatorname{Tr} \rho(x_1x_2) = \operatorname{Tr} AB = z$. This implies that the map f_0 is surjective.

Let $\eta \in H_{1,\pi}$. This implies that there exists an irreducible representation $\rho \in \mathcal{R}(Y_1)$ such that $\rho|_{\pi_1(\partial Y_1)} \equiv \eta$ and $\rho(h_1) = -1$. Since q_1 and q_2 are odd, we obtain that

$$\rho(x_1)^{p_1} = -1 \quad \text{and} \quad \rho(x_2)^{p_2} = -1.$$

As a result of Remark 3.3.11, there exist two odd integers k_1 and k_2 such that $\operatorname{Tr} \rho(x_1) = \frac{\pi k_1}{p_1}$ and $\operatorname{Tr} \rho(x_2) = \frac{\pi k_2}{p_2}$. Proposition 3.3.2 implies that

$$\operatorname{Tr} \eta(x_1x_2) = \operatorname{Tr} \rho(x_1x_2) \in I\left(\frac{k_1}{2p_1}, \frac{k_2}{2p_2}\right) \subseteq J_\pi(p_1, p_2).$$

This implies that f_π is well defined.

Let $z \in J_\pi(p_1, p_2)$. As a result of Remark 3.3.11, there exist two odd integers k_1 and k_2 such that $z \in I({}^{k_1}\!/_{2p_1}, {}^{k_2}\!/_{2p_2}) \subseteq J_\pi(p_1, p_2)$. According to Proposition 3.3.2, there exist two matrices A and B of $SU(2)$ such that

$$\operatorname{Tr} A = 2\cos\left(\frac{\pi k_1}{p_1}\right), \quad \operatorname{Tr} B = 2\cos\left(\frac{\pi k_2}{p_2}\right), \quad \operatorname{Tr} AB = z, \quad \text{and} \quad AB \neq BA.$$

Let $\rho\colon \pi_1(Y_1) \to SU(2)$ be the representation determined by the following:

$$\rho(x_1) = A, \quad \rho(x_2) = B, \quad \text{and} \quad \rho(h_1) = -1.$$

Since A and B do not commute, ρ is irreducible. Up to conjugation, we can suppose that $\rho|_{\pi_1(\partial Y_1)} \in \mathcal{R}_{U(1)}(\partial Y_1)$. In particular, this implies that $\rho|_{\pi_1(\partial Y_1)} \in H_{1,\pi}$. Moreover, $f_\pi(\rho|_{\pi_1(\partial Y_1)}) = \operatorname{Tr} \rho(x_1x_2) = \operatorname{Tr} AB = z$. This implies that f_π is surjective. □

Before proving the next corollary, we need to mention a property of $\pi_1(\mathbb{D}^2({}^{p_1}\!/_{q_1}, \cdots, {}^{p_n}\!/_{q_n}))$, which is considered presented as in (2.3.3). We notice that

$$\pi_1\big(\mathbb{D}^2({}^{p_1}\!/_{q_1}, \cdots, {}^{p_n}\!/_{q_n})\big) = \frac{\pi_1(\mathbb{D}^2({}^{p_1}\!/_{q_1}, \cdots, {}^{p_{n-1}}\!/_{q_{n-1}})) * \mathbb{Z}}{\langle\!\langle x_n^{p_n} h_1^{q_n} \rangle\!\rangle}.$$

Thus,

$$\pi_1\big(\mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_{n-1}}/_{q_{n-1}})\big) \leq \pi_1\big(\mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_n}/_{q_n})\big).$$

In particular if μ_{n-1} is the fibration meridian of $\mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_{n-1}}/_{q_{n-1}})$ and μ_n is the one of $\mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_n}/_{q_n})$, then

$$\mu_n = \mu_{n-1}x_n \in \pi_1\big(\mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_n}/_{q_n})\big).$$

Corollary 3.3.13. *Let* $Y_1 = \mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_n}/_{q_n})$ *with* $n \geq 3$. *Let* $\pi_1(Y_1)$ *be presented as in* (2.3.3) *with* q_1 *and* q_2 *odd. Then the maps*

$$f_0\colon H_{1,0} \to J_0(p_1, \cdots, p_n) \qquad \textit{and} \qquad f_\pi\colon H_{1,\pi} \to J_\pi(p_1, \cdots, p_n)$$
$$\eta \mapsto \operatorname{Tr}\eta(x_1 \cdots x_n) \qquad\qquad \eta \mapsto \operatorname{Tr}\eta(x_1 \cdots x_n)$$

are well-defined and surjective.

Proof. We prove the conclusion by induction on n. The case $n = 2$ holds by Lemma 3.3.12. Let us suppose that the conclusion holds for $n-1$ fibres. We define $J_{n-1} \coloneqq J_0(p_1, \cdots, p_{n-1})$. We first prove that f_0 is well-defined, then we prove that f_0 is surjective.

According to Lemma 3.3.10, the $J_0(2, \cdots, 2, p_n)$ is empty. Let $\rho\colon \pi_1(\mathbb{D}^2({}^2/_1, \cdots, {}^2/_1, {}^{p_n}/_{q_n})) \to SU(2)$ be a representation such that $\rho(h_1) = 1$. This implies that $\rho(x_i) \in \mathcal{Z}(SU(2))$ for $i \in \{1, \cdots, n-1\}$. Therefore,

$$\operatorname{Im}\rho \subset \Lambda_{\rho(x_n)},$$

where $\Lambda_{\rho(x_n)}$ is centralizer of $\rho(x_n) \in SU(2)$. Hence, ρ is abelian and $H_{1,0} = \emptyset$. This implies that the the map f_0 is well-defined and surjective for the manifold $\mathbb{D}^2({}^2/_1, \cdots, {}^2/_1, {}^{p_n}/_{q_n})$.

Let $\rho\colon \pi_1(Y_1) \to SU(2)$ be an irreducible representation such that $\rho(h_1) = 1$. By Definition 2.1.5, we have that $\rho|_{\pi_1(\partial Y_1)} \in H_1$. We define ρ_{n-1} as the restriction of ρ on the subgroup $\pi_1(\mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_{n-1}}/_{q_{n-1}})$. The representation ρ_{n-1} is either abelian of irreducible.

If ρ_{n-1} is irreducible, then

$$\operatorname{Tr}\rho_{n-1}(x_1 \cdots x_{n-1}) \coloneqq c \in J_{n-1},$$

by induction. By the presentation of $\pi_1(Y_1)$ in (2.3.3), we have that $\rho(x_n)^{p_n} = 1$. Thus, according to Remark 3.3.11,

$$\operatorname{Tr}\rho(x_n) = 2\cos\frac{2\pi k_n}{p_n},$$

with $k_n \in \mathbb{Z}$. Therefore, the matrix $\rho(x_1 \cdots x_{n-1})$ either commute or does not commute with $\rho(x_n)$. By Proposition 3.3.2 and Corollary 3.3.4,

$$\operatorname{Tr} \rho(x_1 \cdots x_{n-1} x_n) = \operatorname{Tr} \rho(x_1 \cdots x_{n-1}) \rho(x_n) \in \bigcup_{c \in J_{n-1}} \overline{I}\left(2 \cos \frac{2\pi k_n}{p_n}, c\right)$$

Therefore, $\operatorname{Tr} \rho(x_1 \cdots x_n) \in J_0(p_1, \cdots, p_n)$.

If ρ_{n-1} is abelian, then as an application of Corollary 3.3.4 $\rho(x_1 \cdots x_{n-1}) \in \overline{J}_{n-1}$ and $\rho(x_1 \cdots x_{n-1})$ does not commute with $\rho(x_n)$. This implies that

$$d \in \bigcup_{c \in \overline{J_{n-1}}} I^\circ\left(2 \cos\left(\frac{2\pi k_n}{p_n}, c\right)\right).$$

Therefore, $\operatorname{Tr} \rho(x_1 \cdots x_n) \in J_0(p_1, \cdots, p_n)$ and f_0 is well-defined.

Let $d \in J_0(p_1, \cdots, p_n)$. Thus, there exists a $k_n \in \mathbb{Z}$ such that either

$$d \in \bigcup_{c \in J^\circ_{n-1}} \overline{I}\left(2 \cos\left(\frac{2\pi k_n}{p_n}, c\right)\right) \quad \text{or} \quad d \in \bigcup_{c \in \partial J_{n-1}} I^\circ\left(2 \cos\left(\frac{2\pi k_n}{p_n}, c\right)\right).$$

We recall that $J_{n-1} = J_0(p_1, \cdots, p_{n_1})$. If

$$d \in \bigcup_{c \in J^\circ_{n-1}} \overline{I}\left(2 \cos\left(\frac{2\pi k_n}{p_n}, c\right)\right),$$

then there exists two matrices $C, X_n \in SU(2)$ such that

$$\operatorname{Tr} C = c \in J^\circ_{n-1}, \quad \operatorname{Tr} X_n = 2 \cos \frac{2\pi k_n}{p_n} \quad \text{and} \quad \operatorname{Tr}(C X_n) = d.$$

Since $c \in J^\circ_{n-1}$, by induction there exists an irreducible representation $\rho_{n-1} \colon \pi_1(\mathbb{D}^2(p_1/q_1, p_{n-1}/q_{n-1})) \to SU(2)$ such that

$$\rho_{n-1}(h_1) = 1 \quad \text{and} \quad \operatorname{Tr} \rho_{n-1}(x_1 \cdots x_{n-1}) = c.$$

Up to conjugation, we can suppose that $\rho_{n-1}(x_1 \cdots x_{n-1}) = C$. We define now $\rho_n \colon \pi_1(Y_1) \to SU(2)$ as the representation such that

$$\rho(x) \coloneqq \begin{cases} 1 & \text{if } x = h_1, \\ \rho_{n-1}(x) & \text{if } x \in \pi_1\big(\mathbb{D}^2(p_1/q_1, p_{n-1}/q_{n-1})\big), \\ X_n & \text{if } x = x_n. \end{cases}$$

Clearly ρ_n is irreducible and

$$\operatorname{Tr}\rho_n(x_1\cdots x_n) = \operatorname{Tr}(\rho_{n-1}(x_1\cdots x_{n-1})\rho_n(x_n)) = \operatorname{Tr}(CX_n) = d.$$

Similarly, let us suppose that

$$d \in \bigcup_{c\in\partial J_{n-1}} I^\circ\left(2\cos\left(\frac{2\pi k_n}{p_n}, c\right)\right).$$

Thus, there exist two matrices $C, X_n \in SU(2)$ such that

$$\operatorname{Tr} C = c \in \partial J^\circ_{n-1}, \quad \operatorname{Tr} X_n = 2\cos\frac{2\pi k_n}{p_n}, \quad \operatorname{Tr}(CX_n) = d,$$

and $CX_n \neq X_nC$. Since $c \in \partial J^\circ_{n_1}$, then there exists an abelian representation

$$\rho_{n-1}\colon \pi_1\left(\mathbb{D}^2({}^{p_1}/_{q_1}, {}^{p_{n-1}}/_{q_{n-1}})\right) \to SU(2)$$

such that

$$\rho_{n-1}(h_1) = 1, \quad \text{and} \quad \operatorname{Tr}\rho_{n-1}(x_1\cdots x_{n-1}) = c.$$

This implies that the map f_0 is surjective.

The conclusion for f_π holds by a similar analysis, therefore it is left to the reader. □

Lemma 3.3.14. *The following hold:*

- $J_0(2,4,4) = J_\pi(2,4,4) = (-2,2)$,
- $J_0(3,3,3) = (-2,2]$ *and* $J_\pi(3,3,3) = [-2,2)$.

Proof. Let us start with $J_0(2,4,4)$. It is easy to see that

$$J_0(2,4) = \bigcup_{k\in\mathbb{Z}} I\left(\pm 2, 2\cos\left(\frac{\pi}{4}k\right)\right) = (-2,-2)\cup(0,0)\cup(2,2).$$

We remind the reader that $\partial(x,x) = \{x\}$. This implies that

$$J_0(2,4,4) = \bigcup_{\substack{k\in\mathbb{Z}\\ x\in\{2,0,-2\}}} I\left(2\cos\left(\frac{\pi}{4}k\right), x\right) = I(0,0) = (-2,2)$$

One notices that $J_\pi(2,4) = (2\cos{}^{\pi}/_4, 2\cos{}^{3\pi}/_4) = (-\sqrt{2}, \sqrt{2})$. Therefore

$$J_\pi(2,4,4) = \bigcup_{k=1,3}\left(\bigcup_{x\in(-\sqrt{2},\sqrt{2})} \overline{I}\left(2\cos\left(\frac{\pi k}{4}\right), x\right) \cup \bigcup_{x\in\left\{-\sqrt{2},\sqrt{2}\right\}} I^\circ\left(2\cos\left(\frac{\pi k}{4}\right), x\right)\right).$$

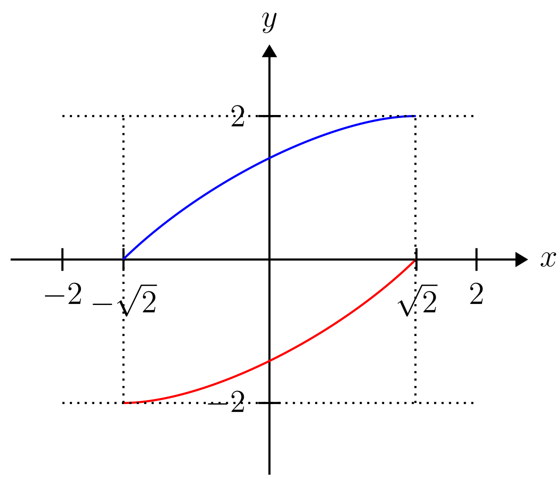


FIGURE 3.1. The plots of the functions $y = \frac{1}{2}\left(x\sqrt{2} - \sqrt{2(4-x^2)}\right)$ and $y = \frac{1}{2}\left(x\sqrt{2} + \sqrt{2(4-x^2)}\right)$ in blue and red with $x \in (-\sqrt{2}, \sqrt{2})$, as in the intervals in (3.3.4).

A direct computation shows that

$$J_\pi(2,4,4) = \bigcup_{x\in(-\sqrt{2},\sqrt{2})} \left[\frac{1}{2}\left(x\sqrt{2} - \sqrt{2(4-x^2)}\right), \frac{1}{2}\left(x\sqrt{2} + \sqrt{2(4-x^2)}\right)\right]. \tag{3.3.4}$$

This implies that $J_\pi(2,4,4) = (-2,2)$, as shown in Figure 3.1

Let us focus on $J_0(3,3,3)$. We see that $J_0(3,3) = (2\cos{}^{2\pi}/_3, 2) = (-1,2)$. As before, a direct computation and Figure 3.2 show that,

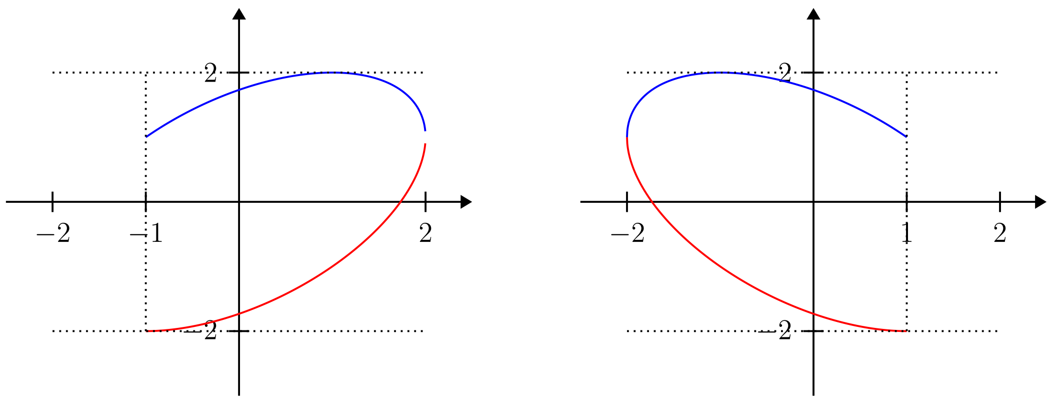
$$J_0(3,3,3) = \bigcup_{x\in(-1,2)} \left[\frac{1}{2}\left(x - \sqrt{3(4-x^2)}\right), \frac{1}{2}\left(x + \sqrt{3(4-x^2)}\right)\right] = (-2,2] \tag{3.3.5}$$

Similarly, $J_\pi(3,3) = (2\cos({}^{2\pi}/_3), 2) = (-1,2)$. Therefore,

$$J_\pi(3,3,3) = \bigcup_{x\in(-1,2)} \left[\frac{1}{2}\left(-x - \sqrt{3(4-x^2)}\right), \frac{1}{2}\left(-x + \sqrt{3(4-x^2)}\right)\right] = [-2,2) \tag{3.3.6}$$

□

Example 3.3.15. Let us suppose that $Y_1 = \mathbb{D}^2({}^3/_{q_1}, {}^3/_{q_2}, {}^3/_{q_3})$ with q_i odd. Corollary 3.3.13 and Lemma 3.3.14 imply that there exists an irreducible representation $\rho : \pi_1(Y_1) \to SU(2)$ such that $\rho(h_1) = 1$ and $\rho(\mu_1) = -1$. We are going to construct such a representation.



(A) Case $J_0(3,3,3)$ (B) Case $J_\pi(3,3,3)$

FIGURE 3.2. In red and blue are the endpoints of the intervals in (3.3.5) and (3.3.6) with the corresponding domains.

We notice that

$$2\cos\frac{\pi}{3} = 1 \in (-1,2) = I\left(2\cos\frac{2\pi}{3}, 2\cos\frac{2\pi}{3}\right) = J_0(3,3).$$

Proposition 3.3.2 implies that there exist two matrices $X_1, X_2 \in SU(2)$ such that

$$\operatorname{Tr} X_1 = \operatorname{Tr} X_2 = 2\cos\frac{2\pi}{3}, \quad X_1X_2 \neq X_2X_1, \quad \text{and} \quad \operatorname{Tr} X_1X_2 = 2\cos\frac{\pi}{3}$$

Up to conjugation we can suppose that the product X_1X_2 is in $U(1)$, this means that

$$X_1X_2 = e^{i\frac{\pi}{3}} = \begin{bmatrix} e^{i\frac{\pi}{3}} & 0 \\ 0 & e^{-i\frac{\pi}{3}} \end{bmatrix}.$$

Let $X_3 = e^{i\frac{2\pi}{3}}$. Remark 3.3.11 states that $X_i^3 = 1$ for all $i \in \{1,2,3\}$. We define the irreducible representation $\rho\colon \pi_1(Y_1) \to SU(2)$ as

$$\rho(x_i) = X_i, \quad \text{and} \quad \rho(h_1) = 1.$$

A direct computation shows that

$$\rho(\mu_1) = \rho(x_1x_2x_2) = (X_1X_2)X_3 = e^{i\frac{\pi}{3}}e^{i\frac{2\pi}{3}} = -1.$$

More explicitly, the matrices X_1, X_2, and X_3 can be chosen in the following way:

$$X_1 = \begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{6} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & -\frac{1}{2} + \frac{i\sqrt{3}}{6} \end{bmatrix} \quad X_2 = \begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{6} & -\frac{\sqrt{6}}{6} + i\frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{6} + i\frac{\sqrt{2}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{6} \end{bmatrix}, \quad X_3 = \begin{bmatrix} e^{i\frac{2\pi}{3}} & 0 \\ 0 & e^{-i\frac{2\pi}{3}} \end{bmatrix}.$$

We use the rest of the section for understanding more the case $Y_1 = \mathbb{D}^2({}^{p_1}/_{q_1}, {}^{p_2}/_{q_2})$. Let us define

$$S : \mathbb{N}_{\geq 2} \longrightarrow \mathbb{N}$$

as the function that maps a natural number $n \geq 2$ into the smallest number $1 \leq k \leq n$ which maximizes the quantity $\left|\sin\left(\frac{2\pi k}{n}\right)\right|$. It is easy to see that

$$S(n) = \frac{n + x_n}{4} \quad \text{with} \quad x_n = \begin{cases} 0 & \text{if } n \equiv_4 0, \\ -1 & \text{if } n \equiv_4 1, \\ -2 & \text{if } n \equiv_4 2, \\ 1 & \text{if } n \equiv_4 3. \end{cases} \tag{3.3.7}$$

Let us assume that $3 \leq p_1 \leq p_2$. The interval $J_0(p_1, p_2)$ is, by definition, the union of the subintervals $I({}^{k_1}/_{p_1}, {}^{k_2}/_{p_2})$. According to the (3.3.2), the interval $I({}^{S(p_1)}/_{p_1}, {}^{S(p_2)}/_{p_2})$ is the one of maximum length among these.

Definition 3.3.16. Let $I = (a_1, a_2) \subset [-2, 2]$. Let $\theta_1, \theta_2 \in [0, \pi]$ such that $\Re(2e^{i\theta_j}) = 2\cos(\theta_j) = a_j$. We say that the interval I is *supported* by the angle $\alpha(I) \coloneqq |\theta_2 - \theta_1|$. See Figure 3.3.

Let $I \subseteq [-2, 2]$ be the interval $(2\cos\theta_1, 2\cos\theta_2)$ as Figure 3.3. Let $\alpha \in [0, \pi]$ be the angle that supports I. We recall that $0 \leq \arccos x \leq \pi$ and that if $\theta \in [\pi, 2\pi]$ and $\psi = \theta - \pi$, then

$$\cos\theta = \cos(\pi - \psi).$$

Hence, it is straightforward to see that

$$\alpha = \left|\arccos\left(\frac{2\cos\theta_1}{2}\right) - \arccos\left(\frac{2\cos\theta_2}{2}\right)\right|.$$

Lemma 3.3.17. *Let $3 \leq p_1$ and $\alpha \in [0, \pi]$ be the angle that supports the interval $I\left(\frac{S(p_1)}{p_1}, \frac{S(p_2)}{p_2}\right)$ as in Definition 3.3.16, then $\alpha \geq \frac{2\pi}{3}$.*

Proof. According to Lemma 3.3.9, the interval $I({}^{S(p_1)}/_{p_1}, {}^{S(p_2)}/_{p_2})$ is nonempty. Let x_{p_i} be defined as in (3.3.7) for $i \in \{1, 2\}$. Definition 3.3.3 implies that the end points the interval

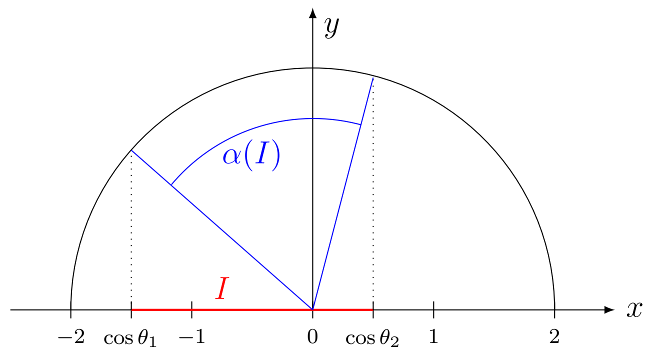


FIGURE 3.3. The angle (in blue) that supports the interval I (in red).

$I\left(\frac{S(p_1)}{p_1}, \frac{S(p_2)}{p_2}\right)$ are

$$2\cos\left(\frac{2\pi S(p_2)}{p_2} + \frac{2\pi S(p_1)}{p_1}\right) = 2\cos\left(\pi + \frac{\pi}{2}\left(\frac{x_{p_2}}{p_2} + \frac{x_{p_1}}{p_1}\right)\right) \quad \text{and}$$

$$2\cos\left(\frac{2\pi S(p_2)}{p_2} - \frac{2\pi S(p_1)}{p_1}\right) = 2\cos\left(\frac{\pi}{2}\left(\frac{x_{p_2}}{p_2} - \frac{x_{p_1}}{p_1}\right)\right)$$

Let (c_1, c_2) be an interval in $[-2, 2]$. The angle that supports the interval (c_1, c_2) is equal to $|\arccos(c_1/2) - \arccos(c_2/2)|$. If $c_1 = 2\cos(\pi \pm \gamma_1)$ and $c_2 = 2\cos(\pm\gamma_2)$ with $\gamma_1, \gamma_2 \in [0, \pi/2]$, then the angle that supports the interval (c_1, c_2) is $\pi - \gamma_1 - \gamma_2$. Hence, we obtain

$$\begin{aligned}\alpha\left(I\left(\frac{S(p_1)}{p_1}, \frac{S(p_2)}{p_2}\right)\right) &= \pi - \frac{\pi}{2}\left|\frac{x_{p_2}}{p_2} + \frac{x_{p_1}}{p_1}\right| - \frac{\pi}{2}\left|\frac{x_{p_2}}{p_2} - \frac{x_{p_1}}{p_1}\right| \\ &= \pi - \pi\frac{|x_{p_1}p_2 + x_{p_2}p_1| + |x_{p_1}p_2 - x_{p_2}p_1|}{2p_1p_2} \\ &\geq \min\left\{\pi - \pi\frac{|x_{p_1}|}{p_1}, \pi - \pi\frac{|x_{p_2}|}{p_2}\right\} \geq \frac{2\pi}{3}.\end{aligned}$$

□

Lemma 3.3.18. *Let $3 \leq p_1$, then $J_0(p_1, p_2)$ is connected.*

PROOF. The intervals $J_0(3,3) = I(1/3, 1/3)$, $J_0(3,4) = I(1/3, 1/4)$, and $J_0(4,4) = I(1/4, 1/4)$ are connected. Hence, let us suppose that $p_2 \geq 5$. We define the interval J_1 as

$$J_1 := \left(2\cos\left(\pi - \frac{2\pi}{p_2}\right), 2\cos\left(\frac{2\pi}{p_2}\right)\right) = \left(-2\cos\left(\frac{2\pi}{p_2}\right), 2\cos\left(\frac{2\pi}{p_2}\right)\right).$$

Claim 2. The interval $J_0(p_1, p_2)$ contains the connected interval J_1

We prove the conclusion assuming Claim 2, and then we prove Claim 2. Note that, since $p_2 \geq 5$, the interval J_1 is nonempty.

Let us assume Claim 2 and suppose, by contradiction, that $J_0(p_1, p_2)$ is not connected. Let J_2 be a connected component of $J_0(p_1, p_2)$ disjoint from J_1. Then

$$\text{either } J_2 \subseteq \left[-2, -2\cos\left(\frac{2\pi}{p_2}\right)\right) \quad \text{or } J_2 \subseteq \left(2\cos\left(\frac{2\pi}{p_2}\right), 2\right].$$

This means that $m(J_2) \leq 2 - 2\cos\left(\frac{2\pi}{p_2}\right) = 4\sin^2\left(\frac{\pi}{p_2}\right)$. Let $i, j \in \mathbb{Z}$ be such that the interval $I(i/p_1, j/p_2)$ has non-zero length. Let $n \in \mathbb{N}$, we remark that if k is an integer such that $k \notin \frac{n}{2}\mathbb{Z}$, then $|\sin(2\pi k/n)| \geq |\sin(\pi/n)|$. According to the (3.3.2), this implies that

$$m\left(I\left(\frac{i}{p_1}, \frac{j}{p_2}\right)\right) = \left|\sin\left(\frac{2\pi i}{p_1}\right)\sin\left(\frac{2\pi j}{p_2}\right)\right| \geq 4\left|\sin\left(\frac{\pi}{p_1}\right)\sin\left(\frac{\pi}{p_2}\right)\right| \geq 4\sin^2\left(\frac{\pi}{p_2}\right).$$

If $p_1 < p_2$, then this last inequality is strict. In this case, we get a contradiction since we supposed that the connected component J_2 has length smaller or equal than $4\sin^2\left(\frac{\pi}{p_2}\right)$.

Suppose than that $p_1 = p_2 \geq 5$. The end points of the interval $I(k-1/p_1, 1/p_1)$ are

$$2\cos\left(2\pi\frac{k}{p_1}\right) \quad \text{and} \quad 2\cos\left(2\pi\frac{k-2}{p_1}\right).$$

This implies that

$$J_3 := \bigcup_{k\in\mathbb{Z}} I\left(\frac{k-1}{p_1}, \frac{1}{p_1}\right) = \bigcup_{k'=1}^{p_1}\left(2\cos\left(\frac{2\pi k'}{p_1}\right), 2\right) \subset J_0(p_1, p_2)$$

is a connected subinterval in $J_0(p_1, p_2)$. If k_1 and k_2 are both not in $\frac{p_1}{2}\mathbb{Z}$, then the end points of the interval $I(k_1/p_1, k_2/p_1)$ are

$$2\cos\left(2\pi\frac{k_1+k_2}{p_1}\right) \quad \text{and} \quad 2\cos\left(2\pi\frac{k_1-k_2}{p_1}\right).$$

This implies that the end points of the interval $I(k_1/p_1, k_2/p_1)$ are both of the form $2\cos(2\pi k'/p_1)$, with $k' \in \mathbb{Z}$. This implies that for every $k_1, k_2 \in \mathbb{Z}$, we have $I(k_1/p_1, k_2/p_2) \subseteq J_3$. And therefore $J_0(p_1, p_1) \subseteq J_3$ that brings us to $J_0(p_1, p_2) = J_3$. The conclusion holds by the fact that J_3 is connected.

PROOF OF CLAIM 2. We shall prove that $J_1 \subseteq J_0(p_1, p_2)$ for $p_2 \geq 5$. Consider the union

$$\bigcup_{k_2\in\mathbb{Z}} I\left(\frac{S(p_1)}{p_1}, \frac{k_2}{p_2}\right) = \bigcup_{j\in\mathbb{Z}} I\left(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j}{p_2}\right). \tag{3.3.8}$$

Let $\theta_1, \theta_2 \in [0,\pi]$, be such that $I({}^{S(p_1)}/{}_{p_1}, {}^{S(p_2)}/{}_{p_2}) = (2\cos\theta_1, 2\cos\theta_2)$. The end points of $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j}{p_2}\Big)$ are:

$$\left\{2\cos\left(2\pi\frac{p_1S(p_2)+p_2S(p_1)}{p_1p_2}+\frac{2\pi j}{p_2}\right), 2\cos\left(2\pi\frac{p_1S(p_2)-p_2S(p_1)}{p_1p_2}+\frac{2\pi j}{p_2}\right)\right\} =$$

$$= \left\{2\cos\left(\theta_1+\frac{2\pi j}{p_2}\right), 2\cos\left(\theta_2+\frac{2\pi j}{p_2}\right)\right\}.$$

By Lemma 3.3.17, $|\theta_1-\theta_2| \geq \frac{2\pi}{3} > \frac{2\pi}{p_2}$. In particular $\pi \geq \theta_1 > \theta_2 + {}^{2\pi}/{}_{p_2} > \theta_2 \geq 0$. This implies that

$$2\cos\theta_2 > 2\cos\left(\theta_2+\frac{2\pi}{p_2}\right) > 2\cos\theta_1.$$

Furthermore, if $\theta_1 < \pi - {}^{\pi}/{}_{p_2}$, then $\pi - {}^{\pi}/{}_{p_2} < \theta_1 + {}^{2\pi}/{}_{p_2} < \pi + {}^{\pi}/{}_{p_2}$. Thus,

$$2\cos\theta_2 > 2\cos\left(\theta_2+\frac{2\pi}{p_2}\right) > 2\cos\theta_1 > 2\cos\left(\theta_1+\frac{2\pi}{p_2}\right).$$

Let $j \in \mathbb{N}$ be such that $\theta_1 + {}^{2\pi j}/{}_{p_2} < \pi - {}^{\pi}/{}_{p_2}$, then

$$2\cos\left(\theta_2+\frac{2\pi(j+1)}{p_2}\right) > 2\cos\left(\theta_1+\frac{2\pi j}{p_2}\right) > 2\cos\left(\theta_1+\frac{2\pi(j+1)}{p_2}\right). \tag{3.3.9}$$

Notice that the left and right sides of identity (3.3.9) are the end points of $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j+1}{p_2}\Big)$. Furthermore, the central term the in identity (3.3.9) is an end point of $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j}{p_2}\Big)$. Hence, identity (3.3.9) implies that $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j}{p_2}\Big) \cap I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j+1}{p_2}\Big)$ is nonempty. Thus, this implies that $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j}{p_2}\Big) \cup I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j+1}{p_2}\Big)$ is connected. Similarly, let $j \in \mathbb{N}$ be such that $\theta_2 + {}^{2\pi j}/{}_{p_2} \geq {}^{\pi}/{}_{p_2}$, then

$$2\cos\left(\theta_1+\frac{2\pi(j-1)}{p_2}\right) > 2\cos\left(\theta_2+\frac{2\pi j}{p_2}\right) > 2\cos\left(\theta_2+\frac{2\pi(j-1)}{p_2}\right).$$

As before, this implies that one of the end points of $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j}{p_2}\Big)$ is in $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j-1}{p_2}\Big)$. So $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j}{p_2}\Big) \cap I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j-1}{p_2}\Big)$ is nonempty. Furthermore, $I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j}{p_2}\Big) \cup I\Big(\frac{S(p_1)}{p_1}, \frac{S(p_2)+j-1}{p_2}\Big)$ is connected. Thus, the union in (3.3.8) contains the connected interval J_1. □

□

Lemma 3.3.19. *Let* $3 \leq p_1$*, the set* $J_\pi(p_1,p_2)$ *does not contain the element* $0 \in [-2,2]$ *if and only if* $p_1 = p_2 = 4$.

PROOF. The interval $I({}^{k_1}/_{2p_1}, {}^{k_2}/_{2p_2})$ has length equal to

$$m\left(I\left(\frac{k_1}{2p_1}, \frac{k_2}{2p_2}\right)\right) = 4\left|\sin\left(\frac{\pi k_1}{p_1}\right)\sin\left(\frac{\pi k_2}{p_2}\right)\right|.$$

It is easy to see that if $p_1 \geq 2$ there exists an odd integer k in the range ${}^{p_1}/_4 \leq k \leq {}^{3p_1}/_4$. For such a k we have

$$\left|\sin\left(\frac{2\pi k}{2p_1}\right)\right| = \left|\sin\left(\frac{\pi k}{p_1}\right)\right| \geq \frac{\sqrt{2}}{2}.$$

Moreover, if $p_1 \neq 4$, we can choose k to make this inequality strict. Thus, if $(p_1, p_2) \neq (4,4)$, then there exists an interval $I({}^{k_1}/_{2p_1}, {}^{k_2}/_{2p_2}) \subseteq J_\pi(p_1, p_2)$ whose length is strictly bigger than 2. This implies that $0 \in I({}^{k_1}/_{2p_1}, {}^{k_2}/_{2p_2})$. A direct computation shows that $0 \notin J_\pi(4,4)$ and this gives the conclusion. □

Lemma 3.3.20. *Let* $3 \leq p_1$ *and let* $t_1 = \gcd(p_1, p_2)$*, then*

$$\mathcal{S}(p_1,p_2) \coloneqq \left\{k_1p_2 + k_2p_1 \,\middle|\, k_i \in \mathbb{Z}, k_i \notin \frac{p_i}{2}\mathbb{Z}\right\} = \begin{cases} \mathbb{Z} \setminus \left(\frac{p_1}{2}\mathbb{Z} \cup \frac{p_2}{2}\mathbb{Z}\right) & \textit{if } t_1 = 1, \\ 2\mathbb{Z} \setminus (p_1\mathbb{Z} \cup p_2\mathbb{Z}) & \textit{if } t_1 = 2, \\ t_1\mathbb{Z} & \textit{if } t_1 \geq 3. \end{cases}$$

PROOF. Let us start with $t_1 = 1$. Let $x = k_1p_2 + k_2p_1 \in \mathcal{S}(p_1,p_2)$. Let us suppose by contradiction that $x \in \frac{p_1}{2}\mathbb{Z}$. This implies that $k_1p_2 \in \frac{p_1}{2}\mathbb{Z}$. Since $\gcd(p_1,p_2) = 1$, we obtain that $k_1 \in \frac{p_1}{2}\mathbb{Z}$ that is a contradiction. Similarly, if $x \in \frac{p_2}{2}\mathbb{Z}$, then $k_2p_1 \in \frac{p_2}{2}\mathbb{Z}$. Hence, $k_2 \in \frac{p_2}{2}\mathbb{Z}$ that is a contradiction. Conversely, let $x \in \mathbb{Z} \setminus \left(\frac{p_1}{2}\mathbb{Z} \cup \frac{p_2}{2}\mathbb{Z}\right)$. Let us write $x = k_1p_2 + k_2p_1$. Let us suppose, by contradiction, that $k_1 \in \frac{p_1}{2}\mathbb{Z}$. This implies that $x \in \frac{p_1}{2}\mathbb{Z}$, that is a contradiction.

Let us suppose that $t_1 = 2$. We write $p_1 = 2n_1$ and $p_2 = 2n_2$, with $\gcd(n_1,n_2) = 1$. Let $x = 2k_1n_2 + 2k_2n_1 \in \mathcal{S}(p_1,p_2)$. We note that since $p_1 \geq 3$, the set $\mathcal{S}(p_1,p_2)$ is nonempty. Let us suppose, by contradiction, that $x \equiv_{2n_1} 0$. This implies that $2k_1n_2 \equiv_{2n_1} 0$. We obtain that $k_1 \equiv_{n_1} 0$, this is a contradiction. Similarly, if $x \equiv_{2n_2} 0$, then $2k_2n_1 \equiv_{2n_2} 0$. Thus, $k_2 \equiv_{n_2} 0$ and this is a contradiction. Conversely, let $x \in 2\mathbb{Z} \setminus (p_1\mathbb{Z} \cup p_2\mathbb{Z})$. Let us write x as $k_1p_2 + k_2p_1$ and suppose that $k_1 \equiv_{n_1} 0$. This assumption implies that $2k_1 \equiv_{2n_1} 0$ and hence $x \equiv_{p_1} 0$ that is a contraction. Similarly, if $k_2 \equiv_{n_2} 0$, then $x \equiv_{p_2} 0$, and here is our contraction.

Let us move to the case $t_1 \geq 3$. Notice that $\mathcal{S}(p_1,p_2) \subseteq t_1\mathbb{Z}$. Let us suppose that $x \in t_1\mathbb{Z}$ and let us write $x = p_2k_1 + p_1k_2$. More generally, we have that for every $n \in \mathbb{N}$

$$x = p_1k_{2,n} + p_2k_{1,n} \quad \text{with} \quad k_{1,n} = k_1 + n\left(\frac{p_1}{t_1}\right) \quad \text{and} \quad k_{2,n} = k_2 - n\left(\frac{p_2}{t_1}\right).$$

We need to show that there exists a $n \in \mathbb{N}$ such that

$$k_{1,n} \notin \frac{p_1}{2}\mathbb{Z} \quad \text{and} \quad k_{2,n} \notin \frac{p_2}{2}\mathbb{Z}. \tag{3.3.10}$$

If $k_{1,0} = k_1$ and $k_{2,0} = k_2$ are not in $\frac{p_1}{2}\mathbb{Z}$ and $\frac{p_2}{2}\mathbb{Z}$ respectively, we get the conclusion. Suppose then without loss of generality that $k_{1,0} \in \frac{p_1}{2}\mathbb{Z}$. Then $k_{1,1} = k_1 + \frac{p_1}{t_1} \notin \frac{p_1}{2}\mathbb{Z}$. If the couple $(k_{1,1}, k_{2,1})$ has the property in (3.3.10), then we get the conclusion. Let us suppose that $k_{2,1} = k_2 + \frac{p_2}{t_1} \in \frac{p_2}{2}\mathbb{Z}$. Then, since $t_1 \geq 3$, we obtain $k_{1,2} \notin \frac{p_2}{2}\mathbb{Z}$ and $k_{2,2} \notin \frac{p_2}{2}\mathbb{Z}$. This implies that $x \in \mathcal{S}(p_1,p_2)$ and this completes the proof. □

Let $\mathcal{S}(p_1,p_2)$ be as in Lemma 3.3.20, we define $x_{\min}, x_{\max} \in [-2,2]$ as

$$x_{\min} = \min_{x\in\mathcal{S}(p_1,p_2)}\left\{2\cos\left(\frac{2\pi x}{p_1p_2}\right)\right\} \quad \text{and} \quad x_{\max} = \max_{x\in\mathcal{S}(p_1,p_2)}\left\{2\cos\left(\frac{2\pi x}{p_1p_2}\right)\right\}. \tag{3.3.11}$$

Lemma 3.3.21. *Let $x_{\min}, x_{\max} \in [-2,2]$ be as in (3.3.11), then $J_0(p_1,p_2) = (x_{\min}, x_{\max})$.*

Proof. Let $k_1, k_2 \in \mathbb{Z}$ be two integers such that $k_i \notin \frac{p_i}{2}\mathbb{Z}$. As we proved in Lemma 3.3.9, the interval $I(k_1/p_1, k_2/p_2)$ is nonempty and $I(k_1/p_1, k_2/p_2) \subseteq J_0(p_1,p_2)$. In particular,

$$\partial I\left(\frac{k_1}{p_1}, \frac{k_2}{p_2}\right) = \left\{2\cos\left(\frac{2\pi(k_1p_2 \pm k_2p_1)}{p_1p_2}\right)\right\} \subset \overline{J_0(p_1,p_2)},$$

Thus, the interval $\overline{J_0(p_1,p_2)}$ must contain $2\cos(2\pi x/p_1p_2)$ for all $x \in \mathcal{S}(p_1,p_2)$. Since $J_0(p_1,p_2)$ is connected by Lemma 3.3.18, we obtain that $\overline{J_0(p_1,p_2)} = [x_{\min}, x_{\max}]$ and hence $J_0(p_1,p_2) = (x_{\min}, x_{\max})$. □

Corollary 3.3.22. *Suppose $p_1 \geq 3$ and let α be the angle that supports the interval $J_0(p_1,p_2)$, then $\alpha \geq \pi - \frac{4\pi t_1}{p_1p_2} \geq \pi - \frac{4\pi}{p_2}$.*

Lemma 3.3.23. *If $p_2 \geq 2$, then $J_\pi(2,p_2) = I\left(\frac{1}{4}, \frac{k}{2p_2}\right)$, where k is an odd number satisfying $1 \leq k \leq p_2$ that maximizes $|\sin(\pi k/p_2)|$. In particular $J_\pi(2,p_2)$ is connected and supported by an angle greater than or equal to $\pi - \frac{2\pi}{p_2}$.*

PROOF. Let $I\left(\frac{1}{4}, \frac{k_2}{2p_2}\right)$ be an interval of $J_\pi(2, p_2)$. The end points of $I\left(\frac{1}{4}, \frac{k_2}{2p_2}\right)$ are

$$\begin{aligned}\partial I\left(\frac{1}{4}, \frac{k_2}{2p_2}\right) &= \left\{2\cos\left(2\pi\frac{2p_2 \pm 4k_2}{8p_2}\right)\right\} \\ &= \left\{2\cos\left(\frac{\pi}{2} + \frac{\pi k_2}{p_2}\right), 2\cos\left(\frac{\pi}{2} - \frac{\pi k_2}{p_2}\right)\right\} = \left\{\pm 2\sin\left(\frac{\pi k_2}{p_2}\right)\right\}.\end{aligned} \tag{3.3.12}$$

Hence, all intervals in $J_\pi(2, p_2)$ are nested, and we conclude that $J_\pi(2, p_2)$ is connected. Let $\alpha \in [0, \pi]$ be the angle supporting $J_\pi(2, p_2)$. The identity (3.3.12) implies that

$$\alpha \geq \pi - \frac{\pi}{p_2} - \frac{\pi}{p_2} = \pi - \frac{2\pi}{p_2}.$$

□

Corollary 3.3.24. *If $p_2 \geq 2$, then $J_\pi(2, p_2)$ is supported by an angle greater than or equal to $\pi/2$. Moreover, $J_\pi(2, 4) \subseteq J_\pi(2, p_2)$ for every $p_2 \geq 2$. This last inclusion is strict if $p_2 \neq 4$.*

PROOF. Let $\alpha_{p_2} \in [0, \pi]$ be the angle supporting $J_\pi(2, p_2)$. A direct computation shows that $\alpha_{p_2} > \pi/2$ if $p_2 = 2, 3$. Moreover, if $p_2 \geq 4$, then $\alpha_{p_2} \geq \pi/2$ by Lemma 3.3.23.

Let $p_2 \geq 2$. Lemma 3.3.23 implies that there exists a real number $x \in (0, 2]$ such that $J_\pi(2, p_2) = (-x, x)$. Since $\alpha(J_\pi(2, p_2)) \geq \pi/2$ and $\alpha(J_\pi(2, 4)) = \pi/2$, we obtain that $J_\pi(2, 4)$ is the smallest possible interval among the ones of the form $J_\pi(2, p_2)$. This also implies that $J_\pi(2, 4) \subseteq J_\pi(2, p_2)$. The identity (3.3.12) implies that if $p_2 \neq 4$, then $J_\pi(2, 4) \neq J_\pi(2, p_2)$. Thus if $p_2 \neq 4$, then $J_\pi(2, 4) \subset J_\pi(2, p_2)$. □

Chapter 4

The Intersections

Let $Y = Y_1 \cup_\Sigma Y_2$ be a graph manifold rational homology sphere such that $\Sigma \subset Y$ is the unique JSJ torus of Y as in (2.3.1). We recall that, as an application of Corollary 2.2.9, we can suppose that Y_1 and Y_2 both admit a fibration with disk base space. In particular we supposed that Y_1 has $n \in \mathbb{N}$ singular fibres and Y_2 has $m \in \mathbb{N}$ singular fibres. Without loss of generality, we suppose that $n \leq m$.

We remind the reader that we lighten the notation by naming the set $A(Y_i)$ (resp. $H(Y_i)$ and $P(Y_i)$) as A_i (resp. H_i and P_i). In this chapter we study the intersections

$$A_1 \cap H_2, \quad H_1 \cap A_2, \quad H_1 \cap H_2, \quad \text{and} \quad P_1 \cap P_2,$$

in order to be able to apply Theorem 2.1.8 and determine the $SU(2)$-abelian status of $Y = Y_1 \cup_\Sigma Y_2$.

We recall that $\mathcal{O}(Y_1)$ indicates the vector whose entries are the orders of the nontrivial singular fibres of Y_1 in ascending order. Thus if $Y_1 = \mathbb{D}^2(p_1/q_1, \cdots, p_n/q_n)$ with $2 \leq p_1 \leq \cdots \leq p_n$, then $\mathcal{O}(Y_1) = (p_1, \cdots, p_n)$. Corollary 2.2.9 implies that if Y_1 has $n \geq 3$ singular fibres, then

$$\mathcal{O}(Y_1) \in \{(2,4,4), \cdots, (3,3,3), (2, \cdots, 2, p_n)\}.$$

4.1. The intersections between P_1 and P_2

As we stated in Chapter 2, if

$$Y = Y_1 \cup_\Sigma Y_2 = Y_1 \cup_\varphi Y_2,$$

then we represented the induced isomorphism $\varphi_*\colon \pi_1(\partial Y_1) \to \pi_1(\partial Y_2)$ with respect to the order basis $\{\mu_1, h_1\}$ and $\{\mu_2, h_2\}$ by the matrix

$$\varphi_* = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL_2(\mathbb{Z}).$$

In particular, $|\beta| = \Delta(h_1, h_2)$. Since Y is not a Seifert fibred manifold, $|\beta| = \Delta(h_1, h_2)$ is bigger then or equal to 1.

We prove now a lemma that leads us to an obstruction on the number of the fibres of Y_1 and Y_2.

Lemma 4.1.1. *If Y_1 is fibred over a disk and $\mathcal{O}(Y_1) \in \{(2,4,4), (3,3,3), (2,\cdots,2,p_n)\}$, then*

$$\left\{\eta \in \mathcal{R}_{U(1)}(\partial Y_1) \middle| \eta(h_1) = -1\right\} \subseteq H_1 \cup P_1.$$

Proof. In all three cases, we suppose that $\pi_1(Y_1)$ is presented as in (2.3.3) with q_i odd and we assume $\mathcal{R}_{U(1)}(\partial Y_1)$ to have coordinates (θ_1, ψ_1) as in (2.3.7). Thus,

$$\left\{\eta \in \mathcal{R}_{U(1)}(\partial Y_1) \middle| \eta(h_1) = -1\right\} = \{\psi_1 = \pi\}.$$

If $\mathcal{O}(Y_1)(2,\cdots,2,p_n)$, then the conclusion holds by Lemma 3.3.10 and Corollary 3.3.13.

If $\mathcal{O}(Y_1) = (2,4,4)$, then $J_\pi(2,4,4) = (-2,2)$. Corollary 3.3.13 implies that

$$H_{1,\pi} = \left\{(\theta_1, \psi_1) \in \mathcal{R}_{U(1)}(\partial Y_1) \middle| \psi_1 = \pi, \theta_1 \notin \pi\mathbb{Z}\right\} = \{\psi_1 = \pi\} \setminus \{0,\pi\}^2.$$

According to Lemma 3.2.7, $P_1 = \{0,\pi\}^2$. Therefore,

$$\{\psi_1 = \pi\} \subseteq H_{1,\pi} \cup \{0,\pi\}^2 \subseteq H_1 \cup P_1.$$

If $\mathcal{O}(Y_1) = (3,3,3)$, then $J_\pi(3,3,3) = (-2,2]$. As an application of Corollary 3.3.13

$$H_{1,\pi} = \left\{(\theta_1, \psi_1) \in \mathcal{R}_{U(1)}(\partial Y_1) \middle| \psi_1 = \pi, \theta_1 \neq \pi\right\} = \{\psi_1 = \pi\} \setminus \{(\pi,\pi)\}.$$

Lemma 3.2.6 implies that the representation $\eta : \pi_1(\partial Y_1) \to SU(2)$ with $\eta(h_1) = \eta(\mu_1) = -1$ is in P_1. The representation η has coordinates (π,π) and therefore,

$$\{\psi_1 = \pi\} \subseteq H_{1,\pi} \cup P_1 \subseteq H_1 \cup P_1.$$

□

Lemma 4.1.2. *Let $Y = Y_1 \cup_\Sigma Y_2$ be the graph manifold as above. If Y_1 has $n \geq 3$ singular fibres and Y_2 has $m \geq 3$ singular fibres, then manifold Y is not $SU(2)$-abelian.*

PROOF. According to Corollary 2.2.9, we can suppose that

$$\mathcal{O}(Y_1), \mathcal{O}(Y_2) \in \{(2,4,4),(3,3,3),(2,\cdots,2,p_n)\}.$$

For $i \in \{1,2\}$, we assume $\mathcal{R}_{U(1)}(\partial Y_i)$ to have coordinates (θ_i, ψ_i) as in (2.3.7). We recall that in Y we have that $\partial Y_1 = \partial Y_2 = \Sigma$. Therefore, the space $\mathcal{R}_{U(1)}(\partial Y_1)$ coincide with $\mathcal{R}_{U(1)}(\partial Y_2)$. Thus, (θ_1, ψ_1) and (θ_2, ψ_2) are two parameterizations of the space space. The presentation (2.3.6) implies that

$$\theta_1 = \alpha\theta_2 + \gamma\psi_2 \quad \text{and} \quad \psi_1 = \beta\theta_2 + \delta\psi_2.$$

We recall that $|\beta| \geq 1$. The line

$$L := \left\{(\alpha\theta_2 + \gamma\pi, \beta\theta_2 + \delta\pi) \in \mathcal{R}_{U(1)}(\partial Y_1) \middle| \theta_2 \in [0,2\pi]\right\} \subset \mathcal{R}_{U(1)}(\partial Y_1)$$

is equal to the line $\{\psi_2 = \pi\} \subset \mathcal{R}_{U(1)}(\partial Y_2)$ in (θ_1, ψ_1) coordinates. Since $|\beta| \geq 1$,

$$\{\psi_1 = \pi\} \cap L = \left\{\left(\alpha\left(\frac{\pi(\delta-1)}{|\beta|} + \frac{2\pi k}{|\beta|}\right) + \gamma\pi, \pi\right), k \in \{1,\cdots,|\beta|\}\right\} \subset \mathcal{R}_{U(1)}(\partial Y_1)$$

In particular, this intersection is not empty. By Lemma 4.1.1,

$$\{\psi_1 = \pi\} \subseteq H_1 \cup P_1, \quad \text{and} \quad L \subseteq H_2 \cup P_2.$$

Thus $H_1 \cup P_1$ has a nonempty intersection with $H_2 \cup P_2$. We recall that $P_i \subset A_i$ by Definition 2.1.5. Therefore, either $H_1 \cap H_2$, $A_1 \cap H_2$, $H_1 \cap P_2$, or $P_1 \cap P_2$ is not empty. The conclusion holds by Theorem 2.1.8. □

We are going to focus on the case $n = m = 2$. Thus $Y_1 = \mathbb{D}^2({}^{p_1}/_{q_1}, {}^{p_2}/_{q_2})$ and $Y_2 = \mathbb{D}^2({}^{r_1}/_{s_1}, {}^{r_2}/_{s_2})$. We recall that t_i is the order of the torsion subgroup of $H_1(Y_i;\mathbb{Z})$. We assume that $t_1 = \gcd(p_1,p_2) \leq t_2 = \gcd(r_1,r_2)$.

Proposition 4.1.3. *Let $Y = Y_1 \cup_\Sigma Y_2$ be the graph manifold as above. Let us suppose that both Y_1 and Y_2 have exactly two singular fibres. The set $P_1 \cap P_2$ is empty if and only if one of the following conditions holds:*

(1) $t_1 = 1$*;*

(2) $t_1 = 2$ *and* $o_1 = 1$*;*

(3) $t_2 = 2$ *and* $o_2 = 1$*;*

(4) $t_1 = 2$*,* o_2 *odd and* $\Delta(h_1, \lambda_2)$ *is even.*

Proof. Let us suppose that either the condition (1) or (2) holds. By Lemma 3.2.5, the set P_1 is empty. Thus, $P_1 \cap P_2 = \emptyset$. Similarly, if the condition (3) holds, then Lemma 3.2.5 implies that $P_2 = \emptyset$ and then, $P_1 \cap P_2 = \emptyset$.

Let us suppose that the condition (4) holds. Without loss of generality, we can suppose that conditions (1), (2), and (3) do not hold. This implies that $t_1 = o_1 = 2$. Let $\rho \in \mathcal{R}(Y)$ be such that $\rho(\pi_1(\Sigma)) \subset \mathcal{Z}(SU(2))$. We will show that $\rho_1(h_1) = 1$. According to Lemma 3.2.5, this implies that ρ_1 is a central. Thus, $P_1 \cap P_2 = \emptyset$.

Since $\rho(\pi_1(\Sigma)) \subset \mathcal{Z}(SU(2))$, the restrictions $\rho_1 = \rho|_{\pi_1(Y_1)}$ and $\rho_2 = \rho|_{\pi_1(Y_2)}$ are abelian. Let ξ be an oriented simple closed curve in ∂Y_2 such that $\{\lambda_2, \xi\}$ generates $\pi_1(\partial Y_2)$. The representation ρ maps both λ_2 and ξ into ± 1. Since ρ_2 is abelian, as a consequence of Corollary 2.1.12, we obtain that

$$\rho(\lambda_2)^{o_2} = \rho_2(\lambda_2)^{o_2} = 1.$$

Since o_2 is odd by hypothesis, $\rho(\lambda_2) = 1$. There exist integers $n, m \in \mathbb{Z}$, with $|n| = \Delta(\xi, h_1)$ and $|m| = \Delta(\lambda_2, h_1)$, such that

$$h_1 = n \cdot \lambda_2 + m \cdot \xi \subset \partial Y_1, \quad \text{and hence} \quad \rho_1(h_1) = \cancel{\rho_2(\lambda_2)^n}\rho_2(\xi)^m = \rho_2(\xi)^m. \tag{4.1.1}$$

Since $\Delta(h_1, \lambda_2) = |m|$ is even, we get that $\rho_1(h_1) = 1$.

Conversely, let us suppose that $P_1 \cap P_2 = \emptyset$. This implies that if the representation $\rho \in \mathcal{R}(Y)$ is such that $\rho(\pi_1(\Sigma)) \subset \mathcal{Z}(SU(2))$ then, either ρ_1 or ρ_2 is central. Let $\eta \colon \pi_1(\Sigma) \to \{1\}$ be the trivial representation. Then, we have $\eta(\lambda_1)^{o_1} = \eta(\lambda_2)^{o_2} = 1$. According to Corollary 2.1.12, the representation η extends to a representation $\rho \colon \pi_1(Y) \to SU(2)$. Since either ρ_1 or ρ_2 is $SU(2)$-central, Lemma 3.2.3 implies that either $t_1 \leq 2$ or $t_2 \leq 2$. Since we supposed that $t_1 \leq t_2$, we obtain $t_1 \leq 2$.

We complete the proof by proving that if conditions (1), (2), and (3) do not hold, then condition (4) holds. If conditions (1), (2), and (3) do not hold, then, since $t_1 \leq 2$, we get that $o_1 = t_1 = 2$ and either $o_2 = t_2 = 2$ or $t_2 \geq 3$.

Suppose first that $o_2 \equiv_2 0$. Then $t_2 \equiv_2 0$ by Lemma 3.1.1. Since o_1 and o_2 are both even, Corollary 2.1.12 implies that every central representation $\pi_1(\Sigma) \to \mathcal{Z}(SU(2))$ extends to both $\pi_1(Y_1)$ and $\pi_1(Y_2)$. Let $\eta\colon \pi_1(\Sigma) \to \mathcal{Z}(SU(2))$ be a representation such that $\eta(h_1) = \eta(h_2) = -1$. Lemma 3.2.1 implies that there exist two non-central representations $\rho_1 \in \mathcal{R}(Y_1)$ and $\rho_2 \in \mathcal{R}(Y_2)$ such that $\rho_1|_{\pi_1(\Sigma)} \equiv \eta$ and $\rho_2|_{\pi_1(\Sigma)} \equiv \eta$. This implies that $\eta \in P_1$ and $\eta \in P_2$. Thus, $P_1 \cap P_2 \neq \emptyset$, and this is a contradiction. This implies that if $P_1 \cap P_2 = \emptyset$, and the conditions (1), (2), and (3) do not hold, then o_2 is odd.

Suppose next that o_2 is odd. Since $2 = t_1 \leq t_2$ and condition (3) do not hold, we obtain that $t_2 \geq 3$. Lemma 3.2.2 implies that there exists a representation $\eta\colon \pi_1(\Sigma) \to \mathcal{Z}(SU(2))$ that extends to $\pi_1(Y_2)$ in a non-central way. In particular $\eta \in P_2$. Since $o_1 = 2$, the representation η extends to $\pi_1(Y_1)$ as well. Since $P_1 \cap P_2 = \emptyset$ by hypothesis, if $\rho_1\colon \pi_1(Y_1) \to SU(2)$ is an extension of η, then ρ_1 is central. Notice that, since $o_1 = 2$, then η extends to $\pi_1(Y_1)$ by Corollary 2.1.12. Since $t_1 = o_1 = 2$, Lemma 3.2.5 implies that ρ_1 is central if and only if $\rho_1(h_1) = 1$. The identity (4.1.1) implies that this happens if and only if $\Delta(h_1, \lambda_2)$ is even. Thus, $P_1 \cap P_2 = \emptyset$, and the conditions (1), (2), and (3) do not hold, then condition (4) applies. This concludes the theorem. □

It is conjectured that $SU(2)$-abelian rational homology 3-spheres are Heegaard Floer L-spaces (see Conjecture 2). It is known that the converse of this conjecture is false, as shown below. By [BGW11, Theorem 5], the gluing of two twisted I-bundles over the Klein bottle is an L-space if and only if the gluing map does not identify the two rational longitudes. However, we prove in Example 4.1.4 that such a manifold is not $SU(2)$-abelian.

Example 4.1.4. Let Y_1 and Y_2 be two copies of the twisted I-bundle over the Klein bottle. Hence, $Y_1 = Y_2 = \mathbb{D}^2(2/1, 2/1)$. Let $\varphi : \partial Y_1 \to \partial Y_2$ be a diffeomorphism. Proposition 4.1.3 implies that the graph manifold $Y_1 \cup_\Sigma Y_2 = Y_1 \cup_\varphi Y_2$ admits an irreducible $SU(2)$-representation whose restriction to $\pi_1(\Sigma)$ is in $P_1 \cap P_2$. This implies that it does not exist a diffeomorphism $\varphi : \partial Y_1 \to \partial Y_2$ such that $Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian.

A more classical example of a non-$SU(2)$-abelian L-space is the Poincaré homology sphere $\Sigma(2,3,5)$. It is known that the Poincaré homology sphere $\Sigma(2,3,5)$ is the double branched cover of the alternating pretzel knot $P(2,3,5)$. As a consequence of [OS05, Proposition 3.3], since $P(2,3,5)$ is alternating it has an L-space double branched cover and therefore the Poincaré homology sphere $\Sigma(2,3,5)$ is an L-space. Moreover, the Poincaré homology sphere $\Sigma(2,3,5)$ is a Seifert fibred space with base S^2 and with three singular fibres of orders $(2,3,5)$, and according to Theorem 2.2.8, it is not $SU(2)$-abelian.

We now would like to write a result à la Proposition 4.1.3 but for the case where $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$ has two nontrivial singular fibres and $Y_2 = \mathbb{D}^2(p_1/q_1, \cdots, p_m/q_m)$ has $m \geq 3$ singular fibres. We recall that

$$P(Y_1) \subset \{0,\pi\}^2 \subset \mathcal{R}_{U(1)}(\partial Y_1).$$

Proposition 4.1.5. *Let $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$ and $Y_2 = \mathbb{D}^2(p_1/q_1, \cdots, p_m/q_m)$ be as above.*

- *If $\mathcal{O}(Y_2) \in \{(2,4,4),(3,3,3)\}$, then $P_1 \cap (P_2 \cup H_2) = \emptyset$ if and only if either $t_1 = 1$ or $t_1 = 2$ and $o_1 = 1$;*
- *If $\mathcal{O}(Y_2) = (2,\cdots,2,p_m)$ and $\Delta(\lambda_1, h_2) = 0$, then $P_1 \cap (P_2 \cup H_2) = \emptyset$ if and only if either $o_1 \equiv_2 1$.*

Proof. For $i \in \{1,2\}$, we assume $\mathcal{R}_{U(1)}(\partial Y_i)$ to have coordinates (θ_i, ψ_i) as in (2.3.7).

Let us suppose that $\mathcal{O}(Y_2) \in \{(2,4,4),(3,3,3)\}$. According to Lemma 3.2.6, Lemma 3.2.7, and Corollary 3.3.13

$$\{0,\pi\}^2 \subset P_2 \cup H_2.$$

Therefore, $P_1 \cap (P_2 \cup H_2) = \emptyset$ if and only if $P_1 = \emptyset$. By Lemma 3.2.5, the latter occur if and only if either $t_1 = 1$ or $t_1 = 2$ and $o_1 = 1$.

Let us suppose that $\mathcal{O}(Y_2) = (2,\cdots,2,p_m)$. If $o_1 \equiv_2 0$, then by Lemma 3.2.5 we have that $P_1 = \{0,\pi\}^2$. Lemma 3.3.10 and Corollary 3.3.13 implies that H_2 contains two points of $\{0,\pi\}^2$. Thus $P_1 \cap (P_2 \cup H_2) \neq \emptyset$. Conversely, let us suppose that $o_1 \equiv_2 1$. Since $\Delta(\lambda_1, h_2) = 0$ by hypothesis, according to Corollary 2.1.12,

$$A_1 = \left\{\psi_2 = \frac{2\pi k}{o_1}\right\}_{k \in \{1,\cdots,o_2\}} \subset \mathcal{R}_{U(1)}(\partial Y_2).$$

In particular, since $o_1 \equiv_2 1$, the line $\{\psi_2 = \pi\}$ has no intersection with A_1. Since by definition, $P_1 \subseteq A_1 \cap \{0,\pi\}^2$, we have that

$$P_1 \subseteq A_1 \cap \{0,\pi\}^2 \subset \{\psi_2 = 0\}.$$

Lemma 3.2.8 states that if $\eta\colon \pi_1(\partial Y_2) \to \mathcal{Z}(SU(2))$ is such that $\eta(h_2) = 1$, then η is not in P_2. Corollary 3.3.13 and Lemma 3.3.10 imply that H_2 is disjoint from the line $\{\psi_2 = 0\}$. Therefore,

$$P_1 \cap (P_2 \cup H_2) = \emptyset.$$

□

4.2. The intersection between H_1 and A_2 and between H_2 and A_1

Let $Y = Y_1 \cup_\Sigma Y_2$ be our usual graph manifold rational homology 3-sphere. We use the coordinates (θ_1, ψ_1) for the space $\mathcal{R}_{U(1)}(\partial Y_1)$ with respect to the basis $\{\mu_1, h_1\}$ as in (2.3.7). Choose integers $n, m \in \mathbb{Z}$ such that

$$\lambda_2 = nh_1 + m\mu_1 \subset \partial Y_1. \tag{4.2.1}$$

Then $|n| = \Delta(\lambda_2, \mu_1)$ and $|m| = \Delta(\lambda_2, h_1)$. Define a subspace $L \subset \mathcal{R}_{U(1)}(\partial Y_1)$ by

$$L = \{(\theta_1, \psi_1) \,|\, o_2 n\psi_1 + o_2 m\theta_1 \equiv_{2\pi} 0\} \subset \mathcal{R}_{U(1)}(\partial Y_1).$$

Let $\eta\colon \pi_1(\partial Y_1) \to SU(2)$ be a representation such that $\eta \in L$, then

$$\eta(\lambda_2)^{o_2} = (\eta(h_1)^n \eta(\mu_1)^m)^{o_2} = e^{i(o_2 n\psi_1 + o_2 m\theta_1)} = 1.$$

Therefore, as a result of Corollary 2.1.14, we obtain that the representation η is in A_2. In fact, L is the set A_2 in (θ_1, ψ_1) coordinates. Let us suppose that $\Delta(\lambda_2, h_1) \neq 0$, then there are $o_2\Delta(\lambda_2, h_1)$ equally spaced intersections between A_2 and the line $\{\psi_1 = 0\}$. Similarly, if $\Delta(\lambda_2, h_1) \neq 0$, then there are $o_2\Delta(\lambda_2, h_1)$ equally spaced intersection points of A_2 with the line $\{\psi_1 = \pi\}$.

Lemma 4.2.1. *Let $Y = Y_1 \cup_\Sigma Y_2$ be as above. If Y_1 has exactly two singular fibres and $5 \leq o_2\Delta(\lambda_2, h_1)$, then $H_1 \cap A_2$ is not empty.*

Proof. If $p_1 = 2$, then $\alpha(J_\pi(2, p_2)) \geq \pi/2$ by Corollary 3.3.24. Furthermore, if $p_1 \geq 3$, then $\alpha(J_0(p_1, p_2)) \geq 2\pi/3 > \pi/2$ by Lemma 3.3.17. Thus, according to Lemma 3.3.12, either $H_{1,0}$ or $H_{1,\pi}$ contains a connected subset of length at least $\pi/2$.

As we said before, if $\Delta(\lambda_2, h_1) \neq 0$, then A_2 intersects the line $\{\psi_1 = 0\}$ (resp. $\{\psi_\pi\}$) in $o_2\Delta(\lambda_2, h_1)$ equally spaced points. Hence since $o_2\Delta(\lambda_2, h_1) \geq 5$ and either $H_{1,0}$ or $H_{1,\pi}$

contains a connected subset of length $\pi/2$, then A_2 intersects $H_1 = H_{1,0} \cup H_{1,\pi}$ at least once. □

A consequence of Lemma 4.2.1 is that if $H_1 \cap A_2$ is empty, then $o_2\Delta(\lambda_2, h_1) \leq 4$.

Proposition 4.2.2. *Let us suppose that Y_1 has exactly two singular fibres. The intersection $H_1 \cap A_2$ is empty if and only if one of the following holds:*

- $\Delta(\lambda_2, h_1) = 0$, $p_1 = 2$ *and* $o_2 \equiv_2 1$ *;*
- $\Delta(\lambda_2, h_1) = 1$ *and* $o_2 \leq 2$
- $\Delta(\lambda_2, h_1) = 1$, $o_2 = 3$, $p_1 = p_2 = 3$, *and* $\Delta(\lambda_2, \lambda_1) \equiv_2 0$,
- $\Delta(\lambda_2, h_1) = 2$, $p_1 = p_2 = 4$ *and* $o_2 = 1$,
- $\Delta(\lambda_2, h_1) = 3$, $p_1 = p_2 = 3$, $o_2 = 1$, *and* $\Delta(\lambda_2, \lambda_1) \equiv_2 0$,
- $\Delta(\lambda_2, h_1) = 4$, $p_1 = 2$, $p_2 = 4$, *and* $o_2 = 1$.

Proof. We assume that $\pi_1(Y_1)$ and $\pi_1(Y_2)$ are presented as in (2.3.3) with all the q_i odd. We recall that Lemma 3.3.12 states that the maps

$$f_0\colon H_{1,0} \to J_0(p_1,p_2) \qquad \text{and} \qquad f_\pi\colon H_{1,\pi} \to J_\pi(p_1,p_2)$$
$$\eta \mapsto \operatorname{Tr}\eta(\mu_1) \qquad\qquad\qquad\qquad \eta \mapsto \operatorname{Tr}\eta(\mu_1)$$

are surjective. We recall that, $\eta \in \mathcal{R}_{U(1)}(\partial Y_1)$ has coordinates equal to (θ_1, ψ_1), then $\operatorname{Tr}\eta(\mu_1) = 2\cos\theta_1$.

As we stated before, if $o_2\Delta(\lambda_2, h_1) \geq 5$, then $H_1 \cap A_2$ is nonempty. Hence, we can suppose that $o_2\Delta(\lambda_2, h_1) \leq 4$. Thus, $0 \leq \Delta(\lambda_2, h_1) \leq 4$ and if this latter is positive, $o_2 \leq \frac{4}{\Delta(\lambda_2,h_1)}$. Notice that the quantities $\Delta(\lambda_2, h_1)$ and $\Delta(\lambda_2, \mu_1)$ are coprime.

If $o_2\Delta(\lambda_2, h_1) = 3$, then either $\Delta(\lambda_2, h_1) = 1$ and $o_2 = 3$ or $\Delta(\lambda_2, h_1) = 3$ and $o_2 = 1$. We divide the proof in five cases:

i) $\Delta(\lambda_2, h_1) = 0$,
ii) $\Delta(\lambda_2, h_1) = 1$ and $o_2 \neq 3$,
iii) $\Delta(\lambda_2, h_1) = 2$,
iv) $\Delta(\lambda_2, h_1) = 4$,
v) $o_2\Delta(\lambda_2, h_1) = 3$.

Case i) $\boldsymbol{\Delta(\lambda_2, h_1) = 0}$. In this case

$$A_2 = \{(\theta_1, \varphi_1) | \theta_1 \in [0, 2\pi] \text{ and } o_2\Delta(\lambda_2, \mu_1)\psi_1 \equiv_{2\pi} 0\} \subset \mathcal{R}_{U(1)}(\partial Y_1).$$

In particular $A_2 \cap H_{1,0} = H_{1,0}$. Thus, $A_2 \cap H_{1,0} = \emptyset$ if and only if $H_{1,0}$ is empty. By Lemma 3.3.9 and Lemma 3.3.12, this happens if and only if $p_1 = 2$. If $p_1 = 2$, then $H_1 \cap A_2$ is empty if and only if $H_{1,\pi} \cap A_2$ is empty as well. Since, by Lemma 3.3.9 and Lemma 3.3.12, the set $H_{1,\pi}$ is not empty, $H_{1,\pi} \cap A_2$ is empty if and only if $\psi_1 = \pi$ is not a solution of

$$o_2\Delta(\lambda_2, \mu_1)\psi_1 \equiv_{2\pi} 0.$$

Since $\Delta(\lambda_2, h_1)$ is zero, $\Delta(\lambda_2, \mu_1)$ is odd. Therefore, $A_2 \cap H_1 = \emptyset$ if and only if $p_1 = 2$ and $o_2 \equiv_2 1$.

Case ii) $\boldsymbol{\Delta(\lambda_2, h_1) = 1}$ and $\boldsymbol{o_2 \neq 3}$. Thus, $o_2 \leq 4$. The intersection between $\{\psi_1 = 0\}$ and A_2 are

$$(\theta_1, \psi_1) = \left(\frac{2\pi k}{o_2}, 0\right) \quad \text{with} \quad k \in \{1, \cdots, o_2\}.$$

If $o_2 \leq 2$, then $H_{1,0} \cap A_2 = \emptyset$ as a consequence of Corollary 3.3.1.

If $p_1 \geq 3$, then $J_0(p_1, p_2)$ is supported by an angle $\alpha \geq {}^{2\pi}/_3$ by Lemma 3.3.17. Thus, $0 \in J_0(p_1, p_2)$. Lemma 3.3.12 implies that the set $H_{1,0}$ contains at least one of the points $({}^{\pi}/_2, 0)$ and $({}^{3\pi}/_2, 0)$. This implies that if $o_2 = 4$, then $H_{1,0} \cap A_2 = \emptyset$ if and only if $H_{1,0} = \emptyset$. This happens if and only if $p_1 = 2$.

The intersection between $\{\psi_1 = \pi\}$ and A_2 are

$$(\theta_1, \psi_1) = \left(\pi\Delta(\lambda_2, \mu_1) + \frac{2\pi k}{o_2}, \pi\right).$$

If $o_2 \leq 2$, then $H_{1,\pi} \cap A_2 = \emptyset$ since $H_{1,\pi}$ does not contain the points $(0, \pi)$ and (π, π) by of Corollary 3.3.1. We conclude that, if $o_2 \leq 2$, then $A_2 \cap H_1 = \emptyset$.

As a result of Lemma 3.3.19, if $o_2 = 4$, then $H_{1,\pi}$ is disjoint from A_2 if and only if $p_1 = 4$ and $p_2 = 4$. Thus, if $o_2 = 4$, then either $H_{1,0} \cap A_2 \neq \emptyset$ or $H_{1,\pi} \cap A_2 \neq \emptyset$. This implies that if $o_2 = 4$, then $H_1 \cap A_2 \neq \emptyset$.

Case iii) $\boldsymbol{\Delta(\lambda_2, h_1) = 2}$. In this case $\Delta(\lambda_2, \mu_1) \equiv_2 1$ and $o_2 \leq 2$. The intersection points of A_2 with $\{\psi_1 = 0\}$ are

$$(\theta_1, \psi_1) = \left(\frac{\pi k}{o_2}, 0\right) \quad \text{with} \quad k \in \{1, \cdots, 2o_2\}.$$

According to Corollary 3.3.1, if $o_2 = 1$ then $H_{1,0} \cap A_2$ is empty.

As we said before, if $p_1 \geq 3$, then at least one of the points $(\pi/2, 0)$ and $(3\pi/2, 0)$ is in $H_{1,0}$. Hence, if $o_2 = 2$, then $H_{1,0} \cap A_2$ is empty if and only if $p_1 = 2$.

Let us focus on the line $\{\psi_1 = \pi\}$. We see that the points in $A_2 \cap \{\psi_1 = \pi\}$ are

$$(\theta_1, \psi_1) = \begin{cases} \left(\frac{\pi}{2} + k\pi, \pi\right)_{k=0,1} & \text{if } o_2 = 1, \\ \left(\frac{\pi k}{2}, \pi\right)_{k=1,\cdots,4} & \text{if } o_2 = 2. \end{cases}$$

If $o_2 = 1$, then $H_{1,\pi} \cap A_2$ is empty if only if $p_1 = p_2 = 4$, as a result of Lemma 3.3.12 and Lemma 3.3.19. If $o_2 = 2$, then, since we suppose that $p_1 = 2$, we have that $H_{1,\pi} \cap A_2$ is nonempty by Lemma 3.3.19. We conclude that $A_2 \cap H_1$ is empty if and only if $p_1 = p_2 = 4$ and $o_2 = 1$.

Case iv) $\boldsymbol{\Delta(\lambda_2, h_1) = 4}$. As before, $\Delta(\lambda_2, \mu_1) \equiv_2 1$ and $o_2 = 1$. The intersection points of A_2 with $\{\psi_1 = 0\}$ are

$$(\theta_1, \psi_1) = \left(\frac{\pi}{2}k, 0\right) \quad \text{with} \quad k \in \{1, \cdots, 4\}.$$

Lemma 3.3.17 implies that if $p_1 \geq 3$, then $0 \in J_0(p_1, p_2)$. Furthermore, Lemma 3.3.12 implies that either $(\pi/2, 0)$ or $(3\pi/2, 0)$ is in $H_{1,0}$. Thus, $H_{1,0} \cap A_2 = \emptyset$ if and only if $H_{1,0} = \emptyset$. This latter happens if and only if $p_1 = 2$.

Let us suppose that $p_1 = 2$. The intersection points of A_2 with $\{\psi_1 = \pi\}$ are

$$(\theta_1, \psi_1) = \left(\frac{\pi}{4} + \frac{\pi k}{2}, \pi\right) \quad \text{with} \quad k \in \{1, \cdots, 4\}.$$

If $p_2 = 4$, then $J_\pi(p_1, p_2) = J_\pi(2, 4) = (-\sqrt{2}, \sqrt{2})$. Lemma 3.3.12 implies that $H_{1,\pi} \cap A_2 = \emptyset$. Conversely, if $p_2 \neq 4$ Corollary 3.3.24 implies that there exists an $\varepsilon > 0$ such that

$$(-\sqrt{2} - \varepsilon, \sqrt{2} + \varepsilon) \subseteq J_\pi(2, p_2)$$

Thus, by Lemma 3.3.12 $H_{1,\pi} \cap A_2 \neq \emptyset$. This leads us to the conclusion that if $\Delta(\lambda_1, h_1) = 4$, then $A_2 \cap H_1$ is empty if and only if $p_1 = 2$, $p_2 = 4$, and $o_2 = 1$.

Case v) $\boldsymbol{o_2\Delta(\lambda_2, h_1) = 3}$. This case includes the ones for which either $\Delta(\lambda_2, h_1) = 3$ and $o_2 = 1$ or $\Delta(\lambda_2, h_1) = 1$ and $o_2 = 3$. In particular, $o_2 \equiv_2 1$. We need to prove that $H_1 \cap A_2$ if and only if $p_1 = p_2 = 3$ and $\Delta(\lambda_2, \lambda_1) \equiv_2 0$.

The intersections between $\{\psi_1 = 0\}$ and A_2 are

$$(\theta_1, \psi_1) = \left(\frac{2\pi k}{3}, 0\right) \quad \text{with} \quad k \in \{1, 2, 3\}.$$

If $p_1 = 2$, then $H_{1,0} = \emptyset$ and this implies that $H_{1,0} \cap L = \emptyset$. If $3 \leq p_1$, then Lemma 3.3.17 and Lemma 3.3.12 imply that $H_{1,0}$ is disjoint from A_2 if and only if

$$J_0(p_1, p_2) = \left(2\cos\left(\frac{2\pi}{3}\right), 2\right) = J_0(3, 3).$$

Lemma 3.3.21 implies that $J_0(p_1, p_2) = J_0(3, 3)$ if and only if $p_1 = p_2 = 3$. Thus, $H_{1,0} \cap A_2 = \emptyset$ if and only if either $p_1 = 2$ or $p_1 = p_2 = 3$.

The intersection of A_2 with $\{\psi_1 = \pi\}$ contains either

$$(\theta_1, \psi_1) = \left(\frac{2\pi}{3}, \pi\right) \quad \text{or} \quad (\theta_1, \psi_1) = \left(\frac{\pi}{3}, \pi\right),$$

depending if the integer n is even or odd, where $n = \pm\Delta(\lambda_2, h_1)$. If $p_1 = 2$, then Corollary 3.3.24 states that for every $p_2 \geq 2$, the interval $J_\pi(2, p_2)$ contains $J_\pi(2, 4) = (-\sqrt{2}, \sqrt{2})$. This implies that $2\cos(2\pi/3) \in J_\pi(2, p_2)$ and $2\cos(\pi/3) \in J_\pi(2, p_2)$. As a result of Lemma 3.3.12, if $p_1 = 2$, then $A_2 \cap H_{1,\pi} = A_2 \cap H_1$ is nonempty.

Let us now assume that $p_1 = p_2 = 3$. Lemma 3.1.1 implies that $o_1 \equiv_2 1$. Since $J_\pi(p_1, p_2) = \left(2\cos\left(\frac{2\pi}{3}\right), 2\right)$, Lemma 3.3.12 implies that

$$H_{1,\pi} = \left\{(\theta_1, \pi) \middle| \theta_1 \in \left(0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right)\right\} \subset \mathcal{R}_{U(1)}\partial Y_1. \tag{4.2.2}$$

Lemma 3.1.1 implies that o_1 divides $\frac{p_2 q_1 + p_1 q_2}{t_1}$. Since we supposed that $p_1 = p_2 = 3$, the quantity $\frac{q_1 + q_2}{o_1}$ is an integer. Furthermore, $\frac{q_1 + q_2}{o_1}$ is even as we suppose that q_1 and q_2 are odd. Let $\eta \colon \pi_1(\partial Y_1) \to SU(2)$ so that $\eta(h_1) = -1$ and $\eta(\mu_1) = 1$. Lemma 3.1.1 implies that

$$\eta(\lambda_1) = \cancel{\eta(\mu_1)^3}\eta(h_1)^{\frac{q_1 + q_2}{o_1}} = (-1)^{\frac{q_1 + q_2}{o_1}} = 1.$$

The representation η has (θ_1, ψ_1)-coordinates equal to $(0, \pi)$. The (4.2.2) implies that η is in $\overline{H_{1,\pi}} \setminus H_{1,\pi}$. We note that the set $\overline{H_{1,\pi}} \setminus H_{1,\pi}$ consists in three points, see for instance Figure 2.3.C. Since the intersections between $\{\psi_1 = \pi\}$ and A_2 are three equally spaced point, $H_{1,\pi} \cap A_2$ is empty if and only if $\{\psi_1 = \pi\} \cap A_2 = \overline{H_{1,\pi}} \setminus H_{1,\pi}$. This latter happens if and only if $\eta \in A_2$. As a result of Proposition 2.1.11, $\eta \in A_2$ if and only if $\eta(\lambda_2)^{o_2} = 1$. Since o_2 is odd by assumption and η is a central representation, this is equivalent to say that

$\eta(\lambda_2) = 1$. Let ξ be a simple closed curve in ∂Y_1 so that $\{\lambda_1, \xi\}$ is a basis of $\pi_1(\partial Y_1)$, then

$$\lambda_2 = a\xi + b\lambda_1 \in \pi_1(\partial Y_2),$$

where $|a| = \Delta(\lambda_2, \lambda_1)$ and $|b| = \Delta(\lambda_2, \xi)$. We notice that, since $\eta(h_1) = -1$, this is not the trivial representation. Therefore, since $\eta(\lambda_2) = 1$, then $\eta(\xi) = -1$. This brings us to the following:

$$\eta(\lambda_2)^{o_2} = \eta(\lambda_2) = \eta(\xi)^a \cancel{\eta(\lambda_1)^b} = (-1)^{\Delta(\lambda_2,\lambda_1)}.$$

Thus, $\eta(\lambda_2)^{o_2} = 1$ if and only if $\Delta(\lambda_2, \lambda_1) \equiv_2 0$. Summing up, if $o_2\Delta(\lambda_2, h_1) = 3$, then $H_1 \cap A_2$ is empty if and only if $p_1 = p_2 = 3$ and $\Delta(\lambda_1, \lambda_2)$ is even. □

The following is the symmetric version of Proposition 4.2.2.

Proposition 4.2.3. *Let us suppose that Y_2 has exactly two singular fibres. The intersection $H_2 \cap A_1$ if empty if and only if one of the following holds:*

- $\Delta(\lambda_1, h_2) = 0$, $r_1 = 2$ *and* $o_1 \equiv_2 1$*;*
- $\Delta(\lambda_1, h_2) = 1$ *and* $o_1 \leq 2$*,*
- $\Delta(\lambda_1, h_2) = 1$, $r_1 = r_2 = 3$, $o_1 = 3$, *and* $\Delta(\lambda_2, \lambda_1) \equiv_2 0$*,*
- $\Delta(\lambda_1, h_2) = 2$, $r_1 = r_2 = 4$ *and* $o_1 = 1$*,*
- $\Delta(\lambda_1, h_2) = 3$, $o_1 = 1$, $r_1 = r_2 = 3$, *and* $\Delta(\lambda_2, \lambda_1) \equiv_2 0$*,*
- $\Delta(\lambda_1, h_2) = 4$, $r_1 = 2$, $r_2 = 4$ *and* $o_1 = 1$*.*

We show now a result as Proposition 4.2.2 but supposing that $\mathcal{O}(Y_2)$ is either $(2, 4, 4)$, $(3, 3, 3)$, or $(2, \cdots, 2, r_m)$.

Lemma 4.2.4. *Let us suppose that Y_2 has at least three singular fibres. The intersection $A_1 \cap H_2$ is empty if and only if one of the following holds:*

- *If* $\mathcal{O}(Y_2) = (3, 3, 3)$*, then* $\Delta(\lambda_1, h_2) = 1$ *and* $o_1 = 1$ *and* $\Delta(\lambda_1, \lambda_2) \equiv_2 0$*;*
- *If* $\mathcal{O}(Y_2) = (2, 4, 4)$*, then* $\Delta(\lambda_1, h_2) = 1$ *and* $o_1 \leq 2$*;*
- *If* $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$*, then* $\Delta(\lambda_1, h_2) = 0$ *and* $o_1 \equiv_2 1$*.*

Proof. As is shown in the proof of Corollary 2.2.9, implies if

$$(\mathcal{O}(Y_2), \Delta(\lambda_1, h_2)) \notin \{((2, 4, 4), 1), ((3, 3, 3), 1), ((2, \cdots, 2, r_m), 0)\},$$

then $Y_2(\lambda_1)$ is not $SU(2)$-abelian and therefore $A_1 \cap H_2 \neq \emptyset$. Thus, we can prove the conclusion only in the three cases above.

Let us suppose that $\mathcal{O}(Y_2) = (3,3,3)$ and $\Delta(\lambda_1, h_2) = 1$. We assume that $A_1 \cap H_2$ is empty and we shall prove that $o_1 = 1$ and $\Delta(\lambda_1, \lambda_2) \equiv_2 0$.

Corollary 3.3.13 implies that

$$H_{2,0} = \{\psi_2 = 0\} \setminus (0,0) \subset \mathcal{R}_{U(1)}(\partial Y_2) \quad \text{and} \quad H_{2,\pi} = \{\psi_2 = \pi\} \setminus \{\pi, \pi\} \subset \mathcal{R}_{U(1)}(\partial Y_2). \tag{4.2.3}$$

Thus, if $o_1 \geq 2$, then A_1 intersects $\{\psi_2 = 0\}$ and $\{\psi_2 = \pi\}$ at least twice. This implies that there exists an intersection between H_2 and A_1. Therefore, if $A_1 \cap H_2 = \emptyset$, then $o_1 = 1$. We prove now that $\Delta(\lambda_1, \lambda_2) \equiv_2 0$. We follow that we did in the proof of Proposition 4.2.2, case (v). As we said before,

$$\overline{H_{2,\pi}} \setminus H_{2,\pi} = \{(\pi, \pi)\} \subset \mathcal{R}_{U(1)}(\partial Y_2)$$

Let $\eta\colon \pi_1(\partial Y_2) \to \mathcal{Z}(SU(2))$ be the representation with coordinates (π, π). Since $\eta(h_2) = -1$, the representation η is nontrivial. In particular, since $\eta \in P_2$ by Lemma 3.2.6, we have that $\eta(\lambda_2)^{o_2} = 1$. As an application of Lemma 3.1.2, o_2 is odd and therefore $\eta(\lambda_2) = 1$.

If $A_1 \cap H_2 = \emptyset$, then A_1 must pass through $\eta = (\pi, \pi)$ and therefore $\eta \in A_1$. This latter holds if and only if $\eta(\lambda_1)^{o_1} = 1$ and, since $o_1 = 1$, this happens if and only if $\eta(\lambda_1) = 1$. Let $\xi \subset \partial Y_2$ be a slope such that $\{\xi, \lambda_2\}$ is a basis for $\pi_1(\partial Y_2)$. Since $\eta(\lambda_2) = 1$ and η is not the trivial representation, $\eta(\xi) = -1$. Let $a, b \in \mathbb{Z}$ be such that

$$\lambda_1 = a\xi + b\lambda_2 \subset \partial Y_2,$$

with $|a| = \Delta(\lambda_1, \lambda_2)$. Therefore

$$1 = \eta(\lambda_1) = \eta(\xi)^a \cancel{\eta(\lambda_2)^b} = (-1)^a.$$

This implies that $\eta(\lambda_1)^{o_1} = 1$ if and only if $|a| = \Delta(\lambda_1, \lambda_2)$ is even. This concludes that if $A_1 \cap H_2$ is empty, then $o_1 = 1$ and $\Delta(\lambda_1, \lambda_2) \equiv_2 0$.

Conversely, let us suppose that $o_1 = 1$ and $\Delta(\lambda_1, \lambda_2) \equiv_2 0$ and we prove that $A_1 \cap H_2$ under the hypothesis that $\Delta(\lambda_1, h_2) = 1$. Since $o_1 \Delta(\lambda_1, h_2) = 1$, the set A_1 intersects both $\{\psi_1 = 0\}$ and $\{\psi_1 = \pi\}$ only once. As proven above, since $\Delta(\lambda_1, \lambda_2) \equiv_2 0$ then A_1 contains

(π, π). Therefore, $A_1 \cap H_{2,\pi} = \emptyset$. Similarly, since $(0,0) \in A_1$, the intersection $A_1 \cap H_{2,0}$ is empty. This shows that if $o_1 = 1$ and $\Delta(\lambda_1, \lambda_2) \equiv_2 0$, then $A_1 \cap H_2 = \emptyset$.

Let us suppose that $\mathcal{O}(Y_2) = (2,4,4)$ and $\Delta(\lambda_1, h_2) = 1$. Corollary 3.3.13 implies that

$$H_2 = \{\psi_2 = 0\} \cup \{\psi_2 = \pi\} \setminus \{0, \pi\}^2 \subset \mathcal{R}_{U(1)}(\partial Y_2).$$

Thus, if $o_1 \geq 3$, then there exists an intersection between A_1 and H_2. Thus, using the contrapositive implication, if $A_1 \cap H_2$ is empty, then $o_1 \leq 2$.

Conversely, we suppose that $o_1 \leq 2$. Since $\Delta(\lambda_1, h_2) = 1$ by hypothesis, there exists an $n \in \mathbb{N}$ such that

$$\lambda_1 = \mu_2 + n h_2 \subset \partial Y_2.$$

Corollary 2.1.12 states that

$$A_1 = \Big\{(\theta_2, \psi_2) \in \mathcal{R}_{U(1)}(\partial Y_2) \Big| o_1(\theta_2 + n\psi_2) \equiv_{2\pi} 0\Big\} \subset \mathcal{R}_{U(1)}(\partial Y_2).$$

It is easy to see that

$$A_1 \cap (\{\psi_2 = 0\} \cup \{\psi_2 = \pi\}) = \{0, \pi\}^2 \subset \mathcal{R}_{U(1)}(\partial Y_2).$$

Therefore, $A_1 \cap H_2 = \emptyset$.

Let us suppose that $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$ and $\Delta(\lambda_1, h_2) = 0$. By Lemma 3.3.10 and Corollary 3.3.13, we obtain that

$$H(Y_2) = \{\psi_2 = \pi\} \subset \mathcal{R}_{U(1)}(\partial Y_2).$$

We recall that

$$A_1 = \left\{\psi_2 = \frac{2\pi k}{o_1}\right\}_{k \in \{1, \cdots, o_1\}} \subset \mathcal{R}_{U(1)}(\partial Y_2).$$

We conclude that $A_1 \cap H_2 = \emptyset$ if and only if $\Delta(\lambda_1, h_2) = 0$ and it does exists a $k \in \{1, \cdots, o_1\}$ for which

$$\frac{2\pi k}{o_1} = \pi.$$

This latter happens if and only if $o_1 \equiv_2 1$. □

4.3. The intersection between H_1 and H_2

In this section we shall prove that $H_1 \cap H_2 = \emptyset$ if and only if $|\beta| \coloneqq \Delta(h_1, h_2) \leq 2$ up to some exceptions. Lemma 4.1.2 implies that if $Y = Y_1 \cup_\Sigma Y_2$ is $SU(2)$-abelian, then either Y_1 or Y_2 have exactly two singular fibres. Without loss of generality, we assume that Y_1 has two nontrivial singular.

Let us fix the presentations for $\pi_1(Y_1)$ and $\pi_1(Y_2)$ as in (2.3.3). We choose $\{\mu_1, h_1\}$ and $\{\mu_2, h_2\}$ as bases for the groups $\pi_1(\partial Y_1)$ and $\pi_1(\partial Y_2)$. We recall that, with respect to the chosen bases for $\pi_1(\partial Y_1)$ and $\pi_1(\partial Y_2)$, the map $\varphi^*\colon \pi_1(\partial Y_1) \to \pi_1(\partial Y_2)$ is represented by the matrix

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{with} \quad \alpha\delta - \beta\gamma = -1. \tag{4.3.1}$$

In particular $|\beta| = \Delta(h_1, h_2)$. For $i \in \{1, 2\}$, we are going to fix the coordinates for the torus $\mathcal{R}_{U(1)}(\partial Y_i)$ in the usual way: the point $(\theta_i, \psi_i) \in [0, 2\pi]^2/_\sim$ corresponds to the representation in $\mathcal{R}_{U(1)}(\partial Y_i)$ such that

$$\mu_i \mapsto \begin{bmatrix} e^{i\theta_i} & 0 \\ 0 & e^{-i\theta_i} \end{bmatrix} \quad \text{and} \quad h_i \mapsto \begin{bmatrix} e^{i\psi_i} & 0 \\ 0 & e^{-i\psi_i} \end{bmatrix}.$$

We recall that, since in Y the boundary ∂Y_1 coincides with ∂Y_2, the space $\mathcal{R}_{U(1)}(\partial Y_1)$ coincides with $\mathcal{R}_{U(1)}(\partial Y_2)$. Consequently, (θ_1, ψ_1) and (θ_2, ψ_2) are two ways of parameterizing the space $\mathcal{R}_{U(1)}(\Sigma)$, where Σ is the torus corresponding to $\partial Y_1 = \partial Y_2$.

Let us define the the lines L_0 and L_π of $\mathcal{R}_{U(1)}(\partial Y_1)$ as

$$\begin{aligned} L_0 &\coloneqq \{(\theta_1, \psi_1) = (\alpha\theta_2, \beta\theta_2) \mid \theta_2 \in [0, 2\pi]\} \subset \mathcal{R}_{U(1)}(\partial Y_1) \quad \text{and} \\ L_\pi &\coloneqq \{(\theta_1, \psi_1) = (\alpha\theta_2 + \pi\gamma, \beta\theta_2 + \pi\delta) \mid \theta_2 \in [0, 2\pi]\} \subset \mathcal{R}_{U(1)}(\partial Y_1). \end{aligned} \tag{4.3.2}$$

Here the quantities α, β, γ and δ are as in (4.3.1).

Let $\rho\colon \pi_1(Y) \to SU(2)$ be a representation and let ρ_1 and ρ_2 the restrictions on $\pi_1(Y_1)$ and $\pi_1(Y_2)$ respectively. The group presentation (2.3.6) of $\pi_1(Y)$ gives that

$$\rho_2(\mu_2)^\alpha \rho_2(h_2)^\gamma = \rho_1(\mu_1) \quad \text{and} \quad \rho_2(\mu_2)^\beta \rho_2(h_2)^\delta = \rho_1(h_1). \tag{4.3.3}$$

Therefore, L_0 is the line $\{\psi_2 = 0\} \subset \mathcal{R}_{U(1)}(\partial Y_2)$ in (θ_1, ψ_1) coordinates. Similarly, L_π is the line $\{\psi_2 = \pi\} \subset \mathcal{R}_{U(1)}(\partial Y_2)$ in (θ_1, ψ_1) coordinates. Hence, L_0 and L_π respectively contain the sets $H_{2,0}$ and $H_{2,\pi}$ in the torus $\mathcal{R}_{U(1)}(\partial Y_1)$. Thus,

$$H_1 \cap H_2 \subseteq (\{\psi_1 = 0\} \cup \{\psi_1 = \pi\}) \cap (L_0 \cup L_\pi).$$

We define $4P \coloneqq \{0, \pi\}^2 \subset \mathcal{R}_{U(1)}(\partial Y_1)$. Lemma 3.3.1 states that $H_1 \cap 4P = \emptyset$.

Corollary 4.3.1. *Let Y_2 be a Seifert space fibred over a disk. If $\mathcal{O}(Y_2)$ is either $(2,4,4)$, $(3,3,3)$, or $(2,\cdots,2,n)$, then*

$$\{\psi_1 = \pi\} \setminus \{(0,\pi),(\pi,\pi)\} \subseteq H_{2,\pi}.$$

Proof. The conclusion holds by Corollary 3.3.13 applied to Lemma 3.3.10 and Lemma 3.3.14. □

We recall the presentation (2.3.2). Since we supposed that Y_1 has two nontrivial singular fibres,

$$Y_1 = \mathbb{D}^2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right), \quad \text{and} \quad Y_2 = \mathbb{D}^2\left(\frac{r_1}{q_1}, \cdots, \frac{r_m}{q_m}\right),$$

with $m \geq 2$. Throughout this section $Y = Y_1 \cup_\varphi Y_2$ and m is the number of nontrivial singular fibres of Y_2. Accoring to Corollary 2.2.9, whenever $m \geq 3$ we can suppose that

$$\mathcal{O}(Y_2) \in \{(2,4,4),(3,3,3),(2,\cdots,2,r_m)\}.$$

Lemma 4.3.2. *Let Y be the manifold $Y_1 \cup_\varphi Y_2$ as above with $m \geq 2$. Let us further suppose than neither $(\mathcal{O}(Y_1), |\beta|)$ nor $(\mathcal{O}(Y_2), |\beta|)$ are in the set $\{((2,4),4),((3,3),3)\}$. The intersection $H_1 \cap H_2$ is empty if and only if $m = 2$ and one of the following condition holds:*

- $|\beta| = 1$,
- $|\beta| = 2$, $m = 2$, *and* $p_1 = r_1 = 2$,
- $|\beta| = 2$, $m \geq 3$, $\mathcal{O}(Y_2) = (2,\cdots,2,r_m)$, *and* $p_1 = 2$,
- $|\beta| = 2$ *and* $p_1 = p_2 = r_1 = r_2 = 4$.

Proof. The conclusion holds by Lemma 4.3.3, Lemma 4.3.4, Lemma 4.3.5, and Lemma 4.3.6. □

Lemma 4.3.3. *Let $Y = Y_1 \cup_\varphi Y_2$ be as above with $m \geq 2$. If $|\beta| = 1$, then $H_1 \cap H_2$ is empty.*

PROOF. We recall that we supposed that Y_1 has two nontrivial singular fibres. The intersection of $L_0 \cup L_\pi$ with $\{\psi_1 = 0\} \cup \{\psi_1 = \pi\}$ is contained in $4P$. Since H_1 is disjoint from $4P$ by Lemma 3.3.1, we get the conclusion. □

Lemma 4.3.4. *Let $Y = Y_1 \cup_\varphi Y_2$ be as above, with $m = 2$ and that $|\beta| = 2$. The set $H_1 \cap H_2$ is empty if and only if either $p_1 = r_1 = 2$ or $p_1 = p_2 = r_1 = r_2 = 4$.*

PROOF. We are going to use the lines L_0 and L_π of $\mathcal{R}_{U(1)}(\partial Y_1)$ as in (4.3.2). Since $\gcd(\alpha, \beta) = 1$ and $\gcd(\delta, \beta) = 1$, the integers α and δ are odd. We note that $\{\psi_1 = 0\} \cap L_0$ and $\{\psi_1 = \pi\} \cap L_\pi$ are in $4P$. By Lemma 3.3.1, $H_{1,0} \cap H_{2,0} = H_{1,\pi} \cap H_{2,\pi} = \emptyset$.

We recall that the intersection $\{\psi_1 = 0\} \cap L_\pi$ contains $H_{1,0} \cap H_{2,\pi}$. The $\{\psi_1 = 0\} \cap L_\pi$ is equal, in (θ_1, ψ_1) and (θ_2, ψ_2) coordinates, to

$$(\theta_1, \psi_1) \in \left\{ \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right) \right\} \quad \text{and} \quad (\theta_2, \psi_2) \in \left\{ \left(\frac{\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, \pi\right) \right\}.$$

According to Lemma 3.3.12, the points

$$\left\{ \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right) \right\} \subset \mathcal{R}_{U(1)}(\partial Y_1)$$

are in $H_{1,0}$ if and only if $0 \in J_0(p_1, p_2)$. Similarly, Lemma 3.3.12 and Corollary 3.3.13 state that the points

$$\left\{ \left(\frac{\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, \pi\right) \right\} \subset \mathcal{R}_{U(1)}(\partial Y_2)$$

are in $H_{2,\pi}$ if and only if $0 \in J_\pi(r_1, r_2)$. Thus, $H_{1,0} \cap H_{2,\pi} \neq \emptyset$ if and only if $0 \in J_0(p_1, p_2)$ and $0 \in J_\pi(r_1, r_2)$.

As a consequence of Lemma 3.3.17, if $J_0(p_1, p_2)$ is nonempty, then it contains 0. We conclude, by Lemma 3.3.19, that $H_{1,0} \cap H_{2,\pi}$ is empty if and only if either $p_1 = 2$ or $r_1 = r_2 = 4$.

Let us focus on $H_{1,\pi} \cap H_{2,0}$, we recall that it is contained in $\{\psi_1 = \pi\} \cap L_0$. This intersection is equal, in (θ_1, ψ_1) and (θ_2, ψ_2) coordinates, to

$$(\theta_1, \psi_1) \in \left\{ \left(\frac{\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, \pi\right) \right\} \quad \text{and} \quad (\theta_2, \psi_2) \in \left\{ \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right) \right\}.$$

By a similar analysis, we can conclude that $H_{1,\pi}\cap H_{2,0}$ is empty if and only if either $p_1 = p_2 = 4$ or $r_1 = 2$. Hence, we have that $H_{1,\pi} \cap H_{2,0} = \emptyset$ and $H_{1,0} \cap H_{2,\pi} = \emptyset$ if and only if either $p_1 = r_1 = 2$ or $p_1 = p_2 = r_1 = r_2 = 4$. □

Lemma 4.3.5. *Let $Y = Y_1 \cup_\varphi Y_2$ be as above, with $m \geq 3$ and that $|\beta| = 2$. The set $H_1 \cap H_2$ is empty if and only if $p_1 = 2$ and $\mathcal{O}(Y_2) = (2, \cdots, 2, p_n)$.*

Proof. The proof is similar to the one of Lemma 4.3.4. Since $\gcd(\alpha, \beta) = \gcd(\beta, \delta) = 1$, then α and δ are odd. As before, $\{\psi_1 = 0\} \cap L_0$ and $\{\psi_1 = \pi\} \cap L_\pi$ are in $4P$. By Lemma 3.3.1, we obtain $H_{1,0} \cap H_{2,0} = H_{1,\pi} \cap H_{2,\pi} = \emptyset$. The intersection $\{\psi_1 = \pi\} \cap L_0$ is equal, in (θ_1, ψ_1) and (θ_2, ψ_2) coordinates, to

$$(\theta_1, \psi_1) \in \left\{\left(\frac{\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, \pi\right)\right\} \quad \text{and} \quad (\theta_2, \psi_2) \in \left\{\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)\right\}.$$

According to Lemma 3.3.12, the points

$$\left\{\left(\frac{\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, \pi\right)\right\} \subset \mathcal{R}_{U(1)}(\partial Y_1)$$

are not $H_{1,\pi}$ if and only if $0 \notin J_\pi(p_1, p_2)$. We recall that by Lemma 3.3.19, this happen if and only if $p_1 = p_2 = 4$. Similarly, Lemma 3.3.14 and Lemma 3.3.10 state that the points

$$\left\{\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)\right\} \subset \mathcal{R}_{U(1)}(\partial Y_2)$$

are not in $H_{2,0}$ if and only if $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$. This implies that $H_{1,0} \cap H_{2,\pi} = \emptyset$ if and only if either $p_1 = p_2 = 4$ or $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$.

Similarly, the intersection $\{\psi_1 = 0\} \cap L_\pi$ is equal, in (θ_1, ψ_1) and (θ_2, ψ_2) coordinates, to

$$(\theta_1, \psi_1) \in \left\{\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)\right\} \quad \text{and} \quad (\theta_2, \psi_2) \in \left\{\left(\frac{\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, \pi\right)\right\}.$$

Corollary 4.3.1 states that the points

$$\left\{\left(\frac{\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, \pi\right)\right\} \subset \mathcal{R}_{U(1)}(\partial Y_2)$$

are in $H_{1,\pi}$. This implies that $H_{1,0} \cap H_{1,\pi} = \emptyset$ if and only if the points

$$\left\{\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)\right\} \subset \mathcal{R}_{U(1)}(\partial Y_1)$$

are not in $H_{1,0}$. According to Lemma 3.3.12, this happens if and only if $0 \notin J_0(p_1, p_2)$. As we saw in Lemma 4.3.4, this happens if and only if $p_1 = 2$. This only implies that $H_{1,0} \cap H_{2,\pi} = \emptyset$ if and only if $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$. □

The next lemma uses four claims that are proved in next subsection.

Lemma 4.3.6. *Let us suppose that Y_1 and Y_2 have both exactly two singular fibres. If $|\beta| \geq 3$ and $(\mathcal{O}(Y_1), |\beta|), (\mathcal{O}(Y_2), |\beta|) \notin \{(2,4,4),(3,3,3)\}$, then $H_1 \cap H_2 \neq \emptyset$.*

Proof. With an abuse of notation, we will write β for its absolute value. The proof is divided in 4 cases:

i) $p_1 \geq 3$ and $r_1 \geq 3$,
ii) $p_1 \geq 3$ and $r_1 = 2$,
iii) $p_1 = 2$ and $r_1 \geq 3$,
iv) $p_1 = 2$ and $r_1 = 2$.

Case i) $\boldsymbol{p_1 \geq 3}$ and $\boldsymbol{r_1 \geq 3}$. We will prove that $H_{1,0} \cap H_{2,0} \neq \emptyset$. We define the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows:

$$\begin{aligned} \mathcal{S}_1 &:= \left\{ 2\cos\left(\frac{2\pi k}{\beta}\right) \middle| k \in \{1, \cdots, \beta\}, 2\cos\left(\frac{2\pi k}{\beta}\right) \in J_0(p_1,p_2) \right\} \quad \text{and} \\ \mathcal{S}_2 &:= \left\{ 2\cos\left(\frac{2\pi \alpha k}{\beta}\right) \middle| k \in \{1, \cdots, \beta\}, 2\cos\left(\frac{2\pi k}{\beta}\right) \in J_0(r_1,r_2) \right\}. \end{aligned} \tag{4.3.4}$$

The two sets in (4.3.5) are both subsets of

$$\left\{ 2\cos\left(\frac{2\pi k}{\beta}\right) \right\}_{k \in \{1, \cdots, \beta\}}.$$

Since the intervals $J_0(p_1,p_2)$ and $J_0(r_1,r_2)$ do not contain ± 2, the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ do not contain ± 2 by Lemma 3.3.12. Hence,

$$|\mathcal{S}_1| \leq \frac{\beta}{2} - 1 \quad \text{and} \quad |\mathcal{S}_2| \leq \frac{\beta}{2} - 1.$$

According to Lemma 3.3.17, $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) is empty if and only if $J_0(p_1,p_2) = J_0(3,3)$ (resp. $J_0(r_1,r_2) = J_0(3,3)$) and $\beta = 3$. Lemma 3.3.21 implies that $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) is empty if and only if $p_1 = p_2 = \beta = 3$ (resp. $r_1 = r_2 = \beta = 3$). Thus neither $\mathcal{S}_1$ nor $\mathcal{S}_2$ is empty.

Claim 3. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be as in (4.3.4). Then $H_{1,0} \cap H_{2,0}$ is empty if and only if $\mathcal{S}_1 \cap \mathcal{S}_2$ is empty.

Claim 4. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be as in (4.3.4). If either $2\beta < p_2$ or $2\beta < r_2$, then $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$.

Let α_1 (resp. α_2) be the angle supporting $J_0(p_1, p_2)$ (resp. $J_0(r_1, r_2)$), Corollary 3.3.22 implies that $\alpha_1 \geq \pi - \frac{4\pi}{p_2}$ and $\alpha_2 \geq \pi - \frac{4\pi}{r_2}$. If either $2\beta < p_2$ or $2\beta < r_2$, then the conclusion holds by Claim 3 and Claim 4. Thus, we assume $p_2 \leq 2\beta$ and $r_2 \leq 2\beta$. Lemma 3.3.17 implies that $\alpha_1 \geq 2\pi/3$ and $\alpha_2 \geq 2\pi/3$. Moreover, Corollary 3.3.22 implies that if α_1 and α_2 are both smaller than $4\pi/5$, then $p_2 \leq 20$ and $r_2 \leq 20$. Hence, up to checking by hand the conclusion for those finitely many cases, we can suppose $\alpha_1 \geq 2\pi/3$ and $\alpha_2 \geq 4\pi/5$. Therefore, we have that

$$|\mathcal{S}_1| \geq \left\lfloor \frac{\beta}{3} \right\rfloor - 2 \quad \text{and} \quad |\mathcal{S}_2| \geq \left\lfloor \frac{2\beta}{5} \right\rfloor - 2.$$

Thus, if $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, then $|\mathcal{S}_1| + |\mathcal{S}_2| \leq \frac{\beta}{2} - 1$. Then we get the following:

$$\left(\frac{\beta}{3} - \frac{2}{3}\right) + \left(\frac{2\beta}{5} - \frac{4}{5}\right) - 4 \leq \left\lfloor \frac{\beta}{3} \right\rfloor + \left\lfloor \frac{2\beta}{5} \right\rfloor - 4 \leq |\mathcal{S}_1| + |\mathcal{S}_2| - \cancel{|\mathcal{S}_1 \cap \mathcal{S}_2|} \leq \frac{\beta}{2} - 1$$

$$\left(\frac{\beta}{3} - \frac{2}{3}\right) + \left(\frac{2\beta}{5} - \frac{4}{5}\right) - 4 \leq \frac{\beta}{2} - 1$$

$$\beta \leq 19.$$

Up to checking finitely many cases (i.e. the ones such that $p_1 \leq p_2 \leq 38$ and $r_1 \leq r_2 \leq 38$) we conclude that if $p_1 \geq 3$ and $r_1 \geq 3$, then $H_{1,0} \cap H_{2,0} \neq \emptyset$. The author checked these finitely many cases with Algorithm 1.

Algorithm 1 If $3 \leq \beta \leq 19$, then $H_{1,0} \cap H_{2,0} \neq \emptyset$

for $\beta \leq 19$ **do**
  for $\alpha \leq \beta \wedge \gcd(\alpha, \beta) = 1$ **do**
    for $p_i \leq 2\beta$ **do**
      if $(p_1, p_2, \beta) \neq (3, 3, 3) \wedge (r_1, r_2, \beta) \neq (3, 3, 3)$ **then**
        Compute $J_0(p_1, p_2)$ and $J_0(r_1, r_2)$ using Lemma 3.3.21
        Compute $\mathcal{S}_1$ and $\mathcal{S}_2$ as in (4.3.4)
        Check $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$

Case ii) $\boldsymbol{p_1 \geq 3}$ and $\boldsymbol{r_1 = 2}$. We shall prove that $H_{1,0} \cap H_{2,\pi} \neq \emptyset$. We define the sets $\mathcal{S}_3$ and $\mathcal{S}_4$ as follows:

$$\begin{aligned} \mathcal{S}_3 &:= \left\{ 2\cos\left(\frac{\pi k}{\beta} - \gamma\pi\right) \middle| k \in \{1, 3, \cdots, 2\beta - 1\}, 2\cos\left(\frac{\pi k}{\beta}\right) \in J_0(p_1, p_2) \right\} \quad \text{and} \\ \mathcal{S}_4 &:= \left\{ 2\cos\left(\frac{2\pi\alpha k}{\beta} - \frac{\delta\alpha\pi}{\beta}\right) \middle| k \in \{1, \cdots, \beta\}, 2\cos\left(\frac{2\pi k}{\beta} - \frac{\delta\pi}{\beta}\right) \in J_\pi(2, r_2) \right\}. \end{aligned} \tag{4.3.5}$$

Since $(p_1, p_2, |\beta|), (r_1, r_2, |\beta|) \notin \{(2,4,4),(3,3,3)\}$, neither $\mathcal{S}_3$ nor $\mathcal{S}_4$ is empty. The sets $\mathcal{S}_3$ and $\mathcal{S}_4$ are both subsets of either

$$\left\{2\cos\left(\frac{\pi(2k+1)}{\beta}\right)\right\}_{k\in\mathbb{Z}} \quad \text{or} \quad \left\{2\cos\left(\frac{2\pi k}{\beta}\right)\right\}_{k\in\mathbb{Z}},$$

depending if $\beta\gamma \equiv_2 0$ or $\beta\gamma \equiv_2 1$. In particular,

$$|\mathcal{S}_3| \leq \frac{\beta}{2} - 1 \quad \text{and} \quad |\mathcal{S}_4| \leq \frac{\beta}{2} - 1.$$

Claim 5. Let $\mathcal{S}_3$ and $\mathcal{S}_4$ as in (4.3.5). Then $\mathcal{S}_3 \cap \mathcal{S}_4$ is empty if and only if $H_{1,0} \cap H_{2,\pi}$ is empty.

Claim 6. Let $\mathcal{S}_3$ and $\mathcal{S}_4$ be as in (4.3.5). If either $r_2 > \beta$ or $p_2 > 2\beta$, then $\mathcal{S}_3 \cap \mathcal{S}_4 \neq \emptyset$.

Without loss of generality, we can assume that $r_2 \leq \beta$ and $p_2 \leq 2\beta$. Let α_1 and α_2 be the angles supporting $J_0(p_1, p_2)$ and $J_\pi(2, r_2)$ respectively, then, according to Lemma 3.3.17 and Corollary 3.3.24, $\alpha_1 \geq {}^{2\pi}/_3$ and $\alpha_2 \geq {}^{\pi}/_2$. Lemma 3.3.23 and Corollary 3.3.22 imply that if $\alpha_2 < {}^{4\pi}/_5$ and $\alpha_1 < {}^{4\pi}/_5$ then, $p_2 \leq 20$ and $r_2 \leq 10$. Up to checking by hand the conclusion for these finitely many cases, we can suppose that $\alpha_2 \geq {}^{4\pi}/_5$. Therefore, we have that

$$|\mathcal{S}_3| \geq \left\lfloor \frac{\beta}{3} \right\rfloor - 2 \quad \text{and} \quad |\mathcal{S}_4| \geq \left\lfloor \frac{2\beta}{5} \right\rfloor - 2.$$

Thus, if $\mathcal{S}_3 \cap \mathcal{S}_4 = \emptyset$, then $|\mathcal{S}_3| + |\mathcal{S}_4| \leq \frac{\beta}{2} - 1$. As we stated before, if $\mathcal{S}_3 \cap \mathcal{S}_4 = \emptyset$, then

$$\begin{aligned}
\left(\frac{2\beta}{5} - \frac{4}{5}\right) + \left(\frac{\beta}{3} - \frac{2}{3}\right) - 4 \leq \left\lfloor \frac{2\beta}{5} \right\rfloor + \left\lfloor \frac{\beta}{3} \right\rfloor - 4 \leq |\mathcal{S}_1| + |\mathcal{S}_2| - \cancel{|\mathcal{S}_3 \cap \mathcal{S}_4|} &\leq \frac{\beta}{2} - 1 \\
\left(\frac{2\beta}{5} - \frac{4}{5}\right) + \left(\frac{\beta}{3} - \frac{2}{3}\right) - 4 &\leq \frac{\beta}{2} - 1 \\
\beta &\leq 19.
\end{aligned}$$

Again, up to checking by hand the remaining cases, namely the ones such that $p_1 \leq p_2 \leq 38$ and $r_2 \leq 19$, we get the conclusion. The author checked these finitely many cases with Algorithm 2.

Case iii) $p_1 = 2$ and $r_1 \geq 3$. This case is proven as in the previous case with the roles of H_1 and H_2 switched.

Algorithm 2 If $3 \leq \beta \leq 19$, then $H_{1,0} \cap H_{2,\pi} \neq \emptyset$

```
for β ≤ 19 do
    for α ≤ β ∧ gcd(α, β) = 1 do
        for p1, p2 ≤ 2β and r2 ≤ β do
            if (p1, p2, β) ≠ (3, 3, 3) ∧ (r1, r2, β) ≠ (2, 4, 4) then
                Compute J0(p1, p2) and Jπ(2, r2) with Lemma 3.3.21 and Lemma 3.3.23
                Compute S3 and S4 as in (4.3.5)
                Check S3 ∩ S4 ≠ ∅
```

Case iv) $\boldsymbol{p_1 = 2}$ and $\boldsymbol{r_1 = 2}$. We shall prove that $H_{1,\pi} \cap H_{2,\pi} \neq \emptyset$. We define the sets $\mathcal{S}_5$ and $\mathcal{S}_6$ as follows:

$$\mathcal{S}_5 := \left\{ 2\cos\left(\frac{2\pi k}{\beta} + \frac{\pi(\alpha+1)}{\beta} - \pi\gamma\right) \middle| k \in \{1, \cdots, \beta\}, 2\cos\left(\frac{2\pi k}{\beta} + \frac{\pi(\alpha+1)}{\beta}\right) \in J_\pi(2, p_2) \right\} \quad \text{and}$$
$$\mathcal{S}_6 := \left\{ 2\cos\left(\frac{2\pi\alpha k}{\beta} + \frac{\pi\alpha(\delta+1)}{\beta}\right) \middle| k \in \{1, \cdots, \beta\}, 2\cos\left(\frac{2\pi k}{\beta} + \frac{\pi(\delta+1)}{\beta}\right) \in J_\pi(2, r_2) \right\}. \tag{4.3.6}$$

Since $(p_1, p_2, |\beta|) \neq (2, 4, 4)$ and $(r_1, r_2, |\beta|) \neq (2, 4, 4)$ by hypothesis, neither $\mathcal{S}_5$ nor $\mathcal{S}_6$ is empty. The sets $\mathcal{S}_5$ and $\mathcal{S}_6$ are both subsets of either

$$\left\{ 2\cos\left(\frac{\pi(2k+1)}{\beta}\right) \right\}_{k\in\mathbb{Z}} \quad \text{or} \quad \left\{ 2\cos\left(\frac{2\pi k}{\beta}\right) \right\}_{k\in\mathbb{Z}},$$

if either $\alpha \equiv_2 1$ and $\delta \equiv_2 0$ or $\beta\gamma\alpha \equiv_2 0$ respectively. In particular,

$$|\mathcal{S}_5| \leq \frac{\beta}{2} - 1 \quad \text{and} \quad |\mathcal{S}_6| \leq \frac{\beta}{2} - 1.$$

Claim 7. Let $\mathcal{S}_5$ and $\mathcal{S}_6$ as in (4.3.6). Then $\mathcal{S}_5 \cap \mathcal{S}_6$ is empty if and only if $H_{1,\pi} \cap H_{2,\pi}$ is empty.

Claim 8. Let $\mathcal{S}_5$ and $\mathcal{S}_6$ be as in (4.3.6). If either $r_2 > \beta$ or $p_2 > \beta$, then $\mathcal{S}_3 \cap \mathcal{S}_4 \neq \emptyset$.

Without loss of generality, we can assume that $r_2 \leq \beta$ and $p_2 \leq 2\beta$. Let α_1 and α_2 be the angles supporting $J_\pi(2, p_2)$ and $J_\pi(2, r_2)$ respectively. According to Corollary 3.3.24, $\alpha_1 \geq \frac{\pi}{2}$ and $\alpha_2 \geq \frac{\pi}{2}$. Corollary 3.3.22 imply that if $\alpha_2 < {}^{4\pi}/_5$ and $\alpha_1 < {}^{4\pi}/_5$ then, $p_2 \leq 10$ and $r_2 \leq 10$. Up to checking by hand the for these finitely many cases, we can suppose that $\alpha_2 \geq {}^{4\pi}/_5$. Thus,

$$|\mathcal{S}_5| \geq \left\lfloor \frac{\beta}{4} \right\rfloor - 2 \quad \text{and} \quad |\mathcal{S}_6| \geq \left\lfloor \frac{2\beta}{5} \right\rfloor - 2.$$

Therefore, if $\mathcal{S}_5 \cap \mathcal{S}_6 = \emptyset$, then $|\mathcal{S}_1| + |\mathcal{S}_2| \leq \frac{\beta}{2} - 1$. If $\mathcal{S}_5 \cap \mathcal{S}_6 = \emptyset$, then

$$\left(\frac{\beta}{4} - \frac{3}{4}\right) + \left(\frac{2\beta}{5} - \frac{4}{5}\right) - 4 \leq \left\lfloor \frac{\beta}{4} \right\rfloor + \left\lfloor \frac{2\beta}{5} \right\rfloor - 4 \leq |\mathcal{S}_1| + |\mathcal{S}_2| - \cancel{|\mathcal{S}_3 \cap \mathcal{S}_4|} \leq \frac{\beta}{2} - 1$$

$$\left(\frac{\beta}{4} - \frac{3}{4}\right) + \left(\frac{2\beta}{5} - \frac{4}{5}\right) - 4 \leq \frac{\beta}{2} - 1$$

$$\beta \leq 30.$$

Again, up to checking by hand the remaining cases, namely the ones such that $p_2 \leq 30$ and $r_2 \leq 30$, we get the conclusion. The author checked these finitely many cases with Algorithm 3. □

Algorithm 3 If $3 \leq \beta \leq 30$, then $H_{1,\pi} \cap H_{2,\pi} \neq \emptyset$

for $\beta \leq 30$ **do**
    for $\alpha \leq \beta \wedge \gcd(\alpha, \beta) = 1$ **do**
        for $p_2 \leq \beta$ and $r_2 \leq \beta$ **do**
            if $(p_1, p_2, \beta) \neq (2, 4, 4) \wedge (r_1, r_2, \beta) \neq (2, 4, 4)$ **then**
                Compute $J_\pi(2, p_2)$ and $J_\pi(2, r_2)$ Lemma 3.3.23
                Compute $\mathcal{S}_5$ and $\mathcal{S}_6$ as in (4.3.5)
                Check $\mathcal{S}_5 \cap \mathcal{S}_6 \neq \emptyset$

Corollary 4.3.7. *Let $Y = Y_1 \cup_\varphi Y_2$ be as above, with $m \geq 3$. Let us further suppose that $(\mathcal{O}(Y_1), |\beta|)$ is not in $\{((2,4),4), ((3,3),3)\}$. If $|\beta| \geq 3$, then $H_1 \cap H_2 \neq \emptyset$.*

Proof. As we said at the beginning of the section, if Y_2 has three or more singular fibres, then $\mathcal{O}(Y_2) \in \{(2,4,4), (3,3,3), (2, \cdots, 2, r_m)\}$.

As an application of Lemma 3.3.14 and Lemma 3.3.10 imply that

$$J_0(2,4,4), \quad J_\pi(2,4,4), \quad J_0(3,3,3), \quad J_\pi(3,3,3), \quad \text{and} \quad J_\pi(2, \cdots, 2, r_m)$$

all contain the interval $(-2, 2)$. Therefore, the sets $\mathcal{S}_2$, $\mathcal{S}_4$, and $\mathcal{S}_6$ or Lemma 4.3.6 are never empty. In particular, if $\mathcal{O}(Y_2)$ is either $(2,4,4)$ or $(3,3,3)$, then the conclusion holds as in case (i) and (iii) of Lemma 4.3.6. If $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$, then the conclusion holds as in case (ii) and (iv). □

4.3.1. Proof of the claims. In this section we prove the claims we used in Lemma 4.3.6.

PROOF OF CLAIM 3. Let us suppose that $\mathcal{S}_1 \cap \mathcal{S}_2$ is nonempty. Then, there exist two integers $k_1, k_2 \in \{1, \cdots, \beta\}$ such that

$$2\cos\left(\frac{2\pi k_1}{\beta}\right) \in J_0(p_1, p_2), \quad 2\cos\left(\frac{2\pi k_2}{\beta}\right) \in J_0(r_1, r_2), \quad \text{and} \quad 2\cos\left(\frac{2\pi k_1}{\beta}\right) = 2\cos\left(\frac{2\pi \alpha k_2}{\beta}\right). \tag{4.3.7}$$

By Lemma 3.3.12, there exist two irreducible representations $\rho_1 \in \mathcal{R}(Y_1)$ and $\rho_2 \in \mathcal{R}(Y_2)$ such that $\rho_1(h_1) = \rho_2(h_2) = 1$,

$$\operatorname{Tr}\rho_1(a_1b_1) = 2\cos\left(\frac{2\pi k_1}{\beta}\right), \quad \text{and} \quad \operatorname{Tr}\rho_2(a_2b_2) = 2\cos\left(\frac{2\pi k_2}{\beta}\right).$$

The third condition in (4.3.7) implies that $\rho_1(a_1b_1) = \rho_2(a_2b_2)^\alpha$ up to conjugation. Thus, the representations ρ_1 and ρ_2 satisfy the conditions in (4.3.3). This guarantees the existence of an irreducible representation $\rho\colon \pi_1(Y) \to SU(2)$ such that $\rho|_{\pi_1(Y_1)} = \rho_1$ and $\rho|_{\pi_1(Y_2)} = \rho_2$. This implies that the intersection $H_{1,0} \cap H_{2,0}$ contains a representation of $\pi_1(\partial Y_1) = \pi_1(\partial Y_2)$ that extends to $\pi_1(Y)$. This implies that $H_{1,0} \cap H_{2,0}$ is nonempty.

Conversely, if $H_{1,0} \cap H_{2,0}$ is nonempty, then there exists and irreducible representation $\rho\colon \pi_1(Y) \to SU(2)$ such that $\rho|_{\pi_1(Y_1)}$ (resp. $\rho|_{\pi_1(Y_2)}$) is irreducible and $\rho(h_1) = \rho(h_2) = 1$. The (4.3.7) implies that $\rho_2(a_2b_2)^\beta = 1$. Moreover, the (4.3.7) implies that $\operatorname{Tr}\rho(a_1b_1) = \operatorname{Tr}\rho(a_2b_2)^\alpha$. In particular, we have that $\rho(a_1b_1)^\beta = 1$. Hence, there exists two integers $k_1, k_2 \in \{1, \cdots \beta\}$ such that

$$\operatorname{Tr}\rho(a_1b_1) = 2\cos\left(\frac{2\pi k_1}{\beta}\right) \in J_0(p_1, p_2), \quad \operatorname{Tr}\rho(a_2b_2) = 2\cos\left(\frac{2\pi k_2}{\beta}\right) \in J_0(r_1, r_2),$$
$$\text{and} \quad 2\cos\left(\frac{2\pi k_1}{\beta}\right) = 2\cos\left(\frac{2\pi \alpha k_2}{\beta}\right).$$

We conclude that $\mathcal{S}_1 \cap \mathcal{S}_2$ is not empty. □

PROOF OF CLAIM 4. If $2\beta < p_2$, meaning that $\pi - {}^{4\pi}/_{p_2} > \pi - {}^{2\pi}/_{\beta}$, then $2\cos({}^{2\pi k}/_{\beta}) \in \mathcal{S}_1$ if and only if $k \notin \frac{\beta}{2}\mathbb{Z}$. Let $k' \in \{1, \cdots, \beta\}$ be such that $2\cos({}^{2\pi k'\alpha}/_{\beta}) \in \mathcal{S}_2$. Such k' exists as $\mathcal{S}_2$ is nonempty, in particular $k' \notin \frac{\beta}{2}\mathbb{Z}$. Since α and β are coprime, we have that $k'\alpha \notin \frac{\beta}{2}\mathbb{Z}$. This means that $2\cos({}^{2\pi k'\alpha}/_{\beta}) \in \mathcal{S}_1$. Hence, $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$. If $2\beta < r_2$, then $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ by an identical strategy. □

PROOF OF CLAIM 5. Let Σ be the torus $\partial Y_1 = \partial Y_2 \subset M$. Let us suppose that there exists a representation $\rho \in \mathcal{R}(Y)$ so that $\rho|_{\pi_1(\Sigma)} \in H_{1,0} \cap H_{2,\pi}$, then $\rho(h_1) = 1$ and $\rho(h_2) = -1$.

The relations in (4.3.3) imply that

$$\rho_2(a_2b_2)^{\beta}(-1)^{\delta} = 1, \quad \text{and} \quad \rho_1(a_1b_1)^{\beta} = \rho(a_2b_2)^{\alpha\beta}(-1)^{\gamma\beta} = (-1)^{-\alpha\delta+\gamma\beta} = -1.$$

In particular, $\rho_2(a_2b_2)^{\beta} = (-1)^{\delta}$. This implies that there exists two integers $1 \leq k_1 \leq 2\beta - 1$ and $1 \leq k_2 \leq \beta$ with k_1 odd, such that

$$\operatorname{Tr}\rho_1(a_1b_1) = 2\cos\left(\frac{\pi k_1}{\beta}\right) \in J_0(p_1, p_2), \quad \text{and} \quad \operatorname{Tr}\rho_2(a_2b_2) = 2\cos\left(\frac{2\pi k_2}{\beta} - \frac{\pi\delta}{\beta}\right) \in J_{\pi}(2, p_2).$$

Furthermore, the (4.3.3) implies that $\operatorname{Tr}((-1)^{\gamma}\rho_1(a_1b_1)) = \operatorname{Tr}\rho_2(a_2b_2)^{\alpha}$ and this implies that

$$2\cos\left(\frac{\pi k_1}{\beta} - \gamma\pi\right) = 2\cos\left(\frac{2\pi\alpha k_2}{\beta} - \frac{\delta\alpha}{\beta}\right).$$

Conversely, let us suppose that $\mathcal{S}_3 \cap \mathcal{S}_4 \neq \emptyset$. Following a strategy similar to Claim 3, we construct the irreducible representations ρ_1 and ρ_2 of $\pi_1(Y_1)$ and $\pi_1(Y_2)$ respectively, such that

$$\rho_1(h_1) = 1, \quad \rho_2(h_2) = -1, \quad \text{and} \quad \rho_1|_{\pi_1(\Sigma)} \equiv \rho_2|_{\pi_1(\Sigma)}.$$

This implies that $H_{1,0} \cap H_{2,\pi} \neq \emptyset$. □

Proof of Claim 6. The proof follows from the same strategy of Claim 4. □

Proof of Claim 7. Let us suppose that $\eta \in H_{1,\pi} \cap H_{2,\pi}$. Then, there exists a representation $\rho\colon \pi_1(Y) \to SU(2)$ such that $\rho|_{\pi_1(Y_1)}$ (resp. $\rho|_{\pi_1(Y_1)}$) is irreducible and $\rho(h_1) = \rho(h_2) = -1$. The relations (4.3.3) become

$$\rho_2(a_2b_2)^{\beta} = (-1)^{\delta+1} \quad \text{and} \quad \rho_1(a_1b_1) = \rho_2(a_2b_2)^{\alpha}(-1)^{\gamma}.$$

Hence, $\rho_1(a_1b_1)^{\beta} = (-1)^{\alpha+1}$. According to Remark 3.3.11 we obtain that there exist two integers k_1 and k_2 so that

$$2\cos\left(\frac{2\pi k_1}{\beta} + \frac{\pi(\alpha+1)}{\beta}\right) \in J_{\pi}(2, p_2) \quad \text{and} \quad 2\cos\left(\frac{2\pi k_2}{\beta} + \frac{\pi(\delta+1)}{\beta}\right) \in J_{\pi}(2, r_2).$$

The condition $\rho_1(a_1b_1) = \rho_2(a_2b_2)^{\alpha}(-1)^{\gamma}$ implies that

$$\operatorname{Tr}\rho_1(a_1b_1) = 2\cos\left(\frac{2\pi k_1}{\beta} + \frac{\pi(\alpha+1)}{\beta} - \pi\gamma\right) = 2\cos\left(\frac{2\pi\alpha k_2}{\beta} + \frac{\pi\alpha(\delta+1)}{\beta}\right) = \operatorname{Tr}\rho(a_2b_2)^{\alpha}.$$

Therefore $\mathcal{S}_5 \cap \mathcal{S}_6 \neq \emptyset$.

Conversely, let us suppose that $\mathcal{S}_5 \cap \mathcal{S}_6 \neq \emptyset$. Let Σ be the torus corresponding to $\partial Y_1 = \partial Y_2 \subset M$ As in Claim 3, this implies the existence of the irreducible representations ρ_1 and ρ_2 of $\pi_1(Y_1)$ and $\pi_1(Y_2)$ respectively, such that

$$\rho_1(h_1) = \rho_2(h_2) = -1 \quad \text{and} \quad \rho_1|_{\pi_1(\Sigma)} \equiv \rho_2|_{\pi_1(\Sigma)}.$$

This implies that $H_{1,\pi} \cap H_{2,\pi} \neq \emptyset$. □

Proof of Claim 8. The proof follows from the same strategy of Claim 4. □

Chapter 5

The Classification

5.1. The main theorem

In this chapter we will provide the promised complete classification of $SU(2)$-abelian graph manifold rational homology 3-sphere with a single JSJ torus $Y = Y_1 \cup_\Sigma Y_2$. We will do that by applying results of the previous chapter. Before that, we need to prove Corollary 5.1.5, which states that if $\Delta(h_1, h_2) \geq 2$, then Y is not $SU(2)$-abelian.

We recall that by Corollary 2.2.9 we can suppose that Y_1 and Y_2 both admit a fibration with disk base space. Furthermore, we recall that we use the matrix

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{with} \quad \alpha\delta - \beta\gamma = -1 \tag{5.1.1}$$

to represent the map $\varphi_*\colon \pi_1(\partial Y_1) \to \pi_1(\partial Y_2)$ with respect to the bases $\{\mu_1, h_1\} \subset \pi_1(\partial Y_1)$ and $\{\mu_2, h_2\} \subset \pi_1(\partial Y_2)$. In particular $|\beta| = \Delta(h_1, h_2)$.

Lemma 5.1.1. *Let us suppose that Y_1 and Y_2 both have exactly two singular fibres. If $\Delta(h_1, h_2) = 2$ and $p_1 = r_1 = 2$, then $Y = Y_1 \cup_\Sigma Y_2$ is not $SU(2)$-abelian.*

Proof. Let us choose two presentations so that $Y_1 = \mathbb{D}^2(2/1, p_2/q_2)$ and $Y_2 = \mathbb{D}^2(2/1, r_2/s_2)$. Notice that t_1 and t_2 are both either 1 or 2. We suppose that $t_1 \leq t_2$. We use the same notation as in Section 4.3. Lemma 3.1.1 implies that

$$\Delta(\lambda_2, h_1) = \frac{2}{o_2 t_2}|r_2\delta - 2s_2 - r_2| \quad \text{and} \quad \Delta(\lambda_1, h_2) = \frac{2}{o_1 t_1}|p_2\alpha + 2q_2 + p_2|.$$

Since $t_1 \leq t_2 \leq 2$, we divide the proof in three cases:

i) $t_1 = t_2 = 1$,

ii) $t_1 = 1$ and $t_2 = 2$,

iii) $t_1 = t_2 = 2$.

Case i) $\boldsymbol{t_1 = t_2 = 1}$. According to Lemma 3.1.1, $o_1 = o_2 = 1$. In particular $\Delta(\lambda_1, h_2)$ is even (or zero). Proposition 4.2.3 implies that $H_2 \cap A_1$ is empty if and only if $\Delta(\lambda_1, h_2) = 0$. However, since $t_1 = 1$, the distance $\Delta(\lambda_1, h_2) = 0$ if and only if p_2 divides q_2. This cannot happen since $\gcd(p_2, q_2) = 1$ and $p_2 \geq 2$. We conclude that $H_1 \cap A_2$ is nonempty.

Case ii) $\boldsymbol{t_1 = 1}$ and $\boldsymbol{t_2 = 2}$. Again, $\Delta(\lambda_1, h_2)$ is even and cannot be zero, Proposition 4.2.3 states that A_1 and H_2 are disjoint if and only if $\Delta(\lambda_1, h_2) = 4$, $r_1 = 2$ and $r_2 = 4$. This implies that $o_2 = 1$. Thus, $\Delta(\lambda_2, h_1) = 2|2\delta - s_2 - 2|$ is even. According to Proposition 4.2.2, $A_2 \cap H_1 = \emptyset$ if and only if $\Delta(\lambda_2, h_1) = 0$. Since $|2\delta - s_2 - 2|$ is an odd number, we conclude that $\Delta(\lambda_2, h_1) \neq 0$ and that $A_2 \cap H_1 \neq \emptyset$.

Case iii) $\boldsymbol{t_1 = t_2 = 2}$. We study the cases $o_1 = 1$ and $o_1 = 2$ separately.

If $o_1 = 1$, then $\Delta(\lambda_1, h_2) = |p_2\alpha + 2q_2 + p_2|$ is even. Proposition 4.2.3 implies that $H_2 \cap A_1$ is empty if and only if $\Delta(\lambda_1, h_2)$ is either 0 or 4.

If $\Delta(\lambda_1, h_2) = 0$, then p_2 divides $2q_2$ and then $p_2 = 2$. This is a contradiction because if $p_1 = p_2 = 2$ then the order of its rational longitude is 2 according to Lemma 3.1.1.

If $\Delta(\lambda_1, h_2) = 4$ then $r_1 = 2$ and $r_2 = 4$. Moreover, $\Delta(\lambda_2, h_1) = 2|2\delta - s_2 - 2|$ is even. Proposition 4.2.2 implies that $\Delta(\lambda_2, h_1)$ has to be either 0 or 4. Both cases are are not possible since $|2\delta - s_2 - 2|$ is an odd number. We conclude that if $t_1 = t_2 = 2$ and $o_1 \equiv_2 1$, then Y is not $SU(2)$-abelian.

If $o_1 = 2$, Lemma 3.1.1 implies that $p_2 = 2$. Proposition 4.2.2 implies that $\Delta(\lambda_2, h_1)$ is either 0 or 1. If $\Delta(\lambda_2, h_1) = 0$, then r_2 divides 2, hence $r_2 = 2$. If $\Delta(\lambda_2, h_1) = 1$, since $(r_2\delta - 2s_2 - r_2)$ is an even number, we have that $o_2 = 2$ and again $r_2 = 2$. Hence, if $t_1 = t_2 = o_1 = 2$, then $p_1 = p_2 = r_1 = r_2 = 2$. The conclusion holds by Example 4.1.4. □

Lemma 5.1.2. *Let us suppose that $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$. If $\Delta(h_1, h_2) = 2$ and $p_1 = 2$, then $Y = Y_1 \cup_\Sigma Y_2$ is not $SU(2)$-abelian.*

Proof. Lemma 4.2.4 implies $A_1 \cap H_2$ is empty if and only if $\Delta(\lambda_1, h_2) = 0$ and $o_1 \equiv_2 1$. Without loss of generality, we can suppose that $o_1 \equiv_2 1$. As an application of Lemma 3.1.1, the order o_1 divides $t_1 = \gcd(2, p_2) \in \{1, 2\}$. Thus, $o_1 \in \{1, 2\}$. Therefore, $o_1 \equiv_2 1$ if and only if $o_1 = 1$. Lemma 3.1.1 also implies that

$$\lambda_1 = \frac{2p_2}{t_1}\mu_1 + \frac{2q_1 + p_2}{t_1}h_1.$$

We recall that $\varphi\colon \partial Y_1 \to \partial Y_2$ can be represented by the matrix (5.1.1). In particular $|\beta| = \Delta(h_1, h_2) = 2$. Thus, $\beta = \pm 2$. The condition $\Delta(\lambda_1, h_2) = 0$ implies that

$$\varphi(\lambda_1) = \begin{bmatrix} \alpha & \pm 2 \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \frac{2p_2}{t_1} \\ \frac{2q_1+p_2}{t_1} \end{bmatrix} = \begin{bmatrix} \alpha\frac{2p_2}{t_1} \pm 2\frac{2q_1+p_2}{t_1} \\ \gamma\frac{2p_2}{t_1} + \delta\frac{2q_1+p_2}{t_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = h_2.$$

In particular, we obtain that

$$\alpha\frac{2p_2}{t_1} = \pm 2\frac{2q_1 + p_2}{t_1}.$$

We recall that $\gcd({}^{2p_2}/_{t_1}, {}^{2q_1+p_2}/_{t_1}) = 1$. Since $\gcd(\alpha, \beta) = \gcd(\alpha, 2) = 1$, we obtain that ${}^{2p_2}/_{t_1} = \pm 2$. Therefore, $p_2 = 2$ and $o_1 = 2$. Lemma 4.2.4 implies that $H_1 \cap A_2$ is not empty, thus Theorem 2.1.8 implies that that Y is not $SU(2)$-abelian. □

Lemma 5.1.3. *Let us suppose that* Y_2 *has extracly two singular fibres. If either* $(\mathcal{O}(Y_1), |\beta|)$ *or* $(\mathcal{O}(Y_2), |\beta|)$ *is in the set* $\{((2,4),4), ((3,3),3)\}$*, then* $Y = Y_1 \cup_\Sigma Y_2$ *is not* $SU(2)$*-abelian.*

Proof. In this proof use the same notation we used in Lemma 4.3.2. We divide the proof in four cases:

i) $(p_1, p_2, |\beta|) = (3, 3, 3)$,
ii) $(p_1, p_2, |\beta|) = (2, 4, 4)$,
iii) $(r_1, r_2, |\beta|) = (2, 4, 4)$,
iv) $(r_1, r_2, |\beta|) = (3, 3, 3)$.

Case i) $\boldsymbol{(p_1, p_2, |\beta|) = (3, 3, 3)}$. Since $p_1 = p_2 = 3$, $t_1 = 3$. Proposition 4.1.3 and Theorem 2.1.8 implies that Y is not $SU(2)$-abelian.

Case ii) $\boldsymbol{(p_1, p_2, |\beta|) = (2, 4, 4)}$. Since β is even, γ and δ are odd. The intersection points between $\{\psi_1 = \pi\}$ and L_π have coordinates

$$(\theta_1, \psi_1) = \left(\frac{\pi}{2}k\alpha + \gamma\pi, \pi\right) \quad \text{and} \quad (\theta_2, \psi_2) = \left(\frac{\pi}{2}k, \pi\right),$$

with $k \in \{0, 1, 2, 3\}$. Lemma 3.3.12 implies that the points $\left(\frac{\pi}{2}k\alpha + \gamma\pi, \pi\right)$ are contained in $H_{1,\pi}$. Thus, the set $H_{1,\pi} \cap H_{2,\pi}$ is empty if and only if $r_1 = r_2 = 4$ as a consequence of Lemma 3.3.12 and Lemma 3.3.19. If $r_1 = r_2 = 4$, then Proposition 4.2.3 implies that $A_1 \cap H_2$ is empty if and only if $\Delta(\lambda_1, h_2) = 2$ and $o_1 = 1$. Since $p_1 = 2$ and $p_2 = 4$, we obtain that

$$\Delta(\lambda_1, h_2) = 4|\alpha + q_2 + 2q_1| \neq 2.$$

Thus, $A_1 \cap H_2 \neq \emptyset$. The conclusion follows from Theorem 2.1.8.

Case iii) $(r_1, r_2, |\beta|) = (2, 4, 4)$. This case is proven as in the previous one by switching the roles of (p_1, p_2) and (r_1, r_2).

Case iv) $(r_1, r_2, |\beta|) = (3, 3, 3)$. The intersection points in $\{\psi_1 = \pi\} \cap L_0$ have coordinates

$$(\theta_1, \psi_1) = \left(\frac{\pi\alpha}{3} + \frac{2\pi\alpha k}{3}, \pi\right) \quad \text{and} \quad (\theta_2, \psi_2) = \left(\frac{\pi}{3} + \frac{2\pi k}{3}, 0\right),$$

with $k \in \{0, 1, 2\}$. Since $J_0(r_1, r_2) = J_0(3, 3) = (-1, 2)$, Lemma 3.3.12 implies that the point $(\theta_2, \psi_2) = (\pi/3, 0)$ is in $H_{2,0}$. Lemma 3.3.20 implies that, $H_{1,\pi} \cap H_{2,0}$ is empty if and only if $p_1 = p_2 = 3$ and $\alpha = 2$. As we stated before, this would imply that $3 = t_1 \leq t_2$ and, as a consequence of Proposition 4.1.3, the existence of an irreducible $SU(2)$-representation of $\pi_1(M)$. □

Lemma 5.1.4. *Let us suppose that $\mathcal{O}(Y_2) \in \{(2, 4, 4), (3, 3, 3), (2, \cdots, 2, r_m)\}$. If $(\mathcal{O}(Y_1), |\beta|)$ is either $((2, 4), 4)$ or $((3, 3), 3)$, then $Y_1 \cup_\Sigma Y_2$ is not $SU(2)$-abelian.*

Proof. The proof uses the same notation as Lemma 5.1.3. By Corollary 4.3.1,

$$\{\psi_2 = \pi\} \setminus \{(0, \pi), (\pi, \pi)\} \subseteq H_{2,\pi}.$$

We split the proof in two: we first show the conclusion assuming that $(p_1, p_2, |\beta|) = (3, 3, 3)$ and then that $(p_1, p_2, |\beta|) = (2, 4, 4)$. We recall that in both cases φ_* is the matrix in (2.3.4), therefore $\alpha\delta - \beta\gamma = -1$ and $\gcd(\alpha, \beta) = \gcd(\gamma, \delta) = 1$.

Let us suppose that $(p_1, p_2, |\beta|) = (3, 3, 3)$. In particular, $\beta = \pm 3$. We remind the reader that $H_{1,0} \subset \{\psi_1 = 0\}$ and that $L_\pi \subset \mathcal{R}_{U(1)}(\partial Y_1)$ is the set $\{\psi_2 = \pi\}$ in (θ_1, ψ_1)-coordinates. Thus, $H_{2,\pi} \subset L_\pi$. Therefore the intersection $\{\psi_1 = 0\} \cap L_\pi$ equals, in (θ_1, ψ_1) and (θ_2, ψ_2) coordinates, to

$$(\theta_1, \psi_1) \in \left\{\frac{\pi}{3} + \frac{2\pi k}{3}, 0\right\}_{k \in \{1,2,3\}} \quad \text{and} \quad (\theta_2, \psi_2) \in \left\{-\frac{\pi\delta}{3} + \frac{2\pi k}{3}, \pi\right\}_{k,\pi \in \{1,2,3\}} \tag{5.1.2}$$

A direct computation shows that $J_0(3, 3) = (-1, 2)$. Therefore, as an application of Lemma 3.3.12, there exists a $k \in \{2, 3\}$ for which the corresponding point in (5.1.2) is simultaneously in $H_{1,0}$ and $H_{2,\pi}$. This implies that $H_1 \cap H_2 \neq \emptyset$ and that Y is not $SU(2)$-abelian by Theorem 2.1.8.

Let us suppose that $(p_1, p_2, |\beta|) = (2, 4, 4)$. We recall that $J_\pi(2,4) = (-\sqrt{2}, \sqrt{2})$. Since $\beta = \pm 4$ and $\gcd(\alpha, \beta) = \gcd(\delta, \beta) = 1$, both α and δ are odd. The intersection $\{\psi_1 = \pi\} \cap L_\pi$ equals, in (θ_1, ψ_1) and (θ_2, ψ_2) coordinates, to

$$(\theta_1, \psi_1) \in \left\{ \frac{\pi}{4} + \frac{\pi k}{2}, \pi \right\}_{k \in \{1,2,3,4\}} \quad \text{and} \quad (\theta_2, \psi_2) \in \left\{ \frac{\pi}{4} + \frac{\pi k}{2}, \pi \right\}_{k, \pi \in \{1,2,3,4\}}. \tag{5.1.3}$$

We notice that for all $k \in \{1, 2, 3, 4\}$, the point in (5.1.3) is $H_{2,\pi}$ by Corollary 4.3.1. Let $(\theta_1, \psi_1) \in \mathcal{R}_{U(1)}(Y_1)$ be a point in the intersection $\{\psi_1 = \pi\} \cap L_\pi$ and $\eta \colon \pi_1(\partial Y_1) \to SU(2)$ the corresponding intersection. We notice that

$$\operatorname{Tr} \eta(\mu_1) = 2 \cos\left(\frac{\pi}{4} + \frac{\pi k}{2} \right) \pm \sqrt{2} = \partial J_\pi(2,4)$$

Corollary 3.3.4 implies that $\eta \in A_1$. We conclude that

$$(\theta_1, \psi_1) \in A_1 \cap H_2,$$

this implies that $A_1 \cap H_2 \neq \emptyset$ and that Y is not $SU(2)$-abelian by Theorem 2.1.8. □

Corollary 5.1.5. *If $Y = Y_1 \cup_\Sigma Y_2$ is $SU(2)$-abelian, then $\Delta(h_1, h_2) = 1$.*

Proof. The conclusion is a consequence of the former chapter. We will explain how by showing the contrapositive implication.

If $|\beta| = 2$, then Lemma 4.3.4 and Lemma 4.3.5 state that Y is not $SU(2)$-abelian, except in some cases. These cases are studied in Lemma 5.1.1 and Lemma 5.1.2.

If $|\beta| \geq 3$, then Lemma 4.3.6 and Corollary 4.3.7 imply that Y is not $SU(2)$-abelian, except in some cases. Thess cases are studied in Lemma 5.1.3 and Lemma 5.1.4.

This implies that if $|\beta| \neq 1$, then Y is not $SU(2)$-abelian. □

Theorem 5.1.6. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a graph manifold rational homology 3-sphere such that $\Sigma \subset Y$ is the only JSJ torus of Y. If Y is $SU(2)$-abelian, then all the following hold:*

- *both Y_1 and Y_2 admit a Seifert fibration with disk base space;*
- *up to swapping the two JSJ pieces, Y_1 has exactly two singular fibres;*
- *if $h_1, h_2 \subset \Sigma = \partial Y_1 = \partial Y_2$ are regular fibres of Y_1 and Y_2, then $\Delta(h_1, h_2) = 1$.*

PROOF. The JSJ pieces Y_1 and Y_2 admit a fibration with disk base space as an application of Corollary 2.2.9. Either Y_1 or Y_2 has exactly two singular fibres by Lemma 4.1.2. The distance $\Delta(h_1, h_2)$ equals 1 by Corollary 5.1.5. □

Henceforth Y_1 is always considered to have exactly two singular fibres.

Remark 5.1.7. Let $Y = Y_1 \cup_\Sigma Y_2$ be a graph manifold as in Thereom 5.1.6. If $\Delta(h_1, h_2) = 1$ and $\Delta(\lambda_1, h_2) = 0$, then $\alpha p_1 p_2 \pm (p_1 q_2 + p_2 q_1) = 0$. This implies that $p_1 = p_2$. Similarly, if Y_2 has exactly two singular fibres, $\Delta(h_1, h_2) = 1$ and $\Delta(\lambda_2, h_1) = 0$, then $r_1 = r_2$.

We now report a classification theorem that is less clean than Theorem 5.1.14, but is more algorithmic: we shall give sufficient conditions for the manifold $Y = Y_1 \cup_\Sigma Y_2$ to be $SU(2)$-abelian. These conditions are the following.

Condition A. *This condition holds if $\Delta(h_1, h_2) = 1$ and up to swapping the two manifolds,*

$$Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2) \quad \textit{and} \quad Y_2 = \mathbb{D}^2(r_1/s_1, \cdots, r_m/s_m),$$

where $m \geq 2$. If $m \geq 3$, then this condition holds if $\mathcal{O}(Y_2)$ is either $(2,4,4)$, $(3,3,3)$, or $(2, \cdots, 2, r_m)$.

If Condition A holds and $m = 2$, then, without loss of generality, we further suppose that $t_1 \leq t_2$.

Condition B. *One of the following holds:*

- *If $m = 2$, then one of the following holds:*
  - ◇ $t_1 = 1$;
  - ◇ $t_1 = 2$ *and* $o_1 = 1$;
  - ◇ $t_2 = 2$ *and* $o_2 = 1$;
  - ◇ $t_1 = 2$, $o_2 \equiv_2 1$, *and* $\Delta(\lambda_2, h_1) \equiv_2 0$.
- *If $\mathcal{O}(Y_2) \in \{(2,4,4), (3,3,3)\}$, then of the following holds:*
  - ◇ $t_1 = 1$;
  - ◇ $t_1 = 2$ *and* $o_1 = 1$.
- *If $\mathcal{O}(Y_2) = (2, \cdots, 2, p_n)$, then $o_1 \equiv_2 1$.*

Condition C. *One of the following holds:*

- $\Delta(\lambda_2, h_1) = 0$, $p_1 = 2$, *and* $o_2 \equiv_2 1$;
- $\Delta(\lambda_2, h_1) = 2$, $p_1 = p_2 = 4$, $o_2 = 1$ *and* $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$;
- $\Delta(\lambda_2, h_1) = 1$, *and* $o_2 \leq 2$;
- $\Delta(\lambda_2, h_1)o_1 = 3$, $p_1 = p_2 = 3$, $\Delta(\lambda_1, \lambda_2) \equiv_2 0$, *and* $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$;
- $\Delta(\lambda_2, h_1) = 4$, $p_1 = 2$, $p_2 = 4$, *and* $o_2 = 1$.

Condition D. *One of the following holds:*

- *If* $m = |Y_2| = 2$, *then one of the following holds:*
  - ⋄ $\Delta(\lambda_1, h_2) = 1$ *and* $o_1 \leq 2$;
  - ⋄ $\Delta(\lambda_1, h_2) = 2$, $p_3 = p_4 = 4$, *and* $o_1 = 1$;
  - ⋄ $\Delta(\lambda_1, h_2) = 3$, $p_3 = p_4 = 3$, $o_1 = 1$, *and* $\Delta(\lambda_1, \lambda_2) \equiv_2 0$;
  - ⋄ $\Delta(\lambda_1, h_2) = 4$, $p_3 = 2$, $p_4 = 4$, *and* $o_1 = 1$.
- *If* $\mathcal{O}(Y_2) = (2, 4, 4)$, *then* $\Delta(\lambda_1, h_2) = 1$;
- *If* $\mathcal{O}(Y_2) = (3, 3, 3)$, *then* $\Delta(\lambda_1, h_2) = 1$ *and* $\Delta(\lambda_1, \lambda_2) \equiv_2 0$;
- *If* $\mathcal{O}(Y_2) = (2, \cdots, 2, p_n)$, *then* $\Delta(\lambda_1, h_2) = 0$.

Let us now give a short summary that exposes where the conditions above come from. If the readers wants a short answer, we can say that such conditions are the consequences of the results of previous chapters.

Condition A comes from Proposition 2.2.7, Lemma 4.1.2 and Corollary 2.2.9. In particular, if Y_1 and Y_2 are the JSJ pieces of Y, then if one of has Möebius band base space, then $Y = Y_1 \cup_\Sigma Y_2$ is not $SU(2)$-abelian as an application of Proposition 2.2.7. Therefore, we can present Y_1 and Y_2 as in (2.3.2), namely:

$$Y_1 = \mathbb{D}^2\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right), \quad Y_2 = \mathbb{D}^2\left(\frac{r_1}{s_1}, \cdots, \frac{r_m}{r_m}\right).$$

Since neither Y_1 nor Y_2 is a solid torus, $n \geq 2$ and $m \geq 2$. Lemma 4.1.2 implies that if both Y_1 and Y_2 have at least three singular fibres, then Y is not $SU(2)$-abelian. Thus, up to swapping the two JSJ pieces, we suppose that $n = 2$ and $m \geq 2$ and if $n = m = 2$, we suppose that $t_1 \leq t_2$. We recall that t_i is the order of the torsion subgroup of $H_1(Y_i; \mathbb{Z})$. If $m \geq 3$, then this condition is a consequence of Corollary 2.2.9.

Condition B holds if and only the hypotheses of Proposition 4.1.3 and of Proposition 4.1.5 are satisfied.

Let us suppose that Y_2 has two nontrivial singular fibres. Since $t_1 \leq t_2$ by Condition A, we can assume that $t_1 \leq 2$ by Condition B. Condition C holds if and only if hypotheses of Proposition 4.2.2 are satisfied with the additional requirement that $t_1 = \gcd(p_1, p_2) \leq 2$. If Y_2 has at least three nontrivial singular fibres, then by Condition A, we have that $\mathcal{O}(Y_2)$ is either $(2,4,4)$, $(3,3,3)$, or $(2,\cdots,2,r_m)$. Thus, Condition C holds if and only if the hypotheses of Lemma 4.2.4 are respected.

Remark 5.1.7 states that if $\Delta(h_1,h_2) = 1$ and $\Delta(\lambda_1,h_2) = 0$, then $p_1 = p_2$. In particular, if $t_1 \leq 2$, then $p_1 = p_2 = 2$. The order of the rational longitude of the manifold $\mathbb{D}^2(2/q_1, 2/q_2)$ is 2. Hence, Condition D holds if and only if hypoteses of Proposition 4.2.3 are satisfies with the additional requirements that $t_1 \leq 2$ and $\Delta(h_1,h_2) = 1$.

Theorem 5.1.8. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a graph manifold rational homology 3-sphere such that $\Sigma \subset Y$ is the only JSJ torus of Y. Then Y is $SU(2)$-abelian if and only if Condition A, Condition B, Condition C, and Condition D all hold.*

Proof. Let us suppose that Condition A, Condition B, Condition C, and Condition D all hold. Condition A implies that $H_1 \cap H_2 = \emptyset$ by Lemma 4.3.3. Condition B implies that $P_1 \cap P_2 = \emptyset$ as an application of Proposition 4.1.3 and Proposition 4.1.5. Condition C implies that $A_1 \cap H_2 = \emptyset$ by Proposition 4.2.3. Similarly, Condition D implies that $A_2 \cap H_1 = \emptyset$ by Propostion 4.2.2 and Lemma 4.2.4. Therefore,

$$A_1 \cap H_2 = H_1 \cap A_2 = H_1 \cap H_2 = P_1 \cap P_2 = \emptyset$$

and the manifold Y is $SU(2)$-abelian as a consequence of Theorem 2.1.8.

Conversely, if Y is $SU(2)$-abelian, then Corollary 5.1.5 implies that $\Delta(h_1,h_2) = 1$. Theorem 2.1.8 implies the sets $P_1 \cap P_2$, $H_1 \cap H_2$, $H_1 \cap A_2$, and $A_1 \cap H_2$ are empty. By the previously mentioned results, Condition A, Condition B, Condition C, and Condition D all hold. □

Let us suppose $Y = Y_1 \cup_\Sigma Y_2$ is an $SU(2)$-abelian manifold, we want to give a description of Y_1 and Y_2 in terms of the (non-unique) Seifert coefficients.

Lemma 5.1.9. *Let $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$. Therefore,*

$$\gcd(p_1 p_2, p_1 q_2 + p_2 q_1) \leq 2$$

if and only if either $t_1 = 1$ or $t_1 = 2$ and $o_1 = 1$.

Proof. If either $t_1 = 1$ or $t_1 = 2$ and $o_1 = 1$, then the conclusion holds by Lemma 3.1.1. Conversely, as an application of Lemma 3.1.1, the product $t_1 o_1$ divides the quantity $\gcd(p_1p_2, p_1q_2 + p_2q_1)$ and o_1 divides t_1. Therefore, if $\gcd(p_1p_2, p_1q_2 + p_2q_1) \leq 2$, then either $t_1 = 1$ or $t_1 = 2$ and $o_1 = 1$. □

Lemma 5.1.10. *Let $Y_1 = \mathbb{D}^2({}^{p_1}/_{q_1}, {}^{p_2}/_{q_2})$ and $Y_2 = \mathbb{D}^2({}^{r_1}/_{s_1}, {}^{r_2}/_{s_2})$ and let $\varphi : \partial Y_1 \to \partial Y_2$ be an orientation reversing diffeomorphism. If $Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian, then*

$$\pm o_1t_1\Delta(\lambda_1, h_2) \equiv_{p_1p_2} p_1q_2 + p_2q_1 \quad \textit{and} \quad \pm\, o_2t_2\Delta(\lambda_2, h_1) \equiv_{r_1r_2} r_1s_2 + r_2s_1.$$

Proof. Lemma 3.1.1 implies that

$$o_1t_1\Delta(\lambda_1, h_2) = \Delta\left(\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{pmatrix}p_1p_2\\ p_1q_2 + p_2q_1\end{pmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = |\alpha p_1p_2 + \beta(p_1q_2 + p_2q_1)|.$$

Corollary 5.1.5 implies that $\beta = \pm 1$. Hence, $\pm o_1t_1\Delta(\lambda_1, h_2) \equiv_{p_1p_2} p_1q_2 + p_2q_1$. The second half of the conclusion holds similarly. □

Let $Y_1 = \mathbb{D}^2({}^{p_1}/_{q_1}, {}^{p_2}/_{q_2})$ be as in the previous lemma. Let $\mu_1 \subset \partial Y_1$ be the fibration meridian as in Definition 1.3.2. Lemma 3.1.1 implies that

$$\Delta(\lambda_1, h_1) = \frac{p_1p_2}{o_1t_1} \quad \text{and} \quad \Delta(\lambda_1, \mu_1) = \frac{p_1q_2 + p_2q_1}{o_1t_1}.$$

Therefore, Lemma 5.1.10 implies that if $Y_1 \cup_\Sigma Y_2$ is $SU(2)$-abelian, then

$$\pm\Delta(\lambda_1, h_2) \equiv_{\Delta(\lambda_1, h_1)} \Delta(\lambda_1, \mu_1).$$

A similar conclusion can be done for Y_2 when it has three or more singular fibres, as shown in the following proposition.

Lemma 5.1.11. *Let $Y_1 = \mathbb{D}^2({}^{p_1}/_{q_1}, {}^{p_2}/_{q_2})$ and $Y_2 = \mathbb{D}^2({}^{r_1}/_{s_1}, \cdots, {}^{r_m}/_{s_m})$ with $m \geq 3$ and let $\varphi : \partial Y_1 \to \partial Y_2$ be an orientation reversing diffeomorphism. Let $\mu_2 \subset \partial Y_2$ be the fibration meridian on the choosen fibration of Y_2. If $Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian, then*

$$\pm o_1t_1\Delta(\lambda_1, h_2) \equiv_{p_1p_2} p_1q_2 + p_2q_1 \quad \textit{and} \quad \pm\, \Delta(\lambda_2, h_1) \equiv_{\Delta(\lambda_2, h_2)} \Delta(\lambda_2, \mu_2).$$

Proof. The proof follows the same strategy of Lemma 5.1.10, therefore the details are left to the reader. □

Proposition 5.1.12. *Let* $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$ *and* $Y_2 = \mathbb{D}^2(r_1/s_1, r_2/s_2)$. *If there exist two integers* n *and* m *such that*

$$o_1 t_1 n \equiv_{p_1p_2} p_1q_2 + p_2q_1 \quad \textit{and} \quad o_2t_2m \equiv_{r_1r_2} r_1s_2 + r_2s_1,$$

then there exists an orientation reversing diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$ *such that*

$$\Delta(\varphi(h_1), h_2) = 1, \quad \Delta(\varphi(\lambda_1), h_2) = |n|, \quad \textit{and} \quad \Delta(\varphi(h_1), \lambda_1) = |m|.$$

Proof. By hypothesis there exist two integers k_1 and k_2 such that

$$o_1t_1n = k_1p_1p_2 + p_1q_2 + p_2q_1 \quad \text{and} \quad o_2t_2m = k_2r_1r_2 + r_1s_2 + r_2s_1.$$

Let us define the matrix A as

$$A = \begin{bmatrix} -k_1 & -1 \\ k_1k_2 - 1 & k_2 \end{bmatrix}.$$

The matrix A has determinant equal to -1. Hence, there exists an orientation reversing diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$ such that $\varphi_* = A$ with respect to the ordered bases $\{\mu_1, h_1\}$ and $\{\mu_2, h_2\}$. In particular $\Delta(\varphi(h_1), h_2) = 1$. By Lemma 3.1.1 we get that

$$\Delta(\varphi(\lambda_1), h_2) = \Delta\big(\lambda_1, \varphi^{-1}(h_2)\big) = \frac{1}{o_1t_1}|k_1p_1p_2 + (p_1q_2 + p_2q_1)| = |n|$$

Similarly, we have that

$$\Delta(\varphi(h_1), \lambda_2) = \frac{1}{o_2t_2}|k_2r_1r_2 + (r_1s_2 + r_2s_1)| = |m|$$

□

Proposition 5.1.13. *Let* $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$ *and* $Y_2 = \mathbb{D}^2(r_1/s_1, \cdots, r_m/s_m)$ *with* $m \geq 2$. *For* $i \in \{1, 2\}$, *let* $\mu_i \subset \partial Y_i$ *be the fibration meridian of the chosen fibration of* Y_i. *If there exist two integers* n *and* m *such that*

$$n \equiv_{\Delta(\lambda_1, h_1)} \Delta(\lambda_1, \mu_1) \quad \textit{and} \quad m \equiv_{\Delta(\lambda_2, h_2)} \Delta(\lambda_2, \mu_2),$$

then there exists an orientation reversing diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$ *such that*

$$\Delta(\varphi(h_1), h_2) = 1, \quad \Delta(\varphi(\lambda_1), h_2) = |n|, \quad \textit{and} \quad \Delta(\varphi(h_1), \lambda_1) = |m|.$$

Proof. As we saw in Chapter 2, we can suppose without loss of generality that $s_j \geq 0$ and $q_i \geq 0$ for every i and j. By hypothesis there exists two integers k_1 and k_2 such that

$$n = k_1\Delta(\lambda_1, h_1) + \Delta(\lambda_1, \mu_1) \quad \text{and} \quad m = k_2\Delta(\lambda_2, h_2) + \Delta(\lambda_2, \mu_2),$$

Let us define the matrix A as

$$A = \begin{bmatrix} -k_1 & -1 \\ k_1k_2 - 1 & k_2 \end{bmatrix}.$$

The matrix A has determinant equal to -1. Hence, there exists an orientation reversing diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$ such that $\varphi_* = A$ with respect to the ordered bases $\{\mu_1, h_1\}$ and $\{\mu_2, h_2\}$. In particular $\Delta(\varphi(h_1), h_2) = 1$. Let a_1, b_1, a_2, b_2 be integers such that

$$\lambda_1 = a_1\mu_1 + b_1h_1 \subset \partial Y_1 \quad \text{and} \quad \lambda_2 = a_2\mu_2 + b_2h_2 \subset \partial Y_2.$$

Thus, $|a_i| = \Delta(\lambda_i, h_i)$ and $|b_i| = \Delta(\lambda_i, h_i)$. Since we assumed that $q_i \geq 0$, Lemma 3.1.1 implies that a_1 and b_1 are positive. Similarly, it is straightforward to see that since $s_j \geq 0$, then a_2 and b_2 are positive as well. Thus,

$$a_i = \Delta(\lambda_i, h_i) \quad \text{and} \quad b_i = \Delta(\lambda_i, h_i).$$

We notice that

$$\Delta(\varphi(\lambda_1), h_2) = |a_1k_1 + b_1| = |n|$$

Similarly, we have that

$$\Delta(\varphi(h_1), \lambda_2) = |a_2k_2 + b_2| = |m|$$

Therefore, the conclusion is given by the diffeomorphism $\varphi\colon \partial Y_1 \to \partial Y_2$ as above. □

If $Y = Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian, then $\Delta(h_1, h_2) = 1$ by Corollary 5.1.5. Thus, the set of the homotopy classes of the regular fibres h_1 and h_2 forms a basis for $\pi_1(\partial Y_1) = \pi_1(\partial m_2)$. Therefore, there exist integers n_1, m_1, n_2, m_2 so that

$$\lambda_1 = n_1h_1 + m_1h_2 \subset \partial Y_1 \quad \text{and} \quad \lambda_2 = n_2h_1 + m_2h_2 \subset \partial Y_2.$$

Then $|n_i| = \Delta(\lambda_i, h_2)$ and $|Y_i| = \Delta(\lambda_i, h_1)$ for $i \in \{1, 2\}$. Thus,

$$\Delta(\lambda_1, \lambda_2) = \big|n_1m_2 - n_2m_1\big|.$$

If Y_1 and Y_2 are presented as in (2.3.3), then, according to Lemma 3.1.1, $\Delta(\lambda_1, h_1) = \frac{p_1 p_2}{o_1 t_1}$ and $\Delta(\lambda_2, h_2) = \frac{r_1 r_2}{o_2 t_2}$. Therefore,

$$\Delta(\lambda_1, \lambda_2) \equiv_2 \Delta(\lambda_1, h_2)\Delta(\lambda_2, h_1) + \frac{p_1 p_2 r_1 r_2}{o_1 t_1 o_2 t_2}. \tag{5.1.4}$$

We now state one of the main results of this work. For compactness reasons, we will split it in two.

Theorem 5.1.14. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a graph manifold rational homology 3-sphere such that Σ is the unique JSJ torus. The manifold Y is $SU(2)$-abelian if and only if it is contained in one of the classes in Table 1 and Table 2.*

Proof. By Theorem 5.1.6, the manifolds Y_1 and Y_2 both have disk base space. According to Theorem 5.1.6, we can suppose that the space Y_1 has two singular fibres. If Y_2 has two singular fibres too, then the conclusion holds by Theorem 5.1.15. If Y_2 has at least three singular fibres, then the conclusion is given by Theorem 5.1.16. □

Theorem 5.1.15. *Let Y_1 and Y_2 be two Seifert spaces both fibreed on a disk with two cone points as in (2.3.2). Let us further suppose that Y_1 and Y_2 are presented in such a way that $0 < q_i < p_i$, $0 < s_i < r_i$, and $t_1 \leq t_2$. The manifold $Y = Y_1 \cup_\Sigma Y_2$ is an $SU(2)$-abelian rational homology 3-sphere if and only if $\Delta(h_1, h_2) = 1$ and it lays in one of the seven classes in Table 1.*

Proof. If a manifold Y is homeomorphic to a manifold in one of the seven classes in Table 1 and $\Delta(h_1, h_2) = 1$, then conditions A, B, C, and D hold. Hence, Y is $SU(2)$-abelian by Theorem 5.1.8.

Conversely, let us suppose that Y is $SU(2)$-abelian. Corollary 5.1.5 implies that $\Delta(h_1, h_2) = 1$. The conditions A, B, C, and D follow by Theorem 5.1.8. In particular, Condition C implies that $\Delta(\lambda_2, h_1)$ is in $\{0, 1, 4\}$. We divide the proof in three cases:

i) $\Delta(\lambda_2, h_1) = 0$,
ii) $\Delta(\lambda_2, h_1) = 4$,
iii) $\Delta(\lambda_2, h_1) = 1$.

Case i) $\boldsymbol{\Delta(\lambda_2, h_1) = 0}$. According to Condition B, $p_1 = 2$ and $o_2 \equiv_2 1$. Remark 5.1.7 implies that $r_1 = r_2$ and in particular $o_2 = \gcd(r_1, s_1 + s_2)$. Lemma 5.1.10 implies $s_1 + s_2 \equiv_{r_1} 0$

and hence that $o_2 = r_1$. Since o_2 is odd by assumption, r_1 is odd as well. Since $r_1 = r_2$, Condition D states that $\Delta(\lambda_1, h_2)$ is either 1, 2 or 3.

If $\Delta(\lambda_1, h_2) = 1$, then the manifold Y is in the class (1).

If $\Delta(\lambda_1, h_2) = 2$, then $r_1 = r_2 = 4$ by Condition D. Thus, according to Lemma 3.1.1, $o_2 \equiv_2 0$. This is a contradiction since we supposed that $o_2 \equiv_2 1$.

If $\Delta(\lambda_1, h_2) = 3$, then Condition D implies that $r_1 = r_2 = 3$. Lemma 5.1.10 implies that $3(s_1 + s_2) \equiv_9 0$, and hence $s_1 \equiv_3 1$ and $s_2 \equiv_3 2$. Therefore, the manifold Y is in the class (2).

Case ii) $\boldsymbol{\Delta(\lambda_2, h_1) = 4}$. As a consequence of Condition C, $p_1 = 2$, $p_2 = 4$, and $o_2 = 1$. Lemma 3.1.1 implies that $o_1 = 1$ and

$$\Delta(\lambda_1, h_2) = \frac{1}{o_1 t_1} |\alpha p_1 p_2 + \beta(p_1 q_2 + p_2 q_1)| = |4\alpha + \pm(q_2 + 2q_1)| \equiv_2 1.$$

Hence, by Condition D, $\Delta(\lambda_1, h_2)$ is either 1 or 3.

If $\Delta(\lambda_1, h_2) = 1$ then Y lies in the class (3). If $\Delta(\lambda_1, h_2) = 3$, then Condition D implies that $r_1 = r_2 = 3$. Lemma 5.1.10 implies that

$$3(s_1 + s_2) \equiv_9 \pm 12 \equiv_9 \mp 3.$$

Therefore, $s_1 + s_2 \equiv_3 \pm 1$. This implies that $s_1 \equiv_3 s_2$ and that Y is in the class (4).

Case iii) $\boldsymbol{\Delta(\lambda_2, h_1) = 1}$. Condition B implies that $o_2 \leq 2$. Condition D states that $\Delta(\lambda_1, h_2) \in \{1, 2, 3, 4\}$.

If $\Delta(\lambda_1, h_2) = 4$, then, up to switching Y_1 and Y_2, the manifold Y is in the class (3).

If $\Delta(\lambda_1, h_2) = 3$, then as a consequence of Condition C, $r_1 = r_2 = 3$, $o_1 = 1$ and $\Delta(\lambda_1, \lambda_2) \equiv_2 0$. Since o_2 divides $t_2 = 3$ by Lemma 3.1.1, $o_2 = 1$ and hence $\gcd(3, s_1 + s_2) = 1$. Thus, $s_1 \equiv_3 s_2$. The identity (5.1.4) shows that

$$\Delta(\lambda_1, \lambda_2) \equiv_2 \frac{3p_1 p_2}{t_1} + 3.$$

Thus, $\Delta(\lambda_1, \lambda_2)$ is even if and only if $p_1 p_2 / t_1$ is odd and hence if and only if $p_1 p_2 \equiv_2 1$ and $t_1 = 1$. This implies that the manifold lies in the class (5).

Let us suppose that $\Delta(\lambda_1, h_2) = 2$. As before, Condition C imposes that $r_1 = r_2 = 4$ and $o_1 = 1$. In this case the manifold Y is in class (6). Finally, let $\Delta(\lambda_1, h_2) = 1$. According to Lemma 3.2.5 $t_1 \leq 2$. Since o_1 divides t_1 by Lemma 3.1.1, $o_1 \leq 2$ and the manifold Y is in class (7). $\square$

We recall that, as an application of Lemma 5.1.9, we have that

$$\gcd(p_1p_2, p_1q_2 + q_1p_2) \leq 2,$$

if and only if either $t_1 = 1$ or $t_1 = 2$ and $o_1 = 1$.

Theorem 5.1.16. *Let Y_1 and Y_2 be two Seifert spaces both with disk base space as* (2.3.2)*. Let us suppose that Y_1 has exactly two singular fibres and Y_2 has at least three. Let us further suppose that Y_1 and Y_2 are presented in such a way that $0 < q_i < p_i$ and $0 < s_i < r_i$. The manifold $Y = Y_1 \cup_\Sigma Y_2$ is an $SU(2)$-abelian rational homology 3-sphere if and only if $\Delta(h_1, h_2) = 1$ and it lays in one of the classes in Table 2.*

Proof. If $Y_1 \cup_\Sigma Y_2$ is in one of the classes in Table 2 and $\Delta(h_1, h_2) = 1$, then the Condition A, Condition B, Condition C, and Condition D all hold. Therefore $Y_1 \cup_\Sigma Y_2$ is $SU(2)$-abelian by Theorem 5.1.8. Therefore, we focus on the converse direction.

Let $Y = Y_1 \cup_\Sigma Y_2$ be an $SU(2)$-abelian manifold. Therefore Condition A, Condition B, Condition C, and Condition D all hold. By Condition A we have that $\Delta(h_1, h_2) = 1$ and that

$$\mathcal{O}(Y_2) \in \{(2, 4, 4), (3, 3, 3), (2, \cdots, 2, r_m)\}.$$

Let us suppose that $\mathcal{O}(Y_2) = (2, 4, 4)$. Condition B implies that either

$$t_1 = 1 \quad \text{or} \quad t_1 = 2 \text{ and } o_1 = 1.$$

According to Lemma 3.1.3

$$o_2 = \gcd(4, 2 + r_2 + r_3) = 2.$$

Thus $o_2 \equiv_2 0$. Therefore, Condition C implies that $\Delta(\lambda_2, h_1) = 1$ and $o_2 = 2$. This latter implies that $r_2 = 1$ and $r_3 = 3$. Condition D implies that $\Delta(\lambda_1, h_2) = 1$. Therefore, $Y = Y_1 \cup_\Sigma Y_2$ is of class (8).

Let us suppose that $\mathcal{O}(Y_2) = (3, 3, 3)$. Condition B implies that either

$$t_1 = 1 \quad \text{or} \quad t_1 = 2 \text{ and } o_1 = 1.$$

Condition D implies that $\Delta(\lambda_1, h_2) = 1$ and $\Delta(\lambda_1, \lambda_2) \equiv_2 0$. Since $Y = Y_1 \cup_\Sigma Y_2$ is a rational homology 3-sphere, we have that $\Delta(\lambda_1, \lambda_2) \neq 0$. According to Lemma 3.1.2 the order $o_2 \in \{1, 3\}$. Therefore, Condition C implies that of the following holds:

- $\Delta(\lambda_2, h_1) = 0$ and $p_1 = 2$;
- $\Delta(\lambda_2, h_1) = 1$ and $o_2 = 1$;
- $\Delta(\lambda_2, h_1) = 4$, $p_1 = 2$, $p_2 = 4$, and $o_2 = 1$.

Let us suppose that $\Delta(\lambda_2, h_1) = 0$ and $p_1 = 2$. By Lemma 5.1.11 and Lemma 3.1.2, we obtain that

$$\frac{r_1 + r_2 + r_3}{o_2} \equiv_3 0.$$

Is is easy to see that this never happen and therefore $\Delta(\lambda_2, h_1) \neq 0$.

Let us suppose that $\Delta(\lambda_2, h_1) = 1$ and $o_2 = 1$. Lemma 3.1.2 states that, up to reindexing, $o_2 = 1$ if and only if $r_1 = r_2$ and $r_3 \neq q_1$. Therefore, the manifold $Y = Y_1 \cup_\Sigma Y_2$ is in class (9).

Let us suppose that $\Delta(\lambda_2, h_1) = 4$ and $o_2 = 1$. As in the previous case, $o_2 = 1$ if and only if $q_1 = q_2$ and $q_3 \neq q_1$. Therefore, the manifold $Y = Y_1 \cup_\Sigma Y_2$ is in class (10).

Let us suppose that $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$. Condition B implies that $o_1 \equiv_2 1$. Condition D implies that $\Delta(\lambda_1, h_2) = 0$. According to Remark 5.1.7, we obtain that $p_1 = p_2$. Furthermore, the Lemma 5.1.11 implies that

$$p_1 q_2 + p_2 q_1 = p_1(q_2 + q_1) \equiv_{p_1 p_2} 0.$$

Therefore, $q_1 \equiv_{p_1} = -q_2$. Lemma 3.1.1 implies that $o_1 = p_1$. Since $o_1 \equiv_2 1$, we obtain that

$$Y_1 = \mathbb{D}^2\left(\frac{p_1}{q_1}, \frac{p_1}{p_1 - q_1}\right),$$

with $p_1 = o_1 \geq 3$. According to Condition C, we obtain that one of the following holds

- $\Delta(\lambda_2, h_1) = 1$, and $o_2 \leq 2$;
- $\Delta(\lambda_2, h_1) o_1 = 3$, $p_1 = p_2 = 3$, $\Delta(\lambda_1, \lambda_2) \equiv_2 0$, and $\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$;

If $\Delta(\lambda_2, h_1) = 1$ and $o_2 \leq 2$ then $Y_1 \cup_\Sigma Y_2$ is in class (11).

If $\Delta(\lambda_2, h_1) = 1$ and $o_2 = 3$, then $Y_1 \cup_\Sigma Y_2$ is in class (12).

Finally, if $\Delta(\lambda_2, h_1) = 3$ and $o_2 = 1$, then $Y_1 \cup_\Sigma Y_2$ is in class (13). □

We conclude this section with an explicit description of the class (4) in Table 1. A combinatorial calculation gives that if Y is of class (4) in Table 1, then it is one of the following manifolds:

$$\mathbb{D}^2\left(\frac{2}{1},\frac{4}{1}\right) \bigcup_{\pm\left[\begin{smallmatrix}0 & 1\\ 1 & -2\end{smallmatrix}\right]} \mathbb{D}^2\left(\frac{3}{1},\frac{3}{1}\right), \qquad \mathbb{D}^2\left(\frac{2}{1},\frac{4}{3}\right) \bigcup_{\pm\left[\begin{smallmatrix}-2 & 1\\ 5 & -2\end{smallmatrix}\right]} \mathbb{D}^2\left(\frac{3}{1},\frac{3}{1}\right),$$
$$\mathbb{D}^2\left(\frac{2}{1},\frac{4}{3}\right) \bigcup_{\pm\left[\begin{smallmatrix}-2 & 1\\ 1 & 0\end{smallmatrix}\right]} \mathbb{D}^2\left(\frac{3}{2},\frac{3}{2}\right), \qquad \mathbb{D}^2\left(\frac{2}{1},\frac{4}{1}\right) \bigcup_{\pm\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]} \mathbb{D}^2\left(\frac{3}{2},\frac{3}{2}\right). \tag{5.1.5}$$

It can be proven the manifold in the right hand corner of (5.1.5) has positive Betti number. Furthermore, the remaining manifolds in (5.1.5) are rational homology spheres.

5.2. Some applications

Let $p, q, r, s \in \mathbb{Z}$ be such that $\gcd(p, q) = \gcd(r, s) = 1$. Let $E(T_{p,q})$ be the exterior of a open tubular neighborhood of the torus knot $T(p, q) \subset S^3$ with knot meridian $\mu_{p,q}$. It is known that $E(T_{p,q})$ is a Seifert fibred space, let us denote its regular fibre by $h_{p,q}$. We denote by $Y_{(p,q),(r,s)}$ the graph manifold obtained by gluing $E(T_{p,q})$ and $E(T_{r,s})$ along the diffeomorphism $\varphi : \partial E(T_{p,q}) \to \partial E(T_{r,s})$ such that $\varphi(\mu_{p,q}) = h_{r,s}$ and $\varphi(h_{p,q}) = \mu_{r,s}$. In [Mot88] Motegi proved that the manifold $Y_{(p,q)(r,s)}$ is $SU(2)$-abelian.

Theorem 5.2.1. *Let Y_1 and Y_2 be as in Theorem 5.1.14 and let us suppose that $t_1 = t_2 = 1$. If $Y = Y_1 \cup_\Sigma Y_2$ is $SU(2)$-abelian, then Y is diffeomorphic to $Y_{(p,q),(r,s)}$, for suitable $p, q, r, s \in \mathbb{Z}$ with* $\gcd(p, q) = \gcd(r, s) = 1$.

Proof. As a consequence of Corollary 2.2.9, if Y_2 has three or more fibres, then $t_2 \geq 2$. Therefore, since $t_2 = 1$, the Seifert space Y_2 has exactly two singular fibres.

Theorem 5.1.14 implies that $\Delta(\lambda_1, h_2) = 1$ and $\Delta(\lambda_2, h_1) = 1$. Lemma 5.1.10 states that

$$p_1q_2 + p_2q_1 \equiv_{p_1p_2} \pm 1 \quad \text{and} \quad r_1s_2 + r_2s_1 \equiv_{r_1r_2} \pm 1.$$

Hence, there exist two torus knots T_1 and T_2 embedded in S^3 such that $Y_1 = S^3 \setminus \nu(T_1)$ and $Y_2 = S^3 \setminus \nu(T_2)$, where $\nu(K)$ is small open neighbourhood of the knot $K \subset S^3$. Let us denote as μ_{T_1} (resp. μ_{T_2}) the meridian of $T_1 \subset S^3$ (resp. $T_2 \subset S^3$). Up to changing the presentations of Y_1 and Y_2, we can suppose that

$$p_1q_2 + p_2q_1 = \pm 1 \quad \text{and} \quad r_1s_2 + r_2s_1 = \pm 1. \tag{5.2.1}$$

Let μ_1 and μ_2 be the corresponding fibration meridians of the chosen presentations. Notice that $\mu_1 = \mu_{T_1}$ and $\mu_2 = \mu_{T_2}$ in $\pi_1(Y_1)$ and $\pi_1(Y_2)$ respectively. Let φ^* be the matrix in 5.1.1. Lemma 3.1.1 implies that

$$\Delta(\lambda_1, h_2) = |\alpha p_1 p_2 \pm 1| = 1 \quad \text{and} \quad \Delta(\lambda_2, h_1) = |\delta r_1 r_2 \pm 1| = 1.$$

Since $|p_1 p_2| \geq 6$ and $|r_1 r_2| \geq 6$, we obtain that $|\alpha| = \Delta(\mu_{T_1}, h_2) = 0$, $|\delta| = \Delta(\mu_{T_2}, h_1) = 0$. This implies that $\varphi(h_1) = \mu_{T_2}$ and $\varphi(h_2) = \mu_{T_1}$. In other words, if $T_1 = T(p, q)$ and $T_2 = T(r, s)$, then $\varphi(\mu_{p,q}) = h_{r,s}$ and $\varphi(h_{p,q}) = \mu_{r,s}$. Thus, $Y_1 \cup_\Sigma Y_2$ is diffeomorphic to $Y_{(p,q),(r,s)}$ □

Corollary 5.2.2. *For $i \in \{1, 2\}$, let $E(T_i)$ be the exterior of a open tubular neighborhood of the torus knot $T_i \subset S^3$. We denote by λ_i and μ_i the null-homologous longitude and the meridian of T_i. The manifold $E(T_1) \cup_\Sigma E(T_2)$ is $SU(2)$-abelian if and only if $\Delta(\lambda_1, \mu_2) = 0$ and $\Delta(\lambda_2, \mu_1) = 0$.*

Proof. If $E(T_1) \cup_\Sigma E(T_2)$ is $SU(2)$-abelian, then the conclusion holds by Theorem 5.2.1. Conversely, if $\Delta(\lambda_1, \mu_2) = 0$ and $\Delta(\lambda_2, \mu_1)$ then the conclusion holds by [Mot88]. □

Given two Seifert fibred spaces Y_1 and Y_2 with $t_1 = t_2 = 1$, Theorem 5.2.1 implies that there exists a unique $SU(2)$-abelian 3-manifold obtained by gluing Y_1 and Y_2. In general, it is not true that for a given couple of Seifert fibred manifolds there exists a unique gluing that produces an $SU(2)$-abelian manifold, as is shown in the following example.

Example 5.2.3. Let $Y_1 = \mathbb{D}^2(4/1, 5/4)$ and $Y_2 = \mathbb{D}^2(2/1, 2/1)$. Let us choose the usual basis $\{\mu_1, h_1\}$ and $\{\mu_2, h_2\}$ for $\pi_1(\partial Y_1)$ and $\pi_1(\partial Y_2)$. Let φ_1 and φ_2 be the two orientation reversing diffeomorphisms $\partial Y_1 \to \partial Y_2$ such that

$$\varphi_1^* = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \varphi_2^* = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}.$$

Theorem 5.1.8 implies that $Y_1 \cup_{\varphi_1} Y_2$ and $Y_1 \cup_{\varphi_2} Y_2$ are $SU(2)$-abelian. Theorem 5.1.14 implies that $Y_1 \cup_{\varphi_1} Y_2$ and $Y_1 \cup_{\varphi_2} Y_2$ are both in class (7) of Table 1. It can be proven that $\Delta(\varphi_1(\lambda_1), \lambda_2) = 19$ and that $\Delta(\varphi_2(\lambda_1), \lambda_2) = 21$. Thus, $Y_1 \cup_{\varphi_1} Y_2$ is not diffeomorphic to $Y_1 \cup_{\varphi_2} Y_2$.

Chapter 6

Applications and L-spaces

6.1. L-spaces

Defining Heegaard Floer L-space is beyond the scope of this thesis. Roughly speaking, Heegaard Floer homology is the infinite dimensional version of the Morse homology of a 3-manifold and the former, unlike Morse's version, produces a single $\mathbb{Z}$-module which is called the *Heegaard Floer group*. Let Y be a closed 3-manifold, we denote by $\widehat{HF}(Y)$ the (hat) Heegaard Floer homology group of Y. We say that Y is a Heegaard Floer L-space if and only if Y is a rational homology sphere and

$$\widehat{HF}(Y) = \text{rank}|H_1(Y;\mathbb{Z})|.$$

When there is not risk of confusion, if Y is a Heegaard Floer L-space we make the notation a little lighter by dropping the *Heegaard Floer* part and by saying that Y is an L-space.

In this work we define a lens space to be a prime 3-manifold with *finite* cyclic fundamental group. This means that $S^1 \times S^2$ is not considered a lens space. A classic argument of 3-manifold topology states that Y is a lens space (or a copy of $S^1 \times S^2$) if and only if there exists an embedded torus $\Sigma \subset Y$ that splits Y in two solid tori:

$$\overline{Y \setminus \Sigma} = S^1 \times \mathbb{D}^2 \cup S^1 \times \mathbb{D}^2.$$

It is well known, for instance see [OS04, Theorem 7.1], that lens spaces are L-spaces, this is indeed the origin of the name of the latter.

Let p,q,r, and s be integers whose absolute values are not one and such that

$$\gcd(p,q) = \gcd(r,s) = 1.$$

Let $Y_{(p,q)(r,s)}$ be as in Section 5.2. It is proven in [Zen17] that if the number $pqrs$ is even, then the graph manifold $Y_{(p,q)(r,s)}$ is the double branched cover of an alternating knot. By [Gre13, Corollary 3.6], $Y_{(p,q)(r,s)}$ is a (strong) L-space. Moreover, in [Zha20] Zhang proved that the manifold $Y_{(p,q)(r,s)}$ is an L-space for every couple of two coprime integers as above.

Let $Y = Y_1 \cup_\varphi Y_2$ be a manifold as in Theorem 5.1.8, we are going to prove that if Y is an $SU(2)$-abelian rational homology sphere, then Y is a Heegaard Floer L-space.

Definition 6.1.1. Let Y be a compact oriented 3-manifold with torus boundary. We denote by $\mathscr{S}(\partial Y)$ the set of slopes in ∂Y. We define the L-space interval of Y to be

$$\mathcal{L}(Y) = \{\gamma \in \mathscr{S}(\partial Y) \,|\, Y(\gamma) \text{ is an L-space}\}.$$

We define $CFG(Y)$ as the set of slopes $\gamma \subset Y$ such that $Y(\gamma)$ has cyclic fundamental group.

For a more detailed description of $\mathcal{L}(Y)$ we give [Ras17] as a reference.

Let Y be a 3-manifold with torus boundary. Let $\mathcal{P}$ be a property of a 3-manifold, we say that the slope $\gamma \subset \partial Y$ is a $\mathcal{P}$ slope if $Y(\gamma)$ has the property $\mathcal{P}$. For example, we say that $\gamma \subseteq \partial Y$ is an L-space slope (resp. lens space slope) if $Y(\gamma)$ is an L-space (resp. a lens space).

This implies that if Y is a 3-manifold with torus boundary and $\gamma \subset \partial Y$ is a lens space slope, then γ is an L-space slope. In our notation we can write that

$$\gamma \in CFG(Y) \cap \mathcal{L}(Y).$$

Since the only closed manifold with cyclic and infinite fundamental group is $S^1 \times S^2$, a slope $\gamma \in CFG(Y)$ is an L-space slope if and only if $Y(\gamma)$ is not $S^1 \times S^2$.

Let Y be a 3-manifold with torus boundary. The set $\mathscr{S}(\partial Y)$ can be identified as the projectivization of the first homology $\mathbb{P}(H_1(\partial Y;\mathbb{Z}))$. Thus, there exists a canonical embedding

$$\mathscr{S}(\partial Y) \hookrightarrow \mathbb{P}(H_1(\partial Y;\mathbb{R})) \cong S^1 = \mathbb{R} \cup \{\infty\}.$$

We identify the compactification $\mathbb{R} \cup \{\infty\}$ with S^1. With an abuse of notation, when we say that the L-space interval $\mathcal{L}(Y)$ contains (equals) an interval $I \subseteq S^1$ we mean that $\mathcal{L}(Y)$

contains (resp. equals) the set $I \cap \mathscr{S}(\partial Y)$. Using the same logic, we will say that $\mathcal{L}(Y)$ is connected if there exists a connected set $A \subseteq S^1$ such that $\mathcal{L}(Y) = A \cap \mathscr{S}(\partial Y)$. Let A be a subset of S^1 and let us suppose that $\mathcal{L}(Y) = A \cap \mathscr{S}(\partial Y)$. We define the *interior* of $\mathcal{L}(Y)$ as

$$\mathcal{L}(Y)^\circ = A^\circ \cap \mathscr{S}(Y).$$

The following is, with a slightly different notation, [Ras17, Proposition 1.5] or more in general a consequence of [HRW23, Theorem 1.14]. The author understands that the latter quoted result is a consequence of the combination of several projects of Hanselman, S.D. Rasmussen, J. Rasmussen, and Watson. The author hopes that none has been excluded by citing these two former articles.

Theorem 6.1.2. *If Y_1 and Y_2 are non-solid-torus graph manifolds with torus boundary, then the union $Y_1 \cup_\varphi Y_2$, with gluing map $\varphi\colon \partial Y_1 \to -\partial Y_2$, is an L-space if and only if*

$$\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2) = \mathbb{P}(H_1(\partial Y_2; \mathbb{Q})) = \mathscr{S}(\partial Y_2).$$

Let $Y = \mathbb{D}^2({}^{p_1}/_{q_1}, \cdots {}^{p_n}/_{q_n})$ with $p_i \geq 2$ and $n \geq 2$. Let $\mu \subset \partial Y$ be the fibration meridian of Y as in Definition 1.3.2 and let $h \subset \partial Y$ be a regular fibre of Y. Let the fraction ${}^{p}/_{q} \in \mathbb{Q} \cup \{{}^{1}/_{0}\}$ correspond to the slope $ph + q\mu \subset \partial Y$. Let $y_-, y_+ \in \mathbb{Q} \cup \{{}^{1}/_{0}\}$ be defined as

$$y_- := \max_{k>0} -\frac{1}{k}\left(1 + \sum_{i=1}^{n} \left\lfloor -k\frac{q_i}{p_i} \right\rfloor\right) \quad \text{and} \quad y_+ := \min_{k>0} -\frac{1}{k}\left(-1 + \sum_{i=1}^{n} \left\lceil -k\frac{q_i}{p_i} \right\rceil\right). \tag{6.1.1}$$

Here $k > 0$ in an integer.

Theorem 6.1.3 ([RR17, Proposition 3.9]). *Let $Y = \mathbb{D}^2({}^{p_1}/_{q_1}, \cdots, {}^{p_n}/_{q_n})$ be a Seifert space fibred over a disk, let h and μ be the regular fibre and the fibration meridian as above. Let us use the convention according to which the fraction ${}^{p}/_{q} \in \mathbb{Q} \cup \{{}^{1}/_{0}\}$ corresponds to the slope $ph + q\mu$. Then $\mathcal{L}(Y) = [\![y_-, y_+]\!]$, with the interval $[\![y_-, y_+]\!]$ is as in Definition 6.1.4 and $y_\pm$ as in* (6.1.1).

Definition 6.1.4 ([Ras17, Definition 3.7]). If $y_-, y_+ \in \mathbb{Q} \cup \{1/0\}$, then we define the interval $[\![y_-, y_+]\!] \subset \mathbb{Q} \cup \{\infty\}$, as follows:

$$[\![y_-, y_+]\!] := \begin{cases} \langle -\infty, +\infty \rangle & \infty = y_- \ , \ y_+ = \infty \\ \langle y_-, +\infty] \cup [-\infty, y_+\rangle & \mathbb{Q} \ni y_- = y_+ \in \mathbb{Q} \\ [y_-, +\infty] \cup [-\infty, y_+] & \mathbb{Q} \ni y_- > y_+ \in \mathbb{Q} \\ [y_-, y_+] & \mathbb{Q} \ni y_- < y_+ \in \mathbb{Q} \\ [-\infty, y_+] & \infty = y_- \ , \ y_+ \in \mathbb{Q} \\ [y_-, +\infty] & \mathbb{Q} \ni y_- \ , \ y_+ = \infty \end{cases} .$$

In other words, $[\![y_-, y_+]\!]$ is the unique interval in $\mathbb{R} \cup \{\infty\}$ with left-hand endpoint y_- and right-hand endpoint y_+ which is closed if $y_- \neq y_+$ and open otherwise.

The author thinks that the following is already proven in the literature but, as they cannot find an explicit proof of this, they report theirs.

Theorem 6.1.5. *Let Y be a Seifert fibred space. If Y is an $SU(2)$-abelian rational homology 3-sphere, then Y is an L-space.*

Proof. If Y is a either lens space or a copy of $\mathbb{RP}^3 \# \mathbb{RP}^3$, then, as we mentioned before, Y is an L-space.

Let us assume that Y is $SU(2)$-abelian. According to Theorem 2.2.8, if Y is neither a lens space nor $\mathbb{RP}^3 \# \mathbb{RP}^3$, then it fibres over either

$$S^2(2,4,4) \quad \text{or} \quad S^2 = (3,3,3).$$

Let us suppose that Y fibres over $S^2(2,4,4)$. The manifold Y can be split in

$$Y = Y_0 \cup S^1 \times \mathbb{D}^2 \quad \text{where} \quad Y_0 = \mathbb{D}^2(2/1, 4/q_2),$$

and $q_2 \in \{1,3\}$. We shall compute the L-space interval of Y_0 using the convention for which the fraction $p/q \in \mathbb{Q} \cup \{1/0\}$ corresponds to the slope

$$ph + q\mu \subset \partial Y_0,$$

where $h \subset \partial Y_0$ is a regular fibre and $\mu \subset \partial Y_0$ is the fibration meridian of Y_0 as in Definition 1.3.2. Remark 2.2.2 implies that Y is obtained as a filling along the slope $n/4$, for an integer $n \in \mathbb{Z}$ coprime with 4. Therefore, Y is an L-space if and only if

$$\frac{n}{4} \in \mathcal{L}(Y_0).$$

Using the algorithm in Theorem 6.1.3, we obtain that

$$\mathcal{L}\left(\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}\right)\right) = \begin{cases} [-\infty, {}^2\!/\!_3] \cup [1, +\infty] & q_2 = 1, \\ [-\infty, 1] \cup [{}^4\!/\!_3, +\infty] & q_2 = 3; \end{cases} \tag{6.1.2}$$

Lemma 3.1.1 implies that the rational longitude of $Y_0 = \mathbb{D}^2({}^2\!/\!_1, {}^4\!/\!_{q_2})$ is equal, with the notation used here, to $(2 + q_2)/4$. Let $n \in \mathbb{Z}$ with $\gcd(n, 4) = 1$, Figure 6.1 and Figure 6.2 show that

$$\frac{n}{4} \in \mathcal{L}^\circ(Y_0) \Longleftrightarrow \frac{n}{4} \text{ is not the rational longitude of } Y_0.$$

Thus, as Y is a rational homology 3-sphere, it is is a an L-space.

Let us suppose now that Y fibres over $S^2(3, 3, 3)$. Thus, $Y = S^2({}^3\!/\!_{q_1}, {}^3\!/\!_{q_2}, {}^3\!/\!_{q_3})$. We notice that, up to reindexing, we can suppose that $q_1 \equiv_3 q_2$. The manifold Y can be split in

$$Y = Y_0 \cup S^1 \times \mathbb{D}^2 \quad \text{where} \quad Y_0 = \mathbb{D}^2({}^3\!/\!_{q_1}, {}^3\!/\!_{q_2}),$$

and $q_1 = q_2 \in \{1, 2\}$. As before, Y is obtained as a filling along the slope $q_3/3$. Thus, Y is an L-space if and only if

$$\frac{s_1}{3} \in \mathcal{L}(Y_0).$$

Using the algorithm in Theorem 6.1.3, we obtain that

$$\mathcal{L}\left(\mathbb{D}^2\left(\frac{3}{q_1}, \frac{3}{q_1}\right)\right) = \begin{cases} [-\infty, {}^1\!/\!_2] \cup [1, +\infty] & q_1 = 1, \\ [-\infty, 1] \cup [{}^3\!/\!_2, +\infty] & q_1 = 2; \end{cases} \tag{6.1.3}$$

Lemma 3.1.1 implies that the rational longitude of $Y_0 = \mathbb{D}^2({}^3\!/\!_{q_1}, {}^3\!/\!_{q_1})$ is ${}^{2q_1}\!/\!_3$. Let $n \in \mathbb{Z}$ with $\gcd(n, 3) = 1$, Figure 6.3 and Figure 6.4 show that

$$\frac{n}{3} \in \mathcal{L}^\circ(Y_0) \Longleftrightarrow \frac{n}{3} \text{ is not the rational longitude of } Y_0.$$

Since Y is a rational homology 3-sphere by hypotesis, Y is an L-space.

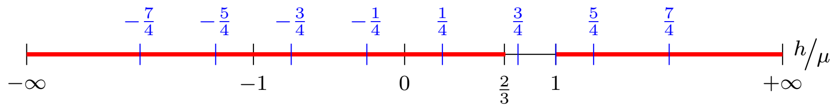


FIGURE 6.1. In red $\mathcal{L}(Y_0)$ with $Y_0 = \mathbb{D}^2(2/1, 4/1)$, in blue the slopes $n/4$ with $\gcd(n, 4) = 1$.

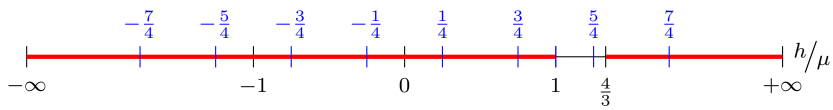


FIGURE 6.2. In red $\mathcal{L}(Y_0)$ with $Y_0 = \mathbb{D}^2(2/1, 4/3)$, in blue the slope $n/4$ with $\gcd(n, 4) = 1$.

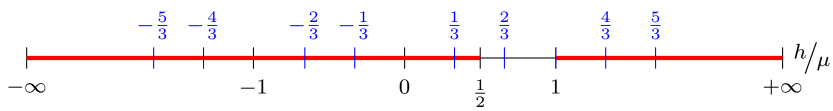


FIGURE 6.3. In red $\mathcal{L}(Y_0)$ with $Y_0 = \mathbb{D}^2(3/1, 3/1)$, in blue the slopes $n/3$ with $\gcd(n, 3) = 1$.

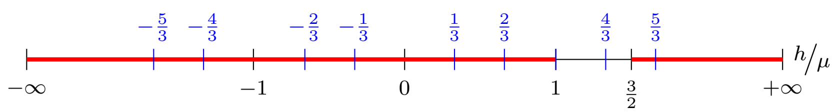


FIGURE 6.4. In red $\mathcal{L}(Y_0)$ with $Y_0 = \mathbb{D}^2(3/2, 3/2)$, in blue the slope $n/3$ with $\gcd(n, 3) = 1$.

This concludes the proof that if Y is an $SU(2)$-abelian Seifert fibred space rational homology 3-sphere, then Y is an L-space. □

It is worth emphasizing that in Theorem 6.1.5 we proved that if Y is a Seifert rational homology 3-sphere fibred $S^2(3, 3, 3)$, then Y is an L-space. In general, such a manifold is not necessarily $SU(2)$-abelian: Theorem 2.2.8 states this latter is $SU(2)$-abelian if and only if it has first homology group $H_1(Y; \mathbb{Z})$ of even order.

Corollary 6.1.6. *Let Y be a Seifert fibred manifold with torus boundary. If $\gamma \subset \partial Y$ is a slope such that $Y(\gamma)$ is an $SU(2)$-abelian rational homology sphere, then $Y(\gamma)$ is an L-space.*

PROOF. According to Lemma 2.2.1, since $Y(\gamma)$ is a rational homology sphere, then Y is a Seifert fibres space with base space either a Möbius band or a disk.

Let us suppose that Y has a Möbius band as base space, we denote by $h \subset \partial Y$ its regular fibre. By Lemma 1.3.4, the slope $h \subset \partial Y$ is the rational longitude of Y. Since $Y(\gamma)$ is a rational homology 3-sphere, $\Delta(\gamma, h) \neq 0$. According to Remark 2.2.2, the filling $Y(\gamma)$ is a Seifert fibred space whose base space is a projective plane $\mathbb{RP}^2$. As an application of [SZ21, Proposition 3.5], the manifold $Y(\gamma)$ is $SU(2)$-abelian if and only if it is a lens space or a copy of $\mathbb{RP}^3 \# \mathbb{RP}^3$. Since both are an L-spaces, we get the conclusion.

Let us suppose that Y has a disk as base space. If Y has exactly one singular fibre, then $Y = S^1 \times \mathbb{D}^2$ and the conclusion is trivial. Let us suppose that Y has at least two singular fibres, we denote by $h \subset \partial Y$ its regular fibre. If $\Delta(\gamma, h) \neq 0$, then $Y(\gamma)$ is a Seifert fibred manifold and the conclusion is given by Theorem 6.1.5.

If $\Delta(\gamma, h) = 0$, then $Y(\gamma)$ is a connected sum of lens spaces, and therefore it is an L-space. Thus, if $Y(\gamma)$ is $SU(2)$-abelian, then $Y(\gamma)$ is an L-space. □

Proposition 6.1.7. *Let Y_1 be a Seifert fibred space fibred over a disk with two cone points. There exists a slope $\gamma \subset \partial Y_1$ so that $Y_1(\gamma) = S^1 \times S^2$ if and only if Y_1 is diffeomorphic to $\mathbb{D}^2(p_1/q_1, p_1/-q_1)$.*

Proof. If Y_1 is diffeomorphic to $\mathbb{D}^2(p_1/q_1, p_1/-q_1)$, then the fibration meridian defined in Definition 1.3.2 is an $S^1 \times S^2$-filling.

Conversely, let $\gamma \subset \partial Y_1$ be a slope such that $Y_1(\gamma) = S^1 \times S^2$. Let us present $\pi_1(Y_1)$ as in (2.3.3). Let us consider $\pi_1(\partial Y_1)$ generated by the usual basis $\{\mu_1, h_1\}$, with μ_1 the fibration meridian as in Definition 1.3.2. As a result of Remark 2.2.2, we obtain that $\gamma \in CFG(Y_1)$ if and only if $\Delta(\gamma, h_1) = 1$. Thus, there exists an $n \in \mathbb{Z}$ so that

$$\gamma = \mu_1 + nh_1 \subset \partial Y_1.$$

The fundamental group of $Y_1(\gamma)$, that is abelian by hypothesis, equals to

$$\frac{\pi_1(Y_1)}{\langle\langle \gamma \rangle\rangle} = \operatorname{coker} \begin{bmatrix} p_1 & 0 & 1 \\ 0 & p_2 & 1 \\ q_1 & q_2 & n \end{bmatrix}.$$

Since $Y_1(\gamma) = S^1 \times S^2$, the determinant of the above matrix is zero. Hence,

$$np_1p_2 = p_1q_2 + p_2q_1.$$

This means that $p_1 = p_2$ and $q_2 = -q_1 + np_1$. The conclusion holds by the classification of Seifert fibred manifold with boundary, see Section 1.3. □

Corollary 6.1.8. *Let Y_1 be a Seifert fibred space fibred over a disk with two cone points. Then $CFG(Y_1) \subseteq \mathcal{L}(Y_1)$ if and only if Y_1 is not diffeomorphic to $\mathbb{D}^2({}^{p_1}\!/\!{}_{q_1}, {}^{p_1}\!/\!{}_{-q_1})$.*

Proof. It is known that $Y_1(\lambda_1)$ has positive first Betti number. In particular, $Y_1(\lambda_1)$ is not a rational homology sphere and, therefore, it is not an L-space. If $Y_1 \cong \mathbb{D}^2({}^{p_1}\!/\!{}_{q_1}, {}^{p_1}\!/\!{}_{-q_1})$, then $\lambda_1 \in CFG(Y_1)$ and $\lambda_1 \notin \mathcal{L}(Y_1)$. Thus, $CFG(Y_1)$ is not contained in $\mathcal{L}(Y_1)$. This implies that if $CGF(Y_1) \subseteq \mathcal{L}(Y_1)$, then $Y_1 \neq \mathbb{D}^2({}^{p_1}\!/\!{}_{q_1}, {}^{p_1}\!/\!{}_{-q_1})$.

Conversely, if $Y_1 \neq \mathbb{D}^2({}^{p_1}\!/\!{}_{q_1}, {}^{p_1}\!/\!{}_{-q_1})$ then Proposition 6.1.7 implies that Y_1 has no $S^1 \times S^2$ filling. Therefore, every cyclic Dehn filling of Y_1 is an lens space. Since lens spaces are L-spaces, every cyclic fundamental group filling of Y_1 is an L-space filling. □

Lemma 6.1.9. *If $Y_1 = \mathbb{D}^2({}^{p_1}\!/\!{}_{q_1}, {}^{p_1}\!/\!{}_{p_1-q_1})$, then*

$$\mathcal{L}(Y_1) = \mathscr{S}(Y_1) \setminus \{\lambda_1\}.$$

Proof. The manifold Y_1 admits an $S^1 \times S^2$ filling by Proposition 6.1.7. Every slope $\alpha \subset \partial Y_1$, with $\Delta(\alpha, h_1) = 1$ and $\alpha \neq \lambda_1$ is a lens space filling of Y_1. This implies that Y_1 is a complement of a knot in $S^1 \times S^2$ and it admits an L-space filling. The conclusion holds by [RR17, Corollary 7.8]. □

We can now prove that the first step of Theorem 6.1.19.

Proposition 6.1.10. *If $Y = Y_1 \cup_\varphi Y_2$ is of class (1) or (2) in Table 1 then Y is an L-space.*

Proof. If Y is of class (1) or (2), then $Y_2 \cong \mathbb{D}^2({}^{r_1}\!/\!{}_{s_1}, {}^{r_1}\!/\!{}_{r_1-s_1})$ with $0 < s_1 < r_1$. Moreover, $Y_1 = \mathbb{D}^2({}^{2}\!/\!{}_{1}, {}^{p_2}\!/\!{}_{q_2})$ with $p_2 \geq 2$ and $0 < q_2 < p_2$. By Lemma 6.1.9, we obtain that $\mathcal{L}(Y_2) = \mathscr{S}(Y_2) \setminus \{\lambda_2\}$. According to Theorem 6.1.2 we have that $Y = Y_1 \cup_\varphi Y_2$ is an L-space if and only if $\varphi^{-1}(\lambda_2) \in \mathcal{L}(Y_1)^\circ$. We observe that if Y is of class (1) or (2), then $p_1 = 2$ and $\varphi^{-1}(\lambda_2) = h_1$. Hence, $Y = Y_1 \cup_\varphi Y_2$ is an L-space if and only if $h_1 \in \mathcal{L}(Y_1)^\circ$.

If $p_2 = 2$, then Y_1 is diffeomorphic to $\mathbb{D}^2({}^{2}\!/\!{}_{1}, {}^{2}\!/\!{}_{1})$. Again, Lemma 6.1.9 implies that

$$\mathcal{L}(Y_1) = \mathscr{S}(Y_1) \setminus \{\lambda_1\}.$$

We note that $h_1 \neq \lambda_1$. Thus $h_1 \in \mathcal{L}(Y_1)^\circ$ and therefore we get the conclusion.

Let us suppose now that $p_2 \geq 3$. As a consequence of Proposition 6.1.7 and Corollary 6.1.8, we have that

$$CFG(Y_1) \subseteq \mathcal{L}(Y_1), \tag{6.1.4}$$

and $\lambda_1 \notin CFG(Y_1)$. Let suppose that $\pi_1(\partial Y_1)$ is generated by the basis $\{\mu_1, h_1\}$, where μ_1 is the fibration meridian of the chosen presentation of $\pi_1(Y_1)$ as in Definition 1.3.2. We use the convention that the fraction $p/q \in \mathbb{Q} \cup \{1/0\}$ corresponds to the slope $p\mu_1 + qh_1$. Let γ be a slope in ∂Y_1. Remark 2.2.2 implies that $\gamma \in CFG(Y_1)$ if and only if $\Delta(\gamma, h_1) = 1$. Equation (6.1.4) implies that

$$CFG(Y_1) = \{\mu_1 + nh_1\}_{n\in\mathbb{Z}} = \left\{\frac{1}{n}\right\}_{n\in\mathbb{Z}} \subset \mathcal{L}(Y_1).$$

As a result of Theorem 6.1.3, the L-space interval $\mathcal{L}(Y_1)$ is connected. Consequently, there exists $m \in \mathbb{Z}$ such that

$$CFG(Y_1) \subset \left[-\infty, \frac{1}{m+1}\right] \cup \left[\frac{1}{m}, +\infty\right] \subset \mathcal{L}(Y_1).$$

In particular $h_1 = 0 \in \mathcal{L}(Y_1)^\circ$. □

According to Theorem 5.1.8, if $Y = Y_1 \cup_\varphi Y_2$ is $SU(2)$-abelian, then $\Delta(h_1, h_2) = 1$. Let us call Σ the embedded torus in $Y = Y_1 \cup_\varphi Y_2$ corresponding to $\partial Y_1 = \partial Y_2$. We recall that, for the 2-torus Σ, we indicate by $\mathscr{S}(\Sigma)$ the set of slopes in Σ. Let us fix the convention according to which the fraction $p/q \in \mathbb{Q} \cup \{1/0\}$ corresponds to the slope $ph_1 + qh_2 \subset \Sigma$. If we compute $\mathcal{L}(Y_1)$ and $\mathcal{L}(Y_2)$ using this convention, then Theorem 6.1.2 is equivalent to say that the manifold $Y = Y_1 \cup_\varphi Y_2$ is an L-space if and only if

$$\mathcal{L}(Y_1)^\circ \cup \mathcal{L}(Y_2)^\circ = \mathscr{S}(\Sigma).$$

The manifolds in classes (3), (4), (5) and (6) in Table 1 are such that either Y_1 or Y_2 is diffeomorphic to one of the following manifolds with boundary:

$$\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}\right), \quad \mathbb{D}^2\left(\frac{3}{s_1}, \frac{3}{s_1}\right), \quad \text{or} \quad \mathbb{D}^2\left(\frac{4}{s_1}, \frac{4}{s_1}\right). \tag{6.1.5}$$

Proposition 6.1.12 computes the L-space interval for these manifolds in terms of the slopes $\{h_1, h_2\}$ and the fraction convention above.

Before moving on, we point out a technique that we will use often for the rest of the section.

Remark 6.1.11. Let Σ be a 2-torus. Let γ_1 and γ_2 be two slopes of Σ with $\Delta(\gamma_1, \gamma_2) = 1$. Then, the fundamental group $\pi_1(\Sigma)$ is generated by $\{\gamma_1, \gamma_2\}$. Let γ_3 be a third slope of Σ such that $\Delta(\gamma_1, \gamma_3) = 1$. Then $\{\gamma_1, \gamma_3\}$ is a second basis of $\pi_1(\Sigma)$. Let

$$f\colon \pi_1(\Sigma) = \mathbb{Z}^2 \to \pi_1(\Sigma) = \mathbb{Z}^2$$

be an isomorphism that transforms the first basis into the second one. In particular f is a matrix of $SL_2(\mathbb{Z})$ and, since $f(\gamma_1) = \gamma_1$, there exists an integer $m \in \mathbb{Z}$ such that

$$f = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}.$$

This also implies that

$$\gamma_3 = f(\gamma_2) = m\gamma_2 + \gamma_1.$$

Proposition 6.1.12. *Let $Y = Y_1 \cup_\varphi Y_2$ be of class either* (3), (5) *or* (6) *in Table 1. If $Y = Y_1 \cup_\varphi Y_2$ is of class* (3), *then either*

$$\mathcal{L}(Y_1) = [-\infty, {}^{-1}/_{3}] \cup [0, +\infty] \quad \text{or} \quad \mathcal{L}(Y_1) = [-\infty, 0] \cup [{}^{+1}/_{3}, +\infty].$$

If $Y = Y_1 \cup_\varphi Y_2$ is of class (5), *then either*

$$\mathcal{L}(Y_2) = [-\infty, 2] \quad \text{or} \quad \mathcal{L}(Y_2) = [-2, +\infty].$$

If $Y = Y_1 \cup_\varphi Y_2$ is of class (6), *then either*

$$\mathcal{L}(Y_2) = [-\infty, {}^{3}/_{2}] \quad \text{or} \quad \mathcal{L}(Y_2) = [-{}^{3}/_{2}, +\infty].$$

In all three we use convention according to which the fraction ${}^{p}/_{q} \in \mathbb{Q} \cup \{{}^{1}/_{0}\}$ corresponds to the slope $ph_1 + qh_2 \subset \Sigma = \partial Y_1 = \partial Y_2$.

Proof. According to Corollary 5.1.5, $\Delta(h_1, h_2) = 1$. Hence, if Y is $SU(2)$-abelian, then $\{h_1, h_2\}$ is a basis for $\pi_1(\Sigma)$, where Σ is the torus embedded in Y corresponding to $\partial Y_1 = \partial Y_2$.

Let Y be of class (3). Thus, $Y_1 = \mathbb{D}^2({}^2\!/_1, {}^4\!/_{q_2})$ and $q_2 \in \{1,3\}$ and $\Delta(\lambda_1, h_2) = 1$. Using the algorithm in Theorem 6.1.3, we obtain that

$$\mathcal{L}\left(\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{q_2}\right)\right) = \begin{cases} [-\infty, {}^2\!/_3] \cup [1, +\infty] & q_2 = 1, \\ [-\infty, 1] \cup [{}^4\!/_3, +\infty] & q_2 = 3; \end{cases} \tag{6.1.6}$$

In (6.1.6) we used the convention for which the fraction ${}^p\!/_q$ corresponds to the slope $ph_1 + q\mu_1$ with μ_1 the fibration meridian of $Y_1 = \mathbb{D}^2({}^2\!/_1, {}^4\!/_{q_2})$ as in Definition 1.3.2. Since $\Delta(h_1, h_2) = 1$ and $\Delta(\mu_1, h_1) = 1$, there exists an $N_1 \in \mathbb{Z}$ such that

$$\mu_1 = h_2 + N_1 h_1 \subset \Sigma.$$

Lemma 3.1.1 implies that

$$\lambda_1 = 4\mu_1 + (2 + q_2)h_1 = h_1(2 + q_2 + 4N_1) + 4h_2.$$

The condition $\Delta(\lambda_1, h_2) = 1$ implies that

$$|2 + q_2 + 4N_1| = 1,$$

and, as $q_2 \in \{1,3\}$, that $N_1 = -1$. We obtain that

$$ph_1 + q\mu_1 = ph_1 + qh_2 - qh_1 = (p - q)h_1 + qh_2.$$

The change of basis $\{h_1, \mu_1\} \to \{h_1, h_2\}$ induces the bijection

$$f : \mathbb{Q} \cup \{{}^1\!/_0\} \to \mathbb{Q} \cup \{{}^1\!/_0\}, \quad \text{with} \quad \frac{p}{q} \mapsto \frac{p}{q} - 1.$$

The conclusion holds by applying to (6.1.6) the change of basis ${}^p\!/_q \mapsto {}^p\!/_q - 1$.

Let Y be of class (5). Thus, $Y_2 = \mathbb{D}^2({}^3\!/_{s_1}, {}^3\!/_{s_1})$ with $s_1 \in \{1,2\}$ and $\Delta(\lambda_2, h_1) = 1$. Using the algorithm in Theorem 6.1.3, we obtain that

$$\mathcal{L}\left(\mathbb{D}^2\left(\frac{3}{q_1}, \frac{3}{q_1}\right)\right) = \begin{cases} [-\infty, {}^1\!/_2] \cup [1, +\infty] & s_1 = 1, \\ [-\infty, 1] \cup [{}^3\!/_2, +\infty] & s_1 = 2; \end{cases} \tag{6.1.7}$$

In (6.1.7) we used the convention that the fraction ${}^p\!/_q$ corresponds to the slope $ph_2 + q\mu_2$ with μ_2 the fibration meridian of Y_2 as in Definition 1.3.2. Since $\Delta(h_1, h_2) = 1$ and $\Delta(\mu_2, h_2) = 1$, there exists an $N_2 \in \mathbb{Z}$ such that

$$\mu_2 = h_1 + N_2 h_2.$$

Lemma 3.1.1 implies that

$$\lambda_2 = 3\mu_2 + 2s_1 h_2 = 3h_1 + (2s_1 + 3N_2)h_2.$$

The condition $\Delta(\lambda_2, h_1) = 1$ implies that

$$|2s_1 + 3N_2| = 1,$$

and hence that $N_2 = -1$. We obtain that

$$ph_2 + q\mu_2 = ph_1 + qh_1 - qh_2 = ph_1 + (p-q)h_2.$$

The change of basis $\{h_2, \mu_2\} \to \{h_1, h_2\}$ induces the bijection

$$\widehat{f} : \mathbb{Q} \cup \{1/0\} \to \mathbb{Q} \cup \{1/0\}, \quad \text{with} \quad \frac{p}{q} \mapsto \frac{q}{p-q} = \left(\frac{p}{q} - 1\right)^{-1}.$$

The conclusion holds by applying to (6.1.7) the map $\widehat{f}$.

Let Y be of class (6). Thus, $Y_2 = \mathbb{D}^2(4/s_1, 4/s_1)$ and $s_1 \in \{1, 3\}$. Using the algorithm in Theorem 6.1.3, we obtain that

$$\mathcal{L}\left(\mathbb{D}^2\left(\frac{4}{s_1}, \frac{4}{s_1}\right)\right) = \begin{cases} [-\infty, 1/3] \cup [1, +\infty] & s_1 = 1, \\ [-\infty, 1] \cup [5/4, +\infty] & s_1 = 3; \end{cases} \tag{6.1.8}$$

In (6.1.8) we used the convention that the fraction p/q corresponds to the slope $ph_2 + q\mu_2$ with μ_2 the fibration meridian of Y_2 as in 1.3.2. The details are left to the reader since the proof strategy is the similar to the previous case. □

Proposition 6.1.13. *Let $Y = Y_1 \cup_\varphi Y_2$ be of class either (3), (4), (5) or (6) in Table 1. If Y is a rational homology sphere, then Y is an L-space.*

Proof. Let Y be in class (4), then it is one of the eight manifolds in (5.1.5). The conclusion is obtain by applying Proposition 6.1.12 and Theorem 6.1.2 to these manifolds. Details are left to the reader.

In what follows we will use the convention that the fraction $p/q \in \mathbb{Q} \cup \{\infty\}$ is the slope $ph_1 + qh_2$.

Let us suppose that $Y = Y_1 \cup_\varphi Y_2$ is of class (3). In particular, $Y_1 = \mathbb{D}^2(2/1, 4/q_2)$. The L-space interval $\mathcal{L}(Y_1)$ is computed in Proposition 6.1.12. As a result of Corollary 6.1.8, we have that

$$CFG(Y_2) = \{h_1 + nh_2\}_{n\in\mathbb{Z}} = \frac{1}{\mathbb{Z}} \subset \mathcal{L}(Y_2). \tag{6.1.9}$$

Since $\lambda_2 \notin CFG(Y_2)$ by Proposition 6.1.7, there exists an $m \in \mathbb{Z}$ so that $\lambda_2 \in (1/m+1, 1/m)$. Since $\mathcal{L}(Y_2)$ is connected by Theorem 6.1.3, we obtain the following inclusion:

$$CFG(Y_2) \subset \left[-\infty, \frac{1}{m+1}\right] \cup \left[\frac{1}{m}, +\infty\right] \subseteq \mathcal{L}(Y_2).$$

Since $\Delta(h_1, h_2) = 1$ and $\Delta(\mu_2, h_2) = 1$, there exists an $N_2 \in \mathbb{Z}$ such that $\mu_2 = h_1 + N_2 h_2$. Lemma 3.1.1 implies that

$$\lambda_2 = \frac{r_1 r_2}{t_2}\mu_2 + \frac{r_1 s_2 + r_2 s_1}{t_2} h_2 = \frac{r_1 r_2}{t_2} h_1 + \frac{r_1 s_2 + r_2 s_1 + r_1 r_2 N_2}{t_2} h_2,$$

where $t_2 = \gcd(r_1, r_2)$. The condition $\Delta(\lambda_2, h_1) = 4$ implies that $|\frac{r_1 s_2 + r_2 s_1}{t_2}| = 4$ and therefore

$$\lambda_2 = \frac{r_1 r_2}{t_2} h_1 \pm 4h_2 = \pm\frac{r_1 r_2}{4t_2}.$$

Since $o_2 = 1$ and $r_2 \geq r_1$, we have that $r_2 \geq 3$ by Lemma 3.1.1. Because of that, we get that

$$|\lambda_2| = \left|\pm\frac{r_1 r_2}{4t_1}\right| = \frac{r_1 r_2}{4t_1} \geq \frac{r_2}{4} \geq \frac{3}{4} > \frac{1}{2}.$$

This implies that λ_2 is in either $[1/2, 1]$, $[1, +\infty[$, $[-1, -1/2]$, or $[-\infty, -1]$. Hence, according to (6.1.9), we obtain that one of the following inclusion holds:

$$\begin{gathered} [-\infty, 1/2] \cup [1, +\infty] \subseteq \mathcal{L}(Y_2), \quad [-\infty, 1] \subseteq \mathcal{L}(Y_2), \\ [-\infty, -1] \cup [-1/2, +\infty] \subseteq \mathcal{L}(Y_2), \quad [-1, \infty] \subseteq \mathcal{L}(Y_2). \end{gathered} \tag{6.1.10}$$

Figure 6.5 shows the possibilities of $\mathcal{L}(Y_1)$ form Proposition 6.1.12 and the possibilities of $\mathcal{L}(Y_2)$ form (6.1.10). Therefore, we see that

$$\mathcal{L}(Y_1)^\circ \cup \mathcal{L}(Y_2)^\circ = \mathscr{S}(\Sigma).$$

Thus, $Y = Y_1 \cup_\varphi Y_2$ of class (3) is an L-space as a result of Theorem 6.1.2.

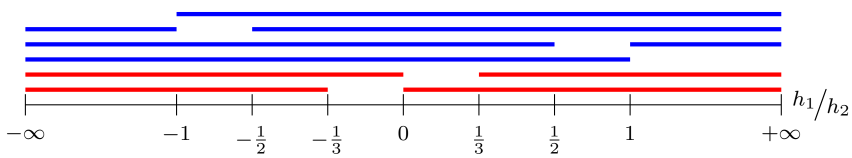


FIGURE 6.5. Let $Y = Y_1 \cup_\varphi Y_2$ be in class (3) of Table 1. The two possibilities of $\mathcal{L}(Y_1)$ are in red, the four possibilities of $\mathcal{L}(Y_2)$ are in blue.

Let us suppose that $Y = Y_1 \cup_\varphi Y_2$ is of class (5). In particular, $Y_2 = \mathbb{D}^2(3/s_1, 3/s_1)$ with $s_1 \in \{1, 2\}$. As a result of Proposition 6.1.12 we obtain that

$$CFG(Y_1) = \{nh_1 + h_2\}_{n\in\mathbb{Z}} = \mathbb{Z} \subset \mathcal{L}(Y_2).$$

Since $\lambda_1 \notin CFG(Y_1)$, there exists an $m \in \mathbb{Z}$ so that $\lambda_1 \in (m, m+1)$. Theorem 6.1.3 implies that

$$[-\infty, m] \cup [m+1, +\infty] \subseteq \mathcal{L}(Y_1).$$

As we saw before, there exists a $N_1 \in \mathbb{Z}$ so that $\mu_1 = h_2 + N_1 h_1$. Lemma 3.1.1 implies that

$$\lambda_1 = \mu_1 p_1 p_2 + h_1(p_1 q_2 + p_2 q_1) = h_1(p_1 q_2 + p_2 q_1 + N_1 p_1 p_2) + h_2 p_1 p_2 = \frac{p_1 q_2 + p_2 q_1 + N_1 p_1 p_2}{p_1 p_2}.$$

The conditions $\Delta(\lambda_1, h_2) = 3$ implies that $|p_1 q_2 + p_2 q_1 + N_1 p_1 p_2| = 3$. Furthermore, the conditions $p_1 p_2 \equiv_2 1$ and $t_1 = \gcd(p_1, p_2) = 1$ give that

$$|\lambda_2| = \left| \pm \frac{3}{p_1 p_2} \right| = \frac{3}{p_1 p_2} \leq \frac{1}{5}.$$

This latter implies that either

$$[-\infty, -1] \cup [0, +\infty] \subseteq \mathcal{L}(Y_1) \quad \text{or} \quad [-\infty, 0] \cup [1, +\infty] \subseteq \mathcal{L}(Y_1). \tag{6.1.11}$$

Figure 6.6 shows the two possibilities of $\mathcal{L}(Y_1)$ as in (6.1.11) and $\mathcal{L}(Y_2)$ as in Proposition 6.1.12. Therefore,

$$\mathcal{L}^\circ(Y_1) \cup \mathcal{L}(Y_2)^\circ = \mathscr{S}(\Sigma),$$

the conclusion holds by Theorem 6.1.2.

The computation for the class (6) holds essentially by the computation we have just shown for class (5) and therefore the details are left to the reader. □

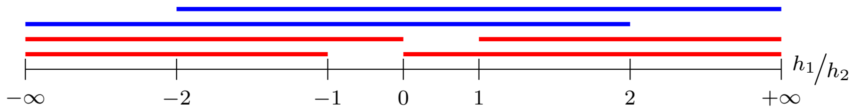


FIGURE 6.6. Let $Y = Y_1 \cup_\varphi Y_2$ be in class (5) of Table 1. The two possibilities of $\mathcal{L}(Y_1)$ are in 6.1.11 in red, the two possibilities of $\mathcal{L}(Y_2)$ are in blue.

Lemma 6.1.14. *Let $Y_1 = \mathbb{D}^2(p_1/q_1, p_2/q_2)$ with $2 \leq p_1 \leq p_2$. If $t_1 = \gcd(p_1, p_2) \leq 2$ and $(p_1, p_2) \neq (2, 2)$, then $p_1p_2/o_1t_1 \geq 3$.*

PROOF. If $t_1 = 1$, then $o_1t_1 = 1$ by Lemma 3.1.1 and $p_1p_2/o_1t_1 = p_1p_2 \geq 6$.

Let us suppose that $t_1 = 2$, this implies that $o_1 \leq 2$. We write $p_1 = 2n_1$ and $p_2 = 2n_2$, with $\gcd(n_1, n_2) = 1$. If $n_1 = 1$ and $n_2 = 2$, hence $(p_1, p_2) = (2, 4)$, then $o_1 = 1$. Thus, $p_1p_2/o_1t_1 = 4 > 3$. Without loss of generality, we can suppose that $n_1n_2 \geq 3$. We conclude that

$$\frac{p_1p_2}{o_1t_1} \geq \frac{4n_1n_2}{4} = n_1n_2 \geq 3.$$

□

In the next lemma we use the following notation: for a given manifold Y with torus boundary, two slopes γ_1, γ_2 forming a basis for $\pi_1(\partial Y)$ and an interval $I \subseteq \mathscr{S}(\partial Y)$, we denote by I_{γ_1/γ_2} the interval I computed with the convention that the fraction $p/q \in I$ corresponds to the slope $p\gamma_1 + q\gamma_2$. More explicitly,

$$\frac{p}{q} \in I_{\gamma_1/\gamma_2} \quad \text{implies that} \quad p\gamma_1 + q\gamma_2 \in I.$$

For instance, the L-space interval of Theorem 6.1.3 is written as $[\![y_-, y_+]\!]_{h_1/\mu_1}$.

Lemma 6.1.15. *Let $Y = Y_1 \cup_\varphi Y_2$ be a manifold in class (7). Let us further suppose that neither Y_1 not Y_2 is diffeomorphic to $\mathbb{D}^2(p/q, p/p-q)$ for some $p \geq 2$, then Y is an L-space.*

PROOF. According to Proposition 6.1.7 and Corollary 6.1.8, we have that $\lambda_1 \notin \mathcal{L}(Y_1)$, $\lambda_2 \notin \mathcal{L}(Y_2)$,

$$CFG(Y_1) = \{nh_1 + h_2\}_{n\in\mathbb{Z}} = \mathbb{Z} \subseteq \mathcal{L}(Y_1), \quad \text{and} \quad CFG(Y_2) = \{h_1 + nh_2\}_{n\in\mathbb{Z}} = \frac{1}{\mathbb{Z}} \subseteq \mathcal{L}(Y_2).$$

Since $\mathcal{L}(Y_1)$ is connected by Theorem 6.1.3, there exists $n \in \mathbb{Z}$ so that

$$([-\infty, n] \cup [n+1, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1), \tag{6.1.12}$$

and hence $\lambda_1 \in (n, n+1)_{h_1/h_2}$. Similarly, there exists an $m \in \mathbb{Z}$ so that $\lambda_1 \in (1/m+1, 1/m)_{h_1/h_2}$ and

$$([-\infty, 1/m+1] \cup [1/m, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_2).$$

The condition $\Delta(\lambda_1, h_2) = 1$ implies that there exists an $n' \in \mathbb{Z}_{\neq 0}$ so that

$$\lambda_1 = h_1 + n'h_2 \quad \text{and hence} \quad \lambda_1 = 1/n'. \tag{6.1.13}$$

The (6.1.12) and (6.1.13) imply that $n \in \{0, -1\}$. Similarly, condition $\Delta(\lambda_2, h_1) = 1$ implies that $\lambda_2 = m'$ for some $m' \in \mathbb{Z}_{\neq 0}$, and therefore, $m \in \{0, -1\}$. In a nutshell, we have just proven that

$$\text{either} \quad ([-\infty, -1] \cup [0, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1) \quad \text{or} \quad ([-\infty, 0] \cup [1, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1),$$

and

$$\text{either} \quad [-1, +\infty]_{h_1/h_2} \subseteq \mathcal{L}(Y_2) \quad \text{or} \quad [-\infty, 1]_{h_1/h_2} \subseteq \mathcal{L}(Y_2).$$

If $([-\infty, -1] \cup [0, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1)$ and $[-\infty, 1]_{h_1/h_2} \subseteq \mathcal{L}(Y_2)$, then the conclusion holds by Theorem 6.1.2. Similarly, if $([-\infty, 0] \cup [1, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1)$ and $[-1, +\infty]_{h_1/h_2} \subseteq \mathcal{L}(Y_2)$, then the conclusion holds by the same.

Up to inverting the orientation of $Y = Y_1 \cup_\varphi Y_2$, we suppose that

$$([-\infty, -1] \cup [0, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1) \quad \text{and} \quad [-1, +\infty]_{h_1/h_2} \subseteq \mathcal{L}(Y_2).$$

This implies that

$$\lambda_1 \in (-1, 0)_{h_1/h_2} \subset \mathscr{S}(Y_1) \quad \text{and} \quad \lambda_2 \in (-\infty, -1)_{h_1/h_2} \subset \mathscr{S}(Y_2).$$

Claim 9. There exists an $\varepsilon \in (0, 1)$ such that

$$([-\infty, -1+\varepsilon) \cup (0, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1)^\circ.$$

Proof. We prove the claim by contradiction. Since we have proven that

$$([-\infty, -1) \cup (0, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1)^\circ,$$

we suppose by contradiction that there exists an $\varepsilon \in [0, 1]$ such that

$$([-\infty, -1) \cup (-\varepsilon, +\infty])_{h_1/h_2} = \mathcal{L}(Y_1)^\circ,$$

Lemma 5.1.10 states that

$$p_1q_2 + p_2q_1 \equiv_{p_1p_2} \pm o_1t_1.$$

Let us choose q_1 and q_2 such that

$$p_1q_2 + p_2q_1 = \delta o_1t_1.$$

where $\delta \in \{\pm 1\}$. Hence,

$$\lambda_1 = \frac{p_1p_2}{o_1t_1}\mu_1 + \frac{p_1q_2 + p_2q_1}{o_1t_1}h_1 = \frac{p_1p_2}{o_1t_1}\mu_1 + \delta h_1,$$

where μ_1 is the fibration meridian of the chosen presentation. The identity $\Delta(\lambda_1, h_2) = 1$ states that

$$1 = \left|\frac{p_1p_2}{o_1t_1}\alpha + \delta\beta\right|,$$

where $\alpha = \pm\Delta(\mu_1, h_2)$ and $\beta = \pm\Delta(h_1, h_2)$. Lemma 6.1.14 implies that $\Delta(\mu_1, h_2) = 0$. Thus, either $\mu_1 = h_2$ or $\mu_1 = -h_2$.

Case $\mu_1 = h_2$. This implies that

$$\lambda_1 = \frac{p_1p_2}{o_1t_1}\mu_1 + \frac{p_1q_2 + p_2q_1}{o_1t_1}h_1 = \frac{p_1p_2}{o_1t_1}h_2 + \delta h_1,$$

since we supposed that $\lambda_1 \in (-1, 0)_{h_1/h_2}$, we obtain that $\delta = -1$. Hence

$$p_1q_2 + p_2q_1 = -o_1t_1, \quad \text{or equivalently} \quad \frac{q_1}{p_1} + \frac{q_2}{p_2} = -\frac{o_1t_1}{p_1p_2}.$$

Since $h_2 = \mu_1$, the identity $\mathbb{Q} \cup \{1/0\} \to \mathbb{Q} \cup \{1/0\}$ represents the basis change $h_1/h_2 \to h_1/\mu_1$. Therefore, since we supposed that

$$\mathcal{L}(Y_1)^\circ = ([-\infty, -1) \cup (-\varepsilon, +\infty])_{h_1/h_2} = ([-\infty, -1) \cup (-\varepsilon, +\infty])_{h_1/\mu_1}.$$

Theorem 6.1.3 implies that

$$-1 = y_+ \leq y_- \leq 0.$$

We remind the reader that for a fraction $q/p \in \mathbb{Q}$ the following hold:

$$\left\lceil \frac{q}{p} \right\rceil \leq \frac{q}{p} + \frac{p-1}{p} \quad \text{and therefore} \quad -\frac{q}{p} - \frac{p-1}{p} \leq -\left\lceil \frac{q}{p} \right\rceil.$$

We recall that (6.1.1) defines $y_\pm$ and $k > 0$ is considered to be an integer. We obtain that

$$\begin{aligned}
-1 = y_+ = &\min_{k>0} \frac{1}{k}\left(1 - \left\lceil -\frac{q_1 k}{p_1}\right\rceil - \left\lceil -\frac{q_2 k}{p_2}\right\rceil\right) \geq \\
&\min_{k>0} \frac{1}{k}\left(1 - \left(-\frac{q_1 k}{p_1} + \frac{p_1 - 1}{p_1}\right) - \left(-\frac{q_2 k}{p_2} + \frac{p_2 - 1}{p_2}\right)\right) = \\
&\min_{k>0} \frac{1}{k}\left(1 + \frac{q_1 k}{p_1} - \frac{p_1 - 1}{p_1} + \frac{q_2 k}{p_2} - \frac{p_2 - 1}{p_2}\right) = \\
&\min_{k>0} \frac{1}{k}\left(1 + \frac{q_1 k}{p_1} - \frac{p_1 - 1}{p_1} - \frac{q_1 k}{p_1} - \frac{o_1 t_1 k}{p_1 p_2} - \frac{p_2 - 1}{p_2}\right) = \\
&\min_{k>0} \frac{1}{k}\left(1 - \frac{p_1 - 1}{p_1} - \frac{o_1 t_1 k}{p_1 p_2} - \frac{p_2 - 1}{p_2}\right) = \\
&-\frac{o_1 t_1}{p_1 p_2} + \min_{k>0} \frac{1}{k}\left(1 - \frac{p_1 - 1}{p_1} - \frac{p_2 - 1}{p_2}\right) = \\
&-\frac{o_1 t_1}{p_1 p_2} + 1 - \frac{p_1 - 1}{p_1} - \frac{p_2 - 1}{p_2}.
\end{aligned}$$

Claim 10. The last line of the above is strictly bigger than -1.

Proof. The following inequalities are equivalent.

$$\begin{aligned}
&-\frac{o_1 t_1}{p_1 p_2} + 1 - \frac{p_1 - 1}{p_1} - \frac{p_2 - 1}{p_2} > -1 \\
&\frac{o_1 t_1}{p_1 p_2} + \frac{p_1 - 1}{p_1} + \frac{p_2 - 1}{p_2} < 2 \\
&o_1 t_1 + (p_1 - 1)p_2 + (p_2 - 1)p_1 < 2p_1 p_2 \\
&o_1 t_1 - p_2 - p_1 < 0.
\end{aligned}$$

We prove the claim by showing that $o_1 t_1 - p_2 - p_1 < 0$. As Y is of class (7) in Table 1, $t_1 = \gcd(p_1, p_2) \leq 2$. Since o_1 divides t_1, $o_1 t_1 \leq 4$ and $p_1 + p_2 \geq 4$. Since $(p_1, p_2) \neq (2, 2)$ by hypothesis, we get the conclusion. □

Claim 10 implies that $y_+ > -1$, this contradicts the assumption that $y_+ = -1$.

Case $\mu_1 = -h_2$. This implies that

$$\lambda_1 = \frac{p_1 p_2}{o_1 t_1}\mu_1 + \frac{p_1 q_2 + p_2 q_1}{o_1 t_1} h_1 = -\frac{p_1 p_2}{o_1 t_1} h_2 + \delta h_1,$$

since we supposed that $\lambda_1 \in (-1,0)_{h_1/h_2}$, we obtain that $\delta = 1$. Hence

$$p_1q_2 + p_2q_1 = o_1t_1, \quad \text{or equivalently} \quad \frac{q_1}{p_1} + \frac{q_2}{p_2} = \frac{o_1t_1}{p_1p_2}.$$

Since $h_2 = -\mu_1$, the map

$$\mathbb{Q} \cup \{1/0\} \to \mathbb{Q} \cup \{1/0\}, \quad \text{with} \quad \frac{p}{q} \mapsto -\frac{p}{q}$$

represents the basis change $h_1/h_2 \to h_1/\mu_1$. Therefore, the L-space interval

$$\mathcal{L}(Y_1)^\circ = ([-\infty,-1) \cup (-\varepsilon,+\infty])_{h_1/h_2}$$

becomes

$$\mathcal{L}(Y_1)^\circ = ([-\infty,\varepsilon) \cup (1,+\infty])_{h_1/\mu_1}$$

Theorem 6.1.3 implies that

$$0 \le y_+ \le y_- = 1.$$

We remind the reader that for a fraction $q/p \in \mathbb{Q}$ the following hold:

$$\frac{q}{p} - \frac{p-1}{p} \le \left\lfloor \frac{q}{p} \right\rfloor \quad \text{and therefore} \quad -\left\lfloor \frac{q}{p} \right\rfloor \le -\frac{q}{p} + \frac{p-1}{p}.$$

We obtain that

$$\begin{aligned}
1 = y_- = &\max_{k>0} \frac{1}{k}\left(-1 - \left\lceil -\frac{q_1k}{p_1} \right\rceil - \left\lceil -\frac{q_2k}{p_2} \right\rceil\right) \ge \\
&\max_{k>0} \frac{1}{k}\left(-1 - \left(-\frac{q_1k}{p_1} + \frac{p_1-1}{p_1}\right) - \left(-\frac{q_2k}{p_2} + \frac{p_2-1}{p_2}\right)\right) = \\
&\max_{k>0} \frac{1}{k}\left(-1 + \frac{q_1k}{p_1} - \frac{p_1-1}{p_1} + \frac{q_2k}{p_2} - \frac{p_2-1}{p_2}\right) = \\
&\max_{k>0} \frac{1}{k}\left(-1 + \frac{q_1k}{p_1} + \frac{p_1-1}{p_1} - \frac{q_1k}{p_1} + \frac{o_1t_1k}{p_1p_2} - \frac{p_2-1}{p_2}\right) = \\
&\max_{k>0} \frac{1}{k}\left(-1 + \frac{q_1k}{p_1} + \frac{p_1-1}{p_1} - \frac{q_1k}{p_1} + \frac{o_1t_1k}{p_1p_2} + \frac{p_2-1}{p_2}\right) = \\
&\max_{k>0} \frac{1}{k}\left(-1 + \frac{p_1-1}{p_1} + \frac{o_1t_1k}{p_1p_2} + \frac{p_2-1}{p_2}\right) = \\
&\frac{o_1t_1}{p_1p_2} + \max_{k>0} \frac{1}{k}\left(-1 + \frac{p_1-1}{p_1} + \frac{p_2-1}{p_2}\right) = \\
&\frac{o_1t_1}{p_1p_2} + -1 + \frac{p_1-1}{p_1} + \frac{p_2-1}{p_2}
\end{aligned}$$

Claim 11. Last line of the above is strictly smaller than 1.

PROOF. The following inequalities are equivalent.

$$\frac{o_1 t_1}{p_1 p_2} - 1 + \frac{p_1 - 1}{p_1} + \frac{p_2 - 1}{p_2} < 1$$
$$o_1 t_1 + (p_1 - 1)p_2 + (p_2 - 1)p_1 < 2p_1 p_2$$
$$o_1 t_1 - p_1 - p_2 < 0$$

The conclusion holds as in Claim 10. □

Claim 11 implies that $y_- < 1$, which contradicts the assumption that $y_- = 1$. □

As a consequence of Claim 9 we obtain that

$$([-\infty, -1-\varepsilon] \cup [0, +\infty])_{h_1/h_2} \subseteq \mathcal{L}(Y_1) \quad \text{and} \quad [-1, +\infty]_{h_1/h_2} \subseteq \mathcal{L}(Y_2).$$

The conclusion is given by Theorem 6.1.2. □

Lemma 6.1.16. *Let $Y = Y_1 \cup_\varphi Y_2$ be a manifold in class (7) then Y is an L-space.*

PROOF. If neither Y_1 not Y_2 is diffeomorphic to $\mathbb{D}^2(p/q, -p/q)$, then the conclusion holds by Lemma 6.1.15.

Let us suppose that either Y_1 or Y_2 is diffeomorphic to $\mathbb{D}^2(p/q, p/-q)$. According to Condition B and Theorem 5.1.8 it is not possible that both Y_1 and Y_2 are of this kind. Without loss of generality, we can suppose that $Y_2 = \mathbb{D}^2(p_2/q_2, p_2/-q_2)$. By Lemma 6.1.9

$$\mathcal{L}(Y_2) = S^1 \setminus \{\lambda_2\}.$$

Theorem 6.1.2 implies that $Y_1 \cup_\varphi Y_2$ is an L-space if and only if $\lambda_2 \in \mathcal{L}(Y_1)^\circ$.

Let μ_2 be the meridian of the chosen presentation for Y_2, Lemma 3.1.1 states that $\lambda_2 = \mu_2$. Since $\{\mu_2, h_2\}$ and $\{h_1, h_2\}$ are both bases for $\pi_1(\partial Y_2)$, we obtain that there exists an $n \in \mathbb{Z}$ so that

$$\lambda_2 = \mu_2 = h_2 + nh_1.$$

The condition $\Delta(\lambda_2, h_1) = 1$ implies that $n \in \{\pm 1\}$. Hence λ_2 is either

$$h_1 + h_2 = \frac{1}{1} = 1 \quad \text{or} \quad -h_1 + h_2 = \frac{-1}{1} = -1.$$

We proved in Lemma 6.1.15, and in particular in Claim 9, that there exists an $\varepsilon \in (0,1)$ such that either

$$((-\infty,-1+\varepsilon)\cup(0,+\infty))_{h_1/h_2} \subseteq \mathcal{L}(Y_1)^\circ \quad \text{or} \quad ((-\infty,0)\cup(1-\varepsilon,+\infty))_{h_1/h_2} \subseteq \mathcal{L}(Y_1).$$

The conclusion holds by Theorem 6.1.2. □

Lemma 6.1.17. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a manifold either of class* (8)*,* (9)*, or* (10)*. The manifold Y is an L-space*

PROOF. Since $\Delta(h_1,h_2)=1$, then $\{h_1,h_2\}$ is a basis for $\pi_1(\Sigma)$. We are going to use the notation, as in Lemma 6.1.16, such that the fraction $p/q \in \mathbb{Q}\cup\{1/0\}$ corresponds to the slope

$$ph_1 + qh_2 \subset \Sigma.$$

According to Theorem 2.2.8 and Theorem 6.1.19, if $\gamma \subset \partial Y_i$ is a slope such that $\Delta(\gamma,h_i) = 1$, then $\gamma \in \mathcal{L}(Y_i)$ for $i \in \{1,2\}$. This implies that, if $\Delta(\lambda_1,h_1) \neq 1$, then there exists an $n \in \mathbb{Z}$ such that

$$[-\infty,n]\cup[n+1,+\infty] \subseteq \mathcal{L}(Y_1).$$

Similarly, either

$$[-\infty,1] \subseteq \mathcal{L}(Y_2), \quad [-1,+\infty] \subseteq \mathcal{L}(Y_2), \quad \text{or}$$

there exists an $m \in \mathbb{Z}$ such that

$$\left[-\infty,\frac{1}{m+1}\right]\cup\left[\frac{1}{m},+\infty\right] \subseteq L(Y_2).$$

We define $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{Z}$ be four integers such that

$$\lambda_1 = \alpha_1 h_1 + \beta_1 h_2 \quad \text{and} \quad \lambda_2 = \alpha_2 h_1 + \beta_2 h_2$$

in Σ. This implies that $|\alpha_i| = \Delta(\lambda_i,h_2)$ and $|\beta_i| = \Delta(\lambda_i,h_1)$.

Let us suppose that $Y = Y_1 \cup_\Sigma Y_2$ is of class (8). This and Lemma 3.1.3 imply that

$$|\alpha_1| = \Delta(\lambda_1,h_2) = |\beta_2| = \Delta(\lambda_2,h_1) = 1 \quad \text{and} \quad |\alpha_2| = \Delta(\lambda_2,h_2) = 4.$$

Therefore $\lambda_2 = \pm 4$ in the h_1/h_2 coordinates. Therefore, either

$$[-\infty,1] \subseteq \mathcal{L}(Y_2), \quad \text{or} \quad [-1,+\infty] \subseteq \mathcal{L}(Y_2).$$

Since either $o_1 = t_1 = 1$ or $t_1 = 2$ and $o_1 = 1$, then

$$\Delta(\lambda_1, h_1) = |\beta_1| = \frac{p_1 p_2}{o_1 t_1} \geq 4.$$

This implies that $\lambda_1 = {}^1\!/_{\beta_1}$ and either

$$[-\infty, 0] \cup [1, +\infty] \subseteq \mathcal{L}(Y_1) \quad \text{or} \quad [-\infty, -1] \cup [0, +\infty] \subseteq \mathcal{L}(Y_1).$$

As we proven in Lemma 6.1.15, we can suppose without loss of generality that

$$[-\infty, -1] \cup [0, +\infty] \subset \mathcal{L}(Y_1) \quad \text{and} \quad [-1, +\infty] \subset \mathcal{L}(Y_2).$$

As we proved in Lemma 6.1.15 and in particular in Claim 9, there exists an $\varepsilon \in (0, 1)$ such that

$$[-\infty, -1 + \varepsilon) \cup (0, +\infty] \subset \mathcal{L}^\circ(Y_1).$$

The conclusion holds by Theorem 6.1.2.

Case (9) holds similarly and the proof is left to the reader.

Let Y be in class (10). Therefore $Y_1 = \mathbb{D}^2({}^2\!/_1, {}^4\!/_{q_2})$, with $q_2 \in 1, 3$ and $Y_2 = \mathbb{D}^2({}^3\!/_{q_1}, {}^3\!/_{q_1}, {}^3\!/_{q_2})$, with $(q_1, q_2) \in \{(1, 2), (2, 1)\}$.

For a given manifold in class (10), Proposition 5.1.13 gives the matrix representing the gluing $\varphi\colon \partial Y_1 \to \partial Y_2$. In particular, this shows that $Y_1 \cup_\Sigma Y_2$ is one of the following manifolds

$$\begin{aligned} &\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{1}\right) \bigcup_{\pm\left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right]} \mathbb{D}^2\left(\frac{3}{1}, \frac{3}{1}, \frac{3}{2}\right), \qquad &&\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{1}\right) \bigcup_{\pm\left[\begin{smallmatrix} -1 & 1 \\ -2 & 3 \end{smallmatrix}\right]} \mathbb{D}^2\left(\frac{3}{1}, \frac{3}{2}, \frac{3}{2}\right), \\ &\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{3}\right) \bigcup_{\pm\left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right]} \mathbb{D}^2\left(\frac{3}{1}, \frac{3}{1}, \frac{3}{2}\right), \qquad &&\mathbb{D}^2\left(\frac{2}{1}, \frac{4}{3}\right) \bigcup_{\pm\left[\begin{smallmatrix} -1 & 1 \\ -2 & 3 \end{smallmatrix}\right]} \mathbb{D}^2\left(\frac{3}{1}, \frac{3}{2}, \frac{3}{2}\right). \end{aligned} \tag{6.1.14}$$

The conclusion is given by Theorem 6.1.3 and Theorem 6.1.2, the explicit computation is left to the reader. □

Lemma 6.1.18. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a manifold either of class (11), (12), or (13). The manifold Y is an L-space.*

Proof. If Y is either of class (11), (12), or (13), then by Lemma 6.1.9 we have that

$$\mathcal{L}(Y_1) = \mathcal{L}^\circ(Y_1) = \mathscr{S}(Y_1) \setminus \{\lambda_1\}.$$

As an application of Theorem 6.1.2, the manifold $Y = Y_1 \cup_\Sigma Y_2$ is an L-space if and only if $\lambda_1 \in \mathcal{L}^\circ(Y_2)$.

We recall that if Y is either of class (12), (13), or (14), then

$$\mathcal{O}(Y_2) = (2, \cdots, 2, r_m)$$

and $\Delta(\lambda_1, h_2) = 0$. This means that $\lambda_1 = h_2$ as slopes of ∂Y_2. According to Corollary 2.1.14, the filling $Y_2(\lambda_1)$ is $SU(2)$-abelian. Therefore, Corollary 6.1.6 implies that $h_2 = \lambda_1 \in \mathcal{L}(Y_2)$.

Using the notation of Theorem 6.1.3, the slope h_2 corresponds to the fraction $1/0 = \pm\infty$. Theorem 6.1.2 implies that $Y_1 \cup_\Sigma Y_2$ is an L-space if and only if $h_2 \in \mathcal{L}(Y_2)^\circ$. Since $h_2 \in \mathcal{L}(Y_2)$, therefore $h_2 \in \mathcal{L}^\circ(Y_2)$ if and only if neither y_- nor y_+ equals $\infty = 1/0$. And indeed we prove the latter below.

We remind the reader that for $x \in \mathbb{R}$ with $x \neq 0$ the followings hold:

$$x \leq \lceil x \rceil \leq x + 1 \qquad \text{and} \quad x - 1 \leq \lfloor x \rfloor \leq x$$

and therefore

$$-x - 1 \leq -\lceil x \rceil \leq -x \qquad \text{and} \quad -x \leq -\lfloor x \rfloor \leq -x + 1.$$

We recall that by (6.1.1) the quantities y_- and y_+ are

$$y_- := \max_{k>0} -\frac{1}{k}\left(1 + (m-1)\left\lfloor -\frac{k}{2} \right\rfloor + \left\lfloor -k\frac{s_m}{r_m} \right\rfloor\right) \quad \text{and}$$
$$y_+ := \min_{k>0} -\frac{1}{k}\left(-1 + (m-1)\left\lceil -\frac{k}{2} \right\rceil + \left\lceil -k\frac{s_m}{r_m} \right\rceil\right).$$

Here $k > 0$ is an integer. We obtain that

$$\frac{m-1}{2} + \frac{s_m}{r_m} \leq \min_{k>0}\left(\frac{m-1}{2} + \frac{s_m}{r_m} + \frac{m+1}{k}\right) \leq y_+ \quad \text{and}$$
$$y_+ \leq \min_{k>0}\left(\frac{1}{k} + \frac{m-1}{2} + \frac{s_m}{r_m}\right) \leq \frac{m-1}{2} + \frac{s_m}{r_m}$$

This implies that $y_+ = \frac{m-1}{2} + \frac{s_m}{r_m} \neq \infty$. Similarly,

$$\frac{m-1}{2} + \frac{s_m}{r_m} \leq \max_{k>0} -\frac{1}{k} + \frac{m-1}{2} + \frac{s_m}{r_m} \leq y_- \quad \text{and}$$
$$y_- \leq \max_{k>0}\left(\frac{m-1}{2} + \frac{s_m}{r_m} - \frac{m-1}{k}\right) \leq \frac{m-1}{2} + \frac{s_m}{r_m}$$

This implies that $y_- = \frac{m-1}{2} + \frac{s_m}{r_m} \neq \infty$. Therefore, $\lambda_1 = h_2 \in \mathcal{L}^\circ(Y_2)$ and $Y = Y_1 \cup_\Sigma Y_2$ is an L-space by Theorem 6.1.2. □

Theorem 6.1.19. *Let $Y = Y_1 \cup_\Sigma Y_2$ be a 3-manifold as in Theorem 5.1.14. If Y is an $SU(2)$-abelian rational sphere, then Y is an L-space.*

Proof. The conclusion holds by Proposition 6.1.10, Proposition 6.1.13 and Lemma 6.1.16. □

Corollary 6.1.20. *Conjecture 2 holds for graph manifolds with at most one JSJ torus.*

Proof. The conclusion holds by Theorem 6.1.5 and Theorem 6.1.19. □

6.2. Instantons

We define the *diagonal* and *parabolic* subgroups of $SL_2(\mathbb{C})$ as:

$$D := \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, a \in \mathbb{C} \setminus 0 \right\} < SL_2(\mathbb{C}) \quad \text{and} \quad P := \left\{ \begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix}, b \in \mathbb{C} \right\} < SL_2(\mathbb{C}).$$

Every matrix in $SL_2(\mathbb{C})$ is conjugate to an element in either D or in P. In particular, the subgroups D and P are abelian.

Lemma 6.2.1. *Let X be a matrix in either D or P. Let $Y \in SL_2(\mathbb{C}) \setminus \mathcal{Z}SL_2(\mathbb{C})$. If X is not central in $SL_2(\mathbb{C})$ and $YXY^{-1}X^{-1} = 1$, then Y is in D or P respectively.*

Proof. Let us suppose that $X_1 \in D$ and $X_2 \in P$. There exist $\alpha \in \mathbb{C}\setminus 0$, $\beta \in \mathbb{C}$, $\varepsilon \in \{\pm 1\}$ and $a, b, c, d \in \mathbb{C}$ such that

$$X_1 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \quad X_2 = \varepsilon \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We suppose that X_1 and X_2 are non-central. Thus, $\alpha \neq \pm 1$ and $\beta \neq 0$. We compute

$$YX_1Y^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} ad\alpha^2 - bc & ab(-\alpha^2 + 1) \\ cd(\alpha^2 - 1) & -bc\alpha^2 + ad \end{bmatrix}.$$

If $YX_1Y^{-1} = X_1$, then $cd(\alpha^2 - 1) = 0$ and $ab(\alpha^2 - 1) = 0$. Thus, $cd = 0$ and $ab = 0$. We recall that $ab - cd = 1$. If $c = 0$, then $b = 0$. Therefore $X_1 \in D$. If $d = 0$, then $a = 0$. This implies that $bc = -1$. Therefore

$$YX_1Y = \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix} = X_1^{-1},$$

therefore $YX_1Y^{-1}X_1^{-1} \neq 1$. This contradicts the assumption we made before. This implies that $Y \in D$.

Similarly,

$$YX_2Y^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \varepsilon \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \varepsilon \begin{bmatrix} ad - c(a\beta + b) & a^2\beta \\ -c^2\beta & -bc + a(c\beta + d) \end{bmatrix}.$$

If $YX_2Y^{-1} = X_2$, then $c^2\beta = 0$. Since we supposed that $\beta \neq 0$, we obtain that $c = 0$. Therefore $X_2 \in P$. □

Definition 6.2.2. [BN90] A finitely generated group G is cyclically finite if each normal subgroup with finite cyclic quotient, other than the ones of maximal even index (on which there is no condition), has finite abelianization.

Theorem 6.2.3. *[BN90, Theorem A] Let G be a finitely generated group with finite abelianization. Then it is cyclically finite if and only if*

$$A(G; SL_2(\mathbb{C})) := \{\rho \in \mathit{Hom}(G; SL_2(\mathbb{C})) | \rho \text{ has abelian image}\}$$

is a union of irreducible components of the algebraic variety $\mathit{Hom}(G; SL_2(\mathbb{C}))$.

Theorem 6.2.4. *[BS18, Theorem 4.6] Let us suppose that Y is a rational homology sphere with $\pi_1(Y)$ cyclically finite. If Y is not an instanton L-space, then there is an irreducible representation $\pi_1(Y) \to SU(2)$.*

Theorem 6.2.4 states that $SU(2)$-abelian rational homology 3-sphere with cyclically finite fundamental group are instanton L-spaces.

Let $T_1 \subset S^3$ and $T_2 \subset S^3$ be two nontrivial torus knots. For $i \in \{1, 2\}$, let $Y_i = S^3 \setminus \nu(T_i)$ be the knot exterior of T_i. We denote by $\mu_i, \lambda_i \subset \partial Y_i$ the knot meridian and the null-homologous longitude of T_i. We say that $Y = Y_1 \cup_\Sigma Y_2$ is the *splicing* of T_1 and T_2 if

$$\Delta(\lambda_1, \mu_2) = 0 \quad \text{and} \quad \Delta(\mu_1, \lambda_2) = 0.$$

If T_1 and T_2 are torus knots of types (p, q) and (r, s) respectively, then $Y = Y_1 \cup_\Sigma Y_2$ is the manifold $Y_{(p,q),(r,s)}$ of [Mot88] and of Theorem 5.2.1.

Proposition 6.2.5. *Let $Y = Y_1 \cup_\Sigma Y_2$ be splicing of the two nontrivial torus knot $T_1 \subset S^3$ and $T_2 \subset S^3$. Then every representation $\rho\colon \pi_1(Y) \to SL_2(\mathbb{C})$ has abelian image. In particular, $\pi_1(Y)$ is cyclically finite.*

Proof. For $i \in \{1, 2\}$, let $h_i \subset \partial Y_i$ be a regular fibre of Y_i. Is it known that h_i is in the center of $\pi_1(Y_i)$.

The knot group $\pi_1(Y_i)$ is normally generated by the meridian $\mu_i \in \pi_1(Y_i)$. This means that for every $x \in \pi_1(Y_i)$, there exists a $y \in \pi_1(Y_i)$ such that $y\mu_i y^{-1} = x$. This implies that $\rho(\pi_1(Y_i))$ is central if and only if $\rho(\mu_i) = \pm 1$. We recall that $\{\pm 1\} = \mathcal{Z}SL_2(\mathbb{C})$.

Up to conjugation, $\rho(h_1) \in \mathcal{S}$ where $\mathcal{S}$ is either D or P. Let us suppose that $\rho(h_1) \neq \pm 1$. Since $h_1 \in \pi_1(Y_1)$ is central, $\rho(h_1)$ commutes with $\rho(x)$ for all $x \in \pi_1(Y_1)$. Lemma 6.2.1 implies that $\rho(x) \in \mathcal{S}$ for all $x \in \pi_1(Y_1)$. Thus, $\rho(\pi_1(Y_1)) \subseteq \mathcal{S}$. In particular, $\rho(h_2) = \rho(\mu_1) \in \mathcal{S}$. Since $\rho(h_1) \neq \pm 1$, then $\rho(\mu_1) \neq \pm 1$. This implies that $\rho(h_2)$ is not central as well. Thus, again by Lemma 6.2.1, the images $\rho(\pi_1(Y_1))$ and $\rho(\pi_1(Y_2))$ are both in $S \in \{D, P\}$. Since the groups D and P are abelian, the representation ρ has abelian image.

If $\rho(h_1) = \pm 1$, then $\rho(\mu_2) = \pm 1$. This implies that $\rho(\pi_1(Y_2))$ is in the center. In particular $\rho(h_2) = \pm 1$. Then, $\rho(\mu_1) = \rho(h_2) = \pm 1$. Therefore $\rho(\pi_1(Y_1))$ is in the center too and ρ has image in the center. Hence, ρ has abelian image.

We have just proven that, using the notation of Theorem 6.2.3,

$$A(\pi_1(Y); SL_2\mathbb{C}) = \operatorname{Hom}(\pi_1(Y); SL_2\mathbb{C}).$$

This implies that $A(\pi_1(Y); SL_2\mathbb{C})$ is the union of irreducible components of $\operatorname{Hom}(\pi_1(Y); SL_2\mathbb{C})$. According to Theorem 6.2.3, the group $\pi_1(Y)$ is cyclically finite. □

Corollary 6.2.6. *Let Y be as in Proposition 6.2.5, then it is an instanton Floer L-space.*

Proof. Proposition 6.2.5 implies that $\pi_1(Y)$ is cyclically finite. According to Theorem 5.2.1, the manifold Y is $SU(2)$-abelian. As an application of Theorem 6.2.4, the manifold Y is an instanton L-space. □

Let Y be the splicing of torus knots as in Proposition 6.2.5. Theorem 5.2.1 and Corollary 6.2.6 imply that all the conjectures of Figure 0.1 hold for Y.

Unfortunately, the technique we used in Corollary 6.2.6 to prove that Y is an instanton L-space is not extendable to all $SU(2)$-abelian manifolds we classified. In fact, it is not true that all $SU(2)$-abelian manifold of Table 1 have a cyclically finite fundamental group, as shown in the following example.

Example 6.2.7. Let $Y_1 = \mathbb{D}^2({}^2/_1, {}^4/_1)$ and $Y_2 = \mathbb{D}^2({}^2/_1, {}^2/_1)$. Let $\pi_1(\partial Y_i)$ be generated by the usual basis $\{\mu_i, h_i\}$ and let $\varphi\colon \partial Y_1 \to \partial Y_2$ be the orientation reversing diffeomorphism represented, using this choice of bases, by the matrix

$$\varphi_* = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The manifold $Y = Y_1 \cup_\varphi Y_2$ is of Class (7) in Table 1. Therefore, Y is $SU(2)$-abelian. According to (2.3.3), we have that

$$\pi_1(Y_1) = \langle x_1, x_2, h_1 | x_1^2 h_1 = x_2^4 h_1 = 1, h_1 \text{ central}\rangle \quad \text{and}$$
$$\pi_1(Y_1) = \langle y_1, y_2, h_2 | y_1^2 h_2 = y_2^2 h_2 = 1, h_2 \text{ central}\rangle.$$

For $i \in \{1, 2\}$, we define the surjective homomorphism $\omega_i\colon \pi_1(Y_i) \to {}^{\mathbb{Z}}/_{2\mathbb{Z}}$ as

$$\omega_1(x_1) = \omega_1(x_2) = \omega_2(y_1) = \omega_2(y_2) = 1 \quad \text{and} \quad \omega_1(h_1) = \omega_2(h_2) = 0.$$

We notice that

$$\omega_1(\mu_1) = \omega_1(x_1 x_2) = 0 \quad \text{and} \quad \omega_2(\mu_2) = \omega_2(y_1 y_2) = 0.$$

Therefore, the restriction of ω_i to $\pi_1(\partial Y_i)$ is the trivial homomorphism. Therefore,

$$\ker \omega_1|_{\pi_1(\partial Y_1)} = \pi_1(\partial Y_1) \quad \text{and} \quad \ker \omega_2|_{\pi_1(\partial Y_2)} = \pi_1(\partial Y_2). \tag{6.2.1}$$

Moreover, we define

$$\omega : \pi_1(Y) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \quad \text{as} \quad \omega(x) = \begin{cases} \omega_1(x) & \text{if } x \in \pi_1(Y_1) \\ \omega_2(x) & \text{if } x \in \pi_1(Y_2). \end{cases}$$

Let $p : Z \to Y$ be the connected cover space corresponding to $\ker\omega$. We recall that $\pi_1(Z) = \ker\omega$. It is easy to see that $\deg p = 2$. For $i \in \{1, 2\}$, the restriction

$$p|_{p^{-1}(Y_i)} \colon p^{-1}(Y_i) \to Y_i$$

is a cover of Y_i and corresponds to $\ker\omega_i$. Since ω_i is surjective, $p^{-1}(Y_i)$ is connected. We call $\Sigma \subset Y$ the torus corresponding to $\partial Y_1 = \partial Y_2$. As an application of (6.2.1),

$$p^{-1}(\Sigma) = \Sigma \cup \Sigma \to \Sigma$$

is the trivial disconnected cover of order two. This implies that

$$Z = p^{-1}(Y_1) \cup_{p^{-1}(\Sigma)} p^{-1}(Y_2)$$

has positive first Betti number. Therefore the normal subgroup

$$\pi_1(Z) = \ker\omega < \pi_1(Y)$$

has cyclic finite quotient and does not have a finite abelianization. It can be proven that $H_1(Y; \mathbb{Z})$ has order equal to 24, therefore the commutator subgroup of $\pi_1(Y)$ has even index equal to 24. Thus, $\ker\omega$ is not of maximal even index. By Definition 6.2.2, the group $\pi_1(Y)$ is not cyclically finite.

Let Y be a closed 3-manifold. We say that $p_{ab} : Y_{ab} \to Y$ is the *universal abelian cover* if Y_{ab} is the cover corresponding to the commutator subgroup

$$[\pi_1(Y) : \pi_1(Y)] < \pi_1(Y).$$

Clearly, the universal abelian cover is unique. Details can be found in [Rol03, Appendix A]. It is straightforward to see that if the universal abelian cover Y_{ab} of a closed 3-manifold is a rational homology 3-sphere, then $\pi_1(Y)$ is cyclically finite. So if Y is $SU(2)$-abelian and admits a universal abelian cover which is a rational homology sphere, then Y is an instanton Floer L-space by Theorem 6.2.4.

Obviously there is no hope that all the $SU(2)$-abelian 3-manifolds we have classified in this work have an universal abelian cover that is a rational homology 3-sphere. For instance, the manifold of Example 6.2.7 does not have the universal abelian cover that is a rational homology 3-sphere. In [Ped14] it can be found a way to algorithmatically determine if a graph manifold has an universal abelian cover that is a rational homology sphere.

Corollary 5.2.2 has an interesting implication. Let us suppose that Y is an $SU(2)$-abelian toroidal rational homology 3-sphere. If

$$H_1(Y;\mathbb{Z}) = 5,$$

then, as a consequence of [BS18, Corollary 4.10] and [Coh73], the manifold Y is an instanton L-space. According to [KM10, Conjecture 7.24], we expect Y to be an Heegaard Floer L-space. As an application of [HRW23, Theorem 7.20], if Y is a prime toroidal Heegaard Floer L-space with $|H_1(Y;\mathbb{Z})| = 5$, then Y is a graph manifolds obtained by gluing two copies of a trefoil exterior. That is, $Y = Y_1 \cup_\Sigma Y_2$, where Y_1 and Y_2 are trefoil exteriors. Therefore, as an application of Corollary 5.2.2,

$$Y = Y_1 \cup_\Sigma Y_2, \quad \text{with} \quad \Delta(\mu_1, h_2) = \Delta(h_1, \mu_2) = 0.$$

Therefore, $Y = Y_1 \cup_\varphi Y_2$, where $\varphi\colon \partial Y_1 \to \partial Y_2$ is an orientation revering diffeomorphism such that

$$\varphi_* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here we are using the usual basis $\{\mu_i, h_i\}$ for $\pi_1(\partial Y_i)$. It is known that, up to reversing orientation, $Y_i = \mathbb{D}^2({}^2\!/\!{}_1, {}^3\!/\!{}_{-1})$, therefore $\lambda_i = \pm 6$ by Lemma 3.1.1. This implies that

$$H_1(Y;\mathbb{Z}) = \Delta(\lambda_1,\lambda_2) = \Delta\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 6 \\ \pm 1 \end{pmatrix}; \begin{pmatrix} 6 \\ 1 \end{pmatrix}\right) = \Delta\left(\begin{pmatrix} \pm 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \end{pmatrix}\right) = |36 \pm 1|.$$

Therefore, $H_1(Y;\mathbb{Z})$ has order either 35 or 37, this contradicts the assumption that $H_1(Y;\mathbb{Z})$ has order 5. Consequently, we expect that every toroidal 3-manifold with first homology of order 5 is not $SU(2)$-abelian.

By [HRW23, Theorem 7.20], if Y is a toroidal manifold with $|H_1(Y;\mathbb{Z})| \in \{1,2,3,4,6\}$, then Y is not an L-space. Conjecture 2 predicts that such a manifold is not $SU(2)$-abelian. Therefore, as the conclusion of this work, we propose the following conjecture that is a refinement of [BS22, Conjecture 1.5]:

Conjecture 4. *Every toroidal manifold such that $|H_1(Y;\mathbb{Z})| \leq 6$ is not $SU(2)$-abelian.*

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