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TABLE DES MATIÈRES

TABLE DES FIGURES

RÉSUMÉ

Dans cette thèse nous étudions une action du monoïde libre $\mathcal{A}^*$ sur les tableaux colonnes à l'aide de l'insertion de gauche de Schensted. Cette action définit une relation de congruence $\equiv_{\mathrm{Styl}}$ et ainsi un monoïde quotient de $\mathcal{A}^*$ appelé le monoïde stylique, $\mathrm{Styl}(\mathcal{A})$. Il s'avère être un quotient fini du monoïde plaxique.

Le monoïde stylique présente plusieurs propriétés intéressantes. Nous décrivons la structure de ce monoïde à l'aide d'un algorithme d'insertion de lettre dans des tableaux appelés des N-tableaux, très proche de l'insertion de Schensted. Grâce à cela nous montrons que sa cardinalité est égale au nombre de partitions ensemblistes d'un ensemble à $|\mathcal{A}| + 1$ éléments. Ce monoïde possède une présentation simple, il est quotient du monoïde libre $\mathcal{A}^*$ par l'union des relations plaxiques avec l'idempotence $a^2 = a$ des lettres $a \in \mathcal{A}$. L'anti-automorphisme involutif canonique sur $\mathcal{A}^*$, renversant l'ordre de $\mathcal{A}$, induit une involution sur $\mathrm{Styl}(\mathcal{A})$ qui, tout comme l'involution correspondante dans le monoïde plaxique, peut se calculer à l'aide d'une opération d'évacuation sur les tableaux immaculés standards (de manière analogue à l'involution de Schützenberger sur les tableaux pour le monoïde plaxique). Le monoïde $\mathrm{Styl}(\mathcal{A})$ est $\mathscr{J}$-trivial, et son $\mathscr{J}$-ordre est gradué : le co-rang est le nombre d'éléments dans le N-tableau. Le monoïde $\mathrm{Styl}(\mathcal{A})$ est le monoïde syntaxique pour la fonction associant à chaque mot $W \in \mathcal{A}^*$ la longueur de son plus long sous-mot strictement décroissant.

Au travers de cette étude, nous revisitons deux résultats connus pour lesquels il manque une preuve directe et complète. Le premier est un résultat de Lascoux et Schützenberger (Lascoux, Schüztzenberger, 1981, Théorème 2.15) dans lequel ils affirment que le monoïde plaxique est le monoïde syntaxique pour la fonction associant à chaque mot $W \in \mathcal{A}^*$ la forme de son P-tableau. Cependant, ils énoncent ce théorème sans preuve.

Le deuxième est un résultat dû à Schensted (Schensted, 1961, Lemma 6) dans lequel il affirme la commutativité entre ses insertions à gauche et à droite. Sa preuve est malheureusement incomplète. Nous en donnons une complète en étudiant les chemins d'insertion gauche et droit.

Finalement, nous construisons un système complet d'idempotents primitifs et orthogonaux de l'algèbre du monoïde stylique. Grâce à ces idempotents, nous construisons de manière explicite une présentation de cette algèbre en terme de quotient d'une algèbre de chemin dans un carquois.

INTRODUCTION

Cette thèse est une thèse par articles. Les trois articles, sur lesquels elle est basée, ont été publiés dans Enumerative Combinatorics and Applications, Semigroup Forum et Algebraic Combinatorics. Ils ont été écrits en collaboration avec mon superviseur, Christophe Reutenauer, avec en plus Franco Saliola pour le dernier.

Cette thèse se situe dans le domaine général de l'*insertion de Schensted* et de son avatar algébrique, le *monoïde plaxique*.

Le monoïde plaxique est un objet fondamental en combinatoire, en théorie de la représentation, et en algèbre. Il prend origine dans une bijection de Schensted [72], souvent appelée la *correspondance de Robinson-Schensted (RS)*. Cette bijection envoie les permutations, vues comme des mots, sur une paire de tableaux de Young standards de même forme. Cette correspondance fut premièrement décrite par Robinson [68] mais, sa description étant abstruse, ne reçut pas beaucoup d'attention. Il décrit cette correspondance dans le cadre d'une étude des représentations du groupe symétrique et du groupe général linéaire dans laquelle il présente une preuve incomplète de la règle de Littlewood-Richardson sur le produit de fonctions de Schur. Plusieurs années plus tard, Schensted [72] présente de manière indépendante cette même correspondance dans une étude sur la classification des permutations par leurs plus long sous-mots croissants et décroissants. Pour cela, il introduit un algorithme d'insertion de lettre dans un tableau de Young standard ; maintenant connu sous le nom de l'*algorithme de Schensted*. Grâce à celui-ci, il donne une interprétation de la longueur de la première ligne et de la première colonne du tableau obtenu comme la longueur du plus long sous-mot croissant et strictement décroissant respectivement. Greene [43] généralise les résultats de Schensted en donnant une interprétation complète de la forme des tableaux.

Dans [50], Knuth généralise la correspondance de RS en une correspondance, que l'on nomme *correspondance Robinson-Schensted-Knuth (RSK)*, entre les mots sur un alphabet ordonné $\mathcal{A}$ et une paire de tableaux de Young (P, Q) de même forme, où P est un tableau semi-standard et Q est un tableau standard[1].

On montre que la condition $\mathrm{P}(V) = \mathrm{P}(U)$ définit une congruence $\equiv_{\mathrm{Plax}}$ sur le monoïde libre $\mathcal{A}^*$, et donne

[1] La généralisation de Knuth est en fait plus générale : elle est entre les matrices entières et les paires de tableaux semi-standards de même forme. Elle passe au travers de certains bimots particuliers.

ainsi place à un monoïde quotient, nommé monoïde plaxique par Lascoux et Schützenberger. Ce monoïde a une présentation cubique, donnée par Knuth [50], ayant pour générateurs $\mathcal{A}$ et des relations appelées les *relations plaxiques*.

Dans un article phare [54], Lascoux et Schützenberger font une présentation détaillée du monoïde plaxique, de sa structure et de certaines de ses applications. On trouvera aussi, dans le chapitre 5 du livre de Lothaire [52], écrit par Lascoux, Leclerc et Thibon, une étude du monoïde plaxique et de ses applications.

Depuis lors, plusieurs généralisations de la correspondance RSK ont vu le jour [82, 39, 41, 35]. De plus, nombreux monoïdes définis à l'aide d'algorithmes d'insertions de lettres dans différents objets combinatoires, analogues à celui de Schensted, sont apparus dans la littérature [31, 65, 28, 48, 42] [2]. Ces derniers, en plus d'être étudiés pour leur simple intérêt combinatoire, ont de nombreuses applications dans différents domaines des mathématiques, particulièrement en géométrie, en algèbre et en théorie de la représentation.

Le chapitre 1 offre une présentation sans preuve du monoïde plaxique au travers de trois algorithmes : l'algorithme de Schensted, introduit plus haut, l'*algorithme de Schützenberger* [73], se décrivant à l'aide de glissements de boites, puis l'*algorithme de Fomin* [36], dans lequel on construit ce qu'il appelle des *diagrammes de croissance*.

Le chapitre 2 présente l'article *On a Lemma of Schensted* [5] donnant une démonstration manquante dans la littérature. Le résultat est dû à Schensted [72], et affirme que les insertions par la droite et par la gauche, dans un tableau de Young, commutent. La preuve de Schensted n'est pas convaincante, ni celle reproduite dans le livre de Bruce Sagan [69] (nous avons eu des échanges à ce sujet avec lui). Dans la littérature, on peut trouver ce résultat prouvé indirectement ; par exemple, comme conséquence de la confluence du jeu de taquin de Schützenberger [73, 80]. La preuve présentée dans l'article est directe, par l'analyse fine des chemins d'insertion à gauche et à droite, et leur intersection, si elle existe ; dans ce cas, c'est une unique case.

Le chapitre 3, extrait de l'article présenté au chapitre 4, présente aussi une démonstration manquante dans la littérature : c'est un résultat de Lascoux et Schützenberger, qui affirme que le monoïde plaxique est le

[2] Ils sont appelées dans la littérature les *plactic-like monoids* [27, 29, 9]. Je n'ai malheureusement pas d'équivalent français succinct qui décrit aussi bien cette famille de monoïdes. J'avais pensé aux *monoïdes simili-plaxiques* mais ne suis pas convaincu par cette appellation.

monoïde syntaxique de la fonction "forme", qui associe à tout mot le diagramme de Ferrers du P-symbole du mot. Ce théorème est donné sans preuve dans leur article fondateur “Le monoïde plaxique” [54].

Le monoïde plaxique possède un autre ensemble fini de générateurs, l’ensemble des *colonnes*. Une colonne est un mot strictement décroissant. Avec cette ensemble de générateurs, le monoïde plaxique a une présentation quadratique qui est confluente [21, 26] (remarquons que la présentation standard n’est pas confluente [51]).

Les colonnes jouent un rôle spécial pour le monoïde plaxique, pouvant être très profond comme on le voit dans la première partie de [53]. Clairement la première colonne de $\mathrm{P}(W)$ dépend seulement de sa classe plaxique. On obtient ainsi, par multiplication à gauche, une action du monoïde plaxique (et de $\mathcal{A}^*$) sur l’ensemble fini des colonnes. Nous appelons *monoïde stylique* le monoïde d’endofonctions définies par cette action (pour la terminologie, nous utilisons le mot grec pour colonne). Ainsi, ce monoïde est un quotient fini du monoïde plaxique. On obtient ainsi sans doute la première représentation finie du monoïde plaxique, qui n’est pas un quotient de son abélianisation.

Le chapitre 4 constitue le cœur de la thèse. Il présente l’article sur le monoïde stylique paru dans Semigroup Forum [6].

Le chapitre 5 présente le troisième article [7] dans lequel on y trouve une construction d’idempotents orthogonaux et primitifs de l’algèbre du monoïde stylique. Elle permet de construire le carquois de cette algèbre, et de donner une présentation de l’algèbre.

CHAPITRE 1
LE MONOÏDE PLAXIQUE

1.1 Introduction

La genèse du *monoïde plaxique* prend la forme d'un algorithme, introduit par Schensted [72], associant à toute permutation un *tableau de Young standard*. Il fut généralisé par Knuth [50], envoyant des mots sur des *tableaux de Young semi-standard*. Cette généralisation permit la définition d'une structure de monoïde non-commutatif de tableaux.

Cette structure permit de nombreuses avancées ; la plus fameuse étant l'obtention d'une des premières preuves de la *règle de Littlewood-Richardson* qui décrit combinatoirement le produit de fonctions de Schur [73].

Ce chapitre est une présentation sommaire du monoïde plaxique. Nous ne présentons pas les preuves détaillées des résultats ; nous indiquons au lecteur les ouvrages de références où l'on peut les trouver. La plupart des résultats présentés dans ce chapitre se trouvent dans les livres *Young tableaux* de Fulton [38], *Enumerative Combinatorics Volume 2* de Stanley [69], *The symmetric group* de Sagan [77], et le chapitre 5 de *Algebraic combinatorics on words* de Lothaire, écrit par Lascoux, Leclerc et Thibon [52].

Après avoir présenté différents objets combinatoires qui seront utilisés tout au long du manuscrit, nous présentons trois algorithmes qui permettent une compréhension approfondie du monoïde plaxique. Le premier, l'algorithme de Schensted, qui permit de mettre en lumière une correspondance mise de l'avant par Robinson 30 ans auparavant [68] ; le deuxième, celui Schützenberger [73], qui permit une compréhension plus fine du produit de tableaux ; et le dernier, dû à Fomin [36], qui unifia les deux algorithmes précédents et permit des preuves plus directes de certaines de leurs propriétés.

1.2 Notations

1.2.1 Mots

Un <u>alphabet</u> $\mathcal{A}$ est un ensemble fini totalement ordonné, ses éléments sont appelés des <u>lettres</u>. Un <u>mot</u> sur $\mathcal{A}$ est une séquence finie $W = w_1 \cdots w_k$ de lettres de $\mathcal{A}$. Nous noterons $\mathcal{A}^*$ l'ensemble des mots sur $\mathcal{A}$. La concaténation de deux mots $U = u_1 \cdots u_r$ et $V = v_1 \cdots v_s$ est le mot $U \cdot V = u_1 \cdots u_r v_1 \cdots v_s$. Muni de la concaténation, $\mathcal{A}^*$ est un monoïde ; c'est le monoïde librement engendré par $\mathcal{A}$. Un alphabet que nous utiliserons fréquemment est l'alphabet des entiers de 1 à n que nous noterons $[n]$.

Soit $W = w_1 w_2 \cdots w_k \in \mathcal{A}^*$ un mot. Un sous-mot de W est soit le mot vide, soit un mot de la forme $S = w_{i_1} w_{i_2} \cdots w_{i_j}$ où $i_1 < i_2 < \cdots < i_j$. Un facteur de W est un sous-mot dans lequel toutes les lettres sont consécutives dans W ; i.e. un mot de la forme $F = w_i w_{i+1} \cdots w_j$ avec $1 \leq i \leq j \leq k$. Si $i = 1$ on dit que F est un préfixe et si $j = k$ que F est un suffixe. La longueur de W est notée $|W|$, et pour une lettre $a \in \mathcal{A}$, le nombre d'apparitions de a dans W est noté $|W|_a$. Le support de W est l'ensemble des lettres constituant W et est noté $\mathrm{Supp}(W)$. Le contenu de W est le multi-ensemble contenant exactement $|W|_a$ a pour chaque $A \in \mathcal{A}$. On peut aussi le voir comme la projection de W dans $\mathbf{Com}(\mathcal{A})$, le monoïde libre commutatif engendré par $\mathcal{A}$; le monoïde où toutes les lettres commutent.

On dit qu'un mot W est croissant si $w_1 \leq w_2 \leq \cdots \leq w_k$ et qu'il est strictement croissant si les inégalités sont strictes. Un mot croissant sera parfois appelé une ligne. Similairement, on dit qu'il est décroissant si $w_1 \geq w_2 \geq \cdots \geq w_k$ et qu'il est strictement décroissant si les inégalités sont strictes. Un mot strictement décroissant sera parfois appelé une colonne.

Un ordre (total) naturel sur les mots d'un alphabet totalement ordonné est l'ordre lexicographique défini par $U <_{lex} V$ si U est un préfixe de V ou s'il existe $W, U', V' \in \mathcal{A}^*$ et $x < y \in \mathcal{A}$ tels que $U = WxU'$ et $V = WyV'$.

Un autre ordre sur $\mathcal{A}^*$ qui va nous être utile est l'ordre de dominance. Soit $U = u_1 \cdots u_r$ et $V = v_1 \cdots v_s$ deux mots : on dit que V domine U si $r \geq s$ et que $u_i \leq v_i$ pour tout $1 \leq i \leq s$. Nous noterons cela $U \leq V$. Nous dirons que V domine strictement U si $u_i < v_i$ pour tout i, peu importe si $r > s$ ou $r = s$. Ainsi, dans l'alphabet $\mathcal{A} = \{a < b < c\}$, ac domine abc, bc domine strictement ab, mais par contre ac et bb ne sont pas comparables. Ainsi, cet ordre n'est pas total.

Il est utile de remarquer que tout mot W admet une unique factorisation $W = L_1 \cdot L_2 \cdots L_p$ en lignes de longueurs maximales ainsi qu'une unique factorisation $W = C_1 \cdot C_2 \cdots C_q$ en colonnes de longueurs maximales.

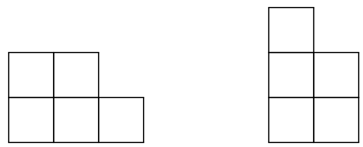

FIGURE 1.1 : Un diagramme de Ferrers de forme $(3, 2)$ et son conjugué.

FIGURE 1.2 : Un tableau de Young semi-standard et standard.

Un mot standard est un mot ayant comme contenu exactement $[n]$. Ainsi, chaque lettre de 1 à n y apparaît une et une seule fois ; par exemple 483925167. Ces mots sont en bijection avec les permutations de $[n]$ de manière très naturelle ; exemple $\frac{123456789}{483925167}$.

Soit $W \in \mathcal{A}^*$ de support $\{a_1 < \cdots < a_k\}$. La standardisation de W est un mot standard obtenu de la manière suivante : soit $\omega_i = \sum_{j=0}^{i-1} |W|_{a_j}$. On remplace les a_i de gauche à droite par les lettres $\omega_i + 1, \omega_i + 2, \cdots, \omega_i + |W|_{a_i}$.

Par exemple, pour $\mathcal{A} = \{a < b < c\}$, la standardisation de $W = abaacbcabcb$ est le mot

$$\mathrm{std}(W) \;=\; 1\,5\,2\,3\,9\,6\,\underline{10}\,4\,7\,\underline{11}\,8.$$

1.2.2 Diagrammes

Considérons le quart de plan discret des entiers naturels $\mathbb{N} \times \mathbb{N}$ munit de son ordre naturel, $(x_1, y_1) \leq (x_2, y_2)$ si $x_1 \leq x_2$ et $y_1 \leq y_2$. Un diagramme de Ferrers D est une section commençante de $\mathbb{N}^2$, ce qui signifie que si $(i, j) \leq (k, l)$ et que $(k, l) \in D$, alors $(i, j) \in D$. Nous les appelons aussi tous simplement des diagrammes, sauf s'il y a une confusion possible avec les *diagrammes gauches*, définis plus bas : nous appelons alors les diagrammes de Ferrers des diagrammes normaux. La taille de D est le nombre d'éléments de D.

On représente ces diagrammes par un empilement de boites justifié à gauche. Sujet à la force gravitationnelle, chaque rangée de boites est alors moins longue que celle en dessous ; voir figure 1.1 Ceci est la convention “française”, mais plusieurs autres conventions existent. Les deux plus connues sont les conventions anglaise et russe. La première provient de la notation matricielle. L'empilement de boites est justifié en haut et à gauche. Pour la deuxième, on effectue une rotation de $\frac{\pi}{4}$ à la convention française. Les boites sont ainsi prises dans un V formé des droites $x = y$ et $x = -y$ et l'on peut faire tomber les boites dans cette cuve.

Il peut être utile de voir les diagrammes de Ferrers à l'aide d'un autre point de vue : celui des partages d'entiers. Un partage λ d'un entier n, noté $\lambda \vdash n$, est une liste décroissante d'entiers naturels $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ telle que $|\lambda| := \sum_{i=1}^{k} \lambda_i = n$. On appelle l'entier $|\lambda|$ le poids de λ et $\ell(\lambda) = k$ la longueur de λ. Parfois, il est utile de voir un partage comme une suite infinie d'entiers naturels décroissants avec un nombre fini de coefficients non-nuls. Avec cette définition, la longueur est simplement le nombre de coefficients non-nuls.

Il y a une bijection naturelle entre les partages d'entiers et les diagrammes de Ferrers. Pour un partage λ, la longueur de la i-ième ligne de son diagramme associé est λ_i, la taille de sa i-ème part. Pour un diagramme de Ferrers D, son partage associé est appelé la forme de D, noté λ_D.

L'ensemble des partitions est ordonné de manière naturelle. Soit deux partages λ, μ, on dit que $\mu \leq \lambda$ si $\mu_i \leq \lambda_i$ pour tout i. Remarquons que cette définition ne fait du sens que si l'on voit les partages comme des suites infinies d'entiers naturels décroissants. On peut décrire facilement les couvertures : λ couvre μ si et seulement s'il existe un unique i tel que $\lambda_i - \mu_i \neq 0$ et que l'on a $\lambda_i - \mu_i = 1$.

Cette ordre possède plus de structure que le simple fait d'être un ordre, c'est en fait un treillis ; i.e. un ordre pour lequel toute paire d'éléments possède un infimum et un supremum. Leurs descriptions sont simples, pour λ, μ des partages, $(\lambda \wedge \mu)_i = \min(\lambda_i, \mu_i)$ et $(\lambda \vee \mu)_i = \min(\lambda_i, \mu_i)$. Avec les diagrammes de Ferrers, ce sont respectivement l'intersection et l'union des deux ensembles de boites qui constituent chacun des diagrammes. On appelle ce treillis le treillis de Young. Les cinq premiers rangs sont représentés à la figure 1.3.

Ce treillis a de nombreuses propriétés intéressantes. Une d'entre elles, que nous utilisons implicitement quand nous parlerons des *diagrammes de croissance*, est qu'il est un treillis *1-différentiel* ([76] Corollary 1.4).

Il y a une involution très naturelle sur les diagrammes de Ferrers, que l'on appelle la conjugaison, consistant à faire une symétrie par rapport à l'axe $x = y$. On note le diagramme obtenu D^t. Ceci induit une involution sur les partages. On note $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_p)$ le partage conjugué de λ. Pour un diagramme D, ceci revient à voir λ^*_D comme étant la taille des colonnes de D. Similairement, λ_D peut alors être vu comme la taille des colonnes de D^t. Voir figure 1.1 pour un exemple.

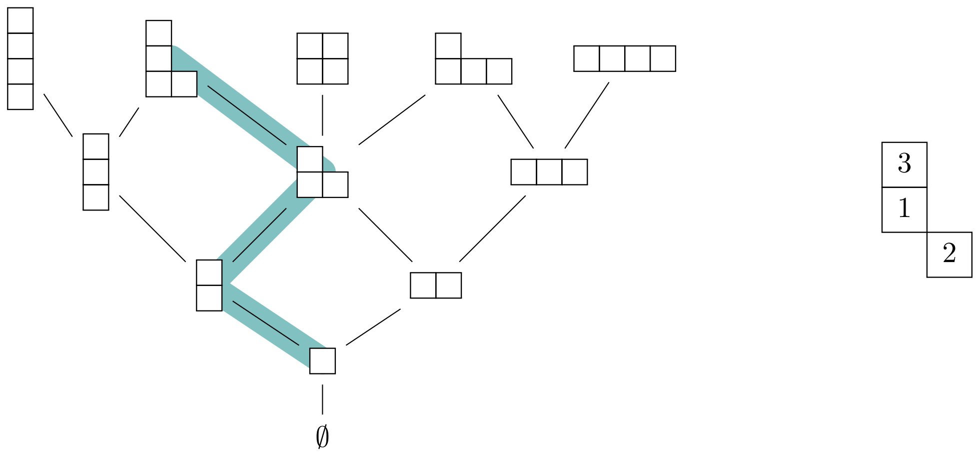

FIGURE 1.3 : Les cinq premiers rangs du diagramme de Hasse du treillis de Young. Le tableau gauche standard associé au chemin dans le treillis de Young.

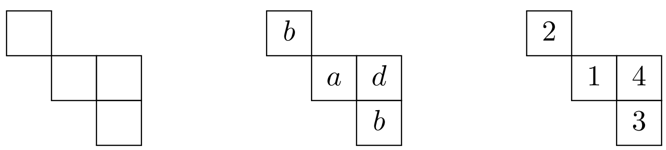

FIGURE 1.4 : Un diagramme gauche de forme $(3,3,1)/(2,1)$, un remplissage semi-standard de ce diagramme ainsi qu'un remplissage standard.

Soit $\lambda > \mu$ deux partages et D_λ, D_μ leurs diagrammes respectifs. Un diagramme gauche de forme λ/μ est le diagramme obtenu en retirant les cases de D_μ du diagramme D_λ. Voir la figure 1.4 pour un exemple de diagramme gauche.

Un coin extérieur d'un diagramme gauche D est une boite (i,j) qui ne fait pas partie de D mais tel que $(i-1,j)$ et $(i,j-1)$ font partie de D. Un coin intérieur est une boite (i,j) qui ne fait pas partie de D mais tel que $(i+1,j)$ et $(i,j+1)$ font partie de D. Remarquons qu'un diagramme gauche n'ayant aucun coin intérieur est un diagramme normal.

1.2.3 Tableaux

Un tableau de Young semi-standard est la donnée d'un diagramme de Ferrers D et d'une fonction croissante f de D vers $\mathcal{A}$ telle que la restriction de f aux sous-ensembles de D ayant la même valeur en x est injective. Autrement dit, la fonction f est un étiquetage croissant sur les lignes et strictement croissant sur colonnes des boites du diagramme de Ferrers . Si la fonction f est une bijection vers $[n]$, le tableau est appelé un tableau de Young standard de taille n. Nous appellerons les tableaux de Young semi-standard simplement des *tableaux semi-standards* et des tableaux de Young standards des *tableaux standards*. Tout comme avec les diagrammes de Ferrers, s'il y a une confusion possible avec les *tableaux gauches*, que nous définissons plus bas, nous les appelons alors des tableaux normaux. Voir figure 1.2 pour des exemples de tableaux semi-standard et standard. Nous appellerons la forme d'un tableau, la forme de son diagramme sous-jacent.

Soit T un tableau semi-standard. Le mot ligne de T est le mot $r(T)$ obtenu en lisant les lignes de T de gauche à droite en partant du haut. Le mot colonne de T est le mot $c(T)$ obtenu en lisant les colonnes de T de haut en bas en partant de la gauche. Par exemple, le tableau de gauche de la figure 1.2 a pour mot ligne $be\,aac$ et pour mot colonne $ba\,ea\,c$.

On peut remarquer qu'un mot W est un mot ligne (resp. un mot colonne) d'un tableau T si et seulement si sa factorisation en lignes $W = L_p \cdot L_{p-1} \cdots L_1$ (resp. sa factorisation en colonnes $W = C_1 \cdot C_2 \cdots C_q$) est telle que $L_i < L_{i+1}$ pour tout $1 \leq i \leq p-1$ (resp. $C_i \leq C_{i+1}$ pour tout $1 \leq i \leq q-1$). Le support d'un tableau est le support de son mot ligne ; il en va de même pour son contenu.

Nous considérerons une colonne, à la fois comme un tableau semi-standard ne possédant qu'une unique colonne, comme un mot strictement décroissant, mais aussi comme un sous-ensemble de $\mathcal{A}$. De même, nous considérerons une ligne, à la fois comme un tableau semi-standard ne possédant qu'une unique ligne, comme un mot croissant, mais aussi comme un multi-sous-ensemble de $\mathcal{A}$.

Un tableau gauche semi-standard (ou simplement *tableau gauche*) est la donnée d'un diagramme gauche D et d'une fonction croissante f de D vers $\mathcal{A}$ telle que la restriction de f aux sous-ensembles de D ayant la même valeur en x est injective. Si f est une bijection avec $[n]$, on dit que le tableau gauche est standard. Voir la figure 1.4 pour des exemples de tableaux gauches semi-standard et standard.

Un coin extérieur d'un tableau est tout simplement un coin extérieur de son diagramme de Ferrers. Il en va de même pour un coin intérieur. Un trou est une boite dans laquelle il n'y a pas d'étiquette. Un tableau troué est un tableau possédant des trous. Cette notion généralise les tableaux gauches car pour un tableau gauche de forme λ/μ, les boites se trouvant des D_μ sont des trous.

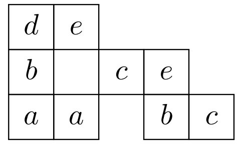

FIGURE 1.5 : Un tableau semi-standard troué avec ses trous aux boites $(3,1)$ et $(2,2)$.

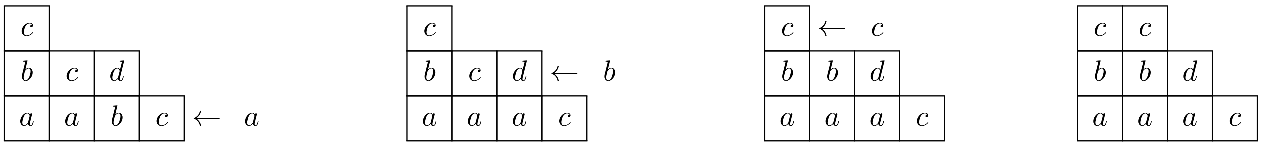

FIGURE 1.6 : Les étapes de l'insertion de a dans le tableau ayant comme mot ligne $c\,bcd\,abbc$.

Il existe une correspondance entre les tableaux standards de forme λ/μ et les chemins croissants dans le diagramme de Hasse du treillis de Young partant de μ et se rendant à λ. Il suffit d'étiqueter par i la i-ème boite ajoutée lors du parcours du chemin. Voir figure 1.3 pour un exemple. Remarquons que si le tableau est normal, $\mu = \emptyset$ et donc le chemin part de l'élément minimal du treillis.

1.3 Algorithme de Schensted

L'algorithme de Schensted [72] associe à chaque mot un tableau standard. Il se définit grâce à une insertion récursive des lettres du mots. Il y a deux insertions, l'*insertion par ligne*, parfois appelée l'insertion à droite, et l'*insertion par colonne*, parfois appelée l'insertion à gauche.

1.3.1 Insertion ligne

Pour définir l'insertion ligne définissons d'abord l'insertion d'une lettre dans une ligne.

Soit L une ligne et x une lettre. L'insertion de x dans L est la ligne L' obtenue en rajoutant x à droite de L si $x \geq \max(\text{Supp}(L))$, sinon $L' = L \backslash y \cup x$ où $y = \min\{z \in \text{Supp}(L) | z > x\}$. Nous dirons de y qu'elle a été éjectée de L par x.

L'insertion à droite de x dans un tableau semi-standard T est définie comme l'insertion de x dans la première ligne, puis de la lettre éjectée dans la ligne suivante et ainsi de suite jusqu'à ce qu'aucune lettre ne soit éjectée. Nous notons cette insertion $T \leftarrow x$. Voir figure 1.6 pour un exemple de l'insertion à droite.

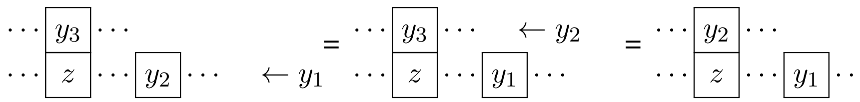

FIGURE 1.7 : La représentation partielle d'une insertion à droite dans un tableau semi-standard.

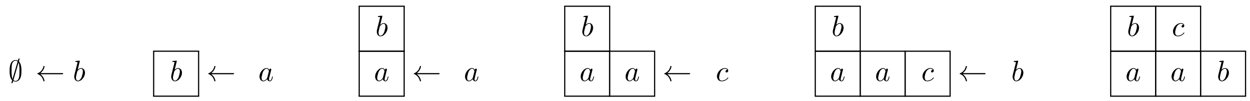

FIGURE 1.8 : Les étapes de calcul de $\mathrm{P}(baacb)$.

Lemme 1.1 ((Schensted) [72] Lemma 1) *Soit T un tableau semi-standard et x une lettre, alors $T \leftarrow x$ est un tableau semi-standard.*

Dans la figure 1.7, par la définition d'un tableau semi-standard et de l'insertion, nous avons $z \leq y_1 < y_2 < y_3$. Il est alors facile de se convaincre que le tableau obtenu est semi-standard aussi.

Soit $W = w_1 w_2 \cdots w_k$ un mot. On appelle P-symbole de W (ou son P-tableau) le tableau semi-standard, noté $\mathrm{P}(W)$, obtenu en insérant à droite récursivement les lettres de W dans le tableau vide en commençant par w_1, i.e. $\mathrm{P}(W) = ((\cdots((\emptyset \leftarrow w_1) \leftarrow w_2)\cdots) \leftarrow w_k)$ où $\emptyset$ représente le tableau vide. Cette procédure est communément appelée l'algorithme de Schensted. Voir figure 1.8 pour un exemple de calcul du P-tableau.

Le mot ligne représente parfaitement un tableau grâce au résultat suivant, corollaire direct des définitions de l'insertion ligne et du P-tableau.

Corollaire 1.2 *Soit T un tableau semi-standard. Alors* $\mathrm{P}(r(T)) = T$.

1.3.2 Insertion colonne

Il est également possible de définir l'algorithme de Schensted à l'aide d'une insertion symétrique, l'*insertion par colonne*, parfois appelée l'insertion à gauche.

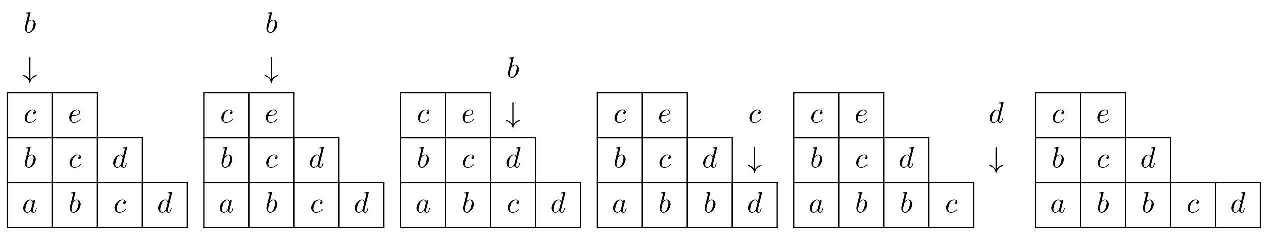

FIGURE 1.9 : Les étapes de l'insertion de b dans le tableau ayant comme mot colonne $cba\,ecb\,db\,c\,d$.

De la même manière, définissons tout d'abord l'insertion d'une lettre dans une colonne.

Soit C une colonne et x une lettre. L'insertion de x dans C est la colonne C' obtenue en rajoutant x en haut de C si $x > \max \mathrm{Supp}(C)$, sinon $C' = C \backslash y \cup x$ où $y = \min\{z \in \mathrm{Supp}(L) | z \geq x\}$. Nous dirons de y qu'elle a été éjectée de C par x.

Remarquons qu'il y a une subtile modification dans les inégalités par rapport à l'insertion ligne : les inégalités strictes deviennent larges et vice-versa. Ceci est due à la croissance stricte des lignes comparativement à la croissance large des lignes.

L'insertion à gauche de x dans un tableau semi-standard T est définie de la même manière que l'insertion à droite : on insère x dans la première colonne, puis la lettre éjectée dans la colonne suivante et ainsi de suite jusqu'à ce qu'aucune lettre ne soit éjectée.

Lemme 1.3 ((Schensted) [72] Lemma 1) *Soit T un tableau semi-standard et x une lettre, alors $x \rightarrow T$ est un tableau semi-standard.*

Le P-tableau peut être calculé aussi à l'aide de l'insertion à gauche grâce au lemme suivant.

Lemme 1.4 ((Schensted) [72] Lemma 7) *Soit $W = w_1 w_2 \cdots w_k$ un mot. Alors,*

$$\mathrm{P}(W) = w_1 \rightarrow (w_2 \rightarrow (\cdots (w_k \rightarrow \emptyset) \cdots)).$$

Ce lemme est prouvé par récurrence en utilisant le résultat suivant sur la commutativité des deux insertions.

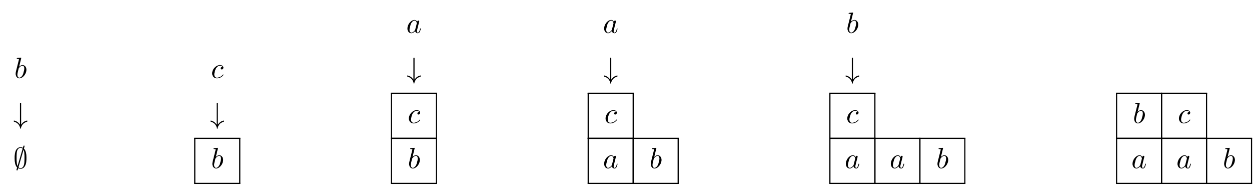

FIGURE 1.10 : Les étapes de calcul de $\mathrm{P}(baacb)$ en utilisant l'insertion colonne.

Lemme 1.5 ((Schensted) [72] Lemma 6) *Soit $x, y \in \mathcal{A}$ et T un tableau de Young semi-standard . Alors*

$$(x \to T) \leftarrow y = x \to (T \leftarrow y).$$

Avec Christophe Reutenauer, nous n'étions pas satisfait de la preuve de ce dernier lemme donnée par Schensted. En plus du fait que certains cas sont manquants et non listés, la récurrence utilisée semble cacher la mécanique de ces deux insertions et la façon dont elles interagissent ensemble.

Nous avons donc écrit conjointement une note dans laquelle nous développons les outils nécessaires pour rigoureusement prouver ce lemme. Cette preuve est présentée au Chapitre 2 de ce présent document.

Le mot colonne représente parfaitement un tableau grâce au résultat suivant, corollaire immédiat de la définition de l'insertion par ligne et du lemme 1.5

Corollaire 1.6 *Soit T un tableau semi-standard. Alors* $\mathrm{P}(c(T)) = T$.

1.3.3 Correspondance de Robinson-Schensted-Knuth

L'algorithme d'insertion de Schensted a mis en lumière une correspondance surprenante entre les paires de tableaux standards de même forme et les permutations, appelée la correspondance de Robinson-Schensted. Knuth la généralisera par la suite à une correspondance entre les matrices sur $\mathbb{N}$ et les paires de tableaux semi-standards.

En regardant une permutation σ comme un mot sur $[n]$, nous pouvons construire à l'aide de l'algorithme de Schensted, un tableau de Young standard. Ce n'est bien évidement pas une correspondance car, par exemple, 132 et 312 nous rendent tous deux le même tableau. Il nous suffit de construire en même temps un tableau d'enregistrement, aussi appelé le Q-tableau (ou Q-symbole) et noté $\mathrm{Q}(\sigma)$. Ce tableau nous indique l'ordre

$$26513784 \quad \longleftrightarrow \quad \left(\begin{array}{|c|c|c|c|} \hline 6 \\ \cline{1-1} 2 & 5 & 7 \\ \cline{1-3} 1 & 3 & 4 & 8 \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|c|} \hline 4 \\ \cline{1-1} 3 & 5 & 8 \\ \cline{1-3} 1 & 2 & 6 & 7 \\ \hline \end{array} \right)$$

FIGURE 1.11 : La correspondance entre 26513784 et la paire de tableaux $(\mathrm{P}(26513784), \mathrm{Q}(26513784))$.

d'apparition des cases dans le tableau $\mathrm{P}(\sigma)$. Ainsi, à l'insertion de $\sigma(i)$, nous mettons l'étiquette i dans la boite nouvellement créée dans $\mathrm{P}(\sigma)$.

Théorème 1.7 ((Schensted) [72] Lemma 3) *L'application RS qui envoie une permutation σ sur $(\mathrm{P}(\sigma), \mathrm{Q}(\sigma))$ est une bijection entre les permutations et les paires de tableaux standards de même forme $\lambda \vdash n$.*

La preuve de ce résultat réside dans la construction d'un algorithme inverse. Le Q-tableau nous indique quelle case a été rajoutée en dernier dans le P-tableau. Soit cette boite est sur la première ligne et ainsi c'est son contenu x qui fut insérée. Soit elle est sur une ligne L_i avec $i > 1$ et son contenu x fut éjecté par une lettre se trouvant à son ancienne place dans la ligne L_{i-1}. C'est en fait précisément y le maximum des lettres strictement plus petites que x se trouvant dans L_{i-1}. Nous pouvons ainsi retracer quelle lettre fut insérée dans le tableau et en procédant de la sorte, nous pouvons reconstruire le mot en partant de la fin.

Ce résultat a pour corollaire l'identité

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!,$$

où f^λ est le nombre de tableaux de Young standard de forme λ.

Nous allons explorer la généralisation de Knuth de cet correspondance.

Pour cela nous avons besoin de la notion de bimots. Une bilettre sur un alphabet $\mathcal{A}$ est une paire de lettres (a, b) de $\mathcal{A}$ souvent représentée de la manière suivante $\frac{a}{b}$. Un bimot est un mot constitué de bilettres. Une permutation écrite à l'aide de la notation à deux lignes est un bimot avec les lettres du haut croissantes.

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \leftrightarrow \begin{smallmatrix} 1111122222333 \\ 2233311233113 \end{smallmatrix} \leftrightarrow \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 & 2 & 2 & 3 \\ \cline{1-4} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ \hline \end{array} , \begin{array}{|c|c|c|c|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 & 2 & 3 & 3 \\ \cline{1-4} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline \end{array} \right)$$

FIGURE 1.12 : Le passage d'une matrice entière 3×3 à un bimot puis à une paire de tableaux semi-standards de même forme par la correspondance RSK.

À toute matrice $M = (a_{i,j})$ carrée à coefficients entiers naturels, nous pouvons lui associer le bimot

$$\prod_{i=1}^{n} \prod_{j=1}^{n} \binom{j}{i}^{a_{ij}} = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \cdots \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \cdots \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \cdots \begin{smallmatrix} 1 \\ n \end{smallmatrix} \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \cdots \begin{smallmatrix} 2 \\ n \end{smallmatrix} \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \cdots \begin{smallmatrix} n \\ n \end{smallmatrix},$$

qui a $a_{i,j}$ copies de la lettre $\begin{smallmatrix} j \\ i \end{smallmatrix}$. On remarque que si la matrice M est la matrice pour une permutation σ, on retrouve exactement σ écrite avec la notation à deux lignes. Voir figure 1.12 pour un exemple de cette correspondance.

Pour construire la paire de tableaux standards, le P-tableau est obtenu en insérant les lettres du mot du bas à l'aide de l'insertion de Schensted et, pour le Q-tableau, on procède comme pour le Q-tableau dans la correspondance de Robinson-Schensted mais, au lieu d'étiqueter les boites ajoutées de 1 à n, on y étiquette la nouvelle boite avec le haut de la bilettre. Voir figure 1.12 pour un exemple de cette procédure.

Théorème 1.8 ((Knuth) [50] Theorem 2) *L'application RSK qui envoie une matrice carrée M à coefficients entiers naturels sur $(\mathrm{P}(M), \mathrm{Q}(M))$ est une bijection entre les matrices carrées M d'ordre n à coefficients entiers naturels et les paires de tableaux semi-standards de même forme λ, où $\ell(\lambda) \leq n$.*

La case qui a été rajoutée en dernier est tout simplement la case qui possède la plus grande étiquette qui est la plus à droite. Le reste de la preuve est exactement la même que pour la correspondance avec les permutations. Un exemple de cette correspondance se trouve à la figure 1.12

1.4 La structure du monoïde plaxique

1.4.1 Monoïde plaxique

Nous avons donc une fonction des mots vers les tableaux semi-standards. Il est intéressant d'étudier la relation d'équivalence induite par cette fonction : deux mots U, V sont équivalents, noté $U \equiv_{\mathrm{Plax}} V$ si

$\mathrm{P}(U) = \mathrm{P}(V)$. Cette relation fut nommée relation plaxique par Lascoux et Schützenberger dans [54]. Grâce à la compatibilité des deux insertions avec la concaténation dans $\mathcal{A}^*$, cette relation est une relation de congruence. Ainsi le quotient de A^* par $\equiv_{\mathrm{Plax}}$ est un monoïde que Lascoux et Schützenberger ont nommé monoïde plaxique.

D'un autre côté, considérons la congruence $\sim$ de $\mathcal{A}^*$ engendrée par les relations suivantes :

$$acb \sim cab, \qquad \text{pour } a \leq b < c, \tag{1.1}$$

$$bac \sim bca, \qquad \text{pour } a < b \leq c. \tag{1.2}$$

Ces relations sont dues à Knuth, qu'il présenta dans [50], et sontc souvent appelées les relations de Knuth.

Théorème 1.9 ((Knuth) [50] Théorème 6) *Soient deux mots $U, V \in \mathcal{A}^*$. Alors $\mathrm{P}(U) = \mathrm{P}(V)$ si et seulement si $U \sim V$.*

Ainsi, la relation plaxique est engendrée par les relations de Knuth. De plus, nous obtenons des représentants en mots des classes plaxiques en les mots lignes et colonnes d'un tableau.

Corollaire 1.10 *Soit W un mot. Nous avons alors*

$$W \sim r(\mathrm{P}(W)) \sim c(\mathrm{P}(W)).$$

1.4.2 Système de réécriture convergent

Une système de réécriture de mots est la paire $(\mathcal{A}^*, \to)$, où $\to$ est une relation binaire sur $\mathcal{A}^*$ appelée règles de réécriture. On dit qu'un système de réécriture est fini s'il y possède un nombre fini de règles de réécriture. On appelle une suite de réécriture une suite $U_0 \to U_1 \to \cdots \to U_n$, et on écrit $U_0 \xrightarrow{*} U_n$ pour indiquer qu'il existe une suite de réécriture allant de U_0 à U_n. Autrement dit, $\xrightarrow{*}$ est la clôture réflexive et transitive de $\to$.

Lorsque l'on a un monoïde quotient **M** de $\mathcal{A}^*$ défini par des relations $\equiv$, nous pouvons orienter ces relations pour obtenir un système de réécriture. Par exemple, comme mentionné plus haut, le monoïde plaxique est le quotient de $\mathcal{A}^*$ par les relations de Knuth. Nous pouvons orienter les relations de Knuth $acb \to cab$ et

FIGURE 1.13 : Représentations diagrammatique de la confluence (à gauche) et de la confluence locale (à droite).

$bac \to bca$ et nous obtenons un système de réécriture plaxique. Ainsi nous avons par exemple la suite de réécritures $21354 \to 23154 \to 23514$ dans laquelle tous les mots sont donc équivalents plaxiquement.

On dit qu'un système de réécriture est <u>noethérien</u> s'il n'existe aucune suite de réécriture infinie $U_0 \to U_1 \to \cdots \to U_n \to \cdots$.

Proposition 1.11 *(i) Le système de réécriture*

$$acb \to cab, \qquad \textit{pour } a \leq b < c, \tag{1.3}$$

$$bac \to bca, \qquad \textit{pour } a < b \leq c. \tag{1.4}$$

est un système noethérien.

(ii) Similairement, le système de réécriture

$$cab \to acb, \qquad \textit{pour } a \leq b < c, \tag{1.5}$$

$$bca \to bac, \qquad \textit{pour } a < b \leq c. \tag{1.6}$$

est un système noethérien.

Ceci est dû au fait que si $U \to V$ alors $U <_{lex} V$ (resp. $U >_{lex} V$) et que chaque classe plaxique ne contient qu'un nombre fini de mots.

Par contre, si nous prenons $acb \to cab$ et $bca \to bac$, le système de réécriture n'est pas noethérien car nous avons la suite $bacb \to b\underline{cab} \to \underline{bac}b$ que nous pouvons répéter autant de fois que nous le voulons.

Un système de réécriture est dit confluent s'il respecte la propriété suivante :
pour tout $U, V \neq V' \in \mathcal{A}^*$ tels que $U \xrightarrow{*} V$ et $U \xrightarrow{*} V'$, il existe $W \in \mathcal{A}^*$ tel que $V \xrightarrow{*} W$ et $V' \xrightarrow{*} W$. Voir figure 1.13 à gauche pour une représentation diagrammatique de la confluence.

En d'autre mots, l'ordre dans lequel on applique les règles de réécritures importe peu.

Une notion moins forte de confluence est la confluence locale qui est définie de la manière suivante : pour tout $U, V \neq V' \in \mathcal{A}^*$ tels que $U \to V$ et $U \to V'$, il existe $W \in \mathcal{A}^*$ tel que $V \xrightarrow{*} W$ et $V' \xrightarrow{*} W$. Voir figure 1.13 à droite pour une représentation diagrammatique de la confluence locale.

Il est évident qu'un système confluent est localement confluent. Par contre, l'inverse n'est pas vrai. Le système, $x \leftarrow y \leftrightarrow z \to t$ est localement confluent car il suffit de vérifier les deux cas suivant :

$$\begin{array}{ccc} y & \longrightarrow & x \\ \downarrow & & \downarrow \\ z \to y & \to & x \end{array} \qquad\qquad \begin{array}{ccc} z & \longrightarrow & t \\ \downarrow & & \downarrow \\ y \to z & \to & t \end{array} .$$

Or il n'est pas confluent car nous avons $y \to x$ et $y \xrightarrow{*} t$ mais aucun w tel que $x \xrightarrow{*} w \xleftarrow{*} t$. Il est important de noter que ce système n'est pas noethérien. En effet, nous avons la séquence infinie $y \to z \to y \cdots$.

Dans le cas des systèmes de réécriture noetherien, Newman démontra qu'il y a une équivalence entre confluence et confluence locale.

Théorème 1.12 ((Newman) [64] Theorem 3) *Un système de réécriture noetherien est confluent si et seulement si il est localement confluent.*

Un système de réécriture est dit convergent s'il est noethérien et confluent. La convergence d'un système de réécriture issu d'une relation de congruence permet de définir des représentants canoniques pour chaque classe d'équivalence ; dans la littérature, l'appelation canonique est parfois utilisée à la place de convergent.

Malheureusement, Kubat et Okniński ont démontré que pour $|\mathcal{A}| > 3$ un système de réécriture convergent issu des relations de Knuth nécessite une infinité de règles de réécriture ([51, Theorem 3]).

Cain, Gray et Malheiro, dans [26], ont défini un système de réécriture différent pour le monoïde plaxique, pour lequel l'ensemble des générateurs est l'ensemble Γ des colonnes. Rappelons que l'on peut écrire de manière unique un tableau semi-standard comme produit croissant de colonnes (vu comme des mots strictement décroissants et l'ordre est l'ordre de dominance). Il est important de noter que Bokut, Chen, Chen et Li ont obtenu la même présentation mais de manière plus explicite.

Pour les relations de réécriture, il faut remarquer que l'insertion d'un mot strictement décroissant dans un tableau colonne nous retourne un tableau ayant au plus deux colonnes.

Lemme 1.13 ((Cain, Gray et Malheiro) [26] Lemma 3.1) *Soit U, V deux mots strictement décroissants. Alors soit UV est strictement décroissant soit il existe U', V' des mots strictement décroissants tels que $c(\mathrm{P}(UV)) = U'V'$.*

Les relations du système de réécriture $(\Gamma, \rightharpoonup)$ sont alors définies de la manière suivante : si deux colonnes U et V sont telles que U n'est pas dominée par V, alors soit $\min(U) > \max(V)$ et $U \cdot V \rightharpoonup W$ pour W la colonne telle que $\mathrm{Supp}(W) = \mathrm{Supp}(U) \cup \mathrm{Supp}(V)$, ou bien il existe des colonnes U', V' telles que $\mathrm{P}(UV) = U'V'$ et $U \cdot V \rightharpoonup U' \cdot V'$.

Rappelons que pour un alphabet fini totalement ordonné $\mathcal{A}$, il n'y a qu'un nombre fini de mots strictement décroissants. Ainsi, il y a un nombre fini de paire de colonnes.

Théorème 1.14 ((Cain, Gray et Malheiro) [26] Theorem 3.4, (Bokut, Chen, Chen, Li) [21] Theorem 4.5) *Le système de réécriture $(\Gamma, \rightharpoonup)$ est convergent et fini.*

Ce qui est d'autant plus intéressant pour ce système de réécriture est que les représentants canoniques sont les factorisations en mots colonnes des tableaux semi-standards.

1.4.3 Sous-mot croissant et strictement décroissant

La première application de l'algorithme de Schensted est le calcul de la longueur du plus long sous-mot croissant ainsi que du plus long sous-mot strictement décroissant d'un mot.

Théorème 1.15 ((Schensted) [72] Theorem 1 & 2) *Pour un mot W, la longueur de son plus long sous-mot croissant est la longueur de la première ligne de son* P*-tableau. La longueur de son plus long sous-mot strictement décroissant est la hauteur de sa première colonne.*

Attention le sous-mot croissant le plus long n'est pas nécessairement la première ligne ! On peut prendre comme exemple 3231. La première ligne de son P-tableau est 13, mais ce n'est pas un sous-mot de 3231.

Greene généralise ce résultat dans son papier [43] apportant une interprétation de la forme du tableau.

Pour énoncer le théorème de Greene nous avons besoin de deux notations. Appelons longueur de k mots la somme de leurs longueurs. Pour W un mot, on note $c_k(W)$ la longueur maximale de k sous-mots croissants disjoints de W et $d_k(W)$ la longueur maximale de k sous-mots strictement décroissant disjoints de W.

Par exemple, si on considère le mot 13434122332, nous avons $c_1(13434122332) = 6$, $c_3(13434122332) = 11$, $d_1(13434122332) = 3$ et $d_3(13434122332) = 8$.

Théorème 1.16 ((Greene) [43] Théorème 3.1) *Soit W un mot, et $\lambda = (\lambda_1, \cdots, \lambda_p)$ la forme de son* P*-tableau. Alors, pour tout $1 \leq k \leq p$,*

$$c_k(W) = \sum_{i=1}^{k} \lambda_i, \tag{1.7}$$

$$d_k(W) = \sum_{i=1}^{k} \lambda'_i. \tag{1.8}$$

Rappelons que $\lambda' = (\lambda_1, \cdots, \lambda_q)$ est le partage conjugué de λ.

Ce résultat est évident sur les mots qui sont respectivement des mots lignes et des mots colonnes d'un tableau semi-standard ; il est même vraiment facile d'extraire un ensemble de k sous-mots donnant le résultat voulu, on peut seulement prendre les k premières lignes (respectivement les k premières colonnes).

Il nous suffit donc de vérifier que les propriétés (1.7) et (1.8) de c_k et d_k sont bien préservées par les relations de Knuth.

Proposition 1.17 *Soit $U \equiv_{\mathrm{Plax}} V$ alors $c_k(U) = c_k(V)$ et $d_k(U) = d_k(V)$.*

1.5 Algorithme de Schützenberger

Schützenberger [73] définit un autre algorithme sur les tableaux qui, à la place de pousser des boites comme le fait Schensted, les fait glisser tel un jeu-de-taquin (d'où son nom). Cet algorithme permit d'exhiber certaines propriétés combinatoires intéressantes du monoïde plaxique. Surtout, il permit de décrire de manière très simple une involution sur les tableaux qui préserve leurs formes et qui décrit un anti-automorphisme induit par un morphisme classique en combinatoire des mots souvent appelé la *double inversion*.

1.5.1 Jeu-de-taquin

Grâce à l'insertion de Schensted, et aux résultats qui en découlent, nous pouvons définir le produit de T, S, deux tableaux de Young semi-standards, de la manière suivante :

$$T \cdot S := \mathrm{P}(UW),$$

où $\mathrm{P}(U) = T$ et $\mathrm{P}(W) = S$. Nous avons les égalités suivantes

$$(T \cdot S) = (T \leftarrow W) = (U \rightarrow S) = (U \rightarrow \emptyset \leftarrow W).$$

Une façon tout à fait ingénieuse de calculer le produit entre deux tableaux, introduite par Schützenberger, consiste à *rectifier* un tableau gauche en utilisant une procédure appelée jeu-de-taquin.

Soit T un tableau et b un coin intérieur. Informellement, un glissement consiste à transformer un coin intérieur en un coin extérieur. Pour cela on échange récursivement b avec celui, entre son voisin nord et est, qui a la plus petite étiquette. S'il y a égalité entre les deux étiquettes, on l'échange avec son voisin nord pour préserver la propriété d'être strictement croissant sur les colonnes. On appelle ces étapes intermédiaires des mouvements extérieurs. Un contre-glissement est l'opération inverse consistant à transformer un coin extérieur en un coin intérieur. Les étapes intermédiaires sont appelées des mouvements intérieurs. Voir figure 1.14 pour un exemple de glissement.

La rectification d'un tableau gauche T consiste à le transformer en un tableau normal. Pour cela, nous itérons le procédé de glissement des coins intérieurs jusqu'à l'obtention d'un tableau normal.

Cette procédure soulève une question de confluence : est-ce que le choix de la séquence de coins intérieurs a une influence sur le tableau normal obtenu ?

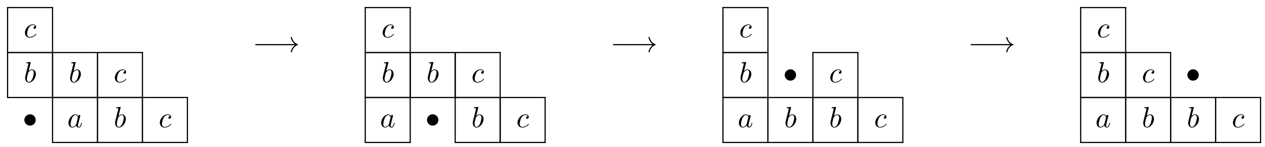

FIGURE 1.14 : Un exemple de glissement avec tous les mouvements extérieurs intermédiaires.

Pour y répondre, définissons la relation d'équivalence que deux tableaux (gauche, troué ou non) T et S sont jeu-de-taquin-équivalents, noté $T \sim_{jdt} S$, s'il existe une série de mouvements extérieurs et intérieurs transformant T en S.

Lemme 1.18 ((Schützenberger) [73] Théorème 2.4) *Soit T et S deux tableaux. Si $T \sim_{jdt} S$ alors $r(T) \equiv_{\text{Plax}} r(S)$.*

Pour cela, il suffit de se restreindre à un simple mouvement. Quand le mouvement est horizontal, il est évident que $r(T) = r(S)$. Quand le mouvement est vertical, on peut se restreindre à des tableaux T, S à deux lignes. Il suffit alors de s'appercevoir que $\mathrm{P}(r(T)) = \mathrm{P}(r(S))$ ce qui implique que $r(T) \equiv_{\text{Plax}} r(S)$.

Le résultat suivant répond à la question de confluence.

Proposition 1.19 ((Thomas) [80] Theorem 3) *Chaque classe d'équivalence du jeu-de-taquin contient un unique tableau semi-standard.*

Ainsi, pour T, S deux tableaux semi-standards, nous pouvons faire le produit $T \cdot S$ simplement en collant le coin sud-est de T avec le coin nord-ouest de S et procéder à la rectification du tableau ainsi construit.

De plus, l'algorithme de Schensted peut être représenté à l'aide de cette procédure. Soit x une lettre et T un tableau semi-standard. Pour procéder à l'insertion à droite de x dans T, il suffit de placer x dans une boite directement au sud-est de T et à procéder à la rectification du tableau ainsi obtenu. Pour l'insertion à gauche, il suffit de placer x directement au nord-ouest de T et à procédé à la rectification de ce tableau. L'insertion de la figure 1.6 calculée à l'aide de cette méthode se trouve à la figure 1.15.

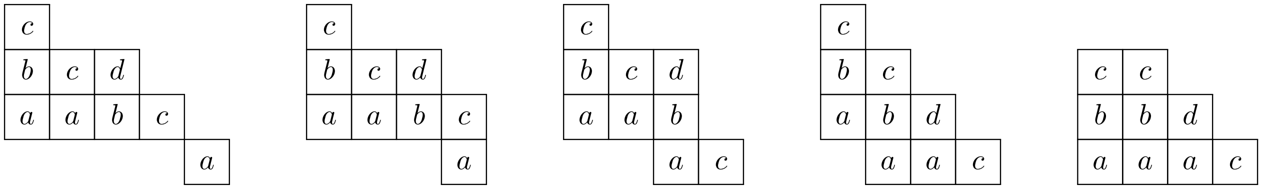

FIGURE 1.15 : L'insertion de la figure 1.6 calculée à l'aide du jeu-de-taquin.

1.5.2 Évacuation

Considérons θ l'unique permutation de $\mathcal{A}$ qui renverse l'ordre de l'alphabet $\mathcal{A}$; c'est une involution. Elle s'étend de manière unique en un anti-automorphisme sur le monoïde libre $\mathcal{A}^*$ que nous dénoterons aussi par θ. Cette anti-automorphisme renverse donc l'ordre de l'alphabet et l'ordre de lecture des mots ; $\theta(aacbdbadb) = cadcacbdd$ dans $\mathcal{A} = \{a < b < c < d\}$. Nous aimons bien l'appeler la double inversion. L'anti-automorphisme θ est clairement une involution.

Attention, θ dépend de l'alphabet sur lequel il est défini. Par exemple :

$$\theta_{\{a,b\}}(ab) = ba, \qquad \theta_{\{a,b,c\}}(ab) = cb.$$

On peut facilement remarquer que θ laisse invariant les relations de Knuth. En effet, pour $a < b < c$, $\theta(bca) = \theta(a)\theta(c)\theta(b)$ avec $\theta(a) > \theta(b) > \theta(c)$. Ainsi, θ induit un anti-automorphisme involutif sur le monoïde plaxique.

Grâce aux glissements, il est possible de décrire cette involution directement sur les tableaux grâce à une procédure, due à Schützenberger, appelée l'évacuation, notée $evac(T)$. Soit T un tableau semi-standard. L'évacuation de T est le tableau T' ayant la même forme que T étiqueté de la manière suivante. On retire récursivement la lettre de la boite la plus au sud-ouest de T puis on procède au glissement du coin intérieur ainsi créé. On appose l'étiquette $\theta(x)$ dans la boite de T' correspondant au coin extérieur créé par ce glissement.

Théorème 1.20 ((Schützenberger) [74]) *Soit un mot $w \in \mathcal{A}^*$, alors* $\mathrm{P}(\theta(w)) = evac(\mathrm{P}(w))$.

Un corollaire direct et bien connu de ce résultat est le suivant.

Corollaire 1.21 *L'évacuation est une involution.*

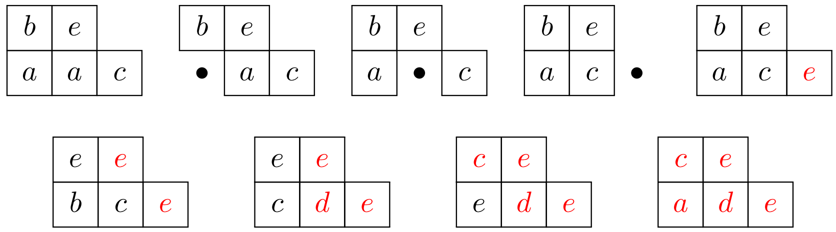

FIGURE 1.16 : L'évacuation du tableau semi-standard de la figure 1.2 sur l'alphabet $\mathcal{A} = \{a < b < c < d < e\}$. Les pas de la première étape sont indiqués dans la première ligne. Remarquons que $\theta(beaac) = ceead \equiv_{\text{Plax}} ceade$.

1.6 Diagramme de Croissance

Fomin introduit dans [36] une façon originale de décrire à la fois l'algorithme de Schensted et celui de Schützenberger de manière unifiée en introduisant les *diagrammes de croissance*[1]. Ils permettent des preuves plus directes de certaines propriétés de RSK et de l'évacuation.

Un diagramme de croissance est un quadrillage carré avec un partage associé à chaque croisement. Les partages formant un carré doivent respecter des propriétés particulières qui seront décrites plus bas.

Dans [36], Fomin a décrit la notion de diagramme de croissance dans un contexte plus large, les *ordres 1-différentiels* ; le treillis de Young en fait partie. Stanley a même conjecturé dans [76] qu'il est le plus petit ordre 1-différentiel si l'on regarde le nombre d'éléments se trouvant sur chaque rang[2]. Dans ce même article, Stanley conjectura que le plus grand ordre 1-différentiel est le treilli de Young-Fibonacci ; ce que prouva Byrnes dans sa thèse en 2012 ([25] Theorem 1.2)[3].

1.6.1 Robinson-Schensted

Pour l'algorithme de Robinson-Schensted, prenons une permutation σ. Soit M_σ la matrice de σ pour laquelle l'ordre des lignes est inversé ; (1,1) est en bas à gauche au lieu d'être en haut à gauche. Pour éviter toute

[1] Nous suivons la présentation de ces objets faite dans [77]. À la section 7.13 Stanley présente les diagrammes de croissance pour l'algorithme de Robinson-Schensted et à l'annexe A, Fomin présente les diagrammes de croissance pour le jeu-de-taquin et l'évacuation.

[2] Les ordres différentiels sont toujours gradués.

[3] Byrnes prouva aussi dans sa thèse que les deux seuls *treillis* 1-différentiels sont les treillis de Young et Young-Fibonacci.

confusion avec les 1 des partages, les entrées non-nulles sont des $\times$. Par exemple la matrice de 43512 est

		$\times$		
$\times$				
	$\times$			
				$\times$
			$\times$	

(1.9)

Cette matrice est le quadrillage du diagramme de croissance de σ. Les partages sur les bords ouest et bas de la matrice sont des partages vides. Pour un carré C

$$\begin{array}{ccc} \mu & \text{---} & \rho \\ | & & | \\ \lambda & \text{---} & \nu \end{array}$$

les partages doivent respecter les règles suivantes :

- si C ne contient pas de $\times$ et $\lambda = \mu = \nu$ alors $\rho = \mu$,
- si C ne contient pas de $\times$ et $\lambda \subset \mu = \nu$ alors μ est obtenu de λ en rajoutant 1 à la part λ_i et ρ est obtenu de μ en ajoutant 1 à la part μ_{i+1},
- si C ne contient pas de $\times$ et $\mu \neq \nu$, alors $\rho_i = \max(\mu_i, \nu_i)$,
- si C contient un $\times$, ceci nous indique que $\lambda = \mu = \nu$. Alors ρ est obtenu de μ en rajoutant 1 à la part μ_1.

Pour un exemple de diagramme de croissance, celui pour la permutation 43512 se trouve à la figure 1.17.

Le parcours de gauche à droite du bord nord du diagramme correspond à un chemin dans le diagramme de Hasse du treillis de Young partant du partage vide et allant vers un certains partage λ. De même, le parcours de bas en haut du côté est du diagramme, nous donne aussi un chemin entre le partage vide et le partage λ. Ainsi, pour toute permutation σ, nous obtenons une paire de tableaux $(\mathrm{P}_\sigma, \mathrm{Q}_\sigma)$ qui sont les tableaux représentés respectivement par les bords nord et est du diagramme de croissance de σ.

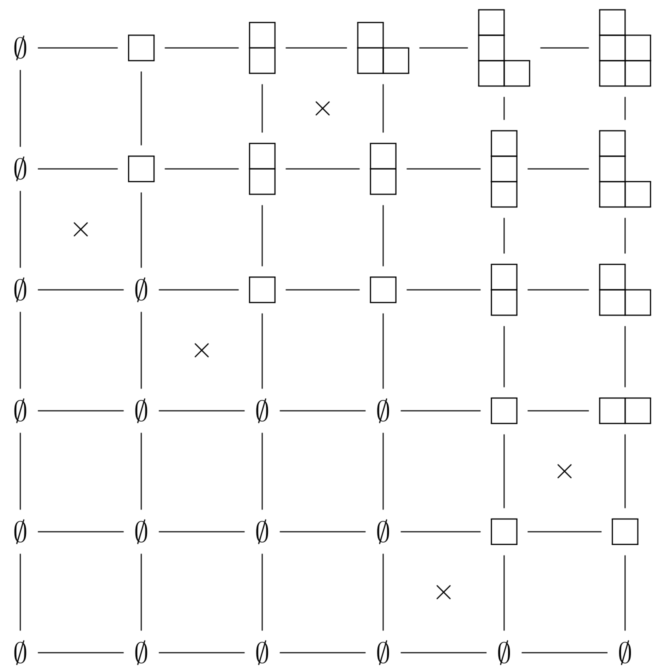

FIGURE 1.17 : Le diagramme de croissance de la permutation 43512.

Nous avons alors

Théorème 1.22 ((Fomin) [36]) *Pour une permutation σ, $\mathrm{P}_\sigma = \mathrm{P}(\sigma)$ et $\mathrm{Q}_\sigma = \mathrm{Q}(\sigma)$.*

Par exemple, pour la permutation 43512, en regardant le diagramme de croissance de la figure 1.17, nous obtenons bien

$$\left(\begin{array}{|c|c|} \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \hline 2 & 5 \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array}. \right)$$

En remarquant que $M_{\sigma^{-1}}$ est tout simplement la symétrie le long de l'axe $x = y$ de la matrice M_σ, nous obtenons une preuve du résultat suivant qui fut prouvé par Schützenberger dans [74].

Corollaire 1.23 ((Schützenberger) [74], (Fomin) [36]) *Soit σ une permutation. Alors $\mathrm{P}(\sigma^{-1}) = \mathrm{Q}(\sigma)$ et $\mathrm{Q}(\sigma^{-1}) = \mathrm{P}(\sigma)$. De plus, σ est une involution si et seulement si $\mathrm{P}(\sigma) = \mathrm{Q}(\sigma)$.*

1.6.2 Jeu-de-taquin

Similairement, nous pouvons calculer la rectification d'un tableau gauche standard T de forme λ/μ à l'aide d'un diagramme de croissance.

Il faut d'abord choisir la séquence de retrait des boites de μ. Cette séquence est associée à un remplissage standard de μ en étiquetant la i-ème case retirée par $|\mu| - i + 1$. Ainsi, on associe à un chemin c_1 dans le treillis de Young qui part de l'élément minimal.

De plus, comme T est standard, il est associé à un chemin c_2 de μ à λ.

Le diagramme de croissance est un rectangle de hauteur $|\mu|$ et de largeur $|\lambda|$. Le côté ouest est étiqueté par c_1 et le nord est étiqueté par c_2.

Pour remplir le centre, nous suivons les règles suivantes :

pour un carré

$$\begin{array}{ccc} \mu & — & \rho \\ | & & | \\ \lambda & — & \nu \end{array} \tag{1.10}$$

(i) si μ est le seul partage respectant $\lambda < \mu < \rho$ alors $\nu = \mu$,

(ii) sinon il en existe un unique autre ν.

Théorème 1.24 ((Fomin) [36]) *Soit T un tableau standard gauche de forme λ/μ et S un tableau standard de forme μ. Alors le côté sud du diagramme de croissance obtenu en plaçant S sur le côté ouest et T sur le côté nord est la rectification de T. Plus encore, les lignes sont les étapes de rectification en suivant la séquence de rectification indiquée par S.*

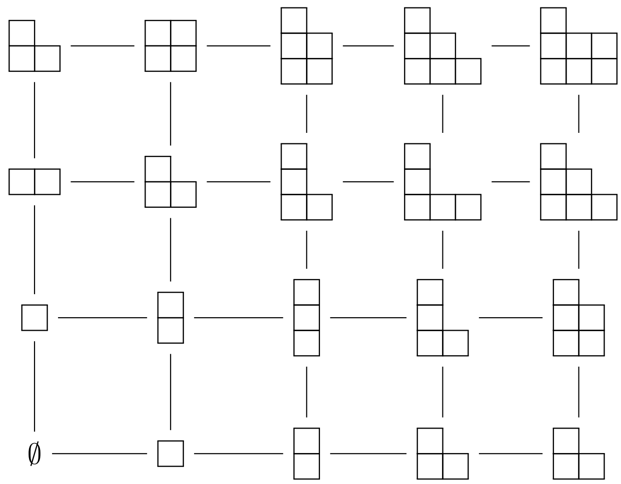

FIGURE 1.18 : Le diagramme de croissance pour la rectification du tableau standard de la figure 1.4.

1.6.3 Évacuation

Nous pouvons aussi décrire l'évacuation d'un tableau à l'aide des diagrammes de croissance. Comme précédemment, soit T un tableau standard et c_1 son chemin associé dans le treillis de Young.

Pour construire le diagramme de croissance d'évacuation de T, prenons un quadrillage carré de taille $|T|$. Sur tous les croisements qui se trouve sur la diagonale partant du nord-ouest, nous plaçons des partages vides. La partie sud-ouest du diagramme n'est pas nécessaire. Le côté nord est étiqueté par c_1.

Tout comme pour la rectification, pour remplir le reste du diagramme il suffit de suivre les règles 1.10 (i) et (ii).

Théorème 1.25 ((Fomin) [36]) *Soit T un tableau standard et D son diagramme de croissance d'évacuation. Alors le côté droit de D est le chemin associé à* $\mathrm{evac}(T)$ *dans le treillis de Young.*

Pour prouver cela, il suffit de prouver que la $(|\lambda_T| - i + 1)$-ème ligne représente le tableau restant après la i-ème étape de l'évacuation.

Avec cette description, il est évident que l'évacuation est bien une involution, comme mentionné au corollaire 1.21, car calculer l'évacuation de $\mathrm{evac}(T)$ revient à faire la symétrie de D le long de l'axe $x = y$. Ceci est dû aux règles de construction.

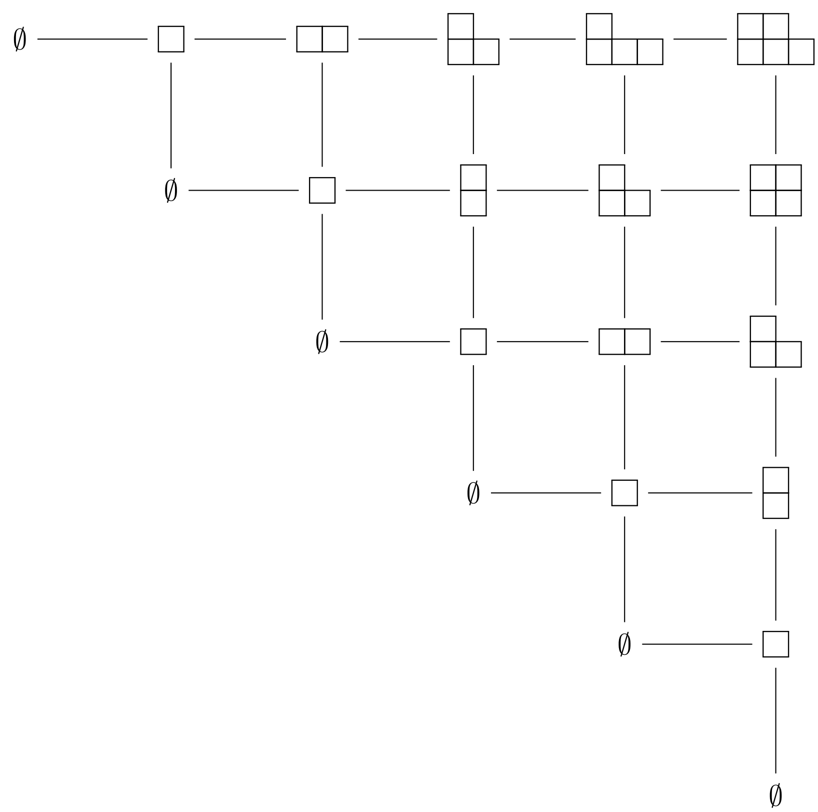

FIGURE 1.19 : Le diagramme de croissance d'évacuation du tableau $\mathrm{P}(35\,124)$.

CHAPITRE 2
SUR UN LEMME DE SCHENSTED

2.1 Avant-Propos et Résumé

Le lemme de Schensted est un résultat de commutation entre deux opérations, l'insertion par ligne et l'insertion par colonne dans un même tableau. Ce Lemme est énoncé par Schensted dans [72] mais sa preuve est incomplète. De même, la preuve dans le livre de Sagan [69] est incomplète ; nous avons d'ailleurs échangé des courriels avec lui à ce sujet. Ce lemme peut être prouvé comme conséquence d'autres résultats de la théorie, voir par exemple [77, Corollaire A.1.2.11]. Il nous a paru intéressant de trouver une preuve complète et directe de ce résultat de commutation. La preuve est un peu technique mais elle met en évidence une propriété géométrique des chemins gauche et droit d'insertion : ils ont au plus une intersection.

À la demande des réviseurs, quelques ajouts on été fait à l'article original pour faciliter la compréhension.

Cet article de 7 pages est publié dans la revue Enumerative Combinatorics and Applications (2023) [5].

2.2 Abstract

We give a direct proof of Schensted's lemma asserting that row and column insertion in a tableau commute.

2.3 Introduction

The purpose of this Note is to give a complete direct proof of Schensted's Lemma 6 in [72]. This result asserts that row insertion and column insertion commute. This nice commutation result, besides its own interest, has many applications. One of them is that reversing the permutation amounts to transpose its P-tableau (Theorem 3.2 in [69], a result which is implicit in Schensted's article). Moreover, many proofs in the theory of the Schensted algorithm (also called Robinson-Schensted, or Robinson-Schensted-Knuth, RSK) use this commutation result.

Schensted gives a direct proof of his lemma; however many cases are not considered and not even listed, so that the proof is incomplete; similarly in [69] (Proposition 3.2.2)[1]. Schensted's lemma has indirect proofs,

[1] The present Note may be seen as an addendum to the very nice book of Bruce Sagan [69].

see for example [77] (Lemma 7.23.14 in Stanley's book, and Corollary A1.2.11 in the Appendix by Fomin), or [55] Theorem 4.1.1. It follows also from the results of Schützenberger relating Schensted's algorithm and jeu de taquin [73]. Moreover, the theory of the plactic monoid also implies Schensted's lemma, since column and row insertions correspond to left and right products in this monoid (see the chapter by Lascoux, Leclerc and Thibon on the plactic monoid in the book by Lothaire [52]).

We sketch now our direct proof of Schensted's lemma. Note that we do not argue by induction using the maximum element of the tableau, as is done by Schensted and Sagan.

The ideas of the proof are as follows. We carefully define the *trail* of an insertion, which is a sequence of boxes with their labels; this is called elsewhere the "insertion path". We consider the trail as embedded in the tableau *before* its modification by the insertion. Note that the modification of the tableau by the insertion is obtained by sliding the labels along the trail.

Now, we have two insertions: a column insertion, and a row insertion; hence two trails, a row trail and a column trail. We study how these two trails intersect. If they do not intersect, the commutation is easy to see.

Suppose now that the two trails intersect: we show that they have exactly one box in common, and do not intersect elsewhere, see Figure 2.2. The tableau obtained by the two insertions, in either order, is obtained as follows. First, observe that if one wants to slide the labels along the two trails (row and column), then at the neighbourhood of the intersection, one has conflicts between boxes. Precisely, the intersection box could get two labels, and there is only one label for the two boxes (in the two trails) after the intersection box. We prove that a simple rule, depending on two cases, solves the conflict (see Figure 2.7, where s is the label of the intersection, and where the labels of the column trail, resp. of the row trail, are $\ldots, a, s, b, \ldots$, resp. $\ldots, i, s, j, \ldots$), and that for the other boxes, one performs the ordinary sliding of the two trails.

2.4 Schensted insertions

Recall that a *tableau* is a finite lower order ideal of $\mathbb{N} \times \mathbb{N}$, where the latter is partially ordered by the usual componentwise order ($(a, b) \leq (u, v)$ if and only if $a \leq u$ and $b \leq v$), together with an increasing injective mapping from this subset into a totally ordered set (usually the latter set is $\mathbb{N}$).

Recall that the *row insertion* of an element x into a row L (with $x \notin L$), is the row obtained by adding x at the end of L if $x > \max(L)$ or if the row is empty, and if this condition is not satisfied, then it is $x \cup L \setminus y$, where y is the smallest upper bound of x in L; in the latter case, y is said to be *bumped*. The new row is denoted by $L \leftarrow x$.

The row insertion of x into a tableau T, denoted by $T \leftarrow x$, is obtained by inserting x into the first row of T, and then inserting the bumped element into the second row, and so on; the process stops when there is no element bumped.

Columns insertion is defined symmetrically, by replacing rows by columns, and the resulting tableau is denoted by $x \rightarrow T$.

For later use, we state without proof the following easy lemma.

Lemma 2.1 *Let L be a row, $x \notin L$, and let y be bumped by x in the row insertion $L \leftarrow x$. Define a row L' by modifying the elements of L, except y, so that no element at the left of y increases; then in the insertion $L' \leftarrow x$, y is also bumped.*

2.5 Schensted's lemma

Below is the statement of the result for which we give a complete direct proof.

Theorem 2.2 *(Schensted [72] Lemma 6) Suppose that T is a tableau, and x, y distinct elements not in T. Then*

$$(x \rightarrow T) \leftarrow y = x \rightarrow (T \leftarrow y).$$

The proof is in Section 2.9; we need some prior results.

2.6 A trail

Given a tableau T and a row insertion $T \leftarrow x$, with x not in T, we call *trail* of this insertion the *sequence* of boxes of T, which are activated in this insertion, with their labels in T, followed by the newly created empty box. Formally, if x is inserted at the end of the first row of T, then the trail is the newly created box, without label. Otherwise, let y be the element bumped from the first row of T, and let T' be the tableau obtained by removing the first row of T; then the trail of $T \leftarrow x$ is the box containing y with its label y, followed by the trail of $T' \leftarrow y$. We call *empty box of the trail* its last box (which is unlabelled).

The following properties follow easily from the definition of row insertion:

- The consecutive boxes on the trail are on consecutive rows, beginning by the first row of T, and they lie on columns that are weakly decreasing; in other words, the trail goes weakly to the north-west[2]; it may go north, but not west.
- If u, v are consecutive labels of the trail, then when u is row-inserted in the row containing v, v is bumped, and this defines v uniquely knowing u and the row of v.
- The labels on a trail are strictly increasing (*trail inequality*).
- The tableau $T \leftarrow x$ is obtained by sliding each label on the trail to the next box in the trail, and by filling the first box by x.

For a column insertion $x \rightarrow T$, the trail is defined symmetrically, and one has the similar properties:

- The consecutive boxes on the trail are on consecutive columns, beginning by the first column of T, and they lie on rows that are weakly decreasing; in other words, the trail goes weakly to the south-east; it may go east, but not south.
- If u, v are consecutive labels of the trail, then when u is column-inserted in the column containing v, v is bumped, and this defines v uniquely knowing u and the column of v;
- The labels on a trail are strictly increasing (*trail inequality*).
- The tableau $x \rightarrow T$ is obtained by sliding each label on the trail to the next box in the trail, and by filling the first box by x.

[2] We take the French representation of tableaux; for the English one, one has to interchange everywhere north and south, and look at the figures upside down.

In Figure 2.8, the boxes with a line below the labels form a trail for a row insertion and the boxes with a line on the left of the labels form a trail for a column insertion.

2.7 Two trails

We consider now a column insertion $x \to T$ and a row insertion $T \leftarrow y$ of two distinct elements in the same tableau, with $x, y \notin T$. We therefore have two trails, called here column trail and row trail. We say that they *strongly intersect* if they have a box in common.

Lemma 2.3 *The column and row trails have at most one box in common.*

Proof. Recall that each trail has an empty box, which is its last box. Suppose that the trails intersect in their last box. Since one trail goes north-west and the other south-east, this box must be the only intersecting box.

Note that if a box is in the intersection, then it cannot be the empty box for one trail, and nonempty for the other. Indeed, each nonempty box of a trail is in T, while an empty one is not in T.

Suppose now that the two trails have two intersecting boxes, which by what precedes, are not empty. Since the labels strictly increase along a trail, and since one goes north-west and the other south-east, we obtain a strictly increasing cycle of numbers; this is impossible. □

In the row trail and in the column trail, we draw a straight segment between the center of any two consecutive boxes; in this way we obtain a broken line in the plane, called the *geometric trail*; it includes as vertices the center of the boxes of the trail. In Figure 2.1, the lines between a and b and, i and j are parts the geometric trails of, respectively, a column insertion and a line insertion.

Observe that if C is any center of a box in the tableau, which lies on the geometric trail, then it is a vertex of the latter (since consecutive boxes of the row trail lie on consecutive rows of the tableau, and similarly for the column trail).

We call an *intersection* of the two trails an intersection of the geometric trails. We say that an intersection is *weak* if it is not strong; that is, in view of what has been just said, if it is not the center of any box in T.

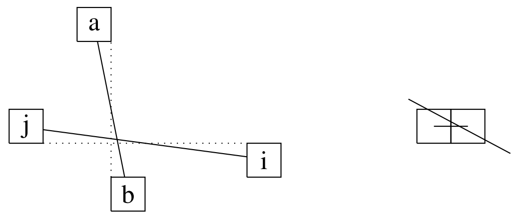

Figure 2.1: Weak intersection

Lemma 2.4 *The trails have no weak intersection.*

Proof. Suppose by contradiction that the geometric trails have a weak intersection M. Then M is not the center of some box in T, by the observation before the statement.

Consider the vertices of the two trails closest to M: A, B in this order for the column trail, and I, J for the row trail. Then A, B are on two consecutive columns and I, J on two consecutive rows.

Suppose by contradiction that A, B are on the same row. Then M is on the open segment (A, B). Thus we have Figure 2.1 (right part) where the oblique segment is part of the geometric row trail. On this row trail there must be a box on the same horizontal line as the boxes of A, B, and its central point must be on the oblique line; but this is impossible.

Hence A, B are not on the same row, and symmetrically, I, J are not on the same column. Thus we see that we must have Figure 2.1 (left) where a, b, i, j are the labels of A, B, I, J. Indeed, the boxes must be nonempty: this is clear for the boxes with i, a, since they are not the last boxes on their trail; and if, for example, the box with b were empty, then it would contradict the tableau property, since we would have this box empty in T, but the box with i is in T.

Now, since the values increase along a trail, we have $i < j, a < b$. By the tableau inequalities, we have $j < a, b < i$. We obtain $a < b < i < j < a$, a contradiction. $\square$

We assume now that the two trails have a strong intersection, which is a box S, which we assume nonempty.

Lemma 2.5 *The part of the row trail (resp. column trail) which is before the intersection is strictly under the part of the column-trail (resp. row-trail) which is after it. See Figure 2.2.*

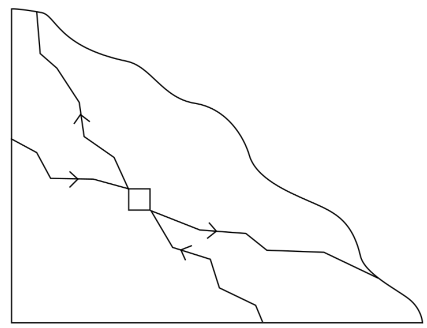

Figure 2.2: Relative position of strongly intersecting row and column trails

Proof. Consider the part of the row trail which is before S, and the part of the column trail which is after it. These two parts have no intersection, by Lemmas 2.3 and 2.4; moreover, the row trail starts from the first row, and the column trail ends at the end of some row. This implies the first statement. The other one is symmetric. □

We denote by s the value in the box S in T and let ℓ be the row number of S. We denote by a, s, b the consecutive values in the trail $x \to T$, and by i, s, j the consecutive values in the trail $T \leftarrow y$. Recall that a, s, b (resp. i, s, j) lie in consecutive colomns (resp. rows). There is a slight abuse of notation, in the sense that the box after S could be the empty box in either trail, in which case we write b or $j = \emptyset$. We also consider the cases where i, resp. a, does not exist, meaning that s is the first label of the trail of $T \leftarrow y$, resp. $x \to T$; we then put $i = x$, resp. $a = y$.

Lemma 2.6 *The configurations in Figure 2.3 are impossible.*

Note that the figure means that a is in the column at the left of s, and higher than s, and so on.

Proof. The cases for $j = \emptyset$ at the left, and $b = \emptyset$ at the right, follow from the properties of Ferrers diagrams Suppose now that $j, b \neq \emptyset$. Suppose that we have the leftmost configuration. Then $j \neq a$ by Lemma 2.3; thus $j < a$ by the tableau inequalities; moreover $a < s$ and $s < j$ by the trail inequalities: a contradiction. The rightmost configuration is treated similarly.

For the central one , we have $i > a$, since by row insertion, i bumps s. Symmetrically, $a > i$: a contradiction. □

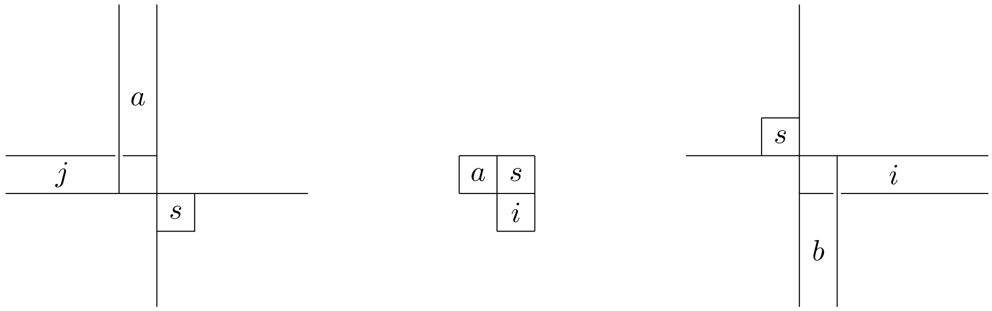

Figure 2.3: Impossible configurations (case b or $j = \emptyset$ included)

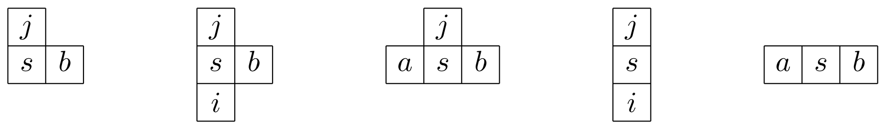

Figure 2.4: Possible configurations

Corollary 2.7 *One has one of the five configurations of Figure 2.4, which indicate the boxes in the trails which have a side in common with S.*

Proof. Excluding the cases listed in Lemma 2.6 (for example, the left part indicates that A or J must have a side in common with S), one verifies that the only possible cases are the five in Figure 2.4. $\square$

2.8 Three trails

We consider the trail of the row insertion $T \leftarrow y$ and the trail of the column insertion $x \rightarrow T$. We assume that the two trails have a strong intersection, which a unique box S labelled s. The third trail that we study in this section is that of the row insertion $(x \rightarrow T) \leftarrow y$.

The notations a, b, i, j, s are the same as before Lemma 2.6, and we denote by A, B, S, I, J the boxes that contain these labels. Say that S is located in row ℓ.

We compare below the two trails obtained by the two row insertions $T \leftarrow y$ and $(x \rightarrow T) \leftarrow y$ (row insertion of y in both cases) and begin the task of showing that these two trails are almost equal. For the rest of the section, when we say *the two trails*, it will always be these two row-trails.

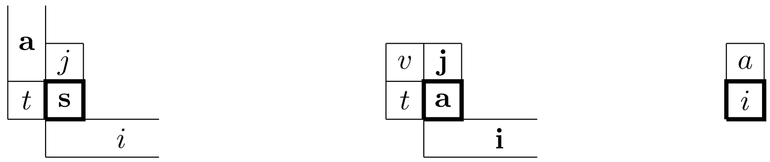

Figure 2.5: Case $i < a$, tableaux T, $x \to T$, $(x \to T) \leftarrow y$

Lemma 2.8 *(i) The part of the two trails in rows* $1, \ldots, \ell - 1$ *are equal.*

(ii) If the first box in row $\ell + 1$ *is the same in the two trails, with the same label, then the parts of the two trails above row* ℓ *are equal.*

Note that it will be shown further that the extra hypothesis (that they should have the same label) in (2) always holds.

Proof. (i). Consider the rows 1 to ℓ of T. When we apply the insertion $x \to T$, which amounts to slide the elements of T in the trail of $x \to T$, we do not modify the part of these rows that are weakly at the left of the trail of $T \leftarrow y$: this follows from Lemma 2.5, see Figure 2.2. Thus (i) then follows from Lemma 2.1.

(ii). Consider the rows $\ell, \ell + 1, \ldots$ of T. When we apply the insertion $x \to T$, which amounts to slide the elements of T in the trail of $x \to T$, we modify the part of these rows which lie strictly at the left of the trail $T \leftarrow y$ only by decreasing elements, or leaving them equal: this follows from Lemma 2.5, see Figure 2.2. Thus (ii) follows from the extra hypothesis and Lemma 2.1. □

Proposition 2.9 *The labels of* S, B, J *in* $(x \to T) \leftarrow y$ *are respectively* i, s, a *if* $i < a$*, and* a, i, s *if* $i > a$*. Moreover, the other labels of this tableau are obtained by sliding the labels of the trails of* $x \to T$ *and* $T \leftarrow y$*, except* i, a, s*, towards the next box of the trail.*

Note that the slidings are unambiguous.

Proof. (1). We assume that i, a exist. Suppose that $i < a$. In T, let t be the label at the left of s; since in $T \leftarrow y$, i bumps s, we have $i > t$, and therefore $t \neq a$; thus a is not at the left of s and it follows by

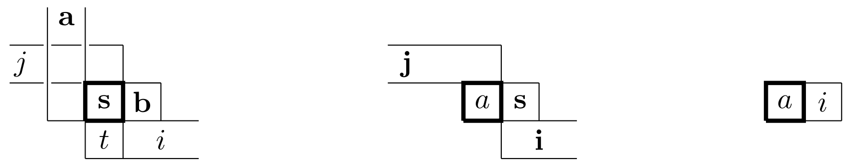

Figure 2.6: Case $i > a$, tableaux T, $x \to T$, $(x \to T) \leftarrow y$

Figure 2.4 that j is above s; moreover a is in the column at the left of s, and higher than s: see Figure 2.5 left part, where the trail of the column insertion of x into T is represented in boldface.

Applying the column insertion of x into T, which amounts to slide the trail $\ldots, a, s, b, \ldots$, we obtain $x \to T$ in the central part of Figure 2.5. Note that s is now in box B. Since $i < a$, and $i > t$ as we just saw, the trail of $(x \to T) \leftarrow y$, which contains i by Lemma 2.8 (i), also contains a. Now, observe that the element v at the left of j in $x \to T$ is $< a$: indeed, either the element u of T in this box is not displaced in the column insertion of x into T, so that $u = v$ and u is under a in the same column of T; or u is displaced and then $u = a$, and the element v replacing u is smaller than a. Hence $a < j$ and $a > v$, and the trail of $(x \to T) \leftarrow y$ contains j.

Applying the row insertion of y into $x \to T$, we obtain the right part of Figure 2.5. Note that s is still in box B, since the trail of $(x \to T) \leftarrow y$ is strictly under B, by Lemmas 2.5 and 2.8 (i). The other assertions follow from Lemma 2.8.

We assume now that $i > a$. Let t be the label right under s in T; since in the insertion $x \to T$, a bumps s, we must have $a > t$, hence $i \neq t$. Thus i is in the row under s, stricly at the right of s, and by Figure 2.4, b is at the right of s; see Figure 2.6 left part; we indicate there that a is in the column at the left of s and in some row weakly above s; similarly, j is in the row above s and in some column weakly at the left of s.

Applying the column insertion x into T, which amounts to slide its trail $\ldots a, s, b, \ldots$, we obtain $x \to T$ in the center of Figure 2.6. Note that j is in box J, both in T and in $x \to T$, by Lemma 2.5. The trail of $(x \to T) \leftarrow y$ contains i by Lemma 2.8 (i), hence also s, since $i > a$. Now, if we denote by v the element at the left of j, then $v < s$: indeed, in the insertion $T \leftarrow y$, s bumps j, hence $s > u$ where u is the element at the left of j in T; and $v \leq u$, since the insertion $x \to T$ does not increase elements. Thus $v < s < j$, hence the trail we consider contains j.

The trail is represented in boldface in the center of Figure 2.6, and we thus obtain $(x \to T) \leftarrow y$ on the right part of Figure 2.6. The other assertions follow from Lemma 2.8.

(2). The cases where a or i does not exists is treated quite similarly, with essentially the same arguments. Observe that if, for example, i does not exist, then s must be in the first row and b at its right. We omit the details of these special cases. □

2.9 Proof of Schensted's lemma

Consider the two trails of the insertions $x \to T$ and $T \leftarrow y$.

If these two trails do not intersect, then the column trail is strictly at the north-west of the row trail, and there is no interference between them: the tableaux $x \to (T \leftarrow y)$ and $(x \to T) \leftarrow y$ are both equal to the tableau obtained from T by sliding the two trails of the insertions $x \to T$ and $T \leftarrow y$.

Suppose now that these two trails have an intersection, hence a strong intersection, which is a unique box (Lemmas 2.4 and 2.3).

Suppose first that this box is the empty box of both trails, denoted by S. Denoting by a and i the last label in the column and row trail respectively, it is easy to see that the two tableaux $x \to (T \leftarrow y)$ and $(x \to T) \leftarrow y$ are both obtained by sliding the labels in the two trails, except a and i which are added as follows: if $i < a$, then i is put in box S and a in the box above; if $i > a$, a is put in box S, and i in the box at the right.

Thus we may assume that the intersecting box is unique and not the empty box of either trail. Then we conclude using Proposition 2.9 and using the symmetry row/column.

2.10 Conclusion

One may state what happens as follows: one has the two trails of $x \to T$ and $T \leftarrow y$. Then $x \to T \leftarrow y$ is obtained by sliding the two trails, except at the boxes where there is a conflict; this is the case for i and a, which both want to slide into S, and for s, which has the choice of sliding into B or J; the conflict is solved, depending on the inequality between a and i, by the first sentence in Proposition 2.9. The two cases are represented in Figure 2.7.

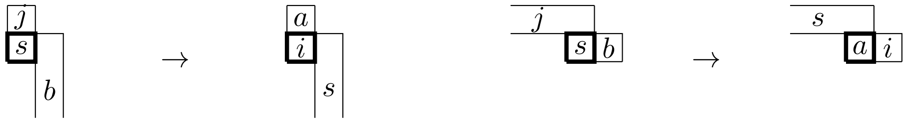

Figure 2.7: the two cases $i < a$ and $i > a$ before and after the row and column insertions

An example is given in the Figure 2.8, where are represented a tableau T, the two trails of $7 \to T$ and $T \leftarrow 8$, and the final tableau $7 \to T \leftarrow 8$; here $i = 10, a = 11, s = 13, j = 18, b = 14$.

$\underline{\emptyset}$					
17	$\underline{19}$				
$\vert 11$	$\underline{18}$				
4	$\vert\underline{13}$	$\vert 14$			
2	6	$\underline{10}$	$\vert 15$	$\vert\emptyset$	
1	3	5	$\underline{9}$	12	16

Figure 2.8: Trails of the insertions $7 \to T$ and $T \leftarrow 8$; marked by lines on the left and below the labels of the boxes respectively.

19					
17	18				
7	11				
4	10	13			
2	6	9	14	15	
1	3	5	8	12	16

Figure 2.9: The tableau obtained by doing both insertions, $7 \to T \leftarrow 8$.

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CHAPITRE 3
UNE PREUVE D'UN RÉSULTAT DE LASCOUX ET SCHÜTZENBERGER

3.1 Avant-Propos et Résumé

Le monoïde syntaxique est un outil fondamental dans la théorie des automates : à tout langage[1] on peut associer son monoïde syntaxique. Ceci relie la théorie des automates finis à la théorie des monoïdes [67].

Plus généralement, on peut associer un monoïde syntaxique à toute fonction sur le monoïde libre. Dans [54, Théorème 2.15], Lascoux et Schützenberger affirment que le monoïde plaxique est le monoïde syntaxique de la fonction qui a tout mot associe la forme de son P-tableau. Cependant, ils énoncent ce théorème sans preuve. Dans ce chapitre, nous en donnons une. Il est l'appendice à l'article the Stylic Monoid [6]. Comme le sujet sort un peu du cadre de cet article, nous en avons séparé le contenu.

À la demande des réviseurs, quelques ajouts on été fait à la version originale de cette annexe pour faciliter la compréhension.

3.2 Définitions

Soit $f : \mathcal{A}^* \to E$, où E un ensemble quelconque. La *congruence syntaxique de* f, notée $\equiv_f$, est définie par

$$U \equiv_f V \Leftrightarrow (\forall X, Y \in \mathcal{A}^*, f(XUY) = f(XVY)).$$

C'est une congruence (bilatère) de $\mathcal{A}^*$; i.e. une relation d'équivalence qui est compatible avec le produit de $\mathcal{A}^*$. C'est la plus grossière congruence $\equiv$ de $\mathcal{A}^*$ qui est compatible avec f ; i.e. satisfaisant $U \equiv V \Rightarrow f(U) = f(V)$. Le *monoïde syntaxique de* f est le monoïde quotient $\mathbf{M}_f = \mathcal{A}^*/\equiv_f$.

Clairement, nous avons $U \equiv_f V \Rightarrow f(U) = f(V)$, faisant en sorte que f induit une fonction $g_f : \mathbf{M}_f \to E$ tel que $f = g_f \circ \mu$, où μ est l'homomorphisme de monoïde canonique $\mathcal{A}^* \to \mathbf{M}_f$.

Similairement, la *congruence syntaxique à gauche de* f, notée $\equiv_f^l$, est définie par

$$U \equiv_f^l V \Leftrightarrow (\forall X \in \mathcal{A}^*, f(XU) = f(XV)).$$

[1] Un langage est une partie d'un monoïde libre.

C'est une congruence à gauche de $\mathcal{A}^*$; i.e. elle est compatible avec la multiplication à gauche de $\mathcal{A}^*$. Nous obtenons ainsi une action à gauche de $\mathcal{A}^*$ sur l'ensemble $\mathcal{A}^*/\equiv_f^l$. La congruence syntaxique à gauche de f est la plus grossière congruence à gauche de $\mathcal{A}^*$ qui est compatible avec f.

Les deux quotients possèdent une propriété universelle respectant f, que nous décrivons seulement pour le monoïde syntaxique. Considérons la catégorie dans laquelle les objets sont les triplets $\mathbf{M}, \nu, g$, où $\mathbf{M}$ est un monoïde, $\nu : \mathcal{A}^* \to \mathbf{M}$ un homomorphisme surjectif de monoïde et $g : \mathbf{M} \to E$ une fonction telle que $f = g \circ \nu$; dans ce cas, on dit que $\mathbf{M}, \nu, g$ (ou simplement $\mathbf{M}$) *reconnaît* f. Les morphismes de cette catégorie sont définies comme des homomorphismes de monoïdes $\phi : \mathbf{M} \to \mathbf{M}'$ telles que $\nu' = \phi \circ \nu$ et $g' \circ \phi = g$. Le triplet $\mathbf{M}_f, \mu, g_f$ est un objet de cette catégorie et c'est en fait un objet terminal de la catégorie. On peut donc dire que "$\mathbf{M}_f$ est le plus petit monoïde reconnaissant f".

3.3 The result

In [54, Théorème 2.15 p.136], Lascoux and Schützenberger state that the plactic congruence is the syntactic monoid of the function λ which associates with each word W the shape ("forme immanente" in their article) of the tableau $\mathrm{P}(W)$. Equivalently: $U \equiv_{\mathrm{Plax}} V \Leftrightarrow (\forall X, Y \in \mathcal{A}^*, \lambda(XUY) = \lambda(XVY))$ (see Section 4.15 for the definitions about syntacticity). Their theorem is given without proof, and we provide a proof below, and a generalization.

Denote by Λ the set of all integer partitions. The *shape* of a tableau is the partition whose parts are the lengths of its rows.

Theorem 3.1 *The left syntactic congruence $\equiv_\lambda^l$ of the function $\lambda : \mathcal{A}^* \to \Lambda$, which associates with each word the shape $\lambda(W)$ of the tableau $\mathrm{P}(W)$, is the plactic congruence.*

The theorem hold also for the right syntactic congruence, as follows from the application of the anti-automorphism θ.

Corollary 3.2 *(Lascoux and Schützenberger) The plactic congruence is the syntactic congruence of the function λ.*

Proof. If $U \equiv_{\text{Plax}} V$, then $XUY \equiv_{\text{Plax}} XVY$, hence $\mathrm{P}(XUY) = \mathrm{P}(XVY)$, for any words X, Y. Thus $\lambda(XUY) = \lambda(XVY)$. Conversely, if $\forall X, Y \in \mathcal{A}^*, \lambda(XUY) = \lambda(XVY)$, then in particular $\forall X \in \mathcal{A}^*, \lambda(XU) = \lambda(XV)$; hence $U \equiv_\lambda^l V$ and therefore $U \equiv_{\text{Plax}} V$ by Theorem 3.1. □

Recall that each plactic equivalence class contains a unique representative which is a product of columns

$$W = \gamma_1 \cdots \gamma_N, \gamma_1 \leq \cdots \leq \gamma_N,$$

where $\leq$ is the order on columns of Section 4.12. This follows from Section 4.5, by considering the column-word of a tableau. We call the *column representative* this representative of the plactic class.

Lemma 3.3 *Let W be as above. Let $b \in \mathcal{A}$ and Y be the strictly decreasing word involving all letters $\geq b$ in $\mathcal{A}$. Let $n \leq N$. Write $\gamma_i = U_i V_i$, where U_i involves only letters $\geq b$, and V_i only letters $< b$. Then the column representative of the plactic class of $Y^n W$ is*

$$m = (YV_1) \cdots (YV_n) V'_{n+1} V'_{n+2} \cdots ,$$

where the V'_i are the (decreasing) reading words of the columns of the rest of the tableau after the insertion of Y^n in $\mathrm{P}(W)$.

Proof. Note the identity $Y^n \equiv_{\text{Plax}} \prod_{t \in \mathcal{A}, t \geq b} t^n$, where the product is strictly decreasing from the largest letter in $\mathcal{A}$ until b: this identity is true because the two sides are the column- and row-words of the rectangular tableau with n columns, all equal to Y. Note that the left product by Y^n (equivalent to Schensted left insertion of Y^n) does not change the letters $\leq b$ in the columns; moreover the product by b^n introduces a b in the n first columns, the product by c^n (with c the next letter in $\mathcal{A}$) introduces a c in them, and so on. Finally, these columns contain all the letters $\geq b$, which proves the lemma. □

Proof. [Proof of Theorem 3.1] If $W \equiv_{\text{Plax}} W'$, we have for any word X, $XW \equiv_{\text{Plax}} XW'$, and therefore $\mathrm{P}(XW) = \mathrm{P}(XW')$ and $\lambda(XW) = \lambda(XW')$.

Conversely, suppose that W, W' are not equivalent modulo the plactic relation. Then for some $n \geq 1$, for $i = 1, \ldots, n-1$, the i-th columns of $\mathrm{P}(W)$ and $\mathrm{P}(W')$ are equal, and their n-th columns differ. If their n-th columns have different heights, then $\lambda(W) \neq \lambda(W')$ and we choose $x = 1$. If their heights are equal, let

$a < b$ be the first letters distinguishing these columns, from left to right (columns being viewed as strictly decreasing words): a appears in the n-th column of W, and b in the n-th column of W', and the letters at the left of a (in W) and b (in W') in the two n-th columns are equal.

Then the plactic classes of W and W' have respectively columns representations of the form given in the displayed equation before the lemma (with primes for W'), and $\gamma_i = \gamma_i'$ for $i = 1, \ldots, n-1$. We may write $\gamma_i = U_i V_i = \gamma_i'$, $i = 1, \ldots, n-1$, where U_i involves only letters $\geq b$, and V_i only letters $< b$. Moreover, by what has been said above, $\gamma_n = U_n V_n = U_n a S_n$, $\gamma_n' = U_n b V_n'$, where U_n involves only letters $> b$ and V_n, V_n' only letters $< b$; moreover, $|V_n'| = |V_n| - 1$. Then by the lemma, the column representatives of $Y^n W$ and $Y^n W'$ are respectively

$$(YV_1) \cdots (YV_n) \cdots$$

and

$$(YV_1') \cdots (YV_n') \cdots$$

Then the n-th column of $\mathrm{P}(Y^n W)$ is longer than the n-th column of $\mathrm{P}(Y^n W')$. Thus $\lambda(Y^n W) \neq \lambda(Y^n W')$, and we take $X = Y^n$. $\square$

CHAPITRE 4
LE MONOÏDE STYLIQUE

4.1 Avant-Propos et Résumé

Ce travail est issu d'une conjecture sur le monoïde plaxique. La conjecture était que l'action à droite du monoïde plaxique sur les lignes, induite par l'insertion de Schensted à droite, est une action fidèle. Cette conjecture s'est avérée fausse, comme nous l'a indiqué Hugh Thomas il y a quelques années. Nous avons alors changé notre fusil d'épaule et avons considéré l'action à gauche sur les colonnes. Sur un alphabet fini, les colonnes, étant un nombre fini, l'action du monoïde plaxique, qui est infini, ne sera donc pas fidèle car cette action définira un monoïde fini. Toutefois, nous obtenons un monoïde quotient fini du monoïde plaxique. Nous l'avons appelé le monoïde stylique du mot grec "stylos", qui signifie colonne, par analogie avec le monoïde plaxique, ainsi appelé par Lascoux et Schützenberger qui ont utilisé le mot grec "plax", qui signifie plaque.

De manière équivalente, on peut voir le monoïde stylique comme le monoïde de transitions d'un automate dont les états sont les colonnes et dont les transitions sont déterminées par l'insertion à gauche de Schensted des lettres. Ce point de vue nous a permis de demander à Jean-Éric Pin d'utiliser pour nous son logiciel de calcul des automates pour des alphabets jusqu'à six lettres. Les états étant les colonnes, ils sont en bijection avec les parties de l'alphabet, il y en a ainsi $2^{|\mathcal{A}|}$, et le calcul explose assez vite. Ces expérimentations ont néanmoins mis en évidence que la cardinalité du monoïde stylique sur un alphabet à n lettres est le nombres de Bell (le nombre de partitions d'un ensemble à $n + 1$ éléments). Nous avons été amenés à introduire la notion de N-tableau, une variante des partitions, et une insertion, variante de celle de Schensted.

Il est apparu que le monoïde stylique a une présentation tout-à-fait remarquable qui est obtenue de la présentation de Knuth du monoïde plaxique en y ajoutant les relations $a^2 = a$ pour chaque lettre $a \in \mathcal{A}$. Ceci fait penser au passage de la présentation du groupe des tresses à la présentation du groupe symétrique (et plus généralement d'un groupe de Tits à son groupe de Coxeter correspondant) : on ajoute à la présentation du groupe des tresses les relations $a^2 = 1$ pour chaque générateur a.

L'évacuation de Schützenberger sur des tableaux induit un anti-automorphisme involutif du monoïde stylique. Nous donnons une construction de cette involution par une opération, semblable à l'évacuation, sur les tableaux immaculés standards. Ces derniers, qui sont en bijection avec les partitions ensemblistes, ont été introduit par Berg, Bergeron, Saliola, Serrano et Zabrocki [15] dans l'étude des fonctions quasi-symétriques, et étudiés par la suite par plusieurs auteurs (voir Campbell [30], Novelli, Thibbon et Toumazet [66] et, Allen, Hallam et Mason [11]).

Il apparaît aussi que le monoïde stylique est $\mathscr{J}$-trivial (une notion importante en théorie des semi-groupes et des automates, liée par le théorème Imre Simon à la notion de sous-mot) ; il hérite ainsi du $\mathscr{J}$-ordre, et nous prouvons que cet ordre est gradué par la taille du N-tableau. La fonction de graduation est quadratique, on obtient ainsi un ordre ayant une hauteur quadratique (en fonction de n) sur les partitions de n alors que l'ordre usuel sur les partitions (le raffinement) possède une hauteur linéaire. La preuve de ceci nous a amené à définir une insertion à gauche dans les N-tableaux, qui n'est pas du tout la symétrique de l'insertion à droite évoquée plus haut (les insertions à gauche et à droite de Schensted sont symétriques).

À la demande des réviseurs, quelques ajouts on été fait à l'article original pour faciliter la compréhension.

Cet article de 45 pages est publié dans la revue Semigroup Forum Volume 105 (2022) [6].

4.2 Abstract

The free monoid $\mathcal{A}^*$ on a finite totally ordered alphabet $\mathcal{A}$ acts at the left on columns, by Schensted left insertion. This defines a finite monoid, denoted $\mathrm{Styl}(\mathcal{A})$ and called the stylic monoid. It is canonically a quotient of the plactic monoid. Main results are: the cardinality of $\mathrm{Styl}(\mathcal{A})$ is equal to the number of onions of a set on $|\mathcal{A}|+1$ elements. We give a bijection with so-called N-tableaux, similar to Schensted's algorithm, explaining this fact. Presentation of $\mathrm{Styl}(\mathcal{A})$: it is generated by $\mathcal{A}$ subject to the plactic (Knuth) relations and the idempotent relations $a^2 = a$, $a \in \mathcal{A}$. The canonical involutive anti-automorphism on $\mathcal{A}^*$, which reverses the order on $\mathcal{A}$, induces an involution of $\mathrm{Styl}(\mathcal{A})$, which similarly to the corresponding involution of the plactic monoid, may be computed by an evacuation-like operation (Schützenberger involution on tableaux) on so-called standard immaculate tableaux (which are in bijection with partitions). The monoid $\mathrm{Styl}(\mathcal{A})$ is $\mathscr{J}$-trivial, and the $\mathscr{J}$-order of $\mathrm{Styl}(\mathcal{A})$ is graded: the co-rank is given by the number of elements in the N-tableau. The monoid $\mathrm{Styl}(\mathcal{A})$ is the syntactic monoid for the the function which associates to each word $W \in \mathcal{A}^*$ the length of its longest strictly decreasing subword.

4.3 Introduction

The plactic monoid is a fundamental object in combinatorics, representation theory, and algebra. It originates in a bijection of Schensted [72], often called the *Robinson-Schensted-Knuth correspondence*. This bijection maps the set $\mathcal{A}^*$ of words on a totally ordered finite alphabet $\mathcal{A}$ onto the the set of pairs (P, Q), where P is a semi-standard Young tableau on $\mathcal{A}$, and Q a standard Young tableau on $\{1, 2, \ldots, n\}$ (n is the length of W), both tableaux having the same shape. It turns out that the condition $\mathrm{P}(W) = \mathrm{P}(W')$ defines a congruence on the free monoid $\mathcal{A}^*$. This congruence was called the *plactic congruence* by Lascoux and Schützenberger, and they studied in [54] the corresponding quotient monoid $\mathcal{A}^*/\equiv_{\mathrm{Plax}}$, called the *plactic monoid*. This monoid has a cubic presentation, given by Knuth [50], with set of generators $\mathcal{A}$, and relations called the *plactic relations*. Besides the mentioned articles, a survey on the plactic monoid and its applications is given by Lascoux, Leclerc and Thibon (Chapter 5 of Lothaire's book [52]).

The plactic monoid has another natural finite generating set, the set of *columns*. A column is a strictly decreasing word. With this generating set, it has a quadratic presentation, which turns out to be confluent [21, 26] (note that the standard presentation is not confluent [51]).

Columns play a special role in the plactic monoid, which may be very deep as is seen in the first section of [53]. Clearly, the first column of $\mathrm{P}(W)$ depends only on the plactic class of w. In that way, one obtains by left multiplication an action of the plactic monoid on the finite set of columns. We call *stylic monoid* the finite monoid of endofunctions of this set obtained by this action (for the teminology, we use the Greek word for columns). Clearly, this monoid is a finite quotient of the plactic monoid.

Note that in the literature, one finds a class of monoids called *partition monoids*, see [47]. They are related to the Temperley-Lieb algebra, and different from the stylic monoids.

The two first main results give the cardinality of this monoid, and a presentation of it (Theorem 4.19). Let n be the cardinality of $\mathcal{A}$. Then the cardinality of $\mathrm{Styl}(\mathcal{A})$ (the stylic monoid on $\mathcal{A}$) is equal to the number of partitions of a set with $n + 1$ elements, the Bell number B_{n+1}. Moreover, the presentation on the set of generators $\mathcal{A}$ is obtained by adding to the plactic relations the *idempotent relations* $a^2 = a$ for each generator $a \in \mathcal{A}$.

In the course of the proof, we establish a bijection between $\mathrm{Styl}(\mathcal{A})$ and a set of semi-standard tableaux that we call N-*tableaux*: they are obtained by the condition that the rows strictly increase, and that each row contains the next one. The bijection is a variant of Schensted right insertion.

Next, we study a natural involution on $\mathrm{Styl}(\mathcal{A})$. It is obtained from the anti-automorphism θ of the free monoid $\mathcal{A}^*$ which reverses words, and reverses the alphabet (for example, $123233 \mapsto 112123$, $\mathcal{A} = \{1, 2, 3\}$). It induces an anti-automorphism of both the plactic monoid and the stylic monoid, as is seen on the plactic relations and idempotent relations. Concerning the plactic monoid, there is a remarkable direct construction on tableaux of this involution by Schützenberger, called *evacuation*.

This leads us to a similar construction for the stylic monoid. First, it is easy to see that N-tableaux are bijectively represented by partitions of subsets of $\mathcal{A}$. Such a partition may be represented by an increasing labeling of a lower ideal of $\mathbb{P}^2$, the latter being ordered as is shown in Figure 4.10. This allows to mimick the classical theory for standard tableaux: tableaux, skew-tableaux, jeu de taquin, evacuation. The third main result is that this modified evacuation corresponds to the involution (Theorem 4.24). The proof is nontrivial, but we followed the classical case (skew diagrams with a hole [73]), as is shown in Sagan's book [69], with the help of Fomin's growth diagrams, which may be extended to our case: partitions are replaced by compositions, appropriately ordered. We use a notion that appeared previously in the literature: composition tableaux of [46, 57] (with one condition removed), and more precisely, standard immaculate tableaux [15] (see also [16], [30], [45], [11], and [66]).

Next, we prove a semigroup-theoretical property of the stylic monoid: it is $\mathscr{J}$-trivial. This follows from the action on columns, and its order properties, once columns are naturally ordered. It is well-known that $\mathscr{J}$-trivial monoids inherit the $\mathscr{J}$-order: $X \leq_{\mathscr{J}} Y$ if X is in the two-sided ideal generated by Y. The fourth main result is that in the stylic monoid, the $\mathscr{J}$-order is graded (Theorem 4.41). For the proof of this, we define the left insertion of a letter in an N-tableau, which corresponds to multiplication at the left in the monoid. Unlike Schensted left and right insertion, which are symmetric, the left and right insertion into N-tableaux are completely asymmetric. The $\mathscr{J}$-order of the stylic monoid induces an order on set partitions, which seems new; in particular, the height of this graded poset is quadratic (unlike the usual refinement order of partitions, whose height is linear).

The fifth main result is an automata-theoretic result: the stylic monoid is syntactic with respect to the function which associates to each word the length of its longest strictly decreasing subsequence, equivalently by Schensted's theorem, the length of the first column of its P-tableau (Theorem 4.48).

We extend the methods to prove this result to give, in the Appendix, a proof of a statement given without proof by Lascoux and Schützenberger [54]: the plactic monoid is syntactic with respect to the function which associates to each word the shape of its P-tableau (Theorem 3.1).

We give also some order-theoretic properties of the action on columns, and as an application, a new proof of the quadratic presentation of the plactic monoid generated by columns, mentioned at the beginning of the introduction (Theorem 4.47, due to [21, 26]).

A remark about terminology, notations and abuse of language: a word $a_1 \cdots a_n, a_i \in \mathcal{A}$, where $\mathcal{A}$ is a totally ordered alphabet, is called *increasing* (resp. *strictly increasing*) if $a_1 \leq \cdots \leq a_n$ (resp. $a_1 < \cdots < a_n$). Similarly for *decreasing*.

We use the notion of *columns*, which are considered simultaneously as Young tableaux, as subsets of $\mathcal{A}$, and as strictly decreasing words on $\mathcal{A}$. We find this more convenient than introducing three different notations.

4.4 Schensted insertions

Let $\mathcal{A}$ be a totally ordered finite *alphabet* (whose elements are called *letters*) and denote by $\mathcal{A}^*$ the set of *words* on $\mathcal{A}$, which is the *free monoid* freely generated by $\mathcal{A}$.

In this article, we call a *tableau* what is called usually a *semi-standard Young tableau*; that is, a finite lower order ideal of the poset $\mathbb{N}^2$, ordered naturally, together with an increasing mapping into $\mathcal{A}$, such that the restriction of this mapping to each subset with given x-coordinate is injective. A tableau is usually represented as in Figure 4.1. The conditions may be expressed by saying that the letters in $\mathcal{A}$ are weakly increasing from left to right in each row, and strictly increasing from the bottom to top in each column.

We call *support* of a word W, and denote it by $\mathrm{Supp}(W)$, the set of letters appearing in W. Similarly for the support of a tableau, denoted likewise.

Call *column* a tableau with only one column, and *row* a tableau with only one row. One may see a column as a subset of $\mathcal{A}$, and a row as a multiset of elements of $\mathcal{A}$. We shall use therefore the symbol $\cup$ to express union of columns, and of rows (for rows, it is the multiset union). The empty column (resp. row) is denoted by $\emptyset$.

Another useful way to view columns is as *decreasing word* (a word whose letters decrease strictly from left to right).

We define now the *column insertion*. Let γ be a column, viewed here as a subset of $\mathcal{A}$, and let $x \in \mathcal{A}$. There are two cases: if $\forall y \in \gamma, x > y$, then define $\gamma' = \gamma \cup x$. Otherwise, let y be the smallest element in γ with $y \geq x$; then define $\gamma' = (\gamma \setminus y) \cup x$. Then γ' is the column obtained by *column insertion of* x *into* γ, and in the second case, y is said to be *bumped*.

One defines the *column insertion of* $x \in \mathcal{A}$ *into a tableau* T recursively as follows: insert x into the first column (the leftmost); in the case no element is bumped, stop; otherwise insert the bumped element in the second column, and so on.

Finally, given a word $W = a_1 \cdots a_n$ on $\mathcal{A}$, and a tableau T, one defines the *column insertion of* W *into* T recursively by inserting a_n into T, then a_{n-1} into the tableau obtained, and so on.

The *insertion into a row of* $x \in \mathcal{A}$ is defined similarly: exchange $>$ and $\geq$ in the definition of the column insertion (for a multiset E containing y, $E \setminus y$ means that one y is removed from E).

The *row insertion in a tableau* is defined similarly to column insertion, by using row insertion and starting from the first row (the one with y-coordinate 0).

Similarly, the *row insertion of a word* W *into* T is obtained recursively by row insertions, starting with a_1, then a_2 and so on.

A fundamental result of Schensted [72] is that inserting a word W into the empty tableau gives the same tableau, by column insertion, or by row insertion. The resulting tableau is denoted by $\mathrm{P}(W)$. See [72], or [69, Chapter 3], for details.

d		
b	b	
a	a	c

Figure 4.1: A tableau

It follows that for words U, V, $\mathrm{P}(UV)$ is equal to the tableau obtained by column insertion of U into $\mathrm{P}(V)$, and also by row insertion of V into $\mathrm{P}(U)$.

Another fundamental result of Schensted [72] states that the maximal length of a strictly decreasing subsequence of the word W is equal to the number of rows of the tableau $\mathrm{P}(W)$. Similarly, the maximal length of a weakly increasing subsequence of W is equal to the number of columns of $\mathrm{P}(W)$.

4.5 The plactic monoid

The condition $\mathrm{P}(U) = \mathrm{P}(V)$ is a monoid congruence on the free monoid, as follows from the previous section. This congruence was called the *plactic congruence*, denoted $\equiv_{\mathrm{Plax}}$, and the quotient monoid $\mathrm{Plax}(\mathcal{A})$ was called the *plactic monoid* by Lascoux and Schützenberger [54]. It follows from the work of Knuth [50] that the plactic congruence is generated by the relations

$$bac \equiv_{\mathrm{Plax}} bca, \quad acb \equiv_{\mathrm{Plax}} cab, \quad baa \equiv_{\mathrm{Plax}} aba, \quad bba \equiv_{\mathrm{Plax}} bab,$$

for all choices of letters $a < b < c$ in the first two relations, and for all choices of letters $a < b$ in the two others.

By definition, the plactic monoid may be identified with the set of tableaux on $\mathcal{A}$, and the surjective monoid homomorphism from $\mathcal{A}^*$ into $\mathrm{Plax}(\mathcal{A})$ is therefore denoted P.

Define for each tableau T its *row-word* to be the word, denoted $r(T)$, obtained by reading its rows from left to right, starting with the row of largest y-coordinate; for example the row-word of the tableau in Figure 4.1 is $dbbaac$. Similarly, its column-word, denoted $c(T)$, is obtained by reading the columns from left to right, each column being read by starting with the box with highest y-coordinate; in the figure, it is $dbabac$.

In particular, the row-word of a column γ is a strictly decreasing word, equal to its column-word. We often identify γ with this word.

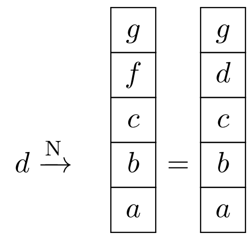

Figure 4.2: Example of the action of d on the column $gfcba$.

It is a well-known result that for each tableau T, one has

$$T = \mathrm{P}(r(T)) = \mathrm{P}(c(T)),$$

and thus

$$r(T) \equiv_{\text{Plax}} c(T).$$

Moreover, for any word u,

$$u \equiv_{\text{Plax}} r(\mathrm{P}(u)).$$

See [69, Lemma 3.6.5], [77, Th. A1.1.6], [52, Theorem 5.2.5 and Problem 5.2.4].

4.6 An action on columns

Denote by $\Gamma(\mathcal{A})$ the set of columns on $\mathcal{A}$. We define a left action of $\mathcal{A}^*$ on $\Gamma(\mathcal{A})$, denoted $U \cdot \gamma$, for each $U \in \mathcal{A}^*$ and each column γ. Since $\mathcal{A}^*$ is the free monoid on $\mathcal{A}$, it is enough to define the action for each letter $a \in \mathcal{A}$. Define

$$a \cdot \gamma = \gamma'$$

if γ' is obtained from γ by column insertion of a into γ.

Proposition 4.1 *Let γ be a column and W be a word. Then $W \cdot \gamma$ is the first column of $\mathrm{P}(Wr(\gamma))$, which is obtained by row insertion of $r(\gamma)$ into $\mathrm{P}(W)$.*

Proof. $\mathrm{P}(Wr(\gamma))$ is the tableau obtained by column insertion of W into $\mathrm{P}(r(\gamma)) = \gamma$ (see Section 4.4). It follows from the definitions of column insertion and the action on columns that its first column is precisely $W \cdot \gamma$. But $\mathrm{P}(Wr(\gamma))$ is also the tableau obtained by row insertion of $\mathcal{R}(\gamma)$ into $\mathrm{P}(W)$, see Section 4.4. □

For a column γ, and a letter x, define $\gamma_x = \{y \in \gamma \mid y < x\}$ and $\gamma^x = \{y \in \gamma \mid y > x\}$.

Lemma 4.2 *Let γ be a column and x be a letter.*

(i) $x \cdot \gamma$ contains x.

(ii) If γ contains x, then $x \cdot \gamma = \gamma$.

(iii) One has $(x \cdot \gamma)_x = \gamma_x$.

Proof. All these statements follows from the definition of the insertion of a letter in a column. □

Corollary 4.3 *Let γ be a column and W be a word.*

(i) If $\mathrm{Supp}(W) \subseteq \gamma$, then $W \cdot \gamma = \gamma$.

(ii) Let ℓ be a letter and $\mathcal{B} = \{x \in \mathcal{A} \mid x \leq \ell\}$. If $\mathcal{B} \subset \gamma$, then $\mathcal{B} \subseteq W \cdot \gamma$.

Proof. (i). It follows from Lemma 4.2 (ii) by induction on the length of W.

(ii). We argue also by induction. The case when W is empty is clear. Suppose that $W = xU$, $x \in \mathcal{A}$, $U \in \mathcal{A}^*$. Then $\mathcal{B} \subseteq U \cdot \gamma = \gamma'$ by induction. We have $W \cdot \gamma = x \cdot \gamma'$. If $x \leq \ell$, then $x \in \mathcal{B} \subset \gamma'$, hence $x \cdot \gamma' = \gamma'$ by Lemma 4.2 (ii) and consequently $\mathcal{B} \subseteq x \cdot \gamma'$. If $x > \ell$, then $\mathcal{B} \subseteq \gamma'_x$; since $(x \cdot \gamma')_x = \gamma'_x$ by Lemma 4.2 (iii), we have $\mathcal{B} \subseteq x \cdot \gamma'$. □

4.7 The stylic monoid

We denote by $\mathrm{Styl}(\mathcal{A})$ the monoid of endofunctions of the set $\Gamma(\mathcal{A})$ of columns obtained by the action defined in the previous section. Since $\Gamma(\mathcal{A})$ is finite, $\mathrm{Styl}(\mathcal{A})$ is finite. Let $\mu : \mathcal{A}^* \to \mathrm{Styl}(\mathcal{A})$ be the canonical monoid homomorphism. We denote by $\equiv_{styl}$ the corresponding monoid congruence of $\mathcal{A}^*$: $U \equiv_{styl} V$, if and only if $\mu(U) = \mu(V)$, if and only if for each column γ, $U \cdot \gamma = V \cdot \gamma$. The monoid $\mathrm{Styl}(\mathcal{A})$ acts naturally on the set of columns, and we take the same notation: $M \cdot \gamma = W \cdot \gamma$ if $M = \mu(w)$.

c		
a	b	d

Figure 4.3

c	d
a	b

Figure 4.4

Proposition 4.4 *If* $\mathrm{P}(U) = \mathrm{P}(V)$*, then for any column* γ*,* $U \cdot \gamma = V \cdot \gamma$*, and in particular,* $U \equiv_{styl} V$*. Thus* $\mathrm{Styl}(\mathcal{A})$ *is naturally a quotient of* $\mathrm{Plax}(\mathcal{A})$*:* $U \equiv_{\mathrm{Plax}} V \implies U \equiv_{styl} V$*.*

Proof. By Proposition 4.1, $U \cdot \gamma$ is the first column of $\mathrm{P}(Ur(\gamma))$; the latter element of $\mathrm{Plax}(\mathcal{A})$ is equal to $\mathrm{P}(U)\,\mathrm{P}(r(\gamma)) = \mathrm{P}(V)\,\mathrm{P}(r(\gamma)) = \mathrm{P}(Vr(\gamma))$, whose first column is by the same result equal to $V \cdot \gamma$. □

Lemma 4.5 *For* $x \in \mathcal{A}$*,* $x^2 \equiv_{styl} x$*.*

Proof. This follows from Lemma 4.2 (i) and (ii). □

Note that one has for any $U \in \mathcal{A}^*$:

$$U \equiv_{\mathrm{Styl}} r(\mathrm{P}(U)),$$

since $U \equiv_{\mathrm{Plax}} r(\mathrm{P}(U))$ (See Section 4.5).

It follows that for each element $M = \mu(U)$ of $\mathrm{Styl}(\mathcal{A})$, one has $M = \mu(r(\mathrm{P}(U)))$. Take U of smallest length. Then no row of $\mathrm{P}(U)$ contains repeated elements, otherwise $r(\mathrm{P}(U))$ contains a factor aa, and by Lemma 4.5, $\mathcal{R}(\mathrm{P}(U)) \equiv_{styl} V$ for some word of shorter length.

Hence each element of $\mathrm{Styl}(\mathcal{A})$ is represented by a tableau which has strictly increasing rows (and columns are evidently strictly increasing, too).

We note that this set of tableaux is not bijectively mapped onto $\mathrm{Styl}(\mathcal{A})$ (only surjectively). Indeed, the smallest example of two such distinct tableaux which are mapped onto the same element of $\mathrm{Styl}(\mathcal{A})$ are shown in Figure 4.3 and 4.4. Their row words are equal modulo $\equiv_{styl}$, since we have the sequence of equivalences, using only the plactic congruence and the relation $cc \equiv_{styl} c$ (Lemma 4.5): $\underline{c}abd \equiv_{styl} c\underline{cab}bd \equiv_{styl} ca\underline{cbd} \equiv_{styl} \underline{cac}db \equiv_{styl} cc\underline{adb} \equiv_{styl} \underline{cc}dab \equiv_{styl} cdab$, where underlines indicate the left-hand side of the relation which is used.

For further use, we state the following lemma.

Lemma 4.6 *If U and V have the same action on the set of columns over $\mathcal{A} = \mathrm{Supp}(U) \cup \mathrm{Supp}(V)$, then* $\mathrm{Supp}(U) = \mathrm{Supp}(V)$.

It follows that the function Supp is well-defined on $\mathrm{Styl}(\mathcal{A})$ (this will be also a consequence of Theorem 4.44).

Proof. Suppose that $\mathrm{Supp}(U) \neq \mathrm{Supp}(V)$. By symmetry, we may assume that there exists a letter ℓ such that $\ell \in \mathrm{Supp}(U), \ell \notin \mathrm{Supp}(V)$. Define the column $\gamma = \mathcal{A} \setminus \ell$. Then, $\mathrm{Supp}(V) \subseteq \gamma$, hence $V \cdot \gamma = \gamma$ by Corollary 4.3 (i), and in particular $\ell \notin V \cdot \gamma$. We may write $U = U_1 \ell U_2$, where $\ell \notin \mathrm{Supp}(U_2)$; then $U_2 \cdot \gamma = \gamma$ by Corollary 4.3 (i); next, $\ell \cdot \gamma = \gamma'$, where γ' has the property that it contains all the letters $\leq \ell$; hence, $U_1 \cdot \gamma'$ has also this property, by Corollary 4.3 (ii). Since $U \cdot \gamma = U_1 \cdot \gamma'$, we have $U \cdot \gamma \neq V \cdot \gamma$, and U, V are not equivalent modulo $\equiv_{styl}$. $\square$

Proposition 4.7 *The monoid $\mathrm{Styl}(\mathcal{A})$ has a zero, which is the image under μ of the decreasing product of all letters in $\mathcal{A}$.*

Proof. [1] Let W be this product, which we view also as column, denoted γ_0: it is the maximal column on $\mathcal{A}$ for the inclusion order. We claim that for any column γ on $\mathcal{A}$, $W \cdot \gamma = \gamma_0$. Hence, for any letter x, $Wx \cdot \gamma = W \cdot (x \cdot \gamma) = \gamma_0 = W \cdot \gamma$; thus $Wx \equiv_{styl} W$. Moreover, $xW \cdot \gamma = x \cdot (W \cdot \gamma) = x \cdot \gamma_0 = \gamma_0 = W \cdot \gamma$; thus $xW \equiv_{styl} W$. Therefore W is the zero of the stylic monoid.

We prove now the claim. Let x any letter; then $W = UxV$ and: $(*)$ each letter in U is greater than x. By Lemma 4.2 (i), $(xV) \cdot \gamma = x \cdot (V \cdot \gamma)$ contains x. Then, an easy induction on the length of U, using $(*)$ and Lemma 4.2 (iii), implies that $U \cdot ((xV) \cdot \gamma)$ also contains x. Hence $W \cdot \gamma$ contains x. Thus $W \cdot \gamma$ contains $\mathcal{A}$, and finally $W \cdot \gamma = \gamma_0$. $\square$

---

[1] We are indebted to the anonymous referee for pointing out an error in an earlier version of this proof.

d	e			
b	d	e		
a	b	c	d	e

Figure 4.5: An N-tableau

4.8 A variant of Schensted row insertion

4.8.1 N-tableaux and right N-insertion

Define an N-*tableau* to be a tableau satisfying the following two conditions:

(i) the rows are strictly increasing;

(ii) each row is contained in the row below.

Note that the support of an N-tableau coincides with its first row. As an example, see Figure 4.5.

To each N-tableau whose support is $\mathcal{A}_1 \subseteq \mathcal{A}$, associate the decreasing sequence of subsets of $\mathcal{A}_1$

$$\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \mathcal{A}_3 \dots \tag{4.1}$$

where $\mathcal{A}_i$ is the i-th row, viewed as a set. One has

$$min(\mathcal{A}_1) < min(\mathcal{A}_2) < min(\mathcal{A}_3) \dots, \tag{4.2}$$

since these elements constitute the first column of the N-tableau. We call N-*filtration on* $\mathcal{A}_1$ a sequence of subsets of $\mathcal{A}_1$ satisfying (4.1) and (4.2); when $\mathcal{A}_1$ is understood, we also say simply N-*filtration*. Note that the condition on the minima implies that the sequence is *strictly decreasing*.

Conversely, given an N-filtration, one associates with it an N-tableau, as is easily verified. Therefore, N-tableaux and N-filtrations are in bijection.

We describe now an algorithm, called the *right* N-*algorithm*, which associates with each word $W \in \mathcal{A}^*$ an N-tableau $\mathrm{N}(W)$. Viewing strictly increasing rows as subsets of $\mathcal{A}$, let $\mathcal{B} \subseteq \mathcal{A}$ be such a row. The *right* N-*insertion of a letter* x *in* $\mathcal{B}$ is equal to $\mathcal{B} \cup x$, and if y is the smallest element of $\mathcal{B}$ which is strictly greater than x, then a copy of y is bumped (and y does not disappear from $\mathcal{B}$). Note that no element is bumped if and only if x is greater than or equal to the elements of $\mathcal{B}$.

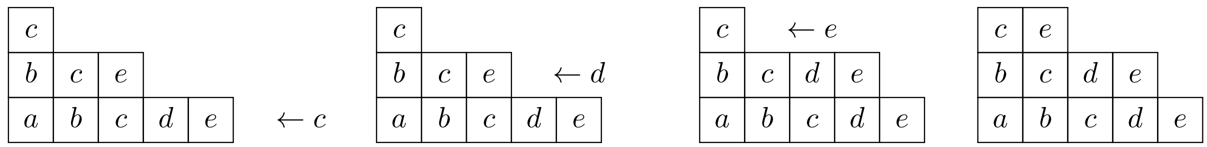

Figure 4.6: Right N-insertion of c into an N-tableau

Now *right* N-*insertion of* x *in an* N-tableau is recursively defined as for the Schensted row insertion: insert x in the first row, then the bumped element, if any, in the second one, and so on. For an example of this, see Figure 4.6.

Proposition 4.8 *The right* N-*insertion of* x *in an* N-*tableau produces an* N-*tableau.*

If T is an N-tableau, we denote by $T \leftarrow x$ the N-tableau obtained by right N-insertion of x into T.

We use in the proof below the fact that if S is a tableau, with S' the tableau obtained by removing the first row of S, assuming that S' is nonempty, then S is an N-tableau if and if the three following conditions are satisfied: S' is an N-tableau; $\min(S) < \min(S')$; $\mathrm{Supp}(S) \supseteq \mathrm{Supp}(S')$.

Proof. If in the N-insertion $T \leftarrow x$, no letter is bumped, then x is greater than or equal to any letter in T. Then $(T \leftarrow x) = T$ if $x \in T$, and otherwise $T \leftarrow x$ is obtained by adding x at the end of the first row of T. Thus $T \leftarrow x$ is clearly an N-tableau.

Otherwise, y is bumped from the first row. Let T' be the N-tableau obtained by removing the first row of T. Then the tableau obtained by removing the first row of $T \leftarrow x$ is the tableau $T' \leftarrow y$. This latter tableau is by induction an N-tableau. By the criterion stated before the proof, it is therefore enough to show that $\min(T \leftarrow x) < \min(T' \leftarrow y)$ and that $\mathrm{Supp}(T \leftarrow x) \supseteq \mathrm{Supp}(T' \leftarrow y)$.

We have $\mathrm{Supp}(T') \subseteq \mathrm{Supp}(T)$, $y \in \mathrm{Supp}(T)$, $\mathrm{Supp}(T \leftarrow x) = \mathrm{Supp}(T) \cup x$ and $\mathrm{Supp}(T' \leftarrow y) = \mathrm{Supp}(T') \cup y$; thus $\mathrm{Supp}(T' \leftarrow y) \subseteq \mathrm{Supp}(T \leftarrow x)$.

We have $\min(T \leftarrow x) = \min(\min(T), x)$ and similarly $\min(T' \leftarrow y) = \min(\min(T'), y)$. Moreover, $\min(T) < \min(T')$ and $x < y$. Thus $\min(T \leftarrow x) < \min(T' \leftarrow y)$ (since $a < a', b < b'$ implies $\min(a, b) < \min(a', b')$). $\square$

c			
a	b	c	d

Figure 4.7: N-tableau of $cabcd$ and $ccabcd$.

c	c		
a	b	c	d

Figure 4.8: P-tableau of $ccabcd$. Its N-tableau is the one in Figure 4.7

Similarly to Schensted row insertion, the *right* N-*insertion of a word* W *into an* N-*tableau* T is obtained by inserting the first letter of W into T, then the second one, and so on. We denote by $\mathrm{N}(W)$ the N-tableau obtained by inserting the word W into the empty N-tableau.

4.8.2 Inflation and simulation by Schensted row insertion

Define an *inflation* of a word $W = a_1 \cdots a_n, a_i \in \mathcal{A}$, to be any word of the form $a_1^{x_1} \cdots a_n^{x_n}$ for some positive exponents $x_i \in \mathbb{N}$.

We show that the right N-algorithm may be simulated by the Schensted row insertion algorithm, in the following sense.

Lemma 4.9 *Each word* W *has an inflation* W' *such that* $\mathrm{N}(W)$ *and* $\mathrm{P}(W')$ *have the same number of rows, and that corresponding rows in* $\mathrm{N}(W)$ *and* $\mathrm{P}(W')$ *have the same support.*

An example will be useful to understand the lemma: the two row-words of the tableaux in Figures 4.3 and 4.4 are $cabd$ and $cdab$. They have the same N-tableau under the N-algorithm, namely the tableau shown in Figure 4.7.

Consider c^3dab, which is an inflation of $W = cdab$. Then it is easily verified that $\mathrm{P}(W')$ is equal to the tableau shown in Figure 4.8. The corresponding rows of $\mathrm{N}(W)$ and $\mathrm{P}(W')$ have the same support.

Proof. [Proof of Lemma 4.9] We consider the following equivalent version of Schensted row insertion of a word W into a tableau T. For a word W, factorized as $W = U_1 \cdots U_k$, one may insert first U_1 in the first row of T, constructing from left to right the word V_1 of bumped letters; then insert V_1 into the second row, and so on until the last row; then continue with the second factor U_2, and so on. We call this *row insertion by factors*.

It may be that each factor U_i is a power of some letter, and also that each bumped word, V_1 and the others, are powers of some letter (not the same letter for all these words). In this case, we say that the insertion by factors *satisfies the block condition*. In order to be such, the necessary and sufficient condition is that each inserted factor is a power a^i and that, when inserted in a row, and if letters are bumped, there must be in this row at least i letters b, with $b =$ the minimum of the letters $> a$ in the row. Note that the bumped word is then b^i, with the same exponent.

Let $W = a_1 \cdots a_n$. We show that for some choice of the exponents x_i, the row insertion by factors of $W' = a_1^{x_1} \cdots a_n^{x_n}$, with the factors $a_i^{x_i}$, satisfies the block condition.

Consider the linear forms $f_i(x) = x_i - \sum_{i<j} x_j$, in the variables $x_1, \ldots, x_n$. Due to their triangularity property, it is clear that the system of inequalities $f_i(x) \geq 1$ has at least one solution $x_1, \ldots, x_n$ in positive integers. We choose these exponents x_i to inflate W.

Denote by T_k the tableau obtained after Schensted row insertion, into the empty tableau, of $a_1^{x_1} \cdots a_k^{x_k}$. We show by induction that the block condition is satisfied, and that each row of T_k, when viewed as a word, is an increasing product of letters with exponents equal to $x_i + \sum_{i<j\leq k} \epsilon_j x_j$, with $\epsilon_j = -1, 0, 1$, for some $i \leq k$. This is clear for $T_1 = a_1^{x_1}$, a tableau with one row.

Now, insert $a_{k+1}^{x_{k+1}}$ into T_k, obtaining T_{k+1}. If nothing is bumped, the block condition is clearly satisfied, as are the exponent conditions for T_{k+1}. Otherwise, some $b^{x_{k+1}}$ is bumped. Moreover, the exponents in the first row are not changed, with the two following exceptions:

1) The exponent of a_{k+1} increases by x_{k+1}.

2) The exponent of b decreases by x_{k+1}; note that this is possible (that is, the block condition is satisfied at this row insertion), since its exponent before bumping is of the form $x_i + \sum_{i<j\leq k} \epsilon_j x_j$, which is greater than x_{k+1}; indeed, this follows from $x_i + \sum_{i<j\leq k} \epsilon_j x_j - x_{k+1} \geq f_i(x) \geq 1$. Now one inserts $b^{x_{k+1}}$ in the second row, and so on, and the argument is similar.

Finally, the tableau T_n, which is $\mathrm{P}(W')$, satisfies the required conditions, since one verifies recursively that each step of the previous insertion by factors corresponds to a step of the N-insertion of W, and that the corresponding rows have the same support. □

4.8.3 The mapping δ

We define a mapping $\delta : \mathcal{A}^* \to \mathcal{A}^*$ as follows. Define for each subset $\mathcal{B}$ of $\mathcal{A}$, and each letter x in $\mathcal{A}$, the element $x^{\uparrow}_{\mathcal{B}} \in \mathcal{B} \cup \{1\}$ to be the smallest letter in $\mathcal{B}$ which is greater than x, and the empty word 1 if such a letter does not exist (that is, if $x \geq \max(\mathcal{B})$). Then we define $\delta(1) = 1$, and $\delta(Wx) = \delta(W)x^{\uparrow}_{\mathrm{Supp}(W)}$, for any word W and any letter x.

Concretely, one scans the letters of W from left to right, at each position one searches at the left the smallest letter which is greater than the letter in the current position (it may not exist), and write these letters form left to right.

Example: let the alphabet be $\{a < b < c < d\}$; then $\delta(acccadbcbac) = ccdcbd$, and the algorithm just described is best seen on a two rows array:

$$\begin{array}{cccccccccccccc} a & c & c & c & a & d & b & c & b & a & c & = & w \\ & & & & c & & c & d & c & b & d & = & \delta(w) \end{array}$$

The following lemma is a direct consequence of the definition of the right N-algorithm; indeed, the sequence of bumped letters from the first row during the right N-algorithm applied to W is precisely the word $\delta(W)$.

Lemma 4.10 *The first row of* $\mathrm{N}(W)$ *is* $\mathrm{Supp}(W)$ *(viewed as a strictly increasing word) and the remaining* N*-tableau is* $\mathrm{N}(\delta(W))$.

Define, for two subsets $\mathcal{B}, \mathcal{C}$ of the alphabet, the set

$$\mathbf{D}_{\mathcal{B}}(\mathcal{C}) = \{c^{\uparrow}_{\mathcal{B}} \mid c \in \mathcal{C}, c^{\uparrow}_{\mathcal{B}} \neq 1\},$$

which is a subset of $\mathcal{B}$. Note that if $\mathcal{B} \subseteq \mathcal{C}$ and $\min(\mathcal{B}) > \min(\mathcal{C})$, then

$$\mathbf{D}_{\mathcal{B}}(\mathcal{C}) = \mathcal{B}. \tag{4.3}$$

We denote by s the natural bijection associating to each subset of $\mathcal{A}$ the increasing product of its elements. Note that if a word U is increasing, then

$$U \equiv_{styl} s(\mathrm{Supp}(U)), \tag{4.4}$$

by Lemma 4.5. For later use, we prove the following lemma.

Lemma 4.11 *Let $U_1, \ldots, U_k$ be strictly increasing words such that their supports $\mathcal{U}_1, \ldots, \mathcal{U}_k$ satisfy $\mathcal{U}_1 \supseteq \cdots \supseteq \mathcal{U}_k$. Let $X \in \mathcal{A}^*$ and $\mathcal{X} = \mathrm{Supp}(X)$. Then*

$$\delta(XU_k \cdots U_1) \equiv_{styl} \delta(X) \prod_{i=k}^{i=1} s(\mathbf{D}_{\mathcal{U}_{i+1} \cup \mathcal{X}}(\mathcal{U}_i)),$$

with the convention that $\mathcal{U}_{k+1} = \emptyset$.

Proof. It follows directly from the definition of δ that for any word W, $\delta(XW) = \delta(X) \prod_{W=UyV} y^{\uparrow}_{\mathrm{Supp}(XU)}$, where the product is over all factorizations $W = UyV, U, V \in \mathcal{A}^*, y \in \mathcal{A}$, and from left to right. Let $W = U_k \cdots U_1$; then $\delta(XW) = \delta(X) \prod_{i=k}^{i=1} \prod_{U_i=UyV} y^{\uparrow}_{\mathrm{Supp}(XU_k \ldots U_{i+1}U)}$. Note that, in the latter product, the letters in U are less than y; hence $y^{\uparrow}_{\mathrm{Supp}(XU_k \ldots U_{i+1}U)} = y^{\uparrow}_{\mathrm{Supp}(XU_k \ldots U_{i+1})}$. Moreover, the supports of the U_i being decreasing from 1 to k in the inclusion order, we have $\mathrm{Supp}(XU_k \ldots U_{i+1}) = \mathrm{Supp}(XU_{i+1}) = \mathcal{U}_{i+1} \cup \mathcal{X}$. Thus $\delta(W) = \delta(X) \prod_{i=k}^{i=1} \prod_{U_i=UyV} y^{\uparrow}_{\mathcal{U}_{i+1} \cup \mathcal{X}}$. Finally, note that if a word M is strictly increasing, and $\mathcal{U}$ a subset of $\mathcal{A}$, then the word $P = \prod_{M=UyV} y^{\uparrow}_{\mathcal{U}}$ is increasing, so that $P \equiv_{styl} s(\mathrm{Supp}(P))$, by (4.4); thus $P \equiv_{styl} s(\mathbf{D}_{\mathcal{U}}(\mathrm{Supp}(M)))$, since $\mathrm{Supp}(P) = \{y^{\uparrow}_{\mathcal{U}} \mid y \in \mathrm{Supp}(M), y^{\uparrow}_{\mathcal{U}} \neq 1\} = \mathbf{D}_{\mathcal{U}}(\mathrm{Supp}(M))$. It follows from this that $\delta(W) \equiv_{styl} \delta(X) \prod_{i=k}^{i=1} s(\mathbf{D}_{\mathcal{U}_{i+1} \cup \mathcal{X}}(\mathcal{U}_i))$.
□

4.9 A bijection

Theorem 4.12 *The mapping $W \mapsto \mathrm{N}(W)$ induces a bijection from the monoid $\mathrm{Styl}(\mathcal{A})$ onto the set of N-tableaux on $\mathcal{A}$.*

The theorem is a consequence of several lemmas.

Lemma 4.13 *Let W be a word and γ be any column. Then*

(i) the tableaux $\mathrm{P}(W)$ and $\mathrm{N}(W)$ have the same first column, which is $W \cdot \emptyset$,

(ii) $W \cdot \gamma$ is equal to the first column of $\mathrm{N}(WU)$, where U is the strictly decreasing word associated to γ.

Proof. (i). We know by Lemma 4.9 that W has some inflation W' such that the corresponding rows in $\mathrm{N}(W)$ and $\mathrm{P}(W')$ have the same support. Hence these two tableaux have the same first column. Moreover $W' \equiv_{styl} W$ by Lemma 4.5; thus $W' \cdot \emptyset = W \cdot \emptyset$. Hence $\mathrm{P}(W')$ and $\mathrm{P}(W)$ have the same first column, by Proposition 4.1.

(ii). We know by Proposition 4.1 that $W \cdot \gamma$ is equal to the first column of $\mathrm{P}(WU)$; hence also to the first column of $\mathrm{N}(WU)$ by (i). □

Let γ be a column on the alphabet $\mathcal{A}$. We denote by γ^- the column obtained by replacing each letter by the previous one in the alphabet $\mathcal{A}$, removing if necessary the smallest letter. The column γ^+ is defined symmetrically.

Lemma 4.14 *Let* $a = \min(\mathcal{A})$ *and* $z = \max(\mathcal{A})$. *Let* γ *be a column on* $\mathcal{A} \setminus z$, *and* $W \in \mathcal{A}^*$ *with* $\mathcal{A} = \mathrm{Supp}(W)$. *Then* $W \cdot \gamma = a \cup \delta(W) \cdot \gamma^+$ *and* $(W \cdot \gamma)^- = \delta(W)^- \cdot \gamma$.

Proof. By Lemma 4.13, $W \cdot \gamma$ is the first column of $\mathrm{N}(WU)$, where U is the strictly decreasing word having same support as γ. Since a appears in W, a appears in $\mathrm{N}(WU)$, necessarily at the bottom of the first column. By Lemma 4.10, the first column of $\mathrm{N}(WU)$ is equal to the first column of $\mathrm{N}(\delta(WU))$ with a added at the bottom.

Now, since U does not involve the letter z and since W involves each letter in $\mathcal{A}$, we have $\delta(WU) = \delta(W)U^+$, where U^+ is obtained by replacing in U each letter by the next one in the alphabet $\mathcal{A}$. Hence the first column of $\mathrm{N}(\delta(WU)) = \mathrm{N}(\delta(W)U^+)$ is by Lemma 4.13 equal to $\delta(W) \cdot \gamma^+$.

It follows from the previous remarks that $W \cdot \gamma = a \cup \delta(W) \cdot \gamma^+$, which implies the lemma. □

Lemma 4.15 $\mathrm{N}(W)$ *depends only on the class of* W *modulo* $\equiv_{styl}$.

Proof. It is enough to show that if W, W' have the same action on the set of columns over $\mathrm{Supp}(W) \cup \mathrm{Supp}(W')$, then $\mathrm{N}(W) = \mathrm{N}(W')$. Note that by Lemma 4.6, the hypothesis implies that they have the same support.

We prove the lemma by induction on $|Supp(W) \cup \mathrm{Supp}(W')|$; the case where it is empty is clear. Suppose now that $\mathcal{A} = \mathrm{Supp}(W) \cup \mathrm{Supp}(W')$ is nonempty and let $a = \min(\mathcal{A})$. By hypothesis, W, W' have the same action on $\Gamma(\mathcal{A})$.

By Lemma 4.10, the first row of $\mathrm{N}(W)$, viewed as a set, is $\mathrm{Supp}(W)$, and the remaining tableau is $\mathrm{N}(\delta(W))$. Hence the first rows of $\mathrm{N}(W)$ and $\mathrm{N}(W')$ are equal. Note that every letter of $\delta(W)$ and $\delta(W')$ is in the alphabet $\mathcal{A} \setminus a$; hence $\mathrm{Supp}(\delta(W)) \cup \mathrm{Supp}(\delta(W')) \subseteq \mathcal{A} \setminus a$.

We claim that the action of $\delta(W)$ on $\Gamma(\mathcal{A} \setminus a)$ depends only on the action of W on $\Gamma(\mathcal{A})$. Indeed, let γ_1 be a column on $\mathcal{A} \setminus a$. Then $\gamma = \gamma_1^-$ is a column on $\mathcal{A} \setminus z$, where $z = \max(\mathcal{A})$; note also that $\gamma^+ = \gamma_1$, hence by Lemma 4.14, $a \cup \delta(W) \cdot \gamma_1 = W \cdot \gamma$, which implies $\delta(W) \cdot \gamma_1 = (W \cdot \gamma) \setminus a$.

The claim is also true for $\delta(W')$, so that $\delta(W)$ and $\delta(W')$ have the same action of $\Gamma(\mathcal{A} \setminus a)$. Hence, they have the same action on the set of columns over $\mathrm{Supp}(\delta(W)) \cup \mathrm{Supp}(\delta(W'))$. By induction $\mathrm{N}(\delta(W)) = \mathrm{N}(\delta(W'))$. It follows that $\mathrm{N}(W) = \mathrm{N}(W')$ by Lemma 4.10. □

Lemma 4.16 *Let T be an* N*-tableau. Then* $\mathrm{N}(r(T)) = T$.

Proof. Let T have k rows, and let $U_1, \ldots, U_k$ be the row-words of the rows from $i = 1$ to $i = k$; moreover, let $\mathcal{U}_i = \mathrm{Supp}(U_i)$. Then $r(T) = U_k \cdots U_1$. By Lemma 4.11, with $x = 1$, we have $\delta(r(T)) = \delta(U_k \cdots U_1) \equiv_{styl} \prod_{i=k-1}^{i=1} s(\mathbf{D}_{\mathcal{U}_{i+1}}(\mathcal{U}_i))$ (since the factor for $i = k$ is the empty word). Now, by (4.3), one has $\mathbf{D}_{\mathcal{U}_{i+1}}(\mathcal{U}_i) = \mathcal{U}_{i+1}$, since $\mathcal{U}_{i+1} \subseteq \mathcal{U}_i$ and $\min(\mathcal{U}_i) < \min(\mathcal{U}_{i+1})$; therefore $s(\mathbf{D}_{\mathcal{U}_{i+1}}(\mathcal{U}_i)) = U_{i+1}$. Hence $\delta(r(T)) \equiv_{styl} U_k \ldots U_2 = r(T')$, the row word of the N-tableau T' obtained by removing the first row from T. It follows from Lemma 4.15 that $\mathrm{N}(\delta(r(T))) = \mathrm{N}(r(T'))$; by induction, this is the N-tableau T'. By Lemma 4.10, we deduce that $\mathrm{N}(r(T))$ is equal to T, since the support of $r(T)$ is equal to that of T and therefore to the first row of T. □

Proof. [Proof of Theorem 4.12] The mapping is well-defined by Lemma 4.15. Surjectivity follows from Lemma 4.16.

The mapping is injective, since, using Lemma 4.9 and its notations, one has $W \equiv_{styl} W'$ by Lemma 4.5. And $W' \equiv_{\mathrm{Plax}} r(\mathrm{P}(W'))$ by Section 4.5, and finally $r(\mathrm{P}(W')) \equiv_{styl} r(\mathrm{N}(W))$, by Lemma 4.5 and Lemma 4.9.

Thus

$$W \equiv_{styl} r(\mathrm{N}(W)), \tag{4.5}$$

which proves injectivity. □

Corollary 4.17 *Let T be an* N*-tableau and x a letter. Then $(T \leftarrow x) = \mathrm{N}(r(T)x)$.*

Proof. By definition of the N-insertion, $\mathrm{N}(r(T)x) = (\mathrm{N}(r(T)) \leftarrow x) = (T \leftarrow x)$, by Lemma 4.16. □

Corollary 4.18 *Let $W \in \mathcal{A}^*$. Then $W \equiv_{styl} \delta(W)s(\mathrm{Supp}(W))$.*

Proof. Let $U, \ldots, U_k$ the increasing words corresponding to the rows of $T = \mathrm{N}(W)$, from the longest row to the shortest. Then $r(T) = U_k \cdots U_1$. Let T' obtained from T by removing the first row; then $r(T') = U_k \cdots U_2$. Moreover, $T' = \mathrm{N}(\delta(W))$ by Lemma 4.10. By (4.5), $W \equiv_{styl} r(\mathrm{N}(W)) = U_k \cdots U_1 = r(T')U_1 \equiv_{styl} \delta(W)s(\mathrm{Supp}(W))$, by (4.5). □

4.10 Cardinality and presentation of the stylic monoid

Recall that the *Bell number* B_n is the number of partitions of a set with n elements. The first few values, starting with $n = 1$, are $1, 2, 5, 15, 52, 203, 877$; see [75, A000110].

Theorem 4.19 *(i) If the cardinality of $\mathcal{A}$ is n, then the cardinality of* $\mathrm{Styl}(\mathcal{A})$ *is B_{n+1}.*

(ii) $\mathrm{Styl}(\mathcal{A})$ *is presented by the plactic relations and the relations $x^2 = x$, $x \in \mathcal{A}$.*

We call *stylic relations* the plactic relations together with the relations $x^2 = x$, $x \in \mathcal{A}$. Denote by $\mathrm{Part}(E)$ the set of partitions on a set E.

Lemma 4.20 *To each* N*-tableau* T *on* $\mathcal{A}$*, associate the partition* R *of the set* $\mathrm{Supp}(T)$ *obtained as follows: denoting the rows of* T *by* R_i*,* $i = 1, \ldots, k$*, from the longest to the shortest, and viewing them as subsets of* $\mathcal{A}$*, the parts of* R *are* $R_k, R_{k-1} \setminus R_k, \ldots, R_1 \setminus R_2$*. This mapping is a bijection from the set of* N*-tableaux on* $\mathcal{A}$ *onto the set* $\bigcup_{\mathcal{B} \subseteq \mathcal{A}} \mathrm{Part}(\mathcal{B})$*. The inverse mapping is defined as follows: let* $R = \{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$*, ordered in such a way that* $\min(\mathcal{B}_1) < \ldots < \min(\mathcal{B}_k)$*; then the rows of the associated* N*-tableau, viewed as subset of* $\mathcal{A}$*, are the sets* $\bigcup_{i \le j \le k} \mathcal{B}_j$*,* $i = 1, \ldots, k$*.*

As an illustration, consider the N-tableau of Figure 4.5, whose rows are $\{a, b, c, d, e\}, \{b, d, e\}, \{d, e\}$: then $R = \{\{a, c\}, \{b\}, \{d, e\}\}$.

Proof. This follows from the bijection between N-tableaux and N-filtrations, as stated at the beginning of Section 4.1. □

Proof. [Proof of Theorem 4.19] (i). The cardinality of $\mathrm{Styl}(\mathcal{A})$ is equal by Theorem 4.12 to the number of N-tableaux on $\mathcal{A}$. This number is by Lemma 4.20 equal to $\sum_{\mathcal{B} \subseteq \mathcal{A}} |\,\mathrm{Part}(\mathcal{B})| = \sum_k \binom{n}{k} B_k$, which is well-known to be equal to B_{n+1}.

(ii). By Corollary 4.4 and Lemma 4.5, the stylic relations are satisfied in $\mathrm{Styl}(\mathcal{A})$.

Conversely, denote by $\equiv$ the congruence of $\mathcal{A}^*$ generated by the stylic relations. Suppose that $U \equiv_{styl} V$; it is enough to show that $U \equiv V$. We have by Lemma 4.15, $\mathrm{N}(U) = \mathrm{N}(V)$. We have $u \equiv U', V \equiv V'$, where U', V' are some inflation of U, V respectively, as indicated in Lemma 4.9; by this lemma, and the idempotence of the generators, we have $r(\mathrm{N}(U)) \equiv r(\mathrm{P}(U')), r(\mathrm{N}(V)) \equiv r(\mathrm{P}(V'))$. We have by Section 4.5, $U' \equiv r(\mathrm{P}(U')), V' \equiv r(\mathrm{P}(V'))$, since $\equiv_{\mathrm{Plax}}$ implies $\equiv$. In conclusion, we have $U \equiv U' \equiv r(\mathrm{P}(U')) \equiv r(\mathrm{N}(U)) = r(\mathrm{N}(V)) \equiv r(\mathrm{P}(V')) \equiv V' \equiv V$. □ The proof also yields the following corollary.

Corollary 4.21 *The set of words of the form* $r(T)$*,* T *an* N*-tableau on* $\mathcal{A}$*, is a set of unique representatives of the stylic classes.*

Corollary 4.22 *Let* $\mathcal{B} \subseteq \mathcal{A}$*. The natural injection* $\mathcal{B}^* \to \mathcal{A}^*$ *induces an injection* $\mathrm{Styl}(\mathcal{B}) \to \mathrm{Styl}(\mathcal{A})$*. In other words, if two words* U, V *in* $\mathcal{B}^*$ *have the same action on* $\Gamma(\mathcal{B})$*, then they have the same action on* $\Gamma(\mathcal{A})$*.*

A direct proof of the latter assertion seems not evident.

Proof. This follows since the presentation is support-preserving: if one applies an elementary plactic move, or a move according to $x^2 \equiv_{styl} x$, the alphabet does not change. Hence the relations $U \equiv_{styl} V$ in the large alphabet imply the relations in the small alphabet. □

We say that an element W of $\mathrm{Styl}(\mathcal{A})$ is *complete* if its support is equal to $\mathcal{A}$. For example, the N-tableau of Figure 4.5 is complete over the alphabet $\mathcal{A} = \{a, b, c, d, e\}$.

Corollary 4.23 *If* $|\mathcal{A}| = n$*, then the number of complete elements in* $\mathrm{Styl}(\mathcal{A})$ *is equal to* B_n*.*

Proof. The complete elements correspond in the bijection of Theorem 4.12 to the N-tableaux whose support is $\mathcal{A}$. Hence their number is B_n by the argument seen in part (i) of the proof of Theorem 4.19. □

4.11 Evacuation of partitions

4.11.1 An involution

Recall that $\mathcal{A}$ is a totally ordered finite alphabet. Denote by θ the unique order-reversing permutation of $\mathcal{A}$. It extends uniquely to an anti-automorphism of the free monoid, that we still denote θ. For example, with $\mathcal{A} = \{a < b < c < d\}$, $\theta(acdaadc) = baddabd$. The mapping θ is clearly an involution.

Strictly speaking, θ depends on $\mathcal{A}$ and we denote it $\theta_{\mathcal{A}}$ if necessary. For later use, we note that if a is the smallest element of $\mathcal{A}$, and denoting by $i_a : (\mathcal{A} \setminus a)^* \to \mathcal{A}^*$ the monoid homomorphism sending each letter x in $\mathcal{A} \setminus a$ onto the letter that precedes x in the total order of $\mathcal{A}$, then

$$\forall w \in (\mathcal{A} \setminus a)^*, \theta_{\mathcal{A}}(w) = i_a \circ \theta_{\mathcal{A} \setminus a}(w). \tag{4.6}$$

Both sides are indeed anti-homomorphisms, which coincide on the alphabet $\mathcal{A} \setminus a$. Likewise, if z is the largest letter of $\mathcal{A}$, and j_z the homomorphism from $(\mathcal{A} \setminus z)^* \to \mathcal{A}^*$ sending each letter to the next one in the order of $\mathcal{A}$, then

$$\forall w \in (\mathcal{A} \setminus z)^*, \theta_{\mathcal{A}}(w) = j_z \circ \theta_{\mathcal{A} \setminus z}(w). \tag{4.7}$$

$$
\begin{array}{ccc}
\mathcal{A}^* & \xrightarrow{\theta} & \mathcal{A}^* \\
\downarrow & & \downarrow \\
\mathrm{Plax}(\mathcal{A}) & \xrightarrow{\theta} & \mathrm{Plax}(\mathcal{A}) \\
\downarrow & & \downarrow \\
\mathrm{Styl}(\mathcal{A}) & \xrightarrow{\theta} & \mathrm{Styl}(\mathcal{A})
\end{array}
$$

Figure 4.9: Commuting homomorphisms and anti-automorphisms

Let us come back to the fixed alphabet $\mathcal{A}$ and $\theta = \theta_{\mathcal{A}}$. Clearly, and as is well-known, the plactic relations (see Section 4.5) are invariant under θ. It follows that θ induces an anti-automorphism of the plactic monoid. Similarly, the stylic relations (see the definition following Theorem 4.19) are invariant under θ, and therefore θ induces an anti-automorphism of the stylic monoid. Both anti-automorphisms are involutions, and we denote them with the same notation θ. We thus obtain the commutative diagram of Figure 4.9, where the vertical mappings are the canonical quotient homomorphisms.

The plactic monoid is in bijection with Young tableaux. The endomorphism θ of the plactic monoid is described directly on the set of tableaux by the *Schützenberger involution* ([74, p.127]), also called *evacuation* (see [69, p. 3.9], [77, p.425]).

We give now a construction on (set-theoretical) partitions, similar to Schützenberger's evacuation, which will be shown to correspond to the involution θ of the stylic monoid.

Fix the alphabet $\mathcal{A}$ and the involution $\theta = \theta_{\mathcal{A}}$. For each nonempty subset $\mathcal{B}$ of $\mathcal{A}$, we define a mapping $\Delta : \mathrm{Part}(\mathcal{B}) \to \mathrm{Part}(\mathcal{B} \setminus \min(\mathcal{B}))$. For this, we order the blocks of each partition on the totally ordered set $\mathcal{B}$, according to the order of the minimum of the blocks. Therefore, we may speak of the j-th block of a partition.

Let $R = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k\} \in \mathrm{Part}(\mathcal{B})$. Let $x_i = \min(\mathcal{B}_i)$; we assume that $x_1 < x_2 < \cdots < x_k$. Let U_i be the strictly increasing word whose support is $\mathcal{B}_i$; then x_i is the first letter of $U_i = x_i V_i$.

Consider the word $W = U_1U_2 \cdots U_k = x_1V_1x_2V_2 \cdots x_kV_k$. We determine an integer $e(R)$ as follows:

- Define first $x := x_1$ and $e := 1$.
- Look for the smallest letter y at the right of x in W: if y is some x_j, let $x := x_j$, $e := j$ and iterate this step. If y is not an x_j, then the algorithms stops.
- Put $e(R) := e$.

Let $e = e(R)$. Define $\mathcal{B}'_j = (\mathcal{B}_j \setminus x_j) \cup x_{j+1}$ for $j = 1, \ldots, e-1$, $\mathcal{B}'_e = \mathcal{B}_i \setminus x_e$ and $\mathcal{B}'_j = \mathcal{B}_j$ for $j > e$. Then $\Delta(R)$ is the partition whose blocks are the nonempty sets $\mathcal{B}'_j$ (only $\mathcal{B}'_e$ may be empty, and in this case e must be equal to k).

For example, with $\mathcal{A} = [8] = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and $R = 13/28/457/6$ (with evident notations), $w = u_1u_2u_3u_4 = (13)(28)(457)(6)$, we have $x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 6$, $e(R) = 3$: indeed, at the end of the algorithm, when x is set to $x_3 = 4$, then y is set to 5, which is not an x_j. Thus $\Delta(R) = 23/48/57/6$, which is a partition of the set $[8] \setminus 1 = \{2, 3, 4, 5, 6, 7\}$. This example is also given in another form in Figures 4.12, 4.20 and 4.21.

For each subset $\mathcal{B}$ of $\mathcal{A}$, the *evacuation mapping* evac, from $\mathrm{Part}(\mathcal{B})$ into itself, is then recursively defined as follows. If $\mathcal{B}$ is empty and $R \in \mathrm{Part}(\mathcal{B})$, then $\mathrm{evac}(R) = R$ (R is here the empty partition). Otherwise, let $R \in \mathrm{Part}(\mathcal{B})$, $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ nonempty. Let $b = \min(\mathcal{B}) = \min(R)$. Then, with the notation $e(R)$ above, $\mathrm{evac}(R)$ is the partition on $\mathcal{B}$, obtained from $\mathrm{evac}(\Delta(R))$ by adding $\theta_{\mathcal{A}}(b)$ to its $e(R)$-th block (and creating this block if necessary; note that it is then the last block).

Note that the definition of evacuation implies that $\theta(b)$ is the largest letter in R, and that

$$\mathrm{evac}(\Delta(R)) = \mathrm{evac}(R) \setminus \theta(b). \tag{4.8}$$

Denote by π the mapping associating to each word w the partition corresponding bijectively to the N-tableau $\mathrm{N}(W)$, as described in Lemma 4.20; see the example following it.

Theorem 4.24 *One has* $\pi(\theta(W)) = \mathrm{evac}(\pi(W))$ *for any word* W.

In other words, the involutive anti-automorphism θ of the stylic monoid corresponds at the level of partitions to evacuation of partitions. We shall prove the theorem in Section 4.11.6, after a détour through a generalization of jeu-de-taquin, which is interesting for itself.

For later use, we note that evacuation, as defined above, depends on the mapping θ, that is, on $\mathcal{A}$, and is therefore denoted $\mathrm{evac}_{\mathcal{A}}$ if necessary. As for θ, we have the following rules. We use the functions i_a and j_z defined around (4.6), naturally extended to partitions.

Lemma 4.25 *Let a (resp. z) be the smallest (resp largest) letter of $\mathcal{A}$.*

$$\forall R \in \mathrm{Part}(\mathcal{B}), \mathcal{B} \subseteq \mathcal{A} \setminus a, \mathrm{evac}_{\mathcal{A}}(R) = i_a \circ \mathrm{evac}_{\mathcal{A}\setminus a}(R). \quad (4.9)$$

$$\forall R \in \mathrm{Part}(\mathcal{B}), \mathcal{B} \subseteq \mathcal{A} \setminus z, \mathrm{evac}_{\mathcal{A}}(R) = j_z \circ \mathrm{evac}_{\mathcal{A}\setminus z}(R). \quad (4.10)$$

Proof. Note that the function Δ is independent of the alphabet. Let $R \in \mathrm{Part}(\mathcal{B})$, $\mathcal{B} \subseteq \mathcal{A}$, $b = \min(\mathcal{B}) = \min(R)$, and let $R' = \Delta(R)$, $e = e(R)$, $R_1 = \mathrm{evac}_{\mathcal{A}}(R)$. Then, by definition of evacuation, R_1 is obtained from $\mathrm{evac}_{\mathcal{A}}(R')$ by inserting $\theta_{\mathcal{A}}(b)$ into its e-th block.

Suppose that $\mathcal{B} \subseteq \mathcal{A} \setminus a$. By definition, $\mathrm{evac}_{\mathcal{A}\setminus a}(R)$ is obtained from $\mathrm{evac}_{\mathcal{A}\setminus a}(R')$ by inserting $\theta_{\mathcal{A}\setminus a}(b)$ into its e-th block. We have clearly $R' \in \mathrm{Part}(\mathcal{B}'), \mathcal{B}' \subseteq \mathcal{A} \setminus a$; hence by induction, $\mathrm{evac}_{\mathcal{A}}(R') = i_a \circ \mathrm{evac}_{\mathcal{A}\setminus a}(R')$; now, since by (4.6), $\theta_{\mathcal{A}}(b) = i_a \circ \theta_{\mathcal{A}\setminus a}(b)$, inserting $\theta_{\mathcal{A}}(b)$ into the e-th block of $\mathrm{evac}_{\mathcal{A}}(R')$ amounts to first inserting $\theta_{\mathcal{A}\setminus a}(b)$ into the e-th block of $\mathrm{evac}_{\mathcal{A}\setminus a}(R')$ and then applying i_a. This proves (4.9), and (4.10) is proved similarly. $\square$

4.11.2 Skew-partitions with a hole

Comparison of the definitions below with Ferrers diagram, lower poset ideals in $\mathbb{N}^2$, Young tableaux, skew Young tableaux, and paths in Young's lattice may be useful (see [69, 77]), since what we do now is very similar, after a change of the order on $\mathbb{N}^2$.

Let $\mathbb{P} = \mathbb{N} \setminus \{0\}$. We consider the order on $\mathbb{P}^2$, denoted $\preceq$, such that the covering relations are $(1, y) \preceq (1, y+1)$ and $(x, y) \preceq (x+1, y)$; its Hasse diagram is represented in Figure 4.10, where one increases in

$$\begin{array}{ccccccc}
(1,4) & - & (2,4) & - & (3,4) & \cdots \\
| & & & & & \\
(1,3) & - & (2,3) & - & (3,3) & \cdots \\
| & & & & & \\
(1,2) & - & (2,2) & - & (3,2) & \cdots \\
| & & & & & \\
(1,1) & - & (2,1) & - & (3,1) & \cdots
\end{array}$$

Figure 4.10

$$\begin{array}{ccccc}
\circ & & & & \\
| & & & & \\
\circ & - & \circ & - & \circ \\
| & & & & \\
\circ & - & \circ & & \\
| & & & & \\
\circ & - & \circ & &
\end{array}$$

Figure 4.11

the order by going up or to the right (north or east). When we speak of the order on $\mathbb{P}^2$, it will be always the order $\preceq$.

A *lower ideal* in a poset E is a subset $I \subseteq E$ such that for any elements $x \leq y$ in E, if $y \in I$, then $x \in I$.

We call $\mathcal{A}$-*labelling* of a finite poset a bijective increasing mapping from the poset into the totally ordered set $\mathcal{A}$. The mapping is indicated by labelling the vertices of the Hasse diagram of the poset.

It is easy to see that a finite lower ideal in $\mathbb{P}^2$ (with the order $\preceq$) corresponds bijectively to a composition: the parts of the composition are the number of points in the ideal with equal y-coordinate, starting from the bottom ($y = 1$); see Figure 4.11 for an example, with the composition $(2, 2, 3, 1)$.

The order induced on compositions[2] by the inclusion of finite lower ideals of $\mathbb{P}^2$ is easily described by its covering relation $\to$: $C \to C'$ if and only if either C' is obtained by increasing one part of C by 1, or if C' is obtained by adding the new part 1 at the end of C (so that the number of covering compositions of C is one more than the number of parts of C). For example, $(2, 2, 3, 1) \to (2, 3, 3, 1)$ and $(2, 2, 3, 1) \to (2, 2, 3, 1, 1)$.

Note that the set of finite lower ideals of $\mathbb{P}^2$, denoted $\mathcal{I}$, is a lattice for the inclusion order. For simplicity, we say *ideal* instead of "finite lower ideal of $\mathbb{P}^2$".

Let I be an ideal of $(\mathbb{P}, \preceq)$, or equivalently, a composition. Consider an $\mathcal{A}$-*labelling* of I, with I considered as a poset with the order $\preceq$. To such a labelling $I \to \mathcal{A}$, we associate the partition $\{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$ of $\mathcal{A}$,

[2] Another order on compositions, with more covering relations, has been considered in [18].

where $\mathcal{B}_i$ is the set of labels of the points in I with y-coordinate i; in the example of Figure 4.12, one has $k = 4$ and $\mathcal{B}_1 = \{1,3\}, \mathcal{B}_2 = \{2,8\}, \mathcal{B}_3 = \{4,5,7\}, \mathcal{B}_4 = \{6\}$. Observe that one has necessarily $\min(\mathcal{B}_1) < \min(\mathcal{B}_2) < \cdots < \min(\mathcal{B}_k)$ since the labelling is increasing.

Note that a (set-theoretical) partition on a finite totally ordered set $\mathcal{A}$ may be uniquely represented by the sequence of its blocks $(\mathcal{B}_1, \dots, \mathcal{B}_k)$ with $\min(\mathcal{B}_1) < \min(\mathcal{B}_2) < \cdots < \min(\mathcal{B}_k)$. It follows that increasing $\mathcal{A}$-labellings of ideals of $\mathbb{P}^2$, of cardinality $|\mathcal{A}|$, correspond bijectively to partitions of $\mathcal{A}$. We call I the *shape* of the partition, if the latter corresponds to an $\mathcal{A}$-labelling of I.

Call *path* in a poset a sequence of elements such that each element covers the previous one. Note that an $\mathcal{A}$-labelling (hence a partition) is equivalent to a path in $\mathcal{I}$, starting from the singleton $\{(1,1)\}$; equivalently, to a path of compositions $C_1 \to \cdots \to C_k$ with $C_1 = (1)$; for example, in Figure 4.12, it is the sequence $(1) \to (1,1) \to (2,1) \to (2,1,1) \to (2,1,2) \to (2,1,2,1) \to (2,1,3,1) \to (2,2,3,1)$.

Given two ideals I, J in $\mathcal{I}$, the set $I \setminus J$ will be called a *skew ideal*; clearly, one may assume that $J \subseteq I$, what we assume in the sequel. If $S = I \setminus J$ is a skew ideal, then a point $H \in S$ such that $S \setminus H$ is still a skew ideal is called a *corner* of S. We call it a *lower corner* if $J \cup H$ is an ideal, and an *upper corner* if $I \setminus H$ is an ideal. For example, in Figure 4.13, with S the set of labelled points, the lower corners are $(1,3), (2,2), (3,1)$ and the upper corners are $(1,4), (3,3), (4,1)$.

A *skew partition* is an $\mathcal{A}$-labelling of a skew ideal; the latter is called its *shape*. Equivalently, a skew partition is an upwards path in the Hasse diagram of $\mathcal{I}$; equivalently, an upwards path in the Hasse diagram of compositions with the order $\preceq$. See Figure 4.13, where the sequence of compositions is $(2,1) \to (2,1,1) \to (3,1,1) \to (3,1,2) \to (3,2,2) \to (3,3,2) \to (4,3,2) \to (4,3,2,1) \to (4,3,3,1)$.

We call *pointed skew ideal* a pair (S, H) of a skew ideal S, together with some point $H \in S$.

Finally, we call *skew partition with a hole* an $\mathcal{A}$-labelling of subset $S \setminus H$, where (S, H) is a pointed skew ideal. We call S the *shape* and H the *hole*; note that the hole has no label. We call the hole *upper* (resp. *lower*) if H is an upper (resp. lower) corner of S; otherwise, the hole is *inner*. For example, in Figure 4.14, the hole is the point of coordinates $H = (1,3)$, indicated by a $\circ$, and is inner.

```
6
|
4 − 5 − 7
|
2 − 8
|
1 − 3
```

Figure 4.12

```
7
|
1 − 3 − 8
|
∗ − 4 − 5
|
∗ − ∗ − 2 − 6
```

Figure 4.13

```
7
|
◦ − 3 − 8
|
1 − 4 − 5
|
∗ − ∗ − 2 − 6
```

Figure 4.14

```
7
|
3 − ◦ − 8
|
1 − 4 − 5
|
∗ − ∗ − 2 − 6
```

Figure 4.15

4.11.3 Jeu de taquin on skew partitions

Given a skew partition with a hole S, we define two types of moves, which change it into another skew partition, with or without hole.

The *downward move* is defined as follows. If H is an upper hole, one removes it and one obtains a skew partition (without hole). If H is not an upper hole, then there may be one or two points in S covering H. In the first case, the point K covering H becomes the new hole, and H gets the label previously on K. In the second case, let K, L be the two points, with respective labels x, y and suppose that $x < y$ in $\mathcal{A}$; then K becomes the new hole, and x becomes the new label of H. One obtains a new skew partition with a hole. For example, the downward move applied in Figure 4.14 gives Figure 4.15. Observe that the hole of the new skew diagram is further from the minimum $(1, 1)$ in the Hasse diagram of $\mathbb{P}^2$.

The *upward move* is defined similarly.

A *downward slide* on a skew partition R is defined as follows; let $I \setminus J$ be its shape. If J is empty, then R is a partition and the slide is completed, producing R. If J is nonempty, choose a point H that is a maximal element in J. Then $(H \cup (I \setminus J), H)$ is a pointed skew diagram, with lower corner H, and R together with H is a skew partition with the hole H. We then apply iteratively downward moves, until one obtains a skew partition without hole. The fact that this ends in finitely many steps follows from the observation above. Observe that the new skew diagram is of the form $I' \setminus J'$, where $J' = J \setminus H$. Each downward slide is determined on the initial skew partition R by a *trail*, which is the set of labels obtained starting form H and choosing iteratively the smallest label among the covering points; see Figure 4.16, where the trail is indicated by bold numbers; the slide is then obtained by sliding downward (in the poset) the labels in the trail, see Figure 4.17.

Finally, *downward jeu de taquin* on a skew partition is applying to it iteratively a sequence of downward slides until a partition is obtained. Note that there are several ways to do it, since one has to choose a point H for each slide, and there may be several choices. The final partition is however unique, as stated below.

An example is given in Figures 4.16 - 4.19. Each slide is indicated by its trail in bold.

$$\begin{array}{ccccccc}
7 & & & & & & \\
| & & & & & & \\
\mathbf{1} & - & \mathbf{3} & - & \mathbf{8} & & \\
| & & & & & & \\
\circ & - & 4 & - & 5 & & \\
| & & & & & & \\
* & - & * & - & 2 & - & 6
\end{array}$$

Figure 4.16

$$\begin{array}{ccccccc}
7 & & & & & & \\
| & & & & & & \\
3 & - & 8 & & & & \\
| & & & & & & \\
1 & - & 4 & - & 5 & & \\
| & & & & & & \\
* & - & \circ & - & \mathbf{2} & - & \mathbf{6}
\end{array}$$

Figure 4.17

$$\begin{array}{ccccc}
\mathbf{7} & & & & \\
| & & & & \\
\mathbf{3} & - & 8 & & \\
| & & & & \\
\mathbf{1} & - & 4 & - & 5 \\
| & & & & \\
\circ & - & 2 & - & 6
\end{array}$$

Figure 4.18

$$\begin{array}{ccccc}
7 & - & 8 & & \\
| & & & & \\
3 & - & 4 & - & 5 \\
| & & & & \\
1 & - & 2 & - & 6
\end{array}$$

Figure 4.19

Define for each word W the increasing rearrangement $\overline{W}$ of W; for example, $\overline{bacbdbc} = abbbccd$. For each skew partition R, with or without hole, we define its *row-word* $r(R)$ as follows: suppose that the shape of R is the skew ideal $I \setminus J$, where the largest y-coordinate of a point in I is k; denote by U_i the word obtained by reading from left to right the labels in R located in the line of y-coordinate i. Then

$$r(R) = U_k \overline{U_{k-1} U_k} \cdots \overline{U_1 \cdots U_{k-1} U_k}.$$

Note that U_k is already increasing, since the labelling is. For example, for the skew partition in Figure 4.13, its row-word is 7 1378 134578 12345678, while the row-word of the skew partition with a hole of Figure 4.14 is 7 378 134578 12345678. Observe that this definition is such that for a partition R, corresponding to the N-tableau T, one has $r(R) = r(T)$, as follows from Lemma 4.20.

Theorem 4.26 *The partition obtained by downward jeu de taquin from a skew partition is independant of the choices of the lower corners during the algorithm.*

Lemma 4.27 *Let $a \in \mathcal{A}, U \in \mathcal{A}^*$ be such that each letter in U is greater or equal than a. Then $aUa \equiv_{styl} Ua$.*

Proof. It is enough to show that for each column γ, $(aUa) \cdot \gamma = (Ua) \cdot \gamma$. This is equivalent to the fact that a fixes $(Ua) \cdot \gamma$, which will follow, by Lemma 4.2 (ii), from the fact that a appears in the column $(Ua) \cdot \gamma$. Now, a appears in $a \cdot \gamma$; and, since the letters in U are all $\geq a$, using recursively Lemma 4.2 (ii) and (iii), we obtain that a appears in $(Ua) \cdot \gamma$. □

Proof. [Proof of Theorem 4.26] Let R be a skew partition and R_0 a partition obtained by downward jeu de taquin applied to R, for some choices of the lower corners. We claim that

$$r(R) \equiv_{styl} r(R_0). \tag{4.11}$$

The claim being admitted, suppose that we obtain another partition R_1 by downward jeu de taquin; by the claim, we have $r(R) \equiv_{styl} r(R_1)$. Let T_i be the N-tableau corresponding to the partition R_i through the natural bijection of Lemma 4.20. Then by the observation before the theorem, we have $r(T_0) \equiv_{styl} r(R_0) \equiv_{styl} r(R) \equiv_{styl} r(R_1) \equiv_{styl} r(T_1)$. Thus $T_0 = T_1$ by Corollary 4.21, and finally $R_0 = R_1$ by Lemma 4.20.

We prove now the claim. It is enough to prove that the stylic class of the row-word is invariant under downward moves of skew partitions with holes. Thus let $R' \to R''$ be such a move. The two cases two consider are: (1) shifting the hole to the right; (2) shifting the hole above.

In case (1), the row word does not change. In case (2), let i and $i+1$ the indices of the rows where the move occurs; note that the hole in R' is then in the first column (x-coordinate 1) and in row i. Denote by U_j the row-word of row j of R'. Then $U_{i+1} = av$, with a smaller than each letter in V, U_i, $U_{i+2}, U_{i+3}, \ldots$. The row-word of the i-th and $i+1$-th rows of R'' are aU_i and V respectively. For $j \neq i, i+1$, the rows of R' and R'' are identical. Let k be the number of rows in R' and R'' (row k of R'' may be empty, when $i+1=k$, but this does not change the argument that follows).

For some words X, Y,

$$r(R') = X(\overline{U_{i+1}U_{i+2}\cdots U_k})(\overline{U_iU_{i+1}\cdots U_k})y,$$

and

$$r(R'') = X(\overline{VU_{i+2}\cdots U_k})(\overline{aU_iVU_{i+2}\cdots U_k})y.$$

Thus it is enough to show that

$$(\overline{U_{i+1}U_{i+2}\cdots U_k})(\overline{U_iU_{i+1}\cdots U_k}) \equiv_{styl} (\overline{VU_{i+2}\cdots U_k})(\overline{aU_iVU_{i+2}\cdots U_k}).$$

But the left word is $(\overline{aVU_{i+2}\cdots U_k})(\overline{U_iaVU_{i+2}\cdots U_k}) = a(\overline{VU_{i+2}\cdots U_k})a(\overline{U_iVU_{i+2}\cdots U_k})$ and the right word is $(\overline{VU_{i+2}\cdots U_k})a(\overline{U_iVU_{i+2}\cdots U_k})$. Thus the congruence follows from Lemma 4.27. $\square$

4.11.4 Properties of the mappings Δ and π.

The operator Δ of Section 4.11.1 may be computed as follows: let R be a partition of a subset of $\mathcal{A}$, viewed as in Section 4.11.2 as an $\mathcal{A}$-labelling of an ideal in $\mathbb{P}^2$. Note that $a = \min(R)$ is in position $(1,1)$; remove it from the labels, obtaining a skew partition $R \setminus a$. Then $\Delta(R)$ is the partition obtained by downward jeu de taquin on $R \setminus a$. See Figures 4.20 and 4.21 for an example, which is the same as the one illustrating the definition of Δ in Section 4.11.1.

Consider a nonempty word $W \in \mathcal{A}^*$, and let x denote a letter appearing in W. Denote by $W \setminus x$ the word obtained by removing all x's from W.

$$\begin{array}{ccccc} 6 & & & & \\ | & & & & \\ \mathbf{4} & - & \mathbf{5} & - & \mathbf{7} \\ | & & & & \\ \mathbf{2} & - & 8 & & \\ | & & & & \\ \circ & - & 3 & & \end{array}$$

Figure 4.20

$$\begin{array}{ccc} 6 & & \\ | & & \\ 5 & - & 7 \\ | & & \\ 4 & - & 8 \\ | & & \\ 2 & - & 3 \end{array}$$

Figure 4.21

Recall that if two words are equal modulo $\equiv_{styl}$, then they have the same underlying alphabet, and in particular the same smallest letter. The next result shows the compatibility of the operations of removing the smallest letter, and the link with Δ. Recall that for any word W, $\pi(W)$ is the partition bijectively associated to the N-tableau $\mathrm{N}(W)$.

Lemma 4.28 *(i) If $U \equiv_{styl} V$, with smallest letter a, then $U \setminus a \equiv_{styl} V \setminus a$; in particular, $\pi(U \setminus a) = \pi(V \setminus a)$. The same holds when removing the largest letter.*

(ii) If a is the smallest letter in W, then $\pi(W \setminus a) = \Delta(\pi(W))$.

(iii) If z is the largest letter of W, then $\pi(W \setminus z) = \pi(W) \setminus z$.

Proof. (i). The stylic congruence is generated by the plactic relations and the idempotence relations. Therefore, it suffices to prove the statement when U, V differ by an elementary step of this congruence, and we may assume that this step involves an a. If it is a plactic step, then since a is the smallest letter, the step amounts to replace aba (resp. bab, resp. acb, resp. bac) by baa (resp. bba, resp. cab, resp. bca) in one of the words U, V, obtaining the other (we have $a < b < c$); this step becomes the identity when the a's are removed. If the step is replacing aa by a, or conversely, then it becomes the identity too, when the a's are removed.

The second assertion follows from the bijection π between the stylic monoid and the set of partitions of subsets of $\mathcal{A}$. The last one by symmetry.

(ii). We have by (4.5) and the definition of the mapping r on partitions, $w \equiv_{styl} r(\mathrm{N}(W)) = r(\pi(W))$. By (i) we have $W \setminus a \equiv_{styl} r(\pi(W)) \setminus a$. Now, by the definition of r, we have $\mathcal{R}(\pi(W)) \setminus a = r(\pi(W) \setminus a)$; here $\pi(W) \setminus a$ denotes the skew partition, obtained by removing a from the partition $\pi(W)$. We now apply downward jeu de taquin to $\pi(W) \setminus a$, obtaining the partition R_0; the latter is by what we have seen above equal to $\Delta(\pi(W))$. By (4.11), $r(\pi(W) \setminus a) \equiv_{styl} r(R_0)$. Thus finally, $W \setminus a \equiv_{styl} r(\Delta(\pi(W)))$, and therefore $\pi(W \setminus a) = \pi(r(\Delta(\pi(W)))) = \Delta(\pi(W))$, the last equality by Lemma 4.16.

(iii). We claim that $\delta(W \setminus z) = \delta(W) \setminus z$. The claim being admitted, (iii) follows by induction from Lemma 4.10.

We prove the claim by induction on $|W|$. If W is empty it is clear. So we may assume that $(*)$ $\delta(W \setminus z) = \delta(W) \setminus z$ and we prove it for Wx, x being some letter. We have $\delta((Wx) \setminus z) = \delta((W \setminus z)(x \setminus z)) = \delta(W \setminus z)t$, where $t = 1$ if $x = z$, and $t = x^{\uparrow}_{\mathrm{Supp}(W \setminus z)}$ if $x < z$. On the other hand, $\delta(Wx) = \delta(W) x^{\uparrow}_{\mathrm{Supp}(W)}$, hence $\delta(Wx) \setminus z = (\delta(W) \setminus z)(x^{\uparrow}_{\mathrm{Supp}(W)} \setminus z)$. By $(*)$, it is therefore enough to show that $t = x^{\uparrow}_{\mathrm{Supp}(W)} \setminus z$. If $x = z$, both sides are equal to 1, since z is the maximum letter. Suppose now that $x < z$. We have to show that $(**)$ $x^{\uparrow}_{\mathrm{Supp}(W \setminus z)} = x^{\uparrow}_{\mathrm{Supp}(W)} \setminus z$. If there exists an element y in $\mathrm{Supp}(W)$ such that $x < y < z$, then, taking y minimum, both sides of $(**)$ are equal to y; if no such y exists, then both sides are equal to 1, because $x^{\uparrow}_{\mathrm{Supp}(W)} = z$ or 1. $\square$

4.11.5 Growth diagram

Recall that a partition on $\mathcal{A}$ is equivalent to a path in the Hasse diagram of the poset of compositions, see Section 4.11.2. Given a partition R on $\mathcal{A}$, consider the sequence of partitions R, $\Delta(R)$, $\Delta^2(R)$,…,$\Delta^n(R)$, with $n = |\mathcal{A}|$; note that these partitions are on different sets. Draw from left to right the n paths of compositions associated with these partitions on a pyramid, each path being represented diagonally upwards, direction north-east; see Figure 4.22, looking only at the north-east arrows $\nearrow$, and disregarding the north-west arrows $\nwarrow$. For example, the path $1 \to 11 \to 12 \to 121 \to 221 \to 222$ is associated with the partition $R = 15/23/46$, and the path $1 \to 11 \to 111 \to 211 \to 212$ is associated with the partition $25/3/46 = \Delta(R)$.

We complete this diagram by adding north-west arrows $\nwarrow$, see the figure; at this point it is not clear that these arrows are also covering relations, but it will be proved soon. We call this the *evacuation pyramid of* R. It follows from the definition of the evacuation that the right side of the pyramid (which goes north-west) represents the path of compositions associated to $\mathrm{evac}(R)$.

Note that the pyramid is formed of rhombuses, that we describe now (the situation, following the work of Sergey Fomin, is quite similar to the one of standard Young tableaux and partitions of integers, see [72, Proposition A1.2.7]).

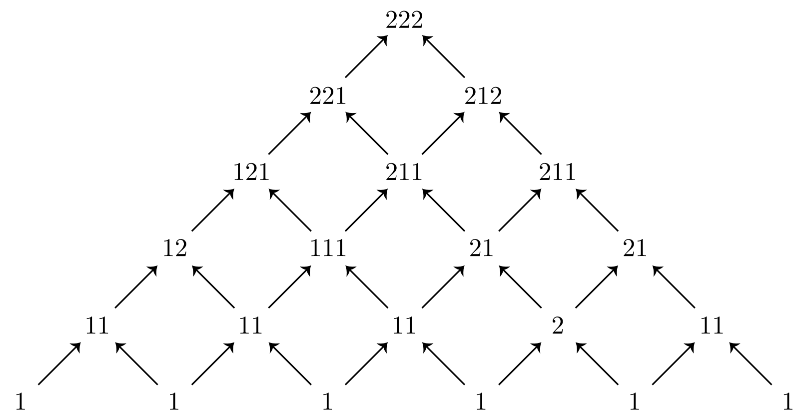

Figure 4.22: Growth diagram: evacuation of partition $15/23/46$

Before that, we describe the intervals of length 2 in the poset of compositions. By inspecting the definition of the covering relation in this poset, one sees that such an interval is always of cardinality 3 or 4; that is, if $C_1 \to C_2 \to C_3$, then either C_2 is unique and we let $C_2' = C_2$, or there is another composition C_2' such that $C_1 \to C_2' \to C_3$. We take this notation below.

Proposition 4.29 *Each arrow in the evacuation pyramid of R is a covering relation of the poset of compositions. The pyramid may be recursively constructed, starting from the bottom row and the leftmost path by applying the following rule: if the two leftmost arrows $C_1 \to C_2 \to C_3$ of a rhombus are known, then the missing composition of the rhombus is C_2'.*

Proof. We claim that: for each rhombus in the pyramid: (i) all its sides are covering relations; and (ii) if its leftmost arrows are $C_1 \to C_2 \to C_3$, then the fourth composition is C_2'.

To prove the claim, by construction of the pyramid, it is enough to prove it for a rhombus located on the two leftmost north-east paths. Also, since the pyramid obtained by removing the largest element of $\mathcal{A}$ is obtained by removing the rightmost north-east sequence, it is enough to prove the claim for the upper rhombus in the pyramid.

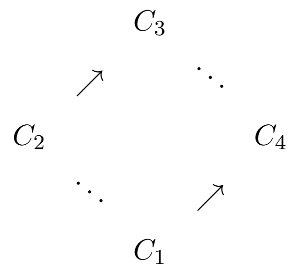

Figure 4.23: The upper rhombus

Denote by C_1, C_2, C_3, C_4 the compositions in this rhombus, as indicated in Figure 4.23.

Let z be the largest letter in $\mathcal{A}$ and x be the letter in $\mathcal{A}$ which is the last label in the trail determined by the downward slide in the computation of $\Delta(R)$ ($x = 7$ in Figure 4.20). Observe that: C_3 is the shape of R; C_2 is the shape of $R \setminus z$; C_4 is the shape of $\Delta(R)$, that is, of $R \setminus x$; C_1 is the shape of $\Delta(R) \setminus z$.

Suppose first that $x \neq z$. Then x, z lie in different parts a, b (respectively) of C_3 (identifying parts of a composition with a horizontal subset with given y-coordinate of an ideal, as in Figure 4.11), and it follows that: C_2 is C_3 with b replaced by $b-1$; C_4 is C_3 with a replaced by $a-1$; C_1 is C_4 with b replaced by $b-1$, hence C_1 is also C_2 with a replaced by $a-1$ (we leave to the reader the subcases where $a = 1$ or $b = 1$; note that $a = b = 1$ cannot occur since $x \neq z$). Thus, the interval $[C_1, C_3]$ is equal to $\{C_1, C_2, C_4, C_3\}$ of cardinality 4, which proves the claim in this case.

Suppose now that $x = z$. Then z lies in a part b of C_3 and it follows that: C_2 is C_3 with b replaced by $b-1$, as is C_4; C_1 is C_4 with the part $b-1$ replaced by $b-2$ (the particular cases where $b = 1$ or 2 are also left to the reader). Thus the interval $[C_1, C_3]$ is equal to $\{C_1, C_2, C_3\}$ of cardinality 3, which proves the claim in this case. □

Corollary 4.30 *Let R be a partition on $\mathcal{A}$ and z be the largest letter in $\mathcal{A}$. Then* $\mathrm{evac}(R \backslash z) = \Delta(\mathrm{evac}(R))$.

Proof. By the previous proposition, the construction of the evacuation pyramid of R is left-right symmetric. Thus, since the first north-west path from the right (the right side of the pyramid) represents $\mathrm{evac}(R)$, the second one represents the partition $\Delta(\mathrm{evac}(R))$. But the evacuation pyramid of $R \backslash z$ is obtained by removing form the whole pyramid its right side. The equality follows. □

4.11.6 Proof of the evacuation theorem

Lemma 4.31 *Let R_1, R_2 be two partitions on $\mathcal{A}$, with largest element z. Suppose that R_1, R_2 have the same number of blocks, that $R_1 \setminus z = R_2 \setminus z$ and that $\Delta(R_1) = \Delta(R_2)$. Then $R_1 = R_2$.*

Proof. Suppose that $R_1 \neq R_2$. Then, identifying partitions and labelled ideals in $\mathbb{P}^2$, z has y-coordinate y_i in R_i ($i = 1, 2$), and $y_1 \neq y_2$. We have $\Delta(R_1) = \Delta(R_2)$. Thus for at least one of the partitions R_i, the y-coordinate of z in R_i and in $\Delta(R_i)$ must differ, and we may assume that $i = 1$; then the trail corresponding to the computation of $\Delta(R_1)$ is the first column of R_1, ending at z, and the corresponding row of R_1 contains only z. Thus $\Delta(R_1)$ has one row less than R_1 and z is in the upper row, and first column, of $\Delta(R_1) = \Delta(R_2)$. Since R_1, R_2 have the same number of rows, $\Delta(R_2)$ has one row less than R_2, too; this is possible if and only if the trail in R_2 is the first column and z is at the top; then $y_1 = y_2$, a contradiction. □

Proof. [Proof of Theorem 4.24] The proof is by double induction on $|\mathcal{A}|$ and $|w|$. The theorem is clear if $\mathcal{A}$ is empty. Now let $\mathcal{A}$ be nonempty, with a, z respectively the smallest and largest element. Let $W \in \mathcal{A}^*$.

(1). We suppose first that a, z appear in W. By induction on the length of W, we have

$$\pi(\theta(W \setminus a)) = \text{evac}(\pi(W \setminus a)), \pi(\theta(W \setminus z)) = \text{evac}(\pi(W \setminus z)).$$

Let $R_1 = \pi(\theta(W))$ and $R_2 = \text{evac}(\pi(W))$. We have to show that $R_1 = R_2$ and do it by verifying the hypothesis of Lemma 4.31.

First, note that, for any word U, the number of blocks of $\pi(U)$ is equal to the length of the first column of $\mathrm{N}(U)$, hence to the length of the first column of $\mathrm{P}(U)$, by Lemma 4.13 (i). This is by Schensted's theorem equal to the length of the longest strictly decreasing subword of U. Now, the lengths of the longest strictly decreasing subword of W and of $\theta(W)$ are clearly equal. It follows that $\pi(W)$ and $\pi(\theta(W))$ have the same number of blocks. Moreover the shape of $\text{evac}(\pi(W))$ is equal to that of $\pi(W)$. Hence R_1 and R_2 have the same number of blocks.

We show now that $R_1 \setminus z = R_2 \setminus z$. We have $\pi(\theta(W)) \setminus z = \pi(\theta(W) \setminus z)$ (by Lemma 4.28 (iii)) $= \pi(\theta(W \setminus a)) = \text{evac}(\pi(W \setminus a))$ (see above) $= \text{evac}(\Delta(\pi(W)))$ (by Lemma 4.28 (ii)) $= \text{evac}(\pi(W)) \setminus z$, by (4.8).

We show now that $\Delta(R_1) = \Delta(R_2)$. By Lemma 4.28 (ii), $\Delta(R_1) = \Delta(\pi(\theta(W))) = \pi(\theta(W) \setminus a)$. This is equal to $\pi(\theta(W \setminus z))$. By the above displayed equation, this is $\mathrm{evac}(\pi(W \setminus z))$. By Lemma 4.28 (iii), this is equal to $\mathrm{evac}(\pi(W) \setminus z)$ and finally, by Corollary 4.30, to $\Delta(\mathrm{evac}(\pi(W))) = \Delta(R_2)$.

(2). Suppose now that a does not appear in W. Then by induction on the cardinality of the alphabet, we have $\pi(\theta_{\mathcal{A}\setminus a}(W)) = \mathrm{evac}_{\mathcal{A}\setminus a}(\pi(W))$. Thus, applying i_a on both sides, using (4.6) and (4.9), and noting that i_a commutes with π (the latter is defined on each alphabet), we obtain the theorem.

If z does not appear in W, the argument is similar. □

4.12 Ordering columns

Following [53], there is a natural order on columns, as follows: $\gamma_1 \leq \gamma_2$ if they are nonempty and if there is a tableau having the two columns γ_1 and γ_2, from left to right. For the empty column $\emptyset$, we define $\gamma \leq \emptyset$ for any column. For example, looking at Figure 4.1, and viewing columns as decreasing words, we see that $dba \leq ba \leq c$.

Equivalently also, $\gamma_1 \leq \gamma_2$ if and only if there is a regressive injective mapping from γ_2 into γ_1 (a function f is *regressive* if $f(x) \leq x$). Note that this order extends the order of $\mathcal{A}$, and also the reverse inclusion order of the subsets of $\mathcal{A}$ [53].

This order on columns is compatible with the action, as follows.

Proposition 4.32 *(i) For each column γ and each word W, one has $W \cdot \gamma \leq \gamma$.*

(ii) For any columns γ_1, γ_2, and each word W, $\gamma_1 \leq \gamma_2$ implies $W \cdot \gamma_1 \leq W \cdot \gamma_2$.

The next lemma is due to Bokut, Chen, Chen, Li ([21, Lemma 4.1]), in a formulation communicated to us by Darij Grinberg; moreover, Lemma 4.34 is due to him, together with the proof of the second part of Proposition 4.32, which simplifies our first version.

For any column γ and any letter $a \in \mathcal{A}$, define $L_a(\gamma)$ to be the number of letters $\leq a$ in γ.

Lemma 4.33 *Two columns γ_1 and γ_2 over $\mathcal{A}$ satisfy $\gamma_1 \leq \gamma_2$ if and only if each $a \in \mathcal{A}$ satisfies $L_a(\gamma_1) \geq L_a(\gamma_2)$.*

Lemma 4.34 *Let $\mathcal{A}$ be an alphabet, γ a column over $\mathcal{A}$ and $x, a \in \mathcal{A}$. Let $y = \max(\ell \in \gamma | \ell \leq a)$, if this set is nonempty, and otherwise, let $y = -\infty$ (smaller than any element in $\mathcal{A}$). Then:*

(i) if $a < x$, then $L_a(x \cdot \gamma) = L_a(\gamma)$;

(ii) if $y < x \leq a$, then $L_a(x \cdot \gamma) = L_a(\gamma) + 1$.

(iii) if $x \leq y$, then $L_a(x \cdot \gamma) = L_a(\gamma)$;

Proof. (i). In this case, x does not bump any $\ell \leq a$ in γ. Therefore, the number of letters $\leq a$ remains the same.

(ii). In this case, x is either going to bump a letter $> a$ or will be added at the top of γ. In either cases, because $x \leq a$, x is added to the count of letters $\leq a$. Therefore $L_a(x \cdot \gamma) = L_a(\gamma) + 1$.

(iii). In this case, x will bump a letter that is $\leq a$. The number of letters $\leq a$ remains the same. □

Proof. [Proof of Proposition 4.32] It is enough to prove both properties when $W = x \in \mathcal{A}$.

(i). We refer to the definition of the column insertion of x into γ in Section 4.4. In the first case, $x \cdot \gamma$ contains γ and the result follows. In the second case, we have, viewing columns as decreasing words, $\gamma = UyV$ and $x \cdot \gamma = UxV$, with $y \in \mathcal{A}$ and $x \leq y$; the result follows.

(ii). Let $a \in \mathcal{A}$. Note that by Lemma 4.34, $L_a(x \cdot \gamma_i) \geq L_a(\gamma_i)$. Using the fact that $\gamma_1 \leq \gamma_2$ and Lemma 4.33, if $L_a(x \cdot \gamma_2) = L_a(\gamma_2)$ we obtain

$$L_a(x \cdot \gamma_2) = L_a(\gamma_2) \leq L_a(\gamma_1) \leq L_a(x \cdot \gamma_1).$$

In the same way, if $L_a(x \cdot \gamma_1) > L_a(\gamma_1)$ we obtain

$$L_a(x \cdot \gamma_2) \leq L_a(\gamma_2) + 1 \leq L_a(\gamma_1) + 1 \leq L_a(x \cdot \gamma_1).$$

There remains only one case to verify: $L_a(x\cdot\gamma_1) = L_a(\gamma_1)$ and $L_a(x\cdot\gamma_2) = L_a(\gamma_2)+1$. Let $y_1 = \max(\ell \in \gamma_1 | \ell \leq a)$ and $y_2 = \max(\ell \in \gamma_2 | \ell \leq a)$, with the same convention as for y in Lemma 4.34.

We know that $L_a(\gamma_1) \geq L_a(\gamma_2)$. If we have strict inequality, then $L_a(\gamma_1) \geq L_a(\gamma_2)+1$, hence $L_a(x\cdot\gamma_1) = L_a(\gamma_1) \geq L_a(\gamma_2) + 1 = L_a(x \cdot \gamma_2)$ and we are done.

Thus we may assume that $L_a(\gamma_1) = L_a(\gamma_2)$; then the height of y_1 in γ_1 is equal to the height of y_2 in γ_2, or they are both $-\infty$. Therefore, because $\gamma_1 \leq \gamma_2$, we have $y_1 \leq y_2$. Now, using Lemma 4.34, $L_a(x \cdot \gamma_2) = L_a(\gamma_2) + 1$ implies that $y_2 < x \leq a$. Thus $y_1 < x \leq a$. We obtain by Lemma 4.34 (ii) that $L_a(x \cdot \gamma_1) = L_a(\gamma_1) + 1$ and

$$L_a(x \cdot \gamma_1) = L_a(\gamma_1) + 1 = L_a(\gamma_2) + 1 = L_a(x \cdot \gamma_2).$$

This conclude the proof. □

4.13 $\mathscr{J}$-relations on the stylic monoid

4.13.1 $\mathscr{J}$-triviality

Recall that a monoid **M** is called *$\mathscr{J}$-trivial* if for any elements $u, v \in \mathbf{M}$ such that $\mathbf{M}U\mathbf{M} = \mathbf{M}V\mathbf{M}$, one has $U = V$.

Theorem 4.35 $\mathrm{Styl}(\mathcal{A})$ *is a $\mathscr{J}$-trivial monoid.*

Proof. We mimick the proof of Proposition 4.15 in [67] of Pin. Suppose that U, V are words such that $\mathbf{M}\mu(U)\mathbf{M} = \mathbf{M}\mu(V)\mathbf{M}$, with $\mathbf{M} = \mathrm{Styl}(\mathcal{A})$. Then for some words X, Y, $V \equiv_{styl} XUY$. For any column γ, we have by Proposition 4.32, $\gamma \geq Y \cdot \gamma$, thus $U \cdot \gamma \geq UY \cdot \gamma \geq XUY \cdot \gamma = V \cdot \gamma$. Symmetrically, $V \cdot \gamma \geq U \cdot \gamma$. Thus $V \cdot \gamma = U \cdot \gamma$. This implies that $U \equiv_{styl} V$ and $\mu(U) = \mu(V)$. □

In a $\mathscr{J}$-trivial monoid, one defines the *$\mathscr{J}$-order* $\leq_{\mathscr{J}}$ by: $U \leq_{\mathscr{J}} V$ if and only if $U \in \mathbf{M}V\mathbf{M}$. We study this order below.

4.13.2 Left N-insertion

We describe now an algorithm which constructs, given a letter x and an N-tableau T, an N-tableau denoted $x \to T$, and which will be shown to correspond to left multiplication by x in the stylic monoid. This will serve us to prove that the $\mathscr{J}$-order is graded (Theorem 4.41).

Let the rows of T be $R_1, \dots, R_k$ (from the lowest one to the highest), which we also view as subsets of $\mathcal{A}$. Let $p_i = \min(R_i)$, the leftmost element in the row R_i; in particular, p_1 is the minimum of all elements in T. For each $i = 1, \dots, k$, let y_i be the smallest element in R_i which is greater than x, if it exists; we write $y_i = \emptyset$ if it does not exist, and $y_i \neq \emptyset$ to express that it exists. Define also r to be the largest i such that $x \in R_i$; if no such i exists, we put $r = 0$.

Case 1: if $x < p_1$, that is, x is smaller than any element in T, then $x \to T$ is obtained by replacing R_1 by $R_1 \cup x$.

Case 2: if x is equal to some p_i, that is, if x appears in the first column of T, then $(x \to T) = T$.

Case 3: we assume now that we are not in Case 1 nor 2. Then we have $x > p_1$.

Subcase 3.1: if $x > p_k$, we let $t = k + 1$ and R_{k+1} be a new empty row.

Subcase 3.2: if $x \leq p_k$, we let t be minimum with $x \leq p_t$. Then $t \leq k$, and $x < p_t$ since we are not in Case 2.

In both subcases $x \notin R_t$. Hence, since any element appearing in a row of an N-tableau also appears in lower rows, we must have $r < t$. Moreover, in both subcases, $p_{t-1} < x$.

In Case 3 (both subcases), $x \to T$ is obtained from T by performing the two following operations (which commute):

(i): add x to the rows $R_{r+1}, \dots, R_t$ (which we call the *active rows*, since only these rows are modified);

Step (ii): for i satisfying $r + 2 \leq i \leq t$, remove y_i from R_i if $\emptyset \neq y_i = y_{i-1} \neq \emptyset$.

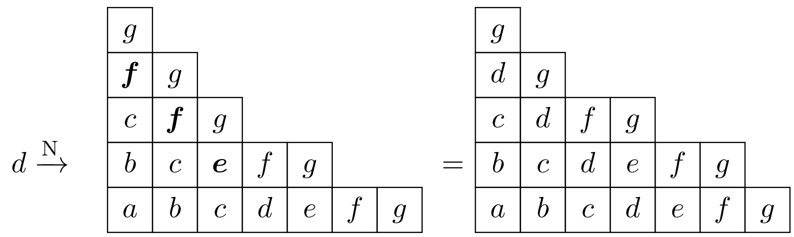

Figure 4.24: Left insertion

See Figure 4.24 for an example: $x = d, r = 1, t = 4$, the active rows are $R_2, R_3, R_4, y_4 = f, y_3 = f, y_2 = e$, hence y_4 disappears after left insertion of d, and d is added in rows 2,3,4.

For later use, define $\mathcal{Y}(x,T) = \{y \mid \exists i, r+1 \leq i \leq t, y = y_i \neq \emptyset\}$. If $\mathcal{Y}(x,T)$ is empty, Step (ii) in Case 3 of the algorithm is empty. If $\mathcal{Y}(x,T)$ is nonempty, let s be the largest i such that $r+1 \leq i \leq t$, and that y_i exists. Then $\mathcal{Y}(x,T) = \{y_i, i = r+1, \ldots, s\}$ and Step (ii) of Case 3 is restricted to the i's satisfying $r+2 \leq i \leq s$.

One notes also that if $t \leq k$, then $p_t > x$, $s = t$, and $y_t = p_t$.

Proposition 4.36 *For T a* N*-tableau and x a letter, we have $x \rightarrow T$ is a* N*-tableau.*

We begin by a simple lemma, whose proof is left to the reader.

Lemma 4.37 *Let $r < s$, and let $E_{r+1} \supseteq E_{r+2} \supseteq \ldots \supseteq E_s$ be a decreasing sequence of subsets of a totally ordered set, with minima $y_{r+1}, \ldots, y_s$. Define $E'_{r+1} = E_{r+1}$, and for $i = r+2, \ldots, s$, $E'_i = E_i$ if $y_i \neq y_{i-1}$, and $E'_i = E_i \setminus y_i$ if $y_i = y_{i-1}$. Then $E'_{r+1} \supseteq E'_{r+2} \supseteq \ldots \supseteq E'_s$.*

Proof. [Proof of Proposition 4.36] The only nontrivial case to consider is Case 3. Recall that the sequence of sets R_i is by definition decreasing. Denote by $R'_1, R'_2, \ldots$ the rows of $x \rightarrow T$. We verify first that this sequence of sets is decreasing. It is enough to show it separately for the three sequences of subsets $R'_i \cap \{c \in \mathcal{A} \mid c < x\}$, $R'_i \cap \{x\}$ and $R'_i \cap \{c \in \mathcal{A} \mid c > x\}$. For the first sequence, it follows from the equality $R'_i \cap \{c \in \mathcal{A} \mid c < x\} = R_i \cap \{c \in \mathcal{A} \mid c < x\}$. For the second, it is by construction the sequence of t sets $\{x\}$, followed by empty sets.

For the third sequence, suppose first that $\mathcal{Y}(x,T)$ is empty; then Step (ii) is empty, and we have $R'_i \cap \{c \in \mathcal{A} \mid c > x\} = R_i \cap \{c \in \mathcal{A} \mid c > x\}$; this implies that the sequence is decreasing. Suppose now that $\mathcal{Y}(x,T)$ is nonempty. Let $E_i = R_i \cap \{c \in \mathcal{A} \mid c > x\}$ and $E'_i = R'_i \cap \{c \in \mathcal{A} \mid c > x\}$; then by construction, for $r+2 \leq i \leq s$, the sets E_i, E'_i satisfy the hypothesis of Lemma 4.37, so that $E_{r+1} = E'_{r+1} \supseteq E'_{r+2} \supseteq \ldots \supseteq E'_s$. Moreover $E_i = E'_i$ for $1 \leq i \leq r+1$ and $E'_i = E_i = \emptyset$ for $i \geq s+1$. Thus the sequence of sets E'_i is decreasing.

We show now that the minima of R'_i strictly increase. This follows from the fact that $p_i = \min(R_i) = \min(R'_i)$, except if $i = t$, in which case $\min(R'_t) = x$. Then the property follows from $p_1 < \ldots < p_{t-1} < x < p_t < p_{t+1} < \ldots$ ($p_t, p_{t+1}, \ldots$ may not exist, in which case the sequence of inequalities stops at x). □

Recall that N-tableaux correspond bijectively to elements in $\mathrm{Styl}(\mathcal{A})$, and that we denote by $r(T)$ the row-word of an N-tableau T: one has $T = \mathrm{N}(r(T))$.

Theorem 4.38 *Let T be an* N*-tableau and x a letter. Then $(x \to T) = \mathrm{N}(xr(T))$.*

In other words, left multiplication by x in the stylic monoid corresponds to the left insertion into N-tableaux; similarly, we already know that right multiplication by x corresponds to right N-insertion.

We need several lemmas. Recall that $\mathbf{D}$ has been defined in Section 4.8.3.

Lemma 4.39 *Let $R_1 \supsetneq \ldots \supsetneq R_k$ be an* N*-filtration, and $x \in \mathcal{A}$. Then one has the sequence of inclusions $R_1 \cup x \supseteq \mathbf{D}_{R_2 \cup x}(R_1) \supsetneq \ldots \supsetneq \mathbf{D}_{R_k \cup x}(R_{k-1}) \supseteq \mathbf{D}_{R_{k+1} \cup x}(R_k)$ (with $R_{k+1} = \emptyset$), and this sequence is an* N*-filtration (the last set may be empty, in which case it is removed).*

An explanation of this technical lemma may be as follows: if R_i are the rows from an N-tableau, then $R_i = \mathbf{D}_{R_i}(R_{i-1})$. Now, the deformation of the latter expression, as it appears in the lemma, gives the rows of the N-tableau corresponding to left multiplication by x. After this lemma, the lemma below computes these deformations.

Proof. (1). Since $\mathbf{D}_{R_2 \cup x}(R_1)$ is a subset of $R_2 \cup x$, the first inclusion follows.

(2). Let $i = 3, \ldots, k+1$, and $d \in \mathbf{D}_{R_i \cup x}(R_{i-1})$. Then there exists $c \in R_{i-1}$ such that $d = c^{\uparrow}_{R_i \cup x} \neq 1$; take c maximum. Since $c \in R_{i-1}$, we have $c \in R_{i-2}$. If $c^{\uparrow}_{R_{i-1} \cup x} = d$, then $d \in \mathbf{D}_{R_{i-1} \cup x}(R_{i-2})$.

Otherwise, we have either $1 \neq c^{\uparrow}_{R_{i-1} \cup x} \neq d$, or $1 = c^{\uparrow}_{R_{i-1} \cup x}$. In the first case, since $c < d$, and $d \in R_i \cup x$ hence $d \in R_{i-1} \cup x$, there exists $z \in R_{i-1} \cup x$ such that $c < z < d$; by maximality of c, we must have $z = x$, and thus $c < x$, hence $c^{\uparrow}_{R_i \cup x} \leq x = z < d = c^{\uparrow}_{R_i \cup x}$, a contradiction.

Thus we have $1 = c^{\uparrow}_{R_{i-1} \cup x}$, which means that $c \geq \max(R_{i-1} \cup x)$; but $R_i \cup x \subseteq R_{i-1} \cup x$, hence $c \geq \max(R_i \cup x)$, hence $c^{\uparrow}_{R_i \cup x} = 1$, a contradiction too.

From all this, the inclusion $\mathbf{D}_{R_i \cup x}(R_{i-1}) \subseteq \mathbf{D}_{R_{i-1} \cup x}(R_{i-2})$ follows.

(3). We now show that the sequence of minima is strictly increasing. If $x \leq \min(R_1)$, then $x = \min(R_1 \cup x)$; therefore, for any $c \in R_1$, $c \geq x$ and $c^{\uparrow}_{R_2 \cup x} > c \geq x$, from which follows that $\min(\mathbf{D}_{R_2 \cup x}(R_1)) > x = \min(R_1 \cup x)$. On the other hand, if $x > \min(R_1)$, then since $\mathbf{D}_{R_2 \cup x}(R_1) \subseteq R_2 \cup x$, we have $\min(\mathbf{D}_{R_2 \cup x}(R_1)) \geq \min(R_2 \cup x) > \min(R_1)$ (because $x, \min(R_2) > \min(R_1)) = \min(R_1 \cup x)$.

Now, let $i = 3, \ldots, k$. We show that: $(*)$ $\min(\mathbf{D}_{R_{i-1} \cup x}(R_{i-2})) < \min(\mathbf{D}_{R_i \cup x}(R_{i-1}))$. Let $d = \min(\mathbf{D}_{R_i \cup x}(R_{i-1}))$. Then there exists $c \in R_{i-1}$ such that $d = c^{\uparrow}_{R_i \cup x}$. It follows from the hypothesis that $R_{i-1} \subseteq R_{i-2}$ and that $\min(R_{i-1}) > \min(R_{i-2})$; thus there exists $b \in R_{i-2}$ such that $c = b^{\uparrow}_{R_{i-1}}$. If $c = b^{\uparrow}_{R_{i-1} \cup x}$, then $c \in \mathbf{D}_{R_{i-1} \cup x}(R_{i-2})$ and we deduce $(*)$, since $c < d$. If on the contrary, $c \neq b^{\uparrow}_{R_{i-1} \cup x}$, then $b^{\uparrow}_{R_{i-1} \cup x} = x$ and we must have $b < x < c$ (otherwise by $c = b^{\uparrow}_{R_{i-1}}$, we have $c = b^{\uparrow}_{R_{i-1} \cup x}$); now $x < c < d$, hence we deduce $(*)$ too.

If $\mathbf{D}_{R_{k+1} \cup x}(R_k)$ is nonempty, then $x > \min(R_k)$, $\mathbf{D}_{R_{k+1} \cup x}(R_k) = \{x\}$, and its minimum is x; since $\min(R_{k-1}) < \min(R_k)$, we have $\min(R_k) \in \mathbf{D}_{R_k \cup x}(R_{k-1})$, thus the minimum of this latter set is $< x$. $\square$

Lemma 4.40 *Let $\emptyset \neq R \subseteq S \subseteq \mathcal{A}$, and $x \in \mathcal{A}$. Let m_R (resp. m_S) be the minimum of R (resp. S) and assume that $m_S < m_R$. Define, if it exists, y_R (resp. y_S) to be the smallest element in R (resp. S) which is greater than x. One has:*

(i) If $x \leq m_S$, then $\mathbf{D}_{R\cup x}(S) = R$.

Suppose now that $m_S < x$. Then one has:

(ii) If $x \in R$ and $x \in S$, then $\mathbf{D}_{R\cup x}(S) = R$.

(iii) If $x \notin R$ and $x \in S$, then $\mathbf{D}_{R\cup x}(S) = R \cup x$.

(iv) If $x \notin R$, $x \notin S$, and if either $y_R = y_S = \emptyset$, or $y_R = \emptyset$ and $y_S \neq \emptyset$, or $\emptyset \neq y_R \neq y_S \neq \emptyset$, then $\mathbf{D}_{R\cup x}(S) = R \cup x$.

(v) If $x \notin R$, $x \notin S$, and if $\emptyset \neq y_R = y_S \neq \emptyset$, then $\mathbf{D}_{R\cup x}(S) = (R \cup x) \setminus y_R$.

Proof. We use several times the fact that $R = \mathbf{D}_R(S)$ (which follows from $R \subseteq S$ and $m_S < m_R$).

(i). If $x \leq m_S$, then for any $c \in S$, one has $c \geq x$, hence $c^{\uparrow}_{R\cup x} = c^{\uparrow}_R$; therefore $\mathbf{D}_{R\cup x}(S) = \mathbf{D}_R(S) = R$, which proves (1).

Assume now that $m_S < x$. We first show that in each of the cases (2) to (5), $x \in \mathbf{D}_{R\cup x}(S)$. Indeed, since $m_S < x$, there is some c in S which is $< x$, and we choose c maximum; then the open interval $]c, x[$ does not intersect S, so does not intersect $R \cup x$ either; thus $c^{\uparrow}_{R\cup x} = x$ and therefore $x \in \mathbf{D}_{R\cup x}(S)$.

Now let $d \in R$, with $d \neq x$ and $d \neq y_R$. We show that in each of the cases (2) to (5), $d \in \mathbf{D}_{R\cup x}(S)$. We have $d \in \mathbf{D}_R(S)$, hence there is some $c \in S$ such that $d = c^{\uparrow}_R$. If x is not between c and d, then $d = c^{\uparrow}_{R\cup x}$. Otherwise we have $c < x < d$, so that y_R exists, and by our assumption, $d > y_R$. Then, since $R \subseteq S$, y_S exists too, $y_S \leq y_R$, and there is some c' in S such that $x < c' < d$ and we choose c' maximum; then $d = c'^{\uparrow}_{R\cup x} \in \mathbf{D}_{R\cup x}(S)$.

Note also that $\mathbf{D}_{R\cup x}(S) \subseteq R \cup x$ (because one has $\mathbf{D}_{\mathcal{B}}(\mathcal{C}) \subseteq \mathcal{B}$), so that in the three cases (2) to (4), the left-hand side of the equality to be proved is contained in the right-hand side.

We now complete the proof in each case.

(ii). We have $\mathbf{D}_{R\cup x}(S) = \mathbf{D}_R(S) = R$.

(iii). We have $y_R = x^{\uparrow}_{R\cup x} \in \mathbf{D}_{R\cup x}(S)$ since $x \in S$.

(iv). Note that, since $R \subseteq S$, and if y_R exists, then y_R, y_S both exist, and $y_S \leq y_R$. Thus either y_R does not exist, which completes this case; or y_R, y_S both exists and $y_S < y_R$ (by the assumption $y_R \neq y_S$), so that there is some $c \in S$ such that $x < c < y_R$, and we choose c maximum; then $y_R = c^{\uparrow}_{R\cup x} \in \mathbf{D}_{R\cup x}(S)$.

(v). We show that $y_R \notin \mathbf{D}_{R\cup x}(S)$. Indeed, otherwise, there is some $c \in S$ such that $c < y_R$, and that $]c, y_R[$ does not intersect $R \cup x$. Then $c \neq x$ since $x \notin S$; and we cannot have $c > x$ since $y_S = y_R$. Thus we must have $c < x$, but then $]c, y_R[$ intersects $R \cup x$, a contradiction. □

Proof. [Proof of Theorem 4.38] Let T be an N-tableau with rows $R_1, \ldots, R_k$, viewed as subsets, with respective minima $p_1, \ldots, p_k$.

Suppose that we are in Case 1: $x < p_1$. Then it is apparent that in the right N-algorithm applied to $xr(T)$, x will appear at the first step in the first row, and the other steps will not involve x; hence $\mathrm{N}(xr(T))$ is obtained from $\mathrm{N}(r(T))$ by adding x in the first row, and therefore $\mathrm{N}(xr(T)) = (x \to T)$, since we are in Case 1. Suppose now that we are in Case 2. Then x appears in the first column of T. Note that if v is a decreasing word containing x, then $xv \equiv_{styl} v$, by Lemma 4.27. Since $\mathcal{R}(T) \equiv_{styl} c(T)$, as follows from Section 4.5 and Proposition 4.4, we obtain that $x\mathcal{R}(T) \equiv_{styl} r(T)$. Hence $\mathrm{N}(xr(T)) = \mathrm{N}(r(T)) = (x \to T)$.

We assume now that we are in Case 3. Define $u_1, \ldots, u_k$ by $s(R_i) = u_i$ (the function s is defined in Section 4.8.3). Then $r(T) = u_k \cdots u_1$. By Lemma 4.39,

$$S_1 = R_1 \cup x \supseteq S_2 = \mathbf{D}_{R_2\cup x}(R_1) \supseteq \ldots \supseteq S_{k+1} = \mathbf{D}_{R_{k+1}\cup x}(R_k)$$

(where $R_{k+1} = \emptyset$) is an N-filtration, $\mathcal{F}$ say. It corresponds to the N-tableau T' whose row-word is $(\prod_{i=k}^{i=1} s(\mathbf{D}_{R_{i+1}\cup x}(R_i)))s(R_1 \cup x)$. By Lemma 4.11 (since $\delta(x) = 1$), this word is congruent modulo $\equiv_{styl}$ to $\delta(xu_k \ldots u_1)s(R_1 \cup x)$. This latter word is congruent to $xr(T)$ by Lemma 4.18. Thus it is enough to show that for $i = 1, \ldots, k+1$, $S_i = R'_i$, where the R'_i are the rows of $x \to T$, with R'_{k+1} possibly empty. We do this by following the algorithm giving $x \to T$, at the beginning of the section, and in particular using the notations there.

Since we are in Case 3, we have $x > p_1$, and either $t = k+1$, $p_k < x$ and R_t empty, or $t \leq k$ and $x < p_t$. We have also $r < t \leq k+1$ and $p_{t-1} < x$, and if s exists, then $r < s \leq t$. Next, for $i = 2, \ldots, t$, we have $p_{i-1} \leq p_{t-1} < x$, so that we may apply Lemma 4.40 (ii), (iii), (iv), and (v) to $S = R_{i-1}, R = R_i$.

(1). If $r = 0$, then $1 = r + 1 \leq t$, hence $R'_1 = R_1 \cup x$; if $r \geq 1$, then $x \in R_1$, hence $R'_1 = R_1 = R_1 \cup x$ too. Thus $S_1 = R'_1$.

(2). Let $i = 2, \ldots, r$; then $x \in R_{i-1}, x \in R_i$, so that by Lemma 4.40 (ii), $S_i = \mathbf{D}_{R_i \cup x}(R_{i-1}) = R_i = R'_i$.

(3). Now let $i = r + 1$; then $x \in R_r = R_{i-1}, x \notin R_{r+1} = R_i$, so that by Lemma 4.40 (iii), $S_i = \mathbf{D}_{R_i \cup x}(R_{i-1}) = R_i \cup x = R_{r+1} \cup x = R'_i$.

(4). Suppose first that s does not exist. Then for $i = r + 1, \ldots, t$, y_i does not exist. Let $i = r + 2, \ldots, t$. Then it follows from Lemma 4.40 (iv) that $S_i = \mathbf{D}_{R_i \cup x}(R_{i-1}) = R_i \cup x = R'_i$.

Suppose now that s exists. Let $i = r + 2, \ldots s$; then y_i, y_{i-1} exist, $x \notin R_{i-1}, x \notin R_i$, so that $S_i = \mathbf{D}_{R_i \cup x}(R_{i-1}) = R_i \cup x$ or $(R_i \cup x) \setminus y_i$, depending on $y_i \neq y_{i-1}$ or $y_i = y_{i-1}$ (by Lemma 4.40 (iv) and (v)) $= R'_i$.

If $i = s + 1 \leq t$, then y_i does not exist, y_{i-1} exists, $x \notin R_i$, $x \notin R_{i-1}$, so that $S_i = \mathbf{D}_{R_i \cup x}(R_{i-1}) = R_i \cup x = R'_i$, by Lemma 4.40 (iv)).

Now let $i = s+2, \ldots, t$. Then $x \notin R_{i-1}, x \notin R_i, y_{i-1}, y_i$ do not exist, so that $S_i = \mathbf{D}_{R_i \cup x}(R_{i-1}) = R_i \cup x$ (by Lemma 4.40 (iv)); hence $S_i = R'_i$.

(5). Finally, suppose that either $t = k + 1, R_t$ empty and $i = k + 1$, or $t + 1 \leq i \leq k + 1$. In the first case, $S_{k+1} = \mathbf{D}_x(R_k) = x$ (since $p_k < x$) $= R'_{k+1}$. In the second case, $x < p_t \leq p_{i-1}$, so that by Lemma 4.40 (i), we have $S_i = \mathbf{D}_{R_i \cup x}(R_{i-1}) = R_i = R'_i$. □

4.13.3 Grading of the $\mathscr{J}$-order

A finite poset P is *graded* if there is a function $\mathrm{rk} : \mathrm{P} \to \mathbb{N}$ such that: if $x < y$ in P, then $\mathrm{rk}(x) < \mathrm{rk}(y)$, and if moreover y covers x, then $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$. The function rk is called the *rank function*. If P has a minimum $\hat{0}$ and a maximum $\hat{1}$, we may assume that $\mathrm{rk}(\hat{0}) = 0$; let $N = \mathrm{rk}(\hat{1})$. We then call the function $P \to \mathbb{N}, x \mapsto N - \mathrm{rk}(x)$ the *co-rank function*.

Theorem 4.41 *The $\mathscr{J}$-order in* $\mathrm{Styl}(\mathcal{A})$ *is graded. The co-rank of an element is given by the number of boxes in its* N*-tableau.*

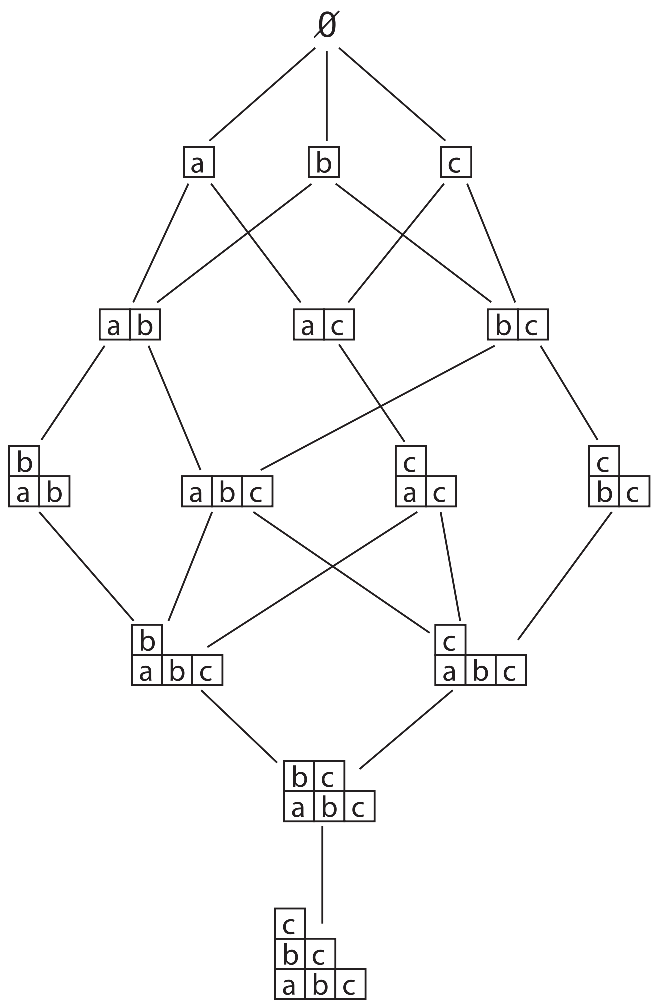

Figure 4.25: $\mathscr{J}$-order for $n = 3$

Although the co-rank function is easy to describe, we do not know a direct criterion that characterizes the $\mathscr{J}$-order directly on the N-tableaux.

We need some preliminary results to prove Theorem 4.41.

For the first result, recall that the *shape* of a *semi-standard tableau* T, denoted $\lambda(T)$, is the integer partition whose parts are the row lengths of T. Each integer partition is classically denoted by the decreasing sequence of its parts. The *Young order* on integer partitions is obtained by the rule $(a_1, a_2, \ldots) \leq_{Young} (b_1, b_2, \ldots) \Leftrightarrow \forall i, a_i \leq b_i$ (where $a_1 \geq a_2 \geq \ldots, b_1 \geq b_2 \geq \ldots$, and where sufficiently many 0's are added to the sequences).

Proposition 4.42 *Let T be an* N*-tableau, x a letter, and let $S = (T \leftarrow x)$ (resp. $S = (x \rightarrow T)$). If $S \neq T$, then $\lambda(T) <_{\mathrm{Young}} \lambda(S)$.*

It follows that the function $S \mapsto \lambda(S)$ is strictly increasing, from the set of N-tableaux with the $\mathscr{J}$-order into the set of integer partitions with the Young order.

Proof. Let $S = T \leftarrow x$ and suppose that $S \neq T$. It follows directly from the algorithm of right N-insertion that, since $S \neq T$, several rows (and at least one row) of T get a new element, and the other rows remain unchanged, producing S. Thus $\lambda(T) <_{Young} \lambda(S)$.

Recall that $\mathrm{Styl}(\mathcal{A})$ is in bijection with partitions, and also with N-tableaux. It follows directly from the definition of evacuation, and of Theorem 4.24, that the anti-automorphism θ of $\mathrm{Styl}(\mathcal{A})$ preserves the shape of a partition, hence of the corresponding N-tableau. Let $y = \theta(x)$. Then the image under θ of $x \rightarrow T$ is $T \leftarrow y$, by Corollary 4.17 and Theorem 4.38. Thus the second assertion follows from the first (it may also be seen directly on the left insertion). $\square$

Lemma 4.43 *Let T be an* N*-tableau, with rows $R_1, \ldots, R_k$, and let R_{k+1} denote the empty row.*

(i) Let c be a letter in R_i, with $c \neq \min(R_i)$. There exists a letter $x \in \mathrm{Supp}(T)$ such that the letter c is bumped from R_i during the right N*-insertion $T \leftarrow x$, and no letter is added in rows $R_1, \ldots, R_i$.*

(ii) Let $c \in R_i$, such that either $c > \max(R_{i+1})$, or $i = k$ and $c \neq \min(R_k)$. Then there exist letters $x_1, \ldots x_{k+1-i} \in \mathrm{Supp}(T)$ such that the $k+1-i$ successive right N*-insertions $(\ldots(T \leftarrow x_1)\cdots) \leftarrow x_{k+1-i}$ produce an* N*-tableau which is obtained from T by adding c in rows $R_{i+1}, \ldots, R_{k+1}$, with one box labelled c added at each insertion.*

Proof. (i). (induction on i) If $i = 1$, then $c \neq \min(R_1)$ implies that some letter $x < c$ is in R_1; we choose x maximum and then in the right N-insertion $T \leftarrow x$, c is bumped from R_1; the second condition holds since $x \in R_1$.

Suppose now that $i \geq 2$; since $\min(R_i) < c$, there is some letter $b < c$ in R_i and we choose b maximum; then the right N-insertion of b into the row R_i bumps c from R_i. Since T is an N-tableau, $\min(R_{i-1}) < \min(R_i) \leq b$ and $b \in R_{i-1}$. Thus, by induction on i, there exists a letter $x \in \mathrm{Supp}(T)$ such that during the right N-insertion $T \leftarrow x$, b is bumped from R_{i-1}, and no letter is added in rows $R_1, \ldots, R_{i-1}$. It follows that during this insertion, c is bumped from R_i. Moreover, no letter is added in R_i, since $b \in R_i$.

(ii). (induction on $k+1-i$) If $i = k$, then by (i) there exists $x_1 \in \mathrm{Supp}(T)$ such that the right N-insertion $T \leftarrow x_1$ bumps c from R_k, producing a new row $\{c\}$, and no letter is added in rows $R_1, \ldots, R_k$.

Suppose now that $i < k$. Then $(*)$ $c > \max(R_{i+1}) \geq \min(R_{i+1}) > \min(R_i)$, so that by (i) there exists $x_1 \in \mathrm{Supp}(T)$ such that $T \leftarrow x_1$ adds c into row R_{i+1}, and since $c > \max(R_{i+1})$, the algorithm stops; denote by T' the resulting tableau, with rows R'_j. We have $c \in R'_{i+1}$ and either: $i+1 < k$, $c > \max(R_{i+1}) \geq \max(R_{i+2})$ (since R_{i+1} contains R_{i+2}) $= \max(R'_{i+2})$ (since $R_{i+2} = R'_{i+2}$); or: $i+1 = k$ and $c \neq \min(R_k)$ as follows from $(*)$

It follows by induction that there exist $x_2, \ldots, x_{k+1-i} \in \mathrm{Supp}(T') = \mathrm{Supp}(T)$ such that $(\ldots(T' \leftarrow x_2)\cdots) \leftarrow x_{k+1-i}$ adds c in rows $R'_{i+2}, \ldots, R'_{k+1}$, and such that each insertion adds exactly one box. This ends the proof. □

Proof. [Proof of Theorem 4.41] For $W \in \mathcal{A}^*$, let $\nu(W)$ denote the number of boxes in the N-tableau $\mathrm{N}(W)$; this function is compatible with the stylic congruence, hence induces a function ν on $\mathrm{Styl}(\mathcal{A})$; we show that it is the co-rank function on the poset $\mathrm{Styl}(\mathcal{A})$ with the $\mathscr{J}$-order.

Since $\mathrm{Styl}(\mathcal{A})$ is generated by the letters, the $\mathscr{J}$-order is defined by the following rule: for $U, V \in \mathrm{Styl}(\mathcal{A})$, $U \leq_{\mathscr{J}} V$ if and only if there exist elements $U_0, \ldots, U_n$ in $\mathrm{Styl}(\mathcal{A})$ such that $U_0 = U$, $U_n = V$ and for each $i = 0, \ldots, n-1$, there exists a letter x such that $U_i = xU_{i-1}$ or $U_i = U_{i-1}x$. Switching to N-tableaux, identified with elements of $\mathrm{Styl}(\mathcal{A})$, this translates into: for any N-tableaux S, T, one has $T \leq_{\mathscr{J}} S$ if and only if there exist N-tableaux $T_0, \ldots, T_n$ such that $T_0 = T$, $T_n = S$ and for each $i = 0, \ldots, n-1$, there exists a letter x such that $T_i = x \to T_{i-1}$ or $T_i = T_{i-1} \leftarrow x$.

We therefore deduce from Proposition 4.42 that for $U, V \in \mathrm{Styl}(\mathcal{A})$ such that $U <_{\mathscr{J}} V$, one has $\nu(U) < \nu(V)$.

It remains to show the following result: if for two N-tableaux S, T, $T \leq_{\mathscr{J}} S$, then there exists a sequence of N-tableaux $T_0, \ldots, T_n$ such that $T_0 = T$, $T_n = S$ and for each $i = 0, \ldots, n-1$, $T_{i-1} <_{\mathscr{J}} T_i$ and $\nu(T_i) = \nu(T_{i-1}) + 1$. It is enough to prove this when S is obtained from T by a left or a right insertion of a letter, and even when it is a left insertion (since the anti-automorphism θ exchanges left and right insertions, and preserves the shape, hence also preserves ν). We show that this left insertion $x \to T$ is equivalent to a sequence of left or right insertions, each adding one box to the shape. We argue by induction on $\nu(S) - \nu(T)$, noting that if this quantity is 0, then $S = T$.

So let $S = (x \to T)$ for some letter x. Referring to the definition of left insertion at the beginning of Section 4.13.2, we see that Cases 1 and 2 give immediately the result. So we may assume that we are in Case 3. There are two cases to consider: $t = k+1$ (subcase 3.1), and $t \leq k$ (subcase 3.2).

(1). Suppose that $t = k+1$; then $R_{k+1} = \emptyset$ and $\min(R_k) = p_k < x$.

(I). Suppose that $\mathcal{Y}(x, T)$ is empty. Then S is obtained from T by adding x in each row $R_{r+1}, \ldots, R_{k+1}$. If $r \geq 1$, we use Lemma 4.43 (i), with $c = x, i = r$: the left insertion $x \to T$ may be simulated by $k+1-r$ right insertions of x, each one increasing the number of boxes by Step (i). If $r = 0$, then the hypothesis $\mathcal{Y}(x, T) = \emptyset$ implies that x is greater than each element in T; then $T \leftarrow x$ adds x in the first row, and nothing else, and we are reduced to $r \geq 1$.

(II). Suppose that $\mathcal{Y}(x, T)$ is nonempty. Then $y_s \in R_s$ and: either $s < k$, and then $y_s > x \geq \max(R_{s+1})$ (the latter inequality since y_{s+1} does not exist), and therefore $y_s > \max(R_{s+1})$; or $s = k$ and $y_s \neq \min(R_k)$ (since $y_s > x > p_k$). Then by Lemma 4.43 (ii) (applied to $c = y_s$, $i = s$), we find a sequence of right insertions, each one of which adds a single box, and whose result is the N-tableau T' obtained from T by adding y_s in rows $R_{s+1}, \dots, R_{k+1}$ (and in particular the $(k+1)$-th row of T' is $\{y_s\}$). Then $\nu(T') > \nu(T)$, since $s+1 \leq k+1$. Now $x \to T'$ adds x in rows $R_{r+1}, \dots, R_t$ and removes the y_s that were just added, together with each y_i in rows $R_{r+2}, \dots, R_s$ if $y_i = y_{i-1}$; thus $(x \to T') = S$. We conclude by induction, since $\nu(S) - \nu(T')$ is smaller than $\nu(S) - \nu(T)$.

(2). Suppose that $t \leq k$. Then $s = t$, $y_{r+1}, \dots, y_t$ exist and $y_t = p_t$; moreover $\mathcal{Y}(x, T) = \{y_{r+1}, \dots, y_t\}$.

(I). Suppose that the set $\mathcal{Y}$ has only one element, which is y_t. Then Step (i) adds x in rows $R_{r+1}, \dots, R_t$ and Step (ii) removes y_t from the rows $R_{r+2}, \dots, R_t$; hence $\nu(S) = \nu(T) + 1$ and we are done.

(II). Suppose that the set $\mathcal{Y}$ has at least two elements, and let $y_u = \max(Y \setminus y_t)$, with u chosen maximum. Let $T' = (y_u \to T)$. We have $p_{t-1} < x < y_u < y_t = p_t$. Hence the left insertion $y_u \to T$ adds y_u in rows $R_{u+1}, \dots, R_t$ and removes y_t from the rows $R_{u+2}, \dots R_t$, and in particular $\nu(T') = \nu(T) + 1$. Now, for the left insertion $x \to T'$, we have $Y(x, T') = Y(x, T) \setminus y_t$. Moreover, the left insertion $x \to T'$: Step (i) adds x in rows $R_{r+1}, \dots, R_t$; and Step (ii) removes y_u from rows $R_{u+1}, \dots, R_t$, and from the rows R_i, $i = r+2, \dots, u$, it removes y_i if $y_i = y_{i-1}$. Thus $(x \to T') = S$, which settles this case by induction, because $\nu(T') = \nu(T) + 1$. □

4.14 Fixpoints and idempotents

Recall that we may view columns as subsets of $\mathcal{A}$. As such, they are ordered by inclusion.

Theorem 4.44 *(i) Let $W \in \mathcal{A}^*$. A column γ is fixed by W if and only if $\mathrm{Supp}(W) \subseteq \gamma$.*

(ii) The support of a word is the smallest fixpoint, in the inclusion order, of its action on the columns.

(iii) The idempotents in $\mathrm{Styl}(\mathcal{A})$ are the images under μ of the strictly decreasing words.

(iv) There are $2^{|\mathcal{A}|}$ idempotents in $\mathrm{Styl}(\mathcal{A})$.

Lemma 4.45 *Let W be a strictly decreasing word and γ a column. Then $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(W \cdot \gamma)$.*

Proof. We show this by induction on the length of W. It is clear if W is empty. Otherwise $W = aV$, with V strictly decreasing. By induction $\mathrm{Supp}(V) \subseteq V \cdot \gamma$. Since a is greater than any letter in V, we also have $\mathrm{Supp}(V) \subseteq (V \cdot \gamma)_a$. We have $(a \cdot (V \cdot \gamma))_a = (V \cdot \gamma)_a$ by Lemma 4.2 (iii). Hence $\mathrm{Supp}(V) \subseteq (a \cdot (V \cdot \gamma))_a \subseteq (aV) \cdot \gamma = W \cdot \gamma$, and since $a \in a \cdot (V \cdot \gamma) = W \cdot \gamma$, we deduce $\mathrm{Supp}(W) \subseteq W \cdot \gamma$, as was to be shown. □

Proof. [Proof of Theorem 4.44] (i). If W is a word such that $\mathrm{Supp}(W) \subseteq \gamma$, W fixes γ by Lemma 4.2 (ii) applied iteratively.

Conversely, let γ be a column fixed by W. If we had $W = UaV$ with $a \notin \gamma$, choose V shortest possible; then $\mathrm{Supp}(V) \subseteq \gamma$, thus by Lemma 4.2 (ii), $V \cdot \gamma = \gamma$. We have $a \cdot \gamma \neq \gamma$, and by Proposition 4.32 (i), $a \cdot \gamma < \gamma$. Therefore $(aV) \cdot \gamma < \gamma$ and finally $W \cdot \gamma = U \cdot ((aV) \cdot \gamma) \leq (aV) \cdot \gamma$ (by the same proposition) $< \gamma$, and we cannot have $W \cdot \gamma = \gamma$, a contradiction.

(ii). Clear by (i).

(iii). Let W be a strictly decreasing word. Then we already know that the fixpoints of W are the columns containing $\mathrm{Supp}(W)$.

Let γ be any column. It follows from Lemma 4.45 and (i), that $W \cdot \gamma$ is a fixpoint of W. Hence $W \cdot (W \cdot \gamma) = W \cdot \gamma$, and W acts as idempotent on the columns.

It remains to prove the converse: each idempotent E in $\mathrm{Styl}(\mathcal{A})$ is equivalent modulo $\equiv_{styl}$ to a strictly decreasing word. For this, let W the strictly decreasing word whose letters are the elements in $\mathrm{Supp}(E)$. Then by (i) E and $f = \mu(W)$ have the same set of fixpoints; moreover, E, F are idempotent, hence their images are contained in this set. It follows by monoid theoretical arguments that $\mu(W) = E$: indeed, for any γ, $F \cdot \gamma$ is in the image of F, hence is a fixpoint of E; hence $EF \cdot \gamma = F \cdot \gamma$; thus $EF = F$; similarly, $FE = E$; hence E, F are $\mathscr{J}$-equivalent, hence equal since $\mathrm{Styl}(\mathcal{A})$ is $\mathscr{J}$-trivial (Theorem 4.35).

(iv). is clear, since the idempotents are in bijection with subsets of $\mathcal{A}$, because two different subsets, viewed as strictly decreasing words, act differently on the empty column. □

4.14.1 Applications to the plactic monoid: a confluent rewriting system

In the next result, columns are also viewed as decreasing words, and as subsets of $\mathcal{A}$.

Proposition 4.46 *Let γ, δ be columns. Let $\gamma' = \gamma \cdot \delta$ and $\delta' = (\gamma \cup \delta) \setminus (\gamma \cdot \delta)$, where this boolean operation is taken as multisets. Then*

(i) $\gamma \subseteq \gamma'$;

(ii) $\gamma' \leq \gamma$;

(iii) $\gamma' = \gamma$ *if and only* $\delta = \delta'$ *if and only if* $\gamma \leq \delta$;

(iv) $\gamma' \leq \delta'$;

(v) $\gamma\delta \equiv_{\mathrm{Plax}} \gamma'\delta'$.

Note that δ' may be the empty column, in which case it is the empty word, according to our conventions.

Proof. (i) Consider the tableau T obtained by column insertion of the word γ into the column δ: its first column is $\gamma' = \gamma \cdot \delta$, by definition of the action on columns, and T has either only one column, or two columns, and by counting letters, the second one must be δ'; in particular, $\gamma' \leq \delta'$ by definition of order. Moreover, $T = \mathrm{P}(\gamma\delta)$. In particular, if $\gamma \leq \delta$, T is the tableau with first column $\gamma = \gamma'$ and second column δ.

(ii). This is Lemma 4.45.

(iii). This follows from (i) and an observation in Section 4.12, relating inclusion of columns, and their order.

(iv). The first equivalence follows from the multiset union $\gamma \cup \delta = \gamma' \cup \delta'$. If $\gamma \leq \delta$, then $\gamma' = \gamma$ by (o). Conversely, if $\gamma' = \gamma$, then $\delta' = \delta$, and we obtain by (o) that $\gamma' \leq \delta'$, hence $\gamma \leq \delta$.

(iv) and (v). These follow from (o) and Section 4.4. □

Theorem 4.47 *([21, Theorem 4.5], [26, Theorem 3.4])*

(i) The plactic monoid has the following presentation: it is generated by the columns, subject to the relations $\gamma\delta = \gamma'\delta'$, for all columns γ, δ, where γ', δ' are defined in Proposition 4.46, with δ' omitted if it is the empty column.

(ii) The rewriting system on the free monoid $\Gamma(\mathcal{A})^*$ given by the rules $\gamma\delta \to \gamma'\delta'$, with the same notations, and where one omits the rules with $\gamma \leq \delta$, is confluent.

Recall that a *rewriting system* on a free monoid $\mathcal{C}^*$, generated by rules $U \to V$, is the least reflexive and transitive binary relation on $\mathcal{C}^*$ which is compatible with left and right multiplication. It is *confluent* if the set of words W which may not be rewritten (that is, do not have as factor any word U which is the left part of a rule) is a set of unique representatives of the congruence generated by this binary relation.

Proof. Consider the order $\leq$ on the set $\Gamma(\mathcal{A})$ of columns, and then order lexicographically the words of equal length in the free monoid $\Gamma(\mathcal{A})^*$, then order this whole free monoid first by length, then lexicographically. We obtain an order on $\Gamma(\mathcal{A})^*$, which is not total, but suffices for our purpose. If we use a rule $\gamma\delta \to \gamma'\delta'$ in a word W, obtaining W', then either W' is shorter than W (in case δ' is the empty column); or W' and W have the same length, and W is smaller for the previous lexicographic order, since $\gamma' < \gamma$, by Proposition 4.46 (ii) and (iii) (because we do not have $\gamma \leq \delta$). Hence $W' < W$. It follows that there is no infinite chain in the rewriting rule, since each such chain decreases for the order, and remains in the finite set of words of bounded length.

As a consequence, each word may be rewritten into a word $\gamma_1 \cdots \gamma_n$ with $\gamma_1 \leq \ldots \leq \gamma_n$. Since tableaux form a set of representatives of the plactic monoid and by Proposition 4.46 (v), we obtain the theorem. □

For the interested reader, note that Bokut et al. give a formula in order to compute γ' and δ', with the notations of Proposition 4.46: see [21, Definition 4.6 and Lemma 4.7]. Note also that when we write $\gamma_1 \leq \gamma_2$, they write $\gamma_1 \rhd \gamma_2$ (and $\gamma_1 \succeq \gamma_2$ in [26]).

4.15 Syntacticity

The syntactic monoid and congruence of a language (a subset of a free monoid) are well-known notions (see for example [67]). As is also well-known, they immediately extend to functions from a free monoid into any set, as follows.

Let $f : \mathcal{A}^* \to \mathcal{E}$, where $\mathcal{E}$ is any set. The *syntactic congruence of* f, denoted $\equiv_f$, is defined by

$$U \equiv_f V \Leftrightarrow (\forall X, Y \in \mathcal{A}^*, f(XUY) = f(XVY)).$$

It is a (two-sided) congruence of $\mathcal{A}^*$, that is, an equivalence relation which is compatible with the product in $\mathcal{A}^*$. It is the coarsest congruence $\equiv$ of $\mathcal{A}^*$ which is compatible with f, that is, satisfying $U \equiv V \implies f(U) = f(V)$. The *syntactic monoid of* f is the quotient monoid $\mathbf{M}_f = \mathcal{A}^*/\equiv_f$. One has clearly $U \equiv_f V \implies f(U) = f(V)$, so that f induces a function $g_f : \mathbf{M}_f \to \mathcal{E}$ such that $f = g_f \circ \mu$, with μ the canonical monoid homomorphism $\mathcal{A}^* \to \mathbf{M}_f$.

Similarly, the *left synctactic congruence of* f, denoted by $\equiv_f^l$, and defined by

$$U \equiv_f^l V \Leftrightarrow (\forall X \in \mathcal{A}^*, f(XU) = f(XV)).$$

It is a left congruence of $\mathcal{A}^*$, that is, compatible with multiplication at the left, and one therefore obtains a left action of $\mathcal{A}^*$ onto the set $\mathcal{A}^*/\equiv_f^l$. The syntactic left congruence of f is the coarsest left congruence of $\mathcal{A}^*$ which is compatible with f.

Both quotients have a universal property with respect to f, which we describe only for the syntactic monoid. Consider the category whose objects are the triples $\mathbf{M}, \mu, g$, where $\mathbf{M}$ is a monoid, μ a surjective monoid homomorphism $\mathcal{A}^* \to \mathbf{M}$, and $g : \mathbf{M} \to \mathcal{E}$ a function, such that $f = g \circ \mu$; in this case, we say that $\mathbf{M}, \mu, g$ (or simply $\mathbf{M}$) *recognizes* f. Morphisms of the category are defined as monoid homomorphisms $\nu : \mathbf{M} \to \mathbf{M}'$ such that $\mu' = \nu \circ \mu$ and $g' \circ \nu = g$. The triple $\mathbf{M}, \mu, g_f$ is an object in the category, and it is a final object in the category. In that sense, we may say that "$\mathbf{M}_f$ is the smallest monoid recognizing f".

Theorem 4.48 *Consider the function* f *which associates to* $W \in \mathcal{A}^*$ *the maximum length of a strictly decreasing subsequence of* W*; equivalently (by Schensted's theorem) the length of the first column of* $\mathrm{P}(W)$.

(i) The syntactic left congruence of f *is determined by:* $U \equiv_f^l V$ *if and only if* $U \cdot \emptyset = V \cdot \emptyset$ *(where* $\emptyset$ *is the empty column).*

(ii) The syntactic monoid of f *is* $\mathrm{Styl}(\mathcal{A})$*, and its syntactic congruence* $\equiv_f$ *coincides with* $\equiv_{styl}$.

Lemma 4.49 *Let $n \geq 2$ and letters $a_n > \ldots > a_2 > a_1$ and U a word such that $a_n \cdots a_3 a_1 U$ is a strictly decreasing word. Viewing columns as strictly decreasing words, let $\gamma = a_n \cdots a_3 a_1 U, \gamma' = a_{n-1} \cdots a_1 U$; then*

$$a_{n-1} \cdots a_2 \cdot \gamma = \gamma'.$$

Proof. For $n = 2$, the equality is $1 \cdot a_1 U = a_1 U$, which is true. Suppose that $n \geq 3$. Let $V = a_1 U$. By induction, $a_{n-1} \cdots a_3 \cdot a_n \ldots a_4 a_2 V = a_{n-1} \cdots a_2 V$. We have $a_2 \cdot a_n \cdots a_3 a_1 U = a_n \cdots a_4 a_2 a_1 U$. By the previous equality, we obtain therefore

$$a_{n-1} \cdots a_2 \cdot \gamma = a_{n-1} \cdots a_2 \cdot a_n \cdots a_3 a_1 U = (a_{n-1} \cdots a_3) \cdot (a_2 \cdot a_n \cdots a_3 a_1 U)$$

$$= (a_{n-1} \ldots a_3) \cdot a_n \ldots a_4 a_2 a_1 U = a_{n-1} \cdots a_2 a_1 U,$$

which was to be shown. □

Proof. [Proof of Theorem 4.48] (i). If $U \cdot \emptyset = V \cdot \emptyset$, then for any word X, $XU \cdot \emptyset = XV \cdot \emptyset$ and therefore $f(XU) = f(xXV)$, since $W \cdot \emptyset$ is the first column of $\mathrm{P}(W)$ by Proposition 4.1, and $f(W)$ is its length. Thus $U \equiv_f^l V$.

Conversely, suppose that $\gamma_1 = U \cdot \emptyset \neq V \cdot \emptyset = \gamma_2$. In order to show that U, V are not equivalent modulo $\equiv_f^l$, it is enough to show the existence of a word X such that $f(XU) \neq f(XV)$, that is: the first columns of $\mathrm{P}(XU)$ and $\mathrm{P}(XV)$ have different lengths. We know by Proposition 4.1 that these columns are $XU \cdot \emptyset$ and $XV \cdot \emptyset$, equivalently $X \cdot \gamma_1$ and $X \cdot \gamma_2$.

If the two columns γ_1, γ_2 have different length, we take $X = 1$. Suppose now that they have the same length. If their largest letter are distinct, we may assume that it is a for γ_1 and b for γ_2 and $a < b$; then $b \cdot \gamma_1 = \gamma_1 \cup b$ and $b \cdot \gamma_2 = \gamma_2$ (since b appears in γ_2) and these columns have different lengths: we then take $X = b$. If their largest letters are equal, we may write (for example) $\gamma_1 = a_n \cdots a_3 a_1 S$, $\gamma_2 = a_n \cdots a_3 a_2 T$, with $n \geq 2$, $a_n > \cdots > a_3 > a_2 > a_1$, and S, T of the same length; let $W = a_{n-1} \cdots a_2$; then by Lemma 4.49, $W \cdot \gamma_1 = a_{n-1} \cdots a_1 S$ and $W \cdot \gamma_2 = \gamma_2 = a_n \cdots a_3 a_2 T$ (by Lemma 4.2 (ii), since $\mathrm{Supp}(W) \subseteq \mathrm{Supp}(\gamma_2)$); these two columns have distinct largest letters, and we are reduced to the previous case.

(ii). The argument we use now is standard in algebraic automata theory. We have $U \equiv_f V \Leftrightarrow (\forall X, Y \in \mathcal{A}^*, f(XUY) = f(XVY)) \Leftrightarrow (\forall Y \in \mathcal{A}^*, UY \equiv_f^l VY) \Leftrightarrow (\forall Y \in \mathcal{A}^*, UY \cdot \emptyset = UY \cdot \emptyset)$ (by (i)) $\Leftrightarrow (\forall \gamma \in \Gamma(\mathcal{A}), U \cdot \gamma = V \cdot \gamma)$ (since $(WY) \cdot \emptyset = W \cdot (Y \cdot \emptyset)$ and since each column is of the form $Y \cdot \emptyset$) $\Leftrightarrow U \equiv_{styl} V$. □

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CHAPITRE 5
LE CARQUOIS DE L'ALGÈBRE STYLIQUE

5.1 Avant-Propos et Résumé

Dans l'étude moderne des algèbres associatives, la construction de leurs carquois est un outil fondamental. Il permet, entre autres, de trouver une présentation de l'algèbre. Cet outil est particulièrement utile dans le cas des algèbres *basiques*[1]. Notons que toute algèbre associative A est *Morita-équivalente* à une algèbre basique B[2].

Il est connu que les algèbres des monoïdes finis $\mathscr{J}$-triviaux sont basiques [60]. Au chapitre précédent nous avons vu que le monoïde stylique est fini et $\mathscr{J}$-trivial, ce qui rend pertinent la recherche du carquois de son algèbre.

C'est ce que nous faisons dans le présent chapitre, inspiré par l'article de Denton, Hivert, Schilling et Thiéry [34]. Nous commençons par construire explicitement un système complet d'idempotents orthogonaux primitifs. [3] De ce système nous déduisons le carquois de l'algèbre du monoïde stylique. Nous construisons un carquois avec relations, ce qui nous donne une présentation de cette algèbre.

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5.2 Abstract

We construct a complete system of primitive orthogonal idempotents and give an explicit quiver presentation of the monoid algebra of the stylic monoid introduced by Abram and Reutenauer.

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[1] Une $\mathbb{K}$-algèbre de dimension finie A est basique si et seulement si $A/\operatorname{rad}(A)$ est isomorphe à $\mathbb{K}^n$ [12].

[2] La catégorie des modules sur A est équivalente à la catégorie des modules sur B.

[3] Steward Margolis et Benjamin Steinberg écrivent dans [60] : "It is notoriously difficult to write down explicit primitive idempotents for monoids algebras and often they have complicated expressions in terms of the monoid basis, making it virtually impossible to determine even the dimension of the corresponding projective indecomposable module let alone construct a matrix representation out of it."

5.3 Introduction

We study the monoid algebra of the stylic monoid $\mathrm{Styl}(\mathcal{A})$ introduced by the first two authors in [6]. We begin by recalling its definition.

Let $\mathcal{A}$ be a totally ordered finite alphabet and $\mathcal{A}^*$ the free monoid that it generates. The Robinson–Schensted–Knuth (RSK) correspondence associates with each word $W \in \mathcal{A}^*$ a semistandard tableau $\mathrm{P}(W)$ with entries in $\mathcal{A}$ called its P-symbol. If W is a decreasing word, then its P-symbol $\mathrm{P}(W)$ is a column, which allows us to identify the set of decreasing words on $\mathcal{A}$ with the set $\Gamma(\mathcal{A})$ of column-shaped tableaux with entries in $\mathcal{A}$. This induces a left action of $\mathcal{A}^*$ on $\Gamma(\mathcal{A})$: for a word $X \in \mathcal{A}^*$ and a column $\gamma \in \Gamma(\mathcal{A})$, take $X \cdot \gamma$ to be the first column of the tableau $\mathrm{P}(XW)$, where W is the decreasing word corresponding to the column γ. (This action can be defined using the Schensted column insertion procedure; see Section 5.4.2.) The finite monoid of endofunctions of $\Gamma(\mathcal{A})$ obtained by this action is the stylic monoid $\mathrm{Styl}(\mathcal{A})$.

It turns out that $\mathrm{Styl}(\mathcal{A})$ is canonically isomorphic to a quotient of the celebrated plactic monoid. Recall that the plactic monoid has appeared in many contexts in algebraic combinatorics and was used to give the first rigorous proof of the Littlewood–Richardson rule [73, 54, 56]. The monoid algebra $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$, where $\mathbb{K}$ is any field, is the first example of a finite dimensional representation of the plactic monoid that does not pass through the abelianisation (to our knowledge). This article is a first step towards understanding the structure of this representation.

Stylic monoids are examples of $\mathscr{J}$-trivial monoids [6], which are a ubiquitous class of monoids that arise naturally in algebraic combinatorics. Other examples include the 0-Hecke monoids associated with finite Coxeter groups, and the monoids of regressive order-preserving functions on a poset; see [34] for many more examples. It follows that the monoid algebra $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$ admits a quiver presentation: that is, $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$ is isomorphic to a quotient of the path algebra $\mathbb{K}Q(\mathcal{A})$ of a canonical quiver $Q(\mathcal{A})$.

Obtaining a quiver presentation is an essential step towards applying the tools and techniques from the modern representation theory of finite dimensional algebras [12]. One of our main results is an explicit presentation of $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$ as a quiver with relations. Our approach is constructive in the sense that we explicitly identify a complete system of primitive orthogonal idempotents in $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$ (Theorem 5.4) that we use to define a quiver $Q(\mathcal{A})$ together with a surjective map $\varphi : \mathbb{K}Q(\mathcal{A}) \to \mathbb{K}\,\mathrm{Styl}(\mathcal{A})$ (Corollary 5.16) whose kernel is an admissible ideal (Theorem 5.18). General theory then implies that $Q(\mathcal{A})$ is the quiver of $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$ (Theorem 5.19).

We remark that the representation theory of finite monoids naturally occurring in algebraic combinatorics, especially in connection with Markov chains, has been investigated by many authors: [24, 19, 20, 23, 22, 71, 70, 17, 40, 62, 61, 63, 49, 44, 14, 58, 60, 78, 59]; see especially Steinberg's recent book and the references therein [79]. Those most closely related to the present work are [34], [79, Chapter 17] and [60], which describe the quiver of the algebra of a $\mathscr{J}$-trivial monoid. While guided by this work, our approach is complementary and completely self-contained as their techniques do not involve constructing primitive orthogonal idempotents or a quiver presentation. In fact, in [60] one reads

> "It is notoriously difficult to write down explicit primitive idempotents for monoids algebras (c.f. [17, 33]) and often they have complicated expressions in terms of the monoid basis, making it virtually impossible to determine even the dimension of the corresponding projective indecomposable module let alone construct a matrix representation out of it."

5.4 Stylic monoid and algebra

We consider a totally ordered finite set $\mathcal{A}$, whose elements are called *letters*, and the free monoid $\mathcal{A}^*$ that it generates. Its elements are called *words*. The *support* of a word X is the set of letters $\mathrm{Supp}(X)$ appearing in X.

5.4.1 Tableaux

We call a *tableau* what is usually called a *semi-standard Young tableau*: a finite lower order ideal (that is, a finite subset $E \subset \mathbb{N}^2$ such that $a \leq b$ and $b \in E$ implies $a \in E$) of the poset $\mathbb{N}^2$, ordered naturally, together with a weakly increasing mapping into $\mathcal{A}$, such that the restriction of this mapping to each subset with given x-coordinate is injective. A tableau is usually represented as in Figure 5.1. The conditions may be expressed by saying that the letters in $\mathcal{A}$ are weakly increasing from left to right in each row, and strictly increasing from the bottom to top in each column.

A *column* is a tableau with only one column. The sets of column on $\mathcal{A}$ is denoted by $\Gamma(\mathcal{A})$. A column is identified naturally to a subset of $\mathcal{A}$, and also to the word in $\mathcal{A}^*$ which is the decreasing product of its elements.

d		
b	b	
a	a	c

Figure 5.1: A tableau

5.4.2 Schensted's column insertion procedure

Let us recall the Schensted *column insertion* algorithm. Let γ be a column, viewed here as a subset of $\mathcal{A}$, and let $x \in \mathcal{A}$. There are two cases: if $\forall y \in \gamma, x > y$, then define $\gamma' = \gamma \cup x$. Otherwise, let y be the smallest element in γ with $y \geq x$; then define $\gamma' = (\gamma \setminus y) \cup x$. Then γ' is the column obtained by *column insertion of* x *into* γ, and in the second case, y is said to be *bumped*.

We define a left action of $\mathcal{A}^*$ on $\Gamma(\mathcal{A})$, denoted $U \cdot \gamma$, for each $U \in \mathcal{A}^*$ and each column γ. Since $\mathcal{A}^*$ is the free monoid on $\mathcal{A}$, it is enough to define the action for each letter $a \in \mathcal{A}$. Define

$$a \cdot \gamma = \gamma'$$

if γ' is obtained from γ by column insertion of a into γ.

For further use, we note that if γ is a column, then we have

$$\gamma \cdot \emptyset = \gamma,$$

where on the left-hand side, γ is viewed as a decreasing word.

5.4.3 Stylic monoid

We denote by $\mathrm{Styl}(\mathcal{A})$ the monoid of endofunctions of the set $\Gamma(\mathcal{A})$ of columns obtained by the action defined above. Thus, a typical element of $\mathrm{Styl}(\mathcal{A})$ is a function

$$\begin{array}{rcl} \mu_W : \Gamma(\mathcal{A}) & \to & \Gamma(\mathcal{A}) \\ \gamma & \mapsto & W \cdot \gamma \end{array}$$

for some word $W \in \mathcal{A}^*$. Since $\Gamma(\mathcal{A})$ is finite, $\mathrm{Styl}(\mathcal{A})$ is finite. Let $\mu : \mathcal{A}^* \to \mathrm{Styl}(\mathcal{A})$ be the canonical monoid homomorphism defined by $\mu(W) = \mu_W$.

We denote by $\equiv_{styl}$ the corresponding monoid congruence of $\mathcal{A}^*$, called the *stylic congruence*:

$$U \equiv_{styl} V \iff \mu(U) = \mu(V) \iff U \cdot \gamma = V \cdot \gamma \text{ for all columns } \gamma.$$

The monoid $\mathrm{Styl}(\mathcal{A})$ acts naturally on the set of columns, and we take the same notation: $M \cdot \gamma = W \cdot \gamma$ if $M = \mu(W)$.

5.4.4 Relationship with the plactic monoid

The *Schensted* P*-symbol* is a mapping that associates with each word W on $\mathcal{A}$ a tableau $\mathrm{P}(W)$, see [69, 56]. The relation $\equiv_{\mathrm{Plax}}$ on $\mathcal{A}^*$, defined by

$$U \equiv_{\mathrm{Plax}} V \iff \mathrm{P}(U) = \mathrm{P}(V),$$

is a congruence of the monoid $\mathcal{A}^*$, called the *plactic congruence*. The quotient $\mathcal{A}^*/\equiv_{\mathrm{Plax}}$ is called the *plactic monoid*.

The *column-reading word* of a tableau is the word obtained by reading the columns from left to right, each column being read as a decreasing word. For example, the column reading word of the tableau from Figure 5.1 is the word $dbabac$. If T is a tableau, with column-reading word W, then

$$\mathrm{P}(W) = T \tag{5.1}$$

by a theorem of Schensted.

The *plactic relations*, due to Knuth, are the following relations:

$$bac \equiv_{\mathrm{Plax}} bca, \qquad acb \equiv_{\mathrm{Plax}} cab \tag{5.2}$$

for any choice of letters $a < b < c$ in $\mathcal{A}$, and

$$bab \equiv_{\mathrm{Plax}} bba, \qquad aba \equiv_{\mathrm{Plax}} baa \tag{5.3}$$

for any choice of letters $a < b$ in $\mathcal{A}$. The plactic congruence is generated by these relations.

By [6, Theorem 8.1], the stylic congruence is generated by the plactic relations (5.2) and (5.3) together with the *idempotent relations* $a^2 = a$ for any letter a in $\mathcal{A}$. It then follows that if $\mathcal{B} \subset \mathcal{A}$, then there is a natural embedding $\mathrm{Styl}(\mathcal{B}) \to \mathrm{Styl}(\mathcal{A})$ [6, Corollary 8.4].

5.4.5 N-tableaux

According to [6, Theorem 7.1], there is a mapping from $\mathcal{A}^*$ into the set of tableaux that induces a bijection from the stylic monoid $\mathrm{Styl}(\mathcal{A})$ onto the set of N-*tableaux* on $\mathcal{A}$. The image of $X \in \mathcal{A}^*$ is denoted $\mathrm{N}(X)$ and is called the N-*tableau* of X. We also denote by N the induced bijection from the stylic monoid onto the set of N-tableaux. The precise definition of N is not needed here, rather we will make use of the following properties of N.

Proposition 5.1 *(i) The first column of* $\mathrm{N}(X)$ *is equal to that of the* P-*symbol* $\mathrm{P}(X)$*, and it is* $X \cdot \emptyset$*, where* $\emptyset$ *denotes the empty column.*

(ii) The set of $a \in \mathcal{A}$ *fixing* $W \in \mathrm{Styl}(\mathcal{A})$ *on the left (*$aW = W$*) is equal to the first column of* $\mathrm{N}(W)$*.*

(iii) If X *is in* $\mathrm{Styl}(\mathcal{A})$*, then the column-reading word* W *of* $\mathrm{N}(X)$ *satisfies*

$$X = \mu(W). \tag{5.4}$$

Proof. The first statement is [6, Lemma 7.2 (i)]. The second statement follows from [6, Theorem 11.4] and definition of the left N-insertion. The third statement follows from the analogous statement for row-reading words, which is [6, Equation 5], and the fact that column-reading and row-reading words of the same tableau are plactic-, hence stylic-, equivalent. □

5.4.6 The anti-automorphism θ

Recall, from [6, Section 9], the involutive anti-automorphism θ of the monoid $\mathcal{A}^*$: when restricted to $\mathcal{A}$, it reverses the order of $\mathcal{A}$. It extends to an endofunction of $\Gamma(\mathcal{A})$, if one identifies as we do columns on $\mathcal{A}$ and subsets of $\mathcal{A}$. Since θ preserves the plactic relations, and the idempotent relations, it induces an anti-automorphism of the monoids $\mathcal{A}^*$, $\mathrm{Plax}(\mathcal{A})$ and of $\mathrm{Styl}(\mathcal{A})$.

5.4.7 Stylic algebra

We denote by $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$ the $\mathbb{Z}$-algebra of the stylic monoid, and we call it the *stylic algebra over* $\mathbb{Z}$. We shall consider also the stylic algebra over a field $\mathbb{K}$, which we denote by $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$.

Lemma 5.2 *Let $X \in \mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$ and let a be a letter such that each letter appearing in X is larger or equal to a.*

(i) $aXa = Xa$;

(ii) $(1-a)Xa = 0$

(iii) $(1-a)X(1-a) = (1-a)X$.

Proof. (i). It follows from Lemma 9.4 in [6].

(ii) and (iii). They follow from an evident computation. □

The next lemma extends the plactic relations in (5.3).

Lemma 5.3 *Let $p, q \geq 1$. Consider letters in $\mathcal{A}$ satisfying $x_1 < \cdots < x_p < y < z_1 < \cdots < z_q$, then*

$$(x_1 \cdots x_p)(z_1 \cdots z_q)y \equiv_{styl} (z_1 \cdots z_q)(x_1 \cdots x_p)y,$$

and

$$y(x_1 \cdots x_p)(z_1 \cdots z_q) \equiv_{styl} y(z_1 \cdots z_q)(x_1 \cdots x_p).$$

Proof. We prove the first identity by double induction. Suppose first that $q = 1$. If $p = 1$, we are reduced to the plactic relation $x_1 z_1 y \equiv_{styl} z_1 x_1 y$. Suppose that $p \geq 2$. Then, by the plactic relations, we have

$$(x_1 \cdots x_{p-1})x_p z_1 y \equiv_{styl} (x_1 \cdots x_{p-1})z_1 x_p y \equiv_{styl} z_1(x_1 \cdots x_{p-1})x_p y$$

by induction on p applied to the product $(x_1 \cdots x_{p-1})z_1 x_p$.

Suppose now that $q \geq 2$. Then, using the congruences $z_q y \equiv_{styl} z_q y y \equiv_{styl} y z_q y$ twice, we have $x_1 \cdots x_p z_1 \cdots z_q y \equiv_{styl} x_1 \cdots x_p z_1 \cdots z_{q-1} y z_q y \equiv_{styl} z_1 \cdots z_{q-1} x_1 \cdots x_p y z_q y$ (by induction on q) $\equiv_{styl}$ $z_1 \cdots z_{q-1} x_1 \cdots x_p z_q y \equiv_{styl} z_1 \cdots z_{q-1} z_q x_1 \cdots x_p y$ (case $q = 1$).

By applying the anti-automorphism θ to the first identity we obtain

$$\theta(y)\theta(z_q) \cdots \theta(z_1)\theta(x_p) \cdots \theta(x_1) \equiv_{\mathrm{Plax}} \theta(y)\theta(x_p) \cdots \theta(x_1)\theta(z_q) \cdots \theta(z_1).$$

Note that

$$\theta(z_q) < \cdots < \theta(z_1) < \theta(y) < \theta(x_p) < \cdots < \theta(x_1).$$

Hence we obtain the second identity of the lemma by a change of variables, after exchanging p and q. $\square$

5.5 Primitive idempotents of the stylic algebra

In this section, we construct a complete system of primitive orthogonal idempotents in the stylic algebra $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$.

Recall that $\Gamma(\mathcal{A})$ denotes the set of columns on the totally ordered finite alphabet $\mathcal{A}$. Let $\gamma \in \Gamma(\mathcal{A})$ be a column and define

$$e_\gamma = \prod_{a \notin \gamma}^{\nearrow} (1-a) \prod_{a \in \gamma}^{\searrow} a \in \mathbb{Z}\,\mathrm{Styl}(\mathcal{A}). \tag{5.5}$$

where the arrows indicate that the first product is in increasing order of letters, and the second in decreasing order.

For example, for $\mathcal{A} = \{t < w < x < y < z\}$ and $\gamma = zxt$, we have

$$e_\gamma = \prod_{a \notin \gamma}^{\nearrow} (1-a) \prod_{a \in \gamma}^{\searrow} a = (1-w)(1-y)zxt = zxt - wzxt - yzxt + wyzxt. \tag{5.6}$$

For future use, we note that the second product in (5.5) is the image of γ (viewed as a word) in $\mathrm{Styl}(\mathcal{A})$; since decreasing words are idempotent in $\mathrm{Styl}(\mathcal{A})$ [6, Theorem 12.1], we have

$$e_\gamma \gamma = e_\gamma. \tag{5.7}$$

Theorem 5.4 *The idempotents e_γ, one for each $\gamma \in \Gamma(\mathcal{A})$, form a complete system of primitive orthogonal idempotents of $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$. Precisely, we have*

(i) $e_\gamma^2 = e_\gamma$ *and* $e_\gamma e_\delta = 0$ *for all* $\gamma, \delta \in \Gamma(\mathcal{A})$ *with* $\delta \neq \gamma$*;*

(ii) $\sum_{\gamma \in \Gamma(\mathcal{A})} e_\gamma = 1$*;*

(iii) for every $\gamma \in \Gamma(\mathcal{A})$*, the idempotent* e_γ *cannot be written as* $e_\gamma = X + Y$ *with* X *and* Y *nonzero orthogonal idempotents in* $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$.

Proof. (i). We show that the elements e_γ are orthogonal idempotents, by induction on the cardinality of the alphabet $\mathcal{A}$. We use the fact that $\mathrm{Styl}(\mathcal{B})$ embeds canonically in $\mathrm{Styl}(\mathcal{A})$ if $\mathcal{B} \subset \mathcal{A}$, and similarly for their monoid algebras.

Let a be the smallest letter in $\mathcal{A}$. Let γ and δ be two columns on $\mathcal{A}$. For $\gamma' \in \Gamma(\mathcal{A} \setminus a)$, we denote by $e'_{\gamma'}$ the elements (5.5) relative to the alphabet $\mathcal{A} \setminus a$. We distinguish four cases:

– If $a \in \gamma \cap \delta$, then by (5.5), $e_\gamma = e'_{\gamma\setminus a}a$ and $e_\delta = e'_{\delta\setminus a}a$. Note that $\gamma = \delta$ if and only if $\gamma \setminus a = \delta \setminus a$, and so, by induction $e'_{\gamma\setminus a}e'_{\delta\setminus a} = e'_{\gamma\setminus a}$ if $\gamma = \delta$, and $e'_{\gamma\setminus a}e'_{\delta\setminus a} = 0$ if $\gamma \neq \delta$. Thus we have $e_\gamma e_\delta = e'_{\gamma\setminus a}ae'_{\delta\setminus a}a = e'_{\gamma\setminus a}e'_{\delta\setminus a}a$ (by Lemma 5.2 (i)), and this is equal to $e'_{\gamma\setminus a}a = e_\gamma$ if $\gamma = \delta$, and to 0 if $\gamma \neq \delta$.

– Suppose now that $a \notin \gamma \cup \delta$. Then $e_\gamma = (1-a)e'_\gamma$ and $e_\delta = (1-a)e'_\delta$. By Lemma 5.2 (iii), we have $e_\gamma e_\delta = (1-a)e'_\gamma(1-a)e'_\delta = (1-a)e'_\gamma e'_\delta$. Thus, $e_\gamma e_\delta$ is e_γ if $\gamma = \delta$, and it is 0 if $\gamma \neq \delta$.

– Suppose that $a \in \gamma, a \notin \delta$. Then $\gamma \neq \delta$ and $e_\gamma e_\delta = e'_{\gamma\setminus a}a(1-a)e'_\delta = e'_{\gamma\setminus a}(a - a^2)e'_\delta = 0$ since a is idempotent.

– Suppose that $a \notin \gamma, a \in \delta$. Then $\gamma \neq \delta$ and $e_\gamma e_\delta = (1-a)e'_\gamma e'_{\delta\setminus a}a = 0$ by Lemma 5.2 (ii).

(ii). We show that the sum in $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$ of all e_γ is equal to 1. Actually we show that this equality holds in the algebra of noncommutative polynomials. By inspection of (5.5), one sees that this sum is equal to a linear combination of all multilinear (without repeated letter) words on $\mathcal{A}$ of the form $W = XY$, where X is strictly increasing, and Y is strictly decreasing. Let W be such a nonempty word; then W has a unique factorization $W = UzV$, where U is strictly increasing, V is strictly decreasing and z is the largest letter in W. Denote by $\mathcal{U}$ the support of U, and by $\mathcal{V}$ that of V. Then the coefficient of W in $e_\mathcal{V}$ is $(-1)^{|\mathcal{U}|+1}$ and in $e_{\mathcal{V}\cup z}$ it is $(-1)^{|\mathcal{U}|}$, while in all other e_γ it is 0 (recall that we identify columns in $\Gamma(\mathcal{A})$ and subsets of $\mathcal{A}$). Thus the coefficient of W in the sum is 0, and therefore the sum is equal to 1.

(iii). We show that the idempotents are primitive. First note that since $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A}) \subset \mathbb{C}\,\mathrm{Styl}(\mathcal{A})$, it suffices to prove it in $\mathbb{C}\,\mathrm{Styl}(\mathcal{A})$. Next, we make use of the following characteristisation: an idempotent e of a finite dimensional $\mathbb{C}$-algebra X is primitive if and only if 0 and e are distinct and are the only idempotents in eXe (see, for instance, [13, Section I.4] [12, Corollary 4.7], or [79, Proposition A.22]). Thus it is enough to prove that $e_\gamma\mathbb{C}\,\mathrm{Styl}(\mathcal{A})e_\gamma = \mathbb{C}e_\gamma$, which we do by induction on the cardinality of $\mathcal{A}$. Let $a = \min(\mathcal{A})$ and $W \in \mathcal{A}^*$.

– Suppose $a \in \gamma$. Then $e_\gamma W e_\gamma = e'_{\gamma\backslash a} a W e'_{\gamma\backslash a} a = e'_{\gamma\backslash a} W' e'_{\gamma\backslash a} a$, by repeated application of Lemma 5.2 (i), where W' is obtained from W by removing all occurrences of a. Hence, $e'_{\gamma\backslash a} W' e'_{\gamma\backslash a} \in \mathbb{C}\,\mathrm{Styl}(\mathcal{A}\backslash a)$, so by induction there exists $z \in \mathbb{C}$ such that $e_\gamma W e_\gamma = (e'_{\gamma\backslash a} W' e'_{\gamma\backslash a})a = (ze'_{\gamma\backslash a})a = ze_\gamma$.

– Suppose $a \notin \gamma$. Then $e_\gamma W e_\gamma = (1-a)e'_{\gamma\backslash a} W (1-a) e'_{\gamma\backslash a}$. This is equal to $(1-a)e'_{\gamma\backslash a} W e'_{\gamma\backslash a}$ by Lemma 5.2 (iii), and by induction there exists $z \in \mathbb{C}$ such that $(1-a)(e'_{\gamma\backslash a} W e'_{\gamma\backslash a}) = (1-a)(ze'_{\gamma\backslash a}) = ze_\gamma$.

To conclude, it is enough to show that the e_γ are nonzero. For this, it suffices to note that each e_γ contains a unique element that is minimal with respect to the $\mathscr{J}$-order on the monoid; we delay the details to the proof of Proposition 5.15 (which will be proved independently), in which we construct a basis of the monoid algebra of $\mathrm{Styl}(\mathcal{A})$ that includes these idempotents. □

5.6 The quiver of the stylic algebra

In this section, we identify the quiver of $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$ over a field $\mathbb{K}$. We do this by defining a quiver $Q(\mathcal{A})$ in Section 5.6.1 together with a $\mathbb{K}$-algebra morphism $\varphi : \mathbb{K}Q(\mathcal{A}) \to \mathbb{K}\,\mathrm{Styl}(Q)$ in Section 5.6.3 that is surjective (proved in Section 5.6.5) and whose kernel is an admissible ideal (proved in Section 5.6.6). Such a morphism uniquely determines the quiver of an algebra; see Section 5.6.7 for details. Most of the results hold over $\mathbb{Z}$, so we work over $\mathbb{Z}$ whenever possible.

5.6.1 A quiver

We define a *right* action of the monoid $\mathcal{A}^*$ on the set $\Gamma(\mathcal{A})$ of columns on $\mathcal{A}$. It is enough to define the action of each letter on each column. Let c be a letter and γ a column. If $c < \min(\gamma)$, we let $\gamma \cdot c = \gamma \cup c$. Otherwise, $c \geq \min(\gamma)$ and we let $b = \max\{x \in \gamma : x \leq c\}$; then $\gamma \cdot c = c \cup (\gamma \setminus b)$; we then say that b is *bumped*. Compactly,

$$\gamma \cdot c = \big(\gamma \setminus \max\{x \in \gamma : x \leq c\}\big) \cup \{c\}.$$

We say that the right action of c on γ is *frank* if $c \geq \min(\gamma)$ and if $c \notin \gamma$. Note that in this case, γ and $\gamma \cdot c$ have the same height.

We define a quiver $Q(\mathcal{A})$ with edges labelled in $\mathcal{A}$: its set of vertices is $\Gamma(\mathcal{A})$; and there is a labelled edge $\gamma \xrightarrow{c} \gamma'$ if $\gamma \cdot c = \gamma'$ and if the action is frank; see Figure 5.2. As usual, the label of a path is the word in $\mathcal{A}^*$ that is the product of the labels of the edges of the path.

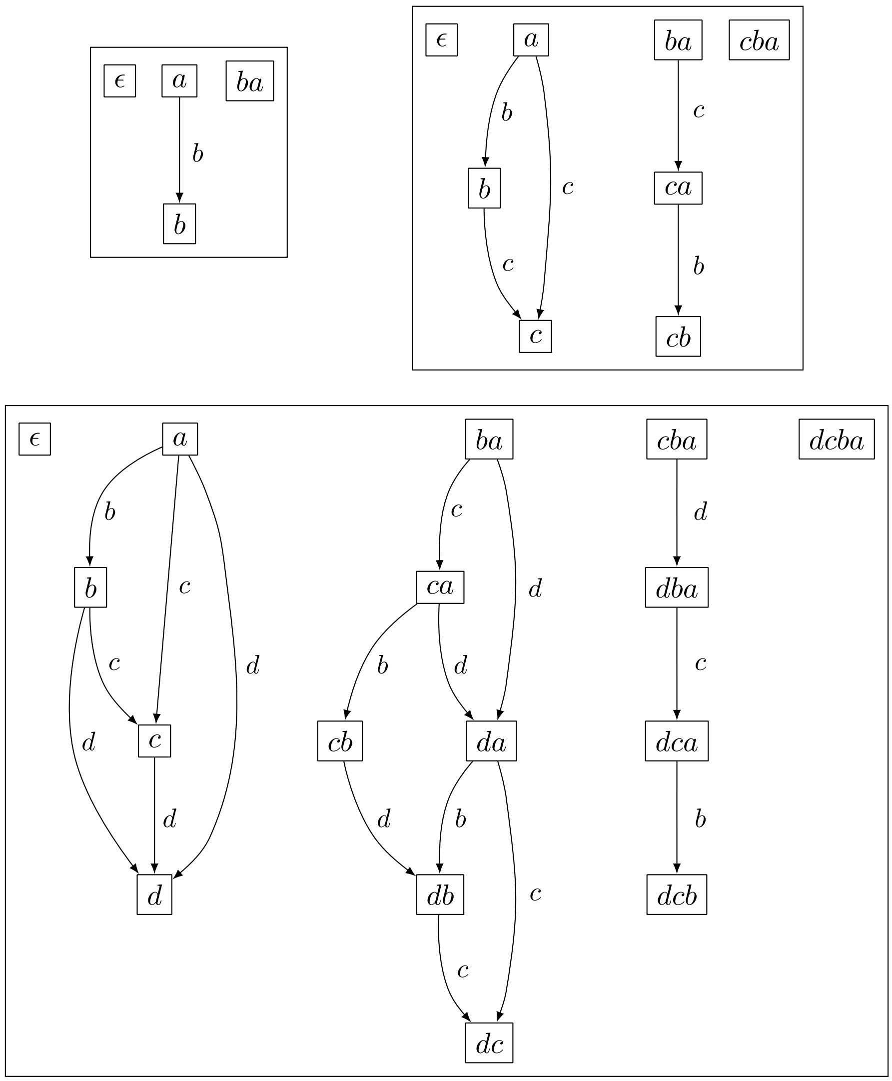

Figure 5.2: The quivers for alphabets of cardinality $2, 3, 4$; the columns are represented by decreasing words and the empty word is denoted ϵ.

For later use, we note the following result relating the left and right actions. The proof is left to the reader.

Lemma 5.5 *For two columns of the same height γ, δ, and two letters b, c, the two following conditions are equivalent:*

(i) $b \cdot \delta = \gamma$ *and* c *is bumped;*

(ii) $\gamma \cdot c = \delta$*, and* b *is bumped.*

5.6.2 A lemma on edges and idempotents

We give a technical, but important, result on the idempotents of the stylic algebra and the quiver introduced previously.

Lemma 5.6 *Let* $\gamma \xrightarrow{c} \delta$ *be an edge in the quiver* $Q(\mathcal{A})$*, and denote by* b *the bumped letter, so that* $\delta = c \cup (\gamma \setminus b)$*, and* $b \in \gamma, c \notin \gamma, b \notin \delta, c \in \delta$*. Then in* $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$

$$be_\gamma c = bce_\delta \qquad \textit{and} \qquad e_\gamma ce_\delta = e_\gamma c.$$

Proof. (1). We prove the first identity. Let $a = \min(\mathcal{A})$. As in the proof of Theorem 5.4, denote by $e'_{\gamma'}$ the idempotents (5.5) relative to the alphabet $\mathcal{A} \setminus a$.

(I). Suppose that $a \in \gamma \cap \delta$. Since the action $\gamma \cdot c$ is frank, and since a cannot be bumped, we have $a < b < c$. Let $\gamma' = \gamma \setminus a$ and $\delta' = \delta \setminus a$. Then $\delta' = \gamma' \cdot c$ and the action is frank. By induction, we deduce that $be'_{\gamma'}c = bce'_{\delta'}$. Note that the minimum of γ' is a letter x such that $a < x < c$; thus $xac = xca$ and since x is the last factor in the product (5.5) defining $e'_{\gamma'}$, we have $e'_{\gamma'}ac = e'_{\gamma'}ca$. We have $e_\gamma = e'_{\gamma'}a$ and $e_\delta = e'_{\delta'}a$. Thus $be_\gamma c = be'_{\gamma'}ac = be'_{\gamma'}ca = bce'_{\delta'}a = bce_\delta$.

(II). Suppose that $a \notin \gamma \cup \delta$. Then $a < b < c$. Moreover $\delta \xrightarrow{c} \delta'$ is an edge in the quiver $Q(\mathcal{A} \setminus a)$ and b is bumped. With notations similar to 1, we have $e_\gamma = (1-a)e'_{\gamma'}$ and $e_\delta = (1-a)e'_{\delta'}$. Since $b(1-a)b = b^2 - bab = b^2 - ba = b(1-a)$ and $bac = bca$, we have $be_\gamma c = b(1-a)e'_{\gamma'}c = b(1-a)be'_{\gamma'}c = b(1-a)bce'_{\delta'}$ (by induction) $= b(1-a)ce'_{\delta'} = bc(1-a)e'_{\delta'} = bce_\delta$.

(III). Suppose that $a \in \gamma$ and $a \notin \delta$. Then the bumped letter is $b = a$. We denote by γ and δ the decreasing words associated with these two columns. Since $c \in \delta$, we have $\delta = \delta_1 c \delta_2$, where each letter in δ_1 is larger than c; hence $c\delta \equiv_{styl} \delta$ by Lemma 5.2 (i). Moreover, $\delta c \equiv_{styl} \delta$ since c is the smallest, hence last, letter of $\delta = \delta' c$ and $c^2 = c$.

We have

$$\gamma c \equiv_{styl} a\delta, \tag{5.8}$$

since this holds even plactically as one sees by computing the image under P of both sides (for $\mathrm{P}(a\delta)$, Schensted left insert a into δ and use Lemma 5.5).

Let $y_1, \dots, y_s, z_1, \dots, z_t$ be the letters in $\mathcal{A}$ that do not appear in γ nor in δ, ordered so that

$$a \ < \ y_1 < \cdots < y_s \ < \ c \ < \ z_1 < \cdots < z_t.$$

Then

$$e_\gamma = \prod_{j=1}^{s}(1 - y_j)(1 - c)\prod_{k=1}^{t}(1 - z_k)\gamma$$
$$e_\delta = (1 - a)\prod_{j=1}^{s}(1 - y_j)\prod_{k=1}^{t}(1 - z_k)\delta.$$

Thus, by (5.8), $ae_\gamma c = a\prod_{j=1}^{s}(1 - y_j)(1 - c)\prod_{k=1}^{t}(1 - z_k)a\delta$. Note that $\prod_{j=1}^{s}(1 - y_j)$ is equal to 1 plus a linear combination of Uy, with

$$a < y < c < z_1 < \cdots < z_t.$$

Therefore, $ae_\gamma c$ is equal to $a(1-c)\prod_{k=1}^{t}(1-z_k)a\delta$ plus a linear combination of $aUy(1-c)\prod_{k=1}^{t}(1-z_k)a\delta$, and we show that each term in the linear combination vanishes.

Note that it suffices to show that $y\prod_{k=1}^{t}(1 - z_k)a\delta = yc\prod_{k=1}^{t}(1 - z_k)a\delta$. We prove this equality, starting from the right-hand side: since $\delta = \delta' c$, we have $yc\prod_{k=1}^{t}(1 - z_k)a\delta = yc\prod_{k=1}^{t}(1 - z_k)a\delta' c = yac\prod_{k=1}^{t}(1-z_k)\delta' c$ (since by the second identity in Lemma 5.3, we have $yac\prod_{k=1}^{t}(1-z_k) = yc\prod_{k=1}^{t}(1-z_k)a) = ya\prod_{k=1}^{t}(1 - z_k)\delta' c$ (by Lemma 5.2 (i), since all letters z_i and in δ' are $> c$) $= y\prod_{k=1}^{t}(1 - z_k)a\delta$, by the same identity in Lemma 5.3.

It follows that $ae_\gamma c = a(1 - c)\prod_{k=1}^{t}(1 - z_k)a\delta = a(1 - c)\prod_{k=1}^{t}(1 - z_k)ac\delta$ (since $c\delta = \delta$) $= a(1 - c)a\prod_{k=1}^{t}(1 - z_k)c\delta$ (by the first identity in Lemma 5.3) $= a(1 - c)ac\prod_{k=1}^{t}(1 - z_k)c\delta$ (by Lemma 5.2 (i)) $= (ac - ca)\prod_{k=1}^{t}(1 - z_k)\delta$.

On the other hand, we have $ace_\delta = ac(1-a)\prod_{j=1}^{s}(1-y_j)\prod_{k=1}^{t}(1-z_k)\delta = (ac-ca)\prod_{j=1}^{s}(1-y_j)\prod_{k=1}^{t}(1-z_k)\delta$. Note that $\prod_{j=1}^{s}(1-y_j)$ is equal to 1 plus a linear combination of yU, with $a < y < c$. Since $(ac-ca)y = acy - cay = 0$ (plactic relation), we obtain $ace_\delta = (ac-ca)\prod_{k=1}^{t}(1-z_k)\delta$.

It follows that $ae_\gamma c = ace_\delta$.

(IV). The last case to consider is when $a \notin \gamma$ and $a \in \delta$; however, it does not occur because the action $\gamma \cdot c$ is frank (in particular, if $a \notin \gamma$, then $a \notin \delta$).

(2). We prove now the second identity. Note that $\gamma = \gamma_1 b \gamma_2$, where each letter in γ_2 is smaller that b; hence $\gamma b = \gamma$, by the dual statement of Lemma 5.2 (i). We have, using the fact that γ is idempotent in $\mathrm{Styl}(\mathcal{A})$ (see the sentence before (5.7)):

$$\begin{aligned} e_\gamma c e_\delta &= e_\gamma b c e_\delta && \text{(since } e_\gamma = e_\gamma\gamma = e_\gamma\gamma b = e_\gamma b\text{)} \\ &= e_\gamma b e_\gamma c && \text{(by the first identity in the lemma, already proved)} \\ &= e_\gamma e_\gamma c && \text{(since } e_\gamma b = e_\gamma\text{)} \\ &= e_\gamma c && \text{(since } e_\gamma \text{ is idempotent).} \end{aligned}$$

□

5.6.3 A quiver map

Let $Q = Q(\mathcal{A})$ be the quiver defined in Subsection 5.6.1. The *path algebra* $\mathbb{Z}Q$ is the free $\mathbb{Z}$-module with basis the set of paths in the quiver, including an empty path around each vertex γ (this empty path is denoted γ); the product is the unique product extending the natural product of paths.

We define a $\mathbb{Z}$-linear mapping $\varphi : \mathbb{Z}Q \to \mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$ as follows:

- if γ is an empty path, then
$$\varphi(\gamma) = e_\gamma;$$
- if
$$\gamma_0 \xrightarrow{c_1} \gamma_1 \xrightarrow{c_2} \cdots \xrightarrow{c_l} \gamma_l \tag{5.9}$$

is a path in Q, then its image under φ is

$$e_{\gamma_0} c_1 e_{\gamma_1} c_2 \cdots c_l e_{\gamma_l}.$$

Note that this mapping is a $\mathbb{Z}$-algebra homomorphism.

Theorem 5.7 *The image under φ of a path from γ to δ with label U is $e_\gamma U$.*

Proof. This is clear if the path is of length 0. Suppose it is true for each path of length $l \geq 0$. Consider a path $\gamma_0 \xrightarrow{c_1} \gamma_1 \xrightarrow{c_2} \cdots \xrightarrow{c_{l+1}} \gamma_{l+1}$. Its image under φ is by definition $x = \varphi(p) e_{\gamma_l} c_{l+1} e_{\gamma_{l+1}}$, where p is the path (5.9). Thus $x = \varphi(p) e_{\gamma_l} c_{l+1}$ (by the second equality in Lemma 5.6) $= \varphi(p) c_{l+1}$ (since $\varphi(p) e_{\gamma_l} = \varphi(p)$ by definition of φ and the idempotence of e_{γ_l}) $= e_{\gamma_0} c_1 \cdots c_l c_{l+1}$ (by induction). □

Corollary 5.8 *Consider two paths in $Q(\mathcal{A})$ starting from the same vertex γ, with labels U, V. If $\gamma U \equiv_{styl} \gamma V$, then these paths have the same image under φ.*

Proof. The images of these paths are $e_\gamma U$ and $e_\gamma V$, respectively. These elements are by (5.7) equal to $e_\gamma \gamma U$ and $e_\gamma \gamma V$. Thus, the lemma follows. □

5.6.4 Extended quiver

The *extended quiver* $Q'(\mathcal{A})$ has the same set of vertices as $Q(\mathcal{A})$, has all edges of $Q(\mathcal{A})$, together with new edges, which are loops: for each column γ and each $c \in \gamma$, we have in $Q'(\mathcal{A})$ the edge

$$\gamma \xrightarrow{c} \gamma.$$

It is clearly a deterministic automaton. Note that if $c \in \mathcal{A}$ and $\gamma \in \Gamma(\mathcal{A})$, there is an edge labelled c starting from γ in $Q'(\mathcal{A})$ if and only if $c \geq min(\gamma)$. Moreover, if for $\gamma, \delta \in \Gamma(\mathcal{A}), W \in \mathcal{A}^*$, there is a path $\gamma \xrightarrow{W} \delta$ in $Q'(\mathcal{A})$, then $\delta = \gamma \cdot W$.

Proposition 5.9 *Let $X \in \mathrm{Styl}(\mathcal{A})$ and denote by γW the column-reading word of the* N*-tableau* $\mathrm{N}(X)$ *of X, with γ being the first column of* $\mathrm{N}(X)$*. Then there is a unique path in the extended quiver $Q'(\mathcal{A})$, starting from γ, with label W.*

Before proving the proposition, we prove a useful lemma, showing that the involution θ defined in Section 5.4.6 conjugates the left and right actions.

Lemma 5.10 *Let $W \in \mathcal{A}^*$ and $\gamma \in \Gamma(\mathcal{A})$. Then*

$$\gamma \cdot W = \theta(\theta(W) \cdot \theta(\gamma)).$$

Proof. For $W \in \mathcal{A}$, the formula follows from the definitions of the left and right actions on columns. To conclude, it is enough to prove that if the formula holds for $U, V \in \mathcal{A}^*$, then also for $W = UV$. We have $\gamma \cdot W = \gamma \cdot (UV) = (\gamma \cdot U) \cdot V = \theta(\theta(V) \cdot \theta(\gamma \cdot U)) = \theta(\theta(V) \cdot (\theta(U) \cdot \theta(\gamma)) = \theta((\theta(V)\theta(U)) \cdot \theta(\gamma)) = \theta(\theta(UV) \cdot \theta(\gamma)) = \theta(\theta(W) \cdot \theta(\gamma))$. □

Proof. [Proof of Proposition 5.9] Uniqueness follows from the deterministic property of $Q'(\mathcal{A})$.

To prove the existence of this path, it is enough, by the definition of the right action and of the extended quiver, to show that the height of $\gamma \cdot P$ is equal to the height k of γ, for each prefix P of W.

Since γW is the column-reading word of $\mathrm{N}(X)$, it follows from (5.1) that the P-tableau of γW is equal to $\mathrm{N}(X)$. Thus, by Schensted's theorem, the height k of $\mathrm{N}(X)$ is equal to the length of the longest strictly decreasing subsequence of γW. Now, the length of the longest strictly decreasing subsequence of $\theta(\gamma W)$ is k, too. Hence, the height of the P-tableau of $\theta(\gamma W)$ is k; by the definition of left action, the first column of this tableau is $\theta(\gamma W) \cdot \emptyset$, and this column is equal to $(\theta(W)\theta(\gamma)) \cdot \emptyset = \theta(W) \cdot (\theta(\gamma) \cdot \emptyset) = \theta(W) \cdot \theta(\gamma)$. Therefore, applying θ and using Lemma 5.10, we see that $\gamma \cdot W$ is of height k.

Since the (left and right) action on columns never decreases the height, it follows that for each prefix P of W, the height of $\gamma \cdot P$ is equal to k. □

Lemma 5.10 has the following corollary.

Corollary 5.11 *Let W be a word and γ be a column. Then $\gamma W \equiv_{styl} U(\gamma \cdot W)$ for some word U.*

Proof. We know that $W \cdot \gamma$ is the first column of $\mathrm{P}(W\gamma)$. It follows by column reading and Schensted's theorem that $W\gamma \equiv_{\mathrm{Plax}} (W \cdot \gamma)U$ for some word U. Applying θ and using Lemma 5.10, we find that for each word W and each column γ, $\gamma W \equiv_{\mathrm{Plax}} U(\gamma \cdot W)$ for some word U; therefore $\gamma W \equiv_{styl} U(\gamma \cdot W)$. □

Each path

$$\gamma \xrightarrow{W} \gamma \cdot W \tag{5.10}$$

in the extended quiver $Q'(\mathcal{A})$, starting from vertex γ and with label W, defines a path

$$\gamma \xrightarrow{W'} \gamma \cdot W \tag{5.11}$$

in the quiver $Q(\mathcal{A})$, by removing the loops. Precisely, we define the label W' of the associated path in $Q(\mathcal{A})$ recursively as follows: if W is empty, $W' = W$; otherwise $W = Uc$, $U \in \mathcal{A}^*$, $c \in \mathcal{A}$, U' is constructed by induction, and then:

- first case: $W' = U'$ if $\gamma \cdot U = (\gamma \cdot U) \cdot c$ (equivalently $c \in \gamma \cdot U$);
- second case: $W' = U'c$ otherwise.

We call this construction *loops removal*.

Lemma 5.12 *With these notations, $\gamma \cdot W' = \gamma \cdot W$.*

Proof. We follow the construction. If W is empty, then W' is empty, and the equality is evident. Suppose now that $W = Uc$. In the first case, $\gamma \cdot W' = \gamma \cdot U' = \gamma \cdot U$ (by induction) $= \gamma \cdot (Uc) = \gamma \cdot W$. In the second case, $\gamma \cdot W' = \gamma \cdot (U'c) = (\gamma \cdot U') \cdot c = (\gamma \cdot U) \cdot c$ (by induction) $= \gamma \cdot (Uc) = \gamma \cdot W$. □

Lemma 5.13 *With these notations, $\gamma W' \equiv_{styl} \gamma W$.*

Proof. (1). Let $\delta \xrightarrow{c} \delta \cdot c$ be a an edge in the extended quiver $Q'(\mathcal{A})$. Then $c \geq \min(\delta)$. Next, $\delta c \equiv_{styl} b(\delta \cdot c)$, with $b \in \mathcal{A}$: this equality holds indeed plactically, as a particular case of the presentation by columns of the plactic monoid due to [21, 26], after applying θ and Lemma 5.10 (see also [6] Proposition 12.3 (v)).

Suppose that moreover $c \in \delta$, equivalently $\delta \cdot c = \delta$. Then $\delta = \delta_1 c \delta_2$ with each letter in δ_2 smaller than c. Then $\delta_2 c \equiv_{styl} \delta_2$, by the dual form of Lemma 5.2 (i), from which follows $\delta c \equiv_{styl} \delta$.

(2). We prove the lemma by following the recursive construction of W'. If W is empty, it is evident. Suppose now that $W = Uc$ and assume by induction that $\gamma U \equiv_{styl} \gamma U'$, where U' is obtained from U by loops removal. By Corollary 5.11, we have $\gamma U \equiv_{styl} V(\gamma \cdot U)$ for some word V. By 1 and 2, we have $(\gamma \cdot U)c \equiv_{styl} b(\gamma \cdot W)$, with $b = 1$ if $\gamma \cdot U = \gamma \cdot (Uc)$, and $b \in \mathcal{A}$ otherwise.

In the first case, we have $\gamma \cdot U = \gamma \cdot (Uc) = \gamma \cdot W$, $W' = U'$, $b = 1$. Then $\gamma W = \gamma Uc \equiv_{styl} V(\gamma \cdot U)c \equiv_{styl} V(\gamma \cdot W)$, and $\gamma W' = \gamma U' \equiv_{styl} \gamma U \equiv_{styl} V(\gamma \cdot U) = V(\gamma \cdot W)$.

In the second case, we have $\gamma \cdot W = \gamma \cdot (Uc) \neq \gamma \cdot U$, $W' = U'c$, $b \in \mathcal{A}$. Then $\gamma W = \gamma Uc \equiv_{styl} V(\gamma \cdot U)c \equiv_{styl} Vb(\gamma \cdot W)$, and $\gamma W' = \gamma U'c \equiv_{styl} \gamma Uc \equiv_{styl} V(\gamma \cdot U)c \equiv_{styl} Vb(\gamma \cdot W)$. □

Corollary 5.14 *The image under φ of the path (5.11) in $Q(\mathcal{A})$, obtained from the path (5.10) in $Q'(\mathcal{A})$ by loops removal, is equal to $e_\gamma W$.*

Proof. Suppose that W is the label of a path in $Q'(\mathcal{A})$ starting form γ; define W' by loops removal. The image under φ of our path of $Q(\mathcal{A})$ is by Theorem 5.7 equal to $y = e_\gamma W'$. By (5.7), we have $y = e_\gamma \gamma W'$. Hence by Lemma 5.13, the corollary follows. □

5.6.5 The surjectivity of the quiver map

Let $X \in \mathrm{Styl}(\mathcal{A})$ and denote by $\eta(X)W_X$ the column-reading word of the N-tableau $\mathrm{N}(X)$ of X, with $\eta(X)$ being the first column of $\mathrm{N}(X)$. Recall from Proposition 5.9, that we have constructed a path, in the extended quiver $Q'(\mathcal{A})$, starting form $\eta(X)$ and with label W_X. From this path in $Q'(\mathcal{A})$, we obtain by loops removal in Section 5.6.4, a path in $Q(\mathcal{A})$ starting form $\eta(X)$ and with label W'_X; we call such a path an N-*path*.

Proposition 5.15 *The set $\{e_{\eta(X)}W_X : X \in \mathrm{Styl}(\mathcal{A})\}$ is a basis of $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$.*

Proof. Recall from [6] that $\mathrm{Styl}(\mathcal{A})$ is a $\mathscr{J}$-trivial monoid, and that it has therefore the $\mathscr{J}$-order $\leq_{\mathscr{J}}$. One has $X \leq_{\mathscr{J}} Y$ if and only if for some U, V, $X = UYV$ (all these elements are in $\mathrm{Styl}(\mathcal{A})$).

Let $X \in \mathrm{Styl}(\mathcal{A})$, with $\eta(X) = \gamma$; then by (5.4), $X = \gamma W_X$ in $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$, and by (5.5),

$$e_{\eta(X)} W_X = \overset{\nearrow}{\prod_{a \notin \gamma}} (1-a)X.$$

Let $a \notin \gamma$; then $aX \leq_{\mathscr{J}} X$; moreover by Proposition 5.1, and since γ is the first column of $\mathrm{N}(X)$, $aX \neq X$ and therefore $aX <_{\mathscr{J}} X$. It follows from the displayed formula that $e_{\eta(X)} W_X$ is equal to X plus a linear combination of elements strictly smaller than X in the $\mathscr{J}$-order. Hence, by triangularity, the elements $e_{\eta(X)} W_X$, $X \in \mathrm{Styl}(\mathcal{A})$, form a basis of $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$. □

Corollary 5.16 *The quiver map φ is surjective.*

Proof. The element $e_{\eta(X)} W_X$ is the image under φ of the path constructed in Corollary 5.14. Hence, by Proposition 5.15, φ is surjective. □

Corollary 5.17 *The* N*-paths are linearly independent modulo* $\ker(\varphi)$.

5.6.6 The kernel of the quiver map

The following result shows that $\ker(\varphi)$ is completely described by Corollary 5.8.

Proposition 5.18 *The kernel of φ is spanned by the elements which are differences of two paths in $Q(\mathcal{A})$ starting from the same vertex γ and having labels U, V satisfying $\gamma U \equiv_{styl} \gamma V$.*

Proof. Denote by H the subspace described in the statement. We know by Corollary 5.8 that H is a subspace of $\ker(\varphi)$.

Consider a path starting from γ and with label U. Let $X = \gamma U$.

(1). We show that the first column of $\mathrm{N}(X)$ is γ.

If $a \in \gamma$, by Lemma 5.2 (i), we have $a\gamma = \gamma$ in $\mathrm{Styl}(\mathcal{A})$; hence, a is in the first column of $\mathrm{N}(X)$ (by Proposition 5.1), and this column therefore contains γ.

Moreover, by definition of the quiver and of paths, the height of $\gamma \cdot U$ is the same as the height h of γ; thus the height of $\theta(\gamma \cdot U)$ is h, and so is that of $\theta(U) \cdot \theta(\gamma)$ by Lemma 5.10; but this column is the first column of $\mathrm{N}(\theta(U)\theta(\gamma)) = \mathrm{N}(\theta(\gamma U)) = \mathrm{N}(\theta(X))$. By Theorem 9.1 in [6], $\mathrm{N}(X)$ and $\mathrm{N}(\theta(X))$ have the same height; the height of $\mathrm{N}(X)$ is therefore h. It follows that its first column is γ.

(2). Consider now the path (5.10) of $Q'(\mathcal{A})$ constructed in Proposition 5.9, and the associated path (5.11) in $Q(\mathcal{A})$, obtained by removing the loops: it starts at γ and has W' as label.

We know that $\gamma U = X \equiv_{styl} \gamma W$, by (5.4) since the latter word is the column-reading word of $\mathrm{N}(X)$. Hence by Lemma 5.13, $\gamma U \equiv_{styl} \gamma W'$ and therefore the two paths of $Q(\mathcal{A})$ starting at γ and with labels U and W' have the same image under φ, by Corollary 5.8.

It follows that each element in the quiver algebra is congruent modulo H to a linear combination of N-paths. Since by Corollary 5.17 these N-paths are linearly independant modulo $\ker(\varphi)$, it follows that $\ker(\varphi) \subset H$. □

5.6.7 The quiver of the stylic algebra

Now, let $\mathbb{K}$ be a field of characteristic 0. We apply a theorem of Auslander, Reiten and Smalø [13], in order to prove that $Q(\mathcal{A})$ is the quiver of $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})$.

Theorem 5.19 *The quiver of the stylic algebra over* $\mathbb{K}$ *is* $Q(\mathcal{A})$.

We first prove the following useful lemma.

Lemma 5.20 *Let* γ *be a column, and* $U, V \in \mathcal{A}^*$. *If* $\gamma U \equiv_{styl} \gamma V$, *then* $\gamma \cdot U = \gamma \cdot V$.

Proof. By Lemma 5.10, it is enough to prove the dual statement: if $U\gamma \equiv_{styl} V\gamma$, then $U \cdot \gamma = V \cdot \gamma$. The hypothesis implies that $(U\gamma) \cdot \emptyset = (V\gamma) \cdot \emptyset$. This implies $U \cdot (\gamma \cdot \emptyset) = V \cdot (\gamma \cdot \emptyset)$, thus $U \cdot \gamma = V \cdot \gamma$. □

Proof. [Proof of Theorem 5.19] According to a theorem in [13], in the formulation of [34, Theorem 3.3.4], it is enough to show that the ideal $\ker(\varphi)$ is *admissible*. This means that $F^m \subset \ker(\varphi) \subset F^2$, where F is the ideal in $\mathbb{K}Q(\mathcal{A})$ generated by the arrows of $Q(\mathcal{A})$.

The first inclusion is clear, since the quiver has no closed path, so that for m large enough, $F^m = 0$.

We know that $\ker(\varphi)$ is spanned by the elements, differences of two paths, described in Proposition 5.18, whose notations we use now. In particular, $\gamma U \equiv_{styl} \gamma V$. Thus it is enough to show that U, V are both of length at least 2. We may assume that the element is nonzero.

Observation 1: the support of γU and γV must be equal, since these words are stylically congruent.

Observation 2: assuming that the alphabet is $1, 2, 3, \ldots$, call *weight* of a column the sum of its elements. Then by definition of frank action, the weight of $\gamma \cdot a$ is larger that the weight of γ, and so the weight of the vertices strictly increases along a path in $Q(\mathcal{A})$.

Suppose by contradiction that U is of length 0 or 1, and we begin by length 0. If V also is of length 0, the element is 0, which was excluded. If V is of positive length then, since the action is frank, the support of γV is strictly larger than that of γ; hence the supports of $\gamma U = \gamma$ and of γV differ, so that by Observation 1, we cannot have $\gamma U \equiv_{styl} \gamma V$.

Thus we may assume that $U = b$ is of length 1. Then V cannot be of length 0, by the same argument just given. If V is of length 1, then by Observation 1, and since the two actions are frank (so that $U, V \notin \gamma$), we must have $U = V$, and the element is 0, which was excluded. Thus V is of length at least 2: $V = cV'$, V' nonempty; then by Observation 1, c appears in γb, but not in γ, since the action $\gamma \cdot c$ is frank, hence $c = b$; but then by Observation 2, the weight of $\gamma \cdot V$ is larger than that of $\gamma \cdot b$, and we cannot have the equality $\gamma \cdot U = \gamma \cdot V$, contradicting $\gamma U \equiv_{styl} \gamma V$ by Lemma 5.20. □

5.6.8 Cartan invariants and Indecomposable Projective Modules

The Cartan invariants of a finite dimensional $\mathbb{K}$-algebra Λ are the numbers $\dim_{\mathbb{K}}(e_i \Lambda e_j)$, where $\{e_1, \ldots, e_n\}$ is a complete system of primitive orthogonal idempotents of Λ. They do not depend on the choice of the complete system.

In the case of the stylic monoid, we are therefore interested in computing the dimension of the subspaces $e_\gamma \mathbb{K}\,\mathrm{Styl}(\mathcal{A}) e_{\gamma'}$ for $\gamma, \gamma' \in \Gamma(\mathcal{A})$.

Proposition 5.21 *For* $\gamma, \gamma' \in \Gamma(\mathcal{A})$,

$$\dim\left(e_\gamma \mathbb{K}\,\mathrm{Styl}(\mathcal{A}) e_{\gamma'}\right) = \left|\left\{X \in \mathrm{Styl}(\mathcal{A}) : \eta(X) = \gamma \textit{ and } \theta(\eta(\theta(X)) = \gamma'\right\}\right|.$$

Proof. By Proposition 5.15, we have that $\{e_{\eta(X)} W_X : X \in \mathrm{Styl}(\mathcal{A})\}$ is a basis of $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$. Moreover, each $e_{\eta(X)} W_X$ is the image under φ of a path in $Q'(\mathcal{A})$ that starts at $\eta(X)$, is labelled W_X, and ends at $\theta(\eta(\theta(X)))$; see Section 5.6.5. Thus,

$$e_{\eta(X)} W_X = e_{\eta(X)} W_X e_{\theta(\eta(\theta(X)))}.$$

It follows that $\{e_{\eta(X)} W_X e_{\theta(\eta(\theta(X)))} : X \in \mathrm{Styl}(\mathcal{A})\}$ is a basis of $\mathbb{Z}\,\mathrm{Styl}(\mathcal{A})$, and that

$$\{e_{\eta(X)} W_X e_{\theta(\eta(\theta(X)))} : X \in \mathrm{Styl}(\mathcal{A}) \text{ with } \eta(X) = \gamma \text{ and } \theta(\eta(\theta(X))) = \gamma'\}$$

is a basis of $e_\gamma \mathbb{Z}\,\mathrm{Styl}(\mathcal{A}) e_{\gamma'}$. □

We remark that an alternative proof of Proposition 5.21 can be obtained by appealing to [34, Theorem 3.20], which gives a formula for the Cartan invariants for any $\mathscr{J}$-trivial monoid **M**. Applied to $\mathrm{Styl}(\mathcal{A})$, the formula says that the Cartan invariants are given by

$$\{X \in \mathrm{Styl}(\mathcal{A}) : \mathrm{lfix}(X) = \gamma \text{ and } \mathrm{rfix}(X) = \gamma'\},$$

where

- $\mathrm{lfix}(X) = \min_{\leq_{\mathscr{J}}} \left\{e \in \mathrm{Styl}(\mathcal{A}) : e^2 = e \text{ and } eX = X\right\}$
- $\mathrm{rfix}(X) = \min_{\leq_{\mathscr{J}}} \left\{e \in \mathrm{Styl}(\mathcal{A}) : e^2 = e \text{ and } Xe = X\right\}$

Proposition 5.21 then follows by observing that $\mathrm{lfix}(X) = \eta(X)$ and $\mathrm{rfix}(X) = \theta(\eta(\theta(X)))$ for all $X \in \mathrm{Styl}(\mathcal{A})$.

Finally, using a similar argument to the proof of Proposition 5.21, one obtains bases for the right and left indecomposable projective $\mathrm{Styl}(\mathcal{A})$-modules.

Proposition 5.22 *For $\gamma \in \Gamma(\mathcal{A})$,*

1. $\{e_{\eta(X)}W_X : X \in \mathrm{Styl}(\mathcal{A})$ *with* $\eta(X) = \gamma\}$ *is basis of* $e_\gamma \mathbb{K}\,\mathrm{Styl}(\mathcal{A})$*, and*
2. $\{e_{\eta(X)}W_X : X \in \mathrm{Styl}(\mathcal{A})$ *with* $\theta(\eta(\theta(X))) = \gamma\}$ *is basis of* $\mathbb{K}\,\mathrm{Styl}(\mathcal{A})e_\gamma$*.*

CONCLUSION

L'étude de l'action des mots sur les colonnes par l'insertion à gauche de Schensted s'est montrée très fructueuse et ouvre la voie à d'autres projets de recherche. L'étude plus approfondie de ce monoïde en fait certes partie. Plusieurs personnes se sont déjà attelés à la tâche en étudiant les identités satisfaites par le monoïde stylique. D'un côté, Volkov [81] étudie de telles identités en exhibant des liens avec les monoïdes de Catalan et de Kiselman. De l'autre, Aird et Ribeiro [10] le font en présentant une représentation fidèle du monoïde stylique en terme de matrices tropicales.

Sinon, dans la même ligne de pensée que pour le monoïde stylique, on peut étudier une action similaire, celle des mots sur les tableaux lignes par l'insertion à droite de Schensted. Christian Choffrut a publié un travail [32] sur le monoïde issu de cette action appelé le *monoïde grammique*. Malgré la cardinalité infinie de l'ensemble des lignes, vérifier si deux mots sont grammiquement équivalents se restreint à regarder l'action de ces mots sur un nombre fini de lignes. Choffrut étudie la congruence grammique dans le cas où $|\mathcal{A}| = 3$ et conjecture une présentation de cette congruence quand $|\mathcal{A}| = 4$. Par contre, il émet des doutes qu'une approche calculatoire puisse permettre une compréhension générale de ce monoïde, exprimant ainsi la complexité de la structure du monoïde grammique.

Dans le même ordre d'idée, il y a l'étude de l'action par insertion à gauche sur l'ensemble des tableaux ayant au plus k colonnes. Pour des raisons similaires au monoïde stylique, le monoïde $\mathrm{Styl}_k(\mathcal{A})$, décrit par cette action, est un quotient fini du monoïde plaxique. Bien évidemment, $\mathrm{Styl}_k(\mathcal{A})$ possède les relations $a^{k+1} = a^k$, mais Florent Hivert et James D. Mitchell ont remarqué que ces relations n'étaient pas suffisantes pour décrire $\mathrm{Styl}_k(\mathcal{A})$. On peut dire un peu plus sur cette famille de monoïdes grâce à la théorie des catégories, elle possède une structure de système projectif. On peut alors se demander si sa limite projective serait le monoïde plaxique.

Dans une tout autre ligne de pensée, les résultats importants qu'a amené le monoïde plaxique en théorie de la représentation et en algèbre, ont pavé la voie à l'étude de nombreux monoïdes définis par un algorithme d'insertion de lettres dans des objets combinatoires variés. Tout comme pour le monoïde plaxique, en plus d'être étudiés pour leur simple intérêt combinatoire, ces monoïdes ont de nombreuses applications dans différents domaines des mathématiques, particulièrement en géométrie, en algèbre et en théorie de la représentation. On peut alors étudier les quotients de tels monoïdes par les relations d'idempotence des lettres, $a^2 = a$, appelés les *2-quotients*. C'est ce qu'ont commencé à faire Hivert, Mitchell, Novelli, Tsalakou et l'auteur de ce manuscrit dans [8][4]. En plus d'être de cardinalité finie, les monoïdes quotients obtenus présentent des propriétés structurelles et combinatoires surprenantes. Ils y étudient aussi des monoïdes quotients plus généraux, les *quotients de puissance*, dans lesquels pour chaque $a \in \mathcal{A}$, on considère la relation $a^n = a$ avec $n \geq 2$. Ces monoïdes quotients s'avèrent être structurellement liés à leurs 2-quotients respectifs.

Par soucis d'unité, nous n'avons pas voulu inclure d'autres travaux et articles [1, 2, 3, 4].

[4] Ceci n'est qu'un résumé présenté lors de la conférence GASCOM 2024. Un ouvrage complet présentant plus en détails ce travail devrait paraître sous peu.

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