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CARACTÉRISATION DE NŒUDS VIA CHIRURGIE DE DEHN

THÈSE

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RÉSUMÉ

Cette thèse s'intéresse aux chirurgies de Dehn caractérisantes pour les nœuds dans la sphère de dimension trois. Une pente p/q est dite caractérisante pour un nœud K si la classe d'homéomorphisme préservant l'orientation de la p/q-chirurgie de Dehn le long de K détermine la classe d'isotopie de K. Étendant des travaux antérieurs de Lackenby et McCoy, nous établissons qu'une pente p/q est caractérisante pour un nœud K donné si le dénominateur $|q|$ est suffisamment grand. Notamment, nous montrons que toute pente non-entière est caractérisante pour les nœuds composés. Dans une collaboration avec Wakelin, nous quantifions effectivement l'énoncé sur les grandes valeurs de $|q|$ en construisant explicitement, pour tout nœud K donné, une borne $C(K)$ telle que si $|q| > C(K)$, alors toute pente p/q est caractérisante pour K. Finalement, nous développons et implémentons un algorithme calculant le complexe de Floer de nœud pour les nœuds d'épaisseur au plus 1. En l'appliquant à l'étude de l'effet du numérateur $|p|$, nous vérifions que pour la vaste majorité des nœuds avec au plus 17 croisements, à l'exception d'au plus un nombre fini de pentes, toutes les pentes non entières sont caractérisantes.

Mots-clés : topologie de basse dimension, théorie des nœuds, chirurgie de Dehn, homologie de Heegaard Floer

ABSTRACT

This thesis focuses on characterizing Dehn surgeries for knots in the three-dimensional sphere. A slope p/q is said to be characterizing for a knot K if the orientation-preserving homeomorphism type of the p/q-Dehn surgery along K determines the isotopy class of K. Extending earlier work of Lackenby and McCoy, we show that a slope p/q is characterizing for a given knot K if the denominator $|q|$ is sufficiently large. Notably, we show that every non-integral slope is characterizing for composite knots. In a collaboration with Wakelin, we effectively quantify the statement about large values of $|q|$ by explicitly constructing, for any given knot K, a bound $C(K)$ such that if $|q| > C(K)$, then any slope p/q is characterizing for K. Finally, we develop and implement an algorithm to compute the knot Floer complex for knots of thickness at most 1. Applying this to study the effect of the numerator $|p|$, we verify that for the vast majority of knots with at most 17 crossings, all but at most finitely many non-integral slopes are characterizing.

Keywords : low-dimensional topology, knot theory, Dehn surgery, Heegaard Floer homology

INTRODUCTION

Gordon et Luecke établissent en 1989 un résultat fondamental en topologie de basse dimension (Gordon et Luecke, 1989): la classe d'isotopie d'un nœud K dans la sphère de dimension trois S^3 est complètement déterminée par la classe d'homéomorphisme préservant l'orientation de son extérieur $S^3_K = S^3 \setminus \mathrm{int}(\nu K)$, obtenu en retirant de la sphère S^3 l'intérieur d'un voisinage tubulaire νK du nœud K. Une question qui en découle est la suivante: quelles modifications peut-on apporter à la variété S^3_K de sorte que l'information du nœud K soit préservée? Une construction d'intérêt est la *chirurgie de Dehn* le long de K. Elle consiste en coller à S^3_K un tore solide selon un paramètre $p/q \in \mathbb{Q} \cup \{1/0\}$, appelé *pente* de chirurgie, produisant ainsi nouvelle variété de dimension trois, notée $S^3_K(p/q)$. Cette opération est prédominante dans la construction de variétés de dimension trois. En effet, toute variété fermée, connexe et orientée peut être obtenue par chirurgie de Dehn le long d'un entrelacs (Lickorish, 1962; Wallace, 1960). De plus, toute telle variété est le bord d'une variété de dimension quatre simplement connexe. Ainsi, plusieurs problèmes en topologie des variétés de dimensions trois et quatre peuvent être formulés en termes de chirurgie de Dehn. Pour comprendre ces variétés, il est donc pertinent de se demander quelle information est retenue à propos d'un nœud après une telle opération.

Cette thèse établit, pour un nœud K quelconque, des conditions sur une pente p/q garantissant que le nœud K soit le seul à produire la variété $S^3_K(p/q)$ via p/q-chirurgie. Plus précisément, une pente p/q est dite *caractérisante* pour un nœud K si l'énoncé suivant est vrai: si K' est un nœud pour lequel il existe un homéomorphisme $S^3_K(p/q) \cong S^3_{K'}(p/q)$ préservant l'orientation, alors K' est forcément équivalent à isotopie près à K.

0.1 Résultats

Le résultat principal de la thèse est un prolongement des travaux de Lackenby et McCoy sur les nœuds hyperboliques et toriques, respectivement (Lackenby, 2019; McCoy, 2020). En étudiant le cas des nœuds satellites, nous obtenons l'énoncé général suivant.

Théorème 1. (Sorya, 2024, Théorème 1) *Soit K un nœud dans S^3. Il existe une constante $C(K) > 0$ telle que si $|q| > C(K)$, alors p/q est une pente caractérisante pour K.*

En particulier, lorsque K est un nœud composé, cette constante $C(K)$ peut être réalisée par le nombre entier 1.

Théorème 2. (Sorya, 2024, Théorème 2) *Toute pente non entière est caractérisante pour tout nœud composé.*

Ce résultat donne lieu à une famille infinie de nœuds pour lesquels l'ensemble des pentes caractérisantes est entièrement connu.

Corollaire 3. (Sorya, 2024, Corollaire 3) *Soient K_1 un nœud non trivial et K_2 un nœud de Baker-Motegi (Baker et Motegi, 2018, Exemple 4.5). Alors l'ensemble des pentes caractérisantes pour $K_1 \# K_2$ est $\mathbb{Q} \setminus \mathbb{Z}$.*

Il s'agit du premier et seul ensemble de pentes caractérisantes entièrement connu à ce jour qui n'est pas égal à $\mathbb{Q}$. En effet, les seuls autres ensembles de pentes caractérisantes entièrement connus sont ceux du nœud trivial (Kronheimer *et al.*, 2007), du nœud de trèfle et du nœud en huit (Ozsváth et Szabó, 2019), pour lesquels toutes les pentes $p/q \in \mathbb{Q}$ sont caractérisantes.

À défaut de connaître l'ensemble des pentes caractérisantes pour les autres cas en général, il est tout de même possible de construire explicitement une borne $C(K)$ réalisant le

Théorème 1 pour tout nœud. En collaboration avec Laura Wakelin, nous établissons le résultat suivant.

Théorème 4. (Sorya et Wakelin, 2024, Théorème 1.1) *Soit K un nœud dans S^3. Une réalisation explicite de la constante $C(K)$ du Théorème 1 peut être obtenue à partir de l'information de la décomposition JSJ de l'extérieur du nœud, c'est-à-dire de la géométrie des pièces JSJ et des applications de recollement entre celles-ci.*

Ces conclusions significatives à propos de l'effet du paramètre q sur la caractérisation de K via chirurgie de Dehn nous orientent naturellement vers la question de l'existence d'une borne analogue pour le numérateur p. La famille de nœuds de Baker et Motegi mentionnée dans le Corollaire 3 démontre qu'il est impossible d'établir une telle borne pour tout nœud. Or, à ce jour, aucun exemple n'est connu pour lequel une infinité de pentes p/q seraient non caractérisantes pour $1 < |q| \leq C(K)$. Cette observation conduit à une conjecture de McCoy stipulant que pour tout nœud, toute pente non entière est caractérisante à l'exception d'au plus un nombre fini de pentes. (McCoy, 2025, Conjecture 1.1).

Nous vérifions la véracité de cette conjecture pour la vaste majorité des nœuds avec au plus 17 croisements.

Théorème 5. *Au moins 95,79% des 9 755 329 nœuds premiers avec au plus 17 croisements ne possèdent qu'au plus un nombre fini de pentes non entières non caractérisantes.*

Ce résultat expérimental est obtenu en vérifiant une condition algébrique formulée par McCoy (McCoy, 2025, Theorem 1.2, Theorem 1.3) concernant des modules d'homologie émanant du *complexe de Floer de nœud* $CFK(K)$ du nœud K étudié. Ce riche invariant développé au début des années 2000 (Rasmussen, 2003; Ozsváth et Szabó, 2004) permet l'étude de diverses propriétés d'un nœud, telles que son genre et son caractére fibré. Cependant son calcul n'est pas aisé: il n'existe pas à ce jour d'algorithme implanté calculant le complexe de Floer de nœud d'un nœud quelconque. Dans le but d'étudier la conjecture de McCoy, nous développons un algorithme calculant le complexe de Floer de nœud pour tout nœud d'épaisseur au plus un, où l'*épaisseur* d'un nœud est l'étendue des écarts entre les bidegrés des générateurs du complexe.

Cet algorithme se base sur le fait que le quotient $CFK(K)/(uv)$ du complexe de Floer de nœud, vu comme un $\mathbb{F}[u, v]$-module où $\mathbb{F}$ est le corps à deux éléments, ne possède qu'un unique relèvement à homotopie filtrée près pour les nœuds d'épaisseur au plus un.

Théorème 6. (Sorya, 2025, Théorème 1) *Soit K un nœud d'épaisseur au plus un. Son complexe de Floer de nœud $CFK(K)$ est complètement déterminé par le quotient $CFK(K)/(uv)$.*

Ainsi, l'algorithme consiste à calculer un relèvement du quotient $CFK(K)/(uv)$, ce dernier étant obtenu via un algorithme développé par Ozváth et Szabó (Ozsváth et Szabó, 2019). En étendant les stratégies derrière ce calcul, nous parvenons également à comprendre les modules d'homologie de Heegaard Floer d'intérêt poue certains nœuds d'épaisseur plus grande que 1. Ainsi, le résultat empirique du Théorème 5 découle du calcul des complexes de Floer pour les nœuds avec au plus 17 croisements.

0.2 Organisation de la thèse et résumés des chapitres

Cette thèse par articles est composée de la présente introduction, de quatre chapitres et d'une conclusion.

Le premier chapitre pose les bases topologiques nécessaires aux développements présentés dans la thèse. Les décompositions première, JSJ et de Heegaard y sont passées en revue. La chirurgie de Dehn y est définie et d'importants résultats sur les pentes de chirurgies contenant certaines surfaces y sont rappelés. Enfin, ce premier chapitre révise la description de l'homologie de Heegaard Floer de variétés obtenues via chirurgie de Dehn.

Le cœur de la thèse consiste en trois articles répartis à travers les chapitres 2, 3 et 4. Le chapitre 2 est constitué de l'article *Characterizing slopes for satellite knots* (Sorya, 2024), publié dans le journal *Advances in Mathematics*. Il comporte la démonstration des Théorème 1 et 2. Le chapitre 3 présente l'article *Effective bounds on characterising slopes for all knots* (Sorya et Wakelin, 2024), rédigé en collaboration avec Laura Wakelin et ayant comme résultat principal le Théorème 4. Le chapitre 4 est basé sur l'article *Computing the knot Floer complex of knots of thickness one* (Sorya, 2025). Il comprend la démonstration du Théorème 6 et les plus récents résultats computationnels menant au Théorème 5.

La conclusion propose une synthèse des réalisations de la thèse et envisage des contributions des méthodes élaborés aux connaissances en topologie de basse dimension.

CHAPITRE 1
PRÉLIMINAIRES

1.1 Géométrie des variétés de dimension trois

Dans cette section, nous rappelons les principales propriétés géométriques des variétés en dimension trois qui seront utilisées dans cette thèse. Tout au long de la discussion, les variétés considérées sont de dimension trois et elles sont compactes, connexes et orientables.

1.1.1 Surfaces essentielles

Les surfaces essentielles jouent un rôle important dans la classification des variétés orientables en dimension trois.

Définition 1.1.1. Une surface S proprement plongée dans une variété de dimension trois M est dite *essentielle* si elle:

1. n'est pas le bord d'une boule dans M;
2. n'est pas parallèle au bord de M;
3. est *incompressible*, c'est-à-dire que le bord ∂D de tout disque $D \subset M$ tel que $D \cap S = \partial D$ est le bord d'un disque dans S;
4. est *incompressible au bord*, c'est-à-dire que pour tout disque D tel que $\partial D = \alpha \cup \beta$ où α, β sont des arcs tels que $\alpha = \partial D \cap S$ et $\beta = \partial D \setminus \alpha \subset \partial M$, il existe un disque $D' \subset S$ tel que $\alpha \subset \partial D'$ et $\partial D' \setminus \alpha \subset \partial M$.

Définition 1.1.2. Une variété est dite

- *réductible* si elle possède une sphère essentielle et *irréductible* sinon;

- *toroïdale* si elle possède un tore essentiel et *atoroïdale* sinon.

1.1.2 Variétés hyperboliques

Le théorème de géométrisation de Thurston démontré par Perelman implique que l'intérieur d'une variété possède une géométrie hyperbolique à volume fini si et seulement si elle est irréductible et atoroïdale, ne possède ni disque ni couronne essentiels, et son groupe fondamental est infini. Dans ce cas, la variété est dite *hyperbolique.*

Toute variété hyperbolique M peut être décrite comme la fermeture d'un quotient $\mathbb{H}^3/\Gamma$ de l'espace hyperbolique $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \,|\, z > 0\}$ par l'action d'un sous-groupe Γ discret et sans torsion du groupe d'isométries de $\mathbb{H}^3$, muni de la métrique $(dx^2 + dy^2 + dz^2)/z^2$. Chaque composante de bord de M est associée à une *cuspide* de $\text{int}(M)$, homéomorphe à $T^2\times[0, \infty)$. Sa préimage par le quotient $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ consiste en des horoboules isométriques à $B_h = \{(x, y, z) \in \mathbb{H}^3 \,|\, z \geq h\}$ pour un $h > 0 \in \mathbb{R}$. Un *voisinage cuspidal maximal* a comme préimage des horoboules B_h où h est le supremum tel que pour tout $h' < h$, ces horoboules s'intersectent dans $\mathbb{H}^3$. Le bord de ce voisinage cuspidal maximal hérite de la métrique euclidienne $(dx^2 + dy^2)/h^2$, qu'on utilise pour définir les notions d'aire d'une cuspide et de longueur d'une courbe le long d'une composante de bord de M.

1.1.3 Espaces fibrés de Seifert

Une autre classe de variétés irréductibles élémentaire est celle des variétés irréductibles possédant une structure de *fibré de Seifert.* Les variétés possédant une telle structure sont appelées des variétés de Seifert ou des *espaces fibrés de Seifert.* Il s'agit de variétés possédant une fibration en cercles où chaque fibre a un voisinage $S^1 \times D^2$ dont la fibration est difféomorphe à celle obtenue de la façon suivante. On considère $[0, 1] \times D^2$ et les segments $[0, 1] \times \{x\}, x \in D^2$. On identifie $\{0\} \times D^2$ à $\{1\} \times D^2$ après avoir appliqué à

D^2 une rotation de $2\pi p/q$, où p et $q \geq 1$ sont premiers entre eux. Ceci donne lieu à une fibration de $S^1 \times D^2$ avec des fibres formées du recollement de q segments $[0,1] \times \{x\}$ et une fibre formée d'un segment $[0,1] \times \{0\}$ avec les bouts recollés.

Une fibre d'une variété de Seifert est dite *exceptionnelle* si la fibration de son voisinage correspond à celle obtenue par une rotation de $2\pi p/q$ où $q \neq 1$, et on dit qu'elle est d'*ordre* q. Les autres fibres sont dites *régulières* et elles sont d'ordre 1.

L'espace quotient obtenu en identifiant chaque fibre d'une variété de Seifert à un point est une *orbivariété*. Il s'agit d'une surface munie de points *cônes*, chacun correspondant à une fibre exceptionnelle. On note $B(a_1, \ldots, a_n)$ l'orbivariété de surface B avec points cônes d'ordres $a_1, \ldots, a_n$, où l'*ordre* d'un point cône est celui de la fibre exceptionnelle correspondante.

1.1.4 Décompositions des variétés de dimension trois

Dans cette sous-section, nous passons en revue les décompositions classiques des variétés de dimension trois. Les surfaces essentielles définissent deux décompositions canoniques, soient la décomposition première le long de sphères essentielles et la décomposition JSJ le long de tores essentiels. Une autre décomposition d'intérêt est la décomposition de Heegaard pour les variétés fermées.

1.1.4.1 Décomposition première

Une variété qui n'est pas S^3 est dite *première* si elle ne contient pas de sphère essentielle séparante. Toute variété irréductible qui n'est pas S^3 est donc première, et l'unique variété à la fois réductible et première est $S^2 \times S^1$. Pour toute variété M, il existe un ensemble de sphères essentielles séparantes tel que le découpage de M le long de ces sphères donne sa *décomposition première*. Il s'agit de l'unique décomposition $M = M_1\#\ldots\#M_n$, où $M_i \ncong S^3$ sont des variétés premières et $X\#Y$ dénote la *somme connexe* de X et Y, obtenue en retirant une boule de X et une boule de Y, puis en les recollant le long des bords sphériques résultants.

1.1.4.2 Décomposition JSJ

Les variétés irréductibles peuvent elles-mêmes êtres subdivisées davantage le long de tores essentiels. Pour toute variété M compacte, irréductible et orientée de dimension trois dont le bord est une union (possiblement vide) de tores, il existe une collection minimale $\mathbf{T}$ de tores essentiels disjoints telle que chaque composante connexe de $M \setminus \mathbf{T}$ est soit une variété hyperbolique, soit un espace fibré de Seifert (Jaco et Shalen, 1979; Johannson, 1979). Cette collection est unique à isotopie près. La décomposition de Jaco-Shalen et Johansson, communément appelée *décomposition JSJ*, est donnée par $M = M_1 \cup \ldots \cup M_m$, où les $M_i, i = 1, \ldots, m$ sont les fermetures des composantes de $M \setminus \mathbf{T}$, et par l'information du recollement le long des composantes de bord toriques des M_i.

Étant donné un tore $T \in \mathbf{T}$ entre deux variétés M_i et M_j de la décomposition JSJ d'une variété M, l'information de recollement le long de T est donnée par un isomorphisme $H_1(T_i;\mathbb{Z}) \to H_1(T_j;\mathbb{Z})$, où T_i et T_j sont respectivement les composantes de bord de M_i et M_j correspondant à T.

1.1.4.3 Décomposition de Heegaard

Une *décomposition de Heegaard* pour une variété M fermée et connexe est une paire de corps à anses H_1, H_2 de même genre g, telle que $M = H_1 \cup H_2$ et $H_1 \cap H_2 = \partial H_1 = \partial H_2$. Cette décomposition peut être obtenue en considérant une triangulation de M. Un voisinage tubulaire du 1-squelette de cette triangulation forme alors le corps à anse H_1 et son complément est le corps à anses H_2. Le recollement entre H_1 et H_2 correspond à la donnée d'un g-tuplet de courbes fermées simples, linéairement indépendantes en homologie et disjointes $\alpha = (\alpha_1, \ldots, \alpha_g)$ plongées dans ∂H_1. Pour chaque courbe $\alpha_i, i = 1 \ldots, g$, on colle une 2-anse $D^2 \times [-1, 1]$ à H_1 le long de $\partial D^2 \times [-1, 1]$ en identifiant $\beta_i = \partial D^2 \times \{0\}$ à α_i. Le bord de la variété résultante est une sphère, à laquelle il existe une unique façon de recoller une boule. Les 2-anses et cette boule forment le corps à anses H_2.

Cette information peut être représentée par une surface $\Sigma \cong H_1 \cap H_2$ de genre g munie des g-tuplets de courbes α et $\beta = (\beta_1, \ldots, \beta_g)$. Le triplet (Σ, α, β) est un *diagramme de Heegaard* pour M et il contient toute l'information nécessaire pour reconstruire la variété M.

La surface $H_1 \cap H_2$ plongée dans M est une *surface de Heegaard* de M. Contrairement aux décompositions présentées précédemment, la décomposition de Heegaard d'une variété n'est pas unique. En fait, il existe un infinité de classes d'isotopie de surfaces de Heegaard pour une variété donnée.

1.1.5 Trichotomie des nœuds

Un nœud K dans la sphère de dimension trois S^3 est un plongement lisse $K : S^1 \hookrightarrow S^3$. Deux nœuds sont considérés équivalents s'il existe une isotopie entre eux. L'*extérieur* d'un nœud est la variété $S^3_K = S^3 \setminus \mathrm{int}(\nu K)$ obtenue en retirant de S^3 l'intérieur d'un voisinage tubulaire νK de K. La classe d'homéomorphisme préservant l'orientation de S^3_K caractérise complètement la classe d'isotopie de K.

Théorème 1.1.1. (Gordon et Luecke, 1989, Theorème du complément du nœud) *Deux nœuds K_1 et K_2 sont équivalents si et seulement si leurs extérieurs $S^3_{K_1}$ et $S^3_{K_2}$ sont homéomorphes via un homéomorphisme préservant l'orientation.*

Les nœuds peuvent donc être décrits selon la topologie de leur extérieur. En tant que cas particulier du théorème de géométrisation, Thurston établit la classification suivante des nœuds dans la sphère de dimension trois.

Théorème 1.1.2. (Thurston, 1982) *Un nœud K dans S^3 appartient à exactement une des familles suivantes:*

- *nœuds hyperboliques, i.e. S^3_K est une variété hyperbolique;*
- *nœuds toriques, i.e. S^3_K est un espace fibré de Seifert;*
- *nœuds satellites, i.e. S^3_K contient un tore essentiel.*

Les nœuds hyperboliques possèdent d'intéressants invariants topologiques en vertu de la rigidité de Mostow-Prasad (Prasad, 1973). Pour K un nœud hyperbolique, le *volume* de K est le volume de la variété hyperbolique S^3_K et la *systole* de K est la longueur de la plus courte géodésique de S^3_K. Cette terminologie pourra aussi être généralisée à des entrelacs L dont l'extérieur $S^3_L = S^3 \setminus \mathrm{int}(\nu L)$ possède une métrique hyperbolique à volume fini.

Les nœuds toriques, en plus d'être les seuls à avoir un extérieur fibré de Seifert, sont aussi caractérisés par le fait qu'ils peuvent être placé sur un tore T dénoué dans S^3. Ainsi, un nœud torique $T_{a,b}$ représente une classe $a\mu + b\lambda \in H_1(T;\mathbb{Z}) = \mathbb{Z}\langle \mu, \lambda \rangle$, où μ et λ bordent chacun un disque essentiel de $S^3 \setminus \mathrm{int}(\nu T)$.

Les nœuds satellites sont quant à eux caractérisés par le fait qu'ils possèdent une décomposition JSJ non triviale. La topologie de l'extérieur de ces nœuds est examinée en détail au chapitre 2.

1.2 Chirurgie et remplissage de Dehn

1.2.1 Définitions et notation

Le remplissage de Dehn d'une variété M le long d'une composante de bord torique T est obtenu de la façon suivante. On fixe d'abord une base $\{\mu, \lambda\}$ de $H_1(T; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Une courbe simple fermée le long de T représente à signe près une classe $p\mu + q\lambda$ où p et q sont premiers entre eux. Cette classe, noté $p/q \in \mathbb{Q} \cup \{1/0\}$ est appelée *pente*. Le p/q-*remplissage de Dehn* de M le long de T est obtenu en collant un tore solide le long de T de sorte que la courbe bordant un disque essentiel du tore solide soit identifiée à un représentant de la pente p/q. On note la variété ainsi obtenue $M(T; p/q)$, ou simplement $M(p/q)$ si M ne possède qu'un seule composante de bord torique. Si M possède plusieurs composantes de bord toriques $T_1, \ldots, T_n$, on peut itérer cette construction afin d'obtenir la variété $M(T_1; p_1/q_1) \ldots (T_n; p_n/q_n)$ qu'on notera $M(T_1, \ldots, T_n; p_1/q_1, \ldots, p_n/q_n)$.

Lorsque M est l'extérieur d'un nœud K, c'est-à-dire $M = S^3_K$, alors un remplissage de Dehn de M est appelé une *chirurgie de Dehn le long de* K. Cette terminologie fait référence au processus de retrait d'un voisinage tubulaire de K dans S^3, homéomorphie à un tore solide, suivi du recollement d'un nouveau tore solide. Dans le cas d'une chirurgie de Dehn, une pente p/q est typiquement exprimée dans la base $\{\mu, \lambda\}$ de $H_1(\partial S^3_K; \mathbb{Z})$ où μ est le bord d'un disque essentiel de νK et λ est le bord d'une surface dans S^3_K. Ces deux courbes sont uniques à isotopie près: il existe un unique disque essentiel à isotopie près dans un tore solide, et pour une courbe simple $\lambda' = a\mu + b\lambda \subset \partial S^3_K$ triviale dans $H_1(S^3_K; \mathbb{Z})$, l'inclusion $i : \partial S^3_K \hookrightarrow S^3_K$ induit $i_*\lambda' = a\mu = 0$ et donc $\lambda' = b\lambda = \lambda$. Les courbes μ et λ sont respectivement appelées le *méridien* et la *longitude* du nœud. Elles sont orientées selon la convention où le méridien poussé dans S^3_K et la longitude ont un nombre d'enlacement de $+1$. Suivant la notation établie plus haut, la variété obtenue est notée $S^3_K(p/q)$.

Cette construction peut être généralisée aux entrelacs $L = L_1 \cup \ldots \cup L_n$ dans S^3, où chaque $L_i, i = 1, \ldots, n$, est un nœud. On notera $S^3_L(L_{i_1}, \ldots, L_{i_k}; p_1/q_1, \ldots, p_k/q_k)$ la variété obtenue par p_j/q_j-chirurgies, $j = 1, \ldots, k$, le long des composantes $L_{i_1}, \ldots, L_{i_k} \subset L$ respectivement.

1.2.2 Surfaces et distance entre pentes

La présence de surfaces essentielles ou de Heegaard dans les variétés obtenues par remplissage de Dehn nous permettra d'obtenir des bornes sur le nombre de points d'intersection entre certains couples de pentes.

Definition 1.2.1. Soient α et β des pentes sur une composante de bord torique T d'une variété de dimension trois. La *distance* entre deux pentes α et β, notée $\Delta(\alpha, \beta)$, est égale à la valeur absolue du nombre d'intersection algébrique entre α et β. Si $\alpha = p\mu + q\lambda$ et $\beta = r\mu + s\lambda$ pour une base $\{\mu, \lambda\}$ de $H_1(T; \mathbb{Z})$, on a alors

$$\Delta(\alpha, \beta) = \Delta(p/q, r/s) = |ps - qr|.$$

Théorème 1.2.1. *Soit M une variété irréductible à composante de bord torique T et soient α_1 et α_2 des pentes le long de T. Soient F_1 et F_2 des surfaces plongées dans $M(T; \alpha_1)$ et $M(T; \alpha_2)$ respectivement. Si M, F_1 et F_2 sont telles qu'indiquées dans le tableau 1.1, alors $\Delta(\alpha_1, \alpha_2) \leq 1$.*

F_1 \ F_2	S^2 essentielle	S^2 de Heegaard	T^2 de Heegaard	T^2 essentiel
S^2 ess.	(Gordon et Luecke, 1987)	(Gordon et Luecke, 1987)	M hyperbolique ou toroïdale (Boyer et Zhang, 1998)	$\lvert\partial M\rvert \geq 2$ (Wu, 1992)
S^2 Heeg.		(Gordon et Luecke, 1989)	M pas fibré de Seifert (Culler *et al.*, 1987)	-
T^2 Heeg.			M pas fibré de Seifert (Culler *et al.*, 1987)	-

Tableau 1.1: Surfaces F_1 et F_2 du Théorème 1.2.1

Le tableau 1.1 est inspiré de (Gordon, 1999), où l'on peut trouver des énoncés analogues à celui du Théorème 1.2.1 pour d'autres types de surfaces.

Notons que le cas où F_1 et F_2 sont toutes deux des sphères de Heegaard correspond au théorème du complément du nœud (Théorème 1.1.1). Les cas où F_1 est une sphère ou un tore de Heegaard et F_2 est un tore de Heegaard sont des cas particulier du théorème de chirurgie cyclique (Culler *et al.*, 1987), dont l'énoncé général concerne tous les remplissages avec groupe fondamental cyclique. Remarquons de plus que le cas où F_1 est une sphère essentielle et F_2 est une sphère de Heegaard est une version faible de la *conjecture du câblage*, qui stipule que seuls les nœuds câbles, qui sont en particulier non hyperboliques, peuvent produire des variétés réductibles via chirurgie de Dehn.

1.3 Homologie de Heegaard Floer

L'homologie de Heegaard Floer d'une variété fermée de dimension trois, définie à partir d'un diagramme de Heegaard de la variété, est un invariant qui nous permettra d'étudier le lien entre un nœud et ses chirurgies sous un point de vue algébrique. Dans cette sous-section, nous rappelons les grandes lignes de la construction d'un complexe de Heegaard Floer et de celle d'un complexe de Floer de nœud, et relions ces deux notions pour les chirurgies de Dehn.

1.3.1 Complexe de Heegaard Floer

Soit Y une variété fermée de dimension trois et un *diagramme de Heegaard pointé* $\mathcal{H} = (\Sigma, \alpha, \beta, w)$ qui consiste en la donnée d'un diagramme de Heegaard (Σ, α, β) pour Y où Σ est de genre g, agrémenté d'un point de base $w \in \Sigma \setminus \alpha \cup \beta$. Dénotons par $\mathbb{F}$ le corps à deux éléments $\mathbb{Z}/2\mathbb{Z}$, et soit U une variable formelle de degré -2. On construit à partir de $\mathcal{H}$ le *complexe de Heegaard Floer* $CF^\infty(\mathcal{H})$ de Y, un $\mathbb{F}[U, U^{-1}]$-module, de la façon suivante.

On considère le g-produit symétrique $\text{Sym}^g(\Sigma) = (\prod_{i=1}^g \Sigma)/S_g$, où le groupe symétrique S_g agit par permutation des composantes Σ. Le complexe $CF^\infty(\mathcal{H})$ est engendré par les points d'intersection entre les sous-espaces $\mathbb{T}_\alpha = (\alpha_1 \times \ldots \times \alpha_g)/S_g$ et $\mathbb{T}_\beta = (\beta_1 \times \ldots \times \beta_g)/S_g$ dans $\text{Sym}^g(\Sigma)$. Puisque $\text{Sym}^g(\Sigma)$ possède une structure complexe, pour $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, on peut considérer l'ensemble $\pi_2(x, y)$ des classes d'homotopie de *disques de Whitney* entre x et y, c'est-à-dire des fonctions holomorphes $\phi : \mathbb{D} \subset \mathbb{C} \to \text{Sym}^g(\Sigma)$ tels que $\phi(i) = x$, $\phi(-i) = y$, $\phi(\{z, \Re(z) > 0\}) \subset \mathbb{T}_\alpha$ et $\phi(\{z, \Re(z) < 0\}) \subset \mathbb{T}_\beta$.

Dénotant par $\mu(\phi)$ la dimension attendue de l'espace de modules $\mathcal{M}(\phi)$ des représentants de ϕ, on a que si $\mu(\phi) = 1$, alors le quotient de $\mathcal{M}(\phi)$ par l'action de translation de $\mathbb{R}$ sur

$\mathbb{D} \setminus \{i, -i\} \cong [-1, 1] \times i\mathbb{R}$ est de dimension nulle. Ceci permet de définir la différentielle

$$dx = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^{n_w(\phi)} y,$$

où $n_w(\phi)$ est le nombre algébrique d'intersection entre $\phi(\mathbb{D})$ et $(\{w\} \times \prod_{i=1}^{g-1} \Sigma)/S_g$.

Les éléments de $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ sont partitionnés parmi les structures Spin^c de Y de sorte que $\pi_2(x, y) \neq \varnothing$ seulement si x et y sont associés à la même structure Spin^c (Ozsváth et Szabó, 2004, Lemma 2.19). Ainsi, $CF^\infty(\mathcal{H})$ admet une décomposition $CF^\infty(\mathcal{H}) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} CF^\infty(\mathcal{H}, \mathfrak{s})$, où $CF^\infty(\mathcal{H}, \mathfrak{s})$ est engendré par les éléments de $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ associés à la structure $\mathfrak{s} \in \text{Spin}^c(Y)$.

1.3.2 Homologie de Heegaard Floer d'une sphère d'homologie rationnelle

Soit $\mathcal{H}$ un diagramme de Heegaard pour Y. Posons $CF^-(\mathcal{H})$ le $\mathbb{F}[U]$-sous-complexe de $CF^\infty(\mathcal{H})$ composé des éléments $U^i x, x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, i > 0$ et $CF^+(\mathcal{H}) = CF^\infty(\mathcal{H})/CF^-(\mathcal{H})$. Les groupes d'homologie de $CF^-(\mathcal{H})$ et $CF^+(\mathcal{H})$ ne dépendent pas du choix de diagramme de Heegaard pointé $\mathcal{H}$ pour Y (Ozsváth et Szabó, 2004, Théorème 11.1). Ces groupes, dénotés $HF^-(Y)$ et $HF^+(Y)$, sont les *groupes d'homologie de Heegaard Floer* « moins » et « plus » de Y respectivement. Ils admettent une décomposition $HF^\pm(Y) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} HF^\pm(Y, \mathfrak{s})$ induite par celle de $CF^\infty(\mathcal{H})$.

Lorsque Y est un sphère d'homologie rationnelle, les groupes $HF^-(Y)$ et $HF^+(Y)$ sont munis d'une $\mathbb{Q}$-gradation absolue (Ozsváth et Szabó, 2006, Théorème 7.1) et les deux groupes contiennent la même information à décalage de degré près. Dans cette thèse, nous étudierons les chirurgies de Dehn $Y = S^3_K(p/q)$, des sphères d'homologie rationnelle, en utilisant le groupe $HF^+(Y)$.

Dans ce contexte, les groupes $HF^+(Y, \mathfrak{s}), \mathfrak{s} \in \text{Spin}^c(Y)$, se décomposent en tant que $\mathbb{F}[U]$-

modules en

$$HF^+(Y,\mathfrak{s}) \cong \mathcal{T}^+ \oplus HF_{red}(Y,\mathfrak{s}),$$

où $\mathcal{T}^+ = \mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$ et $HF_{red}(Y,\mathfrak{s})$, le *groupe d'homologie de Heegaard Floer réduit*, est isomorphe à une somme directe finie $\bigoplus_j \mathbb{F}[U]/U^{n_j}, n_j \geq 1$.

1.3.3 Complexe de Floer de nœud

Lorsque Y est obtenu via chirurgie de Dehn le long d'un nœud K dans S^3, la structure de $HF^+(Y,\mathfrak{s})$ en tant que $\mathbb{F}[U]$-module gradué est déterminée par le complexe de Floer de K, dont nous résumons la construction dans cette sous-section.

Un noeud K dans S^3 donne lieu à un *diagramme de Heegaard doublement pointé* $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ tel que (Σ, α, β) est un diagramme de Heegaard de S^3 et $w, z \in \Sigma \setminus \alpha \cup \beta$. Considérant Σ dans S^3 comme l'intersection des corps en anses H_1 et H_2 de la décomposition de Heegaard donnée par $\mathcal{H}$, on a $K = a \cup b$ où a est un arc dans $\Sigma \setminus \alpha$ reliant w et z, que l'on pousse légèrement dans H_1, et b est un arc dans $\Sigma \setminus \beta$ reliant z à w, que l'on pousse légèrement dans H_2.

Le diagramme pointé $\mathcal{H}_w = (\Sigma, \alpha, \beta, w)$ donne lieu au complexe de Heegaard Floer $CF^\infty(\mathcal{H}_w) = (\mathbb{F}[U,U^{-1}]\langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle, d)$. On obtient le *complexe de Floer de noeud* de K, dénoté $CFK^\infty(K)$, en agrémentant $CF^\infty(\mathcal{H}_w)$ d'une bigradation donnée par l'information du point z de la façon suivante.

Soit $\phi \in \pi_2(x,y)$ tel que $\mu(\phi) = 1$. Les *degrés de Maslov* et *d'Alexander* relatifs de $U^i x$ et $U^i y$ pour tout $i \in \mathbb{Z}$ sont données par $M(U^i x) - M(U^i y) = 1 - 2n_w(\phi)$ et $A(U^i x) - A(U^i y) = n_z(\phi) - n_w(\phi)$ respectivement. De plus, U agit relativement par $M(U^i x) = M(x) - 2i$ et $A(U^i x) = A(x) - i$. Le degré de Maslov absolu est assigné de sorte que l'homologie de $CF^-(\mathcal{H}_w)/U$, isomorphe à $\mathbb{F}$, soit supportée en degré 0. Le degré d'Alexander absolu est quant à lui assigné de sorte que l'*homologie de Floer de noeud* $\widehat{HFK}(K)$ de K, définie comme étant l'homologie du complexe associé gradué de $CF^-(\mathcal{H}_w)/U$, ait des filtrations d'Alexander maximale et minimale égales en valeur absolue. De plus, on munit le complexe d'une filtration sur $\mathbb{Z} \oplus \mathbb{Z}$ telle que pour tout $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $U^i x$ soit au niveau de filtration $(-i, A(U^i x))$.

Tel que la notation le suggère, l'homologie de Floer de noeud $\widehat{HFK}(K)$ est un invariant pour K, alors que le complexe de Floer de noeud $CFK^\infty(K)$ est un invariant pour K à homotopie $\mathbb{Z} \oplus \mathbb{Z}$-filtrée près, tous deux en tant que modules bigradués par les degrés d'Alexander et de Maslov (Ozsváth et Szabó, 2004, Théorème 3.1).

Au chapitre 4, nous donnons une description du complexe de Floer de noeud en tant que $\mathbb{F}[u, v]$-module, ce qui permet de considérer le quotient par l'idéal (uv) mentionné dans l'énoncé du Théorème 6.

1.3.4 Homologie de Heegaard Floer d'une chirurgie de Dehn

Les structures Spin^c de $Y = S^3_K(p/q)$ étant en bijection avec le premier groupe d'homologie de Y, on peut étiqueter chaque structure Spin^c par un élément de $i \in \mathbb{Z}/p\mathbb{Z}$. La structure de $\mathbb{F}[U]$-module gradué de $HF^+(S^3_K(p/q), i)$ pour $p, q > 0$ est déterminée par les entiers p, q, i et le complexe de Floer de noeud $CFK^\infty(K)$. Fixons un représentant de $CFK^\infty(K)$ et dénotons le aussi $CFK^\infty(K)$ par abus de notation.

Posons A^+_k le groupe d'homologie du complexe obtenu en quotientant $CFK^\infty(K)$ par le sous-complexe de filtration strictement plus petite à $(0, k)$.

Le groupe A^+_k se décompose en $\mathcal{T}^+ \oplus A^{red}_k$, où A^{red}_k est tel qu'il existe un V_k tel que pour tout $N \geq V_k$, $U^N A^{red}_k = 0$.

Le $\mathbb{Q}$-degré de $1 \in \mathcal{T}^+$ dans la décomposition

$$HF^+(S^3_K(p/q), i) \cong \mathcal{T}^+ \oplus HF_{\text{red}}(S^3_K(p/q), i)$$

est alors donné par $d(p,q,i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, V_{\lfloor \frac{p+q-1-i}{q} \rfloor}\}$, où $d(p,q,i)$ est un nombre rationnel qui ne dépend que de p, q et i (Ozsváth et Szabó, 2003a, Proposition 4.8).

Dénotant par $\mathcal{T}(N)$ le $\mathbb{F}[U]$-sous-module de $\mathcal{T}^+$ généré par U^{N-1}, la composante $HF_{\text{red}}(S^3_K(p/q), i)$ admet une décomposition

$$\left(\bigoplus_{s\in\mathbb{Z}} A^{red}_{\lfloor \frac{i+ps}{q} \rfloor}\right) \oplus \left(\bigoplus_{n\geq 1+\eta} \mathcal{T}(V_{-\lfloor \frac{i-np}{q} \rfloor})\right) \oplus \left(\bigoplus_{n\geq 1-\eta} \mathcal{T}(V_{\lfloor \frac{i+np}{q} \rfloor})\right)$$

où $\eta = 0$ ou 1 selon que $\left\lfloor \frac{i}{q} \right\rfloor \leq -\left\lfloor \frac{i-p}{q} \right\rfloor$ ou $\left\lfloor \frac{i}{q} \right\rfloor > -\left\lfloor \frac{i-p}{q} \right\rfloor$ respectivement. Les supports des parenthèses du milieu et de droite sont déterminés par p, q, i et les entiers V_k (Gainullin, 2017, Corollaire 14).

1.3.5 Grandes pentes de chirurgie et pentes caractérisantes

Il existe un entier $\nu^+ \geq 0 \in \mathbb{Z}$ avec la propriété que $V_k = 0$ pour tout $k \geq \nu^+$ (Ni et Wu, 2015). Il s'ensuit que lorsque $p/q \geq 2\nu^+ - 1$, les composantes $\mathcal{T}(V_{-\lfloor \frac{i-np}{q} \rfloor})$ et $\mathcal{T}(V_{\lfloor \frac{i+np}{q} \rfloor})$ de la décomposition de $HF_{\text{red}}(S^3_K(p/q), i)$ sont triviales.

De plus, dans le contexte de pentes caractérisantes, lorsque deux noeuds K et K' partagent une même p/q-chirurgie de Dehn avec $p/q \geq q\nu^+(K) + 4q^2 - 2q + 12$, alors $V_k(K) = V_k(K')$ pour tout $k \in \mathbb{Z}$ (McCoy, 2025, Proposition 3.8).

Ceci permet d'étudier des nœuds K et K' partageant une même chirurgie de grande pente en se concentrant sur la structure de leurs modules A^{red}_k. En ce faisant et en utilisant les Théorèmes 1 et 2 de la présente thèse, McCoy démontre que pour tout nœud, seul au plus un nombre fini de pentes p/q avec $|q| \geq 3$ ne sont pas caractérisantes (McCoy, 2025, Théorème 1.2). Pour $|q| = 2$, il obtient un condition suffisante sur la structure des modules A_k^+ d'un noeud (McCoy, 2025, Theorem 1.3) afin que celui-ci ne possède qu'au plus un nombre fini de pentes $p/2$ non caractérisantes, et donc au plus un nombre fini de pentes non entières non caractérisantes.

CHAPITRE 2
PENTES CARACTÉRISANTES DE NŒUDS SATELLITES

Le premier article composant cette thèse est publié dans le journal *Advances in Mathematics* sous le titre *Characterizing slopes for satellite knots*. Une analyse approfondie de la décomposition JSJ de l'extérieur d'un nœud et de ses chirurgies de Dehn y est exposée. En particulier, nous identifions des pentes distinguées le long des tores de la décomposition. Ceux-ci constituent l'outil central de la démonstration du Théorème 1, laquelle occupe la majeure partie de l'article. Nous y démontrons également l'énoncé du Théorème 4 dans le cas des nœuds satellites dont l'extérieur ne contient que des pièces fibrées de Seifert et dans le cas des nœuds composés, ce dernier correspondant au Théorème 2. Basée sur l'idée générale derrière le Théorème 1, la démonstration du Théorème 2 fait de plus appel de façon innovante à plusieurs résultats de la littérature sur les remplissages de Dehn de variétés hyperboliques. Une place substantielle lui est donc consacrée dans ce premier article.

2.0 Abstract

A slope p/q is said to be characterizing for a knot K if the homeomorphism type of the p/q-Dehn surgery along K determines the knot up to isotopy. Extending previous work of Lackenby and McCoy on hyperbolic and torus knots respectively, we study satellite knots to show that for a knot K, any slope p/q is characterizing provided $|q|$ is sufficiently large. In particular, we establish that every non-integral slope is characterizing for a composite knot. Our approach consists of a detailed examination of the JSJ decomposition of a surgery along a knot, combined with results from other authors giving constraints on surgery slopes that yield manifolds containing certain surfaces.

2.1 Introduction

A non-trivial slope p/q is said to be *characterizing* for a knot K in S^3 if whenever there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ between the p/q-Dehn surgery along K and the p/q-Dehn surgery along some knot K', then $K = K'$, where "=" denotes an equivalence of knots up to isotopy. In (Lackenby, 2019), Lackenby proved that every knot has infinitely many characterizing slopes by showing that any slope is characterizing for a knot K, provided $|p| \leq |q|$ and $|q|$ is sufficiently large.

The main theorem of the present paper strengthens this result.

Theorem 2.1.1. *Let K be a knot in S^3. Then any slope p/q is characterizing for K, provided $|q|$ is sufficiently large.*

In (Kronheimer *et al.*, 2007), Kronheimer, Mrowka, Ozsváth and Szabó proved that all non-trivial slopes are characterizing for the unknot. McCoy showed in (McCoy, 2020) that if K is a torus knot, there are only finitely many non-integral slopes that are non-characterizing for K, thus giving the torus knot case of the theorem. Lackenby showed the hyperbolic knot case in (Lackenby, 2019). In this paper, we establish the theorem for any knot by studying the case of satellite knots.

The extension to satellite knots requires a distinct approach, as it cannot be simply derived from the cases of hyperbolic and torus knots. This is due to the presence of essential tori in the exterior of a satellite knot, which lead to a non-trivial JSJ decomposition of the knot's exterior. Hence, Dehn surgery along a satellite knot involves attaching a solid torus to a torus boundary component of a manifold that is not a knot exterior. Our strategy therefore consists of an in-depth analysis of the topology of Dehn fillings of manifolds that arise as JSJ pieces of a knot exterior, along with a description of the gluing between these manifolds through the distance between specific slopes. In particular, we rely on the rigidity of Seifert fibred structures, as well as results pertaining to fillings of non-Seifert fibred manifolds that contain certain surfaces.

Moreover, the ideas employed in the proof of Theorem 2.1.1 can be adapted to derive explicit bounds on $|q|$ for some families of satellite knots. We obtain the following result for composite knots.

Theorem 2.1.2. *If K is a composite knot, then every non-integral slope is characterizing for K.*

Baker and Motegi constructed composite knots for which every integral slope is non-characterizing (Figure 2.1 based on (Baker et Motegi, 2018, Theorem 1.6(2) and Example 4.5)). As a corollary, Theorem 2.1.2 gives the complete list of non-characterizing slopes for these knots.

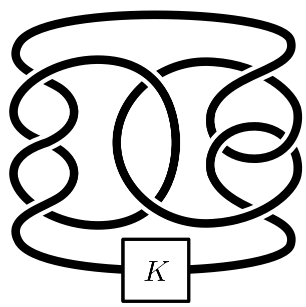

Figure 2.1: The set of characterizing slopes for the connected sum of 9_{42} and any non-trivial knot K is $\mathbb{Q} \setminus \mathbb{Z}$

Corollary 2.1.3. *The set of non-characterizing slopes for Baker and Motegi's composite knots consists of all integral slopes.* □

To the author's knowledge, this yields the first examples of knots for which the complete list of non-characterizing slopes is known and is not empty. Indeed, the other known examples of a complete list are for the unknot, the trefoil and the figure-8 knot, for which all slopes are characterizing (Kronheimer *et al.*, 2007; Ozsváth et Szabó, 2019).

The constraints given by the topology of the exterior of composite knots also lead to the following result.

Theorem 2.1.4. *If K is a knot with an exterior consisting solely of Seifert fibred JSJ pieces, with one of them being a composing space, then any slope that is neither integral nor half-integral is a characterizing slope for K.*

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2.1.1 Structure of paper

After introducing our notation in Section 2.2, the paper is structured into three main parts. The first, covered in Section 2.3, describes the JSJ decomposition of a surgery along a knot. The second, consisting of Sections 2.4, 2.5 and 2.6, presents the proof of Theorem 2.1.1. Finally, Sections 2.7 and 2.8 establish explicit bounds that realize the main theorem for certain families of knots.

2.1.2 Outline of proof

Dehn surgery along a knot K is obtained by gluing a solid torus to the boundary of S^3_K, the exterior of K in S^3. This boundary is contained in a single JSJ piece of the JSJ decomposition of S^3_K. Thus, to understand the topology of a surgery, we must study the fillings of manifolds that arise as JSJ pieces of a knot exterior. We do so in Section 2.3, where we describe the JSJ decomposition of $S^3_K(p/q)$. In particular, when $|q|$ is sufficiently large, there is one JSJ piece that contains the surgery solid torus; we call it the *surgered piece*.

For a fixed non-trivial knot K, suppose there is some knot K' such that there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$. Two scenarios may occur: the surgered piece of $S^3_K(p/q)$ is not mapped to the surgered piece of $S^3_{K'}(p/q)$, or the surgered pieces are mapped one to another. Most of the work towards Theorem 2.1.1 lies in the study of the first case. For each possible description of K as a pattern P and a companion knot J, we demonstrate that there is a lower bound on $|q|$ determined solely by K such that the outermost JSJ piece of S^3_J is not mapped to the surgered piece of $S^3_{K'}(p/q)$. This yields the following proposition, whose proof occupies Sections 2.4 and 2.5.

Proposition 2.1.5. *Let K be a knot. Suppose $|q| > 2$. If there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some knot K', then the homeomorphism sends the surgered piece of $S^3_K(p/q)$ to the surgered piece of $S^3_{K'}(p/q)$, provided $|q|$ is sufficiently large.*

It follows that for $|q|$ sufficiently large, we find ourselves in the situation where an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ must send the surgered pieces one to another. In that case, the JSJ structures of S^3_K and $S^3_{K'}$ agree away from the JSJ pieces that yielded the surgered pieces. Hence, the problem is now reduced to determining a bound on $|q|$ such that the surgered pieces were in fact obtained from the same manifold. This is done in Section 2.6.

In the final two sections of the paper, we outline explicit bounds that realize Theorem 2.1.1 for certain families of knots.

In Section 2.7, we provide a lower bound for $|q|$ that ensures that p/q is a characterizing slope for a cable K whose exterior contains only Seifert fibred JSJ pieces. This bound is obtained from the proof of Theorem 2.1.1. In particular, when K is not an n-times iterated cable of a torus knot, $n \geq 1$, we show that every slope that is not integral or half-integral is characterizing for K.

In Section 2.8, we demonstrate Theorem 2.1.2, which gives a realization of Theorem 2.1.1 for composite knots when $|q| > 1$. Up until this section, we have assumed $|q| > 2$, which guaranteed that hyperbolic fillings of JSJ pieces of a knot exterior were also hyperbolic. To lower the bound to $|q| > 1$, we need to consider the possibility of exceptional fillings of hyperbolic manifolds. We are able to constrain the topology of half-integer fillings of hyperbolic manifolds of interest by relying on various results that provide upper bounds on the distance between surgery slopes yielding manifolds that contain certain surfaces (Wu, 1992; Gordon et Luecke, 1996; Boyer et Zhang, 1998; Wu, 1998). We also use the classification by Gordon and Luecke of hyperbolic knots in S^3 and in $S^1 \times D^2$ that admit half-integral toroidal surgeries (Gordon et Luecke, 2004). As a result, we establish that if a non-integral surgery along a knot is obtained from the filling of a hyperbolic JSJ piece, then it can never be orientation-preserving homeomorphic to a non-integral surgery along a composite knot. Theorem 2.1.2 then follows from the argument in Section 2.6 regarding composing spaces.

2.2 Notation and preliminaries

Let K be a knot in S^3. We denote by S^3_K the exterior of K in S^3, i.e., the manifold obtained by removing the interior of a closed tubular neighbourhood νK of K in S^3. We write $P(J)$ for the satellite knot with pattern P and companion knot J. The *winding number* of P is the absolute value of the algebraic intersection number between P and an essential disc in $V = S^1 \times D^2$. The exterior of the satellite $P(J)$ is a gluing $V_P \cup S^3_J$ where V_P denotes the exterior of P seen as a knot in V. We call V_P the *pattern space* associated to P.

Recall that for any compact irreducible orientable 3-manifold M whose boundary is a (possibly empty) union of tori, there is a minimal collection $\mathbf{T}$ of properly embedded disjoint essential tori such that each component of $M \setminus \mathbf{T}$ is either a hyperbolic or a Seifert fibred manifold, and such a collection is unique up to isotopy (Jaco et Shalen, 1979; Johannson, 1979). The *JSJ decomposition* of M is given by

$$M = M_0 \cup M_1 \cup \ldots \cup M_k,$$

where each M_i is the closure of a component of $M \setminus \mathbf{T}$. A manifold M_i is called a *JSJ piece* of M and a torus in the collection $\mathbf{T}$ is called a *JSJ torus* of M. Any homeomorphism between compact irreducible orientable 3-manifolds can be seen as sending JSJ pieces to JSJ pieces, up to isotopy.

The JSJ piece of S^3_K containing the boundary of νK is said to be *outermost* in S^3_K.

For $\mathcal{T}$ a torus, fix a basis $\{\mu, \lambda\}$ of $H_1(\mathcal{T}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. A simple closed curve on $\mathcal{T}$ represents a class $p\mu + q\lambda$ up to sign, where p and q are coprime.We denote this class by $p/q \in \mathbb{Q} \cup \{1/0\}$ and we call it a *slope.* The *distance* between two slopes p/q and r/s is $\Delta(p/q, r/s) = |ps - qr|$ and it corresponds to the absolute value of the algebraic intersection number between curves representing p/q and r/s.

If M is a 3-manifold with toroidal boundary components $\mathcal{T}_1, \ldots, \mathcal{T}_n$ with fixed bases $\{\mu_i, \lambda_i\}$ for each $H_1(\mathcal{T}_i; \mathbb{Z}), i = 1, \ldots, n$, then

$$M(\mathcal{T}_1, \ldots, \mathcal{T}_n; p_1/q_1, \ldots, p_n/q_n)$$

denotes the Dehn fillings along a simple closed curve representing p_i/q_i on $\mathcal{T}_i$ for each $i = 1, \ldots, n$. If only one boundary component of M is filled, we may simply write $M(p/q)$ if it is clear from context which boundary component is filled. If ∂M is connected, there is a unique slope γ on ∂M that has finite order in $H_1(M; \mathbb{Z})$, called the *rational longitude* of M. We refer to the rational longitude as the *longitude* if it is of order 1 in $H_1(M; \mathbb{Z})$.

When the manifold M is a knot exterior S^3_K, a slope p/q on ∂S^3_K is expressed in terms of the coordinates of $H_1(\partial S^3_K; \mathbb{Z})$ given by the homotopy class of a curve that bounds an essential disc in νK, the *meridian of* S^3_K, and the homotopy class of a curve that bounds a surface in S^3_K, the *longitude of* S^3_K, with orientations following the usual convention (a meridional curve pushed into S^3_K and a longitudinal curve have linking number $+1$). The meridian is well-defined by Gordon and Luecke's knot complement theorem (Gordon et Luecke, 1989, Theorem 1) and the longitude is the unique element of $H_1(\partial S^3_K; \mathbb{Z})$ that is null-homologous in $H_1(S^3_K; \mathbb{Z})$. The slope $1/0$ corresponds to the meridian, while the slope $0/1$ corresponds to the longitude. We will refer to this preferred basis as the one *given by* the knot K.

When K is a satellite, we have the following.

Lemma 2.2.1. *Let K be a satellite knot. For each JSJ torus $\mathcal{T}$ of S^3_K, there is a pattern P and a knot J such that $K = P(J)$ and $\mathcal{T} = V_P \cap S^3_J$.*

Proof. Let $\mathcal{T}$ be a JSJ torus of S^3_K. It separates S^3_K into $A \cup_{\mathcal{T}} B$, where B contains $\mathcal{K} = \partial S^3_K$. Note that $S^3 \cong S^3_K(1/0) \cong A \cup_{\mathcal{T}} B(\mathcal{K}; 1/0)$. By the loop theorem, any torus in S^3 bounds a solid torus, so either A or $B(\mathcal{K};1/0)$ must be a solid torus. Since $\mathcal{T}$ is incompressible in A by definition of a JSJ torus, we have that $B(\mathcal{K};1/0)$ is a solid torus. Its core is a non-trivial knot J in S^3. Thus, A is homeomorphic to S^3_J.

Let $V = B(\mathcal{K};1/0) = B \cup_{\mathcal{K}} (\nu K)$. Then B is the solid torus V with the interior of νK removed. We can thus see K as a knot in V. By the incompressibility of $\mathcal{T}$, K intersects every essential disc in V at least once. Also, K is not the core of V because $\mathcal{T}$ is not boundary parallel in S^3_K. Hence, $V \setminus \mathrm{int}(\nu K)$ is the pattern space for a pattern P. $\square$

Definition 2.2.2. Let $\mathcal{T}$ be a JSJ torus of S^3_K. We say that $\mathcal{T}$ *decomposes* K *into* P *and* J if $\mathcal{T}$ separates S^3_K into V_P and S^3_J as described by Lemma 2.2.1.

If $\mathcal{T}$ decomposes K into P and J, we fix the preferred basis of $H_1(\mathcal{T};\mathbb{Z})$ to be the one given by the meridian μ_J and longitude λ_J of J (Figure 2.2), i.e., a slope p/q along $\mathcal{T}$ corresponds to the class $p\mu_J + q\lambda_J \in H_1(\mathcal{T};\mathbb{Z}) = H_1(\partial S^3_J;\mathbb{Z})$.

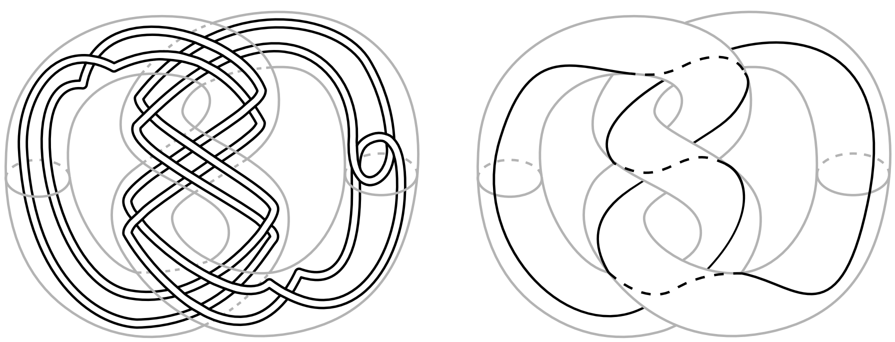

Figure 2.2: Left: A satellite $K = P(J)$ and the JSJ torus $\mathcal{T} = V_P \cap S^3_J$; Right: The longitude on $\mathcal{T}$ given by J

Conversely, a pattern space V_P is the data of a knot P in a solid torus V, along with a slope λ on $\mathcal{T} = \partial V$ that intersects μ once, where μ is the slope that bounds a disc in V. Gluing V_P to a knot exterior S^3_J by respectively identifying μ and λ to the meridian and longitude of J results in the exterior of the knot $K = P(J)$. The preferred basis of $H_1(\mathcal{T};\mathbb{Z})$, where $\mathcal{T}$ is seen as a JSJ torus of S^3_K is $\{\mu, \lambda\}$.

Furthermore, for the boundary component $\mathcal{P} = \partial \nu P$ of V_P, there is a unique class $\lambda_P \in H_1(\mathcal{P};\mathbb{Z})$ that is homologous to $w\lambda \in H_1(\mathcal{T};\mathbb{Z})$ in V_P, where w is the winding number of P (see for instance (Gordon, 1983, p.692)). The preferred basis of $H_1(\mathcal{P};\mathbb{Z})$ is thus given by λ_P and μ_P, the class of a curve that bounds an essential disc in νP.

Lemma 2.2.3. *Let $K = P(J)$ be a satellite knot. The classes λ_P and $\mu_P \in H_1(\mathcal{P};\mathbb{Z})$ defined above coincide with the longitude and meridian of S^3_K.*

Proof. The meridian of S^3_K and the slope μ_P coincide because they both bound an essential disc in νK.

In S^3_K, the class λ_P is homologous to w times the longitude λ of S^3_J, where w is the winding number of P. Let α_P be a curve on $\partial S^3_K = \mathcal{P}$ representing λ_P.

If $w = 0$, then α_P bounds a surface in S^3_K, so λ_P is the longitude of S^3_K.

If $w \neq 0$, then there is a surface F in S^3_K such that

$$\partial F = (\bigsqcup_{i=1}^{w} \alpha_i) \sqcup \alpha_P,$$

where α_i are curves on ∂S^3_J representing λ.

By definition of λ, each α_i bounds a surface $S_i \subset S^3_J$, $i = 1, \ldots, w$. The union of F with the S_i gives a surface in S^3_K whose boundary is α_P. Hence, λ_P coincides with the longitude of S^3_K. □

If a pattern P in a solid torus V intersects an essential disc in V once, then P is a *composing pattern*. Note that if $K = K_1 \# K_2$ is a composite (or connected sum) of knots K_1 and K_2, then $K = P_1(K_2) = P_2(K_1)$ where P_1, P_2 are composing patterns such that $P_1(U) = K_1$ and $P_2(U) = K_2$, where U is the unknot.

The (r, s)-cable of a knot J is denoted by $C_{r,s}(J)$, where s is the winding number of the cable pattern. We may assume that $s > 0$ since the (r, s) and $(-r, -s)$-cable patterns are equivalent. The pattern space $V_{C_{r,s}}$ is an (r, s)-*cable space*. It is the outermost JSJ piece of the exterior of $C_{r,s}(J)$. Further, it admits a Seifert fibration with base orbifold an annulus with one cone point of order s. On its boundary component corresponding to $\partial S^3_{C_{r,s}(J)}$, the (r, s)-cable space has regular fibres of slope $rs/1$. On the other boundary component coinciding with ∂S^3_J, a regular fibre has slope r/s in the coordinates given by J.

We denote by $T_{a,b}$ the (a, b)-torus knot. Its exterior is Seifert fibred, with two exceptional fibres of orders $|a|$ and $|b|$. The regular fibres have slope $ab/1$ on $\partial S^3_{T_{a,b}}$.

2.3 JSJ decompositions and the surgered piece

The JSJ pieces of a non-trivial knot exterior take on one of four special types. Here is a version of this result found in (Budney, 2006).

Theorem 2.3.1. (Budney, 2006, Theorem 4.18) *In the JSJ decomposition of the exterior of a non-trivial knot, the outermost JSJ piece is either*

1. *the exterior of a torus knot;*

2. *a composing space, i.e., a Seifert fibre space with at least 3 boundary components and base orbifold a planar surface with no cone points;*

3. *the exterior of a hyperbolic knot or link such that if the component of the link corresponding to the knot is removed, the resulting link is the unlink;*

4. *a cable space, i.e., a Seifert fibre space with base orbifold an annulus with one cone point.*

By Lemma 2.2.1, a JSJ torus of the exterior S^3_K of a knot K is the boundary of the exterior of a non-trivial knot in S^3. Therefore, each JSJ piece of S^3_K is the outermost piece of some knot exterior, which implies that every JSJ piece of S^3_K belong to one of the types listed in Theorem 2.3.1.

Homological calculations from (Gordon, 1983) lead to the following two results.

Lemma 2.3.2. (Gordon, 1983, Lemma 3.3) *Let $P(J)$ be a satellite knot, where P has winding number w. Denote the boundary components of the pattern space V_P by $\mathcal{P} = \partial\nu P$ and $\mathcal{T} = \partial S^3_J$.*

(i) $H_1(V_P(\mathcal{P}; p/q); \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/g_{p,w}\mathbb{Z})$*, where $g_{p,w}$ is the greatest common divisor of p and w;*

(ii) The kernel of $H_1(\mathcal{T}; \mathbb{Z}) \to H_1(V_P(\mathcal{P}; p/q); \mathbb{Z})$ induced by inclusion is generated by

$$\begin{cases} (p/g_{p,w})\mu + (qw^2/g_{p,w})\lambda & \textit{if } w \neq 0 \\ \mu & \textit{if } w = 0 \end{cases},$$

where $\{\mu, \lambda\}$ is the basis of $H_1(\mathcal{T}; \mathbb{Z})$ given by J.

Proposition 2.3.3. (Gordon, 1983, Corollary 7.3) *Let $K = C_{r,s}(J)$ be cable knot.*

1. *If* $|qrs - p| > 1$*, then* $S^3_K(p/q)$ *is the union along their boundary of* S^3_J *and a Seifert fibre space with incompressible boundary;*

2. *If* $|qrs - p| = 1$*, then* $S^3_K(p/q) \cong S^3_J(p/(qs^2))$.

Note that if $|qrs - p| = 1$, then $g_{p,s} = 1$ and $p/(qs^2)$ is a well-defined slope.

Gordon and Luecke showed that if p/q is not an integer, then the surgery $S^3_K(p/q)$ is irreducible (Gordon et Luecke, 1987, Theorem 1). Thus, it admits a JSJ decomposition. For the rest of this section, we focus our attention on the topology of the JSJ pieces of $S^3_K(p/q)$ when $|q| > 2$. The next theorem from (Lackenby, 2019) combines results from various authors (Gordon et Luecke, 1999; Gordon et Wu, 2000; Scharlemann, 1990; Wu, 1992).

Theorem 2.3.4. (Lackenby, 2019, Theorem 2.8) *Let* M *be the exterior of a hyperbolic link in* S^3 *with components* $L_0, L_1, \ldots, L_n, n \geq 1$*, such that the link formed by the components* $L_1, \ldots, L_n$ *is the unlink. Let* σ *be a slope on* $\mathcal{L}_0 = \partial \nu L_0 \subset \partial M$ *and let* μ *be the slope on* $\mathcal{L}_0$ *that bounds a disc in* νL_0*. If* $\Delta(\sigma, \mu) > 2$*, then* $M(\mathcal{L}_0; \sigma)$ *is hyperbolic.*

Let K be a non-trivial knot and $Y_0 \cup Y_1 \cup \ldots \cup Y_k$ be the JSJ decomposition of its exterior S^3_K, where Y_0 is the outermost piece. The Dehn surgery $S^3_K(p/q)$ is obtained by filling Y_0 along $\mathcal{K} = \partial S^3_K \subset \partial Y_0$.

Proposition 2.3.5. *If* $|q| > 2$*, the filling* $Y_0(\mathcal{K}; p/q)$ *is either a Seifert fibre space or a hyperbolic manifold. In particular,*

1. *If* Y_0 *is the exterior of a hyperbolic link that is not a knot, then* $Y_0(\mathcal{K}; p/q)$ *is hyperbolic;*

2. *If* Y_0 *is a composing space, then* $Y_0(\mathcal{K}; p/q)$ *is Seifert fibred with base orbifold a planar surface with at least two boundary components and one cone point of order* $|q|$*;*

3. *If* Y_0 *is an* (r,s)*-cable space and* $|qrs-p|>1$*, then* $Y_0(\mathcal{K};p/q)$ *is Seifert fibred with base orbifold a disc with two cone points of orders* $|qrs-p|$ *and* s*;*

4. *If* Y_0 *is an* (r,s)*-cable space and* $|qrs-p|=1$*, then* $Y_0(\mathcal{K};p/q)$ *is a solid torus.*

Proof. If $Y_0 = S^3_K$ and K is a hyperbolic knot, then $Y_0(\mathcal{K};p/q) = S^3_K(p/q)$ does not contain an essential sphere or an incompressible torus if $|q|>2$, so it is either hyperbolic or Seifert fibred (Gordon et Luecke, 1995, Theorem 1.1). If $Y_0 = S^3_K$ and K is a torus knot, then $Y_0(\mathcal{K};p/q) = S^3_K(p/q)$ is Seifert fibred if $|q|>1$ (Moser, 1971, Proposition 3.1).

If Y_0 is the exterior of a hyperbolic link, then by Theorem 2.3.4, $Y_0(\mathcal{K};p/q)$ is hyperbolic if $|q|>2$.

If Y_0 is a composing space, a regular fibre on $\mathcal{K}$ has slope $1/0$. If $|q|>1$, we have $\Delta(1/0,p/q) = |q| > 1$, so the surgery slope does not coincide with the regular fibre slope. The Seifert fibred structure of Y_0 thus extends to the surgery solid torus adding an exceptional fibre of order $|q|$. Moreover, a composing space has at least three boundary components, so $Y_0(\mathcal{K};p/q)$ has at least two boundary components.

If Y_0 is an (r,s)-cable space, a regular fibre on $\mathcal{K}$ has slope $rs/1$. If $|q|>1$, we have $\Delta(rs/1,p/q) = |qrs-p| \neq 0$, so the surgery slope does not coincide with the regular fibre slope. The Seifert fibred structure of Y_0 thus extends to the surgery solid torus. If $|qrs-p|>1$, the surgery adds an exceptional fibre of order $|qrs-p|$. If $|qrs-p|=1$, then the surgery solid torus is regularly fibred in $Y_0(\mathcal{K};p/q)$, so $Y_0(\mathcal{K};p/q)$ has base orbifold a disc and one cone point. It is a solid torus. □

Proposition 2.3.6. *Suppose* $|q|>2$*. The JSJ decomposition of* $S^3_K(p/q)$ *is either*

$$Y_0(\mathcal{K};p/q) \cup Y_1 \cup Y_2 \cup \ldots \cup Y_k$$

or

$$Y_1(\mathcal{J}; p/(qs^2)) \cup Y_2 \cup \ldots \cup Y_k,$$

where $\mathcal{J} = Y_0 \cap Y_1$ *and* $s \geq 2$. *The second scenario occurs precisely when* K *is a cable knot* $C_{r,s}(J)$, Y_1 *is the outermost piece of* S^3_J, *and* $|qrs - p| = 1$.

Proof. By the previous proposition, $Y_0(\mathcal{K}; p/q)$ is either Seifert fibred or hyperbolic. If it is hyperbolic or closed, then the result is immediate.

If $Y_0(\mathcal{K}; p/q)$ is Seifert fibred and has boundary, i.e., in cases (2),(3) and (4) of Proposition 2.3.5, then $Y_0(\mathcal{K}; p/q)$ might admit a Seifert structure that extends across adjacent JSJ pieces. By definition of the JSJ decomposition, this structure would have to differ from the one inherited from the Seifert structure on Y_0.

Only cases (3) and (4) of Proposition 2.3.5, which correspond to K being the cable of a knot J, may give rise to manifolds $Y_0(\mathcal{K}; p/q)$ that admit multiple Seifert fibred structures.

In case (3), $Y_0(\mathcal{K}; p/q)$ admits more than one Seifert fibred structure when it is a twisted I-bundle over the Klein bottle. One is inherited from Y_0 and has base orbifold a disc with two cone points each of order 2, and the other has base orbifold a Möbius band with no cone points.

This second structure has regular fibres that are non-meridional and non-integral on $\partial Y_0(\mathcal{K}; p/q)$ if $|q| > 1$. Indeed, a regular fibre of this structure corresponds to the generator of the $\mathbb{Z}/2\mathbb{Z}$ summand of $H_1(Y_0(\mathcal{K}; p/q); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let $p'\mu_J + q'\lambda_J \in H_1(\partial Y_0(\mathcal{K}; p/q); \mathbb{Z})$ be the class of a regular fibre in the coordinates given by J. Let $i : H_1(\partial Y_0(\mathcal{K}; p/q); \mathbb{Z}) \to H_1(Y_0(\mathcal{K}; p/q); \mathbb{Z})$ be induced by inclusion. Then $2(p'\mu_J + q'\lambda_J)$ generates the kernel of i. Since $s = |qrs - p| = 2$, p is even. By Lemma 2.3.2(ii), $\ker i$

is generated by $(p/2)\mu_J + 2q\lambda_J$. Since q and r are odd, we have that p is divisible by 4, from which we deduce that $p'\mu_J + q'\lambda_J = (p/4)\mu_J + q\lambda_J$, which is non-meridional and non-integral.

Therefore, this Seifert fibred structure does not extend to an adjacent Seifert fibred JSJ piece Y_1, because the slope of a regular fibre of Y_1 on the JSJ torus $\mathcal{J} = Y_0 \cap Y_1$ is either meridional (if Y_1 is a composing space) or integral (if Y_1 is a torus knot exterior or a cable space) in the coordinates given by the meridian μ_J and longitude λ_J of the companion knot J.

In case (4), K is a cable knot $C_{r,s}(J)$ such that $|qrs - p| = 1$, and $Y_0(\mathcal{K}; p/q)$ is a solid torus. By Proposition 2.3.3, $S^3_K(p/q) \cong S^3_J(p/(qs^2))$. We have $|qs^2| > |q| > 2$. We iterate the above argument for $S^3_J(p/(qs^2))$ to reduce to case (4) of Proposition 2.3.5 for $S^3_J(p/(qs^2))$. We show that this case does not occur if $|q| > 1$.

Suppose $Y_1(\mathcal{J}; p/(qs^2))$ is a solid torus. Then Y_1 must be an (r', s')-cable space and $|qrs - p| = |qs^2r's' - p| = 1$ (Proposition 2.3.3). Hence, $|q(rs - s^2r's')| = 2$ or 0. As $|q|, s > 1$, the first case does not occur, and the second case happens only if $rs - s^2r's' = 0$, but this contradicts r and s being coprime. □

It follows that the surgery solid torus is contained in exactly one JSJ piece of $S^3_K(p/q)$ when $|q| > 2$.

Definition 2.3.7. Suppose $|q| > 2$. The *surgered piece* of $S^3_K(p/q)$ is the JSJ piece of $S^3_K(p/q)$ that contains the surgery solid torus. It corresponds to either $Y_0(\mathcal{K}; p/q)$ or $Y_1(\mathcal{J}; p/(qs^2))$, as outlined in Proposition 2.3.6.

The topology of the surgered piece is summarized as follows.

Proposition 2.3.8. *Suppose $|q| > 2$. The surgered piece of $S^3_K(p/q)$ is a filling $Y(p/(qt^2))$ of a JSJ piece Y of S^3_K, for some integer $t \geq 1$. In particular,*

1. *$Y(p/(qt^2))$ has non-empty boundary and is hyperbolic if and only if Y is the exterior of a hyperbolic link that is not a knot;*

2. *$Y(p/(qt^2))$ is Seifert fibred with base orbifold a planar surface with at least two boundary components and one cone point of order $|qt^2|$ if and only if Y is a composing space;*

3. *$Y(p/(qt^2))$ is Seifert fibred with base orbifold a disc with two cone points if and only if Y is a cable space. In particular, if Y is an (r, s)-cable space, then the cone points have orders $|qt^2rs - p| > 1$ and s.*

Furthermore, if $|q| > 8$, then

4. *$Y(p/(qt^2))$ is closed and Seifert fibred if and only if Y is the exterior of a torus knot;*

5. *$Y(p/(qt^2))$ is closed and hyperbolic if and only if Y is the exterior of a hyperbolic knot.*

Proof. The converses of (1), (2), (3) follow from Proposition 2.3.5. We deduce the direct implications from Theorem 2.3.1 as follows.

If $Y(p/(qt^2))$ is not closed and is hyperbolic, then Y is hyperbolic with at least two boundary components, so it must be the exterior of a hyperbolic link that is not a knot.

If $Y(p/(qt^2))$ is Seifert fibred and has $n \geq 1$ boundary components, then Y must be Seifert fibred (Theorem 2.3.4) and it has $n + 1$ boundary components. Hence, if $n \geq 2$, Y is a composing space, while if $n = 1$, Y is a cable space.

When $|q| > 8$, it is a result of Lackenby and Meyerhoff (Lackenby et Meyerhoff, 2013, Theorem 1.2) that if Y is the exterior of a hyperbolic knot, then $Y(p/(qt^2))$ must also be hyperbolic. Conversely, if $Y(p/(qt^2))$ is closed and hyperbolic, then Y is the exterior of knot that must be hyperbolic.

If Y is the exterior of a torus knot, then $Y(p/(qt^2))$ is Seifert fibred (Moser, 1971, Proposition 3.1). Conversely, if $Y(p/(qt^2))$ is closed and Seifert fibred, Y is a knot exterior that must be Seifert fibred, by the result of Lackenby and Meyerhoff. The only knots whose exteriors are Seifert fibred are torus knots (Moser, 1971, Theorem 2). □

The five types of surgered pieces described in Proposition 2.3.8 correspond to fillings of distinct types of JSJ pieces of a knot exterior.

Corollary 2.3.9. *Suppose $|q| > 2$ and let K and K' be knots. Suppose further that the surgered piece $Y(p/(qt^2))$ of $S^3_K(p/q)$ is homeomorphic to the surgered piece $Y'(p'/(q'(t')^2))$ of $S^3_{K'}(p'/q')$.*

1. *If $Y(p/(qt^2))$ and $Y'(p'/(q'(t')^2))$ have non-empty boundary, then Y and Y' are of the same type, as listed by Theorem 2.3.1.*
2. *Furthermore, if $|q| > 8$ and if $Y(p/(qt^2))$ and $Y'(p'/(q'(t')^2))$ are closed, then Y and Y' are both torus knots or both hyperbolic knots.* □

Comparing with Theorem 2.3.1, we obtain additional constraints on the structure of the surgered piece.

Proposition 2.3.10. *Suppose $|q| > 2$. Let Y be the JSJ piece of S^3_K such that the surgered piece of $S^3_K(p/q)$ is a filling $Y(p/(qt^2))$ for some integer $t \geq 1$. If $Y(p/(qt^2))$ is homeomorphic to a JSJ piece of a knot exterior, then*

1. *Y is not the exterior of a knot;*

2. *$Y(p/(qt^2))$ is homeomorphic to the exterior of a hyperbolic knot or link such that if a specific component of the link is removed, the resulting link is the unlink, if and only if Y is hyperbolic;*

3. *$Y(p/(qt^2))$ is homeomorphic to an $(r, |qt^2|)$-cable space if and only if Y is a composing space;*

4. *$Y(p/(qt^2))$ is homeomorphic to the exterior of a torus knot if and only if Y is an (r, s)-cable space.*

Proof. For (1), we observe that if Y is the exterior of a knot, then $Y(p/(qt^2))$ is a closed manifold. However, all JSJ pieces of a knot exterior have non-empty boundary.

The implications of (2), (3) and (4) follow from Proposition 2.3.8. We show their converses.

If Y is hyperbolic, then by (1), it is not the exterior of a knot. Hence, $Y(p/(qt^2))$ is hyperbolic by Proposition 2.3.8. By Theorem 2.3.1, a hyperbolic JSJ piece of a knot exterior is as stated in (2).

If Y is a composing space, then $Y(p/(qt^2))$ is Seifert fibred with only one exceptional fibre of order $|qt^2|$ (Proposition 2.3.8). By Theorem 2.3.1, cable spaces are the only Seifert fibred JSJ pieces of a knot exterior with only one exceptional fibre. An (r, s)-cable space has an exceptional fibre of order s, so $Y(p/(qt^2))$ is an $(r, |qt^2|)$-cable space.

If Y is an (r, s)-cable space, $Y(p/(qt^2))$ is Seifert fibred with two exceptional fibres (Proposition 2.3.8). By Theorem 2.3.1, torus knot exteriors are the only Seifert fibred JSJ pieces of a knot exterior with two exceptional fibres. □

2.4 Distinguished slopes

The goal of Sections 2.4 and 2.5 is to prove the following proposition.

Proposition 2.4.1. *Let K be a satellite knot and $\mathcal{T}$ be a JSJ torus of S^3_K that decomposes K into P and J. There exists a constant $L(\mathcal{T})$ with the following property. Suppose $|q| > 2$. If there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some knot K', then the homeomorphism does not map the outermost piece of $S^3_J \subset S^3_K(p/q)$ to the surgered piece of $S^3_{K'}(p/q)$, provided $|q| > L(\mathcal{T})$.*

Note that if $\mathcal{T}$ is compressible in the statement of Proposition 2.4.1, then the outermost piece of S^3_J inside $S^3_K(p/q)$ is not a JSJ piece of $S^3_K(p/q)$, so the conclusion holds with $L(\mathcal{T}) = 2$. Therefore, in the subsequent discussion, we will assume that $\mathcal{T}$ is incompressible in $S^3_K(p/q)$.

2.4.1 Filled patterns and companion knots

Throughout Section 2.4, we will consider the following scenario.

The satellite knot K is fixed. Suppose there is a knot K' such that there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$, mapping JSJ pieces of $S^3_K(p/q)$ to JSJ pieces of $S^3_{K'}(p/q)$.

Let X, X' be the surgered pieces of $S^3_K(p/q)$, $S^3_{K'}(p/q)$ respectively. Suppose that the homeomorphism does not send X to X'. Then X' is the image of a JSJ piece of $S^3_K(p/q)$ that is not X. That JSJ piece in $S^3_K(p/q)$ is the outermost piece of $S^3_J \subset S^3_K$ for some knot J. Let $\mathcal{T} = \partial S^3_J$. This is a JSJ torus of S^3_K. By Lemma 2.2.1, $\mathcal{T}$ decomposes K into a pattern P and the knot J.

The JSJ torus $\mathcal{T}$ is sent by the homeomorphism to a JSJ torus $\mathcal{T}'$ of $S^3_{K'}(p/q)$, which is also a JSJ torus of $S^3_{K'}$ by Proposition 2.3.6. By Lemma 2.2.1, $\mathcal{T}'$ decomposes K' into a pattern P' and a knot J'.

Let $\mathcal{P}$ and $\mathcal{P}'$ respectively denote the boundary components $\partial\nu P$ and $\partial\nu P'$ of V_P and $V_{P'}$, the pattern spaces associated to P and P'. The homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ then restricts to a homeomorphism between $V_P(\mathcal{P}; p/q)$ and $S^3_{J'}$, and between S^3_J and $V_{P'}(\mathcal{P}'; p/q)$ (Figure 2.3).

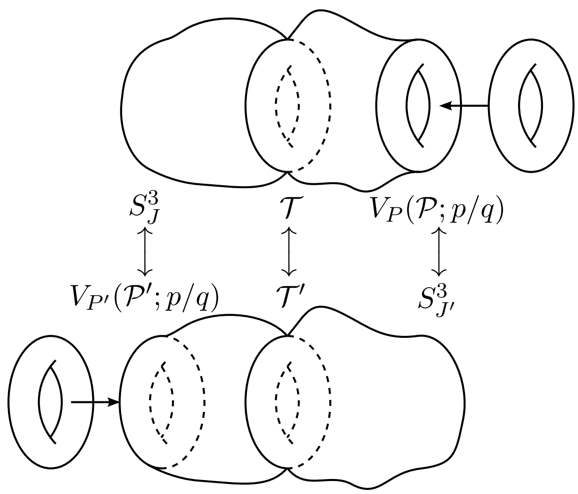

Figure 2.3: Homeomorphism carrying filled pattern space to exterior of companion knot

We will now identify distinguished slopes on the JSJ tori $\mathcal{T} \subset S^3_K(p/q)$ and $\mathcal{T}' \subset S^3_{K'}(p/q)$. Information about the gluing of JSJ pieces along their boundaries will be obtained by analyzing the distances between these slopes. In Section 2.5, we will rely on the fact that distances between slopes are preserved by homeomorphisms to establish constraints on the coefficients p and q.

2.4.2 General pattern case

In the scenario described in Section 2.4.1 and Figure 2.3, we have the following lemma.

Lemma 2.4.2. *The greatest common divisor $g_{p,w}$ of p and w is 1.*

Proof. On one hand, we have $H_1(V_P(\mathcal{P};p/q);\mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Z}/g_{p,w}\mathbb{Z})$ (Lemma 2.3.2(i)). On the other hand, $H_1(S^3_{J'};\mathbb{Z}) = \mathbb{Z}$. Since $S^3_{J'} \cong V_P(\mathcal{P};p/q)$, we conclude that $g_{p,w} = 1$. □

Our first distinguished slope on $\mathcal{T} \subset S^3_K(p/q)$ is the longitude of $V_P(\mathcal{P};p/q)$ seen as a knot exterior. Combining Lemma 2.3.2(ii) with Lemma 2.4.2, we have that this slope corresponds to the class

$$\begin{cases} p\mu_J + qw^2\lambda_J & \text{if } w \neq 0, \\ \mu_J & \text{if } w = 0 \end{cases}$$

in $H_1(\mathcal{T};\mathbb{Z}) = H_1(\partial S^3_J;\mathbb{Z})$, where μ_J and λ_J are the meridian and longitude of S^3_J respectively.

Our second distinguished slope on $\mathcal{T} \subset S^3_K(p/q)$ is the meridian of $V_P(\mathcal{P};p/q)$ seen as a knot exterior. Let $x\mu_J + y\lambda_J$ be a class corresponding to this slope.

On $\mathcal{T}' \subset S^3_{K'}(p/q)$, we have two analogous distinguished slopes: the meridian and the longitude of $V_{P'}(\mathcal{P}';p/q)$ seen as a knot exterior. Let $x'\mu_{J'} + y'\lambda_{J'}$ be a class in $H_1(\mathcal{T}';\mathbb{Z}) = H_1(\partial S^3_{J'};\mathbb{Z})$ corresponding to this meridian, where $\mu_{J'}$ and $\lambda_{J'}$ are the meridian and longitude of $S^3_{J'}$ respectively.

Lemma 2.4.3. *(i) The meridian $x\mu_J + y\lambda_J$ of $V_P(\mathcal{P};p/q)$ is such that $|x| = |q(w')^2|$.*

(ii) The meridian $x'\mu_{J'} + y'\lambda_{J'}$ of $V_{P'}(\mathcal{P}';p/q)$ is such that $|x'| = |qw^2|$.

Proof. The homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ sends the meridian $x\mu_J + y\lambda_J$ of $V_P(\mathcal{P}; p/q)$ to the meridian $\mu_{J'}$ of $S^3_{J'}$, and the longitude λ_J of S^3_J to the longitude $p\mu_{J'} + q(w')^2\lambda_{J'}$ of $V_{P'}(\mathcal{P}'; p/q)$. Since the homeomorphism preserves distances between slopes, we have

$$|x| = \Delta\left(\frac{x}{y}, \frac{0}{1}\right) = \Delta\left(\frac{1}{0}, \frac{p}{q(w')^2}\right) = |q(w')^2|.$$

We obtain (ii) symmetrically. $\square$

Table 2.1 summarizes this discussion.

	$\mathcal{T}$	$\mathcal{T}'$
Meridian of $V_P(\mathcal{P}; p/q) \cong S^3_{J'}$	$q(w')^2\mu_J + y\lambda_J$ if $w' \neq 0$ λ_J if $w' = 0$	$\mu_{J'}$
Longitude of $V_P(\mathcal{P}; p/q) \cong S^3_{J'}$	$p\mu_J + qw^2\lambda_J$ if $w \neq 0$ μ_J if $w = 0$	$\lambda_{J'}$
Meridian of $S^3_J \cong V_{P'}(\mathcal{P}'; p/q)$	μ_J	$qw^2\mu_{J'} + y'\lambda_{J'}$ if $w \neq 0$ $\lambda_{J'}$ if $w = 0$
Longitude of $S^3_J \cong V_{P'}(\mathcal{P}'; p/q)$	λ_J	$p\mu_{J'} + q(w')^2\lambda_{J'}$ if $w' \neq 0$ $\mu_{J'}$ if $w' = 0$

Tableau 2.1: Distinguished slopes on $\mathcal{T}$ and $\mathcal{T}'$

2.4.3 Iterated cable case

In the case where P is an iterated cable, we also distinguish the slopes of regular Seifert fibres in the scenario described in Section 2.4.1 and Figure 2.3.

Lemma 2.4.4. *If P is an iterated cable $C_{r_n,s_n} \dots C_{r_2,s_2} C_{r_1,s_1}, n \neq 1$, then*

1. *The JSJ piece of $V_P(\mathcal{P}; p/q)$ with boundary component $\mathcal{T}$ is Seifert fibred and its regular fibre has slope $r_1\mu_J + s_1\lambda_J$ on $\mathcal{T}$;*
2. *The outermost JSJ piece of $S^3_{J'}$ is Seifert fibred and its regular fibre has integral slope $k\mu_{J'} + \lambda_{J'}$ on $\mathcal{T}'$, for some $k \in \mathbb{Z}$.*

Proof. Let V_i be (r_i, s_i)-cable spaces for $i = 1, \dots, n$. The pattern space V_P has JSJ decomposition $V_1 \cup V_2 \cup \dots \cup V_n$, where $\mathcal{T} \subset \partial V_1$. A regular fibre of V_1 has slope $r_1\mu_J + s_1\lambda_J$ on $\mathcal{T}$.

If $V_P(\mathcal{P}; p/q)$ contains an incompressible torus, then it is clear that the regular fibre slope on $\mathcal{T}$ remains unchanged. Furthermore, V_1 is homeomorphic to the outermost piece of $S^3_{J'}$, which must also be a cable space. Hence, a regular fibre of the outermost piece of $S^3_{J'}$ has integral slope on $\mathcal{T}' = \partial S^3_{J'}$.

If $V_P(\mathcal{P}; p/q)$ contains no incompressible torus, then it is a filling of V_1 by Proposition 2.3.3. By hypothesis (Figure 2.3), this filling is homeomorphic to a JSJ piece of $S^3_{K'}$. By Proposition 2.3.10, this piece is the exterior of a torus knot. It follows that the Seifert fibred structure on $V_P(\mathcal{P}; p/q)$ is unique, and it is the one inherited from V_1. Moreover, the torus $\mathcal{T}' \subset S^3_{K'}(p/q)$ is the boundary of a torus knot exterior, so a regular fibre has integral slope on $\mathcal{T}'$. □

Thus, the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ maps the slope $r_1\mu_J + s_1\lambda_J$ on $\mathcal{T}$ to a slope $k\mu_{J'} + \lambda_{J'}$ on $\mathcal{T}'$, where $k \in \mathbb{Z}$.

	$\mathcal{T}$	$\mathcal{T}'$
Regular fibre in $V_P(\mathcal{P}; p/q) \cong S^3_{J'}$	$r_1\mu_J + s_1\lambda_J$	$k\mu_{J'} + \lambda_{J'}$

Tableau 2.2: Regular fibre slopes on $\mathcal{T}$ and $\mathcal{T}'$

2.5 Surgered pieces are sent to surgered pieces

This section is dedicated to demonstrating Proposition 2.4.1, from which Proposition 2.1.5 follows easily.

Proposition 2.1.5. *Let K be a knot. Suppose $|q| > 2$. If there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some knot K', then the homeomorphism sends the surgered piece of $S^3_K(p/q)$ to the surgered piece of $S^3_{K'}(p/q)$, provided $|q|$ is sufficiently large.*

Proof. If K is not a satellite knot, then the result follows from Proposition 2.3.6, so suppose K is a satellite knot. Set

$$L(K) = \max_{\mathcal{T}}\{L(\mathcal{T}),\ \mathcal{T} \text{ is a JSJ torus of } S^3_K\},$$

where the $L(\mathcal{T})$'s are given by Proposition 2.4.1. Let $|q| > L(K)$.

Suppose there is an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ that does not carry the surgered piece X of $S^3_K(p/q)$ to the surgered piece X' of $S^3_{K'}(p/q)$. Then, as described in Section 2.4.1, X' is the image of a JSJ piece of $S^3_K(p/q)$ that is the outermost piece of the exterior of some knot J such that $S^3_J \subset S^3_K$. Let $\mathcal{T} = \partial S^3_J$.

Proposition 2.4.1 implies that since $|q| > L(K) \geq L(\mathcal{T})$, the outermost piece of S^3_J cannot be mapped to the surgered piece X', a contradiction. Therefore, if $|q| > L(K)$, then any orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ must send the surgered pieces one to another. □

The proof of Proposition 2.4.1 is divided into three cases: composing patterns, once or twice-iterated cables, and other patterns.

For the last two cases, we will need a simplified version of a theorem from Cooper and Lackenby, as well as some related lemmas.

Theorem 2.5.1. (Cooper et Lackenby, 1998, Theorem 4.1) *Let M be a compact orientable 3-manifold with boundary a union of tori. Let $\epsilon > 0$. Then there are finitely many compact orientable hyperbolic 3-manifolds X and slopes σ on some component of ∂X such that $M \cong X(\sigma)$ and where the length of each slope σ is at least $2\pi + \epsilon$, when measured using some horoball neighbourhood of the cusp of X that is being filled.*

Lemma 2.5.2. *Let Y be a hyperbolic JSJ piece of a knot exterior and let $\mathcal{L}_0$ be the cusp of Y along which the trivial filling yields the exterior of an unlink. Let $l(p/q)$ be the length of the slope p/q on $\mathcal{L}_0$, measured in a maximal horoball neighbourhood N of $\mathcal{L}_0$. Then $l(p/q) \geq |q|/\sqrt{3}$.*

Proof. By a geometric argument as in (Cooper et Lackenby, 1998, Lemma 2.1) or (Agol, 2000, Theorem 8.1), the lengths of two slopes σ_1, σ_2 on $\mathcal{L}_0$ satisfy $l(\sigma_1)l(\sigma_2) \geq Area(\partial N) \cdot \Delta(\sigma_1, \sigma_2)$. By Theorem 1.2 of (Gabai *et al.*, 2021)), $Area(\partial N) \geq 2\sqrt{3}$. By taking $\sigma_1 = p/q$ and $\sigma_2 = 1/0$, and by the 6-theorem (Agol, 2000; Lackenby, 2000), we get

$$l(p/q) \geq 2\sqrt{3} \cdot |q|/l(1/0) \geq |q|/\sqrt{3}.$$ □

The next lemma follows the approach of (Lackenby, 2019, Theorem 1.1, Case 2).

Lemma 2.5.3. *Let Y be a JSJ piece of the exterior of a knot. There exists a constant $L(Y)$ with the following property. Let Y' be a hyperbolic JSJ piece of the exterior of a knot, with boundary component $\mathcal{L}_0$ such that $Y'(\mathcal{L}_0; 1/0)$ is S^3 or the exterior of an unlink. If $Y'(\mathcal{L}_0; p/q) \cong Y$, then $|q| \leq L(Y)$.*

Proof. Let $\epsilon = 1/15$. By Theorem 2.5.1, there are finitely many manifolds $\{X_j\}$ that are JSJ pieces of a knot exterior, and finitely many slopes $\{p_{j_i}/q_{j_i}\}$ of length at least $2\pi + 1/15$ (measured in a maximal horoball neighbourhood in X_j) such that $X_j(p_{j_i}/q_{j_i})$ is homeomorphic to Y. Set $L(Y) = \max\{|q_{j_i}|, 11\}$. If $|q| > L(Y)$, then by the previous lemma, $l(p/q) > 11/\sqrt{3} \geq 2\pi + 1/15$, but $p/q \notin \{p_{j_i}/q_{j_i}\}$. It follows that for any hyperbolic JSJ piece Y' as in the statement, the filling $Y'(\mathcal{L}_0; p/q)$ cannot be homeomorphic to Y. □

2.5.1 Composing pattern case

We begin the proof of Proposition 2.4.1 by considering the case of composing patterns.

Lemma 2.5.4. *Let P be a composing pattern and $\mathcal{P} = \partial\nu P \subset \partial V_P$. If $|q| > 1$, then the filling $V_P(\mathcal{P}; p/q)$ is not homeomorphic to a knot exterior.*

Proof. Let $Y \subset V_P$ be the composing space containing $\mathcal{P}$. Let $n+1$ be the number of boundary components of Y.

If $n > 2$, then $Y(\mathcal{P}; p/q)$ is Seifert fibred and has more than two boundary components (proof of Proposition 2.3.5(2)), so it is not a JSJ piece of a knot exterior by Proposition 2.3.10.

Suppose now that $n = 2$. Then $V_P = Y \cup S^3_{K_1}$ for some knot K_1. The filling $Y(\mathcal{P}; p/q)$ is

Seifert fibred (proof of Proposition 2.3.5(2)), and on the JSJ torus $\mathcal{T}_1 = \partial S^3_{K_1}$ of $V_P(\mathcal{P}; p/q)$, a regular fibre of $Y(\mathcal{P}; p/q)$ has meridional slope.

By Lemma 2.3.10, if J' is a knot such that $S^3_{J'}$ has the same JSJ pieces in its decomposition as $V_P(\mathcal{P}; p/q)$, then J' must be a cable of K_1. By the knot complement theorem, if $V_P(\mathcal{P}; p/q)$ were homeomorphic to $S^3_{J'}$, then the meridian on $\mathcal{T}_1$ would be mapped to the meridian on $\mathcal{T}_1' = \partial S^3_{K_1} \subset S^3_{J'}$. Further, regular fibres of $Y(\mathcal{P}; p/q)$ would be mapped to regular fibres of the outermost cable space of $S^3_{J'}$. However, a regular fibre of the outermost cable space of $S^3_{J'}$ does not have meridional slope on $\mathcal{T}_1'$, and a cable space possesses a unique Seifert fibred structure. Hence, $V_P(\mathcal{P}; p/q)$ cannot be homeomorphic to the exterior of a knot. □

Proposition 2.5.5. *Let $K = K_1 \# K_2 \# \dots \# K_n$ be a composite knot, where the K_i's are prime for each $i = 1, \dots, n$. Let $\mathcal{T}$ be a JSJ torus of S^3_K that decomposes K into P, a composing pattern, and K_i for some $i \in \{1, \dots, n\}$. Suppose there is an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some knot K' where $|q| > 2$. Then the homeomorphism does not map the outermost piece of $S^3_{K_i}$ to the surgered piece of $S^3_{K'}(p/q)$.*

Proof. Let $\mathcal{P} = \partial \nu P \subset \partial V_P$. If the homeomorphism maps the outermost piece of $S^3_{K_i}$ to the surgered piece of $S^3_{K'}(p/q)$, then $V_P(\mathcal{P}; p/q)$ is homeomorphic to a knot exterior by the discussion of Section 2.4.1 and Figure 2.3. By Lemma 2.5.4, this cannot happen if $|q| > 2$. □

Proof of Proposition 2.4.1. Let $\mathcal{T}$ be a JSJ torus of S^3_K that decomposes K into P and J. Suppose there exists a knot K' such that there is an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$. By Proposition 2.5.5, if P is a composing pattern, we may take $L(\mathcal{T}) = 2$.

2.5.2 Cable and twice-iterated cable case

We proceed with the case when P is a once or twice-iterated cable.

Proof of Proposition 2.4.1 (continued). We now suppose that P is a cable C_{r_1,s_1} or a twice-iterated cable $C_{r_2,s_2}(C_{r_1,s_1})$. Recall that the JSJ torus $\mathcal{T}$ decomposes K into P and a knot J. Let Y be the outermost piece of S^3_J. Let Y' be the JSJ piece of $S^3_{K'}$ such that the surgered piece of $S^3_{K'}(p/q)$ is $X' = Y'(p/(q(t')^2), t' \geq 1$ (Proposition 2.3.8).

Suppose the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ carries the outermost piece Y of S^3_J to the surgered piece of $S^3_{K'}(p/q)$, as described in Section 2.4.1. We look at each possibility for Y given by Proposition 2.3.10.

If Y is hyperbolic, then Y' is also hyperbolic if $|q| > 2$, according to Proposition 2.3.10(2). By Lemma 2.5.3, there is a constant $L(Y)$ such that $|q| \leq |q(t')^2| \leq L(Y)$.

If Y is an (r, s)-cable space, then Y' is a composing space by Proposition 2.3.10(3) and K' is a composite knot. Using the notation in Figure 2.3, $\mathcal{T}'$ separates K' into a composing pattern P' and some companion knot J'. By the discussion of Section 2.4.1, $V_{P'}(\mathcal{P}'; p/q)$ is homeomorphic to S^3_J, but this contradicts Lemma 2.5.4 applied to P' when $|q| > 2$.

If Y is the exterior of a torus knot $T_{a,b}, |a| > |b| > 1$, then Y' is an (r', s')-cable space by Proposition 2.3.10(4). Since the orders of exceptional fibres in $S^3_{T_{a,b}}$ and X' coincide, we have without loss of generality

$$|a| = |q(t')^2 r' s' - p|. \tag{1}$$

Recall that the JSJ torus $\mathcal{T} \subset \partial Y$ is mapped by the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$

to the JSJ torus $\mathcal{T}' \subset \partial X'$, in the notation of Figure 2.3. As distances are preserved between slopes that are carried one to another, we have the following equality by comparing Table 2.2 (regular fibre slopes) and the last row of Table 2.1 (longitudinal slopes) from Section 2.4:

$$|r_1| = \Delta\left(\frac{r_1}{s_1}, \frac{0}{1}\right) = \Delta\left(\frac{k}{1}, \frac{p}{q(t's')^2}\right) = |q(t's')^2k - p|.$$

Combining this with equation (1) yields

$$|q(t')^2| \cdot |(s')^2k - r's'| = |r_1 \pm a|.$$

Since $r', s' \neq 0$ are coprime, we have $(s')^2k - r's' \neq 0$. This implies that $|q| \leq |r_1| + |a|$.

Summing up, suppose $\mathcal{T}$ decomposes K into P and J, where $P = C_{r_1,s_1}$ or $C_{r_2,s_2}(C_{r_1,s_1})$. Denoting the outermost JSJ piece of S^3_J by Y, we let

$$L(\mathcal{T}) = \begin{cases} L(Y) \text{ from Lemma 2.5.3} & \text{if } Y \text{ is hyperbolic,} \\ 2 & \text{if } Y \text{ is an } (r,s)\text{-cable space,} \\ |r_1| + |a| & \text{if } Y \text{ is the exterior of a torus knot } T_{a,b}. \end{cases}$$

Then if $|q| > L(\mathcal{T})$, and if there exists and orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some knot K', the outermost piece Y of S^3_J is not carried to the surgered piece of $S^3_{K'}(p/q)$.

2.5.3 Other pattern case

To conclude the proof of Proposition 2.4.1, it remains to study patterns that are neither composing patterns nor once or twice-iterated cables. We will be using the Cyclic surgery theorem by Culler, Gordon, Luecke and Shalen.

Theorem 2.5.6. (Culler *et al.*, 1987, Cyclic surgery theorem) *Let M be a compact, connected, irreducible, orientable 3-manifold such that ∂M is a torus. Suppose that M is not a Seifert fibre space. If $\pi_1(M(\sigma_1))$ and $\pi_1(M(\sigma_2))$ are cyclic, then* $\Delta(\sigma_1, \sigma_2) \leq 1$.

We want to apply this theorem to the case where M is a filling of a pattern space. To do so, we must show that this filling is not a Seifert fibre space. We will need the following homological lemma about fillings of composing spaces.

Lemma 2.5.7. *Let Y be a composing space with three boundary components $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$. Denote by h the slope of a regular fibre on each boundary component of Y. Suppose σ_1 and σ_2 are slopes on $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, that are homologous in $Y(\mathcal{T};\sigma)$ for some surgery slope σ on $\mathcal{T}$. Then $\Delta(h,\sigma_1) = \Delta(h,\sigma_2) = k\Delta(h,\sigma)$ for some $k \in \mathbb{Z}$.*

Proof. There are slopes λ_1 and λ_2 on $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively such that $\{h, \lambda_i\}$ generates $H_1(\mathcal{T}_i;\mathbb{Z})$, $i = 1, 2$, and $\{h, \lambda_2 - \lambda_1\}$ generates $H_1(\mathcal{T};\mathbb{Z})$ (Figure 2.4). Further, the images induced by inclusion of h, λ_1, λ_2 into Y generate $H_1(Y;\mathbb{Z})$. Write

$$\begin{aligned} \sigma &= mh + n(\lambda_2 - \lambda_1), \\ \sigma_1 &= a_1 h + b_1 \lambda_1, \\ \sigma_2 &= a_2 h + b_2 \lambda_2. \end{aligned}$$

The σ-surgery along $\mathcal{T}$ adds the relation $mh + n(\lambda_2 - \lambda_1)$ in $H_1(Y(\mathcal{T};\sigma);\mathbb{Z})$. Hence, if σ_1 and σ_2 are homologous in $Y(\mathcal{T};\sigma)$, then

$$(a_1 h + b_1 \lambda_1) + k(mh + n(\lambda_2 - \lambda_1)) = a_2 h + b_2 \lambda_2,$$

for some $k \in \mathbb{Z}$. This implies that $b_1 = b_2 = kn$, giving us

$$\Delta(h,\sigma_1) = \Delta(h,\sigma_2) = kn = k\Delta(h,\sigma).$$

□

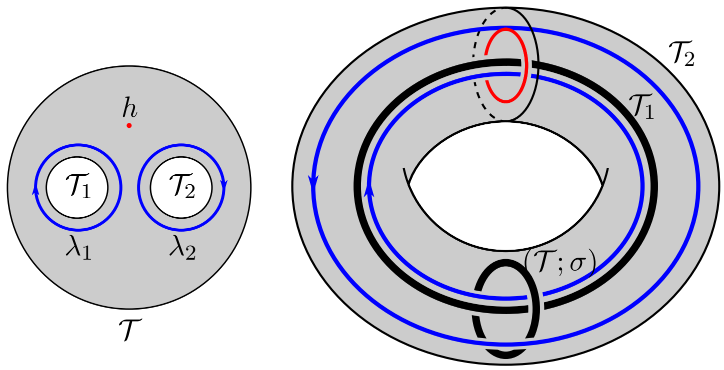

Figure 2.4: Left: Generators of $H_1(Y;\mathbb{Z})$ on the base orbifold of Y; Right: Generators of $H_1(Y(\mathcal{T};\sigma);\mathbb{Z})$ where Y is depicted as a link exterior in a solid torus bounded by $\mathcal{T}_2$

Proposition 2.5.8. *Suppose P is a pattern that is neither a composing pattern, nor a cable C_{r_1,s_1} or a twice-iterated cable $C_{r_2,s_2}(C_{r_1,s_1})$. Let $\mathcal{T}$ be the boundary component of V_P that is not $\partial\nu P$. Then the filling $V_P(\mathcal{T};m/n)$ is not a Seifert fibre space for $|m|$ sufficiently large.*

Proof. The pattern space V_P admits a JSJ decomposition $V_1 \cup V_2 \cup \ldots \cup V_k$ where V_1 is the JSJ piece that contains $\mathcal{T}$. Recall from the discussion following Definition 2.2.2 that we express slopes along $\mathcal{T}$ in the coordinates given by slopes μ and λ such that gluing the exterior of a knot J by respectively identifying μ and λ to the meridian and longitude of J yields the exterior of the knot $P(J)$.

If V_1 is hyperbolic, there are only finitely many slopes m/n such that $V_1(\mathcal{T};m/n)$ is not hyperbolic. So $V_1(\mathcal{T};m/n)$ is hyperbolic for $|m|$ sufficiently large, and $V_P(\mathcal{T};m/n)$ is not a Seifert fibre space.

If V_1 is a Seifert fibre space, then by Theorem 2.3.1, V_1 is either a composing space or an (r_1, s_1)-cable space.

Suppose V_1 is a composing space. For $V_P(\mathcal{T}; m/n)$ to be Seifert fibered, the JSJ pieces adjacent to V_1 in V_P must be Seifert fibred and $V_1(\mathcal{T}; m/n)$ must admit a Seifert fibred structure that differs from the one inherited by the fibration on V_1. The only such possibility is if $V_1(\mathcal{T}; m/n)$ is a trivial I-bundle over the torus. Recall that the regular fibres of V_1 have meridional slopes on each boundary component of V_1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the boundary components of $V_1(\mathcal{T}; m/n)$. As regular fibres are homologous in $V_1(\mathcal{T}; m/n)$, Lemma 2.5.7 says the distance on $\mathcal{T}_1$ between the meridian of $\mathcal{T}_1$ and a regular fibre of $V_1(\mathcal{T}; m/n)$ is equal to the distance on $\mathcal{T}_2$ between the meridian of $\mathcal{T}_2$ and a regular fibre of $V_1(\mathcal{T}; m/n)$.

Suppose the pieces adjacent to V_1 in V_P are Seifert fibred. Note that since P is not a composing pattern, $V_1(\mathcal{T}; m/n)$ shares a boundary component, say $\mathcal{T}_1$, with a cable space V_2 whose regular fibre has non-integral slope on $\mathcal{T}_1$. The other boundary component $\mathcal{T}_2$ of $V_1(\mathcal{T}; m/n)$ is shared with a torus knot exterior or a cable space V_3, whose regular fibre has integral slope on $\mathcal{T}_2$. By Lemma 2.5.7, the Seifert fibred structure of $V_1(\mathcal{T}; m/n)$ cannot extend across both V_2 and V_3, so $V_P(\mathcal{T}; m/n)$ is not Seifert fibred.

Suppose now that V_1 is an (r_1, s_1)-cable space. Let V_2 be the JSJ piece of V_P that shares a boundary component $\mathcal{T}_1$ with V_1.

Suppose that $\mathcal{T}_1$ remains incompressible in $V_P(\mathcal{T}; m/n)$. The pattern space V_P is either the union of V_1 with a hyperbolic V_2, or it decomposes into at least three JSJ pieces. In the first case, $V_P(\mathcal{T}; m/n)$ is clearly not Seifert fibred. In the second case, a Seifert fibred structure on $V_1(\mathcal{T}; m/n)$ might extend across a Seifert fibred structure on V_2. However, a JSJ piece of a knot exterior admits a unique Seifert fibred structure, so the structure on V_2 does not extend across the other JSJ pieces of $V_P(\mathcal{T}; m/n)$.

Suppose now that the torus $\mathcal{T}_1$ is compressed in $V_1(\mathcal{T}; m/n)$. On $\mathcal{T}_1$ and $\mathcal{T}$, the regular fibres of V_1 have respective slopes $r_1 s_1/1$ and r_1/s_1. By a similar reasoning as that of Proposition 2.3.5, cases (3) and (4), we have $|ms_1 - r_1 n| = 1$. For homological reasons (analogous to Lemma 2.3.2), the filling $V_P(\mathcal{T}; m/n)$ is homeomorphic to $(V_P \setminus V_1)(\mathcal{T}_1; ms_1^2/n)$.

If V_2 is hyperbolic or a composing space, we iterate the argument previously given for V_1.

If V_2 is an (r_2, s_2)-cable space, let $\mathcal{T}_2$ be its boundary component that is not $\mathcal{T}_1$. Let V_3 be the JSJ piece of V_P such that $V_3 \cap V_2 = \mathcal{T}_2$. The Seifert fibred structure on $V_2(\mathcal{T}_1; ms_1^2/n)$ might extend across V_3 only if $V_2(\mathcal{T}_1; ms_1^2/n)$ is a solid torus or a twisted I-bundle over the Klein bottle. This occurs when $|ms_1^2 s_2 - r_2 n| = 1$ or 2. Combining this with the fact that $|ms_1 - r_1 n| = 1$, we have that (m, n) must be a solution to the system

$$\begin{pmatrix} s_1 & -r_1 \\ s_1^2 s_2 & -r_2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix} \text{ or } \begin{pmatrix} \pm 1 \\ \pm 2 \end{pmatrix}.$$

As r_2 and s_2 are coprime,

$$\det \begin{pmatrix} s_1 & -r_1 \\ s_1^2 s_2 & -r_2 \end{pmatrix} \neq 0.$$

Therefore, there are only finitely many slopes m/n such that the Seifert fibred structure on $V_2(\mathcal{T}_1; ms_1^2/n)$ extends across V_3.

Consequently, $V_P(\mathcal{T}; m/n)$ is not a Seifert fibre space for $|m|$ sufficiently large. □

Proof of Proposition 2.4.1 (continued). Recall that $\mathcal{T}$ is a JSJ torus of S_K^3 that decomposes K into P and J. Suppose there exists a knot K' such that there is an orientation-preserving homeomorphism $S_K^3(p/q) \cong S_{K'}^3(p/q)$ where $|q| > 2$.

We now suppose that P is neither a composing pattern nor a once or twice-iterated cable. Suppose the homeomorphism $S_K^3(p/q) \cong S_{K'}^3(p/q)$ carries the outermost piece Y of S_J^3 to the surgered piece of $S^3_{K'}(p/q)$, as described in the scenario of Section 2.4.1. By Lemma 2.4.3, there is a slope $x/y = q(w')^2/y$ on $\mathcal{T}$ that is the meridian of $V_P(\mathcal{P}; p/q)$ seen as a knot exterior, where w' is the winding number of P'. See Figure 2.3.

By Lemma 2.5.8, there exists a bound $L(P)$ such that if $|m| > L(P)$, then $V_P(\mathcal{T}; m/n)$ is not a Seifert fibre space. If $w' \neq 0$, suppose that $|q| > L(P)$. The inequality $|x| = |q(w')^2| > |q| > L(P)$ implies that $V_P(\mathcal{T}; x/y)$ is not a Seifert fibre space. On one hand, the filling of $V_P(\mathcal{P}; p/q)$ along the meridian x/y is the trivial filling $V_P(\mathcal{P}, \mathcal{T}; p/q, x/y) \cong S^3$. On the other hand, $V_P(\mathcal{P}; 1/0)$ is the trivial filling of the pattern P, so it is homeomorphic to a solid torus. Consequently, $V_P(\mathcal{P}, \mathcal{T}; 1/0, x/y)$ is homeomorphic to the lens space $L_{y,x}$. We obtain that both the p/q and $1/0$-fillings of the non-Seifert fibred manifold $V_P(\mathcal{T}; x/y)$ yield manifolds with cyclic fundamental groups. By the Cyclic surgery theorem (Theorem 2.5.6), we have $|q| = \Delta(p/q, 1/0) \leq 1$, which contradicts $|q| > 2$.

Suppose now that $w' = 0$. If the surgered piece X' of $S^3_{K'}(p/q)$ were Seifert fibred, then it would be a filling of either a cable space or a composing space (Proposition 2.3.10). In both cases, w' would be non-zero, a contradiction. Therefore, X' is hyperbolic. As $|q| > 2$, X' is the $p/(q(t')^2)$-filling of a hyperbolic JSJ piece of $S^3_{K'}$, $t' \geq 1$ (Proposition 2.3.8(1)). By Lemma 2.5.3, there exists a constant $L(Y)$ such that X' is homeomorphic to the outermost piece Y of S^3_J only if $|q| \leq L(Y)$.

Setting $L(\mathcal{T}) = \max\{L(P), L(Y)\}$ gives the desired bound.

This completes the proof of Proposition 2.4.1. □

2.6 Proof of Theorem 1

Proposition 2.1.5 tells us that if $|q|$ is sufficiently large, an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ restricts to a homeomorphism between the surgered pieces of $S^3_K(p/q)$ and $S^3_{K'}(p/q)$. By the knot complement theorem, this homeomorphism preserves the slopes on the boundary of the surgered pieces. To complete the proof of Theorem 2.1.1, we must show that it further restricts to the JSJ pieces of S^3_K and $S^3_{K'}$ that were filled to produce the surgered pieces.

First, we need the following intermediate results.

Proposition 2.6.1. *Let K and K' be knots such that there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p'/q')$. If the core of the surgery solid torus in $S^3_K(p/q)$ is mapped to the core of the surgery solid torus in $S^3_{K'}(p'/q')$, then $K = K'$.*

Proof. Let v and v' be the cores of the surgery solid tori of $S^3_K(p/q)$ and $S^3_{K'}(p'/q')$ respectively. Since v is sent to v' by the homeomorphism, the neighbourhoods $\nu(v)$ and $\nu(v')$ are also sent one to another by the homeomorphism. Therefore, $S^3_K(p/q) \setminus \text{int}(\nu(v)) \cong S^3 \setminus \text{int}(\nu K)$ is homeomorphic to $S^3_{K'}(p'/q') \setminus \text{int}(\nu(v')) \cong S^3 \setminus \text{int}(\nu K')$, which implies that $K = K'$ by the knot complement theorem. □

Lemma 2.6.2. *If q, p, r, s, r', s' are integers such that $|q| > 2$ and $|qrs - p| = |qr's' - p| = 1$, then $rs = r's'$.*

Proof. We have $|q(rs - r's')| = 0$ or 2. But $|q| > 2$, so $|q(rs - r's')| = 0$ and $rs = r's'$. □

Let K be a non-trivial knot, and suppose there is a knot K' such that there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$. Let X, X' be the surgered pieces of $S^3_K(p/q)$, $S^3_{K'}(p/q)$ respectively. Let Y, Y' be the JSJ pieces of S^3_K, $S^3_{K'}$ such that $X = Y(p/(qt^2))$ and $X' = Y'(p/(q(t')^2))$, for some $t, t' \geq 1$ (Proposition 2.3.8).

We now assume, by Proposition 2.1.5, that $|q|$ is large enough such that X and X' are sent one to another by the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$. We will study each possibility listed in Theorem 2.3.1 for Y and show in each case that $K = K'$ for $|q|$ sufficiently large.

2.6.1 Exterior of a torus knot

Suppose Y is the exterior of a torus knot. In this case, K is a torus knot or a cable of a torus knot (Proposition 2.3.8). By McCoy, if K is a torus knot, we have that $K = K'$ for $|q|$ sufficiently large (McCoy, 2020). So suppose K is a cable knot $C_{r,s}(T_{a,b})$ such that $|qrs - p| = 1$.

By Corollary 2.3.9, Y' is also the exterior of a torus knot if $|q| > 8$. Therefore, K' is either a torus knot or a cable of a torus knot by Proposition 2.3.6.

We have the following corollary of a proposition from McCoy.

Proposition 2.6.3. (McCoy, 2020, Proposition 1.5) *If an (r, s)-cable of a torus knot shares a p/q-surgery with a torus knot where $|q| > 1$, then $|q| = s$.*

It follows that the cable $K = C_{r,s}(T_{a,b})$ cannot share a p/q surgery with a torus knot when $|q| > s$. Hence, if $|q| > s$ and 8, K' is a cable of a torus knot $C_{r',s'}(T_{c,d})$ where $|qr's' - p| = 1$. We thus have an homeomorphism

$$S^3_{T_{a,b}}(p/(qs^2)) \cong S^3_{T_{c,d}}(p/(q(s')^2))$$

by Proposition 2.3.3. This gives the homeomorphism of base orbifolds

$$S^2(|a|, |b|, |qs^2ab - p|) \cong S^2(|c|, |d|, |q(s')^2cd - p|).$$

Comparing orders of cone points, without loss of generality, assume that $|b| = |d|$ and $|a| = |q(s')^2cd - p|$. By combining this with $|qrs - p| = 1$, we find

$$|q| \cdot |(s')^2cd - rs)| = |a \pm 1|.$$

The right-hand side is a non-zero integer since $|a| > 1$, which implies that $|q| \leq |a| + 1$.

Consequently, if $|q| > |a| + 1$, the homeomorphism $S^3_{T_{a,b}}(p/(qs^2)) \cong S^3_{T_{c,d}}(p/(q(s')^2))$ sends the core of the surgery solid torus in $S^3_{T_{a,b}}(p/(qs^2))$ of order $|qs^2ab - p|$ to the core of the surgery solid torus in $S^3_{T_{c,d}}(p/(q(s')^2))$ of order $|q(s')^2cd - p|$. By Proposition 2.6.1, we obtain that $T_{a,b} = T_{c,d}$. Furthermore, the equality of orders yields $qs^2ab - p = \pm(q(s')^2cd - p)$, but since $|q| > 1$ and p and q are coprime, the only possibility is $qs^2ab - p = q(s')^2cd - p$, which in turn gives $s = s'$. By Lemma 2.6.2, since $|q| > |a| + 1 > 2$, we have $C_{r,s} = C_{r',s'}$. Hence, $C_{r,s}(T_{a,b}) = C_{r',s'}(T_{c,d})$, that is, $K = K'$, as desired.

2.6.2 Composing space

Suppose Y is a composing space. By Corollary 2.3.9, Y' is also a composing space if $|q| > 2$. By Proposition 2.3.8(2), X and X' are Seifert fibred, each with one exceptional fibre of order $|qt^2|$ and $|q(t')^2|$ respectively. These exceptional fibres correspond to the cores of the surgery solid tori in X and X'.

Since $X \cong X'$, the unique exceptional fibre of X is sent to the unique exceptional fibre of X' by the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$. This implies that $t = t'$. If $t = 1$, then these exceptional fibres are precisely the cores of the surgery solid tori of $S^3_K(p/q)$ and $S^3_{K'}(p/q)$. Then $K' = K$, by Proposition 2.6.1. If $t > 1$, by Proposition 2.3.6, t is the winding number of cable patterns $C_{r,t}$ and $C_{r',t}$ such that $K = C_{r,t}(J)$ and $K' = C_{r',t}(J')$, where J and J' are composite knots. By Proposition 2.3.3, we have $S^3_J(p/(qt^2)) \cong S^3_{J'}(p/(qt^2))$ and $J = J'$ by Proposition 2.6.1. Since $|qrt - p| = 1$ and $|qr't - p| = 1$, Lemma 2.6.2 tells us that

$r = r'$, and we conclude that $K = K'$.

2.6.3 Exterior of a hyperbolic link

Suppose Y is the exterior of a hyperbolic knot or link. By Corollary 2.3.9, Y' is also the exterior of a hyperbolic knot or link if $|q| > 8$. Recall that $Y(p/(qt^2)) \cong Y'(p/(q(t')^2))$. We will apply the following theorem by Lackenby and use the arguments in his proof of (Lackenby, 2019, Case 2 of Theorem 1.1).

Theorem 2.6.4. (Lackenby, 2019, Theorem 3.1) *Let M be S^3 or the exterior of the unknot or unlink in S^3, and let K be a hyperbolic knot in M. Let $M_K = M \setminus \mathrm{int}(\nu K)$. There exists a constant $C(K)$ with the following property. If $M_K(\sigma) \cong M_{K'}(\sigma')$ for some hyperbolic knot K' in M and some σ' such that $\Delta(\sigma', \mu') > C(K)$, where μ' is the slope that bounds a disc in $\nu K'$, and if the homeomorphism restricted to the boundary of M is the identity, then $(M, K) \cong (M, K')$ and $\sigma = \sigma'$.*

Lemma 2.6.5. *For $|q|$ sufficiently large, $Y \cong Y'$ and $t = t'$.*

Proof. Let $n + 1$ be the number of boundary components of Y and Y'. If $n = 0$, let M be S^3. If $n \geq 1$, let M be the exterior of the unlink with n components. By Theorem 2.3.1, Y and Y' are respectively homeomorphic to exteriors of hyperbolic knots H and H' in M.

By the argument in the last paragraph of (Lackenby, 2019, proof of Theorem 1.1, p.13), there is a knot H'' in M such that $(M, H') \cong (M, H'')$ and there exists an homeomorphism $M_H(p/(qt^2)) \cong M_{H''}(p/(q(t')^2))$ which is the identity when restricted to the boundary of M. Let $C(H)$ be the constant given by Theorem 2.6.4 for H. If $|q| > C(H)$, then $|q(t')^2| > C(H)$. By Theorem 2.6.4, $(M, H) \cong (M, H'')$ and $p/(qt^2) = p/(q(t')^2)$. Therefore, $(M, H) \cong (M, H')$, so $Y \cong Y'$, and $t = t'$. □

In S^3_K and $S^3_{K'}$ respectively, the JSJ pieces Y, Y' are the outermost pieces of the exteriors of knots J, J'. We thus have $S^3_J(p/(qt^2)) \cong S^3_{J'}(p/(qt^2))$ which restricts to $Y(p/(qt^2)) \cong Y'(p/(qt^2))$. This is precisely the scenario of (Lackenby, 2019, Case 2 of Theorem 1.1). We adapt the relevant parts of its proof using our notation to conclude the case when Y is hyperbolic.

Proposition 2.6.6. *Let $C(H)$ be as in the proof of Lemma 2.6.5. If $|q| > \max\{8, C(H)\}$, then $K = K'$.*

Proof. The homeomorphism $(M, H) \cong (M, H')$ from the Lemma 2.6.5 gives a homeomorphism $h : S^3 \setminus (S^3_J \setminus \text{int}(Y)) \to S^3 \setminus (S^3_{J'} \setminus \text{int}(Y'))$ that sends J to J'.

Further, $S^3_K(p/q) \cong S^3_{K'}(p/q)$ restricts to a homeomorphism $S^3_J \setminus \text{int}(Y) \cong S^3_{J'} \setminus \text{int}(Y')$ which agrees with h on the boundary. We can thus extend it to a homeomorphism $(S^3, J) \cong (S^3, J')$. Hence, $J = J'$. If $t = 1$, then $K = J$ and $K' = J'$ and we are done. If $t > 1$, then K and K' are cables of $J = J'$. By Lemma 2.6.2, $K = K'$. $\square$

2.6.4 Cable space

Suppose Y is an (r_1, s_1)-cable space. By Corollary 2.3.9, Y' is also a cable space if $|q| > 2$. Therefore, K and K' are once or twice-iterated cables of knots J and J' respectively. Let us write

$$K = \begin{cases} C_{r_1,s_1}(J) & \text{if } t = 1 \\ C_{r_2,s_2}(C_{r_1,s_1}(J)) & \text{if } t > 1 \end{cases}, \quad K' = \begin{cases} C_{r'_1,s'_1}(J') & \text{if } t' = 1 \\ C_{r'_2,s'_2}(C_{r'_1,s'_1}(J')) & \text{if } t' > 1 \end{cases}.$$

Proposition 2.6.7. *If $|q| > 2$, then $K = K'$. That is:*

(i) $J = J'$;

(ii) $C_{r_1,s_1} = C_{r'_1,s'_1}$;

(iii) $t = t'$, *and* $C_{r_2,s_2} = C_{r'_2,s'_2}$ *if* $t > 1$.

Proof. Since $S^3_K(p/q) \setminus X \cong S^3_J$ and $S^3_{K'}(p/q) \setminus X' \cong S^3_{J'}$, the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ restricts to a homeomorphism between S^3_J and $S^3_{J'}$. This implies (i) by the knot complement theorem.

By (i), the meridians and longitudes forming the bases of $H_1(\partial S^3_J; \mathbb{Z})$ and $H_1(\partial S^3_{J'}; \mathbb{Z})$ are respectively sent one to another. The regular fibres of X and X' have respective slopes r_1/s_1 and r'_1/s'_1 on ∂X and $\partial X'$, and both Seifert fibred structures have base orbifold a disc with two cone points. If a given oriented manifold admits a Seifert fibration with base orbifold a disc and two cone points, then there is no other Seifert fibration on this manifold with the same orbifold structure. It follows that the slopes r_1/s_1 and r'_1/s'_1 are equal. Hence, $C_{r_1,s_1} = C_{r'_1,s'_1}$ showing (ii).

The longitudes of X and X' coincide and have respective slopes $p/(q(ts_1)^2)$ and $p/(q(t's'_1)^2)$, so $q(ts_1)^2 = q(t's'_1)^2$. Since $s_1 = s'_1$ by (ii), we get the equality $t = t'$. If $t > 1$, then $t = s_2 = s'_2$, and $C_{r_2,s_2} = C_{r'_2,s'_2}$ by Lemma 2.6.2, which proves (iii). □

This concludes the proof of Theorem 2.1.1.

2.7 Characterizing slopes for cables with only Seifert fibred pieces

For some specific families of satellite knots, an explicit bound for $|q|$ that realizes Theorem 2.1.1 can be expressed. The following result is obtained from the treatment of Seifert fibred JSJ pieces throughout Sections 2.5 and 2.6.

Theorem 2.7.1. *Let K be a cable knot with an exterior consisting solely of Seifert fibred*

JSJ pieces. A slope p/q is characterizing for K if:

(i) $|q| > 2$ *and K is not an n-times iterated cable of a torus knot,* $n \geq 1$*;*

(ii) $|q| > |r_1| + |a|$ *and K is an n-times iterated cable of* $C_{r_1,s_1}(T_{a,b}), |a| > |b| > 1, n \geq 1$*;*

(iii) $|q| > \max\{8, s_1, |r_1| + |a|\}$ *and K is a cable* $C_{r_1,s_1}(T_{a,b}), |a| > |b| > 1$.

Proof. We first show that Proposition 2.1.5 is realized. Suppose K' is a knot such that there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$. If $S^3_K(p/q)$ is Seifert fibred, Proposition 2.1.5 is immediately realized.

If $S^3_K(p/q)$ contains a JSJ torus, then the surgered piece X' of $S^3_{K'}(p/q)$ is not a filling of a knot exterior. It follows that the JSJ decomposition of $S^3_{K'}$ does not contain hyperbolic pieces if $|q| > 2$. The surgered piece X' is thus Seifert fibred and it is a filling of a JSJ piece Y' of $S^3_{K'}$ that is either a composing space or a cable space (Proposition 2.3.8).

If Y' is a composing space, then by Lemma 2.5.4 and Section 2.4.1, X' must be the image by the homeomorphism of the surgered piece of $S^3_K(p/q)$ if $|q| > 2$.

If Y' is a cable space and if X' is not the image of the surgered piece of $S^3_K(p/q)$, then X' is the image of the exterior of a torus knot $T_{a,b}$ in S^3_K (Proposition 2.3.10). Using the notation introduced in Section 2.4.1, let $\mathcal{T} = \partial S^3_{T_{a,b}}$ and let P be the pattern such that $\mathcal{T}$ decomposes K into P and $T_{a,b}$.

Suppose K is not an n-times iterated cable of $C_{r_1,s_1}(T_{a,b}), n \geq 1$. Then the pattern space V_P contains a composing space that shares all its boundary components with other JSJ pieces of S^3_K. By the proof of Proposition 2.5.8, $V_P(\mathcal{T}; q(w')^2/y)$ is not Seifert fibred. Applying the Cyclic surgery theorem (Theorem 2.5.6) as described in Section 2.5, we obtain that $|q| = 1$, contradicting (i).

If K is an n-times iterated cable of $C_{r_1,s_1}(T_{a,b}), n \geq 1$, we apply the same method as in Section 2.5.2 to compare the distances between the regular fibre slope and the longitudinal slope on $\mathcal{T}$ and $\mathcal{T}'$. This yields the inequality $|q| \leq |r_1| + |a|$, which implies that if (ii) holds, X' must be the image of the surgered piece of $S^3_K(p/q)$.

The theorem now follows, as (i), (ii) and (iii) are greater than or equal to the bounds from Section 2.6. □

Note that Theorem 2.7.1(i) is equivalent to Theorem 2.1.4 when applied to cable knots. If K is not a cable knot in Theorem 2.1.4, then it is a composite knot and the result follows from Theorem 2.1.2, which we prove in the next section.

Theorem 2.1.4. *If K is a knot with an exterior consisting solely of Seifert fibred JSJ pieces, with one of them being a composing space, then any slope that is neither integral nor half-integral is a characterizing slope for K.*

Remark 2.7.2. We relied on the constructive nature of Seifert fibred spaces to compute the above bounds. If S^3_K contains hyperbolic JSJ pieces, the task becomes more difficult. Indeed, for generic cases, we need to determine values that realize Theorems 2.5.1 and 2.6.4. Recently, Wakelin established a lower bound on $|q|$ for a slope p/q to be characterizing for certain hyperbolic patterns (Wakelin, sous presse). In forthcoming work with Wakelin, we combine her findings and our study of Seifert fibred JSJ pieces to obtain further results.

2.8 Characterizing slopes for composite knots

We now turn to the proof of Theorem 2.1.2.

Theorem 2.1.2. *If K is a composite knot, then every non-integral slope is characterizing for K.*

2.8.1 The surgered submanifold

Thus far, we have made the assumption $|q| > 2$, allowing us to define the surgered piece of a surgery along a knot. When $|q| = 2$, the surgered piece may not be defined as in Definition 2.3.7 if the resulting manifold is obtained from filling a hyperbolic JSJ piece. Indeed, the surgery operation can create essential tori, or it might yield a Seifert fibre space which admits a Seifert fibred structure that extends to other JSJ pieces.

Definition 2.8.1. Let $Y_0 \cup Y_1 \cup Y_2 \cup \ldots \cup Y_n$ be the JSJ decomposition of the exterior of a knot K.

If $|q| > 1$, then up to re-indexing the Y_i, the JSJ decomposition of $S^3_K(p/q)$ is of the form

$$(X_0 \cup X_1 \cup \ldots \cup X_m) \cup (Y_i \cup Y_{i+1} \cup \ldots \cup Y_n),$$

for some $1 \leq i \leq n$ and $m \geq 0$, and where none of the X_j's are JSJ pieces of S^3_K. The manifold $X_0 \cup X_1 \cup \ldots \cup X_m$ is the *surgered submanifold* of $S^3_K(p/q)$.

If the surgered submanifold is a JSJ piece of $S^3_K(p/q)$, i.e., $m = 0$, then we may also call it the *surgered piece* of $S^3_K(p/q)$.

Remark 2.8.2. This definition is compatible with Definition 2.3.7. In fact, the surgered submanifold of a surgery $S^3_{K'}(p/q)$ may not be a surgered piece only in the case where the outermost piece of $S^3_{K'}$ is hyperbolic (proof of Proposition 2.3.6 and Proposition 2.3.3).

We obtain an analogue of Proposition 2.1.5 for composite knots when $|q| > 1$.

Proposition 2.8.3. *Let K be a composite knot and suppose there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some knot K'. If $|q| > 1$, then the* *homeomorphism* $S^3_K(p/q) \cong S^3_{K'}(p/q)$ *carries the surgered piece of* $S^3_K(p/q)$ *to a JSJ piece of the surgered submanifold of* $S^3_{K'}$.

Proof. Let $K = K_1 \# K_2 \# \ldots \# K_n$ where the K_i's are prime for each $i = 1, \ldots, n$. Let Y be the outermost composing space of S^3_K. It is homeomorphic to the exterior of the link in S^3 with unknotted components $L_0, L_1, \ldots, L_n$ such that each pair (L_0, L_i) for $i = 1, \ldots, n$, is a Hopf link and the link formed by $L_1, \ldots, L_n$ is the unlink with n components. Let $\mathcal{L}_i = \partial \nu L_i$ be the boundary components of Y (Figure 2.5). By Remark 2.8.2, the surgered piece $X = Y(\mathcal{L}_0; p/q)$ of $S^3_K(p/q)$ is well-defined.

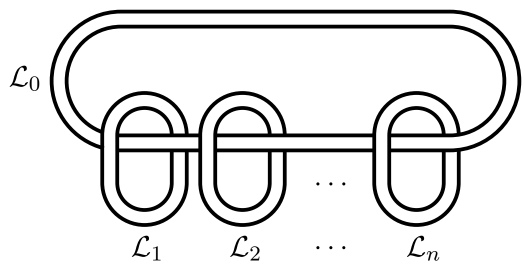

Figure 2.5: Composing space seen as a link complement in S^3

Suppose the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ does not carry X into the surgered submanifold X' of $S^3_{K'}$. Then there is a component K_i of $K, i \in \{1, \ldots, n\}$, whose exterior contains a submanifold homeomorphic to X'. Let $\mathcal{T} = \partial S^3_{K_i} \subset S^3_K$. The JSJ torus $\mathcal{T}$ decomposes K into a composing pattern P and K_i. The homeomorphism sends $\mathcal{T}$ to a JSJ torus $\mathcal{T}'$ of $S^3_{K'}(p/q)$ that separates $S^3_{K'}(p/q)$ into a manifold homeomorphic to $S^3_{K_i}$ and the exterior of a knot J'. As a result, $V_P(\mathcal{P}; p/q)$ is homeomorphic to the exterior of J', which contradicts Lemma 2.5.4. $\square$

2.8.2 Fillings of a hyperbolic piece

In order to prove Theorem 2.1.2, we must demonstrate that the surgered submanifold of $S^3_{K'}(p/q)$ does not result from filling a hyperbolic JSJ piece of $S^3_{K'}$. Therefore, we now focus on the topology of the surgered submanifold of $S^3_{K'}(p/q)$, under the assumption that the outermost piece of $S^3_{K'}$ is hyperbolic. In this subsection, we study the surgery $S^3_{K'}(p/q)$ by itself, without taking into account any constraints arising from an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$.

Recall from Theorem 2.3.1 that if Y' is a hyperbolic JSJ piece in the exterior of a knot, then Y' is either the exterior of a hyperbolic knot L'_0 in S^3 or the exterior of a hyperbolic link in S^3 with components $L'_0, L'_1, \ldots, L'_n$ such that the link formed by $L'_1, \ldots, L'_n$ is the unlink with n components. From now on, we will denote the boundary components of such a hyperbolic piece Y' by $\mathcal{L}'_i = \partial\nu L'_i$, $i = 0, \ldots, n$.

Proposition 2.8.4. *Let Y' be a hyperbolic JSJ piece of a knot exterior. If $Y'(\mathcal{L}'_0; p'/q')$ is homeomorphic to either a composing space or to a p/q-filling of a composing space where $|q| > 1$, then $|q'| \leq 1$.*

Proof. Let Y be a composing space with $n+1$ boundary components labelled as in Figure 2.5. Whitout loss of generality, suppose that $Y'(\mathcal{L}'_0; p'/q')$ is homeomorphic to $Y(\mathcal{L}_0; p/q)$, a Seifert fibre space with one exceptional fibre of order $|q| > 1$. Then Y' also has $n+1$ boundary components. Up to permuting indices, we can assume that for each $i = 1, \ldots, n$, the homeomorphism maps $\mathcal{L}'_i$ to $\mathcal{L}_i$.

There exists infinitely many slopes σ_1 on $\mathcal{L}'_1$ such that the cores of the surgery solid torus is an exceptional fibre in $Y'(\mathcal{L}'_0, \mathcal{L}'_1; p'/q', \sigma_1)$. There also exists infinitely many slopes σ_i on each $\mathcal{L}'_i$, $i = 2, \ldots, n$, such that the core of the surgery solid tori are regular fibres in $Y'(\mathcal{L}'_0, \mathcal{L}'_i; p'/q', \sigma_i)$. Since hyperbolic manifolds possess only finitely many exceptional surgery slopes on each of their torus boundary components, we can choose $\sigma_1, \ldots, \sigma_n$ such that $\widetilde{Y'} = Y'(\mathcal{L}'_1, \ldots, \mathcal{L}'_n; \sigma_1, \ldots, \sigma_n)$ is hyperbolic.

Now, $\widetilde{Y'}(\mathcal{L}'_0; p'/q')$ has base orbifold S^2 with two exceptional fibres, which means that it has cyclic fundamental group. On the other hand, $Y'(\mathcal{L}'_0; 1/0)$ is homeomorphic to the exterior of the unlink with n components. Therefore, $\widetilde{Y'}(\mathcal{L}'_0; 1/0)$ is a connected sum of manifolds with cyclic fundamental groups. By Boyer and Zhang (Boyer et Zhang, 1998, Corollary 1.4), or the Cyclic surgery theorem (Theorem 2.5.6) if the connected sum is trivial, we must have $|q'| = \Delta(p'/q', 1/0) \leq 1$ since $\widetilde{Y'}$ is not Seifert fibred.

Suppose now that $Y'(\mathcal{L}'_0; p'/q')$ is homeomorphic to a composing space with n boundary components. The argument is similar to that above. There are infinitely many slopes σ_i on each component $\mathcal{L}'_i$ of $\partial Y'(\mathcal{L}'_0; p'/q')$ such that the cores of the surgery solid tori corresponding to σ_1, σ_2 are exceptional fibres and the cores of the surgery solid tori corresponding to $\sigma_3, \ldots, \sigma_n$ are regular fibres. We can choose the σ_i's so that $\widetilde{Y'} = Y'(\mathcal{L}'_1, \ldots, \mathcal{L}'_n; \sigma_1, \ldots, \sigma_n)$ is hyperbolic. Then $\widetilde{Y'}(\mathcal{L}'_0; p'/q')$ and $\widetilde{Y'}(\mathcal{L}'_0; 1/0)$ are fillings of a hyperbolic manifold that are respectively a manifold with cyclic fundamental group and a connected sum of manifolds with cyclic fundamental groups. We conclude as before with (Boyer et Zhang, 1998, Corollary 1.4) or Theorem 2.5.6. □

Lemma 2.8.5. *Let K' be a knot such that the outermost piece Y' of $S^3_{K'}$ is hyperbolic. If $|q| > 1$, then the JSJ tori of $S^3_{K'}$ are incompressible in $S^3_{K'}(p/q)$.*

Proof. We may assume that K' is a satellite knot as otherwise, the statement is vacuously true. The surgered submanifold of $S^3_{K'}(p/q)$ contains $Y'(\mathcal{L}'_0; p/q)$. Suppose $Y'(\mathcal{L}'_0; p/q)$ has compressible boundary. As $S^3_{K'}(p/q)$ is irreducible, $Y'(\mathcal{L}'_0; p/q)$ is also irreducible, so it must be a solid torus. This implies that Y' has two boundary components and, therefore, as $Y'(\mathcal{L}'_0; 1/0)$ is also a solid torus, a result of Wu (Wu, 1992, Theorem 1) implies that $|q| = \Delta(p/q, 1/0) \leq 1$, which contradicts our assumption $|q| > 1$. Hence, $Y'(\mathcal{L}'_0; p/q)$ has incompressible boundary and as a consequence, the JSJ tori of $S^3_{K'}$ are incompressible in $S^3_{K'}(p/q)$. □

Corollary 2.8.6. *Let K' be a knot such that the outermost piece Y' of $S^3_{K'}$ is hyperbolic. If $|q| > 1$, then the surgered submanifold of $S^3_{K'}(p/q)$ is either $Y'(\mathcal{L}'_0; p/q)$, or the union of $Y'(\mathcal{L}'_0; p/q)$ and some Seifert fibred JSJ pieces of $S^3_{K'}$ sharing a boundary component with Y' in $S^3_{K'}$.* □

2.8.3 Non-integral toroidal surgeries

We now study the surgered submanifold of $S^3_{K'}(p/q)$ in the context of an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ where K is a composite knot.

Recall that a 3-manifold is said to be *toroidal* if it contains an essential torus, and *atoroidal* otherwise. The following proposition narrows down our investigation to hyperbolic manifolds that admit a non-integral toroidal surgery.

Proposition 2.8.7. *Let K be a composite knot. Suppose there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ where K' is such that the outermost piece Y' of $S^3_{K'}$ is hyperbolic. If $|q| > 1$, then $Y'(\mathcal{L}'_0; p/q)$ is toroidal.*

Proof. By Proposition 2.8.3, the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ sends the surgered piece X of $S^3_K(p/q)$ to a JSJ piece of the surgered submanifold X' of $S^3_{K'}(p/q)$.

Suppose $Y'(\mathcal{L}'_0; p/q)$ is atoroidal. Then by Corollary 2.8.6, X' has trivial JSJ decomposition, which means that it is homeomorphic to X, a filling of a composing space. Hence, $Y'(\mathcal{L}'_0; p/q)$ is homeomorphic to a submanifold of a filling of a composing space.

Since $Y'(\mathcal{L}_0'; p/q)$ has incompressible torus boundary components, it must be homeomorphic to either a filling of a composing space or a composing space, since these are the only submanifolds of X' that have such a boundary. However, this contradicts Proposition 2.8.4. $\square$

Lemma 2.8.8. *Let Y' be a hyperbolic JSJ piece of a knot exterior with at least three boundary components. If $|q| > 1$, then $Y'(\mathcal{L}_0'; p/q)$ is atoroidal.*

Proof. The filling $Y'(\mathcal{L}_0'; 1/0)$ is the complement of the unlink and it has compressible boundary. If $Y'(\mathcal{L}_0'; p/q)$ contains an essential torus, then by a result of Wu (Wu, 1998, Theorem 4.1), we have $|q| = \Delta(p/q, 1/0) \leq 1$, contradicting the assumption $|q| > 1$. $\square$

Eudave-Muñoz contructed in (Eudave-Muñoz, 1997) a family of hyperbolic knots that admit half-integral toroidal surgeries. These surgeries produce a union of two Seifert fibre spaces. Gordon and Luecke proved that if a hyperbolic knot admits a non-integral toroidal surgery, then it belongs to Eudave-Muñoz's family and the surgery slope is half-integral (Gordon et Luecke, 2004).

Lemma 2.8.9. *Let K be a composite knot. Suppose there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ where K' is such that the outermost piece of $S^3_{K'}$ is hyperbolic. If $|q| > 1$, then K' is not a hyperbolic knot.*

Proof. By Eudave-Muñoz, Gordon and Luecke, any non-integral surgery along a hyperbolic knot contains at most one essential torus. However, the surgery $S^3_K(p/q)$ contains at least two essential tori, the boundary components of the surgered piece being such tori. $\square$

We obtain the next corollary by combining Proposition 2.8.7 and the two preceding lemmas.

Corollary 2.8.10. *Let K be a composite knot. Suppose there exists an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ where K' is such that the outermost piece Y' of $S^3_{K'}$ is hyperbolic. If $|q| > 1$, then Y' has exactly two boundary components.* □

Gordon and Luecke also classified in (Gordon et Luecke, 2004) all hyperbolic knots in solid tori which admit non-integral toroidal surgeries. They are derived from Eudave-Muñoz's construction mentioned above, and the resulting surgeries have link surgery descriptions as in Figure 2.6. The labels L_i identify the link components and the α_i's are the corresponding surgery slopes, which correspond to the slopes α, β, γ in (Gordon et Luecke, 2004). The component L_4 is left unfilled. The essential torus $\widehat{\mathcal{T}}$ is pictured. If μ_i is the slope on $\partial\nu L_i$ that bounds a disc in νL_i, then $\Delta(\alpha_i, \mu_i) \geq 2$ (Gordon et Luecke, 2004, Proof of Corollary A.2). Hence, the surgery is the union along $\widehat{\mathcal{T}}$ of two Seifert fibre spaces M_1 and M_2, with respective base orbifolds a disc with two cone points of orders $\Delta(\alpha_1, \mu_1)$ and $\Delta(\alpha_2, \mu_2)$, and an annulus with one cone point of order $\Delta(\alpha_3, \mu_3)$.

Proposition 2.8.11. (Gordon et Luecke, 2004, Proof of claim in proof of Corollary A.2) *Let $\mathcal{E}$ be the exterior of a hyperbolic knot K_0 in $S^1 \times D^2$ such that $\mathcal{E}(\mathcal{K}_0; \sigma)$ is toroidal and $\Delta(\sigma, \mu) > 1$, where μ bounds a disc in νK_0. Then $\mathcal{E}(\mathcal{K}_0; \sigma)$ is the union of Seifert fibre spaces M_1 and M_2. Suppose $\partial M_2 = \partial\mathcal{E}(\mathcal{K}_0; \sigma) = \partial(S^1 \times D^2)$. The slope of a regular fibre of M_2 on $\partial(S^1 \times D^2)$ does not coincide with the slope that bounds a disc in $\mathcal{E}(\mathcal{K}_0; \mu) \cong S^1 \times D^2$.*

Proof. We follow (Gordon et Luecke, 2004), in which the solid torus containing K_0 is denoted $L(\alpha, \beta, \gamma, *, 1/2)$ and the non-integral toroidal filling $\mathcal{E}(\mathcal{K}_0; \sigma)$ is denoted $L(\alpha, \beta, \gamma, *, 1/0)$. If $\Delta(\sigma, \mu) > 1$ and $\mathcal{E}(\mathcal{K}_0; \sigma)$ is toroidal, then by the discussion above, $\mathcal{E}(\mathcal{K}_0; \sigma)$ is the union of Seifert fibre spaces M_1 and M_2, and its surgery description is given by Figure 2.6.

Let h be the slope of a regular fibre of M_2 on $\mathcal{S} = \partial(S^1 \times D^2)$. Suppose by contradiction that h bounds a disc in $\mathcal{E}(\mathcal{K}_0; \mu) \cong S^1 \times D^2$. Then $\mathcal{E}(\mathcal{S}, \mathcal{K}_0; h, \mu) \cong S^2 \times S^1$.

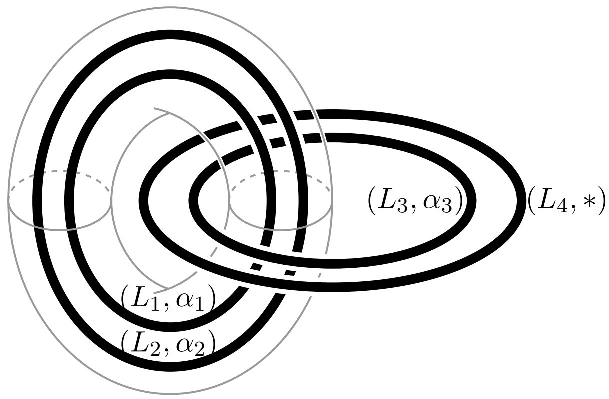

Figure 2.6: Surgery description of a non-integral toroidal surgery along a hyperbolic knot in $S^1 \times D^2$

In the surgery description from Figure 2.6, filling along $\mathcal{S}$ corresponds to filling along L_4. One can see that filling M_2 along a regular fibre yields the connected sum of a lens space and a solid torus whose meridian has distance one with a regular fibre of M_1. Hence, $\mathcal{E}(\mathcal{S}, \mathcal{K}_0; h, \sigma)$ is either a lens space or a connected sum of lens spaces. Since $\Delta(\sigma, \mu) > 1$, this implies that $\mathcal{E}(\mathcal{S}; h)$ is reducible (Gordon et Luecke, 1996, Theorem 1.2; Boyer et Zhang, 1998, Corollary 1.4).

Write $\mathcal{E}(\mathcal{S}; h) = N_1 \# N_2$. Then $S^2 \times S^1 \cong \mathcal{E}(\mathcal{S}, \mathcal{K}_0; h, \mu) \cong N_1 \# N_2(\mathcal{K}_0; \mu)$. But $S^2 \times S^1$ does not contain a separating S^2, which means that $N_1 \cong S^2 \times S^1$ and $N_2(\mathcal{K}_0; \mu) \cong S^3$. It follows that $\mathcal{E}(\mathcal{S}, \mathcal{K}_0; h, \sigma) = (S^2 \times S^1) \# N_2(\mathcal{K}_0; \sigma)$. This contradicts the assertion that $\mathcal{E}(\mathcal{S}, \mathcal{K}_0; h, \sigma)$ is a lens space or a connected sum of lens spaces, as such spaces do not contain a non-separating essential sphere. □

Proposition 2.8.12. *Let K be a composite knot. Let K' be a knot such that the outermost* *piece of $S^3_{K'}$ is hyperbolic. If $|q| > 1$, then there is no orientation-preserving homeomorphism between $S^3_K(p/q)$ and $S^3_{K'}(p/q)$.*

Proof. Suppose by contradiction that there exists an orientation-preserving homeomorphism

$S^3_K(p/q) \cong S^3_{K'}(p/q)$ where K' is such that the outermost piece Y' of $S^3_{K'}$ is hyperbolic, and where $|q| > 1$. By Corollary 2.8.6, the surgered submanifold of $S^3_{K'}(p/q)$ contains $Y'(\mathcal{L}'_0; p/q)$. By Corollary 2.8.10 and Proposition 2.8.7, Y' is the exterior of a knot in a solid torus and $Y'(\mathcal{L}'_0; p/q)$ is a non-integral toroidal filling. By Gordon and Luecke, $Y'(\mathcal{L}'_0; p/q)$ is the union of two Seifert fibre spaces M_1 and M_2, with respective base orbifolds a disc with two cone points and an annulus with one cone point.

Let X be the surgered piece of $S^3_K(p/q)$ and let X' be its image in $S^3_{K'}(p/q)$ by the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$. Since X' is a filing of a composing space, we have $X' \cap Y'(\mathcal{L}'_0; p/q) = M_2$ (proof of Proposition 2.3.5(2)). Let $\mathcal{T}' = \partial Y'(\mathcal{L}'_0; p/q) \subset \partial M_2$. In $S^3_{K'}$, the torus $\mathcal{T}'$ decomposes K' into P' and J'. Let $\mathcal{E} = V_{P'}$ and $\mathcal{P}' = \partial \nu P'$.

The torus $\mathcal{T}'$ is the image by the homeomorphism of an incompressible torus $\mathcal{T}$ in $X \subset S^3_K$. Although this torus might not be a JSJ torus of S^3_K, it separates S^3_K into a pattern space V_P and a knot exterior S^3_J, where P is a composing pattern (and J is a composite knot if $\mathcal{T}$ is not a JSJ torus). Let $\mathcal{P} = \partial \nu P \subset \partial V_P$.

We thus have homeomorphisms $V_P(\mathcal{P}; p/q) \cong \mathcal{E}(\mathcal{P}'; p/q)$ and $S^3_J \cong S^3_{J'}$. These imply that the meridian on $\mathcal{T}$ given by J is sent by the homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ to the meridian on $\mathcal{T}'$ given by J', by the knot complement theorem.

By construction of satellite knots, the meridian on $\mathcal{T}'$ given by J' coincides with the slope that bounds a disc in $\mathcal{E}(\mathcal{P}'; 1/0)$. By Proposition 2.8.11, this slope does not coincide with the slope of a regular fibre of M_2 on $\mathcal{T}'$. On the other hand, a regular fibre on $\mathcal{T}$ in $V_P(\mathcal{P}; p/q)$ comes from the Seifert fibred structure of a composing space, so it has meridional slope.

Hence, regular fibres on $\mathcal{T}$ are not mapped to regular fibres on $\mathcal{T}'$. This contradicts the unicity of the Seifert fibred structure on a Seifert fibre space with base orbifold an annulus and one cone point. □

Proof of Theorem 2.1.2. Let K be a composite knot and suppose there is an orientation-preserving homeomorphism $S^3_K(p/q) \cong S^3_{K'}(p/q)$ for some knot K', where $|q| > 1$. According to Proposition 2.8.3, the surgered piece of $S^3_K(p/q)$ is carried into the surgered submanifold of $S^3_{K'}(p/q)$. Proposition 2.8.12 implies that the surgered submanifold of $S^3_{K'}(p/q)$ is a JSJ piece of $S^3_{K'}(p/q)$, and it is the $p/(qt^2)$-filling of a JSJ piece Y' of $S^3_{K'}$ for some $t \geq 1$.

According to Proposition 2.8.3, this filling $Y'(p/(qt^2))$ is homeomorphic to the surgered piece of $S^3_K(p/q)$, a filling of a composing space with one exceptional fibre of order $|q|$. By Proposition 2.8.4, Y' is Seifert fibred. Since $Y'(p/(qt^2))$ has at least two boundary components, Y' is a composing space by Theorem 2.3.1. The exceptional fibre of $Y'(p/(qt^2))$ has order $|qt^2| = |q|$, so $t = 1$ and K' is not a cable. Therefore, we conclude that $K = K'$. □

CHAPITRE 3
BORNES EFFECTIVES SUR LES PENTES CARACTÉRISANTES POUR TOUT NŒUD

Le deuxième article de cette thèse, ayant pour titre original *Effective bounds on characterising slopes for all knots*, est rédigé en collaboration avec Laura Wakelin. En combinant des idées issues du travail de Wakelin sur la réalisation de la constante $C(K)$ du Théorème 1 pour les nœuds hyperboliques (Wakelin, sous presse) et l'application du Théorème de chirurgie cyclique (Culler *et al.*, 1987) tel qu'employé dans la démonstration du Théorème 1, nous réalisons quantitativement chaque étape menant à ce dernier, donnant ainsi lieu au Théorème 4. De plus, nous développons une stratégie nouvelle permettant de trouver la valeur optimale de $C(K)$ dans certains cas. En s'inspirant d'une construction de Brakes afin d'obtenir une même variété par des chirurgies de Dehn le long de nœuds distincts (Brakes, 1980), nous montrons que pour certains nœuds, on peut constructivement trouver une valeur de $C(K)$ qui réalise le Théorème 1 et pour laquelle il existe un nœud K' différent de K partageant la même $1/C(K)$-chirurgie de Dehn.

3.0 Abstract

A slope p/q is characterising for a knot $K \subset \mathbb{S}^3$ if the orientation-preserving homeomorphism type of the manifold $\mathbb{S}^3_K(p/q)$ obtained by performing Dehn surgery of slope p/q along K uniquely determines the knot K. We combine new applications of results from hyperbolic geometry with previous individual work of the authors to determine, for any given knot K, an explicit bound $\mathcal{C}(K)$ such that $|q| > \mathcal{C}(K)$ implies that p/q is a characterising slope for K.

3.1 Introduction

Given a knot $K \subset \mathbb{S}^3$, one can perform *Dehn surgery* of slope $p/q \in \mathbb{Q} \cup \{1/0\}$ on K to produce a new 3-manifold $\mathbb{S}^3_K(p/q) = \mathbb{S}^3_K \cup \nu(K)$, where $\mathbb{S}^3_K := \mathbb{S}^3 \setminus \mathrm{int}(\nu(K))$ is the exterior of K and the fraction p/q specifies how to glue back in the solid torus $\nu(K) \cong \mathbb{S}^1 \times \mathbb{D}^2$. When $p/q = 1/0$, we have $\mathbb{S}^3_K(1/0) \cong \mathbb{S}^3$ and it is impossible to uniquely determine the knot K from the oriented homeomorphism type of the manifold obtained by this procedure. However, this is not true for a general slope $p/q \in \mathbb{Q}$. We say that a slope $p/q \in \mathbb{Q}$ is *characterising* for a knot $K \subset \mathbb{S}^3$ if the existence of an orientation-preserving homeomorphism $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{K'}(p/q)$ for $K' \subset \mathbb{S}^3$ implies that $K = K'$.

The Dehn surgery characterisation problem asks which slopes $p/q \in \mathbb{Q}$ are characterising for a given knot K. So far, it has been solved for the unknot, the trefoil knots and the figure-eight knot, for which all slopes in $\mathbb{Q}$ are characterising (Kronheimer *et al.*, 2007; Ozsváth et Szabó, 2019), as well as for an infinite family of composite knots, for which the set of characterising slopes is precisely $\mathbb{Q} \setminus \mathbb{Z}$ (Sorya, 2024). An advance towards answering this question for knots in general is a result of Sorya, building on prior work on the subject (Lackenby, 2019; McCoy, 2020), which says that for any knot $K \subset \mathbb{S}^3$, there exists a constant $\mathcal{C}(K)$ such that every slope p/q with $|q| > \mathcal{C}(K)$ is characterising for K (Sorya, 2024). Whilst the proof of existence of $\mathcal{C}(K)$ is non-constructive in general, the present paper provides an explicit value of $\mathcal{C}(K)$ that depends only on the JSJ decomposition of the exterior of K.

Theorem 3.1.1. *Let $K \subset \mathbb{S}^3$ be a knot with exterior $\mathbb{S}^3_K$. Then the JSJ decomposition of $\mathbb{S}^3_K$ – namely, the geometry of the JSJ pieces of $\mathbb{S}^3_K$, together with the gluing maps between them – explicitly determines a constant $\mathcal{C}(K)$ such that if $|q| > \mathcal{C}(K)$, then p/q is a characterising slope for K.*

For torus knots $T_{a,b}$, the constant $\mathcal{C}(K)$ can be realised as $\max\{8, |a|, |b|\}$ (McCoy, 2020). For hyperbolic knots, Wakelin showed that $\mathcal{C}(K)$ can be constructed using the systole of the hyperbolic knot exterior (Wakelin, sous presse). For satellite knots, explicit bounds are known for prime knots whose exteriors consist only of Seifert fibred JSJ pieces and for all composite knots (Sorya, 2024); for the latter, we can take $\mathcal{C}(K) = 1$.

This leaves the case where K is a prime satellite knot whose exterior contains at least one hyperbolic JSJ piece. In this paper, we construct a value for $\mathcal{C}(K)$ for such knots which only depends on the geometry of the hyperbolic JSJ pieces of $\mathbb{S}^3_K$ and how they are glued within the full JSJ decomposition.

3.1.1 Determining a value for $\mathcal{C}(K)$

The remaining case of $\mathcal{C}(K)$ is realised by taking the maximum of three geometric constants, $Q(K)$, $R(K)$ and $S(K)$, which we now introduce and define.

Theorem 3.1.2. *Let K be a prime satellite knot whose exterior is not a graph manifold.*

If $|q| > \max\{Q(K), R(K), S(K)\}$, then p/q is a characterising slope for K.

To define these constants, consider the JSJ decomposition of $\mathbb{S}^3_K$. Each JSJ piece of $\mathbb{S}^3_K$ is homeomorphic to the exterior of a unique link $L = L_0 \cup L_1 \cup \ldots \cup L_{m-1}$ with a distinguished component L_0 such that $L \setminus L_0 = U^{m-1}$ is a (possibly empty) unlink (Budney, 2006, Proposition 2.4). Denote by μ_i the meridian of the i^{th} component of U^{m-1} for $i = 1, \ldots, m-1$. When L is hyperbolic, we set

$$\mathfrak{q}(L) := \left\lfloor \sqrt{6\sqrt{3}\left(1.9793\frac{2\pi}{\operatorname{sys}(\mathbb{S}^3_L)} + 28.78\right)} \right\rfloor,$$

$$\mathfrak{r}(L) := \left\lfloor \sqrt{3}\max\{0, l(\mu_i) \mid L_i \subset L, i \neq 0\} \right\rfloor,$$

$$\mathfrak{s}(L) := \left\lfloor \sqrt{6\sqrt{3}\left(\frac{2\pi}{\operatorname{sys}(\mathbb{S}^3_L)} + 28.78\right)} \right\rfloor,$$

where $\operatorname{sys}(\mathbb{S}^3_L)$ denotes the length of the shortest geodesic in $\mathbb{S}^3_L$ and $l(\mu_i)$ is the length of $\mu_i \subset \partial \mathbb{S}^3_L$; these quantities are described in more detail in Subsection 3.2.4.

Let $\mathcal{X}$ denote the set of hyperbolic JSJ pieces of $\mathbb{S}^3_K$. For each $X \in \mathcal{X}$, let L_X denote the unique hyperbolic link corresponding to X in the satellite construction of K.

To define $Q(K)$, we only need to consider one JSJ piece of $\mathbb{S}^3_K$. If K is a cable of a knot $\widehat{K}$, let Y be the JSJ piece of $\mathbb{S}^3_{\widehat{K}}$ containing $\partial \mathbb{S}^3_{\widehat{K}}$; otherwise, let Y be the JSJ piece of $\mathbb{S}^3_K$ containing $\partial \mathbb{S}^3_K$.

Définition 3.1.3. Define $Q(K) := \max\{34, \mathfrak{q}(L_Y)\}$ if $Y \in \mathcal{X}$ and set $Q(K) = 0$ otherwise.

For the constant $R(K)$, we use $\mathfrak{r}(L_X)$ for $X \in \mathcal{X}$, which can only contribute non-trivially if $|\partial X| \geq 2$.

Définition 3.1.4. Define $R(K) := \max\{1, \mathfrak{r}(L_X) \mid X \in \mathcal{X}\}$.

Finally, the constant $S(K)$ involves every $X \in \mathcal{X}$ which does not contain $\partial \mathbb{S}^3_K$.

Définition 3.1.5. Define $S(K) := \max\{25, \mathfrak{s}(L_X) \mid X \in \mathcal{X},\ \partial \mathbb{S}^3_K \not\subset X\}$.

Combining these new constants with the ones from previous work, we obtain the following explicit description of a constant $\mathcal{C}(K)$ that realises Theorem 3.1.1 for all knots.

Theorem 3.1.6. *Let $K \subset \mathbb{S}^3$ be a knot with exterior $\mathbb{S}^3_K$.*

(i) If K is the unknot, set $\mathcal{C}(K) = 0$.

(ii) If K is a composite knot, set $\mathcal{C}(K) = 1$.

(iii) If K is a prime knot and $\mathbb{S}^3_K$ is a graph manifold, express K as an iterated cable $C_{r_n,s_n} \dots C_{r_2,s_2} C_{r_1,s_1}(J)$, where s_i is the winding number of C_{r_i,s_i} and J is either a torus knot or a composite knot, and set

$$\mathcal{C}(K) = \begin{cases} \max\{8, |a|, |b|\} & \text{if } J \text{ is a torus knot } T_{a,b} \text{ and } n = 0, \\ \max\{8, |s_1|, |r_1| + |a|, |r_1| + |b|\} & \text{if } J \text{ is a torus knot } T_{a,b} \text{ and } n = 1, \\ \max\{|r_1| + |a|, |r_1| + |b|\} & \text{if } J \text{ is a torus knot } T_{a,b} \text{ and } n \geq 2, \\ 2 & \text{if } J \text{ is a composite knot.} \end{cases}$$

(iv) If K is a prime knot and $\mathbb{S}^3_K$ is not a graph manifold, set

$$\mathcal{C}(K) = \max\{Q(K), R(K), S(K)\}.$$

If $|q| > \mathcal{C}(K)$, then p/q is a characterising slope for K.

Proof. The first case is due to (Kronheimer *et al.*, 2007); the second case is (Sorya, 2024, Theorem 2); the third case comes from (McCoy, 2020, Theorem 1.1) and (Sorya, 2024, Theorem 7.1); the fourth case is (Wakelin, sous presse, Theorem 1.3) together with Theorem 3.1.2. □

The most challenging part of the proof of Theorem 3.1.2 lies in showing that the denominator bound $|q| > \max\{R(K), S(K)\}$ ensures that any orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ restricts (possibly after an isotopy) to one between the *surgered pieces* – the JSJ pieces containing the surgery curves – thereby realising (Sorya, 2024, Propostion 4.1). This is encompassed in the following result.

Proposition 3.1.7. *Let K be a prime satellite knot whose exterior is not a graph manifold. Suppose that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ for some knot K'.*

If $|q| > \max\{R(K), S(K)\}$, then f maps the surgered piece of $\mathbb{S}^3_K(p/q)$ to the surgered piece of $\mathbb{S}^3_{K'}(p/q)$.

Once we have proved Proposition 3.1.7, Theorem 3.1.2 will then follow from the techniques of (Sorya, 2024) when the surgered piece is Seifert fibred and from the bound $|q| > Q(K)$ previously introduced in (Wakelin, sous presse) when the surgered piece is hyperbolic.

3.1.2 Winding number zero

Note that in general, the bound provided by Theorem 3.1.2 may not be optimal. However, in certain cases of prime satellite knots whose exteriors are not graph manifolds, we can use a slightly different approach via a new constant $T(K)$ to find a refined value for $\mathcal{C}(K)$.

Theorem 3.1.8. *Let K be a satellite knot such that for every choice of satellite description $K = P(J)$, the pattern P has winding number zero.*

If $|q| > \max\{Q(K), T(K)\}$, then p/q is a characterising slope for K.

With this strategy, we may be able to improve the bound used to ensure that an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ restricts to one between surgered pieces. Namely, instead of assuming that $|q| > \max\{R(K), S(K)\}$, we write $K = P(J)$ and use the fact that the pattern P has winding number zero to deduce that f restricts to a homeomorphism between the surgered pieces unless $|p| = 1$. If we write $K' = P'(J')$ and suppose that $f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(\pm 1/q)$ and $f(\mathbb{S}^3_P(\pm 1/q)) = \mathbb{S}^3_{J'}$, we can express these $\pm 1/q$-surgeries as

Rolfsen $\mp q$-twists. We obstruct this by introducing a constant $T(K)$ encoding a certain unknotting property of the companion J. This leads to the following proposition.

Proposition 3.1.9. *Let K be a satellite knot such that for every choice of satellite description $K = P(J)$, the pattern P has winding number zero. Suppose that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ for some knot K'.*

If $|q| > \max\{2, T(K)\}$, then f maps the surgered piece of $\mathbb{S}^3_K(p/q)$ to the surgered piece of $\mathbb{S}^3_{K'}(p/q)$.

Combining Proposition 3.1.9 with Wakelin's previous work (Wakelin, sous presse) leads to Theorem 3.1.8. Furthermore, we show that sometimes this bound is optimal.

3.1.3 Outline

The paper is organised as follows. Section 3.2 reviews the structure of satellite knots, as well as JSJ decompositions of knot exteriors and their Dehn fillings, and recaps useful results from hyperbolic geometry. Section 3.3 presents the proof of Theorem 3.1.2. Section 3.4 contains the proof of Theorem 3.1.8. Section 3.5 comprises applications of our main results: we give examples showing how to compute the bound from Theorem 3.1.2 and investigate some special cases where the bound from Theorem 3.1.8 is optimal.

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3.2 Preliminaries

In this section, we review satellite knots, JSJ decompositions and hyperbolic geometry in the context of this article.

3.2.1 Satellite knots

Let $L = L_0 \cup L_1 \cup \ldots \cup L_{m-1}$ be an m-component link in $\mathbb{S}^3$, where each component L_i is a knot. We denote by $\mathbb{S}^3_L$ the exterior of L in $\mathbb{S}^3$: the manifold obtained by removing the interior of a closed tubular neighbourhood $\nu(L)$ of L in $\mathbb{S}^3$.

A *slope* on a boundary component $\partial\nu(L_i)$ of $\mathbb{S}^3_L$ is an isotopy class of essential simple closed curves representing an element of $H_1(\partial\nu(L_i);\mathbb{Z})$ up to sign. The *meridian* μ_i on $\partial\nu(L_i)$ is the unique slope that bounds a disc in $\nu(L_i)$. Let M_i be the manifold obtained by Dehn filling, for each $j \neq i$, the boundary component $\partial\nu(L_j)$ of $\mathbb{S}^3_L$ along μ_j, so that $M_i \cong \mathbb{S}^3_{L_i}$. The *longitude* λ_i on $\partial\nu(L_i) = \partial M_i$ is the unique slope that is trivial in $H_1(M_i;\mathbb{Z})$. Fixing $\{\mu_i, \lambda_i\}$ as a basis for $H_1(\partial\nu(L_i);\mathbb{Z})$, the *slope* $p/q \in \mathbb{Q} \cup \{1/0\}$ refers to $p\mu_i + q\lambda_i$ up to sign, where p_i, q_i are coprime.

We write $P(J)$ for the satellite knot with pattern P and non-trivial companion knot J. The pattern P can be described as a 2-component link $Q \cup U$, where U is the unknot and Q is a knot inside the solid torus $\mathbb{S}^3_U$ which is neither its core nor a local knot. The satellite knot $P(J)$ is obtained by *splicing* U with J; the exterior of $P(J)$ is the *splice* of $\mathbb{S}^3_P$ and

$\mathbb{S}^3_J$ along $\partial\nu(U)$ and $\partial\nu(J)$, i.e. the meridian μ_U and longitude λ_U of U are identified with the longitude λ_J and meridian μ_J of J, respectively.

The *winding number* of the pattern $P = Q \cup U$ is the absolute value of the algebraic intersection number between Q and an essential disc in the solid torus $\mathbb{S}^3_U$. If Q happens to be unknotted, then we can swap the components of P via an isotopy and observe that the definition of the winding number respects this exchange for homological reasons. If a pattern $P = Q \cup U$ is such that Q intersects an essential disc in $\mathbb{S}^3_U$ geometrically once, then P is called a *composing pattern* and $P(J)$ is precisely the *composite knot* $Q\#J$ (also called the *connected sum* of Q and J). If a pattern $P = Q \cup U$ is such that Q is isotopic in $\mathbb{S}^3_U$ to a torus knot, then P is a *cable pattern* or *cable link* and $P(J)$ is a *cable knot*.

3.2.2 JSJ decompositions

Recall that for any compact orientable irreducible 3-manifold M whose boundary is a (possibly empty) union of tori, there is a minimal collection $\mathbf{T}$ of properly embedded disjoint essential tori such that each component of $M \backslash \mathbf{T}$ is either a hyperbolic 3-manifold or a Seifert fibre space; such a collection is unique up to isotopy (Jaco et Shalen, 1979; Johannson, 1979). The *JSJ decomposition* of M is

$$M = M_0 \cup M_1 \cup \ldots \cup M_k,$$

where each M_i is the closure of a component of $M \setminus \mathbf{T}$. Each M_i is called a *JSJ piece* of M and each torus in the collection $\mathbf{T}$ is called a *JSJ torus* of M. If all of the JSJ pieces of M are Seifert fibred, then M is said to be a *graph manifold*. Any homeomorphism between compact orientable irreducible 3-manifolds must preserve the JSJ decomposition up to isotopy.

We will be most interested in the case where M is the exterior $\mathbb{S}^3_K$ of a knot K. The unique

JSJ piece of $\mathbb{S}^3_K$ containing the boundary of $\nu(K)$ is said to be the *outermost* JSJ piece of $\mathbb{S}^3_K$.

When K is a torus knot or a hyperbolic knot, its exterior $\mathbb{S}^3_K$ contains no JSJ tori. When K is a satellite knot, each JSJ torus splits $\mathbb{S}^3_K$ into a pattern space $\mathbb{S}^3_P$ and a knot exterior $\mathbb{S}^3_J$ such that K can be described as $P(J)$ (Sorya, 2024, Lemma 2.1). In particular, this description may not be unique.

The JSJ pieces of a knot exterior take on one of four special types, which are all homeomorphic to the exterior of a certain type of link in $\mathbb{S}^3$. The isotopy class of this link is uniquely determined by gluing maps arising in the satellite construction.

Theorem 3.2.1. (Budney, 2006, Proposition 2.4, Theorem 4.18) *Let K be a non-trivial knot and let Y be a JSJ piece of the exterior of K. Then Y is homeomorphic to the exterior of a link $L = L_0 \cup L_1 \cup \ldots \cup L_{m-1}$ with a distinguished component L_0 such that $L \backslash L_0 = U^{m-1}$ is the unlink.*

Furthermore, this link L is unique up to isotopy, given the condition that for each $i \neq 0$, the gluing map used in the satellite construction of K corresponds to splicing L_i with a non-trivial knot J_i.

This unique link L corresponding to Y can be classified into one of the following four types.

(i) L is a torus knot $T_{a,b}$, i.e. $\mathbb{S}^3_L$ is a Seifert fibre space whose base orbifold is a disc with two cone points of orders $|a|$ and $|b|$.

(ii) L is a cable link $C_{r,s}$, i.e. $\mathbb{S}^3_L$ is a Seifert fibre space whose base orbifold is an annulus with one cone point of order $|s|$ (where s is the winding number of the cable pattern).

(iii) L is a composing link, i.e. $\mathbb{S}^3_L$ is a Seifert fibre space with at least three boundary

components whose base orbifold is a planar surface with no cone points.

(iv) L is a hyperbolic link, i.e. $\mathbb{S}^3_L$ is a hyperbolic 3-manifold.

This is summarised in the table below.

Case	Link L	JSJ piece $Y = \mathbb{S}^3_L$	SFS orbifold	K when Y is outermost
(i)	$T_{a,b}$ torus knot	$\mathbb{S}^3_{T_{a,b}}$ torus knot exterior	$\mathbb{D}^2(\lvert a\rvert, \lvert b\rvert)$	$T_{a,b}$ torus knot
(ii)	$C_{r,s}$ cable link	$\mathbb{S}^3_{C_{r,s}}$ cable space	$\mathbb{A}^2(\lvert s\rvert)$	$C_{r,s}(\widehat{K})$ cable knot
(iii)	L composing link	$\mathbb{S}^3_L$ composing space	$\Sigma, \lvert\partial\Sigma\rvert \geq 3$	$K_1 \# K_2$ composite knot
(iv)	L hyperbolic link	$\mathbb{S}^3_L$ hyperbolic link exterior	–	K knot of hyperbolic type

Tableau 3.1: JSJ pieces of the exterior of a knot K.

The distinguished component L_0 of the link L in the statement of Theorem 3.2.1 is said to be *outermost*; similarly, the *outermost* boundary component of $Y = \mathbb{S}^3_L$ refers to $\partial\nu(L_0) \subset \partial Y$.

3.2.3 Dehn surgery

Let M be a 3-manifold and let $T_0, \dots, T_{m-1}$ denote the toroidal boundary components of ∂M. For fixed bases $\{\mu_i, \lambda_i\}$ for each $H_1(T_i; \mathbb{Z}), i = 0, \dots, m-1$, let

$$M(T_i; p_i/q_i)$$

denote the manifold obtained by Dehn filling M along a simple closed curve representing $p_i\mu_i + q_i\lambda_i$ up to sign on T_i. If it is clear from context which boundary component of M is filled, then we may simply write $M(p/q)$.

If M is a link exterior $\mathbb{S}^3_L$ and $L_0, \dots, L_{m-1}$ are the components of L, we may write

$$\mathbb{S}^3_L(L_i; p_i/q_i)$$

instead of $\mathbb{S}^3_L(\partial\nu(L_i); p_i/q_i)$. Furthermore, the slopes p_i/q_i are assumed to be expressed with respect to the basis of $H_1(\partial\nu(L_i);\mathbb{Z})$ given by the meridian and longitude of L_i, as defined earlier.

Performing Dehn surgery on a knot K to obtain the manifold $\mathbb{S}^3_K(p/q)$ corresponds to Dehn filling the outermost JSJ piece of $\mathbb{S}^3_K$ along a slope p/q on the boundary component corresponding to $\partial\nu(K)$. The following proposition shows that if $|q| > 2$, then the core of the surgery solid torus is contained inside a single JSJ piece of $\mathbb{S}^3_K(p/q)$. We call this piece the *surgered piece*.

Proposition 3.2.2. (Sorya, 2024, Proposition 3.6) *Let K be a non-trivial knot with exterior $\mathbb{S}^3_K$ and let $Y_0 \cup Y_1 \cup \ldots \cup Y_k$ be the JSJ decomposition of $\mathbb{S}^3_K$, where Y_0 is the outermost piece of $\mathbb{S}^3_K$.*

If $|q| > 2$, then the JSJ decomposition of $\mathbb{S}^3_K(p/q)$ takes one of the following forms:

(i) $Y_0(\partial\nu(K); p/q) \cup Y_1 \cup Y_2 \cup \ldots \cup Y_k$,

(ii) $Y_1(Y_0 \cap Y_1; p/qs^2) \cup Y_2 \cup \ldots \cup Y_k$,

where case (ii) occurs precisely when $K = C_{r,s}(\widehat{K})$ is a cable knot, Y_1 is the outermost piece of $\mathbb{S}^3_{\widehat{K}}$, $|s| \geq 2$ and $|p - qrs| = 1$.

Therefore the surgered piece can be described as $Y(p/qt^2)$, where either $Y = Y_0$ and $t = 1$ or $Y = Y_1$ and $t > 1$.

3.2.4 Hyperbolic geometry

When considering the hyperbolic JSJ pieces contributing to the constants in Theorem 3.1.2, we will require some quantitative results from hyperbolic geometry.

Recall that a hyperbolic 3-manifold M is one whose interior admits a complete finite-volume hyperbolic metric. A toroidal boundary component $T \subset \partial M$ is the boundary at infinity of a *cusp* of the interior of M. Each of these has a well-defined *maximal horocusp neighbourhood* $N(T)$, whose boundary $\partial N(T)$ inherits a unique Euclidean metric from the hyperbolic metric (see for instance (Lackenby, 2019, Section 2)).

Définition 3.2.3. Let σ be a slope on a toroidal boundary component $T \subset \partial M$.

- The *area* $A(T)$ of T is the Euclidean area of $\partial N(T)$.
- The *length* $l(\sigma)$ of σ is the Euclidean length of a geodesic representative of σ on $\partial N(T)$.
- The *normalised length* $\hat{l}(\sigma)$ of σ is given by $\hat{l}(\sigma) = l(\sigma)/\sqrt{A(T)}$.

Slope length is related to the geometry of Dehn fillings. The 6-theorem (Agol, 2000; Lackenby, 2000) states that filling a hyperbolic 3-manifold along any slope σ of length $l(\sigma) > 6$ must give a hyperbolic 3-manifold.

The lengths (or normalised lengths) of a pair of slopes on the same boundary component $T \subset \partial M$ can be related to the distance Δ between them (the absolute value of their algebraic intersection number), as presented in (Wakelin, sous presse, Lemma 4.2).

Lemma 3.2.4. *Let σ and σ' be slopes on the same toroidal boundary component T of a hyperbolic 3-manifold.*

- *The area $A(T)$ of T satisfies the universal bound $A(T) \geq 2\sqrt{3}$.*
- *The length $l(\gamma)$ of γ satisfies $l(\gamma)l(\mu) \geq \Delta(\gamma, \mu) \cdot A(T)$.*
- *The normalised length $\hat{l}(\gamma)$ of γ satisfies $\hat{l}(\gamma)\hat{l}(\mu) \geq \Delta(\gamma, \mu)$.*

Proof. The first inequality comes from $\sqrt{3}$ being a universal lower bound for the volume of a maximal horocusp neighbourhood N (Gabai *et al.*, 2021, Theorem 1.2) and the fact that the area of ∂N is equal to twice the volume of N (Gabai *et al.*, 2021, Section 1). The others follow from (Cooper et Lackenby, 1998, Lemma 2.13) or (Agol, 2000, Theorem 8.1). □

Now we will move on to consider closed curves in the interior of M. We denote the length of any geodesic $\gamma \subset M$ with respect to the hyperbolic metric by $l(\gamma)$.

Définition 3.2.5. The *systole* $\text{sys}(M)$ of a hyperbolic 3-manifold M is the length $l(\gamma_0)$ of a shortest geodesic $\gamma_0 \subset M$.

There is a quantitative relationship between the length $l(\gamma)$ of a simple geodesic $\gamma \subset M$ and the normalised length $\hat{l}(\mu)$ of its meridian μ considered as a slope on the boundary torus of the maximal horocusp neighbourhood of $M \setminus \nu(\gamma)$ corresponding to γ.

Theorem 3.2.6. (Futer *et al.*, 2022, Corollary 6.13) *Let M be a hyperbolic 3-manifold and let $\gamma \subset M$ be a simple geodesic with meridian μ. If $\hat{l}(\mu) \geq 7.823$, then*

$$l(\gamma) < \frac{2\pi}{\hat{l}(\mu)^2 - 28.78}.$$

When we perform Dehn filling along a slope $\sigma \subset \partial M$ such that $M(\sigma)$ is a hyperbolic manifold, the hyperbolic metric on M deforms to a complete hyperbolic metric on the surgered manifold $M(\sigma)$, in which the core v of the surgery solid torus becomes a geodesic.

Moreover, the meridian of v is precisely the slope σ. Therefore we can apply this result to relate the normalised length $\hat{l}(\sigma)$ of the filling slope σ and the hyperbolic length $l(v)$ of the core curve v of the filling.

3.3 Determining a value for $\mathcal{C}(K)$

We now turn to the proof of Theorem 3.1.2. Let $K \subset \mathbb{S}^3$ be a prime knot whose exterior $\mathbb{S}^3_K$ is not a graph manifold. Since hyperbolic knots have already been studied (Wakelin, sous presse), we may also assume that K is a satellite knot.

Suppose that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ for some other knot $K' \subset \mathbb{S}^3$, where $|q| > 2$. Then K' is also a satellite knot by Proposition 3.2.2. Let Y and Y' be the JSJ pieces of $\mathbb{S}^3_K$ and $\mathbb{S}^3_{K'}$ such that $Y(p/qt^2)$ and $Y'(p/qt'^2)$ are the surgered pieces of $\mathbb{S}^3_K(p/q)$ and $\mathbb{S}^3_{K'}(p/q)$, respectively (where $t, t' \geq 1$).

The main part of our argument lies in proving Proposition 3.1.7. This tells us that if $|q| > \max\{R(K), S(K)\}$, then the initial orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ restricts to one between the surgered pieces, so that $f(Y(p/qt^2)) = Y'(p/qt'^2)$.

To prove Proposition 3.1.7, we will go through each possible case from Theorem 3.2.1 for Y'. Firstly, note that Y' cannot be the exterior of a torus knot, otherwise $\mathbb{S}^3_{K'}(p/q) \cong Y'(p/qt'^2)$ would contain no JSJ tori. Secondly, if Y' is a composing space, then $\mathbb{S}^3_{K'}(p/q)$ contains the filling $\mathbb{S}^3_{P'}(p/qt'^2)$ of a composing pattern P', which must be homeomorphic to a knot exterior if $f(Y(p/qt^2)) \neq Y'(p/qt'^2)$; according to (Sorya, 2024, Lemma 5.4), this cannot happen for $|q| > 1$. This leaves us with two cases: either Y' is hyperbolic or Y' is a cable space.

3.3.1 The hyperbolic case

Suppose now that Y' is hyperbolic.

Recall that for a hyperbolic JSJ piece $X \subset \mathbb{S}^3_K$, we defined

$$\mathfrak{s}(L_X) = \left\lfloor \sqrt{6\sqrt{3}\left(\frac{2\pi}{\mathrm{sys}(\mathbb{S}^3_{L_X})} + 28.78\right)} \right\rfloor,$$

where L_X is the unique link corresponding to X in the satellite construction of K. We claim that taking $|q| > \max\{25, \mathfrak{s}(L_X)\}$ obstructs the filling $Y'(p/qt'^2)$ of our hyperbolic JSJ piece Y' from becoming homeomorphic to another hyperbolic JSJ piece $X \subset \mathbb{S}^3_K \setminus Y$, i.e. $f(X) \neq Y'(p/qt'^2)$.

Note that the definition of $\mathfrak{s}(L)$ for a hyperbolic link L only depends on the homeomorphism type of $\mathbb{S}^3_L$, so we may also define $\mathfrak{s}(X)$ for any hyperbolic 3-manifold X, with $\mathrm{sys}(X)$ taking the place of $\mathrm{sys}(\mathbb{S}^3_L)$, to obtain a more general statement.

Lemma 3.3.1. *Let Y' be a hyperbolic JSJ piece of a knot exterior. Consider a non-trivial slope $\sigma' = a'/b'$ on the outermost boundary component of Y'. Let X be any hyperbolic 3-manifold.*

If $|b'| > \max\{25, \mathfrak{s}(X)\}$, then $Y'(a'/b') \ncong X$.

Proof. By the 6-theorem (Agol, 2000; Lackenby, 2000), the meridional slope $\mu = 1/0$ of the outermost boundary component of Y' has length $l(\mu) \leq 6$. Applying Lemma 3.2.4 to the slopes $\mu = 1/0$ and $\sigma' = a'/b'$ gives

$$\hat{l}(\sigma') \geq \frac{|b'|}{\hat{l}(\mu)} \geq \frac{|b'|}{\sqrt{6\sqrt{3}}}.$$

Suppose that $|b'| > \max\{25, \mathfrak{s}(X)\}$. Then

$$\hat{l}(\sigma') \geq \frac{26}{\sqrt{6\sqrt{3}}} \geq 7.823$$

and hence we can apply (Futer *et al.*, 2022, Corollary 6.13). We also have

$$\hat{l}(\sigma') \geq \frac{\mathfrak{s}(X)+1}{\sqrt{6\sqrt{3}}} > \sqrt{\frac{2\pi}{\mathrm{sys}(X)}+28.78}$$

and so Theorem 3.2.6 gives us the following bound on the length of the core curve v' of the filling $Y'(a'/b')$:

$$l(v') < \frac{2\pi}{\hat{l}(\sigma')^2 - 28.78} < \mathrm{sys}(X).$$

Since v' is a geodesic in $Y'(a'/b')$ with length strictly less than $\mathrm{sys}(X)$, we have $Y'(a'/b') \ncong X$. □

We now apply Lemma 3.3.1 to the setting of Proposition 3.1.7. Recall that

$$S(K) = \max\{25, \mathfrak{s}(L_X) \mid X \in \mathcal{X},\ \partial\mathbb{S}^3_K \not\subset X\},$$

where $\mathcal{X}$ is the set of hyperbolic JSJ pieces $X \subset \mathbb{S}^3_K$.

Proof of Proposition 3.1.7 (hyperbolic case). Suppose that $f(X) = Y'(p/qt'^2)$ for some JSJ piece X of $\mathbb{S}^3_K(p/q)$ which is not its surgered piece. Since $|qt'^2| \geq |q| > 2$, X must have been a hyperbolic JSJ piece of $\mathbb{S}^3_K$, so $X \in \mathcal{X}$ and $\partial\mathbb{S}^3_K \not\subset X$. Let L_X be the unique link corresponding to X in the satellite construction of K. By Lemma 3.3.1, we have $|q| \leq |qt'^2| \leq \mathfrak{s}(L_X) \leq S(K)$. □

3.3.2 The cable case

We are left with the case when Y' is a cable space.

Let X be a hyperbolic JSJ piece of $\mathbb{S}^3_K$ and let $L_X = L_0 \cup L_1 \cup \ldots \cup L_{m-1}$ be the unique link corresponding to X, where L_0 is the outermost component as usual. Recall that we defined

$$\mathfrak{r}(L_X) = \left\lfloor \sqrt{3}\max\{0, l(\mu_i) \mid L_i \subset L, i \neq 0\} \right\rfloor.$$

If Y' is a cable space and the homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ does not restrict to one between the surgered pieces, then the preimage of $Y'(p/qt'^2)$ is the exterior $\mathbb{S}^3_J \subset \mathbb{S}^3_K$ of a torus knot J by Theorem 3.2.1. Let P be the pattern such that $K = P(J)$. The JSJ torus $\partial\mathbb{S}^3_J \subset \mathbb{S}^3_K(p/q)$ is mapped by f to the boundary of $Y'(p/qt'^2)$ in $\mathbb{S}^3_{K'}(p/q)$, which also bounds the exterior $\mathbb{S}^3_{J'} \subset \mathbb{S}^3_{K'}$ of a knot J'. It follows that K' can be described as a satellite $P'(J')$ such that the homeomorphism f restricts to $f(\mathbb{S}^3_P(p/q)) = \mathbb{S}^3_{J'}$ and $f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(p/q)$.

We claim that taking $|q| > R(K)$ obstructs this situation from happening, thus completing the proof of Proposition 3.1.7. A key ingredient is our assumption that the exterior of K is not a graph manifold: in particular, since $\mathbb{S}^3_J$ is a torus knot exterior, the pattern space $\mathbb{S}^3_P$ must contain a hyperbolic JSJ piece X. We'll perform a clever filling of $\mathbb{S}^3_P$ that realises (Sorya, 2024, Proposition 5.8) in our context.

Lemma 3.3.2. *Let K be a satellite knot by a pattern $P = Q \cup U$. Let $\mathcal{X}_P$ be the set of hyperbolic JSJ piece of $\mathbb{S}^3_P$ and, for each $X \in \mathcal{X}_P$, let L_X denote the unique link corresponding to X in the satellite construction of K. Let a/b be a slope on the boundary component T of $\mathbb{S}^3_P$ corresponding to U.*

If $\mathcal{X}_P \neq \varnothing$ and $|b| > \max\{1, \mathfrak{r}(L_X) \mid X \in \mathcal{X}_P\}$, then the filling $\mathbb{S}^3_P(T; a/b)$ is not a Seifert fibre space.

Proof. Let $L_X = L_0 \cup \ldots \cup L_{m-1}$ be the link corresponding to an $X \in \mathcal{X}_P$, where L_0 is outermost.

If $\mathbb{S}^3_P(T;a/b)$ is a Seifert fibre space, then no $X \in \mathcal{X}_P$ can be a JSJ piece of $\mathbb{S}^3_P(T;a/b)$. Since $|b| > 1$, the proof of (Sorya, 2024, Proposition 5.8) implies that there is some $X \in \mathcal{X}_P$ and some $i \neq 0$ such that $\partial\nu(L_i)$ either is T itself or is glued to the exterior of an iterated cable pattern containing T inside $\mathbb{S}^3_P$. In both cases, we see that $\mathbb{S}^3_P(T;a/b)$ contains a manifold $X(L_i;a/bc^2), X \in \mathcal{X}_P$, such that $|bc^2| \geq |b| > \mathfrak{r}(L_X)$.

We claim that the assumption $|b| > \mathfrak{r}(L_X)$ ensures that the filling $X(L_i;a/bc^2)$ remains hyperbolic. Applying Lemma 3.2.4 to the distinct pair of slopes $\sigma = a/bc^2$ and $\mu_i = 1/0$ along L_i gives

$$l(\sigma) \geq \frac{2\sqrt{3}|bc^2|}{l(\mu_i)} \geq \frac{2\sqrt{3}(\mathfrak{r}(L_X)+1)}{l(\mu_i)} > \frac{2\sqrt{3}(\sqrt{3}l(\mu_i))}{l(\mu_i)} = 6.$$

By the 6-theorem, $X(L_i;a/bc^2)$ is hyperbolic. Therefore $\mathbb{S}^3_P(T;a/b)$ cannot be a Seifert fibre space. $\square$

We'll apply this to solve our last remaining case of Proposition 3.1.7. Recall that

$$R(K) = \max\{1, \mathfrak{r}(L_X) \mid X \in \mathcal{X}\},$$

where $\mathcal{X}$ is the set of hyperbolic JSJ pieces $X \subset \mathbb{S}^3_K$.

Proof of Proposition 3.1.7 (cable case). Write $K = P(J)$ and $K' = P'(J')$ as in the discussion at the beginning of this subsection. Let $T = \mathbb{S}^3_P \cap \mathbb{S}^3_J$. Since Y' is a cable space, P' has winding number $w' \neq 0$. By (Sorya, 2024, Lemma 4.3), the preimage of the meridian $\mu_{J'}$ of J' has slope y/qw'^2 along T, for some integer y, in the coordinates given by the link component U of $P = Q \cup U$ which corresponds to T. Let $M = \mathbb{S}^3_P(T;y/qw'^2)$. Observe that

$$M(p/q) \cong \mathbb{S}^3_{J'}(\mu_{J'}) = \mathbb{S}^3,$$

$$M(1/0) \cong \mathbb{S}^3_U(\mu_{J'}) = L(y, qw'^2).$$

By Lemma 3.3.2, the manifold M is not a Seifert fibre space. However, $M(p/q)$ and $M(1/0)$ are cyclic surgeries for M such that $|q| > 1$, contradicting the cyclic surgery theorem (Culler *et al.*, 1987). □

3.3.3 Proof of Theorem 3.1.2

Having completed the proof of Proposition 3.1.7, we see that the orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ restricts to $f(Y(p/qt^2)) = Y'(p/qt'^2)$. We claim that in fact $Y = Y'$ as JSJ pieces.

Proposition 3.3.3. *Let K be a prime satellite knot whose exterior is not a graph manifold. Suppose that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ for some knot K'.*

Let Y and Y' be the JSJ pieces such that f restricts to a slope-preserving homeomorphism between the surgered pieces $Y(p/qt^2)$ and $Y'(p/qt'^2)$. Let L and L' be the links corresponding to Y and Y' in the satellite constructions of K and K', respectively.

If $|q| > Q(K)$, then $Y = Y'$, in the sense that $Y \cong Y'$ and $L = L'$.

Proof. We run through the cases in Theorem 3.2.1. Note that Y cannot be a torus knot exterior, as the exterior of K is assumed to contain at least one hyperbolic JSJ piece. If Y is a composing space or a cable space, then (Sorya, 2024, Sections 6.2 and 6.4) imply that $Y = Y'$. If Y is a hyperbolic JSJ piece, then taking $|q| > Q(K) = \max\{34, \mathfrak{q}(L)\}$ and applying (Wakelin, sous presse, Proposition 4.8) implies that $Y = Y'$. □

Finally, we return to the caveat that the outermost JSJ pieces of $\mathbb{S}^3_K$ and $\mathbb{S}^3_{K'}$ may or may not be cable spaces that become solid tori after filling, as in case (ii) of Proposition 3.2.2. This can be resolved by considering cosmetic surgeries.

Proof of Theorem 3.1.2. Let K be a prime satellite knot whose exterior is not a graph manifold. Suppose that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ for some knot K'. By Proposition 3.1.7, this restricts to a slope-preserving homeomorphism between the surgered pieces: $f(Y(p/qt^2)) = Y'(p/qt'^2)$, where $t, t' \geq 1$. By Proposition 3.3.3, it follows that $Y = Y'$. Therefore we can write K and K' as (possibly trivial) cables of the same non-trivial knot $\widehat{K}$ whose exterior has outermost piece $Y = Y'$. Observe that we can write:

$$\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{\widehat{K}}(p/qt^2),$$
$$\mathbb{S}^3_{K'}(p/q) \cong \mathbb{S}^3_{\widehat{K}}(p/qt'^2).$$

If $t \neq t'$, then $\widehat{K}$ has a pair of distinct cosmetic surgery slopes of the same sign, contradicting (Ni et Wu, 2015, Theorem 1.2). Therefore $t = t'$ and, by (Sorya, 2024, Lemma 6.2), we deduce that $K = K'$. □

3.4 Winding number zero

The goal of this section is to prove Theorem 3.1.8. Recall that K will now be a satellite knot for which every satellite description $K = P(J)$ is by a pattern P with winding number zero. We will show that a different strategy can be used to obstruct the swapping of JSJ pieces in an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$: instead of assuming that $|q| > \max\{R(K), S(K)\}$ as in Proposition 3.1.7, we will take $|q| > \max\{2, T(K)\}$ and prove Proposition 3.1.9. The new bound $T(K)$ arises from showing that for such a knot K, the swapping of JSJ pieces can only occur for finitely many possibilities, which can sometimes be identified and avoided directly.

3.4.1 Splicifiable knots

Recall that a *nullhomologous Rolfsen t-twist* on a knot J refers to performing $-1/t$-surgery along a nullhomologous unknot in the exterior of J, thus adding t full twists to J in this location. We call the integer t involved in this process the *nullhomologous Rolfsen twist coefficient*.

Définition 3.4.1. Let $K = P(J)$ be a satellite knot. If the pattern $P = Q \cup U$ has winding number zero, Q is unknotted and J can be unknotted by some nullhomologous Rolfsen t-twist, then we say that K is *t-splicifiable* with respect to the pair (P, J).

The reason for this choice of terminology is illustrated in the following result.

Proposition 3.4.2. *Let $K = P(J)$ be a satellite knot by a pattern P with winding number zero.*

There exists a satellite knot $K' = P'(J')$ and an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p'/q')$ with $f(\mathbb{S}^3_P(p/q)) = \mathbb{S}^3_{J'}$ and $f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(p'/q')$ if and only if $|p| = |p'| = 1$ and K is $\pm q'$-splicifiable with respect to (P, J).

Furthermore, $\mathbb{S}^3_K(\mp 1/q) \cong \mathbb{S}^3_{K'}(\mp 1/q')$ is homeomorphic to the splice of the knot exteriors $\mathbb{S}^3_J$ and $\mathbb{S}^3_{J'}$ and K' is $\pm q$-splicifiable with respect to (P', J').

Remark 3.4.3. Brakes (Brakes, 1980) provides a general method for constructing an orientation-preserving homeomorphism $f : \mathbb{S}^3_{P(J)}(p/q) \to \mathbb{S}^3_{P'(J')}(p'/q')$ such that $f(\mathbb{S}^3_P(p/q)) = \mathbb{S}^3_{J'}$ and $f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(p'/q')$. Proposition 3.4.2 says that if such a manifold is obtained from a knot $K = P(J)$, where P is a pattern with winding number zero, then Brakes' construction is in fact the only possibility. We will soon see that this restricts the existence of non-characterising slopes for K with large denominator.

The following lemma is key to the proof of Proposition 3.4.2.

Lemma 3.4.4. *Let $K = P(J)$ and $K' = P'(J')$ be satellite knots and let p/q and p'/q' be non-trivial slopes along K and K', respectively. Suppose that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p'/q')$ with $f(\mathbb{S}^3_P(p/q)) = \mathbb{S}^3_{J'}$ and $f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(p'/q')$.*

Then P has winding number $w = 0$ if and only if P' has winding number $w' = 0$. Moreover, if these winding numbers are zero, then $|p| = |p'| = 1$, $P(U) = Q$ and $P'(U') = Q'$ are both unknotted patterns and $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{K'}(p'/q')$ is homeomorphic to the splice of the knot exteriors $\mathbb{S}^3_J$ and $\mathbb{S}^3_{J'}$.

Proof. Write $P = Q \cup U$ and $P' = Q' \cup U'$. From our satellite constructions, we have the following identifications between meridians and longitudes:

$$K = P(J) \iff \begin{cases} \mu_J = \lambda_U \\ \lambda_J = \mu_U \end{cases};$$

$$K' = P'(J') \iff \begin{cases} \mu_{J'} = \lambda_{U'} \\ \lambda_{J'} = \mu_{U'} \end{cases}.$$

Given the homeomorphism f between fillings, recall from (Sorya, 2024, Lemma 4.3) that the meridians and longitudes of companion knots can be expressed as follows (for some integers y and y'):

$$f(\mathbb{S}^3_P(p/q)) = \mathbb{S}^3_{J'} \implies \begin{cases} \mu_{J'} = y'\mu_U + q'w'^2\lambda_U \\ \lambda_{J'} = qw'^2\mu_U + p\lambda_U \end{cases};$$

$$f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(p'/q') \implies \begin{cases} \mu_J = y\mu_{U'} + qw^2\lambda_{U'} \\ \lambda_J = q'w^2\mu_{U'} + p'\lambda_{U'} \end{cases}.$$

We observe that

$$w' = 0 \iff \begin{cases} \mu_{J'} = \mu_U \\ \lambda_{J'} = \lambda_U \end{cases} \iff \begin{cases} \mu_{J'} = \lambda_J \\ \lambda_{J'} = \mu_J \end{cases} \iff \begin{cases} \lambda_{U'} = \lambda_J \\ \mu_{U'} = \mu_J \end{cases} \iff w = 0.$$

Moreover, this gluing map produces precisely the splice of the knot exteriors $\mathbb{S}^3_J$ and $\mathbb{S}^3_{J'}$.

Consider the manifolds $\mathbb{S}^3_P(U; \mu_{J'})$ and $\mathbb{S}^3_{P'}(U'; \mu_J)$. Since $\mu_{J'} = \mu_U$ and $\mu_J = \mu_{U'}$, these manifolds are in fact the knot exteriors $\mathbb{S}^3_Q$ and $\mathbb{S}^3_{Q'}$, respectively. These both have non-trivial $\mathbb{S}^3$-fillings:

$$\mathbb{S}^3_Q(p/q) \cong \mathbb{S}^3_{J'}(\mu_{J'}) \cong \mathbb{S}^3,$$
$$\mathbb{S}^3_{Q'}(p'/q') \cong \mathbb{S}^3_J(\mu_J) \cong \mathbb{S}^3.$$

We conclude that we must have unknotted patterns, $P(U) = Q$ and $P'(U) = Q'$, and that $|p| = |p'| = 1$, as required. □

Combining this lemma with Brakes' construction, we complete the proof of Proposition 3.4.2.

Proof of Proposition 3.4.2. Let $K = P(J)$ be $\pm q'$-splicifiable with respect to (P, J) and write $P = Q \cup U$. Since Q is unknotted and P has winding number zero, we can now perform a nullhomologous Rolfsen $\pm q$-twist to U along Q, for any choice of coefficient with $|q| \geq 1$, to obtain a knot J' with $f(\mathbb{S}^3_P(\mp 1/q)) = \mathbb{S}^3_{J'}$. Since J can be unknotted by a Rolfsen $\pm q'$-twist, there must also be a pattern P' for which $f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(\mp 1/q')$. Thus Brakes' construction gives a $\pm q$-splicifiable knot $K' = P'(J')$ as in the statement of the theorem. Lemma 3.4.4 gives the other direction. □

By specialising q to be a nullhomologous Rolfsen twist coefficient which unknots the companion of a splicifiable knot, we obtain non-characterising slopes.

Corollary 3.4.5. *Let $K = P(J)$ be a satellite knot which is $-t$-splicifiable with respect to (P, J).*

Then there exists a satellite knot $K' = P'(J')$ which is $-t$-splicifiable with respect to (P', J') such that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(1/t) \to \mathbb{S}^3_{K'}(1/t)$ with $f(\mathbb{S}^3_P(1/t)) = \mathbb{S}^3_{J'}$ and $f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(1/t)$.

Furthermore, if $K \neq K'$, then $1/t$ is a non-characterising slope for K and K'. □

This generalises (Wakelin, sous presse, Theorem 1.8), which corresponds to the special case when $K = P(J)$ is a multiclasped Whitehead double of a double twist knot, demonstrating that this process can be used to realise non-characterising slopes with arbitrarily high denominator.

3.4.2 Maximal nullhomologous Rolfsen twist coefficient

We are now ready to define the constant $T(K)$ appearing in Theorem 3.1.8.

First, we observe that the construction leading to the non-characterising slopes in Corollary 3.4.5 bounds an unknotting nullhomologous Rolfsen twist coefficient. To the best of the authors' knowledge, this is the first such proof which does not require extra conditions on either the knot or the twist.

Corollary 3.4.6. *Let J be a non-trivial knot which can be unknotted by a nullhomologous Rolfsen t-twist for some $t \in \mathbb{Z}$. There there is a maximal possible value $\mathfrak{t}(J) > 0$ for $|t|$.*

Proof. Choose a satellite knot $K = P(J)$, where P has winding number zero, $P(U) = U$ and $\mathbb{S}^3_P(\mp 1/q) \not\cong \mathbb{S}^3_J$ for all $q \in \mathbb{Z}$. For instance, if J is not a twist knot, take the Whitehead double $K = W(J)$; if J is a twist knot, take a multiclasped Whitehead double $K = W^n(J)$ with $|n| \geq 2$.

Suppose for contradiction that no such $\mathfrak{t}(J)$ exists. Then there are infinitely many $q \in \mathbb{Z}$ such that J can be unknotted by a single nullhomologous Rolfsen $\pm q$-twist. Thus K is $\pm q$-splicifiable for infinitely many $q \in \mathbb{Z}$. By Corollary 3.4.5, there are infinitely many knots $K'_{\pm q} \neq K$ such that $\mathbb{S}^3_K(\mp 1/q) \cong \mathbb{S}^3_{K'_{\pm q}}(\mp 1/q)$. This contradicts (Sorya, 2024, Theorem 1.1). □

Given such a knot J, we call the integer $\mathfrak{t}(J)$ its *maximal nullhomologous Rolfsen twist coefficient*.

Définition 3.4.7. Define $T(K) := \max\{0, \mathfrak{t}(J) \mid K \text{ is splicifiable with respect to a pair } (P, J)\}$.

We are now in a position to prove Theorem 3.1.8.

3.4.3 Proof of Theorem 3.1.8

We begin with the proof of Proposition 3.1.9, which uses the bound $T(K)$ to obstruct the swapping of JSJ pieces in a homeomorphism between surgeries. Recall that we are in the case where K is a satellite knot for which every satellite description $K = P(J)$ is by a pattern P with winding number zero.

Proof of Proposition 3.1.9. Suppose that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ for some slope p/q with $|q| > \max\{2, T(K)\}$. Since we have $|q| > 2$, Proposition 3.2.2 ensures that the JSJ pieces of the surgered manifolds are well-defined.

Suppose that the surgered pieces of $\mathbb{S}^3_K(p/q)$ and $\mathbb{S}^3_{K'}(p/q)$ are not mapped to one another by f. Then, by the same reasoning as in Subsection 3.3.2, we can describe the knots K, K' as satellites $P(J), P'(J')$, respectively, such that $f(\mathbb{S}^3_P(p/q)) = \mathbb{S}^3_{J'}$ and $f(\mathbb{S}^3_J) = \mathbb{S}^3_{P'}(p/q)$. By Proposition 3.4.2, the knot K must be $\pm q$-splicifiable, which implies that J can be unknotted by a nullhomologous Rolfsen $\pm q$-twist. By Definition 3.4.7, we have $|q| \leq \mathfrak{t}(J) \leq T(K)$, a contradiction. □

The orientation-preserving homeomorphism f thus restricts to one between the surgered pieces. It remains to deduce that $K = K'$.

Proof of Theorem 3.1.8. Let K be a satellite knot such that for every choice of satellite description $K = P(J)$, the pattern P has winding number zero. Suppose that there is an orientation-preserving homeomorphism $f : \mathbb{S}^3_K(p/q) \to \mathbb{S}^3_{K'}(p/q)$ for some knot K' and slope p/q with $|q| > \max\{Q(K), T(K)\}$. By Proposition 3.1.9, this restricts to a slope-preserving homeomorphism between the surgered pieces. Since K is a satellite knot of hyperbolic type, we can now apply Proposition 3.3.3 and the cosmetic surgery argument used in the proof of Theorem 3.1.2 to deduce that $K = K'$. □

3.5 Examples

We conclude this article with a series of examples exhibiting the utility of our main results.

We will begin by showcasing Theorem 3.1.2 through some illustrative examples. Recall that the bound $\mathcal{C}(K) = \max\{Q(K), R(K), S(K)\}$ only depends on the hyperbolic JSJ pieces of the knot exterior $\mathbb{S}^3_K$. We will find a value for this $\mathcal{C}(K)$ using the computer programme SnapPy (Culler *et al.*, 2024).

For certain knots, Theorem 3.1.8 will give a refinement $\mathcal{C}(K) = \max\{Q(K), T(K)\}$. This depends on the maximal nullhomologous Rolfsen twist coefficient of a companion for K, which is generally harder to compute, but we will use the fact that $T(K)$ is known (often to be just 0 or 1) in many cases.

3.5.1 Examples of Theorem 3.1.2

We will first give an example which simply demonstrates how to compute the bound in Theorem 3.1.2. We will then see how this is affected by making modifications to the knot.

Example 3.5.1. Let $K = B(W(3_1), 4_1\#6_1)$ be the satellite knot of hyperbolic type constructed by splicing the Borromean rings B with $W(3_1)$ (the Whitehead double of the right-handed trefoil) and $4_1\#6_1$ (the connected sum of the figure-eight knot and the stevedore knot).

By Theorem 3.1.2, only the hyperbolic JSJ pieces of $\mathbb{S}^3_K$ contribute to $\mathcal{C}(K) = \max\{Q(K), R(K), S(K)\}$. We have

$$\begin{aligned} Q(K) &= \max\{34, \mathfrak{q}(B)\} = \max\{34, 18\} = 34; \\ R(K) &= \max\{1, \mathfrak{r}(B), \mathfrak{r}(W), \mathfrak{r}(4_1), \mathfrak{r}(6_1)\} = \max\{1, 2, 2, 0, 0\} = 2; \\ S(K) &= \max\{25, \mathfrak{s}(W), \mathfrak{s}(4_1), \mathfrak{s}(6_1)\} = \max\{25, 18, 18, 22\} = 25; \end{aligned}$$

which gives $\mathcal{C}(K) = \max\{34, 2, 25\} = 34$.

Remark 3.5.2. Let $\widehat{K}$ be a knot of hyperbolic type with hyperbolic outermost JSJ piece Y and corresponding link L_Y. Let $K = C_{r,s}(\widehat{K})$ be any cable of $\widehat{K}$. The only change from the bound $\mathcal{C}(\widehat{K})$ to the bound $\mathcal{C}(K)$ is the extra contribution of $\mathfrak{s}(L_Y)$, as Y is no longer the outermost JSJ piece. However, it is easy to see from the formulae that $\mathfrak{s}(L_Y) \leq \mathfrak{q}(L_Y)$. Therefore we can in fact take $\mathcal{C}(K) = \mathcal{C}(\widehat{K})$.

Example 3.5.3. Let $K = C_{1,2}(B(W(3_1), 4_1\#6_1))$ be the $(1,2)$-cable of the knot in Example 3.5.1. Then we can take $\mathcal{C}(K) = 34$ by Remark 3.5.2.

Remark 3.5.4. Let $L = L_0 \cup U^{m-1}$ be a hyperbolic link and consider any link obtained by adding a nullhomologous Rolfsen twist to L along a component of U^{m-1}. Performing such a twist does not change the homeomorphism type of the link exterior, so its systole is unchanged and hence both $\mathfrak{q}(L)$ and $\mathfrak{s}(L)$ are unaffected. However, $\mathfrak{r}(L)$ is defined in terms of the meridians of the link components, so any such change to L may affect this.

Example 3.5.5. Let $K = B_{-5,2}(W_{-7}(3_1), 4_1\#6_1)$ be the knot constructed in almost the same way as the one in Example 3.5.1, but with a -7-twisted Whitehead link and with $B_{-5,2}$ denoting the Borromean rings twisted along two of its unlink components -5 and 2 times, respectively. Then we can compute

$$R(K) = \max\{1, \mathfrak{r}(B_{-5,2}), \mathfrak{r}(W_{-7}), \mathfrak{r}(4_1), \mathfrak{r}(6_1)\} = \max\{1, 24, 36, 0, 0\} = 36$$

and take $\mathcal{C}(K) = \max\{Q(K), R(K), S(K)\} = \max\{34, 36, 25\} = 36$.

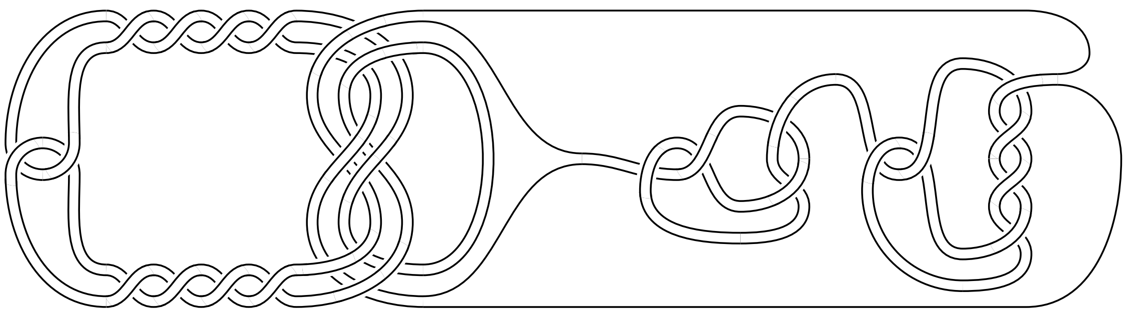

Figure 3.1: The knot $B_{-5,2}(W_{-7}(3_1), 4_1\#6_1)$ from Example 3.5.5.

3.5.2 Examples of Theorem 3.1.8

In many cases, the alternative swapping obstruction used in the proof of Theorem 3.1.8 allows us to refine the realisation of $\mathcal{C}(K)$. Not only do the non-characterising slopes in Corollary 3.4.5 give a lower bound on the optimal value for $\mathcal{C}(K)$, but Theorem 3.1.8 may also improve the realisation obtained by Theorem 3.1.2.

Although it might not always be easy to check whether a knot can be unknotted by a single nullhomologous Rolfsen twist, nor to determine the constant $T(K)$, we will show that $T(K) = 0$ for certain satellites of knots with large signature and $T(K) \leq 1$ for certain satellites of knots which are composite or fibred. Furthermore, we'll see that when $T(K) \geq Q(K)$, we obtain an optimal value for $\mathcal{C}(K)$.

3.5.2.1 Satellites of knots with large signature

In the following situation, we will see that $T(K) = 0$.

Corollary 3.5.6. *Let K be a satellite knot such that for every choice of satellite description $K = P(J)$, the winding number of P is zero but K is not splicifiable with respect to (P, J).*

If $|q| > Q(K)$, then p/q is a characterising slope for K.

Proof. By definition, we have $T(K) = 0$ because there is no satellite description $K = P(J)$ such that K is splicifiable with respect to (P, J). By Theorem 3.1.8, we have that every slope p/q with $|q| > \max\{Q(K), T(K)\} = Q(K)$ is characterising for K. $\square$

Recall that the *surgery description number* $sd(K)$ of a knot K is defined to be the minimum number of regions required to unknot K via nullhomologous Rolfsen twists. Note that $\mathfrak{t}(K)$ is only defined when $sd(K) = 1$. The surgery description number is related to several other knot invariants (Allen *et al.*, 2024). Here, we observe that the signature of a knot gives a lower bound for its surgery description number.

Lemma 3.5.7. *Let K be a knot. Then its signature $\sigma(K)$ satisfies $\frac{|\sigma(K)|}{2} \leq sd(K)$.*

Proof. The signature $\sigma(K)$ is a lower bound for twice the topological 4-genus, $2g_4^{\mathrm{TOP}}(K)$ (Kauffman et Taylor, 1976). Let $g_a(K)$ be the minimal difference between the genera of a Seifert surface F for K and a subsurface $F' \subset F$ bounded by a knot K' with Alexander polynomial $\Delta_{K'} = 1$. We have $g_4^{\mathrm{TOP}}(K) \leq g_a(K)$ since a knot K' with $\Delta_{K'} = 1$ has $g_4^{\mathrm{TOP}}(K) = 0$ (Freedman, 1982). Performing the nullhomologous Rolfsen twists relating K to the unknot in succession, we obtain a sequence of knots $K_i, i = 0, \ldots, sd(K)$, where $K_0 = K$ and $K_{sd(K)} = U$ is the unknot. Using (McCoy, 2021, Theorem 1.1), we see that

$$g_a(K) = |g_a(K_0) - g_a(K_{sd(K)})| = \sum_{i=0}^{sd(K)-1} |g_a(K_i) - g_a(K_{i+1})| \leq sd(K).$$

Combining this with the earlier inequalities gives the result. $\square$

By definition, the companion J of a satellite knot $P(J)$ that is splicifiable with respect to (P, J) must have $sd(J) = 1$. Combining Corollary 3.5.6 with Lemma 3.5.7 yields the following.

Corollary 3.5.8. *Let K be a satellite knot such that for every choice of satellite description $K = P(J)$, the winding number of P is zero and the companion J has signature satisfying $|\sigma(J)| \geq 4$.*

If $|q| > Q(K)$, then p/q is a characterising slope for K.

Proof. Write $K = P(J)$. Since $|\sigma(J)| \geq 4$, Lemma 3.5.7 implies that J cannot be unknotted via a nullhomologous Rolfsen twist so K is not splicifiable with respect to (P, J). This being true for every satellite description $K = P(J)$, we apply Corollary 3.5.6. $\square$

Below is an example of a knot for which this result yields a better bound than the one given by Theorem 3.1.2.

Example 3.5.9. Let J be the hyperbolic knot pictured in Figure 3.2 and let $K = W(J)$ be its Whitehead double. We have $Q(K) = \max\{34, \mathfrak{q}(W)\} = 34$ and $R(K) = \max\{1, \mathfrak{r}(W)\} = 1$. Moreover, SnapPy tells us that $\text{sys}(\mathbb{S}^3_J) \approx 0.0141687$, so $S(K) = \max\{25, \mathfrak{s}(J)\} = 70$. Theorem 3.1.2 then gives a realisation of $\mathcal{C}(K)$ as $\max\{34, 1, 70\} = 70$. However, we also have $\sigma(J) = -38$, so $T(K) = 0$ and our bound can be improved to $\max\{Q(K), T(K)\} = \max\{34, 0\} = 34$ by Corollary 3.5.8.

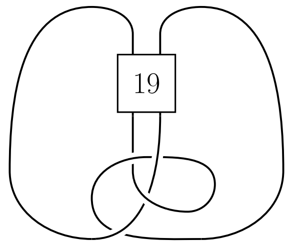

Figure 3.2: A knot with "small" systole and "large" signature.

3.5.2.2 Satellites of composite and fibred knots

Lackenby (Lackenby, 1997) showed that for any composite or fibred knot J, if $sd(J) = 1$ then in fact $\mathfrak{t}(J) = 1$. This yields the following corollary for certain satellites of such knots.

Corollary 3.5.10. *Let K be a satellite knot such that for every choice of satellite description $K = P(J)$, the winding number of $P = Q \cup U$ is zero, and if Q is unknotted, then the companion J is either composite or fibred.*

If $|q| > Q(K)$, then p/q is a characterising slope for K.

Proof. By assumption, K may be splicifiable with respect to a pair (P, J) only when J is either a composite knot or a fibred knot. Hence any knot J that contributes to $T(K)$ has maximal nullhomologous Rolfsen twist coefficient $\mathfrak{t}(J) = 1$ according to (Lackenby, 1997). Therefore $Q(K) \geq 34 > 1 \geq T(K)$ and we apply Theorem 3.1.8. □

Corollary 3.5.10 provides more examples of knots for which the bound obtained from this result is an improvement on the bound coming from Theorem 3.1.2. First, we make the following simple observation.

Lemma 3.5.11. *Let K be a satellite knot of hyperbolic type. If*

$$\operatorname{sys}(X) \leq \frac{12\sqrt{3}\pi}{Q(K)^2 - 172.68\sqrt{3}}$$

for every hyperbolic JSJ piece X which is not outermost in $\mathbb{S}^3_K$, then $Q(K) \leq S(K)$.

Proof. The hypothesis implies that $Q(K) \leq \mathfrak{s}(L_X)$ for every non-outermost hyperbolic JSJ piece X, where L_X is the link corresponding to X in the satellite construction of K. Hence $Q(K) \leq S(K)$. $\square$

This allows us to construct examples where the bound from Theorem 3.1.2 is $\mathcal{C}(K) = S(K)$ but Theorem 3.1.8 gives an improved bound $\mathcal{C}(K) = Q(K)$.

Example 3.5.12. Let J be a fibred hyperbolic knot and let $K = W(J)$ be its Whitehead double. Applying Corollary 3.5.10, we obtain a realisation of $\mathcal{C}(K)$ as $Q(K) = 34$. By Lemma 3.5.11, this is an improvement of the realisation obtained by Theorem 3.1.2 whenever

$$\operatorname{sys}(\mathbb{S}^3_J) \leq \frac{12\sqrt{3}\pi}{34^2 - 172.68\sqrt{3}} \approx 0.0762003.$$

For instance, take J to be the fibred pretzel knot $P(-2,-77,77)$ (Gabai, 1986). We have

$$\operatorname{sys}(\mathbb{S}^3_J) \approx 0.0035737 \leq 0.0762003.$$

Whilst Corollary 3.5.10 realises the bound $\mathcal{C}(K)$ as $Q(K) = 34$, Theorem 3.1.2 realises $\mathcal{C}(K)$ as $\max\{Q(K), R(K), S(K)\} = \max\{34, 1, 136\} = 136 > 34$.

Example 3.5.13. Let J be the connected sum of two simple knots J_1 and J_2 and let $K = W(J)$ be its Whitehead double. There are three possible satellite descriptions of K, each corresponding to a JSJ torus of $\mathbb{S}^3_K$: $W(J), P_2(J_1)$ and $P_1(J_2)$, where P_i is the composing pattern $W(J_i) \cup U$ for $i = 1, 2$. Since the winding number of W is zero, the winding number of P_i is also zero. Furthermore, the component $W(J_i)$ of P_i is knotted. Therefore K satisfies the conditions of Corollary 3.5.10, and we obtain a realisation of $\mathcal{C}(K)$ as 34. If J_1 or J_2 is hyperbolic, suppose that the condition of Lemma 3.5.11 is satisfied. Then, as in Example 3.5.12, this is an improvement of the bound coming from Theorem 3.1.2.

3.5.2.3 Optimal bounds

The bound from Theorem 3.1.2 is unlikely to be optimal due to the nature of its construction. However, in some cases the refined bound from Theorem 3.1.8 is truly optimal.

Corollary 3.5.14. *Let K be a satellite knot such that for every choice of satellite description $K = P(J)$, K is splicifiable with respect to (P, J). Suppose that $T(K) \geq Q(K)$.*

If $|q| > T(K)$, then p/q is a characterising slope for K and this is the optimal such bound.

Proof. First, observe that $1/T(K)$ is a non-characterising slope for K. Lemma 3.4.4 tells us that any non-characterising slope with larger denominator would have to correspond to an orientation-preserving homeomorphism between surgeries which restricts to one between the surgered pieces. However, if $|q| > Q(K)$, then no such non-characterising slope can exist. Hence our bound is optimal when $T(K) \geq Q(K)$. □

Whilst $Q(K)$ is computable (Hodgson et Weeks, 1994), it is generally harder to find an explicit value for $T(K)$. In all of our previous examples, we had $T(K) < Q(K)$. Suppose that $K = P(J)$ is a satellite of a simple knot J by a hyperbolic pattern P such that K is splicifiable with respect to (P, J). If it is known that $\mathfrak{t}(J) > 1$, then one can follow Lackenby's algorithm in (Lackenby, 2003) to find the exact value of $\mathfrak{t}(J)$ and hence $T(K)$. The following example shows that this can be made arbitrarily high, so that $T(K) \geq Q(K)$.

Example 3.5.15. Let $K = W^n(T_t^m)$ be a multiclasped Whitehead double of a double twist knot with $\max\{m, t\} > 1$. Both $1/m$ and $1/t$ are non-characterising slopes for K which can be realised by nullhomologous Rolfsen twists unknotting T_t^m (Wakelin, sous presse, Theorem 1.8). Since $\mathfrak{t}(T_t^m) \geq \max\{m, t\} > 1$, we may choose m, n, t such that $T(K) \geq Q(K) = \max\{34, \mathfrak{q}(W^n)\}$ and we can apply Corollary 3.5.14. For instance, if $n = 1$, then for any $m, t \geq 34$, we have that $T(K) \geq Q(K) = 34$. Thus $T(K)$, which can be obtained by Lackenby's algorithm, is optimal.

CHAPITRE 4
CALCUL DU COMPLEXE DE FLOER DE NŒUD POUR LES NŒUDS D'ÉPAISSEUR UN

Le troisième article de cette thèse, dont le titre original est *Computing the knot Floer complex of knots of thickness one*, est divisé en deux volets principaux. Dans le premier volet, nous présentons l'algorithme de calcul du complexe de Floer de nœud pour les nœuds d'épaisseur au plus un. Nous y établissons les bases théoriques, dont le Théorème 6, et nous décrivons son implémentation dans le logiciel de calcul formel SageMath. Le second volet est consacré à l'étude des chirurgies de Dehn caractérisantes, en s'appuyant sur l'algorithme précédemment développé. Nous y décrivons les étapes théoriques et computationnelles menant au Théorème 5.

4.0 Abstract

We develop and implement an algorithm that computes the full knot Floer complex of knots of thickness one. As an application, by extending this algorithm to certain knots of thickness two, we show that all but finitely many non-integral Dehn surgery slopes are characterizing for most knots with up to 17 crossings.

4.1 Introduction

Knot Floer homology, introduced by Rasmussen (Rasmussen, 2003) and independently by Ozsváth and Szabó (Ozsváth et Szabó, 2004), is a knot invariant that has proven to be effective for studying various topological properties of knots in S^3, such as fibredness, genus and concordance. It can be obtained from a richer algebraic structure, the *knot Floer complex*. This complex retains more data about the knot, providing further invariants, some of which are particularly useful for the study of Dehn surgeries.

While there are available algorithms for computing knot Floer homology, there is currently no implemented algorithm that effectively outputs the knot Floer complex of an arbitrary knot in S^3. The grid diagram algorithm of Manolescu, Ozsváth and Sarkar (Manolescu *et al.*, 2009) has led to a program that calculates knot Floer homology (Baldwin et Gillam, 2012), but the high number of generators it considers makes it impractical for the computation of the full knot Floer complexes. Another knot Floer homology calculator, developed by Ozsváth and Szabó (Ozsváth et Szabó, 2019), uses bordered algebras to provide more information about the knot Floer complex, but it only yields a quotiented version rather than the full complex.

In this paper, we present and implement an algorithm that recovers the full knot Floer complex of any knot of thickness at most one in S^3, from the quotiented complex of Ozsváth and Szabó.

Theorem 4.1.1. *The full knot Floer complex of a knot of thickness at most one is determined by the data of its horizontal and vertical arrows.*

The algorithm is grounded in the work of Popović (Popović, 2025b) who classified the direct sum components of knot Floer complexes of knots of thickness one. The proof of this classification has Theorem 4.1.1 as a consequence.

We apply our algorithm to the study of characterizing Dehn surgeries. We show that for the vast majority of knots with up to 17 crossings, all but finitely many non-integral Dehn surgeries are characterizing. This supports McCoy's conjecture asserting the same statement for all knots (McCoy, 2025, Conjecture 1.1).

Theorem 4.1.2. *Out of the 9 755 329 prime knots with at most 17 crossings, at least*

95.79% admit only finitely many non-integral non-characterizing Dehn surgeries.

This result is achieved by computationally verifying an algebraic condition formulated by McCoy, *property Spli**F**f*, concerning the homology modules A_k^+ of the knot Floer complex. We first identify knots whose knot Floer homology is simple enough to guarantee this condition, by using McCoy's previous work for knots of thickness at most one (McCoy, 2025, Corollary 1.4, Proposition 1.6) and the following proposition for thickness-two knots.

Proposition 4.1.3. *Let K be a knot of thickness two. Let ρ be an integer such that for all s, the knot Floer homology group $\widehat{HFK}_d(K, s)$ is non-zero only for gradings $d \in \{s + \rho, s + \rho - 1, s + \rho - 2\}$.*

*Suppose $\rho \in \{0, 1, 2\}$. If for each $k \geq 0$, at least one of the groups $\widehat{HFK}_{k+\rho}(K, k)$ or $\widehat{HFK}_{k+\rho-2}(K, k)$ is trivial, then K and its mirror both satisfy property Spli**F**f. Therefore, K admits only finitely many non-integral non-characterizing Dehn surgeries.*

We then compute the structure of the modules A_k^+ for most of the knots that do not verify (McCoy, 2025, Proposition 1.6) or Proposition 4.1.3. For thickness-one knots, this is done by using our algorithm to compute the full knot Floer complex, from which we extract the modules A_k^+. For thickness-two knots, we adapt the algorithm to recover sufficient information about the modules A_k^+ and apply it to cases within our computational capabilities. In particular, for all knots with up to 16 crossings, our strategy yields the full knot Floer complex due to the work of Hanselman who computationally verified, using immersed curves, that the statement of Theorem 4.1.1 holds for these knots (Hanselman, 2023, Corollary 12.6). We note that Hanselman's computation also provides a description of their knot Floer complex, as immersed curves turn out to capture the necessary structure for these knots.

Furthermore, our computation showcases the limitations of McCoy's algebraic condition in addressing (McCoy, 2025, Conjecture 1.1), with the remaining 4.21% of unresolved cases providing examples of knots that do not satisfy property Spli**F**f. Notably, this includes knots of thickness one, whereas previously identified examples had thickness at least two (McCoy, 2025, Proposition 3.3(ii), Example 3.4).

4.1.1 Structure of paper

The paper is organized as follows. In Section 4.2, we introduce the algebraic settings in which knot Floer complexes will be studied. Section 4.3 contains the proof of Theorem 4.1.1. In Section 4.4, we present an overview of the algorithm for computing the knot Floer complex of knots of thickness at most one. Section 4.5 translates the problem into a computational framework where the differential map is encoded as a matrix. In Section 4.6, we show that certain degree constraints reduce the problem to a system of linear equations. Section 4.7 describes the SageMath implementation of the algorithm. In Sections 4.8, 4.9 and 4.10 we extend and apply our algorithm to study characterizing Dehn surgeries.

4.1.2 Acknowledgements

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4.2 Algebraic setting

Knot Floer complexes come in a variety of algebraic flavours. We are interested in the *full knot Floer complex*, from which all other variants can be derived. This full complex can itself be described in different algebraic settings. We present two such settings and we show that the data they encode is equivalent.

4.2.1 Basic construction

We first recall the basics of the construction of a knot Floer complex. From a knot K in S^3, we obtain a doubly pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$, where Σ is a genus-g surface, α and β are sets of g curves on Σ and w, z are the two basepoints. A knot Floer complex for K associated to $\mathcal{H}$ is generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = (\alpha_1 \times \ldots \times \alpha_g) \cap (\beta_1 \times \ldots \times \beta_g)$ in the g-fold symmetric product $\mathrm{Sym}^g(\Sigma)$. The differential of a knot Floer complex counts certain representatives of Whitney discs $\phi \in \pi_2(x, y)$ between two generators $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and their intersections with certain auxiliairy submanifolds associated to the basepoints.

In this section, we will assume that any knot Floer complex mentioned refers to a fixed knot K and is obtained from a fixed Heegaard diagram $\mathcal{H}$ for K. The knot Floer complex is an invariant of K up to filtered chain homotopy equivalence and does not depend on the choice of $\mathcal{H}$. Therefore, instead of writing $CFK^\infty(\mathcal{H})$ for instance, we may simply write $CFK^\infty(K)$.

4.2.2 Knot Floer complex as an $\mathbb{F}[U, U^{-1}]$-module

We now recall the classical presentation of the knot Floer complex $CFK^\infty(K)$ as an $\mathbb{F}[U, U^{-1}]$-module from (Ozsváth et Szabó, 2004), a knot invariant up to filtered homotopy equivalence. $\mathbb{F}$ denotes the field with two elements and U is a formal variable. Let

$CFK^-(K)$ be the chain complex generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ over the ring $\mathbb{F}[U]$ with differential given by

$$dx = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^{n_w(\phi)} y,$$

where $\mathcal{M}(\phi)$ is the moduli space of holomorphic representatives of the Whitney disc ϕ, $\mu(\phi)$ is the expected dimension of $\mathcal{M}(\phi)$, and $n_w(\phi)$ is the algebraic intersection number of ϕ with $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma)$. The chain complex $CFK^\infty(K)$ is defined as $CFK^-(K) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$.

We may visually depict a representative of $CFK^\infty(K)$ in a $\mathbb{Z} \oplus \mathbb{Z}$ lattice as follows. An element $U^i x, x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, has position $(-i, A(U^i x))$, where $A(U^i x)$ is the Alexander grading of $U^i x$. We have in fact $A(U^i x) = A(x) - i$, so all elements $U^i x, i \in \mathbb{Z}$ are represented on a diagonal line of slope 1 intersecting the vertical axis at $A(x)$. If there is a Whitney disc $\phi \in \pi_2(x, y), \mu(\phi) = 1$, then $A(U^i x) - A(U^i y) = n_z(\phi) - n_w(\phi)$.

Homogeneous elements of $CFK^\infty(K)$ are endowed with an additional grading called the Maslov grading. The action of multiplication by U, modifies this grading by -2, i.e. $M(U^i x) = M(x) - 2i$. If there is a Whitney disc $\phi \in \pi_2(x, y), \mu(\phi) = 1$, then $M(U^i x) - M(U^i y) = 1 - 2n_w(\phi)$. Thus, the differential lowers the Maslov grading by 1, making it the homological degree on $CFK^\infty(K)$. Therefore, we may interchangeably use *grading* and *degree* to refer to the Maslov grading.

Arrows are drawn between generators to indicate the differential. Arrows are said to be *horizontal*, *vertical* or *diagonal* with respect to this visual representation. The position of an element in the $\mathbb{Z} \oplus \mathbb{Z}$ lattice indicates its filtration level, with respect to the partial order on $\mathbb{Z} \oplus \mathbb{Z}$ given by

$$(i, j) \leq (i', j') \iff i \leq i' \text{ and } j \leq j',$$

with a strict inequality if $i < i'$ or $j < j'$.

The filtered chain homotopy class of $CFK^\infty(K)$ can be represented by a reduced chain complex (see for instance (Hedden et Watson, 2018, Section 2.1)). Let x and y be generators of a reduced representative (C,d) of $CFK^\infty(K)$ such that $U^k y$ has non-zero coefficient in $d(U^i x)$. Since the differential d strictly lowers the filtration, we have $U^k y < U^i x$. Therefore, $-k \leq -i$ and $A(y) - k \leq A(x) - i$, where $-k < -i$ or $A(y) - k < A(x) - i$.

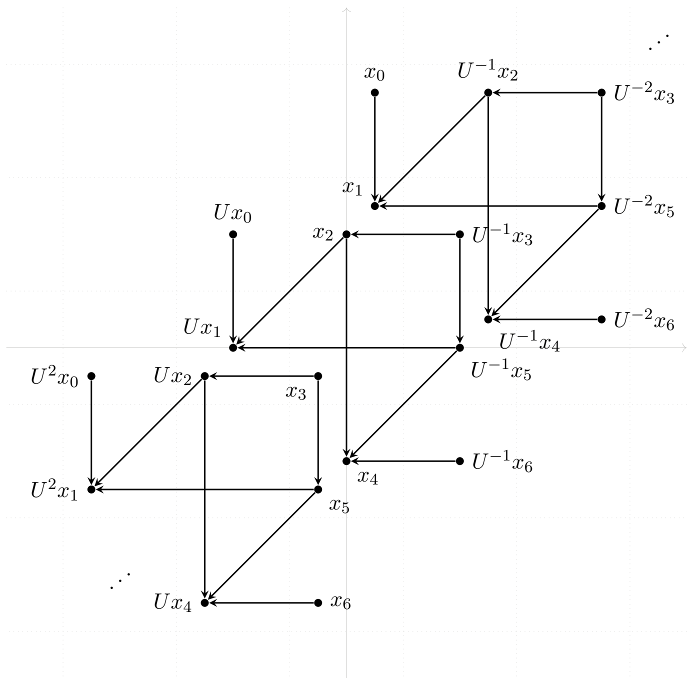

Figure 4.1: The complex $CFK^\infty(K)$ for the $(2,-1)$-cable of the left-handed trefoil

4.2.3 Knot Floer complex as an $\mathbb{F}[u,v]$-module

We also recall the presentation of the knot Floer complex $CFK_{\mathbb{F}[u,v]}(K)$ as an $\mathbb{F}[u,v]$-module, also a knot invariant up to homotopy equivalence, as introduced in (Zemke, 2017) and summarized in (Hom, 2020). As before, $\mathbb{F}$ is the field with two elements and u, v are formal variables. The ring $\mathbb{F}[u,v]$ is bigraded by a u-grading gr_u and a v-grading gr_v such that $(gr_u(u), gr_v(u)) = (-2, 0)$ and $(gr_u(v), gr_v(v)) = (0, -2)$. The complex $CFK_{\mathbb{F}[u,v]}(K)$

is generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ over $\mathbb{F}[u,v]$ and the differential is given by

$$d_{\mathbb{F}[u,v]}x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot u^{n_w(\phi)} v^{n_z(\phi)} y.$$

A representative of $CFK_{\mathbb{F}[u,v]}(K)$ admits a decomposition into direct summands $\mathcal{A}_s(K), s \in \mathbb{Z}$, consisting of all $\mathbb{F}$-linear combinations of elements $u^i v^j x, x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, (i,j) \in \mathbb{Z} \oplus \mathbb{Z}$, that have $\mathcal{A}$-grading $s \in \mathbb{Z}$, where $\mathcal{A}(u^i v^j x) = (gr_u(u^i v^j x) - gr_v(u^i v^j x))/2$. The $\mathcal{A}$-grading $\mathcal{A}(x)$ of a generator $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ agrees with its Alexander grading $A(x)$. Since the action of multiplication by u modifies the u-grading by -2 and the multiplication by v leaves it untouched, and vice versa for the v-grading, we have $gr_u(u^i v^j x) = gr_u(x) - 2i$ and $gr_v(u^i v^j x) = gr_v(x) - 2j$. The u-grading of $u^i v^j x$ agrees with the Maslov grading of $U^i x \in CFK^\infty(K)$ described above.

In a visual representation of $\mathcal{A}_s(K)$ for some $s \in \mathbb{Z}$, an element $u^i v^j x, x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ has relative position $(-i,-j)$ in the $\mathbb{Z} \oplus \mathbb{Z}$ lattice. Arrows are drawn between generators to indicate the differential. This complex has an implicit filtration given by the powers of u and v, since by definition, the differential always increases these powers. This agrees with the partial order on $\mathbb{Z} \oplus \mathbb{Z}$ mentioned above. We may extend this visual representation to the tensor product $\mathcal{A}_s(K) \otimes_{\mathbb{F}[uv]} \mathbb{F}[uv,(uv)^{-1}]$, which we denote by $CFK^\infty_{\mathbb{F}[u,v],s}(K)$.

4.2.4 Equivalence between algebraic settings

The two algebraic settings contain the same information for a given knot, as given by the next proposition.

Proposition 4.2.1. (Zemke, 2017, Section 1.5) *Let $CFK^\infty_{\mathbb{F}[u,v],s}(\mathcal{H}),\ s \in \mathbb{Z}$ and $CFK^\infty(\mathcal{H})$ be representatives of $CFK^\infty_{\mathbb{F}[u,v],s}(K)$ and $CFK^\infty(K)$ respectively, obtained from the same Heegaard diagram $\mathcal{H}$. Then each complex $CFK^\infty_{\mathbb{F}[u,v],s}(\mathcal{H}),\ s \in \mathbb{Z}$ is isomorphic to $CFK^\infty(\mathcal{H})$ by an isomorphism that respects both the filtration up to translation and the $\mathbb{F}[U,U^{-1}]$-*

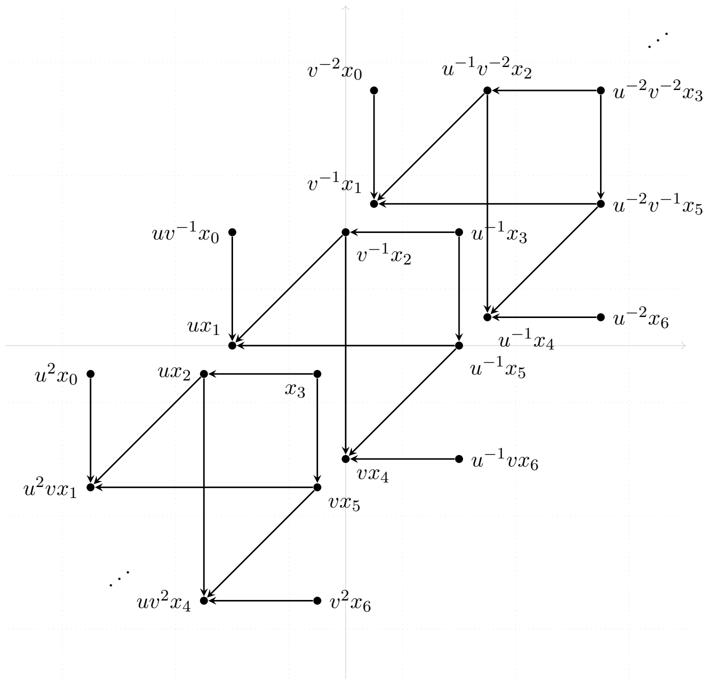

Figure 4.2: The complex $CFK^{\infty}_{\mathbb{F}[u,v],0}(K)$ for the $(2,-1)$-cable of the left-handed trefoil

module structure, by setting $U = uv$.

Proof. We define a map $\varphi_s : CFK^{\infty}_{\mathbb{F}[u,v],s}(\mathcal{H}) \to CFK^{\infty}(\mathcal{H})$ in the following way.

Let $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, so that x is a generator of both $CFK_{\mathbb{F}[u,v]}(\mathcal{H})$ and $CFK^{\infty}(\mathcal{H})$. Set $\varphi_s(u^i v^j x) = U^i x$. We show that φ_s is a $U^{\pm 1}$-equivariant filtered chain isomorphism realizing the proposition.

1) *φ_s is injective:*

For a fixed i, there is only one possible power j of v such that $u^i v^j x \in CFK^{\infty}_{\mathbb{F}[u,v],s}(\mathcal{H})$, i.e. $\mathcal{A}(u^i v^j x) = s$. Indeed,

$$\begin{aligned} s &= \mathcal{A}(u^i v^j x) \\ &= (gr_u(u^i v^j x) - gr_v(u^i v^j x))/2 \\ &= (gr_u(x) - 2i - gr_v(x) + 2j)/2 \\ &= (j - i) + A(x) \end{aligned}$$

implies that $-j = A(x) - i - s$.

2) φ_s *is surjective:*

An element $U^i x \in CFK^\infty(\mathcal{H})$ has antecedent $u^i v^{i-A(x)} x \in CFK^\infty_{\mathbb{F}[u,v],s}(\mathcal{H})$.

3) φ_s *preserves the filtration up to translation:*

The element $u^i v^{s+i-A(x)} x \in CFK^\infty_{\mathbb{F}[u,v],s}(\mathcal{H})$ and its image $U^i x \in CFK^\infty(\mathcal{H})$ have respective filtration levels $(-i, A(x)-i-s)$ and $(-i, A(x)-i)$. Therefore, φ_s translates filtration levels by $(0, s)$.

4) φ_s *is* $U^{\pm 1}$*-equivariant:*

We have $\varphi_s((uv)^{\pm 1} \cdot u^i v^j x) = \varphi_s(u^{i\pm 1} v^{j \pm 1} x) = U^{i\pm 1} x = U^{\pm 1} \cdot U^i x = U^{\pm 1} \varphi_s(u^i v^j x)$.

5) φ_s *is a chain map:*

By definition of the differentials, we have

$$\begin{aligned} \varphi_s(d_{\mathbb{F}[u,v]} u^i v^j x) &= \varphi_s \Bigg(\sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot u^{i+n_w(\phi)} v^{j+n_z(\phi)} y \Bigg) \\ &= \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot \varphi_s(u^{i+n_w(\phi)} v^{j+n_z(\phi)} y) \\ &= \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^{i+n_w(\phi)} y \\ &= d(U^i x) \end{aligned}$$

$$= d_{\mathbb{F}[u,v]}\varphi_s(u^i v^j x) \qquad \square$$

Applying the reduction lemma of (Hedden et Watson, 2018, Section 2.1) in a mirrored way to $CFK^{\infty}_{\mathbb{F}[u,v],s}(\mathcal{H})$ and $CFK^{\infty}(\mathcal{H})$, we obtain from Proposition 4.2.1 a $U^{\pm 1}$-equivariant filtered chain isomorphism $\varphi_s : \mathcal{C}_s \to \mathcal{C}$ between reduced representatives of $CFK^{\infty}_{\mathbb{F}[u,v],s}(K)$ and $CFK^{\infty}(K)$ respectively. We can recover the (reduced) summands $\mathcal{A}_s(K)$ of $CFK_{\mathbb{F}[u,v]}$ from $CFK^{\infty}(K)$ by restricting φ_s^{-1} to elements with filtration $(i,j) \leq (s,0)$.

4.2.5 Thickness

Both algebraic settings contain the data of the *knot Floer homology* $\widehat{HFK}(K)$ of the knot: on one hand, $\widehat{HFK}(K) \cong CFK_{\mathbb{F}[u,v]}(K)/(u,v)$ and on the other hand, $\widehat{HFK}(K)$ is the homology of the associated graded complex of $CFK^{-}(K)/U$. Reduced representatives of $CFK_{\mathbb{F}[u,v]}$ and $CFK^{\infty}(K)$ have generating sets that are in bijection with the generating set of $\widehat{HFK}(K)$. We denote by $\widehat{HFK}(K,a)$ the knot Floer homology of K in Alexander grading a.

The thickness of a knot K is defined from $\widehat{HFK}(K) = \oplus_{a\in\mathbb{Z}}\widehat{HFK}(K,a)$.

Definition 4.2.2. The *thickness* of a knot K is the number

$$th(K) = \max\{|(M(x) - A(x)) - (M(y) - A(y))|, x, y \text{ generators of } \widehat{HFK}(K)\}$$

A low thickness imposes constraints on the possible arrows representing the differential map. We will apply these constraints in the next section, where we focus on knots of thickness one.

4.3 Chain homotopy equivalence of lifts

4.3.1 Horizontal and vertical arrows

The algorithm of Ozsváth and Szabó mentioned in the introduction outputs the quotient of a reduced representative of $CFK_{\mathbb{F}[u,v]}(K)$ by uv, for any knot K given as input (Culler *et al.*, 2024). From now on, we will assume that all chain complexes mentioned are reduced. In this subsection, we recall how the horizontal and vertical arrows of the full complex $CFK_{\mathbb{F}[u,v]}(K)$ are captured by this quotiented complex for any knot K.

Proposition 4.3.1. *Let (C,d) be a reduced representative of $CFK_{\mathbb{F}[u,v]}(K)$. Then $(C,d)/(uv)$ is obtained from the data of the horizontal and vertical arrows of (C,d). Conversely, the data of the horizontal and vertical arrows of (C,d) is contained in $(C,d)/(uv)$.*

Proof. Let x be a generator of C. The differential of $[x]$ in $(C,d)/(uv)$ is given by

$$[dx] = \sum_{\substack{y \text{ generator} \\ \text{of } C}} c_y \cdot [u^{i_y} v^{j_y} y]$$

for some $c_y \in \mathbb{F}$ and $i_y, j_y \geq 0 \in \mathbb{Z}$. If i_y and j_y are both non-zero, then $[u^{i_y} v^{j_y} y] = 0$ and therefore

$$[dx] = \sum_{\substack{y \text{ generator} \\ textofC}} \left(\sum_{j_y=0} c_y [u^{i_y} y] + \sum_{i_y=0} c_y [v^{j_y} y] \right),$$

which is precisely the data of horizontal and vertical arrows leaving x in (C,d).

Since C is generated over $\mathbb{F}[u,v]$, this also gives the data of horizontal and vertical arrows leaving $u^i v^j x$ for all $i,j \in \mathbb{Z}$. □

Note that, due to the isomorphism from the discussion following Proposition 4.2.1, the arrows of the quotient complex $(C,d)/(uv)$ also provide the data of the horizontal and vertical arrows of a reduced representative of $CFK^\infty(K)$.

To recover the full knot Floer complex from $(C, d)/(uv)$, we need to find a lift of $(C, d)/(uv)$ to a chain complex (C', d') over $\mathbb{F}[u, v]$ which is chain homotopy equivalent to (C, d). By Proposition 4.3.1, if this lift (C', d') is reduced, we know that it must contain the same data of horizontal and vertical arrows as (C, d).

4.3.2 Chain homotopy equivalence

The lifts of $(C, d)/(uv)$ to complexes over $\mathbb{F}[u, v]$, for representatives (C, d) of $CFK_{\mathbb{F}[u,v]}(K)$, may belong to distinct chain homotopy classes. However, when the thickness of K is at most one, all such complexes are in fact equivalent.

Theorem 4.3.2. *Let K be a knot of thickness at most one and let (C, d) be a reduced representative of $CFK_{\mathbb{F}[u,v]}(K)$. Then all lifts of $(C, d)/(uv)$ to a reduced complex over $\mathbb{F}[u, v]$ are isomorphic.*

Theorem 4.3.2 combined with Proposition 4.3.1 immediately implies Theorem 4.1.1.

Theorem 4.1.1. *The full knot Floer complex of a knot of thickness at most one is determined by the data of its horizontal and vertical arrows.* □

The case of thickness zero in Theorem 4.3.2 is trivial since all representatives of $CFK_{\mathbb{F}[u,v]}(K)$ contain only horizontal and vertical arrows. In particular, Petkova showed that the chain homotopy class $CFK_{\mathbb{F}[u,v]}(K)$ of a knot of thickness zero is determined by the knot's Alexander polynomial and τ invariant (Petkova, 2013, Theorem 4). For knots of thickness one, this is a consequence of the proof of the following result of Popović.

Theorem 4.3.3. (Popović, 2025b, Theorem 1.1) *Let K be a knot of thickness one. Then $CFK_{\mathbb{F}[u,v]}(K)$ splits uniquely as a direct sum of an $\mathbb{F}[u, v]$-standard complex of thickness at most 1 and trivial local systems, each of which belongs to a specific set of systems $\mathcal{L}$.*

The $\mathbb{F}[u,v]$-standard complex of Theorem 4.3.3 is an $\mathbb{F}[u,v]$-realization of a standard complex as originally defined in (Dai *et al.*, 2021, Definition 4.3). The exact description of the local systems in $\mathcal{L}$ can be found in the statement of (Popović, 2025b, Theorem 1.1), but the key property of $\mathcal{L}$ relevant to our purposes is the following.

Proposition 4.3.4. (Popović, 2025b, Proposition 4.11) *Let C be a chain complex over $\mathbb{F}[u,v]$ of thickness one and let $L \in \mathcal{L}$ be a local system such that $C/(uv) \cong L/(uv) \oplus A/(uv)$ for some $\mathbb{F}[u,v]$-chain complex A. Then $C \cong L \oplus A$.*

Proof of Theorem 4.3.2. Let K be a knot of thickness one. Let (C,d) be a representative of the chain homotopy class of $CFK_{\mathbb{F}[u,v]}$ and let (C',d') be a lift of $(C,d)/(uv)$ over $\mathbb{F}[u,v]$ at the level of chain complexes. Note that $C' = C$ as bigraded $\mathbb{F}[u,v]$-modules, so we may write $(C,d') = (C',d')$.

We decompose the differential map d into $d = H+V+D$, where H, V and D are respectively the horizontal, vertical and diagonal arrows of d. Let $d_{uv} = H + V$. Similarly, we write $d' = H' + V' + D'$ and $d'_{uv} = H' + V'$.

Since $(C,d)/(uv) \cong (C,d')/(uv)$, we have $d_{uv} = d'_{uv}$ by Proposition 4.3.1.

The splitting of Theorem 4.3.3 is realized by a change of basis P such that (C, PdP^{-1}) is a direct sum as in the statement of Theorem 4.3.3 (see proofs of (Popović, 2025b, Lemmas 4.12, 4.13 and 4.14)). Restricting d to d_{uv}, we have that $(C, Pd_{uv}P^{-1})$ is a direct sum of a standard complex of thickness at most one and local systems from $\mathcal{L}$ with the diagonal arrows removed.

Thus, by Proposition 4.3.1, both $(C, PdP^{-1})/(uv)$ and $(C, Pd'P^{-1})/(uv)$ are isomorphic to the same direct sum $L_1/(uv) \oplus \ldots \oplus L_k/(uv) \oplus S/(uv), k \geq 0$, where $L_i \in \mathcal{L}, i = 1, \ldots, k$, and S is an $\mathbb{F}[u,v]$-standard complex of thickness at most one.

By (Popović, 2025a, Algorithm 3.12), the quotiented standard complex $S/(uv)$ has a unique lift over $\mathbb{F}[u,v]$. Applying Proposition 4.3.4 inductively on the number k of local systems in the direct sum, we obtain the isomorphism $(C, PdP^{-1}) \cong (C, Pd'P^{-1})$. Performing the change of basis P^{-1} yields the isomorphism $(C,d) \cong (C,d')$ as desired. □

4.4 Finding a lift: an overview

In this section, we give an overview of our method to find a lift of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$ for knots of thickness one, which we will detail in the following two sections. By Theorem 4.3.2, this leads to an algorithm that determines the full knot Floer complex of knots of thickness one. For computational reasons, we pass to the setting of $CFK^\infty(K)$ over the ring $\mathbb{F}[U,U^{-1}]$, for which we only need to consider a single formal variable U.

Algorithm 4.4.1 $CFK^\infty(K)$ for knots of thickness ≤ 1

Input: Knot K with $th(K) \leq 1$

Output: Filtered homotopy representative of $CFK^\infty(K)$

The main goal is to construct a chain complex $\mathcal{C} = (C,d)$ over $\mathbb{F}[U,U^{-1}]$ such that $\varphi_s^{-1}(\mathcal{C})/(uv) \simeq CFK^\infty_{\mathbb{F}[u,v],s}(K)/(uv)$ for all $s \in \mathbb{Z}$. Here φ_s^{-1} are the isomorphisms from the discussion following Proposition 4.2.1. We say that such a complex $\mathcal{C}$ is a *lift* of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$.

Since the $\mathbb{F}[U,U^{-1}]$-module C and the vertical and horizontal arrows of the differential d are known from Ozsváth and Szabó's algorithm, we only need to find the diagonal arrows of d.

The first main step is to encode the differential map as a matrix. We construct a matrix

d_{var} that contains the data of the known vertical and horizontal arrows, along with entries consisting of unknown variables for possible diagonal arrows, considering constraints given by the Alexander and Maslov gradings. This step does not depend on the thickness of the knot and is described in Section 4.5.

The second main step is to determine a value in $\mathbb{F}[U, U^{-1}]$ for each unknown variable in the matrix d_{var} such that the condition $d_{var}^2 = 0$ of a chain complex is satisfied. We thus rewrite $d_{var}^2 = 0$ as a set of equations to be solved. By construction of d_{var}, a solution to these equations will yield a chain complex $\mathcal{C} = (C, d)$ that respects the filtration and degree constraints expected for a knot Floer complex. The complex $\mathcal{C}$ also has the same data of horizontal and vertical arrows as a reduced representative of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$, making it a lift of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$. A key point for the computational feasibility of our algorithm is that, for thickness one knots, the equations coming from $d_{var}^2 = 0$ are always linear. This is demonstrated in Section 4.6. A solution is then obtained by basic linear algebra, giving the desired lift of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$.

4.5 Matricial representation

Our first main step is to encode the differential map d as a matrix with placeholders for the unknown entries. The $\mathbb{F}[U, U^{-1}]$-module underlying $CFK^\infty(K)$ is generated by the generators $x_0, x_1, \ldots, x_{n-1}$ of $\widehat{HFK}(K)$ over $\mathbb{F}[U, U^{-1}]$. Thus, d can be represented by an $n \times n$ matrix with values in $\mathbb{F}[U, U^{-1}]$: the (i, j) entry of this matrix is the coefficient $a_{i,j}$ in $d(x_j) = \sum_{i=0}^{n-1} a_{i,j} x_i$. In fact, since d respects the filtration, all entries $a_{i,j}$ take values in $\mathbb{F}[U]$. From now on, we will denote both the differential and its matrix by d.

4.5.1 Entries for horizontal and vertical arrows

We decompose d into $d = H + V + D$, where H, V and D are respectively the horizontal, vertical and diagonal arrows of the differential.

We recover the matrix $H + V$ using the output from Ozsváth and Szabó's algorithm for computing $CFK_{\mathbb{F}[u,v]}(K)/(uv)$. It provides us with the generators $x_0, x_1, \ldots, x_{n-1}$ and their Maslov and Alexander gradings, and tells us if $H + V$ has an arrow from x_j to $U^k x_i$ for some power $k \geq 0$. Since the differential lowers the Maslov grading by 1 and multiplication by U lowers the Maslov grading by 2, we have

$$M(U^k x_i) = M(x_i) - 2k = M(x_j) - 1.$$

Therefore, if Ozsváth and Szabó's algorithm indicates that there is an arrow from a x_j to $U^k x_i$ for some power $k \geq 0$, we set the (i, j) entry of the matrix $H + V$ to be

$$a_{i,j} = U^{(M(x_i) - M(x_j) + 1)/2}.$$

4.5.2 Entries for possible diagonal arrows

Next, we find pairs of generators of $CFK^{\infty}(K)$ that may be connected by a diagonal arrow. We consider how a differential map affects the Maslov and Alexander gradings.

A diagonal arrow from x_j to $U^k x_i$ for some power $k \geq 1$ must meet the conditions $M(U^k x_i) - M(x_j) = -1$ and $A(x_i) - A(x_j) < k$. Thus, for every (i, j) such that

(D1) $(M(x_i) - M(x_j) + 1)/2 \geq 1$ and

(D2) $(M(x_i) - M(x_j) + 1)/2 > A(x_i) - A(x_j)$,

there could be a diagonal arrow from x_j to $U^k a_{i,j}$, where

(D3) $k = (M(x_i) - M(x_j) + 1)/2$.

We construct a placeholder matrix D_{var} in the following way. If (i, j) satisfies both (D1) and (D2), then the (i, j) entry of D_{var} is $U^k a_{i,j}$, where $a_{i,j}$ is an unknown variable with values in $\mathbb{F}$ and k is as in (D3). Otherwise, the entry is zero. We then form the matrix $d_{var} = H + V + D_{var}$ with entries in $\mathbb{F}[U][\{a_{i,j} \mid (i, j) \text{ verify (D1) and (D2)}\}]$. We now want to find the values of $a_{i,j}$ for which d_{var} is a differential map for the $\mathbb{F}[U, U^{-1}]$-module underlying $CFK^\infty(K)$.

4.6 Solving for $d^2 = 0$

Setting $d_{var}^2 = 0$, we obtain equations $[d_{var}^2]_{k,l} = 0$, for each $(k, l) \in \{0, \ldots, n-1\}^2$, where the variables $a_{i,j}$ are the unknowns. Finding these solutions is in general computationally challenging as the equations may involve degree-two polynomials in the ring $\mathbb{F}[\{a_{i,j} \mid (i, j) \text{ verify (D1) and (D2)}\}]$, with a number of variables $a_{i,j}$ that can be quite large. However, it turns out that for knots of thickness one, the system $[d_{var}^2]_{k,l} = 0$ consists only of linear equations, which can be solved easily with basic linear algebra.

4.6.1 Consecutive diagonal arrows

While the methods of Section 4.5 can be applied to any knot, we now restrict our study to knots with low thickness to obtain further constraints on the possible diagonal arrows. The goal of this subsection is to show that given certain degree conditions on $\widehat{HFK}(K)$, there cannot be consecutive diagonal arrows in a reduced chain complex representing $CFK^\infty(K)$.

Proposition 4.6.1. *Suppose K is a knot of thickness at most two such that $\widehat{HFK}(K, a)$ is supported in at most 2 degrees for all $a \in \mathbb{Z}$. Then $d_{var} = H + V + D_{var}$ as constructed above is such that $D_{var}^2 = 0$.*

Note that, by the definition of thickness, knots of thickness at most one verify the condition of Proposition 4.6.1. Although Algorithm 4.4.1 focuses on this case only, the more general statement of Proposition 4.6.1 will be applied in later sections.

Under the condition that the thickness is at most two, we obtain the next three lemmas concerning the Alexander and Maslov gradings of generators connected by a diagonal arrow. We will then use the condition on the support of $\widehat{HFK}(K)$ to prove Proposition 4.6.1.

Lemma 4.6.2. *Suppose K is a knot of thickness at most two and let $U^k a_{i,j}$ be a non-zero entry in D_{var}. Then $|A(x_i) - A(x_j)| \leq 1$.*

Proof. Suppose $A(x_i) - A(x_j) \geq 2$. Then (D3) and (D2) yield

$$\begin{aligned} M(x_i) - A(x_i) &= M(x_j) + 2k - 1 - A(x_i) \\ &\geq M(x_j) + 2(A(x_i) - A(x_j) + 1) - 1 - A(x_i) \\ &\geq M(x_j) - A(x_j) + 3 \end{aligned}$$

which implies that K has thickness at least three.

Suppose $A(x_j) - A(x_i) \geq 2$. Similarly to the argument above, we obtain

$$\begin{aligned} M(x_i) - A(x_i) &= (M(x_j) + 2k - 1) - A(x_i) \\ &\geq M(x_j) + 1 - A(x_j) + 2 \\ &\geq M(x_j) - A(x_j) + 3. \end{aligned}$$

□

Lemma 4.6.3. *Suppose K is a knot of thickness at most two and let $U^k a_{i,j}$ be a non-zero entry in D_{var}. Let $\eta = A(x_i) - A(x_j)$. Then $\eta \in \{-1, 0, 1\}$ by Lemma 4.6.2 and $k = 1$ when $\eta = -1$ or 0, and $k = 2$ when $\eta = 1$.*

Proof. We have $M(x_i) - A(x_i) = M(x_j) + 2k - 1 - A(x_j) - \eta$, which implies that $2k - 1 - \eta \leq 2$, hence $k \leq (3+\eta)/2$. Since $k \geq 1$, replacing the value of η with $-1, 0$ or 1 in $k \leq (3+\eta)/2$ gives the result. □

Lemma 4.6.4. *Suppose K is a knot of thickness at most two and let $U^k a_{i,j}$ be a non-zero entry in D_{var}. Then $M(x_i) - A(x_i) = M(x_j) - A(x_j) + 2$ when $\eta = 1$ or -1, and $M(x_i) - A(x_i) = M(x_j) - A(x_j) + 1$ when $\eta = 0$.*

Proof. Replace k and η in $M(x_i) - A(x_i) = (M(x_j) + 2k + 1) - (A(x_j) + \eta)$ by the pairs given by Lemma 4.6.3. □

Proof of Proposition 4.6.1. Suppose that $D_{var}^2 \neq 0$. This means that there are non-zero entries $U^{k_1} a_{j,k}$ and $U^{k_2} a_{i,j}$ in D_{var} that contribute $U^{k_1+k_2} a_{i,j} a_{j,k}$ to a non-zero entry of $D_{var}^2 \neq 0$.

If $A(x_k) \neq A(x_j)$ or $A(x_j) \neq A(x_i)$, then by Lemma 4.6.4, $(A(x_k) - M(x_k)) - (A(x_i) - M(x_i)) = (A(x_k) - M(x_k)) - (A(x_j) - M(x_j)) + (A(x_j) - M(x_j)) - (A(x_i) - M(x_i)) \geq 3$, which contradicts the thickness of K being at most 2.

If $A(x_k) = A(x_j) = A(x_i)$, then by (D3) we have $M(x_i) = M(x_j) + 2k_2 - 1 = M(x_j) + 1$ and $M(x_j) = M(x_k) + 2k_1 - 1 = M(x_k) + 1$. Hence, the knot Floer homology of K in Alexander grading $A(x_k) = A(x_j) = A(x_i)$ is supported in at least 3 distinct degrees, a contradiction. □

4.6.2 Linear system of equations

We now return to the setting of d_{var} and translate the problem of finding lifts of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$ into a system of linear equations.

Proposition 4.6.5. *Suppose K is a knot of thickness at most two such that $\widehat{HFK}(K,a)$ is supported in at most 2 degrees for all $a \in \mathbb{Z}$. Then the entries of d_{var}^2 are polynomials of degree at most one in the variables $a_{i,j}$ over $\mathbb{F}[U]$.*

Proof. By Proposition 4.6.1,

$$\begin{aligned} d_{var}^2 &= (H+V+D_{var})^2 \\ &= (H+V)^2 + (H+V)D_{var} + D_{var}(H+V) + D_{var}^2 \\ &= (H+V)^2 + (H+V)D_{var} + D_{var}(H+V). \end{aligned}$$

The result follows from the fact that $(H+V)$ has entries in $\mathbb{F}[U]$, for which the variables $a_{i,j}$ have degree zero, and D_{var} has entries of the form $U^k a_{i,j}$, where the variables $a_{i,j}$ have degree one. □

We may view the entries of d_{var}^2 as polynomials in U with coefficients in $\mathbb{F}\langle\{a_{i,j}\}\rangle$. By setting $d_{var}^2 = 0$, we must have that each coefficient $\sum a_{i_k,j_k}$ of a power of U is equal to zero. We thus obtain a linear system of equations $E = \{\sum a_{i_k,j_k} = 0\}$ over $\mathbb{F}$ where the variables $a_{i,j}$ are the unknowns. This system can be represented by a matrix equation $Aa = b$ where a is the vector of variables $a_{i,j}$ to solve for.

Given a solution $a = a_0$, we replace its values into the corresponding entries of D_{var} to obtain a matrix $D_0 = D_{var}(a_0)$. We then build the differential complex $\mathcal{C}_0 = (C, d_0 = H + V + D_0)$, where $C \cong \widehat{HFK}(K) \otimes \mathbb{F}[U, U^{-1}]$ is the $\mathbb{F}[U, U^{-1}]$-module underlying $CFK^\infty(K)$. By Theorem 4.3.2 and Proposition 4.2.1, the complex $\mathcal{C}_0$ is a representative of $CFK^\infty(K)$ if K has thickness at most one.

4.7 Implementation

The previous discussion has been implemented in SageMath, utilizing SnapPy (Culler *et al.*, 2024) as an imported package. SnapPy is used to input the data of a knot, via its integrated census or a planar diagram, and for calling upon the method `knot_floer_homology`, an implementation of Ozsváth and Szabó's algorithm, to obtain the data of $CFK_{\mathbb{F}[u,v]}/(uv)$.

SageMath can generate polynomial rings and handle symbolic computations over them. This allows us to extract the equations to be solved over the ring $\mathbb{F}[U]$, as described in Section 4.6.2, and to translate them into a matrix equation $Aa = b$ over $\mathbb{F}$.

To obtain a solution to the matrix equation $Aa = b$, we use SageMath's matrix equation solver `solve_right` which implements Gaussian elimination over $\mathbb{F}$.

Algorithm 4.4.1 $CFK^\infty(K)$ for knots of thickness ≤ 1

Input: Knot K with $th(K) \leq 1$

Output: Filtered homotopy representative of $CFK^\infty(K)$

1: Obtain $CFK_{\mathbb{F}[u,v]}/(uv)$ and $\widehat{HFK}$ via the `knot_floer_homology(complex=True)` method

2: Let $\{x_0, \ldots, x_{n-1}\}$ be the generators of $\widehat{HFK}$

3: Generate the matrix $H + V \in M_n(\mathbb{F}[U])$ of horizontal and vertical arrows from $CFK_{\mathbb{F}[u,v]}/(uv)$

4: Initiate a zero $n \times n$ matrix D_{var} and populate it
  for $i, j \in \{0, \ldots, n-1\}$ **do**
    if (i, j) satisfies (D1) and (D2) **then** set $[D_{var}]_{i,j} = U^k a_{i,j}$, where k is as in (D3)
    end if
  end for

5: Generate the matrix equation $Aa = b$

- Obtain a set of expressions E from the $\mathbb{F}\langle\{a_{i,j}\}\rangle$ coefficients of non-zero entries of the matrix $(H + V + D_{var})^2$
- Let A be the matrix with each row consisting of the $\mathbb{F}$ coefficients of the $a_{i,j}$ for an entry in E
- Let b be the vector of constant terms for each element in E
- Let a be the vector of unknown variables $a_{i,j}$

6: Find a solution a_0 via `solve_right`

7: Get a matrix $D_0 = D_{var}(a_0)$

8: Construct a chain complex $\mathcal{C}_0 = (C, d_0 = H + V + D_0)$, where $C \cong \widehat{HFK} \otimes \mathbb{F}[U, U^{-1}]$.

9: **return** $\mathcal{C}_0$

4.8 Finiteness of non-integral non-characterizing slopes: an overview

As an application of Algorithm 4.4.1, we investigate the set of characterizing slopes for knots in S^3. A Dehn surgery slope is said to be characterizing for a knot K if the orientation-preserving homeomorphism type of its p/q-Dehn surgery $S^3_K(p/q)$ determines K up to isotopy. That is, if there is some knot K' such that $S^3_{K'}(p/q) \cong S^3_K(p/q)$ via an orientation-preserving homeomorphism, then $K' = K$. Baker and Motegi asked whether a non-integral slope p/q is characterizing for a hyperbolic knot when $|p| + |q|$ is sufficiently large (Baker et Motegi, 2018, Question 5.6). This naturally leads to the question of whether the same holds for any knot in S^3.

Conjecture 4.8.1. (McCoy, 2025, Conjecture 1.1) *Let K be a knot in S^3. Then all but finitely many non-integral slopes are characterizing for K.*

Conjecture 4.8.1 has been shown to hold for thickness-zero knots, L-space knots (McCoy, 2025, Corollary 1.4) and composite knots (Sorya, 2024, Theorem 2). In this paper, we restrict our attention to prime knots of thickness one and two, and show the conjecture to be true for the vast majority of prime knots with at most 17 crossings.

Theorem 4.1.2. *Out of the 9 755 329 prime knots with at most 17 crossings, at least 95.79% admit only finitely many non-integral non-characterizing Dehn surgeries.*

4.8.1 Property Spli**F**f

A key result towards Theorem 4.1.2 is a sufficient condition on the knot Floer complex $CFK^\infty(K)$ formulated by McCoy, which guarantees that the conjecture holds for a given knot K. Let $C_{\{i\geq 0 \vee j \geq k\}}$ be the quotient complex of $CFK^\infty(K)$ represented by homogenous elements with $\mathbb{Z} \oplus \mathbb{Z}$ filtration satisfying $i \geq 0$ or $j \geq k$, and denote its homology by A^+_k. Let $\mathbb{F}_d$ denote an $\mathbb{F}$ summand supported in grading d.

Definition 4.8.1. (McCoy, 2025, Definition 1.5) A knot K has *property Spli**F**f* if for all $k \in \mathbb{Z}$, the graded $\mathbb{F}[U]$-module A_k^+ admits a direct sum decomposition of the form

$$A_k^+ = A' \oplus \mathbb{F}_{d_1}^{n_1} \oplus \mathbb{F}_{d_2}^{n_2}, \tag{4.1}$$

where $n_1, n_1 \geq 0$, d_1 is odd, d_2 is even and the $\mathbb{F}[U]$-module A' does not contain a summand whose elements are all killed by the U-action.

Theorem 4.8.2. (McCoy, 2025, Theorem 1.2, Theorem 1.3) *Let K be a knot in S^3 such that both K and its mirror have property Spli**F**f. Then all but finitely many non-integral slopes are characterizing for K.*

Recall that A_k^+ admits a decomposition $A_k^+ \cong \mathcal{T}_{-2V_k} \oplus A_k^{red}$ for some integer $V_k \geq 0$, where $\mathcal{T}_d = \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$ and 1 has grading d. Since $\mathcal{T}_{-2V_k}$ contains elements that are not killed by U, and since there is an even grading shift $A_k^{red} \cong A_{-k}^{red}[-2k]$, showing that A_k^{red} decomposes as in (4.1) for all $k \geq 0$ is equivalent to saying that K has property Spli**F**f.

Theorem 4.1.2 is thus obtained by computing the complexes $A_k^{red}, k \geq 0$, for knots and their mirrors, and verifying whether they satisfy property Spli**F**f, i.e. they decompose as in (4.1).

4.8.2 Summary of results

4.8.2.1 Thickness-one knots

We applied Algorithm 4.4.1 to all knots obtained from SnapPy's `NonalternatingKnotExteriors` iterator for prime knots with up to 16 crossings and most knots in Regina's database (Burton, 2020) of prime knots with 17 crossings.

Combining the output of Algorithm 4.4.1 and McCoy's work on the structure of the modules A_k^+ of thickness-one knots (McCoy, 2025, Section 3.3), we determine whether property

Spli**F**f is satisfied for each of the 437 982 prime thickness-one knots with at most 16 crossings and their mirrors, and for 2 367 449 of the 2 516 641 prime thickness-one knots with 17 crossings and their mirrors. We found that 2 196 093 pairs of such knots and their mirrors have property Spli**F**f, thus verifying the conjecture for 87.9% of prime thickness-one knots with at most 17 crossings. In particular, Conjecture 4.8.1 is solved for all prime knots up to 11 crossings, and all but 6 prime knots with 12 crossings, listed in Table 4.1 along with their A_k^{red} module which fails to have property Spli**F**f.

Knot	k	A_k^{red}
12n67	0	$\mathbb{F}_0 \oplus \mathbb{F}_2^2$
m12n89	0	$\mathbb{F}_0 \oplus \mathbb{F}_2^2$
m12n134	0	$\mathbb{F}_0 \oplus \mathbb{F}_2^2$
m12n229	0	$\mathbb{F}_0 \oplus \mathbb{F}_2^2$
m12n244	1	$\mathbb{F}_2 \oplus \mathbb{F}_4$
m12n639	0	$\mathbb{F}_0 \oplus \mathbb{F}_2^2$

Tableau 4.1: Knots with 12 crossings for which Conjecture 4.8.1 remains unresolved

4.8.2.2 Thickness-two knots

We also extended the strategy of Algorithm 4.4.1 to thicker knots and check whether property Spli**F**f is satisfied for certain knots of thickness two. To do this, we first establish thickness-two analogues of McCoy's results on the structure of the modules A_k^+. In particular, Proposition 4.1.3 gives a condition on the knot Floer homology $\widehat{HFK}(K)$ of a thickness-two knot K that guarantees that it has property Spli**F**f.

We then apply the extended algorithm to all thickness-two knots with up to 16 crossings and certain thickness-two knots with 17 crossings. Table 4.2 provides a breakdown of the number of thickness-two knots up to 16 crossings according to whether both the knot and its mirror satisfy property Spli**F**f, or whether at least one of them does not.

Crossings	K and mK Spli**F**f	K or mK non-Spli**F**f
13	3	0
14	32	9
15	256	193
16	2058	2578

Tableau 4.2: Thickness-two knots up to 16 crossings and property Spli**F**f

For knots with 17 crossings, 1489 of the 1634 thickness-two knots for which we were able to compute the structure of the modules A_k^+ verified property Spli**F**f. The large number of complexes to generate prevented us from carrying out the computation for the remaining 49 675 thickness-two knots with 17 crossings. This computational limitation, along with the empirical observation that the proportion of knots satisfying property Spli**F**f decreases as the number of crossings increases, suggest that another strategy must be considered to solve Conjecture 4.8.1 for an arbitrary knot.

Combining all this with the fact that thickness-zero knots always have property Spli**F**f (McCoy, 2025, Proposition 1.6) and that all but 7 knots with at most 17 crossings have thickness at most two, we obtain the computational result stated as Theorem 4.1.2.

4.8.3 Organization towards Theorem 4.1.2

Sections 4.9 and 4.10 detail the theoretical results and computational methods required to establish Theorem 4.1.2. Their content is organized as follows. We first explain our strategy to compute A_k^{red} for knots of thickness one in Section 4.9. We then develop the case of thickness-two knots in Section 4.10. We analyze the structure of their modules A_k^+ in Subsection 4.10.1, and in Subsection 4.10.2, we prove Proposition 4.1.3. In Subsection

4.10.3, we explain how Algorithm 4.4.1 was extended to compute the modules A_k^{red} for certain thickness-two knots, and thus obtain the statement of Theorem 4.1.2.

4.9 Finiteness of non-integral non-characterizing slopes: thickness one

4.9.1 Computing A_k^+

Recall that $\mathcal{T}_d = \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$ where 1 has grading d. Let $\mathcal{T}_{d^-}|_{\leq d^+}$ denote the submodule of $\mathcal{T}_{d^-}$ generated by $U^{(d^- - d^+ + \epsilon)/2}$, i.e.

$$\mathcal{T}_{d^-}|_{\leq d^+} = \{0, 1, U^{-1}, U^{-2}, \dots, U^{(d^- - d^+ + \epsilon)/2}\},$$

where $\epsilon = 0$ if d^+ is even and 1 if d^+ is odd. If $d^+ < d^-$, then $\mathcal{T}_{d^-}|_{\leq d^+} = 0$. Otherwise, the element 1 has degree d^- and $U^{(d^- - d^+ + \epsilon)/2}$ has degree $d^+ - \epsilon$. In other words, it is the truncation of the tower $\mathcal{T} = \mathbb{F}[U, U^{-1}]$ with lowest degree d^- and highest degree $d^+ - \epsilon$.

Our main object of interest, the $\mathbb{F}[U]$-module A_k^+, is the homology group of the complex $C_{\{i \geq 0 \vee j \geq k\}}$, represented by homogeneous elements of $CFK^\infty(K)$ whose $\mathbb{Z} \oplus \mathbb{Z}$ filtration level (i, j) satisfies $i \geq 0$ or $j \geq k$. This complex has has infinitely many generators when seen as an $\mathbb{F}$-module, which makes it unpractical for computational manipulation. To address this, we consider instead the quotient complex $C_{\{i<0 \wedge j \geq k\}}$, represented by homogenous elements of $CFK^\infty(K)$ whose $\mathbb{Z} \oplus \mathbb{Z}$ filtration level (i, j) satisfies $i < 0$ and $j \geq k$. This has finitely many generators as an $\mathbb{F}$-module, so it is well suited for computational encoding. Its homology is related to A_k^+ by an $\mathbb{F}[U]$-module isomorphism (see the proof of (Gainullin, 2017, Lemma 29) or (Ni et Zhang, 2014, Lemma 3.2(i)))

$$H_*(C_{\{i<0 \wedge j \geq k\}}) \cong \mathcal{T}_{-2V_k}|_{\leq -2} \oplus A_k^{red}. \tag{4.2}$$

We do not know *a priori* which components of the $\mathbb{F}[U]$-module $H_*(C_{\{i<0 \wedge j \geq k\}})$ are mapped to A_k^{red} under this (non-canonical) isomorphism.

The following structural lemma will allow us to recover enough information about A_k^+ from $H_*(C_{\{i<0\wedge j\geq k\}})$ to conclude whether A_k^+ has property Spli**F**f.

Lemma 4.9.1. (McCoy, 2025, Lemma 3.14) *Let K be a knot of thickness one. Let ρ be an integer such that for all s, the group $\widehat{HFK}_d(K,s)$ is non-zero only for gradings $d\in\{s+\rho,s+\rho-1\}$. Then for all $k\geq 0$, there exist integers $a,b\geq 0$ such that A_k^+ takes the following form*

$$A_k^+ = \mathcal{T}_{\min(0,k+\rho-1\pm\epsilon)} \oplus \mathcal{T}_{2k}|_{\leq k+\rho-2\pm\eta} \oplus \mathbb{F}^a_{k+\rho-1} \oplus \mathbb{F}^b_{k+\rho-2},$$

where $\epsilon=0$ if $k+\rho-1$ is even and 1 otherwise, and $\eta=0$ if $k+\rho-2$ is even and 1 otherwise.

Corollary 4.9.2. *Let K be a knot of thickness one and ρ,ϵ,η be as in Lemma 4.9.1. Then*

$$H_*(C_{\{i<0\wedge j\geq k\}}) \cong \mathcal{T}_{\min(0,k+\rho-1\pm\epsilon)}|_{\leq -2} \oplus \mathcal{T}_{2k}|_{\leq k+\rho-2\pm\eta} \oplus \mathbb{F}^a_{k+\rho-1} \oplus \mathbb{F}^b_{k+\rho-2}$$

*for all $k\in\mathbb{Z}$ and K has property Spli**F**f if and only if $H_*(C_{\{i<0\wedge j\geq\rho-3\}})$ has property Spli**F**f.*

Proof. The isomorphism is a direct consequence of combining (4.2) and Lemma 4.9.1.

Next, we observe that if the even number among $k+\rho-1$ and $k+\rho-2$ is greater than zero, then a component of $H_*(C_{\{i<0\wedge j\geq k\}})$ is mapped by (4.2) into A_k^{red} unless it is supported in negative even degrees.

By (McCoy, 2025, Lemma 3.15), K may fail to have property Spli**F**f only if $\rho\geq 3$ and $A^+_{\rho-3}$ does not have property Spli**F**f. In this case, $k+\rho-1=2\rho-4$ and thus $k+\rho-2=2\rho-5$ are always greater than zero, so $\mathcal{T}_{\min(0,k+\rho-1\pm\epsilon)}|_{\leq -2}=\mathcal{T}_0|_{\leq -2}$ is trivial. Therefore, $H_*(C_{\{i<0\wedge j\geq\rho-3\}})\cong A^{red}_{\rho-3}$, and K has property Spli**F**f if and only if $H_*(C_{\{i<0\wedge j\geq\rho-3\}})$ has property Spli**F**f. ∎

4.9.2 Implementation in SageMath

The complex $C_{\{i<0 \wedge j \geq k\}}$ is generated in the following way. Recall that Algorithm 4.4.1 outputs a matrix for the differential of $CFK^\infty(K)$ in the basis given by that of $\widehat{HFK}(K)$. The basis for $C_{\{i<0 \wedge j \geq k\}}$ is given by $B = \{U^{-i}x \mid A(x) + i \geq k, i < 0, x \in \widehat{HFK}(K)\}$. We index the elements of B by $b_0, \ldots, b_{m-1}$. An element $b_l = U^{-i}x$ is implemented as an object with attributes recording the index $l \in \{0, \ldots, m-1\}$, the power $-i$ of U and the generator $x \in \widehat{HFK}(K)$.

We then construct the matrix $d \in M_m(\mathbb{F})$ of the differential of $C_{\{i<0 \wedge j \geq k\}}$ in this basis, according to the output of Algorithm 4.4.1. To obtain the homology group $H_*(C_{\{i<0 \wedge j \geq k\}})$, we use SageMath's built-in `kernel` and `image` methods. Next, we use SageMath's `basis` and `lift` methods to obtain representatives of the basis elements of $H_*(C_{\{i<0 \wedge j \geq k\}})$ in the coordinates $b_0, \ldots, b_{m-1}$. We then extract the Maslov index of the (homogeneous) element $\sum_{i \in I} U^{k_i} x_i$ corresponding to a representative $\sum_{j \in J} b_j$ via the associated object parameters.

Finally, to check for property Spli**F**f according to Corollary 4.9.2, we need to understand the $\mathbb{F}[U]$-module structure of $H_*(C_{\{i<0 \wedge j \geq \rho - 3\}})$. The latter may fail to have property Spli**F**f only if there are elements in both gradings $2\rho - 4$ and $2\rho - 6$. In this situation, we consider a subset $B' \subset B$ consisting of a representative for each element in grading $2\rho - 4$. We have that $H_*(C_{\{i<0 \wedge j \geq \rho-3\}})$, and thus K, has property Spli**F**f if and only if UB' is not entirely contained in the image of d. This condition is verified by iterating through the elements $b \in B'$, stopping if Ub is not in the image of d.

4.10 Finiteness of non-integral non-characterizing slopes: thickness two

4.10.1 Structure of A_k^+ for thickness-two knots

The aim of this section is to describe the general algebraic structure of the modules A_k^+ for knots of thickness two by establishing the following analogue of Lemma 4.9.1.

Lemma 4.10.1. *Let K be a knot of thickness two. Let ρ be an integer such that for all s, the group $\widehat{HFK}_d(K,s)$ is non-zero only for gradings $d \in \{s+\rho, s+\rho-1, s+\rho-2\}$. Then for all $k \geq 0$, there exist integers $r, a, b, c \geq 0$ such that A_k^+ takes the following form*

$$A_k^+ = \mathcal{T}_{\min(0,k+\rho-\eta\pm1)} \oplus \mathcal{T}_{2k}|_{\leq k+\rho-2\pm1} \oplus (\mathcal{T}_{k+\rho-3}|_{\leq k+\rho-1})^r \oplus \ \mathbb{F}^a_{k+\rho-1} \oplus \mathbb{F}^b_{k+\rho-2} \oplus \mathbb{F}^c_{k+\rho-3},$$

where $\eta = 1$ if $k+\rho$ is even and 2 if $k+\rho$ is odd.

Proof. The proof is modelled on the proofs of (McCoy, 2025, Lemma 3.14) and (Ozsváth et Szabó, 2003b, Theorem 1.4).

Denote by $C_{\{\mathcal{F}\}}$ the quotient of $CFK^\infty(K)$ represented by homogenous elements whose $\mathbb{Z}\oplus\mathbb{Z}$ filtration levels (i,j) satisfy the constraint $\mathcal{F}$.

For a $\mathbb{Z}$-graded module $M = \bigoplus_{s\in\mathbb{Z}} M_s$, let $M|_{\geq k} = \bigoplus_{s\geq k} M_s$, $M|_{\leq k} = \bigoplus_{s\leq k} M_s$ and $M|_k = M_k$.

We have a short exact sequence of complexes

$$0 \to C_{\{i\geq 0 \vee j\geq k\}} \to C_{\{i\geq 0\}} \oplus C_{\{j\geq k\}} \to C_{\{i\geq 0 \wedge j\geq k\}} \to 0.$$

By definition of ρ, elements of $C_{\{i\geq 0\wedge j\geq k\}}$ have degree at least $k+\rho-2$. Therefore, $H_s(C_{\{i\geq 0\wedge j\geq k\}}) = 0$ for all $s \leq k+\rho-3$ and the induced long exact sequence in homology gives isomorphisms

$$H_{s-1}(C_{\{i\geq 0\vee j\geq k\}}) \cong H_{s-1}(C_{\{i\geq 0\}}) \oplus H_{s-1}(C_{\{j\geq k\}})$$

for all $s-1 \leq k+\rho-4$. Thus, we obtain a commutative diagram

$$\begin{array}{ccc} H_*(C_{\{i\geq 0\vee j\geq k\}})|_{\leq k+\rho-4} & \xrightarrow{\cong} & (H_*(C_{\{i\geq 0\}}) \oplus H_*(C_{\{j\geq k\}}))|_{\leq k+\rho-4} \\ = \downarrow & & \downarrow \cong \\ A_k^+|_{\leq k+\rho-4} & \xrightarrow{\cong} & \mathcal{T}_0|_{\leq k+\rho-4} \oplus \mathcal{T}_{2k}|_{\leq k+\rho-4} \end{array}$$

In grading $k+\rho-3$, we have a surjection

$$A_k^+|_{k+\rho-3} \twoheadrightarrow \mathcal{T}_0|_{k+\rho-3} \oplus \mathcal{T}_{2k}|_{k+\rho-3} \tag{4.3}$$

Hence, $A_k^+|_{\leq k+\rho-3}$ surjects onto $\mathcal{T}_0|_{\leq k+\rho-3} \oplus \mathcal{T}_{2k}|_{\leq k+\rho-3}$. In particular, $\mathcal{T}_{-2V_k}|_{\leq k+\rho-3} \subset A_k^+|_{\leq k+\rho-3}$ maps onto $\mathcal{T}_0|_{\leq k+\rho-3}$. Indeed, if $\mathcal{T}_{-2V_k}|_{\leq k+\rho-3} \neq 0$, then its lowest-degree element must be mapped to the lowest-degree element of either $\mathcal{T}_0$ or $\mathcal{T}_{2k}$. Therefore, $-2V_k = 0$ or $-2V_k = 2k \geq 0$. But $V_k \geq 0$, so we must have $-2V_k = 0$.

It follows that A_k^{red} contains $\mathcal{T}_{2k}|_{\leq k+\rho-3}$. If (4.3) is not an isomorphism, then A_k^+ also contains a component $\mathbb{F}^c_{k+\rho-3}$ for some $c \geq 1$, or an element of degree $k+\rho-3$ that is the image by multiplication by U of an element of degree $k+\rho-1$.

Similarly, we can consider the short exact sequence

$$0 \to C_{\{i\leq -1\wedge j\leq k-1\}} \to CFK^\infty(K) \to C_{\{i\geq 0\vee j\geq k\}} \to 0.$$

By definition of ρ, elements of $C_{\{i\leq -1\wedge j\leq k-1\}}$ have degree at most $k+\rho-2$. Therefore, $H_s(C_{\{i\leq -1\wedge j\leq k-1\}}) = 0$ for all $s \geq k+\rho-1$ and the induced long exact sequence in homology gives isomorphisms

$$H_{s+1}(CFK^\infty(K)) \cong H_{s+1}(C_{\{i\geq 0\vee j\geq k\}})$$

for all $s+1 \geq k+\rho$. Thus, we obtain a commutative diagram

$$\begin{array}{ccc} H_*(CFK^\infty(K))|_{\geq k+\rho} & \xrightarrow{\cong} & H_*(C_{\{i\geq 0 \vee j \geq k\}})|_{\geq k+\rho} \\ \cong \big\downarrow & & \big\downarrow = \\ HF^\infty(S^3)|_{\geq k+\rho} \cong \mathcal{T}|_{\geq k+\rho} & \xrightarrow{\cong} & A_k^+|_{\geq k+\rho} \end{array}$$

whose bottom row implies that the elements of A_k^+ of degree at least $k+\rho$ are precisely those in the tower $\mathcal{T}_{-2V_k}$.

In grading $k+\rho-1$, we have an injection

$$\mathcal{T}|_{k+\rho-1} \hookrightarrow A_k^+|_{k+\rho-1}, \tag{4.4}$$

If (4.4) is not an isomorphism, then A_k^+ contains a component $\mathbb{F}^a_{k+\rho-1}$ for some $a \geq 1$, or an element of degree $k+\rho-1$ that is not killed by U.

Since the argument so far says nothing about elements in grading $k+\rho-2$, they may appear in an extra component $\mathbb{F}^b_{k+\rho-2}$ of A_k^{red} for some $b \geq 1$, or at an end of a truncated tower if $k+\rho$ is even.

Combining all this, we have that $\mathcal{T}_{-2V_k} \subset A_k^+$ is of the form

$$\begin{cases} \mathcal{T}_{\min(0,k+\rho-1\pm 1)} & \text{if } k+\rho \text{ is even,} \\ \mathcal{T}_{\min(0,k+\rho-2\pm 1)} & \text{if } k+\rho \text{ is odd,} \end{cases}$$

and $A_k^{red} \subset A_k^+$ is of the form

$$\begin{cases} \mathcal{T}_{2k}|_{\leq k+\rho-3\pm 1} \oplus (\mathcal{T}_{k+\rho-3}|_{\leq k+\rho-1})^r \oplus \ \mathbb{F}^a_{k+\rho-1} \oplus \mathbb{F}^b_{k+\rho-2} \oplus \mathbb{F}^c_{k+\rho-3} & \text{if } k+\rho \text{ is even,} \\ \mathcal{T}_{2k}|_{\leq k+\rho-2\pm 1} \oplus (\mathcal{T}_{k+\rho-3}|_{\leq k+\rho-1})^r \oplus \ \mathbb{F}^a_{k+\rho-1} \oplus \mathbb{F}^b_{k+\rho-2} \oplus \mathbb{F}^c_{k+\rho-3} & \text{if } k+\rho \text{ is odd,} \end{cases}$$

for some $r, a, c \geq 0$. □

4.10.2 Property Spli**F**f for thickness-two knots

Lemma 4.10.1 says that A_k^{red} is of the form

$$\mathcal{T}_{2k}|_{\leq k+\rho-2+\epsilon} \oplus (\mathcal{T}_{k+\rho-3}|_{\leq k+\rho-1})^r \oplus \ \mathbb{F}^a_{k+\rho-1} \oplus \mathbb{F}^b_{k+\rho-2} \oplus \mathbb{F}^c_{k+\rho-3},$$

where $\epsilon = \pm 1$. We now examine each possibility for ϵ, a, b, c and verify whether A_k^+ has property Spli**F**f, i.e. it admits a decomposition as in (4.1). Note that we can ignore r since $U^{-1} \in \mathcal{T}_{k+\rho-3}|_{\leq k+\rho-1}$ is not killed by the U-action. If both a and c are non-zero, then A_k^+ does not have property Spli**F**f. We may thus assume that at least one of a or c is zero.

First, suppose $k+\rho$ is odd. Elements of odd degree may only appear in $\mathbb{F}^b_{k+\rho-2}$, so we are interested only in the values of ϵ, a and c.

- If $a = c = 0$, then A_k^+ has property Spli**F**f.
- If $a \neq 0$ and $c = 0$, then A_k^+ does not have property Spli**F**f if and only if $\mathcal{T}_{2k}|_{\leq k+\rho-2+\epsilon}$ is generated by a unique element of degree $k+\rho-3$. This happens if and only if $k = \rho - 3$ and $\epsilon = -1$.
- If $a = 0$ and $c \neq 0$, then A_k^+ does not have property Spli**F**f if and only if $\mathcal{T}_{2k}|_{\leq k+\rho-2+\epsilon}$ is generated by a unique element of degree $k+\rho-1$. This happens if and only if $k = \rho - 1$ and $\epsilon = +1$.

Suppose now that $k+\rho$ is even. Elements of odd degree may only appear in one of $\mathbb{F}^a_{k+\rho-1}$ or $\mathbb{F}^c_{k+\rho-3}$, so we are interested only in the values of ϵ and b.

- If $b = 0$, then A_k^+ has property Spli**F**f.

- If $b \neq 0$ and $\epsilon = +1$, then A_k^+ has property Spli**F**f because the even elements of A_k^{red} that are not in $\mathbb{F}^b_{k+\rho-2}$ must appear in $\mathcal{T}_{2k}|_{\leq k+\rho-2}$, which has non-zero U-action or is supported in degree $k+\rho-2$ if it is non-trivial.

- If $b \neq 0$ and $\epsilon = -1$, then A_k^+ fails to have property Spli**F**f if and only if $\mathcal{T}_{2k}|_{\leq k+\rho-4}$ is supported only in degree $k+\rho-4$. This happens if and only if $k = \rho - 4$.

This is summarized in Table 4.3.

(a,c)	$k+\rho$ odd		$k+\rho$ even		
	$\epsilon = +1$	$\epsilon = -1$	$b = 0$	$\epsilon = +1, b \neq 0$	$\epsilon = -1, b \neq 0$
$(1,0)$	yes	yes iff $k \neq \rho - 3$	yes	yes	yes iff $k \neq \rho - 4$
$(0,1)$	yes iff $k \neq \rho - 1$	yes	yes	yes	yes iff $k \neq \rho - 4$
$(0,0)$	yes	yes	yes	yes	yes iff $k \neq \rho - 4$
$(1,1)$	no	no	no	no	no

Tableau 4.3: Structure of A_k^+ and satisfaction of property Spli**F**f for knots of thickness two

Using the maps (4.3) and (4.4), we can guarantee that A_k^+ has property Spli**F**f given certain conditions on $\widehat{HFK}(K)$.

Lemma 4.10.2. *Let K be a knot of thickness two. Let ρ be an integer such that for all s, the group $\widehat{HFK}_d(K,s)$ is non-zero only for gradings $d \in \{s+\rho, s+\rho-1, s+\rho-2\}$. Suppose $k \geq 0$.*

*(i) If $k+\rho$ is odd, then A_k^+ has property Spli**F**f if $\widehat{HFK}_{k+\rho}(K,k) = 0$.*

*(ii) If $k+\rho$ is odd and $k \neq \rho - 3$, then A_k^+ has property Spli**F**f if $\widehat{HFK}_{k+\rho-2}(K,k) = 0$.*

*(iii) If $k+\rho$ is even and $k \neq \rho - 4$, then A_k^+ has property Spli**F**f if at least one of the groups $\widehat{HFK}_{k+\rho}(K,k)$ or $\widehat{HFK}_{k+\rho-2}(K,k)$ is trivial.*

Proof. If $\widehat{HFK}_{k+\rho}(K,k) = 0$, then $C_{\{i \leq -1 \wedge j \leq k-1\}}$ does not contain elements of degree $k+\rho-2$, so neither does its homology group. Hence, the injection (4.4) is an isomorphim and A_k^{red} is of the form

$$\mathcal{T}_{2k}|_{\leq k+\rho-3} \oplus \mathbb{F}^b_{k+\rho-2} \oplus \mathbb{F}^c_{k+\rho-3}.$$

This corresponds to columns 3 to 6 of the $(0,1)$ row of Table 4.3. Hence, A_k^+ fails to have property Spli**F**f only if $k+\rho$ is even and $k = \rho - 4$.

If $\widehat{HFK}_{k+\rho-2}(K,k) = 0$, then $C_{\{i \geq 0 \wedge j \geq k\}}$ does not contain elements of degree $k+\rho-2$, so neither does its homology group. Hence, the surjection (4.3) is an isomorphim and A_k^{red} is of the form

$$\mathcal{T}_{2k}|_{\leq k+\rho-2\pm 1} \oplus \mathbb{F}^a_{k+\rho-1} \oplus \mathbb{F}^b_{k+\rho-2}.$$

This corresponds to the $(1,0)$ row of Table 4.3. Hence, A_k^+ fails to have property Spli**F**f only if $k = \rho - 3$ when $k+\rho$ is odd, and only if $k = \rho - 4$ when $k+\rho$ is even. □

We now turn to the proof of Proposition 4.1.3.

Proposition 4.1.3. *Let K be a knot of thickness two. Let ρ be an integer such that for all s, the knot Floer homology group $\widehat{HFK}_d(K,s)$ is non-zero only for gradings $d \in \{s+\rho, s+\rho-1, s+\rho-2\}$.*

*Suppose $\rho \in \{0,1,2\}$. If for each $k \geq 0$, at least one of the groups $\widehat{HFK}_{k+\rho}(K,k)$ or $\widehat{HFK}_{k+\rho-2}(K,k)$ is trivial, then K and its mirror both satisfy property Spli**F**f. Therefore, K admits only finitely many non-integral non-characterizing Dehn surgeries.*

We first show a slightly more general statement for K.

Lemma 4.10.3. *Let K be a knot of thickness two. Let ρ be an integer such that for all s, the knot Floer homology group $\widehat{HFK}_d(K,s)$ is non-zero only for gradings $d \in \{s+\rho, s+\rho-1, s+\rho-2\}$.*

*Suppose $\rho \leq 2$. If for each $k \geq 0$, at least one of the groups $\widehat{HFK}_{k+\rho}(K,k)$ or $\widehat{HFK}_{k+\rho-2}(K,k)$ is trivial, then K has property Spli**F**f.*

Proof. We have $\rho-3, \rho-4 < 0$, so $k \neq \rho-3, \rho-4$ for all $k \geq 0$. By Lemma 4.10.2, A_k^+ has property Spli**F**f for all $k \geq 0$. Hence, K has property Spli**F**f. □

Proof of Proposition 4.1.3. The statement for K follows from Lemma 4.10.3, so we need to show that mK also has property Spli**F**f. Recall the symmetry properties of knot Floer homology (Ozsváth et Szabó, 2004)

(S1) $\widehat{HFK}_d(K,s) \cong \widehat{HFK}_{-d}(mK,-s)$ and

(S2) $\widehat{HFK}_d(K,s) \cong \widehat{HFK}_{d-2s}(K,-s)$.

Let ρ_m denote the integer such that for all s, the group $\widehat{HFK}(mK,s)$ is non-zero only for gradings $d \in \{s+\rho_m, s+\rho_m-1, s+\rho_m-2\}$. By (S1), we have $\rho_m = 2-\rho \in \{0,1,2\}$. Further, by (S1) and (S2), we have isomorphisms $\widehat{HFK}_{k+\rho_m-2}(mK,k) \cong \widehat{HFK}_{k+\rho}(K,k)$ and $\widehat{HFK}_{k+\rho_m}(mK,k) \cong \widehat{HFK}_{k+\rho-2}(K,k)$.

Therefore, mK satisfies the hypotheses of Lemma 4.10.3 and thus has property Spli**F**f. It follows from Theorem 4.8.2 that any knot of thickness two that satisfies the assumptions of Proposition 4.1.3 verifies Conjecture 4.8.1. □

4.10.3 Computations for thickness-two knots

To verify if Conjecture 4.8.1 holds for a knot of thickness two, we need to compute the structure of its modules A_k^+ for all $k \geq 0$ that do not satisfy the conditions of Lemma 4.10.2. To achieve this, one may first compute $CFK^\infty(K)$, following the approach used for thickness-one knots. However, two main issues arise when dealing with knots of thickness greater than one.

4.10.3.1 Computing lifts

First, it is difficult in general to find a lift of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$, in the sense of Section 4.4. We can still exploit the computational effectiveness of solving linear systems, as was done in the case of thickness-one knots, to reduce the number of possibilities for diagonal arrows. Recall that we encode the unknown differential map acting on the underlying $\mathbb{F}[U, U^{-1}]$-module C of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$ as a matrix $d_{var} = H + V + D_{var}$. As in Algorithm 4.4.1, we obtain a system of equations E by setting $d_{var}^2 = 0$, but it may contain non-linear equations if Proposition 4.6.5 is not satisfied. By considering the maximal subsystem of linear equations E' of E, we obtain a matrix equation $Aa = b$ with an initial solution $a = a_0$ and whose set of solutions is $a_0 + \ker A$. If $E' \neq E$, we need to determine which elements of $a_0 + \ker A$ are solutions of the full system E. Indexing the elements of $a_0 + \ker A$ by $a_l, l = 0, \ldots, 2^{\dim \ker A} - 1$, we obtain maps $d_l = H + V + D_{var}(a_l) = d_{var}(a_l)$. If $d_l^2 = 0$, then a_l is a solution to E and the differential complex $\mathcal{C}_l = (C, d_l)$ is a lift of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$. Note that this approach is computationally manageable only when the dimension of $\ker A$ is relatively small, or when Proposition 4.6.5 is satisfied, in which case the set of lifts is $\{\mathcal{C}_l \mid a_l \in a_0 + \ker A\}$.

4.10.3.2 Equivalence of lifts

Second, the computed lifts may not be filtered chain homotopy equivalent to one another. For knots with up to 16 crossings, Hanselman showed that $CFK_{\mathbb{F}[u,v]}(K)$ splits as in Theorem 4.3.3 (Hanselman, 2023, Corollary 12.6; Hanselman, 2025); therefore, any lift of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$ is a genuine representative of the full knot Floer complex. It then suffices to verify property Spli**F**f for the modules A_k^+ of any lift obtained by the method described previously. This is done by using Lemma 4.10.2 and adapting the method described in Section 4.9.2 to thickness-two knots, according to Lemma 4.10.1.

For knots with at least 17 crossings, we may not have such an equivalence between lifts. However, for our application at hand, we are interested only in the modules A_k^+, which are the homology groups of quotients of $CFK^\infty(K)$. Our strategy thus consists in computing all possible lifts of $CFK_{\mathbb{F}[u,v]}(K)/(uv)$ by considering each element in the set $a_0 + \ker A$. We then check that the modules A_k^+ of each of these lifts – which may belong to different homotopy equivalence classes –, verify property Spli**F**f. Table 4.4 summarizes the results of our computation for knots with 17 crossings, carried our for knots and their mirrors whose maximal linear subsystems have kernel of dimension at most 12. For each dimension, the table indicates the number of knots with both the knot and its mirror satisfying property Spli**F**f, and the number with either the knot or its mirror not satisfying property Spli**F**f.

In all computed cases, every lift for a given knot yielded the same outcome for the presence or absence of property Spli**F**f. We did not verify whether the different lifts were chain homotopy equivalent, but inspection of the relevant groups A_k^+ for some of these knots revealed that they have the same graded $\mathbb{F}[U]$-module structure across all lifts. Therefore, for these cases, the quotient complex $CFK_{\mathbb{F}[u,v]}(K)/(uv)$ appears to fully determine the homology groups A_k^+. This observation suggests that this, or the stronger statement of Theorem 4.1.1, may hold not only for thickness-one knots, but also for thicker knots whose quotient complex $CFK_{\mathbb{F}[u,v]}(K)/(uv)$ is sufficiently simple. It remains unclear what precise algebraic conditions would convey this simplicity, or whether such conditions exist at all beyond thickness one.

$\dim \ker A$	K and mK Spli**F**f	K or mK non-Spli**F**f
0	498	0
2	174	6
4	155	20
6	153	21
8	117	39
10	95	26
12	135	31

Tableau 4.4: Thickness-two knots with 17 crossings and property Spli**F**f

CONCLUSION

Dans cette thèse, nous avons exploré la question de caractérisation des nœuds dans la sphère de dimension trois via leurs chirurgies de Dehn. Notre travail a apporté des contributions significatives à la compréhension de l'ensemble des pentes p/q caractérisantes d'un nœud. En établissant l'existence d'une borne inférieure sur le dénominateur $|q|$ garantissant qu'une pente p/q soit caractérisante, et en la construisant explicitement à partir de la géométrie des pièces JSJ et leurs recollements, nous révélons le rôle structurel de la longitude d'un nœud dans la détermination de sa classe d'isotopie. En effet, on obtient qu'en recollant le tore de chirurgie en parcourant la longitude assez de fois, l'information de la topologie du nœud initial est complètement préservée. Lorsque le nœud est hyperbolique, cette heuristique est quantifiée en faisant de l'âme du tore de chirurgie la géodésique la plus courte dans la nouvelle variété; le résultat général et sa démonstration font de l'âme de chirurgie une courbe distinguée de la variété résultante.

Nos résultats précisent les connaissances de l'ensemble exact de pentes caractérisantes d'un nœud. En particulier, nous avons établi l'ensemble exact de ces pentes pour une infinité de nœuds composés: ce sont les seuls exemples connus à ce jour ayant un ensemble de pentes non caractérisantes non vide. De plus, pour la vaste majorité des nœuds avec au plus 17 croisements, cet ensemble restreint aux pentes non entières est maintenant connu comme étant fini, grâce au développement dans cette thèse d'un algorithme calculant le complexe de Floer de nœud de nœuds d'épaisseur un. Outre l'étude de chirurgies caractérisantes, cet algorithme pourra être utile à l'extraction d'invariants de concordance (Hom et Wu, 2016; Ozsváth *et al.*, 2017) et à la construction potentielles de variétés exotiques (Levine *et al.*, 2023), menant à des perspectives vers une réfutation de la conjecture de Poincaré lisse en dimension quatre.

Plusieurs aspects de la caractérisation des nœuds via leurs chirurgies de Dehn restent ouverts. D'abord, un objectif naturel serait de construire explicitement une borne pour le numérateur $|p|$ réalisant la finitude des pentes non caractérisantes non entières, en exploitant les idées menant à la borne effective pour le dénominateur $|q|$. Ensuite, nous avons exposé les limitations de l'homologie de Heegaard Floer dans l'étude de l'effet du paramètre $|p|$. En effet, pour des nœuds ayant un complexe de Floer de nœud algébriquement compliqué, nous ne parvenons pas à obstruer la possibilité d'avoir une infinité de pentes demi-entières non caractérisantes. De plus, les pentes entières $p/1$ caractérisantes demeurent mystérieuses; les nœuds de Baker-Motegi fournissent des exemples pour lesquels tous les entiers ne sont pas caractérisants, suggérant que le méridien n'est pas aussi robuste que la longitude dans la détermination d'un nœud via chirurgie de Dehn. Ainsi, des stratégies alternatives doivent être explorées pour l'étude des pentes entières et demi-entières. La combinaison d'idées de topologie géométrique s'inspirant de nos travaux sur les nœuds composés, et de nouvelles structures comme celles émanant de la topologie de contact (Baldwin *et al.*, 2025), constitue une avenue d'exploration pour des travaux futurs.

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