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ADJOINT LINEAR SYSTEMS AND CANONICAL STRUCTURES ON FIBERED VARIETIES

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SYSTÈMES LINÉAIRES ADJOINTS ET STRUCTURES CANONIQUES SUR LES VARIÉTÉS MUNIES D’UNE FIBRATION

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CONTENTS

RÉSUMÉ

Dans cette thèse, on établit plusieurs nouveaux résultats en géométrie algébrique complexe. La thèse est divisée en cinq chapitres.

Dans le **Chapitre 1**, nous présentons une revue de la littérature sur les travaux de Xiao concernant la géographie des surfaces fibrées, l'application canonique des surfaces de type général, la théorie de l'annulation générique, les problèmes de génération globale, ainsi que quelques notions liées au programme minimal de Mori.

Le **Chapitre 2** explore en détail la méthode de la pente pour les surfaces fibrées. Elle est introduite par Xiao (108) pour démontrer sa célèbre inégalité pour une surface fibrée relativement minimale $f : S \to C$, avec $g(F) \geq 2$:

$$K_{S/C}^2 \geq 4\frac{g-1}{g}\deg f_*\omega_{S/C}.$$

Plus précisément, soit $f : S \longrightarrow C$ un morphisme surjectif à fibres connexes d'une surface projective lisse complexe S vers une courbe projective lisse C, avec fibre générale F. Dans notre article (9), nous développons une version plus générale de l'inégalité de la pente pour des données $(D, \mathcal{F})$, où D est un diviseur relativement effectif arbitraire sur S, et $\mathcal{F}$ est un sous-faisceau localement libre de $f_*\mathcal{O}_S(D)$. Nous analysons comment la spécialité de D, restreinte à la fibre générale, influence les résultats. De plus, nous calculons des exemples naturels et proposons des applications.

Le **Chapitre 3** porte sur la conjecture de Xiao concernant les surfaces canoniquement fibrées (Conjecture 1.2.5). L'auteur résout cette conjecture lorsque la base est une courbe elliptique (Théorème 3.1.2). Plus précisément, nous prouvons qu'il n'existe pas de surfaces de type général canoniquement fibrées $f : S \to C$, avec fibre générale F de genre $g(F) = 5$ et $g(C) = 1$. Dans la section 3.2, nous proposons une méthode pour résoudre la conjecture et formulons une nouvelle conjecture (Conjecture 3.2.5) pour surmonter une difficulté technique. Cette dernière a récemment été résolue dans un article en collaboration (10) avec Chen et Grieve.

Dans le **Chapitre 4**, nous utilisons des techniques issues de la théorie de l'annulation pour obtenir des résultats de génération globale. Nous montrons comment prouver la génération globale des systèmes linéaires adjoints sur des variétés irrégulières de manière inductive. Par exemple, nous prouvons que la conjecture

de Fujita 1.3.1 est valide pour les variétés irrégulières de dimension n avec un fibré anticanonique nef, en supposant qu'elle est vraie pour les variétés de dimension inférieure et sous des hypothèses modérées. Ces résultats proviennent d'un article (pré-publication) de l'auteur (7).

Enfin, dans le **dernier chapitre**, nous établissons certains résultats de positivité. Plus précisément, en appliquant la décomposition de Chen-Jiang, nous démontrons que la Conjecture 5.1.1 de non-annulation est vraie pour une paire lc (X, Δ), où X est une variété irrégulière, à condition qu'elle soit valide pour des variétés de dimension inférieure. De plus, nous étendons la décomposition de Catanese-Fujita-Kawamata au cas klt (X, Δ), ce qui conduit à l'existence de sections de $K_X + \Delta$ dans certaines situations. Ce travail est effectué dans notre pré-publication (8).

ABSTRACT

In this thesis, we prove several new results in complex algebraic geometry. The thesis is divided into five chapters.

In **Chapter 1**, we provide a literature review of Xiao's work on the geography of fibered surfaces, the canonical map of surfaces of general type, generic vanishing theory, global generation problems, and some terminology related to Mori's minimal model program.

Chapter 2 explores in detail the slope method for fibered surfaces introduced by Xiao (108), where he proved his celebrated inequality for a relatively minimal fibered surface $f : S \to C$, with $g(F) \geq 2$:

$$K^2_{S/C} \geq 4\frac{g-1}{g} \deg f_*\omega_{S/C}.$$

More precisely, let $f : S \longrightarrow C$ be a surjective morphism with connected fibers from a smooth complex projective surface S to a smooth complex projective curve C, with general fiber F. In our article (9), we develop a more general version of the slope inequality for datum $(D, \mathcal{F})$, where D is an arbitrary relatively effective divisor on S, and $\mathcal{F}$ is a locally free subsheaf of $f_*\mathcal{O}_S(D)$. We see how the speciality of D, restricted to the general fiber, plays a role in the results. Furthermore, we compute natural examples and provide applications.

Chapter 3 is about Xiao's Conjecture on canonically fibered surfaces (Conjecture 1.2.5). The author settles the conjecture when the base is an elliptic curve (Theorem 3.1.2). More precisely, we prove that there are no canonically fibered surfaces of general type $f : S \to C$ with general fiber F of $g(F) = 5$ and $g(C) = 1$. In Section 3.2, we provide an approach to settle the conjecture and present a new conjecture (Conjecture 3.2.5) to address a technical difficulty. This conjecture has recently been resolved in a joint work (10) with Chen and Grieve.

In **Chapter 4**, we use the techniques from vanishing theory to obtain some global generations results. We show how to prove the global generation of adjoint linear systems on irregular varieties inductively. For instance, we prove that Fujita's conjecture 1.3.1 holds for irregular varieties of dimension n with nef anti-canonical bundle, assuming it holds for lower-dimensional varieties and under mild conditions, the results are from the author's preprint article (7).

In the **last chapter**, we prove certain positivity results. more precisely, by applying the Chen-Jiang decomposition, we prove that the non-vanishing Conjecture 5.1.1 holds for an lc pair (X, Δ), where X is an irregular variety, provided it holds for lower-dimensional varieties. Furthermore, we extend the Catanese-Fujita-Kawamata decomposition to the klt case (X, Δ), which leads to the existence of sections of $K_X + \Delta$ in certain situations. This work is done in our preprint (8).

INTRODUCTION

Throughout this thesis, we work over the field of complex numbers $\mathbb{C}$.

The **first chapter** gives a detailed review of the literature, explaining the main ideas, summarizing earlier results, and pointing out open problems to prepare for the following chapters. Chapters 2, 3, 4, and 5 are written to be independent of one another, allowing each to be read and understood separately. While the overarching theme of the thesis ties these chapters together, each one addresses a distinct problem or topic and develops its own framework, results, and conclusions. Consequently, readers can engage with any of these chapters without requiring prior knowledge of the others.

In the **second chapter**, we explore slope inequalities for an arbitrary relatively effective divisor D on a surface S. Let $f : S \to C$ be a surjective morphism from a smooth complex projective surface S to a smooth complex projective curve C with connected fibers. We call the morphism f a fibration or a fibered surface. We consider the sheaf $\mathcal{E} = f_*\mathcal{O}_S(D)$, which is torsion free because C is a curve. Since a torsion free sheaf on curve is always a locally free sheaf, $\mathcal{E}$ is locally free and its rank is $h^0(F, D_{|_F})$, where F is a general fiber of f of genus $g(F) = g$.

The fibration f is called smooth if all its fibers are smooth, isotrivial if all its smooth fibers are isomorphic to one another, and locally trivial if it is both smooth and isotrivial. Let ω_S (respectively K_S) be the canonical sheaf (respectively a canonical divisor) of S and $\omega_{S/C} = \omega_S \otimes f^*\omega_C^\vee$ (respectively $K_{S/C} = K_S - f^*K_C$) be the relative canonical sheaf (respectively a relative canonical divisor), where ω_C (respectively K_C) is the canonical sheaf of C (respectively a canonical divisor). In particular, if $D = K_{S/C}$, then $\mathcal{E}$ is a nef vector bundle of rank g (41, Theorem 0.6) and degree:

$$\deg(\mathcal{E}) := \deg(f_*\omega_{S/C}) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_F).\chi(\mathcal{O}_C)$$

$$= \chi(\mathcal{O}_S) - (g-1)(b-1),$$

for $b := g(C)$. By the Leray spectral sequence, we note that

$$h^0(C, (f_*\omega_{S/C})^\vee) = h^0(C, \mathcal{R}^1 f_*\mathcal{O}_S) = q(S) - b,$$

where $q(S) := h^1(S, \mathcal{O}_S)$ is the irregularity of the surface S.

In (108), Xiao wrote a fundamental paper on fibered surfaces over curves. He discussed the geometry of fibrations where f is relatively minimal and $g(F) \geq 2$. He proved that if f is relatively minimal and not locally trivial, that is $\deg f_*\omega_{S/C} \neq 0$, then

$$K^2_{S/C} \geq 4\frac{g-1}{g}\deg f_*\omega_{S/C}.$$

The last result is a key ingredient in Pardini's proof (87) of the Severi conjecture (36), (95).

We call such an inequality a slope inequality for the relative canonical divisor. Recall that $K_{S/C}$ is a nef divisor (86, Theorem 1.4).

Independently, Cornalba and Harris (31) proved the above inequality for semi-stable fibrations (that is, fibrations where all the fibers are semi-stable curves in the sense of Deligne and Mumford). Later Stoppino (101) showed that a generalization of the Cornalba-Harris approach gives a full proof of the slope inequality in which all fibrations are treated by the same method. Also, we recall that Yuan and Zhang (115) gave a new approach to prove the slope inequality by giving a sense to the relative Noether inequality using Frobenius iteration techniques. Moreover, there has been interest in giving a bound related to other geometric invariants such as the relative irregularity and the unitary rank of the fibered surface $f : S \to C$. These points have been discussed in several papers, for instance in (79) and (97). Konno, in (70), described $K^2_{S/C}$ as a sum of two parts under some conditions on the fibration f. More precisely, the first part is related to $\deg f_*\omega_{S/C}$ and the second one is described by the Horikawa index (57).

Since we apply Fujita's decompositions in this chapter (see Example 2.4.4) we recall them.

Theorem 0.0.1 (First Fujita decomposition for fibered surfaces (41, Theorem 3.1)) *Let $f : S \to C$ be a fibration from a smooth complex projective surface S to a smooth projective curve C. Then*

$$f_*\omega_{S/C} = \mathcal{O}_C^{q(S)-b} \oplus \mathcal{N},$$

where $\mathcal{N}$ is a nef vector sub-bundle and $h^0(C, \mathcal{N}^\vee) = 0$.

We remark that in the conclusion of Theorem 0.0.1, the trivial part comes from the nonzero global sections of the dual of $f_*\omega_{S/C}$. In other words, it comes from $H^0(C, \mathcal{R}^1 f_*\mathcal{O}_S)$.

Theorem 0.0.2 (Second Fujita decomposition for fibered surface (21), (22), (23, Theorem 1.1), (42)) *Let f : $S \to C$ be a fibration as above. Then*

$$f_*\omega_{S/C} = \mathcal{A} \oplus \mathcal{U},$$

where $\mathcal{A}$ is an ample vector sub-bundle and $\mathcal{U}$ is a unitary flat vector sub-bundle.

In the situation of Theorem 0.0.2, we denote by u_f the rank of $\mathcal{U}$, and call it the *unitary rank* of the fibered surface $f : S \to C$. A proof of the second Fujita decomposition is given by Catanese and Dettweiler, (21), (22), (23). In Theorem 2.4.11 and Theorem 2.4.12, we discuss the first and the second Fujita decompositions for the adjoint cases.

Let $f : S \to C$ be a fibered surface, and let D be a relatively effective divisor on S with a general fiber F. Consider the datum $(D, \mathcal{F})$ where $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$. The goal is to give a lower bound for D^2 in terms of $\deg \mathcal{F}$, even if D is not positive, for instance not nef. We explain how the negative part of D appears in the lower bound of D^2. The method used is due to Xiao (108).

Now, we describe briefly the process of the main results. We start by defining the Miyaoka divisors $(N_i)_{1\le i\le k}$ of a datum $(D, \mathcal{F})$, where k is the length of the Harder-Narasimhan filtration $(\mathcal{F}_i)_{1\le i\le k}$ of $\mathcal{F}$ (Definition 2.2.4). We realize that this sequence of divisors can be divided into two sub-sequences.

The first is a sequence of special Miyaoka divisors, and the second is a sequence of nonspecial Miyaoka divisors. We thus define an important number, $\hat{n}_{(D,\mathcal{F})}$, which tells us the index of the last divisor in the first sequence (Proposition 2.2.10). It is natural to study the sequence of rational numbers

$$\left(\frac{d_i}{h^0(N_{i|_F}) - 1}\right)_{1\le i\le k},$$

where $d_i = N_i.F$. In Theorem 2.2.17, we give a uniform lower bound for this sequence.

Clearly, by Clifford's Theorem, the number 2 is a lower bound of the sub-sequence

$$\left(\frac{d_i}{h^0(N_{i|_F}) - 1}\right)_{1\le i\le \hat{n}_{(D,\mathcal{F})}},$$

and we prove that the number

$$\beta_D := 1 + \frac{g(F)}{h^0(F, D_{|_F}) - 1}$$

is a lower bound of the sub-sequence

$$\left(\frac{d_i}{h^0(N_{i|_F})-1}\right)_{\hat{n}_{(D,\mathcal{F})}+1\leq i\leq k},$$

for a fixed D, and any $\mathcal{F}\subseteq f_*\mathcal{O}_S(D)$. Moreover, we will see that this sub-sequence decreases. Following the Modified Xiao Lemma (Lemma 2.3.1), we observe that we need to find a constant $\alpha_{(D,\mathcal{F})}$ for the datum $(D,\mathcal{F})$ such that

$$d_i \geq \alpha_{(D,\mathcal{F})}(h^0(F,N_{i|_F})-1)\geq \alpha_{(D,\mathcal{F})}(r_i-1)$$

(here $r_i := \operatorname{rk}\mathcal{F}_i$) with the aim that $\deg\mathcal{F}$ appears. Thus, naturally Definition 2.3.6 follows. The main result of Chapter 2 is the following theorem.

Theorem 0.0.3 (= Theorem 2.3.9) *Let $f: S\to C$ be a fibered surface. Consider the datum $(D,\mathcal{F})$, where D is a relatively effective divisor and $\mathcal{F}\subseteq f_*\mathcal{O}_S(D)$ is a locally free sub-sheaf on C. Assume that $\mathcal{F}$ is not semi-stable. Here, we set:*

$$t_{(D,\mathcal{F})} := \begin{cases} \max\{i|\mu_i\geq 0\} & \text{if } \mu_1\geq 0\\ -\infty & \text{otherwise.}\end{cases}$$

(1). If $t_{(D,\mathcal{F})}=1$, then

$$D^2 \geq \frac{2d_1}{r_1}\deg\mathcal{F}_1 + 2\epsilon^*D.Z_1 - Z_1^2 \geq \frac{2d_1}{r_1}\deg\mathcal{F} + 2\epsilon^*D.Z_1 - Z_1^2.$$

(2). If $1< t_{(D,\mathcal{F})}\leq k$, then

$$\begin{aligned} D^2 &\geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}d_{t_{(D,\mathcal{F})}}}{d_{t_{(D,\mathcal{F})}}+\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}}\deg\mathcal{F}_{t_{(D,\mathcal{F})}} + 2\epsilon^*D.Z_{t_{(D,\mathcal{F})}} - Z^2_{t_{(D,\mathcal{F})}}\\ &\geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}d_{t_{(D,\mathcal{F})}}}{d_{t_{(D,\mathcal{F})}}+\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}}\deg\mathcal{F} + 2\epsilon^*D.Z_{t_{(D,\mathcal{F})}} - Z^2_{t_{(D,\mathcal{F})}}.\end{aligned}$$

(3). If $t_{(D,\mathcal{F})}=-\infty$, set:

$$C_{(D,\mathcal{F})} := \begin{cases} \frac{2\alpha_{(D,\mathcal{F})}d_k}{-\alpha_{(D,\mathcal{F})}+2\alpha_{(D,\mathcal{F})}r_k-d_k} & \text{if } \alpha_{(D,\mathcal{F})}-2\alpha_{(D,\mathcal{F})}r_k+2d_k\leq 0\\ 3\alpha_{(D,\mathcal{F})}+2d_k-2\alpha_{(D,\mathcal{F})}r_k & \text{otherwise.}\end{cases}$$

Then

$$D^2\geq C_{(D,\mathcal{F})}.\deg\mathcal{F}+2\epsilon^*D.Z_k - Z_k^2.$$

In addition, if $d_k = \alpha_{(D,\mathcal{F})}(r_k - 1)$*, then we have the following inequality which is independent of* $t_{(D,\mathcal{F})}$*:*

$$D^2 \geq \frac{2d_k}{r_k} \deg \mathcal{F} + 2\epsilon^* D.Z_k - Z_k^2.$$

We make some remarks about the notations introduced in the previous theorem.

The map $\epsilon : \widehat{S} \to S$ is an iterated blow-up constructed in Proposition 2.2.3. By definition, the divisor Z_i on $\widehat{S}$ is a *fixed part* of $\mathcal{F}_i, \forall i; 1 \leq i \leq k$ (Definition 2.2.4). The number $\mu_i := \mu(\mathcal{F}_i/\mathcal{F}_{i-1})$ is the slope of the quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ (Proposition 2.2.1) of the Harder-Narasimhan filtration of $\mathcal{F}$.

We interpret the result of Theorem 0.0.3 (= Theorem 2.3.9) as follows. If we consider the datum $(D, \mathcal{F})$, we estimate the lower bound of D^2 by a sum of two parts. The first one is related to the degree of the sub-bundle $\mathcal{F}_{t_{(D,\mathcal{F})}} \subseteq \mathcal{F}$ and the geometry of the Miyaoka divisors $(N_i)_{1 \leq i \leq t_{(D,\mathcal{F})}}$. The second part captures the negativity of D when D is not positive. The case where $\mathcal{F}$ is semi-stable is discussed in Remark 2.3.8. As an application, we apply the previous theorem to the datum $(D, f_*\mathcal{O}_S(D))$ when D is relatively nef to obtain Corollary 2.3.12.

This new result pertains to the way in which the constant $\alpha_{(D,f_*\mathcal{O}_S(D))}$, defined in Definition 2.3.6, arises in lower bounds for D^2. Moreover, the last paragraph in Corollary 2.3.12 extends the result in (102, Theorem 5), see Example 2.3.14 and Theorem 2.3.15. The form of Corollary 2.3.17 can be compared to (97, Theorem 3.20).

Among other results, our contribution is to give an explicit description of the constant α that arises there. This is achieved by the constant $\alpha_{(D,\mathcal{F})}$ (which is defined in Definition 2.3.6).

Recently, in (30, Theorem E), the authors proved a slope inequality for the datum $(D, f_*\mathcal{O}_X(D))$ where D is a relatively effective divisor on a variety X of dimension $n \geq 2$ equipped with a fibration $f : X \to C$ over a curve, under the assumption that D and $f_*\mathcal{O}_X(D)$ are both nef. If we apply their result to fibered surfaces, we can see that our result is sharper. More precisely, in (30, Theorem E), they proved that if $D_{|F}$ is nef, big, and nonspecial, then

$$D^2 \geq 2\frac{D.F}{D.F + 1} \deg f_*\mathcal{O}_S(D).$$

However, if we apply Corollary 2.3.17 for the datum $(D, f_*\mathcal{O}_S(D))$ where D and $f_*\mathcal{O}_S(D)$ are both nef, and $D_{|_F}$ is nonspecial, then we have

$$D^2 \geq \frac{2\alpha_{(D,f_*\mathcal{O}_S(D))}D.F}{D.F+\alpha_{(D,f_*\mathcal{O}_S(D))}} \deg f_*\mathcal{O}_S(D).$$

Since $\alpha_{(D,f_*\mathcal{O}_S(D))} > 1$ for $g(F) \neq 0$, we see that the constant $\frac{\alpha_{(D,f_*\mathcal{O}_S(D))}D.F}{D.F+\alpha_{(D,f_*\mathcal{O}_S(D))}}$ is strictly bigger than $\frac{D.F}{D.F+1}$ in general.

In the **third chapter**, we study the following conjecture by Xiao.

Conjecture 0.0.4 (= Conjecture 1.2.5 (111, Problem 6)) *There exists a positive integer N such that if $\phi_{|K_S|} : S \dashrightarrow C$ is a canonically fibered surface of general type with general fiber of genus $g(F) = 5$, then $P_g < N$.*

The conjecture has been solved in some cases, we refer to Section 1.2 for a literature review. We resolve the conjecture in the following case:

Theorem 0.0.5 (= Theorem 3.1.2) *There is no canonically fibered general type surface $f : S \to C$ with general fiber F of genus $g(F) = 5$ and $g(C) = 1$.*

For the remaining case, we conjecture the following statement, which automatically implies Xiao's conjecture.

Conjecture 0.0.6 (= Conjecture 3.2.5) *Let $f : S \to C$ be a canonically fibered surface with nonhyperelliptic general fiber F of genus $g(F) = 5$, and*

$$K_S = 8\Gamma + V + f^*D,$$

where $8\Gamma + V$ is the fixed part of K_S (Γ is a section, V is the vertical part), and f^*D is the moving part of K_S with $\deg D := P_g + g(C) - 1$. Then

1. *If $h^0(\mathcal{O}_F(4p)) = 1$, then the map $\phi : S \dashrightarrow \mathbb{P}_C(f_*(5\Gamma))$ defined by the linear system $|5\Gamma + f^*A|$ for a sufficiently ample divisor A is regular.*
2. *If $h^0(\mathcal{O}_F(4p)) = 2$, then the map $\phi : S \dashrightarrow \mathbb{P}_C(f_*(4\Gamma))$ defined by the linear system $|4\Gamma + f^*A|$ is regular.*

Remark 0.0.7 *In the joint work (10) with Chen and Grieve, we recently settle Conjecture 0.0.6 above. The following theorem then follows.*

Theorem 0.0.8 (= Theorem 3.2.9) *There is no canonically fibered general type surface $f : S \dashrightarrow C$ such that the general fiber F is a nonhyperelleptic genus 5 curve with*

$$K_S = 8\Gamma + V + f^*D$$

if $g(C) = 1$ or $P_g > 56$.

Remark 0.0.9 *Applying the log Miyaoka-Yau (82) for (K_S, Γ), we can improve the lower bound of P_g. In other words, we can prove that there is no such canonically fibered surface if $P_g > 50$. This is a joint work with Chen and Grieve (10).*

In the **fourth chapter**, we present some results on Fujita's Conjecture 0.0.10 below. Let D be a positive divisor on a smooth complex projective irregular variety X of dimension n. We shall use the Albanese map to prove that the global generation problem for the adjoint linear system $|K_X + cD|$, where $c \in \mathbb{N}^*$, can be reduced to lower-dimensional cases.

For motivation, let us recall Fujita's conjecture (44) on the global generation of adjoint linear systems.

Conjecture 0.0.10 (= Conjecture 1.3.1 (44, page 167)) *Let X be a smooth projective variety of dimension n and D an ample divisor on X. Then the linear system, $|K_X + mD|$ is basepoint-free $\forall m \geq n + 1$.*

This conjecture is trivial in dimension 1 and was proved by Reider (96, Theorem 1) in dimension 2. For $\dim X = 3$ or 4, it was proved by Ein-Lazarsfeld (37, Corollary 2*) and Kawamata (65, Theorem 4.1), respectively. Recently, the conjecture was settled in dimension 5 by Ye and Zhu, see (114, Page 3). In higher dimensions, there exist some partial results. For instance, Demailly in (32, Page 324) and (33, Theorem 0.2) established certain results using analytic techniques and Monge–Ampère equations, he showed that $2K_X + mD$ is very ample if $m \geq 2 + \binom{3n+1}{n}$ for every ample divisor D on X. In (1, Corollary 0.2), Angehrn and Siu achieved an important result by utilizing multiplier ideal sheaves, Nadel's vanishing theorem, and the Ohsawa–Takegoshi extension theorem. Specifically, they proved that $|K_X + mD|$ is basepoint-free if D

is ample and $m \geq \frac{n(n+1)}{2}$. Later, Heier (54, Theorem 1.4) improved this bound to $O(n^{\frac{4}{3}})$. Some logarithmic bounds were obtained by Ghidelli and Lacini (45, Theorem 1.1). Notably, Helmke (55) and (56, Page 3) proved Conjecture $0.0.10$ under stronger numerical conditions on D.

Recall that by the classical Castelnuovo-Mumford regularity theorem, if D is ample and globally generated, then Conjecture $0.0.10$ holds.

In Chapter 4, we mainly study Fujita's Conjecture for irregular varieties. We first mention that, in (88, Section 5), Pareschi and Popa, for instance, obtained some basepoint-freeness results on varieties whose Albanese morphism is finite. We present new inductive results on Fujita's Conjecture for irregular varieties. The main results are as follows:

Theorem 0.0.11 (= Theorem 4.1.5) *Let X be an irregular variety of dimension $n \geq 2$ with Albanese dimension $1 \leq \alpha(X) < n$. Let $X \xrightarrow{f} Z \xrightarrow{u} \mathrm{alb}(X) \subseteq \mathrm{Alb}(X)$ be the Stein factorization of the Albanese morphism alb, and let F be a general fiber of the morphism f. Let D be an ample divisor on X.*

If Conjecture $0.0.10$ holds in dimension $< n$, then $|K_X + mD + \mathrm{alb}^* p|$ has no basepoint supported on F for all $m \geq n - \alpha(X) + 1$ and for a general $p \in \mathrm{Pic}^0(\mathrm{Alb}(X))$.

Additionally, if the following condition is satisfied:

$(*)$ *There exists an integer r with $1 \leq r \leq \alpha(X)$ such that $|rD_{|_F}|$ is basepoint-free and $rD - K_X$ is nef and big.*

Then $|K_X + mD|$ has no basepoint supported on F for all $m \geq n + 1$.

We can include the case of X of maximal Albanese dimension in the theorem above. However, we decided to separate this case from the inductive case (in other words, from the case in which the image of the Albanese map of X is of smaller dimension than X). Theorem 4.3.1 covers the case of maximal Albanese dimension.

In some special situations, such as in the case of varieties with $-K_X$ nef, we can prove Fujita's Conjecture by induction under mild conditions. Recall that projective varieties with $-K_X$ nef are a larger class than Fano varieties. One of the methods to study these varieties is to analyze their Albanese map if they are irregular. In (18, Theorem 1.2), Cao proved that for an irregular variety X with $-K_X$ nef, the Albanese map $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is a locally trivial fibration. This result was conjectured in (35, Page 221), many authors contributed to solving the problem and proved partial results, for instance, in (77) and (116).

Further, Cao and Höring (19, Theorem 1.4) proved a decomposition structure theorem for varieties with $-K_X$ nef. They showed that the universal cover $\widetilde{X}$ of X decomposes as the following product:

$$\widetilde{X} \simeq \mathbb{C}^d \times \prod Y_j \times \prod S_k \times Z,$$

where Y_j are irreducible projective Calabi-Yau varieties, S_k are irreducible projective hyperkähler varieties, and Z is a projective rationally connected variety with $-K_Z$ nef. Now, we state the following theorem for these type of varieties under extra conditions.

Theorem 0.0.12 (= Theorem 4.1.10) *Let X be an irregular variety of dimension $n \geq 2$ with $-K_X$ nef. Let $\mathrm{alb} : X \to \mathrm{Alb}(X)$ be the Albanese map, and let D be an ample divisor on X. If Conjecture $0.0.10$ holds in dimension $< n$ and there exists an integer r with $1 \leq r \leq \alpha(X)$ such that $|rD_{|_F}|$ is basepoint-free for every fiber F of alb, then Conjecture $0.0.10$ holds for X.*

In the **fifth chapter**, we study the non-vanishing conjecture for irregular varieties, and the structure of the direct image sheaf of a log relative pluricanonical bundle. In (12, Theorem 1.2), the authors proved that if (X, B) is a klt pair and $f : X \to Y$ is a map onto a normal variety Y such that $K_X + B$ is f-big, then (X, B) has a relatively good minimal model. In other words, there exists a klt birational pair $(\widetilde{X}, \widetilde{B})$ of (X, B) such that $K_{\widetilde{X}} + \widetilde{B}$ is f-nef, and moreover, $K_{\widetilde{X}} + \widetilde{B}$ is f-semi-ample. In another significant higher-dimensional result, Fujino (39, Theorem 1.1) proved that if (X, B) is a klt pair and X has maximal Albanese dimension, then (X, B) has a good minimal model. In this result, it is noteworthy that the existence of a minimal model is trivial. In (13, Theorem 1.3), Birkar and Chen generalized the results in (12) and (39). They proved that if (X, B) is a klt pair and $f : X \to Y$ is a surjective morphism such that Y is of maximal Albanese dimension and $K_X + B$ is f-big, then (X, B) has a good minimal model. Later, in (59), the author generalized the work in (13) to the log canonical case by applying the canonical bundle formula (40) and its generalization (51, Theorem 2.1). Furthermore, the condition that $K_X + B$ be f-big was relaxed to the condition that $K_X + B$ have a good minimal model over Y, and more generally to $K_X + B$ being f-abundant. In the same article (59), the author also proved certain forms of semi-ampleness in the context of generalized polarized pairs, which were introduced in (15). In this chapter, we initiate a program for studying irregular varieties to advance Mori's minimal model theory in this context. We prove several theorems, some of which we state below.

Conjecture 0.0.13 (= Conjecture 5.1.1) *Let (X, Δ) be a lc pair. If $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ is Cartier pseudo-effective with $m \in \mathbb{N}$, then $\kappa(D) \geq 0$.*

Theorem 0.0.14 (= Theorem 5.1.2) *Let (X, Δ) be a klt pair such that $q(X) > 0$, and let $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ be a Cartier pseudo-effective divisor. If Conjecture 0.0.13 holds for lower-dimensional klt pairs, then it also holds for (X, Δ).*

Corollary 0.0.15 (= Corollary 5.1.3) *Let (X, Δ) be an lc pair such that $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ is a Cartier pseudo-effective divisor, with $q(X) > 0$. If Conjecture 0.0.13 holds for lower-dimensional varieties, then it also holds for (X, Δ).*

As another notable result, we generalize Fujita's decomposition to the klt case.

Theorem 0.0.16 (= Theorem 5.1.5) *Let $f : X \to Y$ be a surjective morphism, and let (X, Δ) be a klt pair such that $D \sim_{\mathbb{Q}} m(K_{X/Y} + \Delta)$ is Cartier. Then, for every positive integer N that is sufficiently large and divisible such that $f_*\mathcal{O}_X(ND) \neq 0$, the sheaf $f_*\mathcal{O}_X(ND)$ is torsion-free, it has a singular metric with semi-positive curvature, satisfies the minimal extension property, and admits a Catanese-Fujita-Kawamata decomposition*

$$f_*\mathcal{O}_X(ND) = \mathcal{A}_N \oplus \mathcal{U}_N,$$

where $\mathcal{A}_N$ is a generically ample sheaf and $\mathcal{U}_N$ is flat.

CHAPTER 1

LITERATURE REVIEW

1.1 Xiao's work on the geography of fibered surfaces

In this section, we review some of Xiao's work on geography of fibered surfaces. In (108), he addressed Severi's problem for fibered surfaces and proved the celebrated slope inequality.

Theorem 1.1.1 *(108, Theorem 2) Let $f : S \to C$ be a relatively minimal fibration, not locally trivial, with $g \geq 2$. Then*

$$12 \geq \frac{K_{S/C}^2}{\deg f_*\omega_{S/C}} \geq 4\frac{g-1}{g}. \tag{1.1}$$

Further:

1. $K_{S/C}^2 = 12 \deg f_*\omega_{S/C}$ *if every fiber of f is smooth and reduced.*
2. *If $K_{S/C}^2 = 4\frac{g-1}{g} \deg f_*\omega_{S/C}$, then either $f_*\omega_{S/C}$ is semi-stable, or f is hyperelliptic, and $f_*\omega_{S/C}$ is a regular ladder: there are an integer $m > 0$ and a total filtration:*

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq ... \subsetneq \mathcal{F}_{k-1} \subsetneq \mathcal{F}_k = f_*\omega_{S/C},$$

such that $\operatorname{rk} \mathcal{F}_i = i$, and

$$\mu_i(\mathcal{F}_i/\mathcal{F}_{i-1}) - \mu_{i+1}(\mathcal{F}_{i+1}/\mathcal{F}_i) = m$$

for $i = 1, \ldots, k-1$.

Furthermore, in the same article (108), the author conjectured that if the slope is minimal, then f is hyperelliptic. In other words, he proposed the following conjecture.

Conjecture 1.1.2 *(108, Conjecture 1) If $K_{S/C}^2 = 4\frac{g-1}{g} \deg f_*\omega_{S/C}$, then f is hyperelliptic.*

This last conjecture was proved in (68, Proposition 2.6). One of the powerful consequences of the slope inequality is that it provides an upper bound for the relative irregularity:

$$q(f) := q(S) - g(C)$$

of a fixed fibered surface $f : S \to C$. Recall that Xiao, in (108) proved the following (non-sharp) corollary.

Corollary 1.1.3 *(108, Corollary 3) If the fibration $f : S \to C$ is not trivial, then*

$$q(f) \leq \frac{5g+1}{6}. \tag{1.2}$$

We note that f is trivial if and only if $q(f) = g$. The previous inequality (1.2) is a direct consequence of the slope inequality (1.1). It is known that this upper bound in not sharp. Finding a sharp upper bound for the relative irregularity is currently an active area of research. We expect a bound similar to the one proved by Xiao in (109) for surfaces with a linear pencil, that is fibered surfaces with $C = \mathbb{CP}^1$:

Theorem 1.1.4 *(109, Theorem 1) If the fibration $f : S \to \mathbb{CP}^1$ is not trivial, then*

$$q(f) = q(S) \leq \frac{g+1}{2}. \tag{1.3}$$

Xiao asked whether the same upper bound (1.3) holds for any fibered surface $f : S \to C$. The answer is no, as proven in (91, Theorem 2). The modified version of Xiao's conjecture is stated as follows.

Conjecture 1.1.5 (Modified Xiao's conjecture (3, Conjecture 1.1)) *If $f : S \to C$ is not trivial with general fiber of genus g. Then:*

$$q(f) \leq \lceil \frac{g+1}{2} \rceil.$$

Let us summarize the main known results regarding this conjecture.

(1) First, we note that $\frac{g+1}{2}$ differs from $\lceil \frac{g+1}{2} \rceil$ only when g is even.

(2) As stated in Theorem 1.1.4, Xiao, in (109, Theorem 1), proved that $q(f) = q(S) \leq \frac{g+1}{2}$ when the base $C = \mathbb{CP}^1$.

(3) Serrano, in (99, Page 63), proved Conjecture 1.1.5 when f is isotrivial but not trivial.

(4) If f is non-isotrivial and the general fiber is either hyperelliptic or bielliptic, the same bound

$$q(f) \leq \frac{g+1}{2}$$

holds, as shown by Cai in (17, Theorem 0.1).

(5) In (3, Theorem 1.2), the authors tackled the general non-isotrivial case. They proved that if f is non-isotrivial, then

$$q(f) \leq g - c(f), \tag{1.4}$$

where $c(f)$ is the Clifford index of the general fiber F.

(6) In (38, Theorem 1.1), the authors improved upon the result in (3). They proved that if the general fiber F of the fibered surface $f : S \to C$ is a smooth plane curve of degree $d \geq 5$, then

$$q(f) \leq g - c(f) - 1.$$

(7) In private communication, Martin (80, Corollary 1.3) informed the author that he proved the last open case for $g = 5$, namely that $q(f) \leq 3$ for fibered surfaces by trigonal curves with $g = 5$.

In general, Konno, in (69), proved the sharpest known upper bound for the relative irregularity. His idea was to explore how positive the intersection number of $K_{S/C}$ and the fixed part of $f_*\omega_{S/C}$ is, and then improve the slope inequality. In other words, we have the following theorem:

Theorem 1.1.6 *(69, Proposition 2.8) Let $f : S \to C$ is a relatively minimal fibration of genus $g \geq 2$ which is not locally trivial. Then*

$$q(f) \leq g\frac{5g-2}{3(2g-1)}. \tag{1.5}$$

Our paper (9) points out that, if we fix any relatively effective divisor D, then understanding the intersection of D with the fixed part of $f_*\mathcal{O}_S(D)$ is crucial for obtaining a sharp slope inequality for the datum $(D, f_*\mathcal{O}_S(D))$.

One natural method to attack Conjecture 1.1.5 is to improve certain forms of the slope inequality. We believe that a good understanding of the fixed part Z of $f_*K_{S/C}$ (Definition 2.2.4 below) is crucial for making progress. In particular, we hope to gain a deeper understanding of the positive number $K_{S/C} \cdot Z$ when $K_{S/C}$ is relatively nef.

1.2 Canonical maps of general type surfaces

We are interested in the linear system $|mK_S|$ for a general type varieties and in particular for surfaces of general type. By the celebrated works of Bombieri (16, Page 449) (Sakai's article (98) for the log case), we understand the geometry of $|mK_S|$ for $m \geq 3$. More precisely, for minimal surfaces of general type S, Bombieri proved that the linear system $|mK_S|$ is birational for $m \geq 3$, except for some explicitly described cases.

For the bicanonical map, we have by the celebrated result of Xiao (106) the following theorem.

Theorem 1.2.1 *(106, Theorem 1) Let S be a minimal projective surface of general type. Then the bicanonical map $\phi_{|2K_S|}$ of S is generically finite if and only if $p_2(S) > 2$.*

We recall that Xiao's proof relies on the study of genus-2 fibrations over curves and on Horikawa's classification of possible degenerations. Another interesting proof was provided by Chen and Viehweg (28, Theorem 0.1), who applied the $\mathbb{Q}$-divisor method commonly used for problems in higher-dimensional birational geometry, particularly in the context of the minimal model program (MMP).

In (4), Beauville first studied the canonical map $|K_S|$ systematically.

Here we are mainly interested in the case when the image of the canonical map is a curve, in which case we say that S admits a canonical fibration or S is a canonically fibered surface or $|K_S|$ is composed with a pencil.

Let us recall the existing results. We recall that we are interested in canonically fibered general type surface. One interesting boundedness result is the following theorem by Beauville (4).

Theorem 1.2.2 *(4, Proposition 2.1) Let $\phi_{|K_S|} : S \dashrightarrow C$ be a canonically fibered general type surface with general fiber F. If the geometric genus P_g of S is very large, that is $P_g >> 0$, then $2 \leq g(F) \leq 5$.*

Remark 1.2.3 *The proof of the previous theorem is based on the celebrated Miyaoka-Yau inequality (83), (112), (113) (see (75), (76), and (82) for developments). We mention that, in that proof, the understanding of the positive number $K_S \cdot Z$, where Z is the fixed part of $|K_S|$ is overlooked.*

In general, we recall that it is straightforward to obtain $g(F) \leq 36$. In (107), Xiao proved the following:

Theorem 1.2.4 *(107, Page 251) For such surfaces, There are two possible situations:* $q(S) = g(C) = 1$ *or* $0 \leq q(S) \leq 2$ *and* C *is a projective line* $\mathbb{CP}^1$.

In (4, Example 2) Beauville constructs such unbounded families with $g(F) = 2, 3$, and recently in (78, Theorem 1.3), the author constructed an unbounded family of canonically fibered general type surface with $g(F) = 4$, we precise that this last example is for canonically fibered general type surfaces of $g(F) = 4$ over $\mathbb{CP}^1$

One interesting problem is the following boundness conjecture by Xiao (110).

Conjecture 1.2.5 *(110, Problem 6) There exists a positive integer* N *such that if* $\phi_{|K_S|} : S \dashrightarrow C$ *is a canonically fibered surface of general type with general fiber of genus* $g(F) = 5$, *then* $P_g < N$.

This last conjecture was proved by Sun (103) if the general fiber is hyperelliptic and by Chen if the general fiber is nonhyperelliptic and nontrigonal (29). More precisely, they proved the following theorems.

Theorem 1.2.6 *(103, Corollary 1) There is no canonically fibered general type surface* $\phi_{|K_S|} : S \dashrightarrow C$ *with hyperelliptic general fiber of* $g(F) = 5$ *if* $g(C) = 1$ *or* $P_g > 53 - 15q(S)$.

Theorem 1.2.7 *(29, Theorem 1.2) There is no canonically fibered general type surface* $\phi_{|K_S|} : S \dashrightarrow C$ *such that the general fiber* F *is a nontrigonal genus* 5 *curve if* $g(C) = 1$ *or* $P_g > 863$.

In Chapter 3, we give a full proof of Conjecture 1.2.5 if the base is an elliptic curve. Furthermore, we develop a method to completely resolve the problem. The crucial part of this method is in a joint article with Chen and Grieve (10).

1.3 Generic vanishing theory and global generation problems

Independently of Section 1.1 and Section 1.2, we are interested in effective global generation problems, such as Fujita's conjecture below.

Conjecture 1.3.1 *(44, Page 167) Let X be a smooth complex projective variety of dimension n, and let D be an ample divisor on X. Then, the linear system $|K_X + mD|$ is base point free $\forall m \geq n+1$.*

One natural method to attack Conjecture 1.3.1 is to use techniques from generic vanishing theory.

We briefly recall some basic definitions, we refer to (26), (46), (47), (89) and (90) for more details.

Definition 1.3.2 *(89, Definition 2.3) A coherent sheaf $\mathcal{F}$ on an abelian variety Y with $\dim Y = g$ satisfies* IT with index 0 *if*

$$H^i(Y, \mathcal{F} \otimes p) = 0, \ \ \forall p \in \operatorname{Pic}^0(Y), \ \ \forall i > 0.$$

Definition 1.3.3 *(89, Definition 3.5) Let X be an irregular variety and $\mathcal{F}$ be a coherent sheaf on X. We say that $\mathcal{F}$ is* continuously globally generated *if for any non-empty open subset $U \subseteq \operatorname{Pic}^0(X)$, the sum of evaluation maps:*

$$\bigoplus_{p \in U} H^0(X, \mathcal{F} \otimes p) \otimes p^\vee \to \mathcal{F}$$

is surjective.

We recall the Fourier-Mukai setting, we refer to Mukai (84) for more details. We denote by $\mathcal{P}$ the Poincaré line bundle on $Y \times \operatorname{Pic}^0(Y)$, and Y is an abelian variety as before. For any coherent sheaf $\mathcal{F}$ on Y, we can associate the sheaf $p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{P})$ on $\operatorname{Pic}^0(Y)$ where p_1 and p_2 are the natural projections on Y and $\operatorname{Pic}^0(Y)$, respectively. This correspondence gives a functor

$$\widehat{S} : \operatorname{Coh}(Y) \to \operatorname{Coh}(\operatorname{Pic}^0(Y)).$$

If we denote by $\operatorname{D}(Y)$ and $\operatorname{D}(\operatorname{Pic}^0(Y))$ the bounded derived categories of $\operatorname{Coh}(Y)$ and $\operatorname{Coh}(\operatorname{Pic}^0(Y))$, then the derived functor

$$\mathcal{R}\widehat{S} : \operatorname{D}(Y) \to \operatorname{D}(\operatorname{Pic}^0(Y))$$

is defined and called the Fourier-Mukai functor. Similarly, we consider

$$\mathcal{R}S : \operatorname{D}(\operatorname{Pic}^0(Y)) \to \operatorname{D}(Y)$$

in a similar way. According to the celebrated result of Mukai (84), the Fourier-Mukai functor induces an equivalence of categories between the two derived categories $\mathrm{D}(Y)$ and $\mathrm{D}(\mathrm{Pic}^0(Y))$. More precisely, we have

$$\mathcal{R}S \circ \mathcal{R}\widehat{S} \simeq (-1_Y)^*[-g]$$

and

$$\mathcal{R}\widehat{S} \circ \mathcal{R}S \simeq (-1_{\mathrm{Pic}^0(Y)})^*[-g]$$

where $[-g]$ is a shift operation for a complex g places to the right.

Now, we define the cohomology support locus (46).

Definition 1.3.4 *(46, Page 389) Let $\mathcal{F}$ be a coherent sheaf on an abelian variety Y. The set $V^i(\mathcal{F})$ is* the cohomology support locus *and defined as the following:*

$$V^i(\mathcal{F}) := \{p \in \mathrm{Pic}^0(Y) | H^i(Y, \mathcal{F} \otimes p) \neq 0\}.$$

The cohomology support locus are studied very carefully in (46) and (100). There the authors proved a foundational generic vanishing theorem.

Remark 1.3.5 *By base change, there is always the following inclusion Supp$(\mathcal{R}^i\widehat{S}(\mathcal{F})) \subseteq V^i(\mathcal{F})$.*

Definition 1.3.6 *(89, Definition 3.1) A coherent sheaf $\mathcal{F}$ on Y is called* M-regular *if*

$$\textit{codim}_{\mathrm{Pic}^0(Y)}(\textit{Supp}(\mathcal{R}^i\widehat{S}(\mathcal{F}))) > i, \ \ \forall i \geq 1.$$

A coherent sheaf $\mathcal{F}$ on Y is called a generic vanishing sheaf *or a* GV-sheaf *if its cohomology support locus $V^i(\mathcal{F})$ satisfies the following inequality*

$$\textit{codim}_{\mathrm{Pic}^0(Y)}(V^i(\mathcal{F})) \geq i, \ \ \forall i \geq 1.$$

We remark that M-regularity is achieved if in particular

$$\textit{codim}_{\mathrm{Pic}^0(Y)}(V^i(\mathcal{F})) > i, \ \ \forall i \geq 1.$$

Remark 1.3.7 *If $\mathcal{F}$ is an M-regular sheaf on an abelian variety Y, then $\mathcal{F}$ is a GV-sheaf. Furthermore, if $\mathcal{F}$ satisfies IT with index 0, then it is clearly M-regular. In the following proposition, we recall that the M-regular property is stronger than the notion of being continuously globally generated.*

Proposition 1.3.8 *(88, Proposition 2.13) Every M-regular coherent sheaf $\mathcal{F}$ on an abelian variety Y is continuously globally generated sheaf.*

Definition 1.3.9 *(26, Definition 2.2) A coherent sheaf $\mathcal{F}$ on an irregular variety X is said to have an* essential base point *at x if there exist a surjective map $\mathcal{F} \to \mathbb{C}(x)$ such that $\forall p \in \operatorname{Pic}^0(X)$, the induced map $H^0(X, \mathcal{F} \otimes p) \to H^0(X, \mathbb{C}(x))$ is zero.*

In (26), the authors proved the next proposition.

Proposition 1.3.10 *(26, Proposition 2.3) If $\mathcal{F}$ is a coherent sheaf on an abelian variety Y satisfying IT with index 0 condition, then $\mathcal{F}$ has no essential base points.*

We are interested in comparing Definition 1.3.3 and Definition 1.3.9, these are new observations.

Proposition 1.3.11 *Let $\mathcal{F}$ be a coherent sheaf on an irregular variety X. If $\mathcal{F}$ is continuously globally generated, then $\mathcal{F}$ has no essential base points.*

Proof. If $\mathcal{F}$ is continuously globally generated, then for any non-empty open subset $U \subseteq \operatorname{Pic}^0(X)$, the following sum of evaluation maps is surjective:

$$\bigoplus_{p \in U} H^0(X, \mathcal{F} \otimes p) \otimes p^\vee \to \mathcal{F}.$$

Now, fix a point $x \in X$ and suppose we are given a surjective map $\mathcal{F} \to \mathbb{C}(x)$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{p \in U} H^0(X, \mathcal{F} \otimes p) \otimes p^\vee & \longrightarrow & \mathcal{F} \\ & \searrow^{\phi} & \downarrow \\ & & \mathbb{C}(x). \end{array}$$

However, the induced map ϕ defined by composition is surjective and hence not zero. In particular, there exist $p \in U$ such that the map

$$H^0(X, \mathcal{F} \otimes p) \otimes p^\vee \to H^0(X, \mathbb{C}(x)) \simeq \mathbb{C}(x)$$

is not zero, and hence the map

$$H^0(X, \mathcal{F} \otimes p) \to \mathbb{C}(x)$$

is not zero. Thus $\mathcal{F}$ has no essential base points as claimed. □

Example 1.3.12 *It is clear that every ample line bundle L on an abelian variety Y satisfies IT with index 0 condition, and thus is continuously globally generated. Furthermore, by Proposition 1.3.10 we also see that L has no essential base points. On the other hand, the converse of Proposition 1.3.11 is not true. Indeed, for an irregular variety X, $\mathcal{O}_X \in \mathrm{Pic}^0(X)$ has no essential base points, however, it is not continuously globally generated.*

In the next proposition, we provide a sufficient condition for Definition 1.3.3 and Definition 1.3.9 to be equivalent in the case of line bundles.

Proposition 1.3.13 *Let X be an irregular variety, and let D be a divisor on X such that $h^0(X, \mathcal{O}_X(D) \otimes p)$ is constant for all $p \in \mathrm{Pic}^0(X)$. Then, $\mathcal{O}_X(D)$ has no essential basepoints if and only if $\mathcal{O}_X(D)$ is continuously globally generated.*

Proof. Assume that $\mathcal{O}_X(D)$ has no essential basepoints, then for all $x \in X$ and for any surjective map

$$\psi : \mathcal{O}_X(D) \to \mathbb{C}(x),$$

we can find $p_x \in \mathrm{Pic}^0(X)$ (Definition 1.3.9) such that the induced map

$$H^0(X, \mathcal{O}_X(D) \otimes p_x) \to H^0(X, \mathbb{C}(x))$$

is surjective. Thus

$$h^0(X, \mathrm{Ker}\,\psi \otimes p_x) = h^0(X, \mathcal{O}_X(D) \otimes p_x) - 1.$$

By the upper semi-continuity of $h^0(X, \mathrm{Ker}\,\psi \otimes p)$ as p varies in $\mathrm{Pic}^0(X)$, we deduce that

$$h^0(X, \mathrm{Ker}\,\psi \otimes p_x) = h^0(X, \mathrm{Ker}\,\psi \otimes p)$$

for general $p \in \mathrm{Pic}^0(X)$. By assumption, $h^0(X, \mathcal{O}_X(D) \otimes p)$ is constant for all $p \in \mathrm{Pic}^0(X)$. Therefore, we conclude that the map

$$H^0(X, \mathcal{O}_X(D) \otimes p) \to H^0(X, \mathbb{C}(x)) \simeq \mathbb{C}(x) \tag{1.6}$$

is surjective for a general $p \in \mathrm{Pic}^0(X)$.

Now, using the surjectivity of the map (1.6) for a general $p \in \mathrm{Pic}^0(X)$, we will prove that D is continuously globally generated. In other words, we will prove for $x \in X$ and for all $U \subseteq \mathrm{Pic}^0(X)$ nonempty open subset, the sum of evaluation maps

$$\bigoplus_{p \in U} H^0(X, \mathcal{O}_X(D) \otimes p) \otimes p^\vee \to \mathcal{O}_X(D)_{|x} \simeq \mathbb{C}(x) \tag{1.7}$$

is surjective. Indeed, the map (1.6) is surjective for a general $p \in \mathrm{Pic}^0(X)$, in particular for some $p_U \in U$. Thus, twisting the map (1.6) by $p_U^\vee$ and then the following map

$$H^0(X, \mathcal{O}_X(D) \otimes p_U) \otimes p_U^\vee \to \mathbb{C}(x)$$

is surjective. Finally, we conclude that the sum of evaluation maps (1.7) is surjective as desired. The converse is proved in Proposition 1.3.11. □

Example 1.3.14 *As in Example $1.3.12$, if L is an ample line bundle on an abelian variety Y, then $h^0(Y, L \otimes p)$ is constant for all $p \in \mathrm{Pic}^0(Y)$ since $\chi(Y, L \otimes p)$ is constant for all $p \in \mathrm{Pic}^0(Y)$. Moreover, it has no essential basepoints and is continuously globally generated. Hence, Proposition 1.3.13 is not empty.*

However, for an irregular variety X, it can happen that there exists a divisor D such that $h^0(X, \mathcal{O}_X(D) \otimes p)$ is constant for all $p \in \mathrm{Pic}^0(X)$, but $\mathcal{O}_X(D)$ is not continuously globally generated. Indeed, for instance, take $X = \mathbb{CP}^1 \times C$ with C an elliptic curve, and let p_2 be the second projection. Let $D = K_X + B$, where B is a section of p_2. Thus, B is p_2-ample and hence, by the relative Kodaira vanishing theorem, $\mathcal{R}^1(p_2)_*(\mathcal{O}_X(D)) = 0$. Also, $(p_2)_*(\mathcal{O}_X(D)) = 0$ because

$$h^0(\mathbb{CP}^1, \mathcal{O}_X(D)_{|_{\mathbb{CP}^1}}) = h^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(-1)) = 0.$$

It follows that $h^i(X, \mathcal{O}_X(D) \otimes p) = 0$, for all i with $0 \leq i \leq 2$ and for all $p \in \mathrm{Pic}^0(X)$. In particular, $\mathcal{O}_X(D)$ is not continuously globally generated.

We recall the definition of Albanese dimension.

Definition 1.3.15 *We define the Albanese dimension $\alpha(X)$ of an irregular variety X by*

$$\alpha(X) := \dim \mathrm{alb}(X).$$

Here, $\mathrm{alb}(X)$ is the image of the Albanese map $\mathrm{alb} : X \to \mathrm{alb}(X) \subseteq \mathrm{Alb}(X)$, where $\mathrm{Alb}(X)$ is the Albanese variety.

In Chapter 4, we denote the Stein factorization $X \xrightarrow{f} Z \xrightarrow{u} \mathrm{alb}(X)$ of alb, where f is a surjective morphism f with connected fibers. We denote a general fiber of f by F. Thus, we have the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & \underset{\mathrm{alb}}{\searrow} & \downarrow u \\ & & \mathrm{alb}(X) \end{array} \qquad (1.8)$$

where $u : Z \to \mathrm{alb}(X)$ is a finite morphism.

1.4 Terminology on the Mori minimal model program

We refer to (12) and (67) for the notation and terminology introduced below.

Fix $f : X \to Y$ a proper morphism of normal projective varieties. We say that X is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier. We say that a $\mathbb{Q}$-divisor D is $\mathbb{Q}$-Cartier if some integral multiple is Cartier. We say that two $\mathbb{Q}$-divisors D_1, D_2 on X are $\mathbb{Q}$-linearly equivalent (over Y), that is $D_1 \sim_{\mathbb{Q}} D_2$ ($D_1 \sim_{\mathbb{Q},f} D_2$) if their difference is an $\mathbb{Q}$-linear combination of principal divisors (and an $\mathbb{Q}$-Cartier divisor pulled back from Y). D_1 and D_2 are numerically equivalent (over Y), denoted $D_1 \equiv D_2$ ($D_1 \equiv_f D_2$), if their difference is an $\mathbb{Q}$-Cartier divisor such that $(D_1 - D_2).C = 0$ for any curve C (contracted by f). A $\mathbb{Q}$-Cartier divisor D is semi-ample over Y, or f-semi-ample, if D is a $\mathbb{Q}$-linearly equivalent to the pullback of an ample $\mathbb{Q}$-divisor over Y. Equivalently, $f^*f_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective for $m \gg 0$. We say that $\mathbb{Q}$-divisor D is big over Y, or f-big, if

$$\text{lim sup}_{m\to\infty} \frac{h^0(F, \mathcal{O}_F(\llcorner mD \lrcorner))}{m^{\dim F}} > 0$$

for the fibre F over any generic point of Y. Equivalently D is f-big if $D \sim_{\mathbb{Q},f} M + E$, where M is ample and E effective. We define the Kodaira dimension of a $\mathbb{Q}$-divisor $\mathbb{Q}$-Cartier by

$$\kappa(D) := \kappa(mD)$$

for some natural number m such that mD is Cartier.

By a pair (X, Δ), we mean a normal variety X associated with $\mathbb{Q}$-divisor $\Delta := \sum a_i \Delta_i$, which is a formal sum of distinct prime divisors Δ_i with $a_i \in [0, 1]$ such that $K_X + \Delta$ is $\mathbb{Q}$-cartier.

By a klt polarized pair $(X, \Delta + L)$, we mean a klt pair (X, Δ) and L is a nef $\mathbb{Q}$-divisor (see (14, Paragraph 2.2)).

We recall the definition of singularities of pairs, we refer to (67) for more details.

Definition 1.4.1 *(67, Definition 2.3.4) Let (X, Δ) be a pair, we say that:*

- (X, Δ) *is terminal if* $a(E, X, \Delta) > 0$ *for every exceptional divisor* E.
- (X, Δ) *is canonical if* $a(E, X, \Delta) \geq 0$ *for every exceptional divisor* E.
- (X, Δ) *is Kawamata log terminal (or. klt) if* $a(E, X, \Delta) > -1$ *for every divisor* E.
- (X, Δ) *is purely log terminal (or. plt) if* $a(E, X, \Delta) > -1$ *for every exceptional divisor* E.
- (X, Δ) *is divisorial log terminal (or. dlt) if* $a(E, X, \Delta) > -1$ *if center*$_X(E) \subset$ *non-snc*(X, Δ).
- (X, Δ) *is log canonical (or. lc) if* $a(E, X, \Delta) \geq -1$ *for every divisor* E.

Here by $a(E, X, \Delta)$, we mean a log discrepancy of E.

Definition 1.4.2 *(13, Page 210) Let $f : X \to Y$ be a proper morphism of normal projective varieties and D be a $\mathbb{Q}$-cartier divisor on X. A normal variety Z with the following commutative diagram:*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & Z \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\ & & Y \end{array} \tag{1.9}$$

is called a minimal model of D over Y if:

- *The inverse of ϕ does not contract any divisor.*

- *$D_Z := \phi_* D$ is g-nef.*
- *There exist a common resolution $p : W \to X$ and $h : W \to Z$ such that $E := p^* D - h^* D_Z$ is effective and Supp $p_* E$ contains all the exceptional divisors of ϕ.*

Further, we say that Z is a relatively good minimal model if D_Z is g-semi-ample.

We recall certain forms of the canonical bundle formula, originally introduced by Fujino and Mori ((40)).

Theorem 1.4.3 *[(51, Theorem 2.1), (59, Theorem 3.1)] Let $f : (X, \Delta) \to Y$ be a projective morphism from an lc pair to a normal variety Y, such that $N(K_X + \Delta)$ is Cartier and $f_* \mathcal{O}_X(N(K_X + \Delta)) \neq 0$ for some integer $N > 0$. Then, there exists a commutative diagram:*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\psi} & X \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{\phi} & Y \end{array}$$

with the following properties:

(1) ψ is a birational morphism, $\widetilde{X}$ is smooth and $\widetilde{f}$ is an algebraic fiber space.

(2) There exist a $\mathbb{Q}$-divisor $\widetilde{\Delta}$ such that $(\widetilde{X}, \widetilde{\Delta})$ is a $\mathbb{Q}$-factorial dlt pair, and

$$\psi_* \mathcal{O}_{\widetilde{X}}(N(K_{\widetilde{X}} + \widetilde{\Delta})) = N(K_X + \Delta).$$

(3) There exist a $\mathbb{Q}$-factorial dlt polarized pair $(\widetilde{Y}, \Delta_{\widetilde{Y}} + L_{\widetilde{Y}})$ pair such that $K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}$ is big $/Y$.

(4) There exist an effective $\mathbb{Q}$- divisor R on $\widetilde{X}$ such that $\widetilde{f}_* \mathcal{O}_{\widetilde{X}}(mR) = \mathcal{O}_{\widetilde{Y}}$ for all $m \geq 0$,

$$K_{\widetilde{X}} + \widetilde{\Delta} \sim_{\mathbb{Q}} \widetilde{f}^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}) + R,$$

and

$$\widetilde{f}_* \mathcal{O}_{\widetilde{X}}(N(K_{\widetilde{X}} + \widetilde{\Delta})) = N(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}).$$

(5) each component of $\lfloor \Delta_{\widetilde{Y}} \rfloor$ is dominated by a vertical component of $\lfloor \widetilde{\Delta} \rfloor$.

CHAPTER 2

SLOPE INEQUALITY FOR AN ARBITRARY DIVISOR

In this Chapter, $f : S \to C$ is a fibered surface from a smooth complex projective surface S to a smooth complex projective curve C, and D is a relatively effective divisor on S. Also, throughout this chapter, unless stated otherwise, all locally free sheaves are assumed to be nonzero.

2.1 Rational map to a projective bundle

Let $f : S \to C$ be a fibered surface and D be a relatively effective divisor on S. Let $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$ be a locally free sub-sheaf of rank $r_{\mathcal{F}}$. There exist always the following commutative diagrams:

$$\begin{array}{ccc} S & \overset{\psi}{\dashrightarrow} & \mathbb{P}_C(f_*\mathcal{O}_S(D)) \\ & \overset{f}{\searrow} & \downarrow \pi \\ & & C \end{array}$$

and

$$\begin{array}{ccc} S & \overset{\psi_{\mathcal{F}}}{\dashrightarrow} & \mathbb{P}_C(\mathcal{F}) \\ & \overset{f}{\searrow} & \downarrow \pi_{\mathcal{F}} \\ & & C. \end{array}$$

In the above, $\mathbb{P}_C(f_*\mathcal{O}_S(D))$ (respectively $\mathbb{P}_C(\mathcal{F})$) is the projective bundle of one dimensional quotients (Grothendieck's notations) of $f_*\mathcal{O}_S(D)$ (respectively of $\mathcal{F}$) and π (respectively $\pi_{\mathcal{F}}$) is the projective morphism from $\mathbb{P}_C(f_*\mathcal{O}_S(D))$ to C (respectively from $\mathbb{P}_C(\mathcal{F})$ to C). The maps ψ and $\psi_{\mathcal{F}}$ are rational and defined by the following evaluation maps:

$$f^*f_*\mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D),$$

(respectively)

$$f^*\mathcal{F} \longrightarrow \mathcal{O}_S(D).$$

Remark 2.1.1 • *If we assume D is f-globally generated in the sense that*

$$f^*f_*\mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D)$$

is surjective, then ψ is a morphism by (53, Proposition II.7.12).

- *If the map*

$$f^*\mathcal{F} \longrightarrow \mathcal{O}_S(D)$$

is surjective, then $\psi_{\mathcal{F}}$ is also a morphism by (53, Proposition II.7.12).

Take a sufficiently very ample divisor A on C such that $f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)$ is a very ample vector bundle. Then the rank of $f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)$ is

$$r = H^0(F, D_{|_F}),$$

and

$$\deg(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) = \deg f_*\mathcal{O}_S(D) + r\deg(A).$$

Now, $\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$ and $\mathbb{P}_C(f_*\mathcal{O}_S(D))$ are isomorphic by an isomorphism s, ((53, Lemma 7.9)).

The rational map

$$\phi : S \dashrightarrow \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)),$$

defined by

$$f^*(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \longrightarrow \mathcal{O}_S(D) \otimes f^*\mathcal{O}(A)$$

is the rational map given by the linear system $|D + f^*A|$, and restricted to the general fiber F of f, $\phi_{|_F}$ is the map defined by $|D_{|_F}|$.

The line bundle $\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D)\otimes\mathcal{O}(A))}(1)$ on $\mathbb{P}_C(f_*\mathcal{O}_S(D)\otimes\mathcal{O}(A))$ is very ample. Then it gives an embedding of this last projective bundle to a projective space $\mathbb{CP}^N$ for some $N > 0$. We thus have the following commutative diagram:

$$\begin{array}{ccc} S & \overset{\phi}{\dashrightarrow} & \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \subseteq \mathbb{CP}^N \\ & \underset{f}{\searrow} & \downarrow{\scriptstyle \pi_A} \\ & & C \end{array}$$

where π_A is the projection map, again we have $\psi = s \circ \phi$, the rational map ϕ is defined by the complete linear system $|D + f^*A|$. If it has no nontrivial fixed part, then its image is contained in any hyperplane.

We assume that there is a fixed part Z of $|D + f^*A|$. Thus, the linear system $|D - Z + f^*A|$ factorizes the map ϕ defined by $|D + f^*A|$ and the following properties are satisfied:

(1). The fixed part Z of $|D + f^*A|$, restricted to F, is just the fixed part of the complete linear system $|D_{|_F}|$.

(2). The system $|D - Z + f^*A|$ has no fixed part, so it has only a finite number of base points.

(3). The fixed part Z is a divisor such that the morphism:

$$f^*(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \longrightarrow \mathcal{O}_S(D - Z) \otimes f^*\mathcal{O}(A),$$

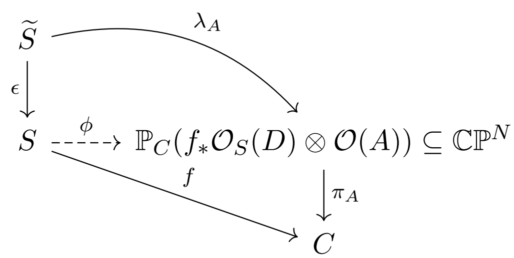
is surjective in codimension 1.

(4). We can assume that Z has no horizontal components because A is sufficiently ample.

Theorem 2.1.2 *There exist a series of blow ups $\epsilon : \widetilde{S} \to S$ and a morphism*

$$\lambda_A : \widetilde{S} \longrightarrow \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$$

such that the following diagram is commutative:



$\phi \circ \epsilon = \lambda_A$ *and*

$$(\lambda_A)^*\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D)\otimes\mathcal{O}(A))}(1) = \epsilon^*(\mathcal{O}(D - Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E),$$

where E is the exceptional divisor of ϵ.

Remark 2.1.3 *$\epsilon^*(\mathcal{O}(D - Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E)$ is globally generated.*

Proof of Theorem 2.1.2. *The linear system $|D - Z + f^*A|$ has at worst finitely many base points. If there are none, ϕ is a morphism and there is nothing to prove. We suppose that there is a base point x in $|D - Z + f^*A|$. We take the blow-up in x defined by ϵ^1, so $|(\epsilon^1)^*(D - Z + f^*A)|$ has a fixed part k_1E_1 with $k_1 \in \mathbb{Z}, k_1 \geq 1$ and $|D_1| = |(\epsilon^1)^*(D - Z + f^*A) - k_1E_1|$ has no fixed part. Hence, it defines a rational map, $\lambda^1 : S_1 \dashrightarrow$*

$\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$ *which is identical to* $\phi \circ \epsilon^1$. *If* λ^1 *is a morphism, then we are done; if not, we repeat the process. Thus, we get by induction a sequence* $\epsilon^i : S_i \longrightarrow S_{i-1}$ *of blow-ups and a linear system* $|D_i|$ *with no fixed part, where* $D_i = (\epsilon^i)^* D_{i-1} - k_i E_i$ *for* $i \geq 1$.

In other words, we arrive at a system D_n *with no base points, which defines a morphism:*

$$\epsilon = \epsilon^1 \circ ... \circ \epsilon^n : \widetilde{S} \longrightarrow S.$$

We conclude that $|\epsilon^*(D - Z + f^*A) - E|$ *defines a morphism*

$$\widetilde{S} \xrightarrow{\lambda_A} \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$$

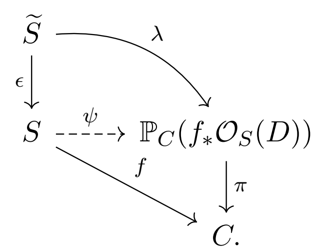
such that

$$\epsilon^*(\mathcal{O}(D - Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E) = (\lambda_A)^* \mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))}(1)$$

where $E = \sum_{i=1}^{i=n} k_i E_i$ *is the exceptional divisor.* □

The last proof is inspired by the proof of (5, Theorem 2.7).

Corollary 2.1.4 *There exists a morphism* $\lambda : \widetilde{S} \to \mathbb{P}_C(f_*\mathcal{O}_S(D))$ *such that the following diagram is commutative:*



Moreover, we have

$$\lambda^*(\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1)) = \epsilon^*(\mathcal{O}(D - Z)) \otimes \mathcal{O}(-E).$$

Proof. By Theorem 2.1.2, there exists a morphism

$$\lambda_A : \widetilde{S} \longrightarrow \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$$

which has the following property:

$$(\lambda_A)^* \mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))}(1) = \epsilon^*(\mathcal{O}(D - Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E).$$

But there exists an isomorphism

$$s : \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \longrightarrow \mathbb{P}_C(f_*\mathcal{O}_S(D))$$

such that

$$\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D)\otimes\mathcal{O}(A))}(1) = s^*\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1) \otimes \pi_A^*\mathcal{O}(A).$$

Therefore,

$$(s \circ \lambda_A)^*\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1) \otimes (\pi_A \circ \lambda_A)^*\mathcal{O}(A) = \epsilon^*(\mathcal{O}(D - Z)) \otimes (f \circ \epsilon)^*\mathcal{O}(A) \otimes \mathcal{O}(-E)$$

implies

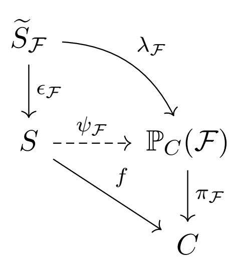
$$(s \circ \lambda_A)^*\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1) = \epsilon^*(\mathcal{O}(D - Z)) \otimes \mathcal{O}(-E).$$

We take

$$\lambda = s \circ \lambda_A.$$

□

Remark 2.1.5 *More generally, for $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$ a locally free sub-sheaf, we take a sufficiently very ample divisor A on C such that $\mathcal{F} \otimes \mathcal{O}(A)$ is very ample. Let $L_\mathcal{F}$ be the linear sub-system of $|D + f^*A|$ which corresponds to the sections of $H^0(\mathcal{F} \otimes \mathcal{O}(A))$. Let $Z_\mathcal{F}$ be the fixed part of $L_\mathcal{F}$, so $L_\mathcal{F} - Z_\mathcal{F}$ has no fixed part and it corresponds to a rational map from S to a projective sub-variety of $\mathbb{P}_C(\mathcal{F} \otimes \mathcal{O}(A))$. By the same arguments as above, $\exists\ \widetilde{S}_\mathcal{F} \xrightarrow{\epsilon_\mathcal{F}} S$ which is a chain of blow ups and $\exists \lambda_\mathcal{F} : \widetilde{S}_\mathcal{F} \longrightarrow \mathbb{P}_C(\mathcal{F})$ such that the following diagram is commutative:*



and

$$(\lambda_\mathcal{F})^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)) = \epsilon_\mathcal{F}^*(\mathcal{O}(D - Z_\mathcal{F})) \otimes \mathcal{O}(-E_\mathcal{F}),$$

where $E_\mathcal{F}$ is the exceptional divisor of $\epsilon_\mathcal{F}$.

2.2 Harder-Narasimhan filtration

In this section, we study the Harder-Narasimhan filtration within the context of fibered surfaces.

Proposition 2.2.1 *(52) Let $\mathcal{F}$ be a vector bundle over a smooth projective curve C. There exists a unique sequence of vector sub-bundles of $\mathcal{F}$:*

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_{k-1} \subsetneq \mathcal{F}_k = \mathcal{F},$$

that satisfies the following conditions:

(1). For $i = 1, \ldots, k$, $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a semi-stable vector bundle.

(2). For any $i = 1, \ldots, k$, setting $\mu_i := \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) = \frac{\deg(\mathcal{F}_i/\mathcal{F}_{i-1})}{\mathrm{rk}(\mathcal{F}_i/\mathcal{F}_{i-1})}$, we have:

$$\mu_1 > \mu_2 > \ldots > \mu_k.$$

In the context of Proposition 2.2.1 above, the filtration is called the Harder-Narasimhan filtration *of $\mathcal{F}$. We set $\mu_f = \mu_k$ and call it the* final slope *of $\mathcal{F}$. The following elementary lemma is important in what follows.*

Lemma 2.2.2 *Let r_i be the rank of $\mathcal{F}_i$. Then*

$$\deg \mathcal{F} = \sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) + r_k \mu_k.$$

Proof. Indeed, we consider the exact sequence:

$$0 \longrightarrow \mathcal{F}_{k-1} \longrightarrow \mathcal{F}_k \longrightarrow \mathcal{F}_k/\mathcal{F}_{k-1} \longrightarrow 0.$$

From the additivity of degree, we have

$$\deg \mathcal{F}_k = \deg \mathcal{F}_{k-1} + \deg \mathcal{F}_k/\mathcal{F}_{k-1}.$$

Similarly, we have

$$\deg \mathcal{F}_{k-1} = \deg \mathcal{F}_{k-2} + \deg \mathcal{F}_{k-1}/\mathcal{F}_{k-2}.$$

And so, by induction, we can conclude that

$$\deg \mathcal{F}_k = \sum_{i=1}^{k} \deg \mathcal{F}_i/\mathcal{F}_{i-1}.$$

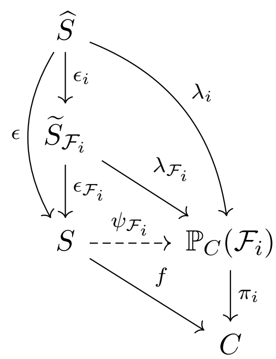
From the definition of slope, for every $i = 1, .., k$ we have:

$$\deg \mathcal{F}_i/\mathcal{F}_{i-1} = \mu_i(r_i - r_{i-1}).$$

Thus, we obtain the desired formula. □

Consider now a fibered surface $f : S \to C$. Let F be its general fiber, let D be a relatively effective divisor on S, and let $(\mathcal{F}_i)_{0\le i\le k}$ be the Harder-Narasimhan filtration of $\mathcal{F} \subseteq \mathcal{E} = f_*\mathcal{O}_S(D)$. By a repeated application of Remark 2.1.5 to each of the $\mathcal{F}_i$'s, we have the following proposition.

Proposition 2.2.3 *There exists a suitable smooth projective surface $\widehat{S}$ and a birational morphism $\epsilon : \widehat{S} \to S$ such that the following diagram is commutative $\forall i; 1 \le i \le k$:*



where $\epsilon_{\mathcal{F}_i}$ is a blow-up morphism along a finite number of points $\{x_{i_1}, \dots, x_{i_{m_i}}\}$, $1 \le i \le k$, as defined in the proof of Theorem 2.1.2 and Remark 2.1.5.

Proof. Set $\bigcup_{1\le i\le k}\{x_{i_1}, \dots, x_{i_{m_i}}\} = \{q_1, \dots, q_m\}$ and let $\widehat{S} = \mathbb{BL}_{q_1,\dots,q_m}(S)$ be the blowing up of S at $q_1, \dots, q_m$. Then there exists a blowing up morphism ϵ_i along $\{q_1, ..., q_m\}\backslash\{x_{i_1}, \dots, x_{i_{m_i}}\}$ which fits into the diagram above and $\epsilon = \epsilon_i \circ \epsilon_{\mathcal{F}_i} : \widehat{S} \to S$. As before, we fix a sufficiently ample divisor A on C. Define the map

$$\lambda_i : \widehat{S} \to \mathbb{P}_C(\mathcal{F}_i)$$

by the linear system $|\epsilon^*(D - Z_{\mathcal{F}_i}) - E|$, and furthermore we have

$$\lambda_i = \lambda_{\mathcal{F}_i} \circ \epsilon_i.$$

Moreover, for any $\mathcal{F}_i$ in the filtration, we have

$$\lambda_i^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1)) = \epsilon^*(\mathcal{O}(D - Z_{\mathcal{F}_i})) \otimes \mathcal{O}(-E).$$

Where $Z_{\mathcal{F}_i}$ is the fixed part of $L_{\mathcal{F}_i} \subseteq |D + f^*A|$ which correspond to sections of $H^0(\mathcal{F}_i \otimes \mathcal{O}(A))$. Here E is the exceptional divisor of ϵ. □

In what follows, we study the fibered surface $\hat{f} := f \circ \epsilon : \widehat{S} \to C$ and we denote by F its general fiber.

Definition 2.2.4 (Compare with (97, Definition 3.11)) *In the setting above, we define the following divisors on $\widehat{S}$:*

- $Z_i := \epsilon^* Z_{\mathcal{F}_i} + E$, the fixed part *of the vector sub-bundle $\mathcal{F}_i, \forall i; 1 \leq i \leq k$.*
- $M_i := \lambda_i^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1))$, the moving part *of the vector sub-bundle $\mathcal{F}_i, \forall i; 1 \leq i \leq k$.*
- *Set $N_i := M_i - \mu_i F, \forall i; 1 \leq i \leq k$. We call this the i^{th}* Miyaoka divisor.

Applying, (83), (85) and (71, Proposition 6.4.11), we prove the following Lemma 2.2.6 using the language of $\mathbb{Q}$-twisted vector bundles. First, we recall the definition of a $\mathbb{Q}$-twisted vector bundle.

Definition 2.2.5 *(See (71, Definition 6.2.1)). A* $\mathbb{Q}$-twisted vector bundle *$\mathcal{E}\langle\delta\rangle$ on a projective variety X consists of a vector bundle $\mathcal{E}$ defined up to isomorphism, and a $\mathbb{Q}$-numerical equivalence class $\delta \in N^1_{\mathbb{Q}}(X)$. If D is a $\mathbb{Q}$-divisor, we write $\mathcal{E}\langle D\rangle$ for the twist $\mathcal{E}$ by the numerical equivalence class of D. We define $\mathbb{Q}$-*isomorphism *of $\mathbb{Q}$-twisted bundles to be the equivalence relation generated by saying that $\mathcal{E}\langle A + D\rangle$ is equivalent to $(\mathcal{E} \otimes \mathcal{O}(A))\langle D\rangle$ for all integral divisors A and $\mathbb{Q}$-divisors D.*

Lemma 2.2.6 (Compare with (85, Corollary IV.3.8)) *$\forall i; 1 \leq i \leq k, \ \ N_i$ are nef divisors on $\widehat{S}$.*

Proof. Fix $i; 1 \leq i \leq k$, let us see that the $\mathbb{Q}$-twisted vector bundle $\mathcal{F}_i\langle -\frac{c_1(\mathcal{F}_i/\mathcal{F}_{i-1})}{rk(\mathcal{F}_i/\mathcal{F}_{k-1})}\rangle$ is nef.

We define the following quotient bundles and $\mathbb{Q}$-divisors:

$$\begin{cases} \mathcal{G}_i = \mathcal{F}_i/\mathcal{F}_{i-1} \\ \delta_i = \frac{c_1(\mathcal{G}_i)}{\mathrm{rk}(\mathcal{G}_i)}. \end{cases}$$

However, $\mathcal{G}_i\langle -\delta_i\rangle$ is a nef vector bundle by (71, Proposition 6.4.11). Furthermore, $\deg\delta_i = \mu_i$. Thus

$$-\deg\delta_1 < -\deg\delta_2 < ... < -\deg\delta_k.$$

Using the following exact sequence of vector bundles:

$$0 \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{G}_i \longrightarrow 0,$$

we next prove that $\mathcal{F}_i\langle -\delta_i\rangle$ is nef by induction, $\forall i; 1 \le i \le k$. For $i = 1$, $\mathcal{F}_1\langle -\delta_1\rangle = \mathcal{G}_1\langle -\delta_1\rangle$ which is nef. Now, assume that $\mathcal{F}_{i-1}\langle -\delta_{i-1}\rangle$ is a nef bundle. Then $\mathcal{F}_{i-1}\langle -\delta_i\rangle$ is $\mathbb{Q}$-ample. Using the above exact sequence and since $\mathcal{F}_{i-1}\langle -\delta_i\rangle$ and $\mathcal{G}_i\langle -\delta_i\rangle$ are nef, we see that $\mathcal{F}_i\langle -\delta_i\rangle$ is a nef bundle.

Thus, $\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i\langle -\delta_i\rangle)}(1)$ is a nef line bundle. So $\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1)) \otimes \pi_i^*\mathcal{O}(-\delta_i)$ is nef, then by the remarks that $\lambda_i^*\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1) = M_i$ and $\lambda_i^*(\pi_i^*\mathcal{O}(-\delta_i)) = \mathcal{O}(-\mu_i F)$, we conclude that N_i is nef, $\forall i; 1 \le i \le k$. □

Remark 2.2.7 *It is noted that the above lemma can be proven by the original Xiao's argument, leveraging the Miyaoka-Nakayama result (85, Corollary IV.3.8). The argument presented in the lemma above serves as an alternative proof using the language of $\mathbb{Q}$-twisted vector bundles.*

Lemma 2.2.8 $\forall i; 1 \le i \le k, \;\; r_i = \operatorname{rk}\mathcal{F}_i \le h^0(F, N_{i|_F})$.

Proof. Let π_i be the projection from $\mathbb{P}_C(\mathcal{F}_i)$ to C. Then,

$$(\pi_i \circ \lambda_i)_*(M_i) = (\pi_i)_*((\lambda_i)_* M_i)$$

$$= (\pi_i)_*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1) \otimes (\lambda_i)_*\mathcal{O}_{\widehat{S}}) \supseteq (\pi_i)_*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1)) = \mathcal{F}_i.$$

Thus,

$$r_i \le h^0(F, M_{i|_F}),$$

which implies

$$r_i \le h^0(F, N_{i|_F}).$$

□

Proposition 2.2.9 *Let $d_i = \deg(N_{i|_F}) = N_i.F$ such that $1 \le i \le k$. Then,*

$$d_k \ge d_{k-1} \ge ... \ge d_1 \ge 0.$$

Proof. *Since F is a fiber, $F^2 = 0$. Then,*

$$d_i = N_i.F = (M_i - \mu_i F).F = M_i.F$$

$$= \lambda_i^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1)).F = (\epsilon^*(D - Z_{\mathcal{F}_i}) - E).F$$

$$= \epsilon^*(D - Z_{\mathcal{F}_i}).F = (D - Z_{\mathcal{F}_i}).F \geq 0.$$

But

$$Z_{\mathcal{F}_i} \geq Z_{\mathcal{F}_{i+1}}.$$

Thus,

$$D - Z_{\mathcal{F}_{i+1}} = D - Z_{\mathcal{F}_i} + (Z_{\mathcal{F}_i} - Z_{\mathcal{F}_{i+1}}).$$

Therefore,

$$d_{i+1} \geq d_i.$$

□

Proposition 2.2.10 *Continuing with the setting as above, we define the following constant for the datum $(D, \mathcal{F})$:*

$$\hat{n}_{(D,\mathcal{F})} := \begin{cases} \max\{i |\ N_{i_{|F}} \text{is special}\} \\ -\infty \text{ otherwise.} \end{cases}$$

If $\hat{n}_{(D,\mathcal{F})} \neq -\infty$, then $N_{j_{|F}}$ is special for $1 \leq j \leq \hat{n}_{(D,\mathcal{F})}$ and $N_{j_{|F}}$ is nonspecial for $\hat{n}_{(D,\mathcal{F})} + 1 \leq j \leq k$. Otherwise, $N_{j_{|F}}$ is nonspecial, $\forall j; 1 \leq j \leq k$.

Proof. *Recall that*

$$Z_{\mathcal{F}_1} \geq ... \geq Z_{\mathcal{F}_k}.$$

Also, we identified the general fiber of S with the general fiber of $\widehat{S}$. Let $\hat{n}_{(D,\mathcal{F})} + 1 \leq j \leq k - 1$, thus

$$N_{j_{|F}} = \epsilon^*(D - Z_{\mathcal{F}_j})_{|F} = (D - Z_{\mathcal{F}_j})_{|F} \leq (D - Z_{\mathcal{F}_{j+1}})_{|F} = N_{j+1_{|F}}.$$

Consider the following short exact sequence:

$$0 \longrightarrow N_{j_{|F}} \longrightarrow N_{j+1_{|F}} \longrightarrow N_{j+1_{|F}}/N_{j_{|F}} \longrightarrow 0$$

which induces a long exact sequence in cohomology:

$$\cdots \longrightarrow H^0(F, N_{j+1_{|F}}/N_{j_{|F}}) \longrightarrow H^1(F, N_{j_{|F}}) \longrightarrow H^1(F, N_{j+1_{|F}}) \longrightarrow 0.$$

So, if $h^1(F, N_{j_{|F}}) = 0$, then $h^1(F, N_{j+1_{|F}}) = 0$. □

To illustrate the above proposition, we consider the following examples.

Example 2.2.11 *Let $D = K_{S/C}, g \geq 1$, and $\mathcal{F} = f_*\omega_{S/C}$. Then we have $N_{k_{|F}} \simeq K_F$, thus $H^1(F, K_F) = 1$. Hence, $N_{k_{|F}}$ is special. Furthermore, it follows that $N_{i_{|F}}$ is special $\forall i; 1 \leq i \leq k$ and $\hat{n}_{(K_{S/C}, f_*\omega_{S/C})} = k$. Conversely, for an arbitrary relatively effective divisor D, if $\hat{n}_{(D,\mathcal{F})} = k$, then $\deg D_{|F} \leq 2g - 2$.*

Example 2.2.12 *Now, let $D = K_{S/C} + \Delta, g \geq 1$, $\mathcal{F} = f_*\mathcal{O}_S(K_{S/C} + \Delta)$, and Δ is an effective divisor on S with $\Delta.F > 0$. Then $N_{k_{|F}} \simeq K_F + \Delta_{|F}$, thus $H^1(F, K_F + \Delta_{|F}) = 0$ since the degree of $K_F + \Delta_{|F}$ is strictly greater than $2g - 2$. So we conclude that $\hat{n}_{(K_{S/C}+\Delta, f_*\mathcal{O}_S(K_{S/C}+\Delta))} < k$.*

Example 2.2.13 *As in Example 2.2.12, let $D = K_{S/C} + \Delta$ and Δ is an effective divisor on S with $\Delta.F > 0$. Assume that $f_*\mathcal{O}_S(D)$ is semi-stable. Then $k = 1$ and $\hat{n}_{(K_{S/C}+\Delta, f_*\mathcal{O}_S(K_{S/C}+\Delta))} = -\infty$.*

Example 2.2.14 *We give another example such that $\hat{n}_{(D,\mathcal{F})} = -\infty$. Let $\mathcal{E} = \mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1)$ and $S = \mathbb{P}_{\mathbb{CP}^1}(\mathcal{E})$. Then we define a natural fibration $f : S \to \mathbb{CP}^1$. S has only one negative curve $E \simeq \mathbb{CP}^1$ such that $E^2 = -1$ and $E = \mathcal{O}_S(1) - F$ in $Pic(S)$. By construction, the trivial part $\mathcal{O}_{\mathbb{CP}^1}$ corresponds to a 0-dimensional linear sub-system L_0 of $|\mathcal{O}_S(1)|$ generated by the effective divisor $E + F$, and the bundle $\mathcal{O}_{\mathbb{CP}^1}(1)$ to the 1-dimensional linear sub-system L_1 generated by the two effective divisor of $|\mathcal{O}_S(1)|$ different from $E + F$. Moreover,*

$$0 \subsetneq \mathcal{O}_{\mathbb{CP}^1}(1) \subsetneq \mathcal{E}$$

is the Harder-Narasimhan filtration of $\mathcal{E}$. Since L_1 has no fixed part, then $N_1 = \mathcal{O}_S(1) - F$ and $N_2 = \mathcal{O}_S(1)$. Thus, $N_{1_{|F}}$ and $N_{2_{|F}}$ are nonspecial divisors and $\hat{n}_{(\mathcal{O}_1(S),\mathcal{E})} = -\infty$.

Example 2.2.15 *More generally than Example 2.2.14, every fibered surface $f : S \to C$ with $g(F) = 0$ has $\hat{n}_{(D,\mathcal{F})} = -\infty$ for every relatively effective divisor D and $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$, since there are no special divisors on $\mathbb{CP}^1$.*

Now, it is natural to ask for some information about the sequence $(\frac{d_i}{h^0(F, N_{i_{|F}}) - 1})_{i \in \{1, \ldots k\}}$. For instance, is it an increasing finite sequence? Is it decreasing? Is it bounded from below by a strictly positive number?

Lemma 2.2.16 *$\forall i; 2 \leq i \leq k$, we have $h^0(F, N_{i|_F}) > 1$.*

Proof. We have $h^0(F, N_{i|_F}) \geq \mathrm{rk}(\mathcal{F}_i)$. So if $h^0(F, N_{i|_F}) = 1$, then the only possibility is $i = 1$. In this case, the degree is $d_1 = g(F) - h^1(F, N_{1|_F})$. □

In the following theorem, we assume that $\mathrm{rk}\,\mathcal{F} \geq 2$.

Theorem 2.2.17 *Let $f : S \to C$ be a fibered surface with general fiber F, and D a relatively effective divisor on S. Consider the Harder-Narasimhan filtration $(\mathcal{F}_i)$ of $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$ such that $\mathrm{rk}\,\mathcal{F} \geq 2$. Define:*

$$\widehat{S}_{(D,\mathcal{F})} := \begin{cases} 1 & \text{if } h^0(F, N_{1|_F}) > 1 \\ 2 & \text{otherwise} \end{cases}$$

and

$$\beta_D := 1 + \frac{g(F)}{h^0(F, D_{|_F}) - 1}.$$

Then, the following result hold:

(1). If $\hat{n}_{(D,\mathcal{F})} = -\infty$, then $N_{i|_F}$ is nonspecial $\forall i; 1 \leq i \leq k$, and

$$\beta_D \leq \frac{d_k}{h^0(N_{k|_F}) - 1} \leq \dots \leq \frac{d_{i+1}}{h^0(N_{i+1|_F}) - 1} \leq$$

$$\frac{d_i}{h^0(N_{i|_F}) - 1} \leq \dots \leq \frac{d_{\widehat{S}_{(D,\mathcal{F})}}}{h^0(N_{\widehat{S}_{(D,\mathcal{F})}|_F}) - 1}.$$

(2). Otherwise:

- *$\forall i; \hat{n}_{(D,\mathcal{F})} + 1 \leq i \leq k$:*

$$\beta_D \leq \frac{d_k}{h^0(N_{k|_F}) - 1} \leq \dots \leq \frac{d_{i+1}}{h^0(N_{i+1|_F}) - 1} \leq$$

$$\frac{d_i}{h^0(N_{i|_F}) - 1} \leq \dots \leq \frac{d_{\hat{n}_{(D,\mathcal{F})}+1}}{h^0(N_{\hat{n}_{(D,\mathcal{F})}+1|_F}) - 1}.$$

- *$\forall i; \widehat{S}_{(D,\mathcal{F})} \leq i \leq \hat{n}_{(D,\mathcal{F})}$:*

$$\frac{d_i}{h^0(N_{i|_F}) - 1} \geq 2.$$

Proof. *For* (1), *if* $\hat{n}_{(D,\mathcal{F})} = -\infty$, *then by definition of* $\hat{n}_{(D,\mathcal{F})}$, $N_{i|_F}$ *is nonspecial* $\forall i; 1 \leq i \leq k$. *Thus, by applying the Riemann-Roch formula:*

$$h^0(N_{i|_F}) = d_i + 1 - g(F).$$

Since

$$h^0(N_{i+1|_F}) = h^0(N_{i|_F}) + d_{i+1} - d_i,$$

it follows that $\forall i; \widehat{S}_{(D,\mathcal{F})} \leq i \leq k-1$*:*

$$\frac{d_{i+1}}{h^0(N_{i+1|_F}) - 1} = \frac{d_i + d_{i+1} - d_i}{h^0(N_{i|_F}) + d_{i+1} - d_i - 1} \leq \frac{d_i}{h^0(N_{i|_F}) - 1}.$$

Moreover, $D_{|_F}$ *is nonspecial, and then by the Riemann-Roch formula we have*

$$h^0(F, D_{|_F}) = D.F + 1 - g(F).$$

Thus,

$$h^0(F, D_{|_F}) = h^0(F, N_{i|_F}) + D.F - d_i, \ \forall i; 1 \leq i \leq k.$$

Then, we deduce the desired lower bound:

$$\frac{d_i}{h^0(N_{i|_F}) - 1} \geq 1 + \frac{g(F)}{h^0(F, D_{|_F}) - 1}, \ \forall i; \widehat{S}_{(D,\mathcal{F})} \leq i \leq k.$$

This proves (1).

For (2), *if* $\hat{n}_{(D,\mathcal{F})} + 1 \leq i \leq k$, *then argue as in* (1). *Suppose now that* $N_{i|_F}$ *is special, that is* $\widehat{S}_{(D,\mathcal{F})} \leq i \leq \hat{n}$. *Then by Clifford Theorem (2), we have*

$$d_i \geq 2(h^0(N_{i|_F}) - 1)$$

which implies

$$\frac{d_i}{h^0(N_{i|_F}) - 1} \geq 2.$$

□

Remark 2.2.18 *If we compare Proposition 2.2.9 and Theorem 2.2.17, then we deduce that* $\forall i; \widehat{S}_{(D,\mathcal{F})} \leq i \leq k$*:*

$$d_k \geq d_{k-1} \geq \ldots \geq d_{\widehat{S}} \geq 1.$$

Now, we are ready to present the technical lemma in the method, we called it the Modified Xiao Lemma. *Note that it is a more general form of Xiao (108, Lemma 2).*

***Lemma 2.3.1* (Modified Xiao Lemma)** *Let $\hat{f}:\widehat{S}\to C$ be a fibered surface with F its general fiber, $\hat{D}$ be a divisor on $\widehat{S}$, and suppose that there exist a sequence of effective divisors:*

$$\mathcal{Z}_1 \geq \mathcal{Z}_2 \geq ... \geq \mathcal{Z}_j,$$

and a sequence of rational numbers:

$$\mu_1 \geq \mu_2 \geq ... \geq \mu_j,$$

such that for every $i \in \{1, ..., j\}$, we have

$$\mathcal{N}_i := \hat{D} - \mathcal{Z}_i - \mu_i F$$

are nef $\mathbb{Q}$-divisors. Then,

$$\hat{D}^2 \geq \sum_{i=1}^{j-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 2\hat{D}.\mathcal{Z}_j - \mathcal{Z}_j^2 + 2\mu_j d_j,$$

where $d_i = \mathcal{N}_i.F$.

Proof. First, observe that $\mathcal{N}_1^2 \geq 0$ by nefness. However,

$$\mathcal{N}_i^2 = \mathcal{N}_i(\mathcal{N}_{i-1} + (\mathcal{Z}_{i-1} - \mathcal{Z}_i) + (\mu_{i-1} - \mu_i)F)$$

$$\geq \mathcal{N}_i(\mathcal{N}_{i-1} + (\mu_{i-1} - \mu_i)F)$$

$$\geq (\mathcal{N}_{i-1} + (\mathcal{Z}_{i-1} - \mathcal{Z}_i) + (\mu_{i-1} - \mu_i)F)(\mathcal{N}_{i-1} + (\mu_{i-1} - \mu_i)F)$$

$$\geq \mathcal{N}_{i-1}^2 + (\mu_{i-1} - \mu_i)(2\mathcal{N}_{i-1}F + (\mathcal{Z}_{i-1} - \mathcal{Z}_i)F)$$

$$= \mathcal{N}_{i-1}^2 + (\mu_{i-1} - \mu_i)(d_{i-1} + d_i).$$

So, by induction we have

$$\mathcal{N}_j^2 \geq \sum_{i=1}^{j-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

Hence,

$$(\hat{D} - \mathcal{Z}_j - \mu_j F)^2 \geq \sum_{i=1}^{j-1} (d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

But,

$$(\hat{D} - \mathcal{Z}_j - \mu_j F)^2 = (\hat{D} - \mathcal{Z}_j)^2 - 2\mu_j(\hat{D} - \mathcal{Z}_j)F$$
$$= \hat{D}^2 - 2\hat{D}.\mathcal{Z}_j + \mathcal{Z}_j^2 - 2\mu_j d_j.$$

Thus,

$$\hat{D}^2 \geq \sum_{i=1}^{j-1} (d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 2\hat{D}.\mathcal{Z}_j - \mathcal{Z}_j^2 + 2\mu_j d_j.$$

□

Remark 2.3.2 *The term* $2\hat{D}.\mathcal{Z}_j - \mathcal{Z}_j^2 + 2\mu_j d_j$ *describes the negativity of* $\hat{D}$.

Example 2.3.3 *In the setting of Lemma 2.3.1, suppose that* $j = k$, $\hat{D}$ *is nef, and* $\mu_k \geq 0$. *Set* $\mathcal{Z}_{k+1} = 0$ *and* $\mu_{k+1} = 0$. *Then apply Lemma 2.3.1 to the sequence of effective divisors:*

$$\mathcal{Z}_1 \geq \mathcal{Z}_2 \geq ... \geq \mathcal{Z}_{k+1} = 0,$$

and a sequence of rational numbers:

$$\mu_1 \geq \mu_2 \geq ... \geq \mu_{k+1} = 0.$$

So, we conclude the original result of Xiao (108, Lemma 2):

$$\hat{D}^2 \geq \sum_{i=1}^{k} (d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

Prior to stating the main results, we apply Lemma 2.3.1 to the datum $(D, \mathcal{F})$ *where* $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$. *Let* $(\mathcal{F}_i)_{0 \leq i \leq k}$ *be the Harder-Narasimhan filtration of* $\mathcal{F}$, *and* $(Z_i, M_i, N_i)_{1 \leq i \leq k}$ *be the triple of fixed parts, moving parts, and the Miyaoka divisors respectively (Definition 2.2.4). Then, we deduce the following inequalities.*

Theorem 2.3.4 *Let* $f : S \to C$ *be a fibered surface with general fiber* F. *Consider the datum* $(D, \mathcal{F})$ *where* D *is a relatively effective divisor on* S *and* $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$ *is a locally free sub-sheaf on* C *with* $\operatorname{rk} \mathcal{F} \geq 2$. *Then we have the following three cases:*

(1). If $\hat{n}_{(D,\mathcal{F})} = -\infty$, then

$$D^2 \geq 2\beta_D \deg \mathcal{F} - \beta_D \mu_1 + (\beta_D - 2\beta_D r_k + 2d_k)\mu_k + 2\epsilon^* D.Z_k - Z_k^2.$$

(2). If $\hat{n}_{(D,\mathcal{F})} = k$, then

$$D^2 \geq 4 \deg \mathcal{F} - 2\mu_1 + 2(1 - 2r_k + d_k)\mu_k + 2\epsilon^* D.Z_k - Z_k^2.$$

(3). If $1 \leq \hat{n}_{(D,\mathcal{F})} < k$, then

$$D^2 \geq 4 \sum_{i=1}^{\hat{n}_{(D,\mathcal{F})}-1} r_i(\mu_i - \mu_{i+1}) + 2\beta_D \sum_{i=\hat{n}_{(D,\mathcal{F})}+1}^{k-1} r_i(\mu_i - \mu_{i+1})$$

$$-2(\mu_1 - \mu_{\hat{n}_{(D,\mathcal{F})}+1}) - \beta_D(\mu_{\hat{n}_{(D,\mathcal{F})}+1} - \mu_k) + (\beta_D + 2) r_{\hat{n}_{(D,\mathcal{F})}}(\mu_{\hat{n}_{(D,\mathcal{F})}} - \mu_{\hat{n}_{(D,\mathcal{F})}+1})$$

$$+2\epsilon^* D.Z_k - Z_k^2 + 2\mu_k d_k.$$

Here ϵ is constructed as in Proposition 2.2.3 and β_D is as in Theorem 2.2.17.

Proof. *Recall that $\hat{f} = f \circ \epsilon : \widehat{S} \to C$ is a fibered surface with F its general fiber and $\hat{D} := \epsilon^* D$ is a relatively effective divisor on $\widehat{S}$. Consider $(\mathcal{F}_i)_{0 \leq i \leq k}$ the Harder-Narasimhan filtration of $\mathcal{F}$, let $(\mu_1, \ldots, \mu_k)$ be the sequence of slopes and $(Z_i, M_i, N_i)_{1 \leq i \leq k}$ the triple of fixed parts, moving parts, and the Miyaoka divisors respectively, recall also that $\hat{D} = Z_i + M_i, \forall i; 1 \leq i \leq k$, and*

$$Z_1 \geq Z_2 \geq \cdots \geq Z_k \geq 0,$$

$$\mu_1 > \mu_2 > \cdots > \mu_k.$$

Therefore, N_i are $\mathbb{Q}$-nef divisors on $\widehat{S}$, where $d_i = N_i.F$. Now, by Lemma 2.3.1, we have the following inequality:

$$\hat{D}^2 \geq \sum_{i=1}^{k-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 2\hat{D}.Z_k - Z_k^2 + 2\mu_k d_k. \tag{2.1}$$

Recall that $\epsilon : \widehat{S} \to S$ is a proper birational morphism, hence $\hat{D}^2 = D^2$. By Theorem 2.2.17, we have the following natural three cases:

(1). If $\hat{n}_{(D,\mathcal{F})} = -\infty$, then $N_{i|_F}$ is nonspecial $\forall i; 1 \leq i \leq k$, and

$$d_i \geq \beta_D(h^0(N_{i|_F}) - 1) \geq \beta_D(r_i - 1).$$

The right hand side inequality follows from Lemma 2.2.8, where $r_i = \mathrm{rk}(\mathcal{F}_i), \forall i; 1 \leq i \leq k$, we also recall that $r_{i+1} \geq r_i - 1, \forall i; 1 \leq i \leq k-1$, hence

$$d_{i+1} \geq \beta_D r_i.$$

We substitule all these information into (2.1)*:*

$$D^2 \geq 2\beta_D \sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) - \beta_D(\mu_1 - \mu_k) + 2\epsilon^* D.Z_k - Z_k^2 + 2\mu_k d_k.$$

By Lemma 2.2.2, we deduce the desired inequality:

$$D^2 \geq 2\beta_D \deg \mathcal{F} - \beta_D \mu_1 + (\beta_D - 2\beta_D r_k + 2d_k)\mu_k + 2\epsilon^* D.Z_k - Z_k^2.$$

(2). If $\hat{n}_{(D,\mathcal{F})} = k$, then $N_{i_{|F}}$ is special $\forall i; 1 \leq i \leq k$ and

$$d_i \geq 2(h^0(N_{i_{|F}}) - 1) \geq 2(r_i - 1),$$

however

$$d_{i+1} \geq 2r_i.$$

This implies

$$D^2 \geq 4 \deg \mathcal{F} - 2\mu_1 + 2(1 - 2r_k + d_k)\mu_k + 2\epsilon^* D.Z_k - Z_k^2.$$

(3). If $1 \leq \hat{n}_{(D,\mathcal{F})} < k$, then

$$\forall i; \hat{n}_{(D,\mathcal{F})} + 1 \leq i \leq k, \quad d_i \geq \beta_D(h^0(N_{i_{|F}}) - 1) \geq \beta_D(r_i - 1),$$

and

$$\forall i; 1 \leq i \leq \hat{n}_{(D,\mathcal{F})}, \quad d_i \geq 2(h^0(N_{i_{|F}}) - 1) \geq 2(r_i - 1).$$

So, we decompose the right hand side of (2.1) *into three parts:*

$$D^2 \geq \sum_{i=1}^{\hat{n}_{(D,\mathcal{F})}-1} (d_i + d_{i+1})(\mu_i - \mu_{i+1}) + \sum_{i=\hat{n}_{(D,\mathcal{F})}+1}^{k-1} (d_i + d_{i+1})(\mu_i - \mu_{i+1})$$
$$+(d_{\hat{n}_{(D,\mathcal{F})}} + d_{\hat{n}_{(D,\mathcal{F})}+1})(\mu_{\hat{n}_{(D,\mathcal{F})}} - \mu_{\hat{n}_{(D,\mathcal{F})}+1})$$
$$+2\hat{D}.Z_k - Z_k^2 + 2\mu_k d_k.$$

Which implies

$$D^2 \geq 4 \sum_{i=1}^{\hat{n}_{(D,\mathcal{F})}-1} r_i(\mu_i - \mu_{i+1}) + 2\beta_D \sum_{i=\hat{n}_{(D,\mathcal{F})}+1}^{k-1} r_i(\mu_i - \mu_{i+1})$$

$$-2(\mu_1 - \mu_{\hat{n}_{(D,\mathcal{F})}}) - \beta_D(\mu_{\hat{n}_{(D,\mathcal{F})}+1} - \mu_k) + (\beta_D + 2) r_{\hat{n}_{(D,\mathcal{F})}}(\mu_{\hat{n}_{(D,\mathcal{F})}} - \mu_{\hat{n}_{(D,\mathcal{F})}+1})$$
$$+2\hat{D}.Z_k - Z_k^2 + 2\mu_k d_k,$$

since

$$d_{\hat{n}_{(D,\mathcal{F})}} \geq 2(r_{\hat{n}_{(D,\mathcal{F})}} - 1), \text{ and } d_{\hat{n}_{(D,\mathcal{F})}+1} \geq \beta_D(r_{\hat{n}_{(D,\mathcal{F})}+1} - 1) \geq \beta_D(r_{\hat{n}_{(D,\mathcal{F})}}).$$

□

Remark 2.3.5 *The lower bound that we obtain for D^2 in Theorem 2.3.4 for $1 \leq \hat{n}_{(D,\mathcal{F})} < k$ is given by a sum of three parts. The first part,*

$$4 \sum_{i=1}^{\hat{n}_{(D,\mathcal{F})}-1} r_i(\mu_i - \mu_{i+1})$$

describes the effect of special Miyaoka divisors. The second part,

$$2\beta_D \sum_{i=\hat{n}_{(D,\mathcal{F})}+1}^{k-1} r_i(\mu_i - \mu_{i+1})$$

explains the impact of the nonspecial Miyaoka divisors in the sequence $(N_i)_{1\leq i\leq k}$. The third part is a transition from special to nonspecial Miyaoka divisors. In general, we could give a lower bound for D^2 by

$$4 \deg \mathcal{F}_{\hat{n}_{(D,\mathcal{F})}}, \ \ 2\beta_D \deg \mathcal{F}/\mathcal{F}_{\hat{n}_{(D,\mathcal{F})}}$$

and other terms, but to formulate the slope inequality for the datum $(D, \mathcal{F})$, it turns out that we should first start by taking $\min(2, \beta_D)$ and write, for instance,

$$d_i \geq \min(2, \beta_D)(r_i - 1).$$

Following this set-up, we will obtain in the next corollary a lower bound for D^2 by

$$2 \min(2, \beta_D) \deg \mathcal{F}$$

and other terms that we will explore in the upcoming paragraphs.

Definition 2.3.6 *Let $f : S \to C$ be a fibered surface, F its general fiber. Consider the datum $(D, \mathcal{F})$ as before, where D is a relatively effective divisor on S and $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$ with $\operatorname{rk} \mathcal{F} \geq 2$. Then we define the following number:*

$$\alpha_{(D,\mathcal{F})} := \begin{cases} \beta_D & \text{if } \hat{n}_{(D,\mathcal{F})} = -\infty \\ 2 & \text{if } \hat{n}_{(D,\mathcal{F})} = k \\ \min(2, \beta_D) & \text{if } 1 \leq \hat{n}_{(D,\mathcal{F})} < k. \end{cases}$$

Corollary 2.3.7 *Let $f : S \to C$ be a fibered surface with general fiber F. Consider the datum $(D, \mathcal{F})$ where D is a relatively effective divisor on S and $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$ is a locally free sub-sheaf on C.*

(1). If $\operatorname{rk}\mathcal{F} \geq 2$, then

$$D^2 \geq 2\alpha_{(D,\mathcal{F})} \deg \mathcal{F} - \alpha_{(D,\mathcal{F})}\mu_1 + (\alpha_{(D,\mathcal{F})} - 2\alpha_{(D,\mathcal{F})}r_k + 2d_k)\mu_k + 2\epsilon^* D.Z_k - Z_k^2.$$

Again, here ϵ is constructed as in Proposition 2.2.3.

(2). Else, if $\operatorname{rk}\mathcal{F} = 1$, then

$$D^2 \geq 2d_1 \deg \mathcal{F} + 2\epsilon^* D.Z_1 - Z_1^2.$$

Proof. First, assuming that $\operatorname{rk}\mathcal{F} \geq 2$, for the cases $\hat{n}_{(D,\mathcal{F})} = -\infty, k$, the inequality is proved in Theorem 2.3.4.

We only need to prove the case $1 \leq \hat{n}_{(D,\mathcal{F})} < k$. By Theorem 2.2.17 and Lemma 2.2.8, we have

$$d_i \geq \alpha_{(D,\mathcal{F})}(h^0(N_{i|_F}) - 1) \;\; \forall i; 1 \leq i \leq k.$$

Putting this last inequality into inequality (2.1), we obtain:

$$D^2 \geq 2\alpha_{(D,\mathcal{F})} \sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) - \alpha_{(D,\mathcal{F})}(\mu_1 - \mu_k) + 2\epsilon^* D.Z_k - Z_k^2 + 2\mu_k d_k.$$

By Lemma 2.2.2, we deduce the desired inequality:

$$D^2 \geq 2\alpha_{(D,\mathcal{F})} \deg \mathcal{F} - \alpha_{(D,\mathcal{F})}\mu_1 + (\alpha_{(D,\mathcal{F})} - 2\alpha_{(D,\mathcal{F})}r_k + 2d_k)\mu_k + 2\epsilon^* D.Z_k - Z_k^2.$$

Now, if $\operatorname{rk}\mathcal{F} = 1$, then $\mathcal{F}$ is a locally free sheaf of rank 1, $k = 1$, and $\mu_k = \mu_1$. Therefore,

$$h^0(N_{1|_F}) = r_1 = 1, \;\; d_k = d_1 = g(F) - h^1(N_{1|_F}).$$

By the inequality (2.1), which is a consequence of Lemma 2.3.1 and does not require the assumption that $\mathcal{F}$ has rank at least 2, we have

$$D^2 \geq 2d_1 \deg \mathcal{F} + 2\epsilon^* D.Z_1 - Z_1^2.$$

□

Remark 2.3.8 *If $\mathcal{F}$ is a semi-stable locally free sheaf, then by Corollary 2.3.7, we have the following inequality:*

$$D^2 \geq 2\frac{d_1}{r_1}\deg \mathcal{F} + 2\epsilon^* D.Z_1 - Z_1^2.$$

Now, the goal is to discuss the first point in Corollary 2.3.7 with respect to μ_1 and μ_k. Without loss of generality, we assume that $\mathcal{F}$ is not a semi-stable locally free sheaf in the next theorem.

Theorem 2.3.9 *Let $f : S \to C$ be a fibered surface. Consider the datum $(D, \mathcal{F})$, where D is a relatively effective divisor and $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$ is a locally free sub-sheaf on C. Assume that $\mathcal{F}$ is not semi-stable. Here, we set:*

$$t_{(D,\mathcal{F})} := \begin{cases} \max\{i | \mu_i \geq 0\} & \text{if } \mu_1 \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

(1). If $t_{(D,\mathcal{F})} = 1$, then

$$D^2 \geq \frac{2d_1}{r_1}\deg \mathcal{F}_1 + 2\epsilon^* D.Z_1 - Z_1^2 \geq \frac{2d_1}{r_1}\deg \mathcal{F} + 2\epsilon^* D.Z_1 - Z_1^2.$$

(2). If $1 < t_{(D,\mathcal{F})} \leq k$, then

$$\begin{aligned} D^2 &\geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} d_{t_{(D,\mathcal{F})}}}{d_{t_{(D,\mathcal{F})}} + \alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}} \deg \mathcal{F}_{t_{(D,\mathcal{F})}} + 2\epsilon^* D.Z_{t_{(D,\mathcal{F})}} - Z^2_{t_{(D,\mathcal{F})}} \\ &\geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} d_{t_{(D,\mathcal{F})}}}{d_{t_{(D,\mathcal{F})}} + \alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}} \deg \mathcal{F} + 2\epsilon^* D.Z_{t_{(D,\mathcal{F})}} - Z^2_{t_{(D,\mathcal{F})}}. \end{aligned}$$

(3). If $t_{(D,\mathcal{F})} = -\infty$, set:

$$C_{(D,\mathcal{F})} := \begin{cases} \frac{2\alpha_{(D,\mathcal{F})} d_k}{-\alpha_{(D,\mathcal{F})} + 2\alpha_{(D,\mathcal{F})} r_k - d_k} & \text{if } \alpha_{(D,\mathcal{F})} - 2\alpha_{(D,\mathcal{F})} r_k + 2d_k \leq 0 \\ 3\alpha_{(D,\mathcal{F})} + 2d_k - 2\alpha_{(D,\mathcal{F})} r_k & \text{otherwise.} \end{cases}$$

Then

$$D^2 \geq C_{(D,\mathcal{F})}. \deg \mathcal{F} + 2\epsilon^* D.Z_k - Z_k^2.$$

In addition, if $d_k = \alpha_{(D,\mathcal{F})}(r_k - 1)$, then we have the following inequality which is independent of $t_{(D,\mathcal{F})}$:

$$D^2 \geq \frac{2d_k}{r_k}\deg \mathcal{F} + 2\epsilon^* D.Z_k - Z_k^2.$$

Proof. We recall that $(\mathcal{F}_i)_{1\leq i\leq k}$ is the sequence of Harder-Narasimhan filtration of $\mathcal{F}$. If we take an integer m such that $1 \leq m \leq k$, then $(\mathcal{F}_i)_{1\leq i\leq m}$ is the sequence of Harder-Narasimhan filtration of $\mathcal{F}_m$. By Corollary 2.3.7, we have the following inequality for the datum $(D, \mathcal{F}_m)$ such that $\mathrm{rk}(\mathcal{F}_m) \geq 2$:

$$D^2 \geq 2\alpha_{(D,\mathcal{F}_m)} \deg \mathcal{F}_m - \alpha_{(D,\mathcal{F}_m)}\mu_1 + (\alpha_{(D,\mathcal{F}_m)} - 2\alpha_{(D,\mathcal{F}_m)} r_m + 2d_m)\mu_m + 2\epsilon^* D.Z_m - Z_m^2. \quad (2.2)$$

(1). If $t_{(D,\mathcal{F})} = 1$, we have $\mu_1 \geq 0$ and starting from $i = 2$, $\mu_i < 0$. Thus, we apply Remark 2.3.8 for $\mathcal{F}_1$:

$$D^2 \geq \frac{2d_1}{r_1} \deg \mathcal{F}_1 + 2\epsilon^* D.Z_1 - Z_1^2 \geq \frac{2d_1}{r_1} \deg \mathcal{F} + 2\epsilon^* D.Z_1 - Z_1^2.$$

(2). If $t_{(D,\mathcal{F})} > 1$, we have $\mu_1, \ldots, \mu_{t_{(D,\mathcal{F})}} \geq 0$. Therefore, by considering that

$$\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} - 2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} r_{t_{(D,\mathcal{F})}} + 2d_{t_{(D,\mathcal{F})}} \geq -\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})},$$

and using inequality (2.2)*, we deduce the following inequality for $m = t_{(D,\mathcal{F})}$:*

$$D^2 \geq 2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} \deg \mathcal{F}_{t_{(D,\mathcal{F})}} - \alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}(\mu_1 + \mu_{t_{(D,\mathcal{F})}}) + 2\epsilon^* D.Z_{t_{(D,\mathcal{F})}} - Z_{t_{(D,\mathcal{F})}}^2. \quad (2.3)$$

Now, we apply Lemma 2.3.1 to the sequences $\{Z_1, Z_{t_{(D,\mathcal{F})}}\}$ and $\{\mu_1, \mu_{t_{(D,\mathcal{F})}}\}$. Thus,

$$D^2 \geq d_{t_{(D,\mathcal{F})}}(\mu_1 + \mu_{t_{(D,\mathcal{F})}}) + d_1(\mu_1 - \mu_{t_{(D,\mathcal{F})}}) + 2\epsilon^* D.Z_{t_{(D,\mathcal{F})}} - Z_{t_{(D,\mathcal{F})}}^2. \quad (2.4)$$

Since $d_1(\mu_1 - \mu_{t_{(D,\mathcal{F})}}) \geq 0$, and by Remark 2.2.18, we have $d_{t_{(D,\mathcal{F})}} \geq 1$. Then we can divide by $d_{t_{(D,\mathcal{F})}}$, thus the following inequality follows:

$$-\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}(\mu_1 + \mu_{t_{(D,\mathcal{F})}}) \geq \frac{-\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}}{d_{t_{(D,\mathcal{F})}}} D^2 + 2\frac{\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}}{d_{t_{(D,\mathcal{F})}}} \epsilon^* D.Z_{t_{(D,\mathcal{F})}} - \frac{\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}}{d_{t_{(D,\mathcal{F})}}} Z_{t_{(D,\mathcal{F})}}^2.$$

Combining this last inequality and the inequality (2.3)*, we deduce the desired result:*

$$\begin{aligned} D^2 &\geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} d_{t_{(D,\mathcal{F})}}}{d_{t_{(D,\mathcal{F})}} + \alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}} \deg \mathcal{F}_{t_{(D,\mathcal{F})}} + 2\epsilon^* D.Z_{t_{(D,\mathcal{F})}} - Z_{t_{(D,\mathcal{F})}}^2 \\ &\geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} d_{t_{(D,\mathcal{F})}}}{d_{t_{(D,\mathcal{F})}} + \alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}} \deg \mathcal{F} + 2\epsilon^* D.Z_{t_{(D,\mathcal{F})}} - Z_{t_{(D,\mathcal{F})}}^2. \end{aligned}$$

(3). If $t_{(D,\mathcal{F})} = -\infty$, so both $\mu_1, \mu_k < 0$. By the fact that

$$\alpha_{(D,\mathcal{F})} - 2\alpha_{(D,\mathcal{F})} r_k + 2d_k \geq -\alpha_{(D,\mathcal{F})},$$

we conclude that

$$(\alpha_{(D,\mathcal{F})} - 2\alpha_{(D,\mathcal{F})} r_k + 2d_k)\mu_1 \leq -\alpha_{(D,\mathcal{F})}\mu_1.$$

We replace this last inequality in (2.2) for the datum $(D, \mathcal{F})$, thus

$$D^2 \geq 2\alpha_{(D,\mathcal{F})} \deg \mathcal{F} + (\alpha_{(D,\mathcal{F})} - 2\alpha_{(D,\mathcal{F})} r_k + 2d_k)(\mu_1 + \mu_k) + 2\epsilon^* D.Z_k - Z_k^2.$$

We set

$$A := \alpha_{(D,\mathcal{F})} - 2\alpha_{(D,\mathcal{F})} r_k + 2d_k.$$

To obtain the desired result, we discuss the above inequality depending on whether $A \leq 0$ or $A > 0$.

(a). If $A \leq 0$, then we apply Lemma 2.3.1 to the sequences $\{Z_1, Z_k\}$ and $\{\mu_1, \mu_k\}$:

$$D^2 \geq d_k(\mu_1 + \mu_k) + 2\epsilon^* D.Z_k - Z_k^2. \tag{2.5}$$

Combining the two last inequalities above, we deduce

$$D^2 \geq \frac{2\alpha_{(D,\mathcal{F})} d_k}{-\alpha_{(D,\mathcal{F})} + 2\alpha_{(D,\mathcal{F})} r_k - d_k} \deg \mathcal{F} + 2\epsilon^* D.Z_k - Z_k^2.$$

(b). If $A > 0$, we note that

$$\deg \mathcal{F} \leq \mu_1 + \mu_k.$$

Then,

$$D^2 \geq (3\alpha_{(D,\mathcal{F})} + 2d_k - 2\alpha_{(D,\mathcal{F})} r_k) \deg \mathcal{F} + 2\epsilon^* D.Z_k - Z_k^2.$$

Now, we prove the last point of the theorem. If $d_k = \alpha_{(D,\mathcal{F})}(r_k - 1)$, we apply inequality (2.2) for $m = k$:

$$D^2 \geq 2\alpha_{(D,\mathcal{F})} \deg \mathcal{F} - \alpha_{(D,\mathcal{F})}(\mu_1 + \mu_k) + 2\epsilon^* D.Z_k - Z_k^2.$$

Combining this last inequality and inequality (2.5), we deduce the desired result:

$$D^2 \geq \frac{2d_k}{r_k} \deg \mathcal{F} + 2\epsilon^* D.Z_k - Z_k^2.$$

□

***Proposition 2.3.10** Let $f : S \to C$ be a fibered surface, and let D be a relatively effective and relatively nef divisor on S. Consider the datum $(D, f_*\mathcal{O}_S(D))$. Then the following number is nonnegative:*

$$\epsilon^* D.Z_k \geq 0.$$

Proof. *We recall that $Z_k = \epsilon^* Z_{\mathcal{F}_k} + E$, where $Z_{\mathcal{F}_k}$ is the fixed part of $|D + f^*A|$ and A is a sufficiently very ample divisor on C. Thus, $Z_{\mathcal{F}_k}$ is an effective divisor supported on fibers. By assumption, D is relatively nef, then we conclude the desired nonnegativity:*

$$\epsilon^* D.Z_k = \epsilon^* D.(\epsilon^* Z_{\mathcal{F}_k} + E) = \epsilon^* D.\epsilon^* Z_{\mathcal{F}_k} = D.Z_{\mathcal{F}_k} \geq 0.$$

□

Proposition 2.3.11 *Let $f : S \to C$ be a fibered surface and D be a relatively effective divisor. Consider the datum $(D, f_*\mathcal{O}_S(D))$. Then $Z_k^2 \leq 0$.*

Proof. *Our aim is to prove that $Z_{\mathcal{F}_k}^2 \leq 0$. By contradiction, we assume that $Z_{\mathcal{F}_k}^2 > 0$. Then by the Hodge index theorem, either*

$$\lim_{n\to\infty} h^0(nZ_{\mathcal{F}_k}) = +\infty \ \text{ or } \ \lim_{n\to\infty} h^0(K_S - nZ_{\mathcal{F}_k}) = +\infty.$$

However, the divisor $K_S - nZ_{\mathcal{F}_k}$ is never effective if $Z_{\mathcal{F}_k}$ is not zero and $n \gg 0$. Thus, the only possibility is

$$\lim_{n\to\infty} h^0(nZ_{\mathcal{F}_k}) = +\infty.$$

This contradicts the fact that $Z_{\mathcal{F}_k}$ is not movable. Therefore, we deduce $Z_{\mathcal{F}_k}^2 \leq 0$. Consequently, by the definition of Z_k:

$$Z_k = \epsilon^* Z_{\mathcal{F}_k} + E,$$

we conclude the nonpositivity of Z_k^2; in other words, $Z_k^2 \leq 0$. □

Corollary 2.3.12 *Let $f : S \to C$ be a fibered surface and D be a relatively effective and relatively nef divisor on S. Consider the datum $(D, f_*\mathcal{O}_S(D))$.*

(1). If $f_*\mathcal{O}_S(D)$ is semi-stable, then:

$$D^2 \geq 2\frac{D.F}{h^0(F, D_{|_F})} \deg f_*\mathcal{O}_S(D).$$

(2). If $f_*\mathcal{O}_S(D)$ is not semi-stable, we have the following cases:

(a). If $t_{(D,f_*\mathcal{O}_S(D))}=k$: D is nef and

$$D^2 \geq \frac{2\alpha_{(D,f_*\mathcal{O}_S(D))}D.F}{D.F+\alpha_{(D,f_*\mathcal{O}_S(D))}}\deg f_*\mathcal{O}_S(D).$$

(b). If $t_{(D,f_*\mathcal{O}_S(D))}=1$:

$$D^2 \geq \frac{2d_1}{r_1}\deg f_*\mathcal{O}_S(D)+2\epsilon^* D.Z_1-Z_1^2.$$

(c). If $1<t_{(D,f_*\mathcal{O}_S(D))}<k$:

$$D^2 \geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,f_*\mathcal{O}_S(D))}})}d_{t_{(D,f_*\mathcal{O}_S(D))}}}{d_{t_{(D,f_*\mathcal{O}_S(D))}}+\alpha_{(D,\mathcal{F}_{t_{(D,f_*\mathcal{O}_S(D))}})}}\deg f_*\mathcal{O}_S(D)+2\epsilon^* D.Z_{t_{(D,f_*\mathcal{O}_S(D))}}-Z^2_{t_{(D,f_*\mathcal{O}_S(D))}}.$$

(d). If $t_{(D,f_*\mathcal{O}_S(D))}=-\infty$:

$$D^2 \geq C_{(D,f_*\mathcal{O}_S(D))}.\deg f_*\mathcal{O}_S(D).$$

In addition, if $D.F=\alpha_{(D,f_*\mathcal{O}_S(D))}(h^0(F,D_{|_F})-1)$, then independently of $t_{(D,f_*\mathcal{O}_S(D))}$ we have:

$$D^2 \geq 2\frac{D.F}{h^0(F,D_{|_F})}\deg f_*\mathcal{O}_S(D).$$

Proof. First, if $f_*\mathcal{O}_S(D)$ is semi-stable, then by Remark 2.3.8 we have:

$$D^2 \geq 2\frac{d_1}{r_1}\deg f_*\mathcal{O}_S(D)+2\epsilon^* D.Z_1-Z_1^2.$$

Since $2\epsilon^* D.Z_1-Z_1^2\geq 0$ by Proposition 2.3.10 and Proposition 2.3.11, where $d_1=D.F$ and $r_1=\mathrm{rk}(f_*\mathcal{O}_S(D))=h^0(F,D_{|_F})$, we deduce the desired inequality:

$$D^2 \geq 2\frac{D.F}{h^0(F,D_{|_F})}\deg f_*\mathcal{O}_S(D).$$

Now, if $f_*\mathcal{O}_S(D)$ is not semi-stable, then we apply Theorem 2.3.9 and we have the following cases:

(a). If $t_{(D,f_*\mathcal{O}_S(D))}=k$:

$$D^2 \geq \frac{2\alpha_{(D,f_*\mathcal{O}_S(D))}d_k}{d_k+\alpha_{(D,f_*\mathcal{O}_S(D))}}\deg f_*\mathcal{O}_S(D)+2\epsilon^* D.Z_k-Z_k^2.$$

(b). If $t_{(D,f_*\mathcal{O}_S(S))}=1$:

$$D^2 \geq \frac{2d_1}{r_1}\deg f_*\mathcal{O}_S(D)+2\epsilon^* D.Z_1-Z_1^2.$$

(c). *If* $1 < t_{(D,f_*\mathcal{O}_S(D))} < k$*:*

$$D^2 \geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,f_*\mathcal{O}_S(D))}})} d_{t_{(D,f_*\mathcal{O}_S(D))}}}{d_{t_{(D,f_*\mathcal{O}_S(D))}} + \alpha_{(D,\mathcal{F}_{t_{(D,f_*\mathcal{O}_S(D))}})}} \deg f_*\mathcal{O}_S(D) + 2\epsilon^* D.Z_{t_{(D,f_*\mathcal{O}_S(D))}} - Z^2_{t_{(D,f_*\mathcal{O}_S(D))}}.$$

(d). *If* $t_{(D,f_*\mathcal{O}_S(D))} = -\infty$*:*

$$D^2 \geq C_{(D,f_*\mathcal{O}_S(D))}. \deg f_*\mathcal{O}_S(D) + 2\epsilon^* D.Z_k - Z_k^2.$$

By Proposition 2.3.10 and Proposition 2.3.11, we have that $2\epsilon^* D.Z_k - Z_k^2 \geq 0$*. Furthermore,* $d_k = D.F$*. So we deduce the desired result:*

- *If* $t_{(D,f_*\mathcal{O}_S(D))} = k$*, equivalently* $\mu_k \geq 0$*, then we have*

$$D^2 \geq \frac{2\alpha_{(D,f_*\mathcal{O}_S(D))} D.F}{D.F + \alpha_{(D,f_*\mathcal{O}_S(D))}} \deg f_*\mathcal{O}_S(D).$$

- *If* $t_{(D,f_*\mathcal{O}_S(D))} = -\infty$*, it is equivalent to* $\mu_1 < 0$*. Then*

$$D^2 \geq C_{(D,f_*\mathcal{O}_S(D))}. \deg f_*\mathcal{O}_S(D).$$

Also, if $\mu_k \geq 0$*, then* $f_*\mathcal{O}_S(D)$ *is nef on* C*, so* M_k *is nef on* $\widehat{S}$*. Since*

$$\hat{D} = Z_k + M_k,$$

$Z_{\mathcal{F}_k}$ *is supported in the fibers, and by assumption* D *is relatively nef, thus we deduce that* D *is nef.* □

Remark 2.3.13 *In general, if* D *is nef, we do not necessarily have* $\mu_k \geq 0$*.*

Example 2.3.14 *In (102, Theorem 5), the authors proved that if* D *is a relatively nef divisor on* S *such that* $D_{|_F}$ *is generated by global sections on a general fiber* F *of* f*,* $D_{|_F}$ *is a special divisor on* F*, and*

$$2h^0(F, D_{|_F}) - D.F - 1 > 0, \quad \text{(P)}$$

then

$$D^2 \geq 2\frac{D.F}{h^0(F, D_{|_F})} \deg f_*\mathcal{O}_S(D).$$

First, we remark that the condition (P) *is equivalent to*

$$2h^0(F, D_{|_F}) - D.F - 1 = 1,$$

since by Clifford's Theorem:

$$D.F \geq 2(h^0(F, D_{|_F}) - 1).$$

Thus, we can assume $D_{|_F}$ has a section and is not necessarily generated by global sections since we can always eliminate the horizontal fixed part of D. By Corollary 2.3.12, we see that whether $D_{|_F}$ is special or nonspecial, we proved the same inequality:

$$D^2 \geq 2\frac{D.F}{h^0(F, D_{|_F})} \deg f_*\mathcal{O}_S(D),$$

if the following more general condition holds:

$$D.F = \alpha_{(D, f_*\mathcal{O}_S(D))}(h^0(F, D_{|_F}) - 1). \tag{Q}$$

Theorem 2.3.15 *let $f : S \to C$ be a fibered surface and F its general fiber. If D is a relatively effective and relatively nef divisor, $D_{|_F}$ is nonspecial with $h^0(F, D_{|_F}) > g$, then*

$$D^2 \geq 2\frac{D.F}{h^0(F, D_{|_F})} \deg f_*\mathcal{O}_S(D).$$

Proof. By assumption, $D_{|_F}$ is nonspecial and $h^0(F, D_{|_F}) > g$. Thus, we have $\beta_D \leq 2$ and $\alpha_{(D, f_*\mathcal{O}_S(D))} = \beta_D$. Then the condition (Q) is satisfied:

$$D.F = \alpha_{(D, f_*\mathcal{O}_S(D))}(h^0(F, D_{|_F}) - 1).$$

Finally, the desired inequality follows from Example 2.3.14. □

Proposition 2.3.16 *Let $f : S \to C$ be a fibered surface and D be a nef divisor on S. Consider the datum $(D, \mathcal{F})$. Then, if $t_{(D,\mathcal{F})} \geq 1$:*

$$2\epsilon^* D.Z_i - Z_i^2 \geq 0, \;\; \forall i; 1 \leq i \leq t_{(D,\mathcal{F})}.$$

Proof. Recall that $\epsilon^* D = M_i + Z_i$, thus $2\epsilon^* D.Z_i - Z_i^2 = (\epsilon^* D + M_i)Z_i$. However, M_i and $\epsilon^* D$ are nef, $\forall i; 1 \leq i \leq t_{(D,\mathcal{F})}$. So, we deduce the nonnegativity of $2\epsilon^* D.Z_i - Z_i^2$. □

Corollary 2.3.17 (Compare with (97, Theorem 3.20)) *Let $f : S \to C$ be a fibered surface and D be a nef divisor on S. Consider the datum $(D, \mathcal{F})$ where $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$. Then*

(1). If $\mathcal{F}$ is semi-stable or $t_{(D,\mathcal{F})} = 1$, then

$$D^2 \geq 2\frac{d_1}{r_1} \deg \mathcal{F}_1 \geq 2\frac{d_1}{r_1} \deg \mathcal{F}.$$

(2). If $1 < t_{(D,\mathcal{F})} \leq k$, then

$$D^2 \geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} d_{t_{(D,\mathcal{F})}}}{d_{t_{(D,\mathcal{F})}} + \alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}} \deg \mathcal{F}_{t_{(D,\mathcal{F})}} \geq \frac{2\alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})} d_{t_{(D,\mathcal{F})}}}{d_{t_{(D,\mathcal{F})}} + \alpha_{(D,\mathcal{F}_{t_{(D,\mathcal{F})}})}} \deg \mathcal{F}.$$

Proof. Apply Proposition 2.3.16 and Theorem 2.3.9. □

2.4 Examples and applications

Example 2.4.1 *Let $D = K_{S/C}$ be the relative canonical divisor of a fibered surface $f : S \to C$ with $g(F) \geq 2$. Thus, by (41), $f_*\omega_{S/C}$ is a nef vector bundle on C. Also, $\hat{n}_{(K_{S/C}, f_*\omega_{S/C})} = k$ since $D_{|F} = K_F$ is a special divisor on F. This implies $\alpha_{(K_{S/C}, f_*\omega_{S/C})} = 2$. We also have $t_{(K_{S/C}, f_*\omega_{S/C})} = k$, where k is the length of the Harder-Narasimhan filtration of $f_*\omega_{S/C}$, and $d_k = 2g - 2$ where $g = g(F)$. Then, by Remark 2.3.8 and Theorem 2.3.9, we have:*

$$K_{S/C}^2 \geq 4\frac{g-1}{g} \deg f_*\omega_{S/C} + 2\epsilon^* K_{S/C}.Z_k - Z_k^2.$$

Recall that

$$2\epsilon^* K_{S/C}.Z_k - Z_k^2 = 2K_{S/C}.Z_{\mathcal{F}_k} - Z_{\mathcal{F}_k}^2 - E^2,$$

*where E is the exceptional divisor of ϵ (Proposition 2.2.3). Here $\mathcal{F}_k = \mathcal{F} = f_*\omega_{S/C}$. In (**?**, Example 2.1), the authors considered the case where f is a relatively minimal nodal fibration. They calculate the term*

$$2K_{S/C}.Z_{\mathcal{F}_k} - Z_{\mathcal{F}_k}^2 - E^2,$$

and prove that if $n \geq 1$ is the total number of disconnecting nodes contained in the fibres, we have

$$2K_{S/C}.Z_{\mathcal{F}_k} - Z_{\mathcal{F}_k}^2 - E^2 \geq n.$$

Therefore,

$$K_{S/C}^2 \geq 4\frac{g-1}{g} \deg f_*\omega_{S/C} + n.$$

This implies if f is relatively minimal and

$$K_{S/C}^2 = 4\frac{g-1}{g}\deg f_*\omega_{S/C},$$

then f is never a relatively minimal nodal fibration with at least 1 disconnected node.

Furthermore, we remark that if f is relatively minimal, then by Corollary 2.3.12, $K_{S/C}$ is nef and therefore

$$2K_{S/C}.Z_{\mathcal{F}_k} - Z_{\mathcal{F}_k}^2 - E^2 \geq 0.$$

Moreover, to have

$$K_{S/C}^2 = 4\frac{g-1}{g}\deg f_*\omega_{S/C},$$

it must be that

$$2K_{S/C}.Z_{\mathcal{F}_k} - Z_{\mathcal{F}_k}^2 - E^2 = 0.$$

Then $|K_{S/C} + f^*\mathcal{O}(A)|$ has no fixed part and is base point free for a sufficiently ample divisor on C.

In general, if f is relatively minimal, we always have the original Xiao's result (108):

$$K_{S/C}^2 \geq 4\frac{g-1}{g}\deg f_*\omega_{S/C}.$$

Example 2.4.2 *Let $D = mK_{S/C}$ be the relative pluricanonical divisor of a fibered surface $f\colon S \to C$, with $g = g(F) \geq 2$, and $m \geq 2$. It is well known that $f_*\omega_{S/C}^{\otimes m}$ is a nef vector bundle on C, see (105, Theorem 1.3) for instance. If $K_{S/C}$ is relatively nef, then by Corollary 2.3.12, we have the following cases:*

- *If $f_*\omega_{S/C}^{\otimes m}$ is semi-stable:*

  $$m^2K_{S/C}^2 \geq \frac{4m}{2m-1}\deg f_*\omega_{S/C}^{\otimes m},$$

  since

  $$\mathrm{rk}(f_*\omega_{S/C}^{\otimes m}) = (2m-1)(g-1).$$

  Therefore,

  $$K_{S/C}^2 \geq \frac{4}{m(2m-1)}\deg f_*\omega_{S/C}^{\otimes m}.$$

- *If $f_*\omega_{S/C}^{\otimes m}$ is not semi-stable, then using the fact that mK_F is a nonspecial divisor, we see that*

  $$\hat{n}_{(mK_{S/C}, f_*\omega_{S/C}^{\otimes m})} < k,$$

and

$$\alpha_{(mK_{S/C}, f_*\omega_{S/C}^{\otimes m})} = 1 + \frac{g}{h^0(F, mK_F) - 1},$$

since

$$h^0(F, mK_F) > g.$$

Moreover,

$$t_{(mK_{S/C}, f_*\omega_{S/C}^{\otimes m})} = k,$$

because $f_*\omega_{S/C}^{\otimes m}$ is a nef vector bundle on C. Then,

$$m^2 K_{S/C} \geq \frac{2\alpha_{(mK_{S/C}, f_*\omega_{S/C}^{\otimes m})} d_k}{d_k + \alpha_{(mK_{S/C}, f_*\omega_{S/C}^{\otimes m})}} \deg f_*\omega_{S/C}^{\otimes m}.$$

Recall that $d_k = 2m(g-1)$ and $h^0(F, mK_F) = (2m-1)(g-1)$, thus

$$\frac{2\alpha_{(mK_{S/C}, f_*\omega_{S/C}^{\otimes m})} d_k}{m^2(d_k + \alpha_{(mK_{S/C}, f_*\omega_{S/C}^{\otimes m})})} = \frac{4}{m(2m-1)}.$$

Therefore,

$$K_{S/C}^2 \geq \frac{4}{m(2m-1)} \deg f_*\omega_{S/C}^{\otimes m}.$$

In this computations, we see that whether $f_*\omega_{S/C}^{\otimes m}$ is semi-stable or not, we have the same lower bound. The reason is that

$$d_k = \alpha_{(mK_{S/C}, f_*\omega_{S/C}^{\otimes m})}.(r_k - 1),$$

as explained in the last item of Corollary 2.3.12. However, in general, for datum $(D, \mathcal{F})$ where $\mathcal{F} \subsetneq f_*\mathcal{O}_S(D)$, the constant $\frac{2\alpha_{(D,\mathcal{F})} d_k}{d_k + \alpha_{(D,\mathcal{F})}}$ is different from $\frac{2d_k}{r_k}$ because the inequality $d_k > \alpha_{(D,\mathcal{F})}(r_k - 1)$ can well happen.

Example 2.4.3 *Now, let $f : S \to C$ be a fibered surface and F its general fiber with $g(F) \geq 2$. We take $D = K_{S/C} + L$ such that L is nef and relatively big with $L.F > 1$. We know that $f_*\mathcal{O}_S(D)$ is a nef vector bundle on C with*

$$\mathrm{rk}(f_*\mathcal{O}_S(D)) = g - 1 + L.F \neq 0.$$

If D is a relatively nef divisor, then by Corollary 2.3.12, we have the following lower bound for $(K_{S/C} + L)^2$:

$$(K_{S/C} + L)^2 \geq 2\left(1 + \frac{g-1}{g-1+L.F}\right) \deg f_*(\omega_{S/C} \otimes \mathcal{O}(L)),$$

because we know the following information for the datum $(D, f_*\mathcal{O}_S(D))$:

$$d_k = \alpha_{(D,f_*\mathcal{O}_S(D))}(r_k - 1), \ \ \hat{n}_{(D,f_*\mathcal{O}_S(D))} < k, \ \alpha_{(D,f_*\mathcal{O}_S(D))} = 1 + \frac{g}{g - 2 + L.F},$$

also the quantity $t_{(D,f_*\mathcal{O}_S(D))}$ is maximal, this means $t_{(D,f_*\mathcal{O}_S(D))} = k$.

Example 2.4.4 *We give a trivial example in which we can see that $t_{(D,f_*\mathcal{O}_S(D))} = -\infty$ can happen. Let $f : S \to C$ be a fibered surface and F its general fiber with $g(F) = 2$. Assume that $K_{S/C}$ is a nef divisor and the second Fujita decomposition (Theorem 0.0.2) of $f_*\omega_{S/C}$ is not trivial. Then there exists an ample line bundle $\mathcal{A}$ on C and a flat line bundle $\mathcal{U}$ such that*

$$f_*\omega_{S/C} = \mathcal{A} \oplus \mathcal{U}.$$

Now, let $D = K_{S/C} + f^*\mathcal{M}$ for a sufficiently negative divisor $\mathcal{M}$ on C such that $\deg(\mathcal{A} \otimes \mathcal{O}(\mathcal{M})) < 0$. In this case, D is a relatively nef divisor on S such that $D_{|_F} = K_F$ and the bundle $f_*\mathcal{O}_S(D)$ decomposes into two parts:

$$f_*\mathcal{O}_S(D) = \mathcal{A} \otimes \mathcal{M} \oplus \mathcal{U} \otimes \mathcal{M}.$$

The Harder-Narasimhan filtration of $f_*\mathcal{O}_S(D)$ is the following:

$$0 \subsetneq \mathcal{A} \otimes \mathcal{M} \subsetneq f_*\mathcal{O}_S(D).$$

We consider the datum $(D, f_*\mathcal{O}_S(D))$. Since $D_{|_F} = K_F$, we see that $D_{|_F}$ is special, $\hat{n}_{(D,f_*\mathcal{O}_S(D))} = 2$, and $\alpha_{(D,f_*\mathcal{O}_S(D))} = 2$. Since $g(F) = 2$, then we have $d_2 = 2$. Also, clearly, we have $t_{(D,f_*\mathcal{O}_S(D))} = -\infty$. Applying Corollary 2.3.12, we deduce the following inequality:

$$D^2 \geq C_{(D,f_*\mathcal{O}_S(D))} \deg f_*\mathcal{O}_S(D).$$

We know that $C_{(D,f_*\mathcal{O}_S(D))} = 2$, so we replace $C_{(D,f_*\mathcal{O}_S(D))}$ by 2 in the above inequality:

$$D^2 \geq 2 \deg f_*\mathcal{O}_S(D).$$

Additionally, we remark that $d_k = \alpha_{(D,f_*\mathcal{O}_S(D))}(r_k - 1)$. Thus, again by Corollary 2.3.12 or Example 2.3.14, we deduce the same lower bound.

Proposition 2.4.5 *Let $f : S \to C$ be a fibered surface and F its general fiber with $g(F) \geq 2$. Then*

$$K_{S/C}^2 = \frac{2}{m(m-1)} \left(\deg f_*\omega_{S/C}^{\otimes m} - \deg f_*\omega_{S/C} \right), \ \ \forall m \geq 2.$$

Remark 2.4.6 *In the setting of Proposition 2.4.5, in particular:*

$$K_{S/C}^2 = \deg f_*\omega_{S/C}^{\otimes 2} - \deg f_*\omega_{S/C}.$$

Proof. Recall that if $S \to C$ is a fibered surface, then for any line bundle $\mathcal{L} = \mathcal{O}_S(L)$ on S, we have the following Grothendieck-Riemann-Roch formula (2, page 333):

$$\deg f_!\mathcal{L} = \deg f_*\mathcal{L} - \deg \mathcal{R}^1 f_*\mathcal{L} = \frac{L(L-K_{S/C})}{2} + \deg f_*\omega_{S/C}.$$

In particular, we apply the above formula to the relative pluricanonical bundle $\omega_{S/C}^{\otimes m}$, thus we obtain

$$\deg f_*\omega_{S/C}^{\otimes m} - \deg \mathcal{R}^1 f_*\omega_{S/C}^{\otimes m} = \frac{m(m-1)K_{S/C}^2}{2} + \deg f_*\omega_{S/C}.$$

But $\mathcal{R}^1 f_*\omega_{S/C}^{\otimes m} = 0$ because $\mathcal{R}^1 f_*\omega_{S/C}^{\otimes m}$ is known to be torsion free and

$$\operatorname{rk} \mathcal{R}^1 f_*\omega_{S/C}^{\otimes m} = h^1(F, \omega_F^{\otimes m}) = h^0(F, \omega_F^{\otimes(1-m)}) = 0,$$

since $g(F) \geq 2$ by assumption. Thus, we obtain the desired formula:

$$K_{S/C}^2 = \frac{2}{m(m-1)}\left(\deg f_*\omega_{S/C}^{\otimes m} - \deg f_*\omega_{S/C}\right), \quad \forall m \geq 2.$$

□

Remark 2.4.7 *Using Noether's Formula, Xiao (108, Theorem 2) remarked that*

$$K_{S/C}^2 \leq 12 \deg f_*\omega_{S/C}.$$

Proposition 2.4.8 *Let $f : S \to C$ be a fibered surface and F its general fiber with $g(F) \geq 2$. Then*

$$K_{S/C}^2 \leq \frac{12}{6m(m-1)+1} \deg f_*\omega_{S/C}^{\otimes m}, \quad \forall m \geq 2.$$

Proof. We recall that in Proposition 2.4.5, we proved that

$$K_{S/C}^2 = \frac{2}{m(m-1)}\left(\deg f_*\omega_{S/C}^{\otimes m} - \deg f_*\omega_{S/C}\right), \quad \forall m \geq 2$$

which implies

$$\frac{m(m-1)}{2}K_{S/C}^2 = \deg f_*\omega_{S/C}^{\otimes m} - \deg f_*\omega_{S/C}, \quad \forall m \geq 2.$$

Thus,

$$\deg f_*\omega_{S/C} = \deg f_*\omega_{S/C}^{\otimes m} - \frac{m(m-1)}{2}K_{S/C}^2, \ \ \forall m \geq 2. \tag{2.6}$$

Now, by Remark 2.4.7, we have

$$K_{S/C}^2 \leq 12 \deg f_*\omega_{S/C}.$$

Combining this inequality with equality (2.6), it follows that

$$K_{S/C}^2 \leq 12\left(\deg f_*\omega_{S/C}^{\otimes m} - \frac{m(m-1)}{2}K_{S/C}^2\right), \ \ \forall m \geq 2,$$

which implies

$$K_{S/C}^2 \leq 12 \deg f_*\omega_{S/C}^{\otimes m} - 6m(m-1)K_{S/C}^2, \ \ \forall m \geq 2.$$

Therefore,

$$(6m(m-1)+1)K_{S/C}^2 \leq 12 \deg f_*\omega_{S/C}^{\otimes m}, \ \ \forall m \geq 2,$$

which further simplifies to the desired inequality:

$$K_{S/C}^2 \leq \frac{12}{6m(m-1)+1} \deg f_*\omega_{S/C}^{\otimes m}, \ \ \forall m \geq 2.$$

□

Remark 2.4.9 *If $K_{S/C}^2 > 0$ and $g(F) \geq 2$, then $\deg f_*\omega_{S/C}^{\otimes m} > 0$, $\forall m \geq 2$.*

Lemma 2.4.10 *Let $f : S \to C$ be a fibered surface, F its general fiber with $g(F) \geq 2$, and $\mathcal{F} \subseteq f_*\omega_{S/C}^{\otimes m}$ with $\deg \mathcal{F} \geq 0$, $m \geq 2$. Then*

$$\mu(\mathcal{F}) \leq \frac{6m}{(6m(m-1)+1)(g-1)} \deg f_*\omega_{S/C}^{\otimes m},$$

where $\mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}}$.

Proof. We consider the datum $(mK_{S/C}, \mathcal{F})$ and let $\mathcal{F}^{'}$ be the maximal destabilizing vector sub-bundle of $\mathcal{F}$. We apply Lemma 2.3.1 to the sequences $\{Z(\mathcal{F}^{'}), 0\}$ and $(\mu(\mathcal{F}^{'}), 0)$. Then, we have

$$K_{S/C}^2 \geq \frac{2}{m}(g-1)\mu(\mathcal{F}^{'}).$$

Here, $Z(\mathcal{F}^{'})$ is the fixed part of the sub-bundle $\mathcal{F}^{'}$ and $\mu(\mathcal{F}^{'}) = \frac{\deg \mathcal{F}^{'}}{\operatorname{rk} \mathcal{F}^{'}}$. Since $\mathcal{F}^{'}$ is the maximal destabilizing vector sub-bundle of $\mathcal{F}$, we have

$$\mu(\mathcal{F}^{'}) \geq \mu(\mathcal{F}).$$

Combining the last two inequalities and Proposition 2.4.8, we deduce the desired inequality:

$$\mu(\mathcal{F}) \leq \frac{6m}{(6m(m-1)+1)(g-1)} \deg f_*\omega_{S/C}^{\otimes m}.$$

$\square$

Now, we state the First Fujita decomposition for the relative pluricanonical bundle and adjoint canonical bundle in the case of a fibered surface.

Theorem 2.4.11 *Let $f : S \to C$ be a fibered surface and F its general fiber, let L be a semi-ample line bundle on S. Then*

$$f_*\omega_{S/C}^{\otimes m} = \mathcal{N}_m \oplus \mathcal{O}_C^{\oplus p_m}, \ \ \forall m \geq 2, \text{ and } f_*(\omega_{S/C} \otimes L) = \mathcal{N}_L \oplus \mathcal{O}_C^{\oplus p_L}.$$

Here, $H^0(C, \mathcal{N}_m^\vee) = 0$ and $H^0(C, \mathcal{N}_L^\vee) = 0$.

Proof. Note $p_m := h^0(C, (f_*\omega_{S/C}^{\otimes m})^\vee) = h^0(C, \mathcal{R}^1 f_*\omega_{S/C}^{\otimes(1-m)})$, since $(f_*\omega_{S/C}^{\otimes m})^\vee \simeq \mathcal{R}^1 f_*\omega_{S/C}^{\otimes(1-m)}$. We take $\{s_1, \dots, s_{p_m}\}$ as a basis of $H^0(C, \mathcal{R}^1 f_*\omega_{S/C}^{\otimes(1-m)}) \simeq Hom(\mathcal{O}_C, \mathcal{R}^1 f_*\omega_{S/C}^{\otimes(1-m)})$. Then $s_1 \oplus \cdots \oplus s_{p_m}$ defines a map:

$$s_1 \oplus \cdots \oplus s_{p_m} : \mathcal{O}_C^{\oplus p_m} \longrightarrow \mathcal{R}^1 f_*\omega_{S/C}^{\otimes(1-m)}$$

which yields the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_C^{\oplus p_m} \longrightarrow \mathcal{R}^1 f_*\omega_{S/C}^{\otimes(1-m)} \longrightarrow \mathcal{Q}_m \longrightarrow 0$$

where $\mathcal{Q}_m$ is the quotient bundle. By duality, we have

$$0 \longrightarrow \mathcal{Q}_m^\vee \longrightarrow f_*\omega_{S/C}^{\otimes m} \longrightarrow \mathcal{O}_C^{\oplus p_m} \longrightarrow 0.$$

We deduce from (49, Theorem 26.4) that the last exact sequence splits since the bundle $f_*\omega_{S/C}^{\otimes m}$ admits a singular hermitian metric with semi-positive curvature and verifies the minimal extension property, see (49) for more details. Hence, the first Fujita decomposition follows:

$$f_*\omega_{S/C}^{\otimes m} = \mathcal{N}_m \oplus \mathcal{O}_C^{\oplus p_m}$$

Here, $\mathcal{N}_m = \mathcal{Q}_m^\vee$. and $h^0(C, \mathcal{Q}_m) = 0$.

For the adjoint case, we note $p_L := h^0(C, (f_*(\omega_{S/C} \otimes L))^\vee)$, the proof is the same as in the relative pluricanonical case, since L admits a smooth hermitian metric h with semi-positive curvature and a trivial multiplier ideal sheaf. Thus, $f_*(\omega_{S/C} \otimes L)$ admits a singular hermitian metric with semi-positive curvature and verifies the minimal extension property. □

In (73), the authors proved that the vector bundles $f_*\omega_{S/C}^{\otimes m}$ and $f_*(\omega_{S/C} \otimes L)$, where L is a semi-ample line bundle on S, admit a Catanese-Fujita-Kawamata decomposition, $\forall m \geq 2$. We state the result in the case of fibered surfaces.

Theorem 2.4.12 *(73, Theorem 2) Let $f : S \to C$ be a fibered surface and F its general fiber. Let L be a semi-ample line bundle on S. Then,*

$$f_*\omega_{S/C}^{\otimes m} = \mathcal{A}_m \oplus \mathcal{U}_m, \ \ \forall m \geq 2, \text{ and } f_*(\omega_{S/C} \otimes L) = \mathcal{A}_L \oplus \mathcal{U}_L.$$

Here, $\mathcal{A}_m$ and $\mathcal{A}_L$ are ample vector sub-bundles of $f_*\omega_{S/C}^{\otimes m}$ and $f_*(\omega_{S/C} \otimes L)$ respectively, $\mathcal{U}_m$ and $\mathcal{U}_L$ are hermitian flat vector sub-bundles of $f_*\omega_{S/C}^{\otimes m}$ and $f_*(\omega_{S/C} \otimes L)$ respectively.

In the following paragraphs, we derive some explicit consequences for the direct image of relative pluricanonical bundles.

Example 2.4.13 *If f is not isotrivial, it is known that $f_*\omega_{S/C}^{\otimes m}$ is ample if not zero $\forall m \geq 2$. This implies that the trivial part in the First Fujita decomposition is zero, and the flat part in the Catanese-Fujita-Kawamata decomposition is zero.*

Example 2.4.14 *If L is an ample line bundle on S, it is well known that $f_*(\omega_{S/C} \otimes L)$ is an ample vector bundle on C. Equivalently, the trivial part in the First Fujita decomposition is zero, and the flat part in the Catanese-Fujita-Kawamata decomposition is zero.*

Recall an important result on the direct image of pluricanonical sheaf over a curve due to Viehweg (105, Theorem 1.3) and (105, Proposition 4.6). We restrict ourselves to a fibered surface case.

Proposition 2.4.15 *Let $f : S \to C$ be a fibered surface and F be its general fiber. The following conditions are equivalent:*

(1). For all $m \geq 2$, the vector bundle $f_*\omega_{S/C}^{\otimes m}$ is ample if not zero.

(2). There exist some $m \geq 1$ such that $f_*\omega_{S/C}^{\otimes m}$ contains an ample sub-sheaf.

(3). There exist some $m \geq 1$ such that $\deg f_*\omega_{S/C}^{\otimes m} > 0$.

Moreover, if f is semi-stable, then the conditions $(1), (2)$, and (3) are equivalent to

(4). f is not isotrivial.

Corollary 2.4.16 (Compare with (73, Corollary 5.2)) *Let $f : S \to C$ be a fibered surface. Then, either $f_*\omega_{S/C}^{\otimes m}$ is ample, $\forall m \geq 2$ when $f_*\omega_{S/C}^{\otimes m} \neq 0$, or $f_*\omega_{S/C}^{\otimes m}$ is hermitian flat $\forall m \geq 2$.*

Proof. Apply Proposition 2.4.15 and Theorem 2.4.12. □

Remark 2.4.17 *If f is semi-stable, then $f_*\omega_{S/C}^{\otimes m}$ being hermitian flat $\forall m \geq 2$ is equivalent to f being isotrivial.*

Remark 2.4.18 *By Corollary 2.4.16, we observe that $f_*\omega_{S/C}^{\otimes m}$ is ample $\forall m \geq 2$ when $f_*\omega_{S/C}^{\otimes m} \neq 0$ or $f_*\omega_{S/C}^{\otimes m}$ is hermitian flat $\forall m \geq 2$. According to Proposition 2.4.15, if $f_*\omega_{S/C}^{\otimes m}$ is flat $\forall m \geq 2$, then $f_*\omega_{S/C}$ is flat. Conversely, if $f_*\omega_{S/C}^{\otimes m}$ is ample $\forall m \geq 2$, $g(F) \geq 2$, and $K_{S/C}$ is nef, then $f_*\omega_{S/C}$ is not flat by Example 2.4.2.*

In this last paragraph, we will explore the relationship between $\deg f_*\omega_{S/C}$ and $\deg f_*\omega_{S/C}^{\otimes m}$ in the case where $f_*\omega_{S/C}^{\otimes m}$ is ample $\forall m \geq 2$.

Corollary 2.4.19 *Let $f : S \to C$ be a fibered surface and F its general fiber with $g(F) \geq 2$. Then:*

$$\deg f_*\omega_{S/C} \leq \frac{6mg}{(6m(m-1)+1)(g-1)} \deg f_*\omega_{S/C}^{\otimes m}, \ \forall m \geq 2.$$

In particular, if $g > 6m(m-1)+1$, then:

$$\deg f_*\omega_{S/C} < \frac{1}{m-1}\deg f_*\omega_{S/C}^{\otimes m}, \ \forall m \geq 2.$$

Proof. Apply Lemma 2.4.10. □

Example 2.4.20 *If $m = 2$ and $g > 13$, then $\frac{12g}{13(g-1)} < 1$, and*

$$\deg f_*\omega_{S/C} \leq \frac{12g}{13(g-1)}\deg f_*\omega_{S/C}^{\otimes 2}.$$

If $m = 3$, then

$$\deg f_*\omega_{S/C} \leq \frac{18g}{37(g-1)}\deg f_*\omega_{S/C}^{\otimes 3}.$$

In particular, if $g > 37$, we deduce that $\frac{18g}{37(g-1)} < \frac{1}{2}$ and

$$\deg f_*\omega_{S/C} < \frac{1}{2}\deg f_*\omega_{S/C}^{\otimes 3}.$$

If $m = 4$, then

$$\deg f_*\omega_{S/C} \leq \frac{24g}{73(g-1)}\deg f_*\omega_{S/C}^{\otimes 4}.$$

Furthermore, if $g > 73$, then $\frac{24g}{73(g-1)} < \frac{1}{3}$ and

$$\deg f_*\omega_{S/C} < \frac{1}{3}\deg f_*\omega_{S/C}^{\otimes 4}.$$

CHAPTER 3

XIAO'S CONJECTURE ON CANONICALLY FIBERED SURFACES

3.1 Proof of the conjecture if the base is an elliptic curve

To prove Conjecture 1.2.5, we may assume that such fibrations exist, and we will find contradictions.

Let $f : S \dashrightarrow C$ be a canonically fibered general type surface of geometric genus P_g and with general fiber F of $g(F) = 5$, by resolving the indeterminacy locus we can assume that f is regular and also K_S is relatively nef since we can replace the surface S by one of its birational models.

By assumption the canonical bundle K_S is composed with pencil, thus it decomposes as a sum of fixed part N and a moving part M coming from the base, that is:

$$K_S = N + M = N + f^*D$$

such that

$$\deg D = P_g + g(C) - 1.$$

To prove the conjecture, it is enough to assume that

$$K_S = 8\Gamma + V + f^*D$$

with Γ is a section, V is the vertical part of N, and $\deg D := P_g + g(C) - 1$ (see (29, Theorem 2.3)). We know the following decomposition of $f_*\omega_S$:

$$f_*\omega_S = \mathcal{O}_C(D) \oplus \mathcal{F},$$

here $\mathcal{F}$ is a rank 4 vector bundle on C with $h^0(C, \mathcal{F}) = 0$. It implies that

$$f_*(8\Gamma + V) = \mathcal{O}_C \oplus \mathcal{F} \otimes \mathcal{O}_C(-D).$$

Remark 3.1.1 *In case of C is an elliptic curve, the sheaf $\mathcal{F}$ is a flat bundle of rank 4.*

As mentioned in Section 1.2, for any nontrivial fibered surface $f : S \to C$ with general fiber F of genus $g := g(F)$, Xiao predicted a sharp upper bound for the relative irregularity

$$q(f) := q(S) - g(C).$$

Specifically, he conjectured that $q(f) \leq \frac{g+1}{2}$. However, this conjecture was disproved by Pirola (91) through a series of counterexamples for $g(F) = 4$. The modified conjecture states that $q(f) \leq \frac{g}{2} + 1$ (Conjecture 1.1.5). This conjecture has been proven for $g(F) \leq 5$ (see (3)), except when the general fiber is trigonal with $g(F) = 5$. Fortunately, this case was recently settled by Martin (80). In other words, it is now known that for any nontrivial fibered surface $f : S \to C$ with general fiber F of genus $g(F) = 5$, we have $q(f) \leq 3$.

Theorem 3.1.2 *There is no canonically fibered general type surface $f : S \to C$ with general fiber F of $g(F) = 5$ and $g(C) = 1$.*

Proof. By contradiction, assume that there exists a canonically fibered surface $f : S \to C$ with general fiber F of genus $g(F) = 5$ and $g(C) = 1$. We know that

$$f_*\omega_S = \mathcal{O}_C(D) \oplus \mathcal{F},$$

where D is ample on C with $\deg D = P_g$, and $\mathcal{F}$ is a unitary flat vector bundle of rank 4. We know that $f_*\omega_S$ is semi-ample. The flat part is decomposed as $\mathcal{F} = \bigoplus \mathcal{L}_i$, where $\mathcal{L}_i$ are torsion line bundles with finite orders. This follows from the so-called Chen-Jiang decomposition (27), (72). Take an isogeny $\sigma : \widetilde{C} \to C$ such that after this base change, the pullback of $\mathcal{F}$ becomes trivial; in other words, $\widetilde{\mathcal{F}} := \sigma^*\mathcal{F} = \mathcal{O}_{\widetilde{C}}^{\oplus 4}$. This base change is described by the following commutative diagram:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\psi} & S \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{C} & \xrightarrow[\sigma]{} & C. \end{array}$$

The pullback nontrivial fibered surface $\widetilde{f} : \widetilde{S} \to \widetilde{C}$ with general fiber $\widetilde{F}$ of $g(\widetilde{F}) = 5$ satisfies the following

$$\widetilde{f}_*\omega_{\widetilde{S}} = \mathcal{L} \oplus \mathcal{O}_{\widetilde{C}}^{\oplus 4}.$$

Here $\mathcal{L}$ is an ample line bundle on $\widetilde{C}$ and $q(\widetilde{f}) = 4$. This is clearly a contradiction. □

3.2 Xiao's conjecture if the base is $\mathbb{CP}^1$

In this section, we explain the method for addressing the open case of Conjecture 1.2.5. Note that this is a joint work with Chen and Grieve. First, we announce the following Proposition.

Proposition 3.2.1 *Let C be a smooth projective curve of genus 5, If there exists a point $p \in C$ such that $\mathcal{O}_C(8p) = K_C$, then $h^0(\mathcal{O}_C(3p)) = 1$ and $h^0(\mathcal{O}_C(5p)) = 2$.*

Proof. If C is nontrigonal, then by definition $h^0(\mathcal{O}_C(3p)) = 1$. So without loss of generality, we can assume that C is trigonal. If $h^0(\mathcal{O}_C(3p)) > 1$, then $|\mathcal{O}_C(3p)|$ is the unique g_3^1 on C. Since C is not hyperelliptic, C can be canonically embedded into $\mathbb{P}^4$ as $C \hookrightarrow \mathbb{P}^4$ with a hyperplane $\Lambda \subset \mathbb{P}^4$ satisfying $\Lambda.C = 8p$.

The base locus of $|I_C(2)|$ is a rational normal scroll $R \subset \mathbb{P}^4$. We know that $C \subset R \cong \mathbb{F}_1$ is a smooth curve in $|3A + 5F|$, where A and F are the effective generators of Pic(R) with $A^2 = -1$, $F^2 = 0$ and $AF = 1$. And $\Lambda \cap R = D \in |A + 2F|$.

Since $\mathcal{O}_C(3p)$ is the unique g_3^1, $\mathcal{O}_C(3p) = \mathcal{O}_C(F)$. So there exists a curve in $|F|$, which we still denote by F, such that $F.C = 3p$ on R. On the other hand, $D.C = 8p$ on R.

Suppose that $D = D_1 + D_2$, where $D_1 \in |A + mF|$ is irreducible for some $0 \le m \le 2$. Then

- $D_1.C = (3m + 2)p$,
- $F.C = 3p$.
- D_1 *and* F *meet transversely at* p, *and*
- C *is smooth at* p.

This is clearly impossible. So $h^0(\mathcal{O}_C(3p)) = 1$.

It follows by an easy application of Riemann-Roch Theorem on curves that $h^0(\mathcal{O}_C(5p)) = 2$. □

Remark 3.2.2 *We inform the reader that the proof of Proposition 3.2.1 is due to Xi Chen in a private communication.*

Recall that for $f : S \to C$ a canonically fibered general type surface of geometric genus P_g and with general fiber F of $g(F) = 5$, we have

$$K_S = 8\Gamma + V + f^*D$$

Theorem 3.2.3 *Let $f : S \to C$ be a canonically fibered surface with nonhyperelliptic general fiber F of $g(F) = 5$, and*

$$K_S = 8\Gamma + V + f^*D.$$

Then

1. *If $h^0(\mathcal{O}_F(4p)) = 1$, then we have the following commutative diagram:*

$$\begin{array}{ccc} S & \overset{\phi}{\dashrightarrow} & \mathbb{P}_C(f_*(5\Gamma)) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle pr} \\ & & C \end{array} \tag{3.1}$$

   ϕ is a rational map regular on the general fiber F and $\mathbb{P}_C(f_*(5\Gamma))$ is a ruled surface.

2. *If $h^0(\mathcal{O}_F(4p)) = 2$, then we have the following commutative diagram:*

$$\begin{array}{ccc} S & \overset{\phi}{\dashrightarrow} & \mathbb{P}_C(f_*(4\Gamma)) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle pr} \\ & & C \end{array} \tag{3.2}$$

   in this case ϕ also is a rational map regular on the general fiber F and $\mathbb{P}_C(f_*(4\Gamma))$ is a ruled surface.

Here $p := \Gamma.F$

Proof. The proof easily follows by Proposition 3.2.1. □

Remark 3.2.4 *We guess that in Proposition 3.2.3, the map ϕ should be regular. So, we state the following conjecture.*

Conjecture 3.2.5 *Let $f : S \to C$ be a canonically fibered surface with nonhyperelliptic general fiber F of $g(F) = 5$, and*

$$K_S = 8\Gamma + V + f^*D.$$

Then

1. If $h^0(\mathcal{O}_F(4p)) = 1$, then the map $\phi : S \dashrightarrow \mathbb{P}_C(f_*(5\Gamma))$ defined by the linear system $|5\Gamma + f^*A|$ for a sufficiently ample divisor A is regular.
2. If $h^0(\mathcal{O}_F(4p)) = 2$, then the map $\phi : S \dashrightarrow \mathbb{P}_C(f_*(4\Gamma))$ defined by the linear system $|4\Gamma + f^*A|$ for a sufficiently ample divisor A is regular.

As we mentioned in the introduction, we settle Conjecture 3.2.5 in the joint paper with Chen and Grieve (10). Thus, we present the method to completely prove Xiao's Conjecture 1.2.5. This provides another proof of Chen's result (29).

Theorem 3.2.6 *Let* $f : S \to C$ *be a canonically fibered surface with geometric genus* P_g*, and nonhyperelliptic general fiber* F *with* $g(F) = 5$*, and*

$$K_S = 8\Gamma + V + f^*D.$$

1. If $h^0(\mathcal{O}_F(4p)) = 1$, then
$$\Gamma^2 \leq -\frac{1}{5}P_g.$$
2. If $h^0(\mathcal{O}_F(4p)) = 2$, then
$$\Gamma^2 \leq -\frac{1}{4}P_g.$$

Proof. It is sufficient to prove the first case, as the proof of the second follows exactly the same steps. Let us assume that $h^0(\mathcal{O}_F(4p)) = 1$*. Thus, we have the following diagram:*

$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & \mathbb{P}_C(f_*(5\Gamma)) \\ & {\scriptstyle f}\searrow & \big\downarrow {\scriptstyle pr} \\ & & C \end{array} \tag{3.3}$$

with ϕ *is a generically finite morphism of degree* 5*. We define the section* $T := \phi(\Gamma)$*. Then* $5\Gamma = \phi^*(T)$ *(because* $p = \Gamma_{|F}$ *is the total ramification point of* $\phi_{|F}$*), and of course*

$$\Gamma^2 = \frac{1}{5}T^2. \tag{3.4}$$

We define the divisor $R := T + pr^*D$ *(recall that* $\deg D := \mathcal{O}_C(P_g + g(C) - 1)$*). Then* T *is the fixed part of* $|R|$ *and*

$$h^0(\mathbb{P}_C(f_*(5\Gamma)), R) = h^0(\mathbb{P}_C(f_*(5\Gamma)), pr^*D) = h^0(C, D) = P_g. \tag{3.5}$$

By the Riemann-Roch theorem, we have:

$$\chi(\mathbb{P}_C(f_*(5\Gamma)), R) = h^0(\mathbb{P}_C(f_*(5\Gamma)), R) - h^1(\mathbb{P}_C(f_*(5\Gamma)), R) + h^2(\mathbb{P}_C(f_*(5\Gamma)), R)$$
$$= \chi(\mathcal{O}_{\mathbb{P}_C(f_*(5\Gamma))}) + \frac{1}{2}(R^2 - R.K_{\mathbb{P}_C(f_*(5\Gamma))}). \quad (3.6)$$

By Serre's duality, we have:

$$h^2(\mathbb{P}_C(f_*(5\Gamma)), R) = h^0(\mathbb{P}_C(f_*(5\Gamma)), K_{\mathbb{P}_C(f_*(5\Gamma))} - R).$$

However, we write the class of $K_{\mathbb{P}_C(f_*(5\Gamma))}$ as the following:

$$K_{\mathbb{P}_C(f_*(5\Gamma))} = -2\mathcal{O}_{\mathbb{P}_C(f_*(5\Gamma))}(1) + (2g(C) - 2 + \deg f_*(5\Gamma))Q,$$

here Q is a fiber of pr. Thus clearly :

$$h^2(\mathbb{P}_C(f_*(5\Gamma)), R) = h^0(\mathbb{P}_C(f_*(5\Gamma)), K_{\mathbb{P}_C(f_*(5\Gamma))} - R) = 0. \quad (3.7)$$

Now, by (3.6) and (3.7) we deduce that

$$h^0(\mathbb{P}_C(f_*(5\Gamma)), R) \geq \chi(\mathcal{O}_{\mathbb{P}_C(f_*(5\Gamma))}) + \frac{1}{2}(R^2 - R.K_{\mathbb{P}_C(f_*(5\Gamma))}).$$

But it is clear that

$$\chi(\mathcal{O}_{\mathbb{P}_C(f_*(5\Gamma))}) = 1 - g(C)$$

(since $h^1(\mathbb{P}_C(f_*(5\Gamma)), \mathcal{O}_{\mathbb{P}_C(f_*(5\Gamma))}) = h^1(C, \mathcal{O}_C) = g(C)$). Then we bound the dimension of the set of sections of R by:

$$h^0(\mathbb{P}_C(f_*(5\Gamma)), R) \geq 1 - g(C) + \frac{1}{2}(R^2 - R.K_{\mathbb{P}_C(f_*(5\Gamma))}). \quad (3.8)$$

By the equality (3.5) and the inequality (3.8) above, we deduce that

$$P_g \geq 1 - g(C) + \frac{1}{2}(R^2 - R.K_{\mathbb{P}_C(f_*(5\Gamma))}). \quad (3.9)$$

We develop the number $R^2 - R.K_{\mathbb{P}_C(f_*(5\Gamma))}$:

$$R^2 - R.K_{\mathbb{P}_C(f_*(5\Gamma))} = T^2 + 2(P_g + g(C) - 1) - K_{\mathbb{P}_C(f_*(5\Gamma))}.T - K_{\mathbb{P}_C(f_*(5\Gamma))}.pr^*D. \quad (3.10)$$

We apply the adjunction formula to explore the equality above:

$$2g(C) - 2 = 2g(T) - 2 = K_{\mathbb{P}_C(f_*(5\Gamma))}.T + T^2. \quad (3.11)$$

By (3.10) and (3.11), we deduce that

$$R^2 - R.K_{\mathbb{P}_C(f_*(5\Gamma))} = 2T^2 + 4(P_g + g(C) - 1) - 2(g(C) - 1) = 2(T^2 + 2P_g + g(C) - 1). \quad (3.12)$$

By (3.9) and (3.12), we have that

$$P_g \geq T^2 + 2P_g, \tag{3.13}$$

and thus

$$T^2 \leq -P_g. \tag{3.14}$$

Finally, by (3.4) and (3.14) we deduce the desired result:

$$\Gamma^2 \leq -\frac{1}{5}P_g.$$

□

Remark 3.2.7 *This last geometric Theorem 3.2.6 is important for completing the proof of the conjecture. We interpret it as follows: If there exist such canonically fibered surface $f : S \to C$, then Γ^2 is too negative when the geometric genus P_g is too large.*

The next Corollary explain the contribution of $K_S.\Gamma$ if such S exist.

Corollary 3.2.8 *Let $f : S \to C$ be a canonically fibered surface with geometric genus P_g, and nonhyperelleptic general fiber F of $g(F) = 5$, and*

$$K_S = 8\Gamma + V + f^*D.$$

1. *If $h^0(\mathcal{O}_F(4p)) = 1$, then*
$$K_S.\Gamma \geq \frac{1}{5}P_g + 2(g(C) - 1).$$
2. *If $h^0(\mathcal{O}_F(4p)) = 2$, then*
$$K_S.\Gamma \geq \frac{1}{4}P_g + 2(g(C) - 1).$$

Proof. It is sufficient to prove the first case. In Theorem 5.1.4, we proved the following inequality:

$$\Gamma^2 \leq -\frac{1}{5}P_g.$$

By adjunction, we have:

$$K_S.\Gamma = 2g(C) - 2 - \Gamma^2.$$

Thus

$$K_S.\Gamma \geq \frac{1}{5}P_g + 2(g(C) - 1)$$

as desired. □

Theorem 3.2.9 *There is no canonically fibered general type surface $f : S \dashrightarrow C$ such that the general fiber F is a nonhyperelleptic genus 5 curve with*

$$K_S = 8\Gamma + V + f^*D$$

if $g(C) = 1$ or $P_g > 56$.

Proof. We have

$$\begin{aligned} K_S^2 &= K_S.(8\Gamma + V + f^*D) \\ &\geq 8K_S.\Gamma + K_S.f^*D \\ &= 8K_S.\Gamma + 8(P_g + g(C) - 1). \end{aligned} \tag{3.15}$$

Furthermore, in Corollary 3.2.8, we proved a lower bound for the positive intersection $K_S.\Gamma$. In other words:

$$K_S.\Gamma \geq \frac{1}{5}P_g + 2(g(C) - 1). \tag{3.16}$$

By (3.15) and (3.16) we deduce:

$$K_S^2 \geq \frac{8}{5}P_g + 16(g(C) - 1) + 8(P_g + g(C) - 1).$$

Therefore

$$K_S^2 \geq \frac{48}{5}P_g + 24(g(C) - 1). \tag{3.17}$$

However, by the Miyaoka-Yau inequality, we have:

$$9(P_g + 1 - q(S)) = 9\chi(\mathcal{O}_S) \geq K_S^2. \tag{3.18}$$

Combining the inequality (3.17) and inequality (3.18), and then

$$9(P_g + 1 - q(S)) \geq \frac{48}{5}P_g + 24(g(C) - 1).$$

We deduce

$$P_g \leq 15(1 - q(S)) - 40(g(C) - 1). \tag{3.19}$$

Finally, we conclude that

1. *If* $g(C) = q(S) = 1$, *then by the inequality (3.19) we have*

$$P_g \leq 0 \quad \textit{(Contradiction).}$$

2. *If* $g(C) = 0$ *and* $0 \leq q(S) \leq 2$, *then*

$$P_g \leq 55.$$

□

CHAPTER 4

GLOBAL GENERATION PROBLEMS AND FUJITA'S CONJECTURE

4.1 Fujita's freeness conjecture on irregular varieties

In this section we present some results in the direction of Conjecture 1.3.1, all of these results can be found in our article preprint (7). In what follows, unless otherwise specified, X *is a smooth complex projective irregular variety of dimension* $n \geq 2$, Y *is an abelian variety of dimension* g, *and* $\mathcal{F}$ *is a nonzero coherent sheaf.*

We first establish some auxiliary lemmas that will be used in the subsequent proofs. These preliminary results serve as key technical ingredients and help clarify the arguments that follow.

Lemma 4.1.1 *Let* X *be an irregular variety of dimension* $n \geq 2$ *with* $h : X \to Y$ *be a morphism to an Abelian variety* Y. *Let* $X \xrightarrow{f} Z \xrightarrow{u} Y$ *be the Stein factorization of* h, *and let* F *be a general fiber of* f. *Let* N *be a divisor on* X. *If* $|N_{|_F}|$ *is basepoint-free and* $f_*\mathcal{O}_X(N)$ *is continuously globally generated, then* $|N + h^*p|$ *has no basepoints supported on* F *for a general* $p \in \mathrm{Pic}^0(Y)$.

Proof. Indeed, take a point $x \in F$ *such that* $z = f(x)$. *By assumption,* $f_*\mathcal{O}_X(N)$ *is continuously globally generated, that is for every nonempty open subset* $U \subset \mathrm{Pic}^0(Z)$, *the following direct sum of evaluation maps*

$$\bigoplus_{p \in U} H^0(Z, f_*\mathcal{O}_X(N) \otimes p) \otimes p^\vee \to f_*\mathcal{O}_X(N)_{|_z} \tag{4.1}$$

is surjective. Since $\mathrm{Pic}^0(Y) \hookrightarrow \mathrm{Pic}^0(Z)$, *we have in particular that for every nonempty open subset* $U \subset \mathrm{Pic}^0(Y)$, *the direct sum of evaluation maps*

$$\bigoplus_{p \in U} H^0(Z, f_*\mathcal{O}_X(N) \otimes u^*p) \otimes u^*p^\vee \to f_*\mathcal{O}_X(N)_{|_z} \tag{4.2}$$

is surjective. Furthermore,

$$f_*\mathcal{O}_X(N)_{|_z} = H^0(F, \mathcal{O}_F(N)) \tag{4.3}$$

and

$$H^0(Z, f_*\mathcal{O}_X(N) \otimes u^*p) \simeq H^0(X, \mathcal{O}_X(N) \otimes h^*p). \tag{4.4}$$

Combining (4.4) with (4.3) and (4.2), we obtain that the following sum of maps

$$\bigoplus_{p\in U} H^0(X,\mathcal{O}_X(N)\otimes h^*p)\otimes h^*p^\vee \to H^0(F,\mathcal{O}_F(N)) \tag{4.5}$$

is surjective. By hypothesis, $|N_{|_F}|$ is basepoint-free. Together with (4.5), this implies that for every nonempty open subset $U\subset \mathrm{Pic}^0(Y)$, the following direct sum of evaluation maps

$$\bigoplus_{p\in U} H^0(X,\mathcal{O}_X(N)\otimes h^*p)\otimes h^*p^\vee \to \mathcal{O}_X(N)_{|x} \tag{4.6}$$

is surjective. In particular, for some $p_U\in U$, the map

$$H^0(X,\mathcal{O}_X(N)\otimes h^*p_U)\otimes h^*p_U^\vee \to \mathcal{O}_X(N)_{|x} \tag{4.7}$$

is surjective. Thus, twisting (4.7) by h^*p_U, we deduce that

$$H^0(X,\mathcal{O}_X(N)\otimes h^*p_U) \to \mathcal{O}_X(N)_{|x} \tag{4.8}$$

is surjective. Hence, the linear system

$$|N+h^*p_U|$$

has no basepoints supported on F. Now, we define the following subset $S\subseteq \mathrm{Pic}^0(Y)$ by

$$S:=\left\{p\in \mathrm{Pic}^0(Y)\;\middle|\; |N+h^*p| \text{ has no basepoints supported on } F\right\}.$$

We claim that S is dense. Indeed, for any nonempty open subset $U\subseteq \mathrm{Pic}^0(Y)$, one can find an element $p_U\in U$ such that $|N+h^*p_U|$ has no basepoints supported on F, as proved. Therefore, $S\cap U\neq\emptyset$ for every nonempty open subset $U\subseteq \mathrm{Pic}^0(Y)$. Hence, S is dense. Finally, we conclude that

$$|N+h^*p|$$

has no basepoints supported on F for general $p\in \mathrm{Pic}^0(Y)$. □

Lemma 4.1.2 *Let Y and Z be irregular varieties and $u: Z\to Y$ be a finite morphism. Let $\mathcal{F}$ be a coherent sheaf on Z.*

1) *If $H^1(Y,u_*\mathcal{F}\otimes p)=0$ for all $p\in \mathrm{Pic}^0(Y)$ and $u_*\mathcal{F}$ has no essential basepoints, then $\mathcal{F}$ has no essential basepoints.*

2) *If $u_*\mathcal{F}$ is continuously globally generated, then $\mathcal{F}$ is continuously globally generated.*

Proof. Assuming that $H^1(Y, u_*\mathcal{F} \otimes p) = 0$ *for all* $p \in \mathrm{Pic}^0(Y)$ *and that* $u_*\mathcal{F}$ *has no essential basepoints, we will prove that* $\mathcal{F}$ *has no essential basepoints. Indeed, we fix* $z \in Z$, $y = u(z) \in Y$, *and assume that there exists the following exact sequence:*

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathbb{C}(z) \to 0. \tag{4.9}$$

By applying u_* *to this exact sequence, we obtain:*

$$0 \to u_*\mathcal{K} \to u_*\mathcal{F} \to u_*\mathbb{C}(z) \to \mathcal{R}^1 u_*\mathcal{K} \to \cdots \tag{4.10}$$

Since the map u *is finite, we have* $\mathcal{R}^1 u_*\mathcal{K} = 0$. *Moreover, it is clear that* $u_*\mathbb{C}(z) = \mathbb{C}(y)$, *so the previous sequence reduces to a short exact sequence of sheaves:*

$$0 \to u_*\mathcal{K} \to u_*\mathcal{F} \to \mathbb{C}(y) \to 0. \tag{4.11}$$

Since $u_*\mathcal{F}$ *has no essential basepoints, there exists* $p_y \in \mathrm{Pic}^0(Y)$ *such that the map*

$$H^0(Y, u_*\mathcal{F} \otimes p_y) \to \mathbb{C}(y) \tag{4.12}$$

is surjective. Twisting the sequence (4.11) by p_y *and passing to the associated long exact sequence in cohomology, we obtain:*

$$0 \to H^0(Y, u_*\mathcal{K} \otimes p_y) \to H^0(Y, u_*\mathcal{F} \otimes p_y) \to \mathbb{C}(y)$$
$$\to H^1(Y, u_*\mathcal{K} \otimes p_y) \to H^1(Y, u_*\mathcal{F} \otimes p_y) \to 0.$$

From the assumption that

$$H^1(Y, u_*\mathcal{F} \otimes p_y) = 0$$

and the surjectivity of the map in (4.12), it follows that

$$H^1(Y, u_*\mathcal{K} \otimes p_y) = 0,$$

which in turn implies

$$H^1(Z, \mathcal{K} \otimes u^*p_y) = 0.$$

We now twist the sequence (4.9) by u^*p_y *and pass to the associated long exact sequence in cohomology, obtaining:*

$$0 \to H^0(Z, \mathcal{K} \otimes u^*p_y) \to H^0(Z, \mathcal{F} \otimes u^*p_y) \to \mathbb{C}(z) \to 0.$$

It follows that the map

$$H^0(Z, \mathcal{F} \otimes u^*p_y) \to \mathbb{C}(z)$$

is surjective. We conclude that $\mathcal{F}$ has no essential basepoints.

Now, assume that $u_*\mathcal{F}$ is continuously globally generated. That is, for every $y \in Y$ and for every nonempty open subset $U \subset \mathrm{Pic}^0(Y)$, the following direct sum of evaluation maps

$$\bigoplus_{p \in U} H^0(Y, u_*\mathcal{F} \otimes p) \otimes p^\vee \to u_*\mathcal{F}_{|_y} \tag{4.13}$$

is surjective. Our goal is to prove that $\mathcal{F}$ is continuously globally generated. Indeed, consider a nonzero element $v \in \mathcal{F}_{|_z} \simeq u_*(\mathcal{F}_{|_z})_{|_y}$ where the isomorphism holds because u is finite. This implies that $v \in u_*\mathcal{F}_{|_y}$. Then, by the surjectivity of the map in (4.13), we can find elements $(p_i)_{1\leq i\leq t}$ in $\mathrm{Pic}^0(Y)$ and section $(s_i)_{1\leq i\leq t}$ in $H^0(Y, u_*\mathcal{F} \otimes p_i)$ such that $s_i(y) \neq 0$ for all i with $1 \leq i \leq t$. By the isomorphism

$$H^0(Y, u_*\mathcal{F} \otimes p_i) \simeq H^0(Z, \mathcal{F} \otimes u^* p_i)$$

which holds for all i with $1 \leq i \leq t$, we can find nonzero sections $(u_i)_{1\leq i\leq t}$ of $H^0(Z, \mathcal{F} \otimes u^*p_i)$ such that v has a preimage under the following sum of evaluation map defined by $(u_i)_{1\leq i\leq t}$:

$$\bigoplus_{p \in U} H^0(Z, \mathcal{F} \otimes u^*p) \otimes u^*p^\vee \to \mathcal{F}_{|_z}. \tag{4.14}$$

Thus, we conclude that $\mathcal{F}$ is continuously globally generated. □

Lemma 4.1.3 *Let X be an irregular variety, and $h : X \to Y$ be a morphism to an Abelian variety Y. Let D be a nef and big divisor on X. If $h_*\mathcal{O}_X(K_X + D) \neq 0$, then $h_*\mathcal{O}_X(K_X + D)$ satisfies IT with index 0.*

Proof. Applying Kawamata-Viehweg's vanishing theorem (63, Theorem 1), we have

$$H^i(X, \mathcal{O}_X(K_X + D) \otimes h^*p) = 0 \text{ for all } i \geq 1 \text{ and for all } p \in \mathrm{Pic}^0(Y).$$

Furthermore, by the relative Kawamata-Viehweg's vanishing theorem (6, Theorem 2.2.1):

$$\mathcal{R}^i h_*(\mathcal{O}_X(K_X + D) \otimes h^*p) = 0 \text{ for all } i \geq 1 \text{ and for all } p \in \mathrm{Pic}^0(Y).$$

Thus, it follows from Leray's spectral sequence argument that

$$H^i(Y, h_*\mathcal{O}_X(K_X + D) \otimes p) = 0 \text{ for all } i \geq 1 \text{ and for all } p \in \mathrm{Pic}^0(Y).$$

Consequently, $h_*\mathcal{O}_X(K_X + D)$ satisfies IT with index 0. □

Proposition 4.1.4 *Let X be an irregular variety, and $h : X \to Y$ be a morphism to an Abelian variety Y. Let N and M be divisors on X. If a point $x \in X$ is not a basepoint of $|N + h^*p|$ for a general $p \in \mathrm{Pic}^0(Y)$, and x is not a basepoint of $|M + h^*p|$ for a general $p \in \mathrm{Pic}^0(Y)$, then x is not a base point of $|N + M|$.*

Proof. By assumption, $x \in X$ is not a base point of $|M + h^*p|$ for a general $p \in \mathrm{Pic}^0(Y)$. Since the map $p \mapsto p^\vee$ is an automorphism of $\mathrm{Pic}^0(Y)$, the image of a general point under this map is again a general point. Therefore, x is not a base point of $|M + h^*p^\vee|$ for all general $p \in \mathrm{Pic}^0(Y)$. Hence, x is not a base point of $|N + M|$. □

Theorem 4.1.5 *Let X be an irregular variety of dimension $n \geq 2$ with Albanese dimension $1 \leq \alpha(X) < n$. Let $X \xrightarrow{f} Z \xrightarrow{u} \mathrm{alb}(X) \subseteq \mathrm{Alb}(X)$ be the Stein factorization of the Albanese morphism alb, and let F be a general fiber of the morphism f. Let D be an ample divisor on X.*

If Conjecture 1.3.1 holds in dimension $< n$, then $|K_X + mD + \mathrm{alb}^*\, p|$ has no basepoint supported on F for all $m \geq n - \alpha(X) + 1$ and for all general $p \in \mathrm{Pic}^0(\mathrm{Alb}(X))$.

Additionally, if the following condition is satisfied:

- $(*)$ *There exists an integer r with $1 \leq r \leq \alpha(X)$ such that $|rD_{|_F}|$ is basepoint-free and $rD - K_X$ is nef and big,*

then $|K_X + mD|$ has no basepoint supported on F for all $m \geq n + 1$.

Proof. From the hypothesis, Conjecture 1.3.1 holds for lower-dimensional varieties, meaning that $|K_F + mD_{|_F}|$ is basepoint-free for all $m \geq n - \alpha(X) + 1$. In particular,

$$h^0(F, \omega_F \otimes \mathcal{O}_F(mD)) \neq 0 \quad \text{for all } m \geq n - \alpha(X) + 1,$$

since $(K_X + mD)_{|_F} = K_F + mD_{|_F}$. Thus, $f_*\mathcal{O}_X(K_X + mD)$ is a nonzero coherent sheaf on Z for all $m \geq n - \alpha(X) + 1$. Therefore, $\mathrm{alb}_*\, \mathcal{O}_X(K_X + mD)$ is a nonzero coherent sheaf for all $m \geq n - \alpha(X) + 1$, as the map u is finite.

Applying Lemma 4.1.3, it follows that $\operatorname{alb}_* \mathcal{O}_X(K_X + mD)$ *satisfies* IT *with index* 0. *By Remark 1.3.7 and Proposition 1.3.8, we deduce that* $\operatorname{alb}_* \mathcal{O}_X(K_X + mD)$ *is continuously globally generated.*
Applying Lemma 4.1.2, we conclude that $f_*\mathcal{O}_X(K_X + mD)$ *is continuously globally generated. Again, by assumption, we know that* $|K_F + mD_{|_F}|$ *is basepoint-free for all* $m \geq n - \alpha(X) + 1$. *Then, applying Lemma 4.1.1, we deduce that* $|K_X + mD + \operatorname{alb}^* p|$ *has no basepoint supported on* F *for all* $m \geq n - \alpha(X) + 1$ *and for all general* $p \in \operatorname{Pic}^0(\operatorname{Alb}(X))$. *This proves the first statement.*

Now, suppose that condition $(*)$ *is satisfied. In particular,* $|rD_{|_F}|$ *is basepoint-free, and*

$$h^0(F, \mathcal{O}_F(rD)) \neq 0.$$

Therefore, $f_*\mathcal{O}_X(rD)$ *is a nonzero sheaf on* Z, *and so is* $\operatorname{alb}_* \mathcal{O}_X(rD)$. *We observe that*

$$rD = K_X + rD - K_X.$$

Thus, applying Lemma 4.1.3, we obtain that $\operatorname{alb}_* \mathcal{O}_X(rD)$ *satisfies* IT *with index* 0, *and therefore,* $\operatorname{alb}_* \mathcal{O}_X(rD)$ *is continuously globally generated. Applying Lemma 4.1.2, we deduce that* $f_*\mathcal{O}_X(D)$ *is continuously globally generated.*
Again, by assumption, we know that $|rD_{|_F}|$ *is basepoint-free. Then, applying Lemma 4.1.1, we conclude that*

$$|rD + \operatorname{alb}^* p|$$

has no basepoint supported on a general fiber F *for all general* $p \in \operatorname{Pic}^0(\operatorname{Alb}(X))$.

Finally, applying Proposition 4.1.4 to $|K_X + mD + \operatorname{alb}^* p|$ *and* $|rD + \operatorname{alb}^* p|$, *we deduce that*

$$|K_X + m' D|$$

has no basepoints supported on F *for all* $m' \geq n + r - \alpha(X) + 1$, *and in particular, for all* $m' \geq n + 1$. □

Remark 4.1.6 *In Theorem 4.1.5, the condition that* $rD - K_X$ *is nef and big for some* r *with* $1 \leq r \leq \alpha(X)$ *is always satisfied if* X *is an irregular variety with a nef anticanonical bundle.*

The following basic example in dimension 2 *illustrates that Condition* $(*)$ *in Theorem 4.1.5 can be weaker than requiring* D *to be globally generated.*

Example 4.1.7 *Let $\mathcal{E}$ be a rank 2-vector bundle on elliptic curve C given by the following non-split short exact sequence:*

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0. \tag{4.15}$$

We take the ruled surface $X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} C$, where F is a fiber. We know that the canonical line bundle $\omega_X = \mathcal{O}_X(-2)$, which means $\omega_X^\vee = \mathcal{O}_X(2)$ is nef. Additionally, as another feature of this example, it turns out that the tangent bundle T_X is nef. Furthermore, the irregularity $q(X) = \alpha(X) = 1$ since $h^1(X, \mathcal{O}_X) = h^1(C, \mathcal{O}_C) = 1$. Let D be a divisor such that $\mathcal{O}_X(D) := \mathcal{O}_X(1) \otimes \mathcal{O}_X(F)$, and we observe that D is ample. However, we claim that it is not basepoint-free. Proof. [Proof of the claim] Since $H^0(X, \mathcal{O}_X(D)) = H^0(C, f_*\mathcal{O}_X(D)) = H^0(C, \mathcal{E} \otimes \mathcal{O}_C(x))$ (where x is a point on C), and by twisting the short exact sequence (4.15) by $\mathcal{O}_C(x)$, we obtain the following long exact sequence of cohomology:

$$0 \to H^0(C, \mathcal{O}_C(x)) \to H^0(C, \mathcal{E} \otimes \mathcal{O}_C(x)) \to H^0(C, \mathcal{O}_C(x)) \to H^1(C, \mathcal{O}_C(x)) \to \dots$$

But since $H^1(C, \mathcal{O}_C(x)) = 0$, we conclude that $h^0(C, \mathcal{E} \otimes \mathcal{O}_C(x)) = 2$. Now, let B be a divisor such that $\mathcal{O}_X(B) = \mathcal{O}_X(1)$. Then, we have the following short exact sequence:

$$0 \to \mathcal{O}_X(F) \to \mathcal{O}_X(D) \to \mathcal{O}_B(D) \to 0,$$

which induces the following long exact sequence on cohomology groups:

$$0 \to H^0(X, \mathcal{O}_X(F)) \to H^0(X, \mathcal{O}_X(D)) \to H^0(B, \mathcal{O}_B(D)) \to H^1(X, \mathcal{O}_X(F)) \to \dots \tag{4.16}$$

By the Kodaira vanishing theorem, we see that

$$h^1(X, \mathcal{O}_X(F)) = h^1(X, \mathcal{O}_X(K_X + 2B + F)) = 0$$

since $2B + F$ is ample. Also, we have

$$h^0(X, \mathcal{O}_X(F)) = h^0(B, \mathcal{O}_B(D)) = 1.$$

The previous sequence (4.16) becomes a short exact sequence:

$$0 \to H^0(X, \mathcal{O}_X(F)) \to H^0(X, \mathcal{O}_X(D)) \to H^0(B, \mathcal{O}_B(D)) \to 0$$

Finally, we fix two linearly independent sections of $H^0(X, \mathcal{O}_X(D))$ and restrict them to B. These sections induce a nonzero section s, which vanishes on $D_{|_B}$. Therefore, D is not basepoint-free. □

We remark that D satisfies the condition $(*)$ in Theorem 4.1.5. Indeed, $\mathcal{O}_F(\alpha(X)D) = \mathcal{O}_{\mathbb{CP}^1}(1)$, which is ample and globally generated. Also,

$$\mathcal{O}_X(\alpha(X)D - K_X) = \mathcal{O}_X(3B + F)$$

is ample on X. This example shows that there exists a divisor D satisfying condition $(*)$ in Theorem 4.1.5 and not necessarily globally generated. Furthermore,

$$K_X + mD = (m-2)B + mF$$

is globally generated, for all $m \geq 3$.

Corollary 4.1.8 *Let X be an irregular variety of dimension $n \geq 3$ with Albanese dimension $1 \leq \alpha(X) < n$ and K_X is ample. Let $X \xrightarrow{f} Z \xrightarrow{u} \mathrm{alb}(X) \subseteq \mathrm{Alb}(X)$ be the Stein factorization of the Albanese morphism alb, and let F be a general fiber of the morphism f. If both $|mK_F|$ and $|rK_F|$ are basepoint-free for all $m \geq n + 2 - \alpha(X)$ and for some r with $1 < r \leq \alpha(X)$, then $|mK_X|$ has no basepoints supported on F for all $m \geq n + 2$.*

Proof. Apply Theorem 4.1.5 for $D = K_X$. □

Remark 4.1.9 *If $r \geq n + 2 - \alpha(X)$ in Corollary 4.1.8, then it suffices to assume the basepoint-freeness of $|mK_F|$ for all $m \geq n + 2 - \alpha(X)$.*

An interesting situation arises when the Albanese map is a locally trivial fibration. In this case, we can simplify the setting of Theorem 4.1.5. For instance, the Stein factorization is trivial, and under the assumption of condition $(*)$ in Theorem 4.1.5, we conclude that the linear system $|K_X + mD|$ is basepoint-free for all $m \geq n + 1$. As an example of this situation, we state the following theorem.

Theorem 4.1.10 *Let X be an irregular variety of dimension $n \geq 2$ with $-K_X$ nef. Let $\mathrm{alb} : X \to \mathrm{Alb}(X)$ be the Albanese map, and let D be an ample divisor on X. If Conjecture 1.3.1 holds in dimension $< n$ and there exists an integer r with $1 \leq r \leq \alpha(X)$ such that $|rD_{|_F}|$ is basepoint-free for every fiber F of alb, then Conjecture 1.3.1 holds for X.*

Proof. *If $-K_X$ is nef, then, by (18, Theorem 1.2), the Albanese map alb is a locally trivial fibration. Hence, all the fibers are isomorphic.*

Note that we may assume the fibers of alb have positive dimension, that is, $\alpha(X) < n$. Indeed, if $-K_X$ is nef and $\alpha(X) = n$ for a variety X, then X is an abelian variety. In that case, $K_X = 0$ and $|2D|$ is basepoint-free.

By assumption, Conjecture 1.3.1 holds in low-dimensions. Then, the linear system

$$|K_F + mD_{|_F}|$$

is basepoint-free for all $m \geq n - \alpha(X) + 1$ and for every fiber F. In particular,

$$h^0(F, \omega_F \otimes \mathcal{O}_F(mD)) \neq 0 \text{ for all } m \geq n - \alpha(X) + 1.$$

Thus, $\mathrm{alb}_* \mathcal{O}_X(K_X + mD)$ is a nonzero coherent sheaf for all $m \geq n - \alpha(X) + 1$.
Furthermore, $\mathrm{alb}_* \mathcal{O}_X(K_X + mD)$ satisfies IT with index 0, by Lemma 4.1.3. Consequently, by Remark 1.3.7 and Proposition 1.3.8, it is continuously globally generated. Applying Lemma 4.1.1, we deduce that

$$|K_X + mD + \mathrm{alb}^* p|$$

is basepoint-free for all general $p \in \mathrm{Pic}^0(\mathrm{Alb}(X))$ and for all $m \geq n - \alpha(X) + 1$.

Since $-K_X$ is nef by assumption, the divisor $rD - K_X$ is ample. Moreover, by assumption, $|rD_{|_F}|$ is basepoint-free for some r with $1 \leq r \leq \alpha(X)$. By the base change theorem, this implies that $\mathrm{alb}_* \mathcal{O}_X(rD)$ is a nonzero locally free sheaf. Furthermore, it satisfies IT with index 0, by Lemma 4.1.3.
Therefore, by Remark 1.3.7 and Proposition 1.3.8, $\mathrm{alb}_* \mathcal{O}_X(rD)$ is continuously globally generated. Applying Lemma 4.1.1 again, we deduce that

$$|rD + \mathrm{alb}^* p|$$

is basepoint-free for all general $p \in \mathrm{Pic}^0(\mathrm{Alb}(X))$. Thus, applying Proposition 4.1.4 to $|K_X + mD + \mathrm{alb}^* p|$ and $|rD + \mathrm{alb}^* p|$, we conclude that Conjecture 1.3.1 holds for X. □

Example 4.1.11 *Since Fujita's Freeness Conjecture 1.3.1 holds for varieties of dimension 5 by (114), then by applying Theorem 4.1.10, it follows that the conjecture holds for irregular varieties of dimension 6 with $-K_X$ nef, provided that $|rD_{|_F}|$ is basepoint-free for some r satisfying $1 \leq r \leq \alpha(X)$. Here, F denotes a fiber of alb.*

Example 4.1.12 *More generally than in Example 4.1.11, if X is an irregular variety of dimension n with $-K_X$ nef, $\alpha(X) \geq n-5$, and if $|rD_{|_F}|$ is basepoint-free for some r such that $1 \leq r \leq \alpha(X)$, then Conjecture 1.3.1 holds for X.*

Example 4.1.13 *Example 4.1.7 is also an instance of Theorem 4.1.10.*

Remark 4.1.14 *In Theorem 4.1.10, the condition that $|rD_{|_F}|$ is basepoint-free for some r satisfying $1 \leq r \leq \alpha(X)$ is equivalent to the induction hypothesis if $\alpha(X) \geq \frac{n+1}{2}$ and the fiber F of $\mathrm{alb} : X \to \mathrm{Alb(X)}$ is a K-trivial variety ($K_F = 0$). Indeed, take $r := n - \alpha(X) + 1$, and note that*

$$(n - \alpha(X) + 1)D_{|_F} = K_F + (n - \alpha(X) + 1)(D_{|_F}).$$

4.2 Basepoint-freeness of adjoint series for varieties fibered over Abelian varieties

In the following proposition, we consider the base to be an abelian variety, not necessarily the Albanese variety. We are interested in studying the linear system defined by the canonical sheaf twisted by an ample line bundle from the base, under the assumption that the morphism is an algebraic fiber space and that a general fiber has a basepoint-free canonical bundle.

Proposition 4.2.1 *Let $h : X \to Y$ be a surjective morphism with connected fibers onto an abelian variety Y of dimension g, and let F be a general fiber of h. If $|K_F|$ is basepoint-free and Θ is an ample divisor on Y, then $|K_X + 2h^*\Theta|$ has no basepoints supported on F.*

Proof. From the hypothesis, $|K_F|$ is basepoint-free. In particular,

$$h^0(F, \omega_F) \neq 0.$$

Thus, $h_*\mathcal{O}_X(K_X)$ is a nonzero coherent sheaf.

By Kollár's vanishing theorem (66, Theorem 2.1), we have

$$H^i(Y, h_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_Y(\Theta) \otimes p) = 0 \text{ for all } i \geq 1 \text{ and for all } p \in \mathrm{Pic}^0(Y).$$

Thus, $h_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_Y(\Theta)$ is a nonzero coherent sheaf satisfies IT with index 0. Therefore, it is continuously globally generated. By assumption, $|K_F|$ is basepoint-free. Thus, applying Lemma 4.1.1, we deduce that

$$|K_X + h^*\Theta + h^*p|$$

has no basepoint supported on F for all general $p \in \mathrm{Pic}^0(Y)$.

Moreover, since Θ is an ample divisor on Y, it satisfies IT with index 0. Furthermore, $h^*\Theta$ is trivial when restricted to any fiber. Thus, applying Lemma 4.1.1 again, we deduce that

$$|h^*(\Theta + p)|$$

is basepoint-free for all general $p \in \mathrm{Pic}^0(Y)$.

Finally, applying Proposition 4.1.4 to $|K_X + h^*\Theta + h^*p|$ and $|h^*(\Theta + p)|$, we conclude that

$$|K_X + 2h^*\Theta|$$

has no basepoints supported on F. □

In the next theorem, we consider the Albanese map and remove condition $(*)$ from Theorem 4.1.5. We observe that if we assume the adjoint linear system of a general fiber is basepoint-free, then the linear system defined by the adjoint canonical bundle twisted with an ample line bundle from the base has no basepoints supported on a general fiber.

Theorem 4.2.2 *Assume that the Albanese map $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is a surjective morphism with connected fibers, and let F be a general fiber. Let D be a nef and big divisor on X, and Θ an ample divisor on $\mathrm{Alb}(X)$. If there exists an integer $c > 0$ such that $|K_F + mD_{|_F}|$ is basepoint-free on F for all $m \geq c$, then $|K_X + mD + \mathrm{alb}^*\Theta|$ has no basepoints supported on a general fiber F for all $m \geq c$.*

Proof. From the hypothesis, $|K_F + mD_{|_F}|$ is basepoint-free for all $m \geq c$. In particular,

$$h^0(F, \omega_F \otimes \mathcal{O}_F(mD)) \neq 0 \ \text{ for all } m \geq c.$$

Thus, $\mathrm{alb}_* \mathcal{O}_X(K_X + mD)$ is a nonzero coherent sheaf for all $m \geq c$.

Applying Lemma 4.1.3, it follows that $\operatorname{alb}_* \mathcal{O}_X(K_X + mD)$ *satisfies* IT *with index* 0. *By Remark 1.3.7 and Proposition 1.3.8, we deduce that* $\operatorname{alb}_* \mathcal{O}_X(K_X + mD)$ *is continuously globally generated. Thus, applying Lemma 4.1.1, we conclude that*

$$|K_X + mD + \operatorname{alb}^* p|$$

has no basepoint supported on F *for all* $m \geq c$ *and for all general* $p \in \operatorname{Pic}^0(\operatorname{Alb}(X))$.

Applying Lemma 4.1.1 to $\operatorname{alb}^*(\Theta)$, *we obtain that*

$$|\operatorname{alb}^*(\Theta + p)|$$

is basepoint-free for all general $p \in \operatorname{Pic}^0(\operatorname{Alb}(X))$.

Finally, applying Proposition 4.1.4, we conclude that

$$|K_X + mD + \operatorname{alb}^* \Theta|$$

has no basepoints supported on F *for all* $m \geq c$. □

4.3 Basepoint-freeness of adjoint series for varieties of maximal Albanese dimension

In what follows, we present some results on the basepoint-freeness of linear series, assuming that X *is an irregular variety of dimension* n *with maximal Albanese dimension, that is,* $\alpha(X) = n$. *Related results on basepoint-freeness, in the setting of varieties whose Albanese morphism is finite, were obtained in (88, Theorem 5.1).*

Theorem 4.3.1 *Let* X *be an irregular variety of maximal Albanese dimension, that is* $\alpha(X) = n$, *and let* D *be a nef and big divisor on* X *such that* $nD - K_X$ *is nef and big, or* nD *is continuously globally generated. Then* $|K_X + mD|$ *is basepoint-free outside the exceptional set of* alb *for all* $m \geq n + 1$.

Proof. Since X *is a variety of maximal Albanese dimension, the Albanese map is generically finite. We take the Stein factorization* $X \xrightarrow{f} Z \xrightarrow{u} \operatorname{alb}(X) \subseteq \operatorname{Alb}(X)$ *of* alb, *where* f *is a birational map. Thus,* $f_*\mathcal{O}_X(K_X + D)$ *is nonzero, and consequently, so is* $\operatorname{alb}_* \mathcal{O}_X(K_X + D)$. *Thus,* $\operatorname{alb}_* \mathcal{O}_X(K_X + D)$ *is a nonzero coherent sheaf that satisfies* IT *with index* 0, *by Lemma 4.1.3. Therefore, it is continuously globally* *generated. By Lemma 4.1.2, this implies that*

$$|K_X + D + \mathrm{alb}^* p|$$

has no basepoints outside the exceptional set of alb *for all general* $p \in \mathrm{Pic}^0(\mathrm{Alb}(X))$.

By assumption $nD - K_X$ *is nef and big. Then, applying Lemma 4.1.3, we deduce that* $\mathrm{alb}_* \mathcal{O}_X(mD)$ *is a nonzero coherent sheaf that satisfies* IT *with index* 0 *for all* $m \geq n$. *Hence, it is continuously globally generated. Thus,*

$$|mD + \mathrm{alb}^* p|$$

is basepoint-free outside the exceptional set of alb *for all* $p \in \mathrm{Pic}^0(\mathrm{Alb}(X))$ *and for all* $m \geq n$, *by Lemma 4.1.2.*

Finally, applying Proposition 4.1.4, we conclude that $|K_X + m' D|$ *is basepoint-free outside the exceptional set of* alb *for all* $m' \geq n+1$. □

Remark 4.3.2 *If we assume that the Albanese map is finite in Theorem 4.3.1, under the same conditions on* D *and* $nD - K_X$, *then* $|K_X + mD|$ *is basepoint-free on* X *for all* $m \geq n+1$.

Corollary 4.3.3 *Let* X *be a minimal variety of general type with maximal Albanese dimension,* $\alpha(X) = n$. *Then,* $|4K_X|$ *is basepoint-free outside the exceptional set of* alb.

Proof. We take the Stein factorization $X \xrightarrow{f} Z \xrightarrow{u} \mathrm{alb}(X) \subseteq \mathrm{Alb}(X)$ *of* alb, *where* f *is a birational map. Thus,* $f_* \mathcal{O}_X(2K_X)$ *is nonzero, and consequently, so is* $\mathrm{alb}_* \mathcal{O}_X(2K_X)$.

Since K_X *is nef and big, we apply Lemma 4.1.3 to obtain that* $\mathrm{alb}_* \mathcal{O}_X(2K_X)$ *is a nonzero coherent sheaf that satisfies* IT *with index* 0. *Consequently, it is continuously globally generated. Thus,*

$$|2K_X + \mathrm{alb}^* p|$$

is basepoint-free outside the exceptional set of alb *for all* $p \in \mathrm{Pic}^0(\mathrm{Alb}(X))$. *Using this fact twice, we conclude that* $|4K_X|$ *is basepoint-free outside the exceptional set of* alb *by Proposition 4.1.4.* □

Remark 4.3.4 *This last corollary is not sharp. Indeed, in (61), the authors proved that $|3K_X|$ is birational under the assumption that X is an irregular variety of general type and of maximal Albanese dimension.*

The next proposition is an analogue of Proposition 4.2.1 for varieties admitting a finite morphism to an abelian variety.

Proposition 4.3.5 *Let $h : X \to Y$ be a surjective finite morphism from X to an abelian variety Y. Let Θ be an ample divisor on Y. Then $|K_X + 2h^*(\Theta)|$ is basepoint-free on X.*

Proof. By Kollár's vanishing theorem (66, Theorem 2.1), we have

$$H^i(Y, h_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_Y(\Theta) \otimes p) = 0 \text{ for all } i \geq 1 \text{ and for all } p \in \mathrm{Pic}^0(Y).$$

Thus, $h_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_Y(\Theta)$ is a nonzero coherent sheaf that satisfies IT with index 0. Therefore, it is continuously globally generated. By Lemma 4.1.2, it follows that

$$|K_X + h^*\Theta + h^*p|$$

is basepoint-free for all general $p \in \mathrm{Pic}^0(Y)$.

Since Θ is an ample divisor on Y, it satisfies IT with index 0. Thus,

$$|h^*(\Theta + p)|$$

is basepoint-free for all general $p \in \mathrm{Pic}^0(Y)$, by Lemma 4.1.2. Applying Proposition 4.1.4, we conclude that

$$|K_X + 2h^*\Theta|$$

is basepoint-free. □

CHAPTER 5
THE DIRECT IMAGE SHEAF OF LOGARITHMIC PLURICANONICAL BUNDLES AND THE NON-VANISHING CONJECTURE

The results on this section are contained in the preprint (8).

5.1 Main results

One crucial step in Mori's MMP program is to prove the Non-Vanishing Conjecture. Indeed, it is the first step towards proving the abundance conjecture.

Conjecture 5.1.1 *Let (X, Δ) be a lc pair. If $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ is Cartier pseudo-effective, then $\kappa(D) \geq 0$.*

Thanks to recent advances in techniques for Generic Vanishing Theory, developed by many authors (50), (88), (89), (90) and originally introduced in (46) and (100), we have gained significant insights into irregular varieties. These advances have also deepened our understanding of the structure of the pushforward of logarithmic pluricanonical bundles under a map from an irregular variety to an abelian variety.

The following theorem and corollary could be deduced using the results and methods from (12), (13), (50), (59). In this note, a simple and short proof is provided.

Theorem 5.1.2 *Let (X, Δ) be a klt pair such that $q(X) > 0$, and let $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ be a Cartier pseudo-effective divisor. If Conjecture 5.1.1 holds for lower-dimensional klt pairs, then it also holds for (X, Δ).*

By an easy application of certain forms of the canonical bundle formula (51) due to (40), we obtain the following corollary.

Corollary 5.1.3 *Let (X, Δ) be an lc pair such that $q(X) > 0$, and let $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ be a Cartier pseudo-effective divisor. If Conjecture 5.1.1 holds for lower-dimensional varieties, then it also holds for (X, Δ).*

A crucial step in proving the results announced above is the application of the so-called Chen-Jiang decomposition. More precisely, given a morphism $f : X \to A$, where A is an abelian variety, the following decomposition is provided by (27) and (72):

$$f_*\omega_X^{\otimes m} = \bigoplus_{i\in I}(\alpha_i \otimes p_i^*\mathcal{F}_i).$$

The morphisms $p_i : A \to A_i$ are algebraic fiber spaces, where A_i are abelian varieties, $\mathcal{F}_i$ are nonzero M-regular coherent sheaves on A_i, and $\alpha_i \in \mathrm{Pic}^0(A)$ are torsion line bundles. Later, in (60) and (81), a Chen-Jiang decomposition is generalized to a klt pair (X, Δ). In these articles, the authors proved that if $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ is Cartier, then for every positive integer N such that $f_*\mathcal{O}_X(ND) \neq 0$, we have

$$f_*\mathcal{O}_X(ND) = \bigoplus_{i\in I}(\alpha_i \otimes p_i^*\mathcal{F}_i).$$

In the same articles (60) and (81), the authors asked whether the previous decomposition is still satisfied for an lc pair. Here, we remark that using a canonical bundle formula from (51), we can find a subsheaf that admits a Chen-Jiang decomposition. More precisely, we have the following theorem.

Theorem 5.1.4 *Let (X, Δ) be an lc pair such that $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ is Cartier, and let $f : X \to A$ be a morphism to an abelian variety. If $\kappa(D_{|_F}) \geq 0$, where F is the general fiber of f, then for every positive integer N that is sufficiently large and divisible such that $f_*\mathcal{O}_X(ND) \neq 0$, there exists a torsion-free subsheaf $\mathcal{F}$ of $f_*\mathcal{O}_X(ND)$ such that $\mathcal{F}$ admits a Chen-Jiang decomposition.*

From the above, we observe that MMP problems are more manageable for irregular varieties due to the extensive development of techniques in this setting. The most challenging aspect, however, lies in working with varieties that have no irregularity, as we cannot use morphisms to abelian varieties. In such cases, it is necessary to explore alternatives, using the so-called Catanese-Fujita-Kawamata decomposition particularly relevant from our perspective.

We know by (24), (25), (43), (48), (74) that if $f : X \to Y$ is a surjective morphism with X and Y are smooth varieties, then $f_*\mathcal{O}_X(mK_{X/Y})$ is a torsion free sheaf, it has a singular metric with semi-positive curvature, satisfies the minimal extention property (74, Definition 2.1), and admits a Catanese-Fujita-Kawamata decomposition, that is

$$f_*\mathcal{O}_X(mK_{X/Y}) = \mathcal{A}_m \oplus \mathcal{U}_m,$$

where A_m is a generically ample sheaf and $\mathcal{U}_m$ is flat. This decomposition holds in the singular case, that is, it holds provided that (X, Δ) is klt.

Theorem 5.1.5 *Let $f : X \to Y$ be a surjective morphism, and let (X, Δ) be a klt pair such that $D \sim_{\mathbb{Q}} m(K_{X/Y} + \Delta)$ is Cartier. Then, for every positive integer N that is sufficiently large and divisible such that $f_*\mathcal{O}_X(ND) \neq 0$, the sheaf $f_*\mathcal{O}_X(ND)$ is torsion-free, it has a singular metric with semi-positive curvature, satisfies the minimal extension property, and admits a Catanese-Fujita-Kawamata decomposition*

$$f_*\mathcal{O}_X(ND) = \mathcal{A}_N \oplus \mathcal{U}_N.$$

Klt polarized pairs are important for the minimal model program, and thus we have the following easy corollary.

Corollary 5.1.6 *Let $f : X \to Y$ be a surjective morphism, and let $(X, \Delta + L)$ be a klt polarized pair such that $D \sim_{\mathbb{Q}} m(K_{X/Y} + \Delta + L)$ is Cartier and f-big. Then for every positive integer N which is sufficiently big and divisible such that $f_*\mathcal{O}_X(ND) \neq 0$, the sheaf $f_*\mathcal{O}_X(ND)$ is torsion free, it has a singular metric with semi-positive curvature, satisfies the minimal extention property, and admits a Catanese-Fujita-Kawamata decomposition.*

$$f_*\mathcal{O}_X(ND) = \mathcal{A}_N \oplus \mathcal{U}_N.$$

Theorem 5.1.7 *Let $f : X \to Y$ be a flat algebraic fiber space of relative dimension p, and let (X, Δ) be a klt pair such that $D \sim_{\mathbb{Q}} m(K_{X/Y} + \Delta)$ is Cartier. Let N be a positive integer that is sufficiently large and divisible such that $f_*\mathcal{O}_X(ND) \neq 0$, and assume that $h^p(F, (1 - Nm)K_F - Nm\Delta_F) \neq 0$ is constant for every fiber F, with $P_N := h^0(\mathcal{R}^p f_*((1 - Nm)K_{X/Y} - Nm\Delta)) > 0$. Then $\mathcal{O}_Y^{\oplus P_N}$ is a direct summand of $f_*\mathcal{O}_X(ND)$.*

This last theorem shows that we can produce sections for $K_{X/Y} + \Delta$. Moreover, under an additional positivity condition for Y, we can produce a section for $K_X + \Delta$. This leads to the following corollary.

Corollary 5.1.8 *Assume the same assumptions of Theorem 5.1.7, and $\kappa(Y) \geq 0$. Then $\kappa(K_X + \Delta) \geq 0$. Furthermore, if $P_N > 1$, then $\kappa(K_X + \Delta) \geq 1$.*

Recall that if $f: X \to A$ is a morphism to an Abelian variety, then for a klt pair (X, Δ), we have a Chen-Jiang decomposition of $f_*\mathcal{O}_X(Nm(K_X+\Delta))$ where $Nm(K_X+\Delta) \sim ND$. It is natural to ask the same question for a Catanese-Fujita-Kawamata decomposition under any surjective morphism $f: X \to Y$. Of course, we cannot make such a base change since it is known that $f_*\mathcal{O}_X(Nm(K_{X/Y}+\Delta))$ is not necessarily semi-ample, and the flat part depends on the monodromy group. However, we still have the following proposition.

Proposition 5.1.9 *Let $f: X \to Y$ be a surjective morphism with $q(Y) \geq 1$, and let (X, Δ) be a klt pair such that $D \sim_{\mathbb{Q}} m(K_X+\Delta)$ is Cartier. Then, for every positive integer N that is sufficiently large and divisible such that $f_*\mathcal{O}_X(ND) \neq 0$, there exist finite maps $p: \widetilde{X} \to X$, $q: \widetilde{Y} \to Y$, and $\phi: \widetilde{A} \to \mathrm{Alb}(Y)$ with a Cartier divisor $\widetilde{D} = p^*D$, and a surjective morphism $\widetilde{f}: \widetilde{X} \to \widetilde{Y}$ such that $(g \circ \widetilde{f})_*\mathcal{O}_{\widetilde{X}}(N\widetilde{D})$ is globally generated, where $g: \widetilde{Y} \to \widetilde{A}$.*

In Section 5.5, we revisit some ideas introduced by Viehweg and explore how they can be used to algebraically derive the existence of a Catanese-Fujita-Kawamata decomposition for a klt pair, assuming we already know the result for the smooth case. For instance, we have Theorem 5.4.1 and Theorem 5.4.3, which are key observations in this context.

5.2 Non-vanishing and the Chen-Jiang decomposition

In the last section, we highlighted that the Albanese map and certain generic vanishing techniques can be used to produce sections of log pluricanonical bundles for irregular varieties. The key element is the application of a Chen-Jiang decomposition.

Proof of Theorem 5.1.2. *We consider the Albanese morphism $\mathrm{alb}: X \to \mathrm{Alb}(X)$. If $\alpha(X) = n$, then the morphism alb is generically finite, and thus $\mathrm{alb}_*(D) \neq 0$. If $\alpha(X) < n$, we take the Stein factorization $f: X \to Y$ of alb and denote its general fiber by F. Clearly, the lower-dimensional pair $(F, \Delta_{|_F})$ is klt, and $Nm(K_F + \Delta|_F)$ has a section for every positive integer N which is sufficiently big and divisible by hypothesis. Thus, $f_*\mathcal{O}_X(Nm(K_X+\Delta)) \neq 0$, which implies $\mathrm{alb}_*\,\mathcal{O}_X(Nm(K_X+\Delta)) \neq 0$. By the decomposition results of (60) and (81), we have*

$$\mathrm{alb}_*\,\mathcal{O}_X(ND) = \bigoplus_{i \in I}(\alpha_i \otimes p_i^*\mathcal{F}_i)$$

The morphisms $p_i: \mathrm{Alb}(X) \to A_i$ are algebraic fiber spaces, where A_i are abelian varieties, $\mathcal{F}_i$ are nonzero M-regular coherent sheaves on A_i, and $\alpha_i \in \mathrm{Pic}^0(\mathrm{Alb}(X))$ are torsion line bundles of finite orders. Choose *any α_j in the decomposition, then*

$$\mathrm{alb}_* \mathcal{O}_X(ND) \otimes \alpha_j^{-1} = p_j^* \mathcal{F}_j \bigoplus_{i \in I - \{j\}} (\alpha_i \otimes \alpha_j^{-1} \otimes p_i^* \mathcal{F}_i).$$

It is clear that any M-regular sheaf has a nonzero section. Indeed, by (88), we know that any M-regular sheaf is continuously globally generated, which implies $h^0(A_j, \mathcal{F}_j \otimes \alpha) \neq 0$ for general $\alpha \in \mathrm{Pic}^0(\mathrm{Alb}(X))$. By semi-continuity, we then have $h^0(A_j, \mathcal{F}_j) \neq 0$.

Now, since p_j is an algebraic fiber space, it follows from the projection formula that $h^0(\mathrm{Alb}(X), p_j^* \mathcal{F}_j) \neq 0$. Thus,

$$h^0(\mathrm{Alb}(X), \mathrm{alb}_* \mathcal{O}_X(ND) \otimes \alpha_j^{-1}) \neq 0,$$

which implies $h^0(X, ND \otimes p_j^* \alpha_j^{-1}) \neq 0$.

Assume that the order of α_j^{-1} is k. Then, $h^0(X, kND) \neq 0$, which completes the proof of the theorem. □

Remark 5.2.1 *If $\Delta = 0$, then by (20), we know that the $C_{n,m}$ conjecture is true for an algebraic fiber space over a variety with maximal Albanese dimension, and of course, we can deduce Theorem 5.1.2. Also, for a klt pair (X, Δ), the $C_{n,m}$ conjecture for the same algebraic fiber space is satisfied by the work of Birkar and Chen (13), but the proof involves many reduction steps, and we should use some technical extension theorems as given in (34).*

Proof of Corollary 5.1.3. *We apply Theorem 1.4.3 to obtain the following commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\psi} & X \\ \big\downarrow \widetilde{f} & & \big\downarrow f = \mathrm{alb} \\ \widetilde{Y} & \xrightarrow{\phi} & Y = \mathrm{Alb}(X) \end{array}$$

such that the properties $(1), \ldots, (5)$ are satisfied. Denote by P the vertical component of $\lfloor \widetilde{\Delta} \rfloor$. Since each component of $\lfloor \Delta_{\widetilde{Y}} \rfloor$ is dominated by P, it is clear that we can find a klt polarized pair $(\widetilde{Y}, \Delta' + L')$ and a $\mathbb{Q}$-Cartier divisor R' such that $K_{\widetilde{Y}} + \Delta' + L'$ is big $/Y$ and for some sufficiently small ϵ

$$K_{\widetilde{X}} + \widetilde{\Delta} - \epsilon P \sim_{\mathbb{Q}} \widetilde{f}(K_{\widetilde{Y}} + \Delta' + L') + R',$$

and

$$\widetilde{f}_* \mathcal{O}_{\widetilde{X}}(Nm(K_{\widetilde{X}} + \widetilde{\Delta} - \epsilon P)) = \mathcal{O}_{\widetilde{Y}}(Nm(K_{\widetilde{Y}} + \Delta' + L')).$$

(Without loss of generality, assume $Nm(K_{\widetilde{X}} + \widetilde{\Delta} - \epsilon P)$ and $Nm(K_{\widetilde{Y}} + \Delta^{'} + L^{'})$ are Cartier). Since $K_{\widetilde{Y}} + \Delta^{'} + L^{'}$ is big $/Y$, we have

$$K_{\widetilde{Y}} + \Delta^{'} + L^{'} \sim_{\mathbb{Q},\phi} M + E,$$

where M is an ample $\mathbb{Q}$-divisor on $\widetilde{Y}$, and E is effective. Then, for some $\delta > 0$, we can find Δ_δ such that $(\widetilde{Y}, \Delta_\delta)$ is klt and

$$K_{\widetilde{Y}} + \Delta_\delta \sim_{\mathbb{Q},\phi} K_{\widetilde{Y}} + \Delta^{'} + L^{'} + \delta E + \delta M \sim_{\mathbb{Q},\phi} (1+\delta)(K_{\widetilde{Y}} + \Delta^{'} + L^{'}). \tag{5.1}$$

Here $K_{\widetilde{Y}} + \Delta_\delta$ is big $/Y$, and for some N which is sufficiently big and divisible, $Nm(K_{\widetilde{Y}} + \Delta_\delta)$ is Cartier. By Theorem 5.1.2, $\kappa(K_{\widetilde{Y}} + \Delta_\delta) \geq 0$, hence $\kappa(K_{\widetilde{Y}} + \Delta^{'} + L^{'}) \geq 0$. Thus $\widetilde{f}_*\mathcal{O}_{\widetilde{X}}(Nm(K_{\widetilde{X}} + \widetilde{\Delta} - \epsilon P))$ has a nonzero section, which implies that $Nm(K_{\widetilde{X}} + \widetilde{\Delta} - \epsilon P)$ has a section. Finally $Nm(K_{\widetilde{X}} + \widetilde{\Delta})$ and $Nm(K_X + \Delta)$ have a section. □

Remark 5.2.2 *We do not know if a Chen-Jiang decomposition holds for a lc pair (X, Δ) since we miss a semi-positivity result for the pushforward of the log pluricanonical bundle. Otherwise, Corollary 5.1.3 would follow automatically without the use of any form of the canonical bundle formula. As we mentioned in the introduction, in (60) and (81), the authors asked if a Chen-Jiang decomposition is satisfied for the pushfoward of a lc pairs. We remark that, by using the canonical bundle formula, we can see that $f_*\mathcal{O}_X(Nm(K_X+\Delta))$ contains a subsheaf that admits a Chen-Jiang decomposition for every positive integer N that is sufficiently large and divisible such that $f_*\mathcal{O}_X(ND) \neq 0$.*

Proof of Theorem 5.1.4. *By assumption, $\kappa(K_F + \Delta|_F) \geq 0$. Thus, we apply Theorem 1.4.3 to the pair (X, Δ), obtaining the following diagram:*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\psi} & X \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{\phi} & Y = A \end{array}$$

such that the properties $(1), \ldots, (5)$ are satisfied. By following the same steps as in the proof of Corollary 5.1.3, we find that for some $\delta > 0$, there exists Δ_δ such that $(\widetilde{Y}, \Delta_\delta)$ is klt. Without loss of generality, we assume that the divisors $Nm(K_{\widetilde{Y}} + \Delta_\delta)$, $Nm(1+\delta)(K_{\widetilde{Y}} + \Delta' + L')$, and $Nm(K_{\widetilde{X}} + \widetilde{\Delta} - \epsilon P)$ are Cartier. It is clear that the following torsion free sheaf is nonzero and admits a Chen-Jiang decomposition:

$$(\phi \circ \widetilde{f})_*\mathcal{O}_{\widetilde{X}}(Nm(K_{\widetilde{X}} + \widetilde{\Delta} - \epsilon P)) = \phi_*\mathcal{O}_{\widetilde{Y}}(Nm(K_{\widetilde{Y}} + \Delta_\delta)) \neq 0,$$

We know that the diagram above is commutative, thus

$$\mathcal{F} := (f\circ\psi)_*\mathcal{O}_{\widetilde{X}}(Nm(K_{\widetilde{X}} + \widetilde{\Delta} - \epsilon P)) \neq 0,$$

and admits a Chen-Jiang decomposition. Note that the sheaf $\mathcal{F}$ is a torsion-free subsheaf of $f_*\mathcal{O}_X(Nm(K_X+\Delta))$. □

5.3 Catanese-Fujita-Kawamata decomposition

As mentioned in Section 5.1, it is not clear how to produce sections of $K_X+\Delta$, even when there exist sections for $K_F+\Delta_{|F}$. In the case where the morphism is to an abelian variety, this follows from the discussion above and is also well known to experts. However, if the base variety is not an abelian variety, it becomes difficult to ensure the existence of sections for $K_X+\Delta$. Exploring a Catanese–Fujita–Kawamata-type decomposition is therefore relevant to our purpose.

Definition 5.3.1 *(74, Definition 1.1) A torsion-free coherent sheaf $\mathcal{F}$ admits a Catanese–Fujita–Kawamata decomposition if it decomposes in the following form*

$$\mathcal{F} \cong \mathcal{U} \oplus \mathcal{A},$$

where $\mathcal{U}$ is a Hermitian flat bundle, and $\mathcal{A}$ is either a generically ample sheaf or the zero sheaf.

Recall the following theorem proven in (74).

Theorem 5.3.2 *(74, Theorem 1.3) Let $\mathcal{F}$ be a torsion-free coherent sheaf on a smooth projective variety Y, endowed with a singular Hermitian metric with semi-positive curvature and satisfying the minimal extension property. Then $\mathcal{F}$ admits a Catanese–Fujita–Kawamata decomposition.*

As an example for the previous theorem, in (74) the authors deduced the decomposition theorem (Definition 5.3.1) for $f_*\mathcal{O}_X(m(K_{X/Y}))$ where $f : X \to Y$ is a surjective morphism between smooth varieties. We remark that the decomposition is satisfied for the klt case.

Proof of Theorem 5.1.5. *The proof is classical, we refer to (94) for details. Indeed, $Nm\Delta$ is Cartier, and by assumption we have $f_*\mathcal{O}_X(Nm(K_{X/Y}+\Delta)) \neq 0$. Then define the divisor*

$$D_N := (Nm-1)K_{X/Y} + Nm\Delta.$$

By (94), the divisor above admits a singular hermitian metric with semi-positive curvature, and the following inclusion is generically an isomorphism

$$f_*\big(\mathcal{O}_X(K_{X/Y}+D_N)\otimes\mathcal{I}(h_N)\big)\hookrightarrow f_*\mathcal{O}_X(K_{X/Y}+D_N)=f_*\mathcal{O}_X(Nm(K_{X/Y}+\Delta)).$$

Here h_N is the metric associated to D_N and $\mathcal{I}(h_N)$ is the multiplier ideal sheaf associated to h_N. We know by the famous result of (94) (see also (11), (48), (58), (93)) that $f_*\big(\mathcal{O}_X(K_{X/Y}+D_N)\otimes\mathcal{I}(h_N)\big)$ admits a singular hermitian metric with semi-positive curvature and satisfies the minimal extention property. The inclusion above is generically an isomorphism. Thus, by (74, Proposition 2.2), the torsion free sheaf $f_*\mathcal{O}_X(K_{X/Y}+D_N)=f_*\mathcal{O}_X(Nm(K_{X/Y}+\Delta))$ is endowed with a singular hermitian metric with semi-positive curvature and satisfies the minimal extention property. Then we can apply Theorem 5.3.2 to conclude. □

Proof of Corollary 5.1.6. *By assumption $K_{X/Y}+\Delta+L$ is big $/Y$. Then we have*

$$K_{X/Y}+\Delta+L\sim_{\mathbb{Q},f}M+E,$$

where M is an ample $\mathbb{Q}$-divisor on Y, and E is effective. Then, for some $\delta>0$, we can find Δ_δ such that (X,Δ_δ) is klt and

$$K_{X/Y}+\Delta_\delta\sim_{\mathbb{Q},f}K_{X/Y}+\Delta+L+\delta E+\delta M\sim_{\mathbb{Q},f}(1+\delta)(K_{X/Y}+\Delta+L). \tag{5.2}$$

Then we apply Theorem 5.1.5 to deduce the decomposition. □

It is natural to ask whether the flat part or the generically ample part have a section. The author believes that a deeper understanding of a Catanese-Fujita-Kawamata decomposition is crucial for making progress on positivity problems.

Proof of Theorem 5.1.7. *By assumption, $h^p(F,(1-Nm)K_F-Nm\Delta_F)\neq 0$ is constant for every fiber F, and the algebraic fiber space is flat. Then, by Grauert's theorem, we deduce that the coherent sheaf $\mathcal{R}^pf_*\mathcal{O}_X((1-Nm)K_{X/Y}-Nm\Delta)$ is locally free. Note*

$$P_N:=h^0(Y,\mathcal{R}^pf_*\mathcal{O}_X((1-Nm)K_{X/Y}-Nm\Delta))=h^0(Y,f_*\mathcal{O}_X(Nm(K_{X/Y}+\Delta))^\vee),$$

since

$$f_*\mathcal{O}_X(Nm(K_{X/Y}+\Delta))^\vee\simeq\mathcal{R}^pf_*\mathcal{O}_X((1-Nm)K_{X/Y}-Nm\Delta).$$

Consider a basis $\{s_1, \ldots, s_{P_N}\}$ of

$$H^0(Y, \mathcal{R}^p f_* \mathcal{O}_X((1-Nm)K_{X/Y} - Nm\Delta)) \simeq Hom(\mathcal{O}_Y, \mathcal{R}^p f_* \mathcal{O}_X((1-Nm)K_{X/Y} - Nm\Delta)).$$

Then $s_1 \oplus \cdots \oplus s_{P_N}$ defines a map

$$s_1 \oplus \cdots \oplus s_{P_N} : \mathcal{O}_Y^{\oplus P_N} \longrightarrow \mathcal{R}^p f_* \mathcal{O}_X((1-Nm)K_{X/Y} - Nm\Delta),$$

which yields the following short exact sequence

$$0 \longrightarrow \mathcal{O}_Y^{\oplus P_N} \longrightarrow \mathcal{R}^p f_* \mathcal{O}_X((1-Nm)K_{X/Y} - Nm\Delta) \longrightarrow \mathcal{Q}_N \longrightarrow 0,$$

where $\mathcal{Q}_N$ is the quotient sheaf. By duality, we have

$$0 \longrightarrow \mathcal{Q}_N^\vee \longrightarrow f_* \mathcal{O}_X(Nm(K_{X/Y} + \Delta)) \longrightarrow \mathcal{O}_Y^{\oplus P_N} \longrightarrow 0.$$

We deduce from (48, Theorem 26.4) that the last exact sequence splits since the bundle $f_* \mathcal{O}_X(Nm(K_{X/Y} + \Delta))$ admits a singular hermitian metric with semi-positive curvature and satisfies the minimal extension property □

Proof of Corollary 5.1.8. *By assumption $\kappa(Y) \geq 0$, then for some positive integer N which is sufficiently big and divisible, we have NmK_Y is effective. By Theorem 5.1.7, we know that $\mathcal{O}_Y^{\oplus P_N}$ is a sub-sheaf of $f_* \mathcal{O}_X(Nm(K_{X/Y} + \Delta))$. Now, by the following multplication map*

$$H^0(Y, f_* \mathcal{O}_X(Nm(K_{X/Y} + \Delta))) \times H^0(Y, NmK_Y) \to H^0(Y, f_* \mathcal{O}_X(Nm(K_X + \Delta))),$$

we produce sections for the torsion free sheaf $f_* \mathcal{O}_X(Nm(K_X + \Delta))$, and of course for the divisor $Nm(K_X + \Delta))$. Thus, we deduce that $\kappa(K_X + \Delta) \geq 0$. The second assertion is clear. □

Proof of Corollary 5.1.9. *By hypothesis $q(Y) \geq 1$, thus $(\mathrm{alb} \circ f)_* \mathcal{O}_X(Nm(K_X + \Delta))$ has a Chen-Jiang decomposition if it is not zero. Furthermore, we can find an isogeny $\phi : \widetilde{A} \to \mathrm{Alb}(Y)$ such that $\phi^*((\mathrm{alb} \circ f)_* \mathcal{O}_X(m(K_X + \Delta)))$ is globally generated, and we have the following diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ p\ } & X \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{\ q\ } & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{alb}} \\ \widetilde{A} & \xrightarrow{\ \phi\ } & \mathrm{Alb}(Y). \end{array}$$

The maps $p : \widetilde{X} \to X$, $q : \widetilde{Y} \to Y$ and $\phi : \widetilde{A} \to \mathrm{Alb}(Y)$ are finites. The morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a surjective morphism. Clearly, the pair $(\widetilde{X}, \widetilde{\Delta})$ is klt such that $K_{\widetilde{X}} + \widetilde{\Delta} := p^*(K_X + \Delta)$. Hence we deduce the result from the commutativity of the diagram above. □

5.4 Viehweg's trick machinery

In this section, we review some techniques introduced by Viehweg, commonly referred to as Viehweg's trick machinery (104) (see also (64)). He developed these techniques to make significant progress in studying the positivity of the direct image sheaf of the pluricanonical bundle, which has direct applications to the $C_{n,m}$ conjecture. For instance, this conjecture was resolved by Cao and Păun (20) in the important case where the variety is fibered over an Abelian variety. They reduced the problem, à la Kawamata (62), to the case where the variety has trivial Kodaira dimension over an Abelian variety. In this setting, they established a crucial positivity result for $f_*\mathcal{O}_X(mK_X)$ ((20)).

We are inspired from (72), (81), (92), (104) to prove the followings.

Theorem 5.4.1 *Let $f : X \to Y$ be a surjective morphism, and let (X, Δ) be a klt pair such that $D \sim_{\mathbb{Q}} (K_{X/Y} + \Delta)$ is Cartier. Then there exist a smooth variety Z with a generically finite map $h : Z \to X$ such that $f_*\mathcal{O}_X(D)$ is a direct summand of $g_*\omega_{Z/Y} := (f \circ h)_*\omega_{Z/Y}$.*

Proof. It is enough to assume that (X, Δ) is a klt log smooth pair. Indeed, take a log resolution of (X, Δ), $\mu : \widetilde{X} \to X$ such that

$$K_{\widetilde{X}/Y} + \Delta_{\widetilde{X}} \sim_{\mathbb{Q}} \mu^*(K_{X/Y} + \Delta) + E,$$

where $\Delta_{\widetilde{X}}$ and E are effective SNC and have no common components, E is the exceptional divisor. Therefore, we have

$$K_{\widetilde{X}/Y} + \Delta_{\widetilde{X}} + \lceil E \rceil - E \sim_{\mathbb{Q}} \mu^*(K_{X/Y} + \Delta) + \lceil E \rceil.$$

We put $\Delta'_{\widetilde{X}} := \Delta_{\widetilde{X}} + \lceil E \rceil - E$, then the pair $(\widetilde{X}, \Delta'_{\widetilde{X}})$ is klt log smooth.

Furthermore

$$(f \circ \mu)_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/Y} + \Delta'_{\widetilde{X}}) = (f \circ \mu)_*\mathcal{O}_{\widetilde{X}}(\mu^*D + \lceil E \rceil) = f_*\mathcal{O}_X(D).$$

Thus, we can work with the log smooth klt pair $(\widetilde{X}, \Delta'_{\widetilde{X}})$, and therefore assume that (X, Δ) is klt log smooth.

Now, we take N such that

$$N(D - K_{X/Y}) \sim N\Delta.$$

Hence, we see that the Cartier divisor $N(D-K_{X/Y})$ is divisible. Then, we can take a finite map $h' : Z' \to X$ ramified along $N\Delta$. By resolving the singularities of Z', we obtain a generically finite map $h : Z \to X$, with Z smooth, such that

$$h_*\omega_Z = \mathcal{O}_X(K_X + (N-1)(D-K_{X/Y})) \otimes \mathcal{O}_X(-\lfloor (N-1)\Delta \rfloor) \bigoplus \dots$$

$$= \mathcal{O}_X((D-K_{X/Y}+K_X)) \otimes \mathcal{O}_X((N-2)(D-K_{X/Y})) \otimes \mathcal{O}_X(-\lfloor (N-1)\Delta \rfloor) \bigoplus \dots$$

But

$$\mathcal{O}_X((N-2)(D-K_{X/Y})) \otimes \mathcal{O}_X(-\lfloor (N-1)\Delta \rfloor) = \mathcal{O}_X.$$

Thus

$$h_*\omega_Z = \mathcal{O}_X((D-K_{X/Y}+K_X)) \bigoplus \dots$$

Hence

$$h_*\omega_{Z/Y} = \mathcal{O}_X(D) \bigoplus \dots$$

Finally, we have

$$g_*\omega_{Z/Y} := (f \circ h)_*\omega_{Z/Y} = f_*\mathcal{O}_X(D) \bigoplus \dots$$

as required. □

Corollary 5.4.2 *Let $f : X \to Y$ be a surjective morphism, and let (X, Δ) be a klt pair such that $D \sim_{\mathbb{Q}} (K_{X/Y} + \Delta)$ is Cartier. Then $f_*\mathcal{O}_X(D)$ admits a Catanese-Fujita-Kawamata decomposition.*

Theorem 5.4.3 *Let $f : X \to Y$ be a surjective morphism, and let (X, Δ) be a klt pair such that $D \sim_{\mathbb{Q}} m(K_{X/Y} + \Delta)$ is Cartier for some $m > 1$. If $f_*\mathcal{O}_X(D)$ is globally generated, then there exist a smooth variety Z with a generically finite map $h : Z \to X$ such that $f_*\mathcal{O}_X(D)$ is a direct summand of $g_*\omega_{Z/Y} := (f \circ h)_*\omega_{Z/Y}$.*

Proof. We have the following evaluation map

$$f^* f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D),$$

and the image is $D \otimes \mathcal{I}$, where $\mathcal{I}$ is the relative base ideal of D. We take a log resolution of (X, Δ) and I, $\mu : \widetilde{X} \to X$ such that

$$K_{\widetilde{X}/Y} + \Delta_{\widetilde{X}} \sim_{\mathbb{Q}} \mu^*(K_{X/Y} + \Delta) + E,$$

where $\Delta_{\widetilde{X}}$ and E are effective SNC and have no common components, E is the exceptional divisor. We put $D_{\widetilde{X}} := \mu^* D$, and $k := f \circ \mu$. Then the image of the following evaluation map

$$k^* k_* \mathcal{O}_{\widetilde{X}}(D_{\widetilde{X}}) \to \mathcal{O}_{\widetilde{X}}(D_{\widetilde{X}})$$

is of the form $\mathcal{O}_{\widetilde{X}}(D_{\widetilde{X}} - F)$ for some effective SNC divisor F. Hence, we define the new boundary divisor

$$\Delta'_{\widetilde{X}} := \Delta_{\widetilde{X}} + \frac{\lceil mE \rceil}{m} - E,$$

and clearly the pair $(\widetilde{X}, \Delta'_{\widetilde{X}})$ is klt log smooth. Therefore

$$m(K_{\widetilde{X}/Y} + \Delta'_{\widetilde{X}}) \sim_{\mathbb{Q}} \mu^*(m(K_{X/Y} + \Delta)) + \lceil mE \rceil \sim_{\mathbb{Q}} \mu^* D + \lceil mE \rceil.$$

Put $G := \mu^* D + \lceil mE \rceil$. Then $k_* \mathcal{O}_{\widetilde{X}}(G) = f_* \mathcal{O}_X(D)$, and $k_* \mathcal{O}_{\widetilde{X}}(G)$ is globally generated. By above, the image of the following evaluation map

$$k^* k_* \mathcal{O}_{\widetilde{X}}(G) \to \mathcal{O}_{\widetilde{X}}(G)$$

is $\mathcal{O}_{\widetilde{X}}(G - F - \lceil mE \rceil)$. We define the divisor $G' := F + \lceil mE \rceil$, and as it pointed in (81), we have $k_*(\mathcal{O}_{\widetilde{X}}(G - G''))= k_*(\mathcal{O}_{\widetilde{X}}(G))$, for any effective divisor $G'' \leq G'$.

Since $k_* \mathcal{O}_{\widetilde{X}}(G)$ is globally generated, then $\mathcal{O}_{\widetilde{X}}(G - G')$ is globally generated, and by Bertini's theorem we can take an effective divisor $H \sim_{\mathbb{Q}} G - G'$, H and $\Delta'_{\widetilde{X}} + G'$ have no comments components, with $H + \Delta'_{\widetilde{X}} + G'$ is SNC.

Now, the goal is to reduce to the structure as in Theorem 5.4.1, we can find a new divisor $T \leq G'$ and a klt pair $(\widetilde{X}, M)$ such that

$$G - T \sim_{\mathbb{Q}} K_{\widetilde{X}/Y} + M,$$

T and M are given by

$$T := \lfloor \Delta'_{\widetilde{X}} + \frac{m-1}{m} G' \rfloor,$$

and

$$M := \frac{m-1}{m} H + \Delta'_{\widetilde{X}} + \frac{m-1}{m} G' - T$$

as proven in (81). The last step is to find a smooth variety Z and a generically finite map $h : Z \to X$ such that $f_* \mathcal{O}_X(D)$ is a direct summand of $g_* \omega_{Z/Y} := (f \circ h)_* \omega_{Z/Y}$. The details are the same as in the proof of Theorem $5.4.1$, so we will leave them to the reader. ☐

Remark 5.4.4 *In Theorem 5.4.3, if* $f_*\mathcal{O}_X(D)$ *is not globally generated, we can twist the bundle with a sufficiently ample line bundle* L *on* Y *to ensure that* $f_*\mathcal{O}_X(D) \otimes L$ *is globally generated on* Y*. In this case, we can prove a similar statement to Theorem 5.4.3; that is, we can find a smooth variety* Z *and a generically finite map* $Z \to X$ *such that* $f_*\mathcal{O}_X(D) \otimes L$ *is a direct summand of* $g_*\omega_{Z/Y} := (f \circ h)_*\omega_{Z/Y}$*. All of these observations provide a way to obtain a Catanese-Fujita-Kawamata decomposition in the logarithmic case.*

CONCLUSION

The author developed some intuition following the work done in this thesis across all these independent subjects. First, looking at Chapter 2 and Chapter 3 of the thesis (or our articles (9) and (10)), we ask the question of sharpness of the slope inequality we obtained. This is always a good question and an active area of research, as it can provide better boundedness results for families of curves. We know that a good slope inequality gives good control over the relative irregularity of a family of curves $f : S \to C$*, yet this remains an open problem. We also notice that to obtain some results on Xiao's canonically fibered surfaces, we should first establish an interesting lower bound for* K_S^2 *(in some sense, a strong slope inequality).*

In Chapter 4, we study Fujita's Conjecture. The author found some interesting results for irregular varieties by induction on dimension, assuming certain conditions. Following this, a natural next step is to try to remove these conditions and extend the results beyond the general fibers, incorporating an analysis of singular fibers.

In the last chapter, the author begins a program to study minimal model theory for irregular varieties. We can easily see that the nonvanishing conjecture can be derived inductively using the Chen–Jiang decomposition and the canonical bundle formula. When the variety is regular but still fibered over some other variety, we have an alternative approach using the Catanese–Fujita–Kawamata decomposition. The first question is: can we produce sections downstairs? In other words, in which cases can the flat or ample part admit a section? An answer to this question could lead to a general proof of the nonvanishing conjecture. The author plans to explore the geometry of this decomposition further. We will continue to develop this program, investigating questions such as: does a minimal model exist for an irregular variety? Is there termination of flips? What about the abundance conjecture?

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