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MEHRDAD NAJAFPOUR GHAZVINI FARDSHAD

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# CONSTANT SCALAR CURVATURE KÄHLER METRICS ON RESOLUTIONS OF AN ORBIFOLD SINGULARITY OF DEPTH 1

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MEHRDAD NAJAFPOUR GHAZVINI FARDSHAD

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**RÉSUMÉ**

Dans cette thèse, nous étudions l'existence de nouvelles métriques de Kähler à courbure scalaire constante (cscK pour faire court) sur une résolution d'une singularité orbifold. Nous considérons $(X, \omega_X)$, un orbifold complexe compact avec un groupe discret d'automorphismes (en particulier, il n'y a pas de champs de vecteurs holomorphes non triviaux sur $X$). On suppose que $X$ a des singularités de type $\mathcal{I}$ le long d'un sous-ensemble $Y$ ayant une codimension $k$ supérieure à 2. Le sous-ensemble $Y$ lui-même est une variété complexe lisse, mais l'inclusion $Y \hookrightarrow X$ dans $X$ est singulière, c'est-à-dire que le fibré normal de $Y$ dans $X$ a des fibres de la forme $\mathbb{C}^k/\Gamma$ avec $\Gamma$ un sous-groupe discret fini de $U(k)$ de type $\mathcal{I}$ de forme $(-w_0, w)$.

Nous utilisons une résolution $\widehat{X}$ de $X$ obtenue en éclatant $Y$ dans $X$ en utilisant l'espace projectif pondéré non-compact introduit par Apostolov et Rollin, et une technique de recollement inspirée par les travaux de Seyyedali et Székelyhidi et de Conlon, Degeratu et Rochon pour démontrer qu'il existe une famille de métriques $\widehat{\omega}_\varepsilon$ sur $\widehat{X}$ proches d'être cscK pour $\varepsilon$ suffisamment petit. Nous établissons alors l'existence d'une fonction potentielle lisse $\phi_\varepsilon$ sur $\widehat{X}$ telle que $\widetilde{\omega}_\varepsilon = \widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}\phi_\varepsilon$ soit cscK sur $\widehat{X}$ en résolvant une équation aux dérivées partielles non-linéaire. En tant qu'application du théorème principal, nous fournissons un nouvel exemple d'un espace projectif pondéré qui admet une métrique cscK sur une résolution de celui-ci.

La résolution $\widehat{X} \to X$ est typiquement encore singulière, mais par définition des singularités de type $\mathcal{I}$, on peut toujours trouver une suite

$$\widehat{X}_l \to \widehat{X}_{l-1} \to \ldots \to \widehat{X}_1 \to X$$

de telles résolutions avec $\widehat{X}_1 = \widehat{X}$ et $\widehat{X}_l$ lisse, et notre résultat principal s'applique successivement à chacune de ces résolutions pour montrer que la résolution lisse $\widehat{X}_l$ admet une métrique cscK.

Mots-clés: Métriques de Kähler à courbure scalaire constante, Orbifold, Résolution, Éclatement, Structures de Lie à l'infini, Espace projectif pondéré, Singularités de type $\mathcal{I}$.

**ABSTRACT**

In this thesis, we study the existence of new constant scalar curvature Kähler (cscK for short) metrics on a resolution of an orbifold singularity. We consider $(X, \omega_X)$, a compact complex orbifold with a discrete group of automorphisms (in particular, there are no non-trivial holomorphic vector fields on $X$). We assume that $X$ has singularities of type $\mathcal{I}$ along a subset $Y$ with codimension $k$ greater than 2. The subset $Y$ itself is a smooth complex manifold, but the inclusion $Y \hookrightarrow X$ in $X$ is singular, i.e., the normal bundle of $Y$ in $X$ has fibers of the form $\mathbb{C}^k / \Gamma$ with $\Gamma$ a discrete finite subgroup of $U(k)$ of type $\mathcal{I}$ of form $(-w_0, w)$ .

We use a resolution $\widehat{X}$ of $X$, obtained by blowing-up $Y$ in $X$ using the non-compact weighted projective space introduced by Apostolov and Rollin, and a gluing technique inspired by the work of Seyyedali and Székelyhidi and of Conlon, Degeratu, and Rochon, to demonstrate that there is a family of metrics $\widehat{\omega}_\varepsilon$ on $\widehat{X}$ close to being cscK for small enough $\varepsilon$. Ultimately, through nonlinear analysis, we establish the existence of a smooth potential function $\phi_\varepsilon$ on $\widehat{X}$ such that $\widetilde{\omega}_\varepsilon = \widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}\phi_\varepsilon$ is cscK on $\widehat{X}$. As an application of the main theorem, we provide a new example of a weighted projective space that admits a cscK metric on a resolution of it.

The resolution $\widehat{X} \to X$ is typically not singular, but by definition of a singularity of type $\mathcal{I}$, we can always find a sequence

$$\widehat{X}_l \to \widehat{X}_{l-1} \to \ldots \to \widehat{X}_1 \to X$$

of such resolutions with $\widehat{X}_1 = \widehat{X}$ and $\widehat{X}_l$ smooth. Our main result can be applied successively to each resolution to show that the smooth resolution $\widehat{X}_l$ admits a cscK metric.



Keywords: Constant scalar curvature Kähler metrics, Orbifold, Resolution, Blow-up, Lie structures at infinity, Weighted projective space, Singularities of type $\mathcal{I}$.

# INTRODUCTION

## 0.1 Motivation

In the 1950s, Eugenio Calabi in [13, 12] proposed a natural notion of canonical Kähler metrics, namely extremal metrics. This involves fixing a Kähler class $\Omega$ and minimizing the Calabi functional:

$$\mathrm{Cal}(\omega) = \int_M S(\omega)^2 \omega^n,$$

where $S(\omega)$ is the scalar curvature, within the space of Kähler metrics whose Kähler form $\omega$ belongs to $\Omega$. Constant scalar curvature Kähler (cscK) metrics are examples of extremal metrics, and Kähler–Einstein metrics are examples of cscK metrics.

The existence of Kähler–Einstein metrics for compact Kähler manifolds depends on the sign of the first Chern class of the Kähler manifold. When the first Chern class is negative, there is always a Kähler–Einstein metric, as independently proved by Thierry Aubin [8] and Shing-Tung Yau [64, 63]. When the first Chern class is zero, there is always a Kähler–Einstein metric, as was shown by Shing-Tung Yau in [64, 63]. However, when the first Chern class is positive (also called Fano), the existence of Kähler–Einstein metric remained a well-known open problem for many years. In 2012, Xiuxiong Chen, Simon Donaldson, and Song Sun [18, 19, 20], as well as independently Gang Tian [59], proved that for the Fano case, an algebraic-geometric criterion called K-stability implies the existence of a Kähler–Einstein metric. Additionally, the converse was proved by Robert Berman [9].

Sixty years after it was proposed, Calabi's program continues to represent the forefront of most active current research in complex geometry, yielding spectacular results. Yau-Tian-Donaldson [26] conjectured more generally that there is an equivalence between the existence of a cscK metric on a polarized projective manifold and the K-polystability of that polarized manifold. Beyond the Kähler–Einstein Fano case, the conjecture was established for toric Kähler surfaces by Donaldson [26] and for general toric varieties by Chen Cheng. This conjecture was recently proven in 2021 by Chen-Cheng [16, 17, 15] in the toric case. In fact, it provides a necessary and sufficient condition, expressed in terms of the corresponding Delzant polytope, for a compact smooth toric manifold to admit a compatible Riemannian metric of constant scalar curvature.

In this thesis, we focus on constant scalar curvature Kähler (cscK) metrics. In 2006, Arezzo and Pacard [5] proved that if a compact manifold or compact orbifold $M$ with isolated singularities and no non-trivial holomorphic vector fields vanishing somewhere, admits a cscK metric, then the blow-up of $M$ at finitely many points also adimt a cscK metric. In 2009 [6], they generalized the statement to situations where there are non-trivial holomorphic vector fields with zeros. In 2011, Arezzo, Pacard, and Singer [7] proved the existence of an extremal metric on the blow-ups of a manifold at certain points, subject to assumptions on the position of the points, such as balancing and genericity conditions. Recently, in 2020, Seyyedali and Székelyhidi [53] extended the results of Arezzo and Pacard to obtain a cscK metric on blow-ups of a manifold along a submanifold. When the extremal metric is cscK and the automorphisms group is trivial, their result can be formulated as follows.

**Theorem A** ([53]). *Let $(X, \omega_X)$ be a compact cscK complex manifold with discrete group of automorphisms (in particular, there are no non-trivial holomorphic vector fields on $X$) and $Y \subset X$ be a submanifold of codimension $k$ greater than 2. Then $\mathrm{Bl}_Y^X$ admits a cscK metric in the class $[\omega_X] - \varepsilon^2[E]$ for sufficiently small $\varepsilon > 0$, where $E$ is the exceptional divisor of the blow-up.*

## 0.2 Main Results

In this thesis, we generalize Arezzo-Pacard-Singer and Seyyedali-Székelyhidi results by constructing cscK metrics on the resolution of a certain orbifolds as follows.

**Theorem B** (Theorem 6.8). *Suppose that $(X, \omega_X)$ is a compact cscK orbifold with no holomorphic vector fields, and such that the set of singular points $Y$ of $X$ is of complex co-dimension $> 2$. Suppose, furthermore, that any point $p \in Y$ has a local orbifold uniformization chart of the form $\mathbb{C}^{n-k} \times (\mathbb{C}^k/\Gamma)$ where $\Gamma$ is a finite linear group of type $\mathcal{I}$ of the form $(-w_0, w)$. If $\pi : \widehat{X} \to X$ is the partial resolution of $X$ obtained by a $(-w_0, w)$-weighted blow-up of $X$ along $Y$, then the class $[\omega_X] - \varepsilon^2[E]$ admits a cscK metric for $\varepsilon > 0$ sufficiently small, where $E = \pi^{-1}(Y)$ is the exceptional divisor of the resolution $\pi : \widehat{X} \to X$.*

Unless the singularity is of type $\mathcal{I}$ and of the form $(-r, 1, \ldots, 1)$, the resolution $\widehat{X}$ is not smooth. However, there is a possibly non-unique sequence of resolutions

$$\widehat{X}_l \to \widehat{X}_{l-1} \to \ldots \to \widehat{X}_1 \to X$$

obtained by a sequence of weighted blow-ups with $\widehat{X}_l$. For such a sequence of resolutions, we show that Theorem B can be applied iteratively to each $\widehat{X}_i$, yielding the following result.

**Corollary C.** *For $(X, \omega_X)$ as in Theorem B, let $\widehat{X}_l \to \widehat{X}_{l-1} \to \ldots \to \widehat{X}_1 \to X$ be a sequence of resolutions* *obtained through a sequence of weighted blow-ups with $\widehat{X}_l$ smooth. Then $\widehat{X}_l$ admits a cscK metric in a suitable Kähler class.*

Our strategy to prove this result consists in adapting the approach of [53] to the singular setting, using a coordinate-free description involving manifolds with corners.

## 0.3 Structure of the thesis

In chapter 1, we describe the basics notions in Kähler geometry. Additionally, we revisit concepts such as Kähler orbifolds, blow-up in complex geometry, weighted projective spaces and their singularities. We finish chapter 1 with the definition of singularities of type $\mathcal{I}$ and resolutions of type $\mathcal{I}$, introduced by Apostolov and Rollin about ten years ago. If we require that $Y$ has codimension at least 2, then the Lie algebra of holomorphic vector fields on the resolution $\widehat{X}$ is identified as a subalgebra of that on $X$, tangent to $Y$ (Proposition 1.148). This first property will later be used to construct an example (Theorem 3.26) to which the theorem applies.

In chapter 2, we describe the analytical tools. We begin by defining manifolds with corners and blow-ups in the Melrose sense. We then define the Lie structure at infinity and Riemannian metrics from them. Finally we introduce asymptotically Euclidean (AE), asymptotically locally Euclidean (ALE), asymptotically conical (AC), scattering (SC), quasi-asymptotically locally Euclidean (QALE), and quasi-asymptotically conical (QAC) metrics within a unified framework.

In chapter 3, we focus on constant scalar curvature Kähler (cscK) metrics. An important example is the notion of constant scalar curvature Kähler metrics is the Kähler-Einstein (KE) metric, which has been the primary focus of Kähler geometry since the inception of the celebrated Calabi conjecture on Kähler-Einstein metrics. We begin this chapter by defining extremal metrics. Then, we briefly study Kähler-Einstein metrics. Following that, we discuss classic results by Matsushima-Lichnerowicz and Arezzo-Pacard for cscK metrics. We finish this chapter by constructing new examples of cscK orbifolds with singularities of type $\mathcal{I}$ and having discete automorphism group (Theorem 3.26).

In chapter 4, we construct a family of Kähler metrics $\widehat{\omega}_\varepsilon$ by gluing tehcnique utilizing the Serre-Swan theorem. Starting from the orbifold $X$, we consider a cornered orbifold $\mathcal{X}$ obtained by blowing-up $X \times [0, \infty)$ along $Y \times \{0\}$. The hypersurface $H_1$ in $\mathcal{X}$ resulting from the blow-up of $Y \times \{0\}$ can be seen as the radial compactification of the normal bundle $N_X(Y)$ of $Y$. During the transition to the resolution $\widehat{X}$, $H_1$ becomes $\widehat{H}_1$ and there is a bundle map $\widehat{\varphi}_1 : \widehat{H}_1 \to \widehat{Y}$ that lifts a map $\varphi_1 : N_X(Y) \to Y$. Theorem 4.2 ensures that there exists a smooth closed $(1,1)$-form $\omega_{\widehat{\varphi}_1}$ on $\widehat{H}_1$ whose restriction to the fibers of $\widehat{\varphi}_1$ is a Kähler form of an asymptotically locally Euclidean (ALE) metric, derived from the combined works of Burns, Eguchi-Hanson, LeBrun, Pedersen-Poon, Simanca and Apostolov-Rollin. There is a rather fine understanding of the global behavior of $\omega_{\widehat{\varphi}_1}$. By examining a level set of $X$ associated with the deformation parameter (denoted by $\varepsilon$) in the real blow-up of Melrose, we construct a smooth Kähler metric $\widehat{\omega}_\varepsilon$ such that on $\widehat{H}_1$,

$$\frac{\widehat{\omega}_\varepsilon}{\varepsilon^2}\Big|_{\widehat{H_1}} = \omega_{\widehat{\varphi}_1} + \frac{\varphi_1^* \omega_{\Sigma_1}}{\varepsilon^2},$$

while for the Melrose blow-up of $X$ along $Y$, denoted by $H_2$, which is a manifold with corners, we have on $\widehat{H}_2$,

$$\widehat{\omega}_\varepsilon\Big|_{\widehat{H}_2} = \omega_X\Big|_{H_2},$$

the cscK metric on the orbifold $X$. The next goal is to perturb the metric $\omega_\varepsilon$ to obtain a true cscK metric on the resolution $\widehat{X}$.

In Chapter 5, we focus on linear analysis through the linearization of constant scalar curvature, which requires considering the Lichnerowicz operator on weighted Hölder spaces using techniques introduced by Mazzeo in his study of conical metrics. The triviality of the kernel of the Lichnerowicz operator is due to the assumption on holomorphic vector fields on $X$ (Lemma 5.7). Based on techniques developed by Seyyedali and Székelyhidi, we proved that the twisted Lichnerowicz operator $\widetilde{L}_\varepsilon$ is indeed boundedly invertible for sufficiently small $\varepsilon$ (Proposition 5.8).

In Chapter 6, we use nonlinear analysis to find a potential $u$ for obtaining a cscK metric $\widetilde{\omega}_\varepsilon = \widehat{\omega}_\varepsilon + \sqrt{-1}\,\partial\bar{\partial}\phi_\varepsilon$ on the resolution. We begin by expressing the first Chern class of the resolution $\widehat{X}$ in terms of that of $X$ (Proposition 6.1). This allows for the explicit calculation of the topological constant in the cscK equation on the resolution that we aim to solve (Proposition 6.2). The final step is to apply Banach's fixed-point theorem by carefully controlling the error term across four different regions, which depend on the distance to $Y$, following a strategy implemented in Székelyhidi's book [57].

# CHAPTER 1
# KÄHLER GEOMETRY

In this chapter, we introduce the fundamental principles of Kähler geometry that constitute the framework of this thesis. For further details, we refer to [60, 54] for the differential Geometry, for the topology [48, 44] and [31, 34, 62, 38, 57] for the complex geometry. Basic knowledge of smooth manifolds is assumed.

## 1.1 Complex Manifolds

Kähler geometry is an important field of mathematics at the intersection of Riemannian and complex geometry introduced by Erich Kähler in 1933. It provides a way to study the geometry of complex manifolds that have a Riemannian structure compatible with the complex structure. Let us recall the definition of a complex manifold.

**Definition 1.1** (Complex manifold)**.** *A complex manifold of complex dimension $n$ is a Hausdorff topological space $M$ together with the following data:*

1. *Atlas of Charts: For every point $p$ in $M$, there exists an open neighborhood $U$ of $p$ and a homeomorphism $\phi : U \to V$ where $V$ is an open subset of $\mathbb{C}^n$.*
2. *Transition Functions: The overlaps between charts are required to be holomorphic functions. More precisely, if $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ are two charts with non-empty intersection $U_1 \cap U_2$, then the map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is holomorphic.*

*These transition functions ensure that the complex structure of the manifold is well-behaved and consistent across different charts, inducing a global complex structures on it.*

The concept of a complex manifold generalizes the notion of complex curves and surfaces to higher dimensions. Complex manifolds provide a framework for studying complex geometry, and they have applications in various mathematical fields, including algebraic geometry, differential geometry and topology.

**Example 1.2.** *Here are a few examples of complex manifolds:*

1. *Complex Euclidean Space:* $\mathbb{C}^n$ *is the simplest example of a complex manifold. Each point in* $\mathbb{C}^n$ *has a natural complex coordinate representation, and the entire space is covered by a single chart.*

2. *Complex Projective Space:* $\mathbb{CP}^n$ *is the space of complex lines through the origin in* $\mathbb{C}^{n+1}$. *It can be defined as a quotient space of* $\mathbb{C}^{n+1}\backslash\{0\}$ *by the rescaling action, i.e.,* $(Z_0, Z_1, \ldots, Z_n) \sim (Z_0', Z_1', \ldots, Z_n')$ *if and only if there exists* $\lambda \in \mathbb{C}$ *such that*

$$(Z_0, Z_1, \ldots, Z_n) = \lambda(Z_0', Z_1', \ldots, Z_n').$$

*We denote by* $[Z_0 : \ldots : Z_n]$ *the point of* $\mathbb{CP}^n$ *corresponding to* $(Z_0, Z_1, \ldots, Z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. *The complex projective space* $\mathbb{CP}^n$ *can be covered by:*

$$U_i = \{[Z_0 : \ldots : Z_n] \in \mathbb{CP}^n : Z_i \neq 0\},$$

*with coordinates* $U_i \ni [Z_0 : \ldots : Z_n] \mapsto (z_1 = \frac{Z_0}{Z_i}, \ldots, \widehat{\frac{Z_i}{Z_i}}, \ldots, z_n = \frac{Z_n}{Z_i}) \in \mathbb{C}^n$. *The complex projective space* $\mathbb{CP}^n$ *is a compact* $n$*-dimensional complex manifold. May also regard* $\mathbb{CP}^n$ *as a quotient of the unit sphere* $\mathbb{S}^{2n+1}$ *in* $\mathbb{C}^{n+1}$ *under the action of* $U(1)$:

$$\mathbb{CP}^n = \mathbb{S}^{2n+1}/U(1),$$

*since every line in* $\mathbb{C}^{n+1}$ *intersects the unit sphere in a circle.*

3. *Complex Torus: The complex torus* $\mathbb{C}^n/\Lambda$, *where* $\Lambda$ *is a lattice of rank* $2n$ *in* $\mathbb{C}^n$, *is an example of a compact complex manifold. It is a higher-dimensional generalization of the notion of elliptic curve.*

4. *Riemann Surfaces: One-dimensional complex manifolds (complex curves) are called Riemann surfaces. The uniformization theorem says that every simply connected Riemann surface is conformally equivalent to one of the following three Riemann surfaces: the open unit disk* $\mathbb{D}$, *the complex plane* $\mathbb{C}$, *or the Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{CP}^1 = \mathbb{S}^2$.

**Remark 1.3.** *Not every even-dimensional manifold has a compatible complex structure. In fact, there are certain topological restrictions that must be satisfied for a manifold to admit a complex structure. The existence of a compatible complex structure is related to the concept of orientability. A complex structure on a smooth manifold* $M$ *of real dimension* $2n$ *implies that* $M$ *is orientable. However, not all orientable manifolds of even dimension admit a complex structure. The sphere* $\mathbb{S}^n$ *does not admit a complex structure* *when $n \neq 2$. However, the Riemann sphere $\mathbb{S}^2$ does admit a complex structure. In fact, every orientable closed 2-manifold has a complex structure as a Riemann surface. As for $\mathbb{S}^6$, it is still unknown whether it admits a complex structure.*

**Definition 1.4** (Holomorphic function)**.** *Let $X$ be a complex manifold of complex dimension $n$. A function $f : X \to \mathbb{C}$ is called holomorphic, if for all local chart $(U, \phi)$ on $X$, $f \circ \phi : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic. The space of holomorphic function from $X$ to $\mathbb{C}$ is denoted by $\mathcal{O}(X)$.*

**Definition 1.5** (Meromorphic function)**.** *Let $X$ be a complex manifold of complex dimension $n$, and $\Omega$ is an open and dense subset of $X$. A function $f : \Omega \to \widehat{\mathbb{C}}$ is called meromorphic if, for any $x_0 \in \Omega$, there exists an open neighborhood $U$ and non-zero holomorphic functions $g$ and $h : U \to \mathbb{C}$ such that for every $x \in U \cap \Omega \setminus \{x_0\}$,*

$$f(x) = \frac{g(x)}{h(x)}.$$

**Definition 1.6** (Almost complex structure)**.** *Let $M$ be a smooth manifold of real dimension $2n$. An almost complex structure on $M$ is a smooth bundle endomorphism $J : TM \to TM$ such that at each point $p \in M$, $J_p^2 = -\operatorname{Id}_{T_pM}$, where $\operatorname{Id}$ is the identity operator.*

**Proposition 1.7.** *Any complex manifold $M$ admits a natural almost complex structure.*

On a complex manifold $M$, the holomorphic charts induce a natural almost complex structure on $TM$ via multiplication by $\sqrt{-1}$. We say in this case that the almost complex structure is integrable (or that is a complex structure). Not all almost complex structures are integrable. The Newlander-Nirenberg theorem provides a necessary and sufficient condition for determining whether an almost complex structure $J$ is integrable in terms of its Nijenhuis tensor $N_J$, which is defined by

$$N_J(V, W) = [V, W] + J([JV, W] + [V, JW]) - [JV, JW],$$

for $V, W$ vector fields on $M$.

**Theorem 1.8** (Newlander-Nirenberg)**.** *Let $M$ be a smooth manifold and $J$ an almost complex structure on $M$. Then $J$ is integrable if and only if the Nijenhuis tensor associated with $J$ vanishes.*

**Definition 1.9** (Holomorphic vector field)**.** *On any almost complex manifold $(M, J)$, a (real) vector field $X$ is said to be (real) holomorphic if*

$$\mathcal{L}_X J = 0,$$

*where $\mathcal{L}_X$ denotes the Lie derivative along $X$.*

**Definition 1.10** (Holomorphic map). *Let $(M, J_M)$, $(N, J_N)$ be two complex manifolds and $f : M \to N$ smooth map. We say that $f$ is a holomorphic map if $J_N \circ df = df \circ J_M$. Furthermore, $f$ is called biholomorphic, if $f$ is a diffeomorphism and its inverse is also a holomorphic map.*

**Definition 1.11** (Automorphism group). *Let $M$ be a complex manifold. An automorphism of $(M, J)$ is a biholomorphic map $\phi : M \to M$. The automorphisms of $M$ form a group $\mathrm{Aut}(M, J)$ or simply $\mathrm{Aut}(M)$ if $J$ is understood, called the automorphism group of $M$. We denote by $\mathrm{Aut}_0(M, J)$ its connected component to the identity.*

**Example 1.12.** *The automorphism group of the complex projective space $\mathbb{CP}^n$ is the projective linear group $\mathbb{P}\mathrm{GL}(n+1, \mathbb{C})$ consisting of all invertible $(n+1) \times (n+1)$ complex matrices up to a scalar factor, i.e,*

$$\mathrm{Aut}(\mathbb{CP}^n) = \mathbb{P}\mathrm{GL}(n+1, \mathbb{C}) := \mathrm{GL}(n+1, \mathbb{C})\,/\mathbb{C}^*.$$

**Remark 1.13.** *The automorphism group $\mathrm{Aut}(M)$ has a structure of complex Lie group. Its Lie Algebra identified with the real smooth vector fields on $M$ whose flow preserves $J$, i.e, real holomorphic vector fields and will be discussed in Example 1.37 on page 15.*

**Definition 1.14** (Complex submanifold). *Let $X$ be a complex manifold of dimension $n$, and $Y \subseteq X$. We call $Y$ a **complex submanifold** of $X$ of codimension $k$, for $0 \leq k \leq n$, if for each $y \in Y$ there exist local holomorphic coordinates $(z_1, \ldots, z_n)$ on $X$ such that $Y$ is locally of the form $z_{n-k+1} = z_{n-k+2} = \ldots = z_n = 0$. i.e, there exist a chart $(U, \phi)$ on $X$ with $y \in \phi(U)$ such that $Y \cap \phi(U) = \phi(\mathbb{C}^{n-k} \cap U)$, where $\mathbb{C}^{n-k} = \{(z_1, \ldots, z_{n-k}, 0, \ldots, 0)\}$. A complex submanifold $Y$ of codimension $k$ is naturally a complex $(n-k)$-manifold.*

**Definition 1.15** (Analytic subvariety). *Let $X$ be a complex manifold of dimension $n$, and $Y \subseteq X$ a closed subset. We call $Y$ an **analytic subvariety** of $X$, if for each $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $Y \cap U$ is the zero set of finitely many holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$. An analytic subvariety is not necessarily smooth. A point $x \in Y$ is a smooth or regular point of $Y$ if the functions $f_1, \ldots, f_k$ can be chosen such that $\phi(x) \in \phi(U)$ is a regular point of the holomorphic map $f := (f_1 \circ \phi^{-1}, \ldots, f_k \circ \phi^{-1}) : \phi(U) \to \mathbb{C}^k$ i.e, its Jacobian has rank $k$. Here, $(U, \phi)$ is a local chart around $X$. A point $x \in Y$ is singular if it is not regular. The set of regular points $Y_{reg} = Y \setminus Y_{sing}$ is a non-empty complex submanifold of $X$. An analytic subvariety in a neighbourhood of a regular point is a complex submanifold. An analytic subvariety $Y$ is irreducible if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two proper analytic subvarieties $Y_i \subset Y$. The dimension of an irreducible analytic subvariety $Y \subset X$ is by definition $\dim(Y) = \dim(Y_{reg})$.*

**Example 1.16** (Hypersurface)**.** *Let $X$ be a complex manifold. An analytic subvariety $Y$ of $X$ of codimension $1$ is called an analytic hypersurface, i.e, for each $y \in X$ there exist an open neighborhood $U \subset X$ and non-zero holomorphic function $f : U \to \mathbb{C}$ such that $U \cap Y = \{u \in U : f(u) = 0\}$. This analytic hypersurface is smooth on $U \cap Y$ if $df$ does not vanishes on $U \cap Y$. Every analytic hypersurface is a locally finite union of irreducible analytic hypersurfaces. If $X$ is compact this union is finite.*

**Definition 1.17** (Projective Variety and Projective Complex Manifold)**.** *A **projective variety** is a subset $X$ of $\mathbb{CP}^n$ which is defined by the vanishing of finitely many homogeneous polynomials $P_1, \dots P_k$, i.e,*

$$X = \{[z_0, z_1, \dots, z_n] \in \mathbb{CP}^n : P_1(z_0, z_1, \dots, z_n) = \dots = P_k(z_0, z_1, \dots, z_n) = 0\}.$$

*Projective varieties are closed in $\mathbb{CP}^n$, and so compact. A projective variety is called a **projective complex manifold** if it is also a complex submanifold of $\mathbb{CP}^n$. A complex manifold is called **algebraic** if it is a projective complex manifold*

**Theorem 1.18** (Chow)**.** *A compact complex submanifold of $\mathbb{CP}^n$ is algebraic.*

There are complex manifolds that are not algebraic; these are studied in Transcendental Geometry. A natural question arises: when a compact complex manifold is algebraic? According to the Kodaira Embedding Theorem, if a compact complex manifold admits an ample line bundle, then it is algebraic. We will demonstrate that any projective complex manifold is Kähler, so by using homogeneous polynomials, we can construct many Kähler manifolds. Furthermore, under some conditions on cohomology, Kähler manifolds are algebraic, allowing us to employ complex algebraic geometry to classify them.

## 1.2 Calculus on Complex Manifolds

Let $M$ be a complex manifold of complex dimension $n$. Then $M$ is a smooth manifold of real dimension $2n$. Let $\{z^i = x^i + \sqrt{-1}y^i\}_{i=1,\dots,n}$ be complex coordinates around $p \in M$ such that $\{x^i, y^i\}_{i=1,\dots,n}$ are the corresponding real coordinates. In these local coordinates the tangent space is given by

$$T_pM = \mathbb{R}\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}_{i=1,\dots,n},$$

and the cotangent space is generated by the dual of this basis, i.e,

$$T_p^*M = \mathbb{R}\left\{dx^i, dy^i\right\}_{i=1,\dots,n}.$$

Each $T_pM$ admits a natural almost complex structure $J : T_pM \to T_pM$ that maps $\frac{\partial}{\partial x^i}$ to $\frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial y^i}$ to $-\frac{\partial}{\partial x^i}$. Its dual $J^* : T_p^*M \to T_p^*M$ maps $dx^i$ to $-dy^i$ and $dy^i$ to $dx^i$. The complexified tangent space $T_p^{\mathbb{C}}M = T_pM \otimes_{\mathbb{R}} \mathbb{C}$ is generated by

$$T_p^{\mathbb{C}}M = \mathbb{C}\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}_{i=1,\dots,n}.$$

Also we can write

$$T_p^{\mathbb{C}}M = \mathbb{C}\left\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\right\}_{i=1,\dots,n},$$

where $\frac{\partial}{\partial z^i} := \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i})$ and $\frac{\partial}{\partial \bar{z}^i} := \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i})$. The condition $J_p^2 = -\mathrm{id}$ means that $J_p : T_pM \to T_pM$ has the minimal polynomial $\lambda^2 + 1 = 0$ for any $p \in M$, so it has two eigenvalues $\pm\sqrt{-1}$. The eigenspace corresponding to the eigenvalue $\sqrt{-1}$ is called the holomorphic tangent space to $M$ at $p$ and is denoted by $T_p^{1,0}M$. The eigenspace corresponding to the eigenvalue $-\sqrt{-1}$ is called the antiholomorphic tangent space to $M$ at $p$ and is denoted by $T_p^{0,1}M$. In terms of $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\}$,

$$T_p^{1,0}M = \mathbb{C}\left\{\frac{\partial}{\partial z^i}\right\}_{i=1,\dots,n},$$
$$T_p^{0,1}M = \mathbb{C}\left\{\frac{\partial}{\partial \bar{z}^i}\right\}_{i=1,\dots,n}.$$

So the complexified tangent bundle $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

such that $T^{1,0}M = \bigsqcup_{p\in M} T_p^{1,0}M$, $T^{0,1}M = \bigsqcup_{p\in M} T_p^{0,1}M$ and the complex linear extension of $J$ satisfies

$$J\big|_{T^{1,0}M} = \sqrt{-1}\,\mathrm{id}_{T^{1,0}M}, \quad J\big|_{T^{0,1}M} = -\sqrt{-1}\,\mathrm{id}_{T^{0,1}M}.$$

Notice that $T^{1,0}M$ is naturally isomorphic to $TM$ as a real vector bundle. For this reason, we say that $T^{1,0}M$ is the complex tanget bundle of $M$ and we will often denote $T^{1,0}M$ simply by $TM$ hoping that will lead to no confusion. Similarly, the complexified cotangent bundle $(T^{\mathbb{C}}M)^* = T^*M \otimes_{\mathbb{R}} \mathbb{C}$ admits an analogous decomposition

$$(T^{\mathbb{C}}M)^* = (T^{1,0}M)^* \oplus (T^{0,1}M)^*,$$

locally trivialized by the dual basis $\{dz^i = dx^i + \sqrt{-1}dy^i\}$ and $\{d\bar{z}^i = dx^i - \sqrt{-1}dy^i\}$ respectively. We say that $(T^{1,0}M)^*$ is the complex cotanget bundle of $M$.

**Remark 1.19.** *Let $(M, J_M)$, $(N, J_N)$ be two almost complex manifolds and $f : M \to N$ be a smooth map. This map induces $\mathbb{R}$-linear map*

$$f_{*p} : T_pM \to \mathrm{T}_{f(p)}\,N,$$

*and $\mathbb{C}$-linear map*

$$f_{*p} : T^{\mathbb{C}}_pM \to \mathrm{T}^{\mathbb{C}}_{f(p)}\,N.$$

*Note that under $f_{*p}$, the tangent space $T^{1,0}_pM$ does not necessarily map to $T^{1,0}_{f(p)}N$. In fact $f_{*p}(T^{1,0}_pM) \subseteq T^{1,0}_{f(p)}N$ if and only if $f$ is a holomorphic map.*

**Definition 1.20** (Differential forms)**.** *Let $M$ be complex manifold.*

1. *Real valued $k$-forms are smooth sections of the real vector bundle $\wedge^kT^*M$. The space of real valued $k$-forms on $M$ is denoted by $\mathcal{A}^k(M, \mathbb{R})$.*
2. *Complex valued $k$-forms are differential forms of the form $\omega = \alpha + \sqrt{-1}\beta$ where $\alpha$ and $\beta$ are real valued $k$-forms. The space of complex valued $k$-forms on $M$ is denoted by $\mathcal{A}^k(M, \mathbb{C})$.*
3. *Complex valued $(p, q)$-forms are complex valued differential forms which in local complex coordinates are of the form*

$$\omega = \sum_{|I|=p,|J|=q} f_{IJ}dz^I \wedge d\bar{z}^J,$$

*where $f_{IJ} : M \to \mathbb{C}$ are smooth functions. The space of $(p, q)$-forms on $M$ is denoted by $\mathcal{A}^{p,q}(M, \mathbb{C})$.*

**Remark 1.21.** *The space of complex valued $k$-forms naturally decomposes in term of complex valued $(p, q)$-forms, i.e,*

$$\mathcal{A}^k(M, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(M, \mathbb{C}).$$

The real exterior derivative is $d : \mathcal{A}^k(M, \mathbb{R}) \to \mathcal{A}^{k+1}(M, \mathbb{R})$ and complexifies as $d : \mathcal{A}^k(M, \mathbb{C}) \to \mathcal{A}^{k+1}(M, \mathbb{C})$ with $d(\alpha + \sqrt{-1}\beta) = d\alpha + \sqrt{-1}d\beta$. On $\mathcal{A}^{p,q}(M, \mathbb{C})$, two differential operators

$$\partial : \mathcal{A}^{p,q}(M, \mathbb{C}) \to \mathcal{A}^{p+1,q}(M, \mathbb{C})$$
$$\bar{\partial} : \mathcal{A}^{p,q}(M, \mathbb{C}) \to \mathcal{A}^{p,q+1}(M, \mathbb{C}),$$

are defined by

$$\partial(\sum_{|I|=p,|J|=q} f_{IJ} dz^I \wedge d\bar{z}^J) = \sum_{|I|=p,|J|=q} \sum_{l=1}^{n} \frac{\partial f_{IJ}}{\partial z^l} dz^l \wedge dz^I \wedge d\bar{z}^J,$$
$$\bar{\partial}(\sum_{|I|=p,|J|=q} f_{IJ} dz^I \wedge d\bar{z}^J) = \sum_{|I|=p,|J|=q} \sum_{l=1}^{n} \frac{\partial f_{IJ}}{\partial \bar{z}^l} d\bar{z}^l \wedge dz^I \wedge d\bar{z}^J.$$

The opertaor $\bar{\partial}$ is called Dolbeault or Cauchy-Riemann operator. The exterior differential

$$d : \mathcal{A}^{p,q}(M,\mathbb{C}) \to \mathcal{A}^{p+1,q}(M,\mathbb{C}) \oplus \mathcal{A}^{p,q+1}(M,\mathbb{C}),$$

decomposes in terms of these operators $d = \partial + \bar{\partial}$. The operators

$$d^c : \mathcal{A}^{p,q}(M,\mathbb{C}) \to \mathcal{A}^{p+1,q}(M,\mathbb{C}) \oplus \mathcal{A}^{p,q+1}(M,\mathbb{C}),$$

is defined by $d^c = -\sqrt{-1}(\partial - \bar{\partial})$. So we could write $\partial = \frac{1}{2}(d + \sqrt{-1}d^c)$ and $\bar{\partial} = \frac{1}{2}(d - \sqrt{-1}d^c)$.

**Proposition 1.22.** *Let $M$ be a complex manifold, $\alpha \in \mathcal{A}^{p,q}(M,\mathbb{C})$ and $\beta \in \mathcal{A}^{r,s}(M,\mathbb{C})$. Then the following Leibniz rules hold:*

$$\partial(\alpha \wedge \beta) = (\partial\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge (\partial\beta),$$
$$\bar{\partial}(\alpha \wedge \beta) = (\bar{\partial}\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge (\bar{\partial}\beta),$$
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge (d\beta).$$

**Remark 1.23.** *Direct computations show that $d^2 = 0$, $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $d^{c2} = 0$. So we get $\partial\bar{\partial} = -\bar{\partial}\partial$, $dd^c = -d^c d$ and*

$$dd^c = 2\sqrt{-1}\partial\bar{\partial}.$$[1]

**Remark 1.24.** *One can check that $\bar{d\omega} = d\bar{\omega}$ and $\bar{d^c\omega} = d^c\bar{\omega}$, so $d$ and $d^c$ are real operators, meaning that they take real forms to real forms. Also $\overline{\partial\omega} = \bar{\partial}\bar{\omega}$ and $\overline{\bar{\partial}\omega} = \partial\bar{\omega}$, so $\partial$ and $\bar{\partial}$ are complex conjugate.*

**Remark 1.25.** *For a smooth function $f : M \to \mathbb{C}$ defined on a complex manifold $M$, the Cauchy-Riemann equations are equivalent to $\bar{\partial}f = 0$. Therefore, $f$ is a holomorphic function if and only if $\bar{\partial}f = 0$.*

**Definition 1.26** (Holomorphic $p$-forms)**.** *On a complex manifold $M$, a $(p,0)$-form $\omega$ is called holomorphic, if $\bar{\partial}\omega = 0$. The space of holomorphic $(p,0)$-forms is denoted by $\Omega^{p,0}(M)$.*

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[1] Some references use $d^c = -\frac{\sqrt{-1}}{2}(\partial - \bar{\partial})$ and so $dd^c = \sqrt{-1}\partial\bar{\partial}$.

The fact that $d^2 = 0$ implies that $\mathrm{Im}(d : \mathcal{A}^{k-1}(M, \mathbb{C}) \to \mathcal{A}^k(M, \mathbb{C})) \subseteq \ker(d : \mathcal{A}^k(M, \mathbb{C}) \to \mathcal{A}^{k+1}(M, \mathbb{C}))$, so we can define the complexified de Rham cohomology

$$H^k_{\mathrm{dR}}(M, \mathbb{C}) = \frac{\ker(d : \mathcal{A}^k(M, \mathbb{C}) \to \mathcal{A}^{k+1}(M, \mathbb{C}))}{\mathrm{Im}(d : \mathcal{A}^{k-1}(M, \mathbb{C}) \to \mathcal{A}^k(M, \mathbb{C}))}.$$

Note that $H^k_{\mathrm{dR}}(M, \mathbb{C}) = H^k_{\mathrm{dR}}(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ where $H^k_{\mathrm{dR}}(M, \mathbb{R})$ is the real de Rham cohomology. So the complex version of de Rham cohomology does not carry more topological information than the real one. For a compact manifold $M$, the de Rham cohomology groups $H^k_{\mathrm{dR}}(M, \mathbb{C})$ are finite-dimensional vector spaces, and their dimensions are called Betti numbers and denoted by $b_k(M)$. We recall the Poincaré lemma.

**Theorem 1.27** ($d$-Poincaré lemma). *Let $U$ be an open disk in $\mathbb{R}^n$. If $\alpha \in \mathcal{A}^k(U, \mathbb{C})$ is closed, i.e $d\alpha = 0$, then $\alpha$ is exact, i.e, $\alpha = d\beta$ for some $\beta \in \mathcal{A}^{k-1}(U, \mathbb{C})$. In terms of the de Rham cohomology,*

$$H^k_{\mathrm{dR}}(U) = 0, k \geq 1.$$

**Definition 1.28** (Dolbeault cohomology). *The property $\bar{\partial}^2 = 0$ implies $\mathrm{Im}(\bar{\partial} : \mathcal{A}^{p,q}(M, \mathbb{C}) \to \mathcal{A}^{p,q+1}(M, \mathbb{C})) \subseteq \ker(\bar{\partial} : \mathcal{A}^{p,q-1}(M) \to \mathcal{A}^{p,q}(M, \mathbb{C}))$, so we can define the $(p, q)$-Dolbeault cohomology group by*

$$H^{p,q}_{\bar{\partial}}(M, \mathbb{C}) = \frac{\ker(\bar{\partial} : \mathcal{A}^{p,q}(M, \mathbb{C}) \to \mathcal{A}^{p,q+1}(M, \mathbb{C}))}{\mathrm{Im}(\bar{\partial} : \mathcal{A}^{p,q-1}(M, \mathbb{C}) \to \mathcal{A}^{p,q}(M, \mathbb{C}))}.$$

For a compact manifold $M$, the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(M, \mathbb{C})$ are finite-dimensional vector spaces, and their complex dimensions are called Hodge numbers and denoted by $h^{p,q}(M)$.

**Remark 1.29.** *Note that Dolbeault cohomology is defined for complex manifolds and not for almost complex manifolds.*

**Theorem 1.30** ($\bar{\partial}$-Poincaré lemma). *Let $U$ be an open polydisk in $\mathbb{C}^n$. If $\alpha \in \mathcal{A}^{p,q}(U, \mathbb{C})$ is $\bar{\partial}$-close, i.e $\bar{\partial}\alpha = 0$, then $\alpha$ is $\bar{\partial}$-exact, i.e, $\alpha = \bar{\partial}\beta$ for some $\beta \in \mathcal{A}^{p,q-1}(U, \mathbb{C})$. In terms of the Dolbeault cohomology,*

$$H^{p,q}_{\bar{\partial}}(U, \mathbb{C}) = 0, q \geq 1.$$

Here we recall the local $\partial\bar{\partial}$-lemma.

**Theorem 1.31** (Local $\partial\bar{\partial}$-lemma). *Let $U$ be an open polydisk in $\mathbb{C}^n$. If $\alpha \in \mathcal{A}^{p,q}(U, \mathbb{C})$ is $d$-closed, i.e $[\alpha] \in H^{p+q}_{\mathrm{dR}}(U, \mathbb{C})$, then $\alpha$ is $\partial\bar{\partial}$-exact, i.e, $\alpha = \partial\bar{\partial}\beta$ for some $\beta \in \mathcal{A}^{p-1,q-1}(U, \mathbb{C})$.*

**Theorem 1.32** (Dolbeault Theorem). *On a compact complex manifold $M$, $H^{p,q}_{\bar{\partial}}(M, \mathbb{C})$ is a finite-dimensional vector space, and the dimension of it is called the $(p, q)$-Hodge number, denoted by $h^{p,q}$.*

**Proposition 1.33.** *There is a natural isomorphism $H^{p,q}_{\bar{\partial}}(M) \cong H^q(M, \Omega^p(M))$. Here $H^q(M, \Omega^p(M))$ is the sheaf cohomology of the complex manifold $M$ with coefficient in the sheaf of holomorphic $p$-forms. In particular $H^{0,q}_{\bar{\partial}}(M) \cong H^q(M, \mathcal{O}_M)$, where $\mathcal{O}_M$ is the sheaf of holomorphic functions on $M$. Sheaf cohomology is typically defined using Čech cohomology, but in this thesis, we will not provides further details.*

## 1.3 Vector Bundles

**Definition 1.34.** *Let $M$ be a complex manifold. A holomorphic (respectively smooth) complex vector bundle $E$ of rank $r$ over $M$ consists of the following data:*

1. *A complex (respectively smooth) manifold $E$ called the total space of the bundle.*
2. *A holomorphic (respectively smooth) surjective map $\pi : E \to M$ called the projection map.*
3. *For each point $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over $p$ is a $r$-dimensional complex vector space.*
4. *For each point $p \in M$, there exists a neighborhood $U$ of $p$ and a biholomorphism (respectively diffeomorphism) $\phi : \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that $\phi\big|_p : E_p \to \mathbb{C}^r$ is linear and the following diagram commutes:*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times \mathbb{C}^r \\ \downarrow \pi & & \downarrow \mathrm{Pr}_1 \\ U & \xrightarrow[\mathrm{id}_U]{} & U, \end{array}$$

*where $\mathrm{Pr}_1$ is the projection onto the first factor in the product space $U \times \mathbb{C}^r$, and $\mathrm{id}_U$ is the identity map on $U$.*

*The map $\phi$ is called a holomorphic (respectively smooth) trivialization of the vector bundle $E$ over $U$, and the pair $(U, \phi)$ is called a holomorphic (respectively smooth) chart for the bundle. The local transition functions between overlapping holomorphic (respectively smooth) charts must be holomorphic (respectively smooth) maps from an open set in $\mathbb{C}^n$ to the general linear group of the vector space, i.e, if $(U_i, \phi_i)$ and $(U_j, \phi_j)$ are two charts the induced transition functions*

$$\phi_{ij}(x) := (\phi_i \circ \phi_j^{-1})(x) : \mathbb{C}^r \to \mathbb{C}^r$$

*are $\mathbb{C}$-linear for all $x \in U_i \cap U_j$.*

**Definition 1.35.** *Let $E$ be a holomorphic vector bundle over complex manifold $M$. A holomorphic (respectively smooth) function $s : M \to E$ is called a holomorphic (respectively smooth) section of $E$, if $\pi \circ s = \mathrm{id}_M$. The space of holomorphic (respectively smooth) sections of the vector bundle $E$ is denoted by $H^0(M, E)$ (respectively $\Gamma(E)$). A function $s : M \to E$ defined on an open and dense subset of $M$ to $E$ is called a meromorphic section of $E$ if $\pi \circ s = \mathrm{id}$ and for any $x \in M$, there exists an open neighborhood $U$, a holomorphic section $t : U \to M$, and a meromorphic function $f : U \to \mathbb{C}$ such that, together, they are non-zero near $x$ and in the domain of $s$, $s = \dfrac{t}{f}$.*

**Example 1.36** (The trivial vector bundle). *Let $M$ be a complex manifold. The product space $E = M \times \mathbb{C}^r$ with the projection onto the first coordinates has the structure of a holomorphic vector bundle of rank $r$ over $M$. This bundle is called the trivial line bundle. For the trivial vector bundle $H^0(M, E) \cong C^\omega(M, \mathbb{C}^r)$ and $\Gamma(E) \cong C^\infty(M, \mathbb{C}^r)$.*

**Example 1.37** (The holomorphic tangent bundle). *The holomorphic tangent bundle of a complex manifold $M$, $T^{1,0}M$, is a holomorphic vector bundle. Its smooth sections are holomorphic vector fields. They forms a vector space denoted by $\mathfrak{h}(M)$. The space of holomorphic vector fields on a compact complex manifold is the Lie algebra of the automorphism group $\mathrm{Aut}(M)$.*

**Definition 1.38** (Picard group). *Holomorphic vector bundles of rank one are called holomorphic line bundles. The Picard group $\mathrm{Pic}(M)$ of a complex manifold $M$ is defined as the set of classes of holomorphic line bundles on $M$. The group operation is given by the tensor product of line bundles, and the identity element is the class of the trivial line bundle.*

**Corollary 1.39.** *There is a natural isomorphism $\mathrm{Pic}(M) \cong H^1(M, \mathcal{O}_M^*)$. Here $H^1(M, \mathcal{O}_M^*)$ is the sheaf cohomology of the complex manifold $M$ with coefficient in the sheaf of non-zero holomorphic functions.*

See Corollary 2.2.10 in [34] for a proof.

Since a line bundle $\mathcal{L} \to M$ is locally trivial, holomorphic sections of $\mathcal{L}$ are locally holomorphic functions $f : M \to \mathbb{C}$, but globally they can be quite different. If $M$ is compact and connected then $\mathcal{O}(M) \cong \mathbb{C}$, i.e., holomorphic functions are constant functions. However, a holomorphic line bundle may have many non-zero holomorphic sections. Therefore, holomorphic sections offer an alternative perspective for studying complex manifolds and vector bundles.

**Example 1.40** (Line bundles over $\mathbb{CP}^n$). *Line bundles over $\mathbb{CP}^n$ play a central role in algebraic geometry. They encode geometric and topological information about complex projective spaces. An important exam-*

*ple is the tautological line bundle, denoted by $\mathcal{O}_{\mathbb{CP}^n}(-1)$. For the tautological line bundle, fibers are lines passing through the origin and so the total space of $\mathcal{O}_{\mathbb{CP}^n}(-1)$ consists of the disjoint union of all lines passing through the origin, i.e, the submanifold of $\mathbb{C}^n \times \mathbb{CP}^n$ given by*

$$\mathcal{O}_{\mathbb{CP}^n}(-1) = \{(z, Z) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n : z_i Z_j = z_j Z_i, \forall i, j\},$$

*where $Z = [Z_0, \dots, Z_n] \in \mathbb{CP}^n$. In terms of bundle, $\mathcal{O}_{\mathbb{CP}^n}(-1)$ is a line bundle over $\mathbb{CP}^n$ and its fiber over a point $[Z_0; \dots; Z_n]$ is the line in $\mathbb{C}^{n+1}$ spanned by $(Z_0, \dots, Z_n)$ and the projection map $\pi : \mathcal{O}_{\mathbb{CP}^n}(-1) \to \mathbb{CP}^n$ assigns to each line its point in $\mathbb{CP}^n$. The dual of the tautological line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$ is called the hyperplane bundle and denoted by $\mathcal{O}_{\mathbb{CP}^n}(1)$. More generally, for any $r \in \mathbb{Z}$, the line bundle $\mathcal{O}_{\mathbb{CP}^n}(r)$ is defined as $\mathcal{O}_{\mathbb{CP}^n}(1)^{\otimes r}$ for $r > 0$ and $\mathcal{O}_{\mathbb{CP}^n}(-1)^{\otimes(-r)}$ for $r < 0$. In a local trivialization over an open set $U_i = \{[Z_0 : \dots : Z_n] \in \mathbb{CP}^n \mid Z_i \neq 0\}$, where $[Z_0 : \dots : Z_n]$ are the homogeneous coordinates of $\mathbb{CP}^n$, $\mathcal{O}_{\mathbb{CP}^n}(r)$ can be represented as $U_i \times \mathbb{C}$, and the transition function $\phi_{ij}$ on the intersection $U_i \cap U_j$ is a holomorphic function defined by:*

$$\phi_{ij} : U_i \cap U_j \to \mathbb{C}, \quad \phi_{ij}([Z_0 : \dots : Z_n]) = \left(\frac{Z_j}{Z_i}\right)^r,$$

*for some $r \in \mathbb{Z}$. In fact, for the complex projective space $\mathbb{CP}^n$, the Picard group is identified with $\mathbb{Z}$ via the isomorphism $\mathbb{Z} \ni r \mapsto \mathcal{O}_{\mathbb{CP}^n}(r) \in \mathrm{Pic}(X)$.*

On $\mathbb{CP}^1$, there is also a simple classification of holomorphic vector bundles.

**Theorem 1.41** (Grothendieck Classification Theorem for Vector Bundles on $\mathbb{CP}^1$)**.** *Let $E$ be a complex vector bundle on $\mathbb{CP}^1$. Then, there exist unique integers $r_1, \dots, r_k$ such that $E$ is isomorphic to a direct sum of line bundles:*

$$E \cong \mathcal{O}_{\mathbb{CP}^1}(r_1) \oplus \mathcal{O}_{\mathbb{CP}^1}(r_2) \oplus \dots \oplus \mathcal{O}_{\mathbb{CP}^1}(r_k).$$

See [32] for a proof.

**Proposition 1.42.** *The behavior of the space of holomorphic sections of $\mathcal{O}_{\mathbb{CP}^n}(r)$ depends on the sign of $r$ as follows.*

1. *When $r > 0$, the space of global holomorphic sections of $\mathcal{O}_{\mathbb{CP}^n}(r)$ is non-trivial and consists of homogeneous polynomials of degree $r$ in $n+1$ complex variables $(Z_0, \dots, Z_n)$.*

2. *When $r = 0$, $\mathcal{O}_{\mathbb{CP}^n}(0)$ is the trivial line bundle and its space of sections identified with $\mathbb{C}$, correspond to the constant functions.*

3. *When $r < 0$, the space of global holomorphic sections of $\mathcal{O}_{\mathbb{CP}^n}(r)$ is trivial, meaning it only contains the zero section, i.e., there are no non-zero global holomorphic functions in $\mathcal{O}_{\mathbb{CP}^n}(r)$.*

Now we define the canonical line bundle of a complex manifold $M$. It will be important in understanding the Ricci curvature of Kähler manifolds. For example, a simply connected Kähler manifold admits a Ricci flat Kähler metric if and only if its canonical bundle is trivial.

**Definition 1.43** (Canonical line bundle)**.** *Given a complex manifold $M$, its canonical bundle $K_M$ is the line bundle corresponding to the top exterior power of the complex cotangent bundle of $M$.*

**Example 1.44** (The canonical bundle of $\mathbb{CP}^n$)**.** *Let $U_i = \{[Z_0, \dots, Z_n] \in \mathbb{CP}^n \mid Z_i \neq 0\}$, where $[Z_0, \dots, Z_n]$ are the homogeneous coordinates of $\mathbb{CP}^n$. Define a meromorphic $n$-form on $U_0$ by*

$$\alpha = \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n},$$

*where $[1, z_1, \dots, z_n] \sim [Z_0, Z_1, \dots, Z_n]$. This form is non-zero on $U_0$ and has poles along the hyperplanes $Z_1 = 0$, $Z_2 = 0$, ..., $Z_n = 0$. Now consider new coordinates on $U_j$ defined by*

$$[W_0, \dots, W_{j-1}, 1, W_{j+1}, \dots, W_n] \sim [Z_0, Z_1, \dots, Z_n].$$

*In these coordinates*

$$[1, z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n] \sim [1, \frac{W_1}{W_0}, \dots, \frac{W_{j-1}}{W_0}, \frac{1}{W_0}, \frac{W_{j+1}}{W_0}, \dots, \frac{W_n}{W_0}].$$

*So we can write*

$$\begin{aligned} \alpha &= (\frac{dW_1}{W_1} - \frac{dW_0}{W_0}) \wedge \dots \wedge (\frac{dW_{j-1}}{W_{j-1}} - \frac{dW_0}{W_0}) \wedge (-\frac{dW_0}{W_0}) \dots \wedge (\frac{dW_n}{W_n} - \frac{dW_0}{W_0}) \\ &= (-1)^j \frac{dW_0}{W_0} \wedge \dots \widehat{\frac{dW_j}{W_j}} \wedge \dots \frac{dW_n}{W_n}, \end{aligned}$$

*this means that $\alpha$ has a single pole along the hyperplane $Z_0 = 0$ as well. As a consequence*

$$K_{\mathbb{CP}^n} = [(\alpha)] = [-(n+1)\mathcal{O}_{\mathbb{CP}^n}(1)] = \mathcal{O}_{\mathbb{CP}^n}(-n-1).$$

**Definition 1.45** (Divisor)**.** *For any complex manifold $M$, the group of divisors, Div$(M)$ is the additive free abelian group whose generators are the connected, irreducible analytic hypersurface of $M$. A non-zero* *element of $D$ is then a formal sum $D = \sum_{i=1}^{k} m_i D_i$, where each $D_i$ is a connected, irreducible, analytic hypersurfaces of $M$ and the coefficients $m_i$ are integers. A divisor $D = \sum_{i=1}^{k} m_i D_i$ is called effective if $m_i \geq 0$ for all $i$.*

Suppose that $M$ is a compact complex manifold, and $f : M \to \mathbb{C}$ is a meromorphic function. Then one can associate a unique divisor

$$D = \operatorname{div}(f) = \sum_{i=1}^{k} m_i V_i,$$

such that $f$ has zeroes of order $m_i$ on $V_i$ when $m_i > 0$, and poles of order $m_i$ on $V_i$ when $m_i < 0$. i.e, each $x \in X$ has an open neighbourhood $U$ in $X$ such that

$$f(x) = g(x) \prod_{i=1}^{k} (f_i(x))^{m_i},$$

where $f_i : U \to \mathbb{C}$ is a holomorphic function with $U \cap V_i = \{x \in U : f_i(x) = 0\}$ and $f_i$ vanishes to order 1 on the smooth part of $U \cap V_i$ , and $g : U \to \mathbb{C} \setminus \{0\}$ is holomorphic.

**Definition 1.46** (Principal divisor)**.** *A divisor $D$ is called principal if $D = \operatorname{div}(f)$ for some meromorphic function $f$. The subset of principal divisors in Div$(M)$ is a subgroup, since $\operatorname{div}(f) + \operatorname{div}(g) = \operatorname{div}(fg)$ and $-\operatorname{div}(f) = \operatorname{div}(f^{-1})$. Two divisors $D_1$ and $D_2$ are called linearly equivalent, written $D_1 \sim D_2$, if $D_1 - D_2 = \operatorname{div}(f)$ for some meromorphic $f$. The quotient group Div$(M)/\sim$ of equivalence classes of $[D]$ is an abelian group as well.*

**Lemma 1.47.** *Let $M$ be a compact complex manifold having a holomorphic line bundle which admits a meromorphic section $s$. Then the class $[\operatorname{div}(s)]$ in Div$(M)$ is independent of the choice of meromorphic section.*

Conversely, given any divisor $D$ on M, one can construct a holomorphic line bundle $\mathcal{L}$ and a meromorphic section $s$ with $\operatorname{div}(s) = D$, and $(\mathcal{L}, s)$ are unique up to isomorphism. Thus the class $[\mathcal{L}] \in$ Pic$(M)$ depends only on the equivalence class $[D]$ of $D$. If $D$ is a smooth analytic hypersurface $Y$, the corresponding line bundle is denoted by $[Y]$ or $\mathcal{L}_Y$.

If $M$ is a compact complex manifold, there is a natural injective morphism $[D] \in$ Div$(M)/\sim \longrightarrow$ Pic$(M) \ni [\mathcal{L}]$, where $\mathcal{L}$ is a holomorphic line bundle with a meromorphic section $s$ with $\operatorname{div}(s) = D$. If $D$ is effective then

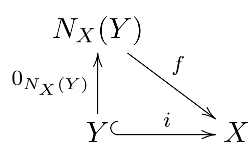
$s$ is holomorphic. The image of this map is the set of $[\mathcal{L}]$ for which $\mathcal{L}$ admits a meromorphic section. One can show that if $X$ is projective then every $\mathcal{L}$ admits meromorphic sections, so the above map is an isomorphism.

**Definition 1.48** (Normal Bundle)**.** *Let $Y$ be a submanifold of a smooth manifold $X$. By considering the canonical embedding $i : Y \to X$, we get a short exact sequence of vector bundles over $Y$:*

$$0 \to TY \to i^*TX \to i^*TX \,/TY \to 0. \quad (1.1)$$

*The quotient bundle $i^*TX\,/TY$ is called the normal bundle of $Y$ in $X$ and denoted by $N_X(Y)$. In fact it represents the directions transversal to the submanifold $Y$ within the ambient manifold $X$.*

**Definition 1.49** (Tubular neighborhood)**.** *Let $Y$ be a submanifold of a smooth manifold $X$. A tubular neighborhood of $Y$ in $X$ is a pair $(\pi : N_X(Y) \to Y, f : N_X(Y) \to X)$ where $\pi : N_X(Y) \to Y$ is vector bundle projection and $f : N_X(Y) \to X$ is a smooth diffeomorphism onto its image called tubular map such that the zero section $0_{N_X(Y)}$ makes the following diagram cummutative,*




**Remark 1.50.** *Let $Y$ be a complex submanifold of a complex manifold $X$, then $N_X(Y) \to Y$ is naturally a holomorpbic vector bundle.*

**Remark 1.51.** *Except from very special cases, it is typically not possible to choose the smooth map $f : N_X(Y) \to X$ to be holomorphic. However, by the tubular neighborhood theorem, a smooth map $f$ satisfying the above conditions always exits.*

Let $Y$ be a complex submanifold of complex codiminsion $k$ of a smooth manifold $X$ of complex dimension $n$. The dual of the short exact sequence 1.1 on page 19 is:

$$0 \to N_X^*(Y) \to (TX\big|_Y)^* \to T^*Y \to 0.$$

So we get

$$\wedge^n((TX\big|_Y)^*) \cong \wedge^k(N_X^*(Y)) \otimes \wedge^{n-k}(T^*Y) \cong N_X^*(Y) \otimes K_Y.$$

On the other hand, $\wedge^n((TX\big|_Y)^*) = K_X\big|_Y$ so this gives the adjunction formula.

**Theorem 1.52** (Adjunction Formula)**.** *Let $Y$ be a submanifold of a complex manifold $X$.*

1. *If $Y$ is a smooth analytic hypersurface of $X$, then*

$$N_X^*(Y) \cong [-Y]\big|_Y,$$

*where $[Y]$ is the corresponding line bundle to divisor $Y$.*

2. *we have*

$$K_X\big|_Y = K_Y \otimes N_X^*(Y) = K_Y \otimes [-Y]\big|_Y.$$

Look at the page 146 of [31] for a proof.

**Example 1.53.** *Let $X = \mathbb{CP}^n$ and $Y$ be a an analytic hypersurface of degree $k$ in $X$, i.e, $Y = s^{-1}(0)$ for $s \in H^0(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k))$. The corresponding line bundle to $Y$ is $\mathcal{L}_Y = \mathcal{O}_{\mathbb{CP}^n}(k)$ and the canonical line bundle of $X$ is $K_X = \mathcal{O}_{\mathbb{CP}^n}(-n-1)$. By the adjunction formula we get*

$$K_Y \cong (K_X \otimes \mathcal{L}_Y)\big|_Y \cong (\mathcal{O}_{\mathbb{CP}^n}(-n-1) \otimes \mathcal{O}_{\mathbb{CP}^n}(k))\big|_Y \cong \mathcal{O}_{\mathbb{CP}^n}(k-n-1)\big|_Y.$$

*There are three possible cases*

1. *If $k = n+1$, then $K_Y \cong \mathcal{O}_{\mathbb{CP}^n}(0)\big|_Y$ is the trivial line bundle and so $Y$ is Calabi–Yau.*

   *As a consequence:*

   - *K3 surfaces[2] (a smooth quartic in $\mathbb{CP}^3$) are a Calabi–Yau.*

2. *If $k < n+1$, then $K_Y$ is a negative line bundle ($Y$ is a Fano manifold).*

3. *If $k > n+1$, then $K_Y$ is a positive line bundle ($Y$ is of general type).*

## 1.4 Hermitian and Kähler Metrics

**Definition 1.54** (Hermitian metric on a complex vector bundle)**.** *A Hermitian metric on a complex vector bundle $E$ over a smooth manifold $M$ is a smoothly varying positive-definite Hermitian form on each fiber, i.e, a smooth global section $h$ of the vector bundle $(E \otimes \bar{E})^*$ such that for every point $p$ in $X$ and any two element $\zeta$, $\eta$ in the fiber $E_p$*

$$h_p(\eta, \bar{\zeta}) = \overline{h_p(\zeta, \bar{\eta})},$$

*and for all nonzero $\zeta$ in $E_p$,*

$$h_p(\zeta, \bar{\zeta}) > 0.$$

[2] The term ”K3” is in honor of Kummer, Kähler, Kodaira, and the beautiful K2 mountain in Kashmir.

**Definition 1.55** (Hermitian metric on a complex manifold). *A Hermitian manifold is a complex manifold with a Hermitian metric on its holomorphic tangent bundle. On a Hermitian manifold the metric can be written in local holomorphic coordinates* $(z^i)$ *as*

$$h = h_{i\bar{j}} dz^i \otimes d\bar{z}^j,$$

*where* $h_{i\bar{j}}$ *are the components of a positive-definite Hermitian matrix.*

**Definition 1.56.** *Let* $h$ *be a Hermitian metric on complex manifold* $M$*, the real part of* $h$*, i.e,*

$$g := \mathrm{Re}(h) = \frac{1}{2}(h + \bar{h}) = h_{i\bar{j}}(dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i),$$

*defines a Riemannian metric on the underlying smooth manifold which is* $J$*-invariant. The imaginary part of* $h$*, i.e,*

$$\mathrm{Im}(h) = -\frac{\sqrt{-1}}{2}(h - \bar{h}) = -\frac{\sqrt{-1}}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

*defines a symplectic form of type* $(1,1)$*. The associate Hermitian form of* $h$ *is defined by*

$$\omega := -\mathrm{Im}(h) = \frac{\sqrt{-1}}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

**Definition 1.57** (Kähler metric). *Let* $M$ *be a complex manifold endowed with a Hermitian metric* $h$*. The metric* $g$ *is called a Kähler metric if the associated Hermitian form* $\omega$ *is closed, i.e,* $d\omega = 0$*. In this case,* $\omega$ *is called the Kähler form and* $(M, h)$ *a Kähler manifold.*

**Definition 1.58** (Kähler potential). *Let* $(M, g)$ *be a Kähler manifold. A smooth real-valued function* $\rho$ *is called a Kähler potential for the Kähler form* $\omega$*, if* $\omega = \dfrac{\sqrt{-1}}{2}\partial\bar{\partial}\rho$*.*

Here is an important differential-geometric property.

**Proposition 1.59.** *Let* $(M, g)$ *be a Kähler manifold, with Kähler form* $\omega$*, and let* $\nabla$ *be the Levi-Civita connection of* $g$*. Then*

$$\nabla g = \nabla J = \nabla \omega = 0.$$

So, $g, J$, and $\omega$ are constant tensors on $(M, g)$. This implies that the holonomy group $(M, g)$ of $g$ is contained in $U(n)$. Kähler metrics are defined by the condition $d\omega = 0$, which is relatively weak and easy to satisfy, resulting in many closed forms. Because of this, there are numerous Kähler manifolds, and examples are readily found. However, $d\omega = 0$ implies the apparently much stronger conditions $\nabla J = 0$ and $\nabla \omega = 0$.

**Remark 1.60.** *Let $M$ be a complex manifold that admits Kähler form $\omega$. Then by the equation $d\omega = 0$, the Kähler class of $\omega$ is the cohomology class $[\omega]$ of $\omega$ in $H^2_{\mathrm{dR}}(M, \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(M)$. When $M$ is compact, $[\omega] \neq 0$ and*

$$[\omega]^n.[M] = \int_M \omega^n = n! \operatorname{Vol}_\omega(M) > 0,$$

*where $[M] \in H_{2n}(M, \mathbb{Z})$ is the top integral homology class of $M$.*

**Example 1.61** (The Euclidean metric on $\mathbb{C}^n$). *The Euclidean space $\mathbb{C}^n$ with the Kähler metric*

$$g_{\mathrm{Euc}} = dz^i \otimes d\bar{z}^i,$$

*has a Kähler form*

$$\omega_{\mathrm{Euc}} = \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \|z\|^2,$$

*is a manifold with Kähler potential $\rho_{\mathrm{Euc}} = \|z\|^2 = \sum_{i=1}^{n} |z_i|^2$, where $(z^1, \ldots, z^n)$ are the standard coordinates of $\mathbb{C}^n$. The Euclidean metric is invariant under unitary transformations, reflecting the isometries of the complex structure.*

**Example 1.62** (The Fubini-Study metric on $\mathbb{CP}^n$). *Let $U \subset \mathbb{CP}^n$ be an open subset. Consider a lift $U$ to $\mathbb{C}^{n+1} \backslash \{0\}$, i.e, a holomorphic map $Z : U \to \mathbb{C}^{n+1} \backslash \{0\}$ such that $q \circ Z = \mathrm{id}_U$ where $q : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$ is the canonical projection and define the Fubini-Study form $\omega_{\mathrm{FS}}$ by*

$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z\|^2.$$

*Note that this $(1,1)$-form is globally well-defined, since for any other lift $Z'$, $Z' = fZ$ for a non-zero holomorphic function $f$ on $U$, and we obtain $\partial\bar{\partial} \log \|Z'\|^2 = \partial\bar{\partial} \log \|Z\|^2$. The unitary group $U(n+1)$ acts transitively on $\mathbb{CP}^n$, so $\omega_{\mathrm{FS}}$ is invariant under this action. Therefore, it's enough to check that $\omega_{\mathrm{FS}}$ is positive at one point. For $Z_0 \neq 0$, we can write $Z = (1, w_1, \ldots, w_n)$ where $w_i = \frac{Z_i}{Z_0}$, and thus, we get*

$$\begin{aligned}
\omega_{\mathrm{FS}} &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z\|^2 \\
&= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(1 + w_i \bar{w}_i) \\
&= \frac{\sqrt{-1}}{2\pi} \partial\left(\frac{w_i dw_i}{1 + w_i \bar{w}_i}\right) \\
&= \frac{\sqrt{-1}}{2\pi} \left(\frac{dw_i \wedge d\bar{w}_i}{1 + w_i \bar{w}_i} - \frac{(\bar{w}_i dw_i) \wedge (w_j d\bar{w}_j)}{(1 + w_i \bar{w}_i)^2}\right) \\
&= \frac{\sqrt{-1}}{2\pi} \left(\frac{\delta_{i\bar{j}}}{1 + \|w\|^2} - \frac{\bar{w}_i w_j}{(1 + \|w\|^2)^2}\right) dw^i \otimes d\bar{w}^j.
\end{aligned}$$

*At the point* $[1 : 0 : \ldots : 0]$,

$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi} dw^i \otimes d\bar{w}^i > 0.$$

*Thus* $\omega_{\mathrm{FS}}$ *defines a Hermitian metric on the complex projective space* $\mathbb{CP}^n$ *which is called the Fubini-Study metric. By construction the Fubini-Study metric is a Kähler metric. In the coordinates above,*

$$g_{\mathrm{FS}} = (\frac{\delta_{i\bar{j}}}{1 + \|w\|^2} - \frac{\bar{w}_i w_j}{(1 + \|w\|^2)^2}) dw^i \otimes d\bar{w}^j,$$

*with Kähler potential* $\rho_{\mathrm{FS}} = -\log(1 + \|w\|^2)$. *The Fubini-Study metric is invariant under the unitary group action, making it a symmetric space.*

**Remark 1.63.** *By direct calculation one can check that the canonical Riemannian volume form of the Kähler manifold* $(M, \omega_M)$ *of dimension* $n$ *in local holomorphic coordinates is given by*

$$\mathrm{Vol}(\omega_M) = \int_M \frac{\omega_M^n}{n!} = \int_M (\frac{\sqrt{-1}}{2})^n \det(g_{\alpha\bar{\beta}}) dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n.$$

*If* $S \subset M$ *is a complex submanifold of dimension* $d$, *then, by the consequence of Wirtinger's inequality we get*

$$\mathrm{Vol}(\omega_S) = \int_S \frac{\omega_M^d}{d!}.$$

**Remark 1.64.** *Not all complex manifolds are Kähler. For example, let* $0 < \lambda < 1$ *and* $\Gamma_\lambda$ *be the infinite cyclic group generated by* $\lambda \, \mathrm{Id}_n$. *Consider the action of* $\Gamma_\lambda \cong \mathbb{Z}$ *on* $\mathbb{C}^n \setminus \{0\}$ *given by*

$$m.(z_1, \ldots, z_n) = (\lambda^m z_1, \ldots, \lambda^m z_n), \forall m \in \mathbb{Z}.$$

*The quotient manifold*

$$\mathrm{CH}_\lambda^n = \mathbb{C}^n \setminus \{0\} / \Gamma_\lambda,$$

*is a compact complex manifold that does not admit any Kähler metric when* $n \geq 2$, *since* $b_1(\mathrm{CH}_\lambda^n) = 1$ *for* $n \geq 2$. *As we will see in Corollary 1.85, for a compact Kähler manifold* $M$, *the first Betti number* $b_1(M)$ *must be an even number. This manifold is called the complex Hopf manifold. As a differentiable manifold, any* $\mathrm{CH}_\lambda^n$ *is diffeomorphic to* $\mathbb{S}^{2n-1} \times \mathbb{S}^1$.

**Lemma 1.65.** *Every complex projective manifold admits a Kähler metric.*

*Proof.* Given a complex projective manifold $X$, there exists an inclusion map $i : X \hookrightarrow \mathbb{CP}^N$ for some $N \in \mathbb{N}$. The pullback metric $i^*\omega_{\mathsf{FS}}$, where $\omega_{\mathsf{FS}}$ is the Fubini-Study metric on $\mathbb{CP}^N$, is a Kähler metric on $X$. This is because the pullback preserves the closedness of the symplectic form, i.e.,

$$d(i^*\omega_{\mathsf{FS}}) = i^*(d\omega_{\mathsf{FS}}) = i^*(0) = 0.$$

□

**Definition 1.66** (Kähler cone). *Let $M$ be a complex manifold. The Kähler cone, denoted by $\mathcal{K}_M$, is the set of all de Rham classes $[\omega]$, where $\omega$ is the Kähler form of a Kähler metric $g$ on $M$. One can check that $\mathcal{K}_M$ is a convex cone and it is open in $H^2_{\mathrm{dR}}(M, \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(M)$.*

The following proposition could be obtained by the Kodaira Embedding Theorem, which we will not discuss in this thesis.

**Proposition 1.67.** *Let $M$ be a compact complex manifold admitting Kähler metrics. Then X is projective if and only if*

$$H^2(M, \mathbb{Q}) \cap \mathcal{K}_M \neq \emptyset.$$

See Corollary 5.3.3 in [34] for a proof.

**Corollary 1.68.** *Let $M$ be a compact complex manifold admitting Kähler metrics. If $H^{2,0}_{\bar{\partial}}(M) = 0$, then $M$ is projective.*

This Corollary shows that, under some conditions, the inverse of Lemma 1.65 is true, i.e, compact Kähler manifolds are projective, and can be studied using complex algebraic geometry.

On a compact Kähler manifold it is never possible to describe a Kähler form globally using a single Kähler potential, but it is possible to describe the difference of two Kähler forms this way, provided they are in the same de Rham cohomology class. This is a consequence of the golbal $\partial\bar{\partial}$-lemma.

**Theorem 1.69** (Global $\partial\bar{\partial}$-lemma). *Let $M$ be a compact Kähler manifold. If $\alpha \in \mathcal{A}^{p,q}(M)$ is $d$-closed, i.e $[\alpha] \in H^{p+q}_{\mathrm{dR}}(M)$, then $\alpha$ is $\partial\bar{\partial}$-exact, i.e, $\alpha = \partial\bar{\partial}\beta$ for some $\beta \in \mathcal{A}^{p-1,q-1}(M)$.*

See page 149 of [31] for a proof.

**Remark 1.70.** *There are examples of compact complex manifolds for which the global $\partial\bar{\partial}$-lemma does not hold.*

**Definition 1.71** (Isometry group). *The isometry group $\mathrm{Iso}(M, g, J)$ of a Kähler manifold $M$ equipped with its Kähler metric $g$ consists of all biholomorphic isometries, which are diffeomorphisms $\phi : M \to M$ that* *preserve both the complex structure and the Kähler metric i.e, for all tangent vector fields $X, Y$ on $M$,*

$$g(\phi_*(X), \phi_*(Y)) = g(X, Y),$$

*and for any tangent vector $X$ on $M$,*

$$\phi_* \ (JX) = J(\phi_* \ X).$$

**Example 1.72.** *The isometry group of the complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric is the projective unitary group $\mathbb{P}\mathrm{U}(n+1) = U(n+1)\,/U(1)$.*

**Definition 1.73** (Curvature of a metric)**.** *Consider a Kähler manifold $X$ equipped with its Hermitian metric $g = \ g_{i\bar{j}} \ \ dz^i \otimes dz^{\bar{j}}$ and let $\omega$ be the corresponding Kähler form. The Riemann curvature, Ricci curvature, and scalar curvature of $g$ can be described as follows in local coordinates:*

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= -\partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{p\bar{q}} (\partial_k g_{i\bar{q}})(\partial_{\bar{l}} g_{p\bar{j}}) \\ R_{i\bar{j}} &= g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} \log \omega^n = -\partial_i \partial_{\bar{j}} \log \det(g_{p\bar{q}}) \\ S(\omega) &= g^{i\bar{j}} R_{i\bar{j}} = -g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log \det(g_{p\bar{q}}) = -\Delta_g \log \det(g_{p\bar{q}}). \end{aligned}$$

*The Ricci form is defined by $\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} \ \ dz^i \wedge d\bar{z}^j$.*

**Remark 1.74.** *The Ricci form $\mathrm{Ric}(\omega)$ is a closed 2-form. Moreover if $\widetilde{\omega}$ is another Kähler form (not necessarily in the same Kähler class), then*

$$\mathrm{Ric}(\omega) - \mathrm{Ric}(\widetilde{\omega}) = \sqrt{-1} \partial \bar{\partial} \log \frac{\det(\widetilde{g}_{p\bar{q}})}{\det(g_{p\bar{q}})},$$

*and then $\dfrac{\det(\widetilde{g}_{p\bar{q}})}{\det(g_{p\bar{q}})} = \dfrac{\widetilde{\omega}^n}{\omega^n}$ and so $[\mathrm{Ric}(\omega)] = [\mathrm{Ric}(\widetilde{\omega})]$.*

**Remark 1.75.** *Note that in the Kähler case, the Ricci form $\mathrm{Ric}(\omega)$ satisfies $J^* \,\mathrm{Ric}(\omega) = \mathrm{Ric}(\omega)$.*

**Definition 1.76** (Kähler–Einstein metric)**.** *A Kähler–Einstein metric $\omega$ on a complex manifold $M$ is a Riemannian metric that is both a Kähler metric, i.e, $d\omega = 0$ and an Einstein metric, i.e, $\mathrm{Ric}(\omega) = \lambda\omega$ for some constant $\lambda$. A manifold is said to be Kähler–Einstein if it admits a Kähler–Einstein metric. The most important special case of these are the Calabi–Yau manifolds, which are Kähler and Ricci-flat. The existence of a Kähler–Einstein metric will be discussed in Chapter 3.*

**Example 1.77.** *The Ricci form of the Euclidian metric $(g_{\mathrm{Euc}})_{i\bar{j}} = \delta_{i\bar{j}}$ in example 1.61 on page 22 is given by*

$$\mathrm{Ric}(\omega_{\mathrm{Euc}}) = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{\mathrm{Euc}}) = -\sqrt{-1} \partial \bar{\partial} \log 1 = 0,$$

*so the Euclidean space $\mathbb{C}^n$ with the standard metric is Ricci flat and Kähler–Einstein as well.*

**Example 1.78.** *The Ricci form of the Fubini-Study metric* $(g_{\mathrm{FS}})_{i\bar{j}} = \dfrac{\delta_{i\bar{j}}}{1+\|z\|^2} - \dfrac{\bar{z}_i z_j}{(1+\|z\|^2)^2}$ *in Example 1.62 is given by*

$$\begin{aligned}\mathrm{Ric}(\omega_{\mathrm{FS}}) &= -\sqrt{-1}\partial\bar{\partial}\log\det((g_{\mathrm{FS}})_{i\bar{j}}) \\ &= -\sqrt{-1}\partial\bar{\partial}\log\frac{1}{(1+\|z\|^2)^{n+1}} \\ &= 2(n+1)(\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(1+\|z\|^2)) = 2(n+1)\omega_{\mathrm{FS}},\end{aligned}$$

*so the complex projective space* $\mathbb{CP}^n$ *with the Fubini-Study metric is Kähler–Einstein.*

Kähler manifolds have some nice topological properties. We will finish this section by introducing Hodge theory.

**Definition 1.79** (Complex Hodge star operator)**.** *Let* $(M, g)$ *be a Kähler manifold of complex dimension* $n$*. The complex Hodge star operator* $\star : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{n-p,n-q}(M)$ *is defined by*

$$\star(\alpha + \sqrt{-1}\beta) = \star_{\mathbb{R}}\alpha - \sqrt{-1}\star_{\mathbb{R}}\beta,$$

*where* $\star_{\mathbb{R}} : \mathcal{A}^k(M,\mathbb{R}) \to \mathcal{A}^{2n-k}(M,\mathbb{R})$ *is the real Hodge star operator corresponding to the Riemannian metric* $g$*, i.e, the Hodge dual of a* $k$*-form* $\beta$*, denoted as* $\star_{\mathbb{R}}\beta$*, as the unique* $(n-k)$*-form satisfying*

$$\alpha \wedge \star_{\mathbb{R}}\beta = g(\alpha,\beta)\,\mathrm{Vol}_g,$$

*for every* $k$*-form* $\alpha$*, where* $g(\alpha,\beta)$ *is a real-valued function on* $M$*, and the volume form* $\mathrm{Vol}_g$ *is induced by the Riemannian metric.*

**Definition 1.80** (The Hodge Laplacian)**.** *Let* $(M, g)$ *be a Hermitian manifold of complex dimension* $n$*. The Hodge Laplacian, also known as the Laplace–de Rham operator, is the second order linear differential operator* $\Delta : \mathcal{A}^k(M) \to \mathcal{A}^k(M)$ *defined by*

$$\Delta = (d + d^\star)^2 = dd^\star + d^\star d,$$

*where* $d^\star = -\star d \star : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ *is the codifferential operator with respect to the metric* $g$*. A* $k-$*form* $\omega$ *is called harmonic, if* $\Delta\omega = 0$*. The space of harmonic* $k-$*forms on* $M$ *with respect to the* $g$ *is denoted by* $\mathcal{H}^k(M)$*. Similarly one can define* $\Delta_\partial = \partial\partial^\star + \partial^\star\partial$ *and* $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^\star + \overline{\partial}^\star\overline{\partial}$ *where* $\partial^\star = -\star\partial\star$ *and* $\overline{\partial}^\star = -\star\overline{\partial}\star$*. The space of* $\partial$*-harmonic* $(p,q)-$*forms on* $M$ *with respect to the* $g$ *is denoted by* $\mathcal{H}^{p,q}_\partial(M)$ *and the space of* $\overline{\partial}$*-harmonic* $(p,q)-$*forms is denoted by* $\mathcal{H}^{p,q}_{\overline{\partial}}(M)$*.*

**Theorem 1.81.** *Let $(M, g)$ be a compact Hermitian manifold. Then there exist two orthogonal decompositions*

$$\mathcal{A}^{p,q}(M) = \mathrm{Im}(\partial_{p,q-1}) \oplus \mathcal{H}^{p,q}_{\partial}(M) \oplus \mathrm{Im}(\partial^{\star}_{p,q+1})$$
$$\mathcal{A}^{p,q}(M) = \mathrm{Im}(\overline{\partial}_{p,q-1}) \oplus \mathcal{H}^{p,q}_{\overline{\partial}}(M) \oplus \mathrm{Im}(\overline{\partial}^{\star}_{p,q+1}).$$

*Furthermore, $\mathcal{H}^{p,q}_{\partial}(M)$ and $\mathcal{H}^{p,q}_{\overline{\partial}}(M)$ are finite dimensional for all $p$ and $q$.*

See Theorem 3.2.8 in [34] for a proof.

**Theorem 1.82** (Hodge Decomposition)**.** *Let $M$ be a compact Kähler manifold. Then we have the direct sum decomposition*

$$H^k_{\mathrm{dR}}(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M, \mathbb{C}).$$

*and the Betti numbers and Hodge numbers of $M$ are related by*

$$b_k(M) = h^{k,0}(M) + h^{k-1,1}(M) + \ldots + h^{1,k-1}(M) + h^{0,k}(M).$$

See Corollary 3.2.12 in [34] for a proof.

**Proposition 1.83.** *For a Kähler manifold $(M, g)$, $\Delta_{\partial} = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta$ and so $\mathcal{H}^{p,q}_{\partial}(M) = \mathcal{H}^{p,q}_{\overline{\partial}}(M)$.*

**Theorem 1.84** (Hodge)**.** *Let $M$ be a compact Kähler manifold. Then we have*

$$H^{p,q}_{\overline{\partial}}(M) \cong \mathcal{H}^{p,q}_{\overline{\partial}}(M).$$

*Furthermore, $H^{p,q}_{\overline{\partial}}(M) = \overline{H^{q,p}_{\overline{\partial}}(M)}$, thus $h^{p,q}(M) = h^{q,p}(M)$.*

**Corollary 1.85.** *Let $M$ be a compact Kähler manifold. Then the even Betti numbers $b_{2k}(M)$ are nonzero, and the odd Betti numbers $b_{2k+1}(M)$ are even for $k = 0, 1, \ldots$.*

Look at the page 117 of [31] for a proof.

Let $M$ be a compact Kähler Ricci-flat manifold. Then $b_1(M) = 0$. For a compact Kähler manifold with negative Ricci curvature, the first Betti number can vary, and the fundamental group can be large. However, for a compact Kähler manifold with positive Ricci curvature, the following obstructions hold.

**Theorem 1.86** (Bochner, Bonnet–Myers)**.** *Let $M$ be a compact Kähler manifold with positive Ricci curvature, then $b_1(M) = 0$ and $\pi_1(M)$ is finite.*

See Theorem 4.5.3 and Corollary 6.3.1 in [35] or Theorem 2.4.2 in [28] for a proof.

## 1.5 Connections and Curvature

In the Definition 1.20 on page 11 the space of complex valued $k$-forms are defined by

$$\mathcal{A}^k(M, \mathbb{C}) = \Gamma(\wedge^k(T^{\mathbb{C}}M)^*),$$

i.e, complex valued $k$-forms are locally $k$-linear alternating function of this form:

$$\omega_p : (T_p^{\mathbb{C}}M)^{*k} \to \mathbb{C}.$$

For a vector bundle $E \to M$, the idea of $E$-valued $k$-form, is a generalization of the above definition to a $k$-linear alternating function of the form:

$$\omega_p : (T_p^{\mathbb{C}}M)^{*k} \to E_p,$$

where $E_p$ is the fiber over $p \in M$. Let us to defined $E$-valued $k$-forms.

**Definition 1.87** (Forms with values in a vector bundle)**.** *If $E$ is a vector bundle on a complex manifold $M$, the space of smooth $E$-valued $k$-forms on $M$ is defined by*

$$\mathcal{A}^k(M, E) := \Gamma((\wedge^k T^*M) \otimes E).$$

*In particular, $E$-valued $0$-forms are just smooth section, i.e, $\mathcal{A}^0(M, E) := \Gamma(E)$.*

**Definition 1.88** (Linear connection)**.** *A linear connection on a holomorphic vector bundle $E$ is a first-order linear differential operator $\nabla : \mathcal{A}^0(M, E) \to \mathcal{A}^1(M, E)$, which satisfies the following Leibniz identity:*

$$\nabla(fs) = df \otimes s + f\nabla s,$$

*for any smooth section $s$ of $E$ and any smooth function $f$. In fact an arbitrary connection could be written as $\nabla = \nabla^0 + \Theta$ where $\nabla^0$ is a fixed connection and $\Theta$ is an $E$-valued 1-forms.*

Any vector field $X \in \mathfrak{X}(M)$ could be considered as a contraction $X : \mathcal{A}^1(M, E) \to \Gamma(E)$ and the covariant derivative with respect to the vector field $X$, $\nabla_X : \Gamma(E) \to \Gamma(E)$ is then defined by the composition of

$\nabla : \Gamma(E) \to \mathcal{A}^1(M,E)$ with the contraction $X : \mathcal{A}^1(M,E) \to \Gamma(E)$. Using $(T^{\mathbb{C}}M)^* = T^*M \otimes \mathbb{C} = (T^{1,0}M)^* \oplus (T^{0,1}M)^*$ a linear connection $\nabla$ could be decomposes as $\nabla = \nabla^{1,0} + \nabla^{0,1}$ where $\nabla^{1,0} : \mathcal{A}^0(M,E) \to \mathcal{A}^{1,0}(M,E)$ and $\nabla^{0,1} : \mathcal{A}^0(M,E) \to \mathcal{A}^{0,1}(M,E)$. The $(1,0)$ and the $(0,1)$-parts are defined by

$$\nabla_X^{1,0 s} = \frac{1}{2}(\nabla_X^s - \sqrt{-1}\nabla_{JX}^s), \nabla_X^{0,1 s} = \frac{1}{2}(\nabla_X^s + \sqrt{-1}\nabla_{JX}^s).$$

**Definition 1.89** (Cauchy-Riemann (Dolbeault) operators)**.** *Let $E$ be a vector bundle on a complex manifold $M$. A Cauchy-Riemann (Dolbeault) operator $\bar{\partial}_E$ on $E$ is defined as a first order $\mathbb{C}$-linear differential operator $\bar{\partial}_E : \Gamma(E) \to \mathcal{A}^{0,1}(M,E)$ satisfying the following Leibniz-like identity*

$$\bar{\partial}_E(fs) = \bar{\partial} f \otimes s + f\bar{\partial}_E s,$$

*where $\bar{\partial}$ is the usual Cauchy-Riemann operator acting on smooth functions.*

Any Cauchy-Riemann operator $\bar{\partial}_E : \Gamma(E) \to \mathcal{A}^{0,1}(M,E)$ could be extended to an operator

$$\bar{\partial}_E^{p,q} : \mathcal{A}^{p,q}(M,E) \to \mathcal{A}^{p,q+1}(M,E),$$

using the Leibniz rule $\bar{\partial}_E(\omega \otimes s) = \bar{\partial}\omega \otimes s$ for $\omega \in \mathcal{A}^{p,q}(M,E)$ and a local holomorphic section $s$. A Cauchy-Riemann operator $\bar{\partial}_E$ is said to be integrable if $\bar{\partial}_E \circ \bar{\partial}_E = 0$.

**Example 1.90.** *Let $E$ be a horlomorphic vector bundle on a complex manifold $M$, then the usual $\bar{\partial}$ is an example of Cauchy-Riemann operator acting on complex valued $k$-forms. In fact on any holomorphic vector bundle $E$, there exist a unique integrable canonical Cauchy-Riemann operator $\bar{\partial}_E : \Gamma(E) \to \mathcal{A}^{0,1}(M,E)$, such that a smooth section $s \in \Gamma(E)$ is holomorphic, if and only if $\bar{\partial}_E s = 0$.*

Here, we define vector bundle cohomology not in the usual way through sheaf cohomology but rather from a differential geometric point of view.

**Definition 1.91** (Vector bundle cohomology)**.** *By using $\bar{\partial}_E^{p,q+1} \circ \bar{\partial}_E^{p,q} = 0$ we can define a generalized Dolbeault cohomology for a holomorphic vector bundle by*

$$H^{p,q}(M,(E,\bar{\partial}^E)) = \frac{\ker(\bar{\partial}_E^{p,q+1} : \mathcal{A}^{p,q}(M,E) \to \mathcal{A}^{p,q+1}(M,E))}{\mathrm{Im}(\bar{\partial}_E^{p,q} : \mathcal{A}^{p,q-1}(M,E) \to \mathcal{A}^{p,q}(M,E))}.$$

Similar to Proposition 1.33 on page 14, we can express this cohomology as a sheaf cohomology.

**Proposition 1.92.** *There is a natural isomorphism $H^{p,q}(M,E) \cong H^q(M, \Omega^p(M,E))$. Here $H^q(M, \Omega^p(M,E))$ is the sheaf cohomology of the complex manifold $M$ with coefficient in the sheaf of $E$-valued holomorphic $p$-forms.*

Like the Levi-Civita connection in Riemannian geometry, there is a natural choice of connection on a Hermitian holomorphic vector bundle.

**Proposition 1.93** (Chern connection). *Let $(E,h)$ be a holomorphic hermitian vector bundle on a complex manifold $M$. Then there is a unique connection $D$ on $E$ compatible with both the metric and the complex structure, i.e, $D^{0,1} = \bar{\partial}_E$ and for any two smooth sections $s$ and $t$ of $E$,*

$$d\langle s,t\rangle_h = \langle Ds,t\rangle_h + \langle s,Dt\rangle_h.$$

*It is called the Chern connection. In fact, we can decompose the Chern connection into $D = D^{1,0} + D^{0,1}$, where the part $D^{0,1}$ depends on the holomorphic structure and is analytic, while the $D^{1,0}$ part depends on the Hermitian metric and is geometric. In particular, if the base manifold is Kähler and the vector bundle is its tangent bundle, then the Chern connection coincides with the Levi-Civita connection of the associated Riemannian metric.*

See Proposition 4.2.14 in [34] for a proof.

Any linear connection $\nabla$ extends into an exterior differentials $d^\nabla$ acting on $E$-valued exterior forms. The exterior differentials $d^\nabla$ acting on $E$-valued exterior forms, corresponds to a linear connection $\nabla$ is the operator $d^\nabla : \mathcal{A}^k(M,E) \to \mathcal{A}^{k+1}(M,E)$, defined by

$$d^\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s,$$

for any $k$-form $\omega$ and any smooth section $s$ of $E$.

**Example 1.94.** *Let $M$ be a smooth manifold, $E = M \times \mathbb{R}$ be the trivial bundle, and $\nabla^0$ be the trivial connection. Take $k = 0$ and $s = f \in C^\infty(M)$. Then, $d^{\nabla_0} f = df$ and $d^{\nabla_0} \circ d^{\nabla_0} = 0$.*

Note that in general, one needs not have $d^\nabla \circ d^\nabla = 0$. In fact, this happens if and only if the connection $\nabla$ is flat. Basically $d^\nabla \circ d^\nabla$ corresponds to the curvature of the connection $\nabla$.

**Definition 1.95** (Curvature homomorphism). *The curvature $R^\nabla$ of a linear connection $\nabla$ on a complex vector bundle $E$ is defined by the composition*

$$R^\nabla := d^\nabla \circ d^\nabla : \mathcal{A}^0(M,E) \to \mathcal{A}^2(M,E).$$

**Remark 1.96.** *By decomposing the curvature form into $R^\nabla = R^{2,0} + R^{1,1} + R^{0,2}$, we can conclude that for the Chern connection on a holomorphic vector bundle, $R^{0,2} = 0$, and the metric-preserving property implies that $R^{2,0} = \overline{R^{0,2}} = 0$, so $R^\nabla = R^{1,1}$.*

**Proposition 1.97** (Curvature form)**.** *The curvature homomorphism $R^\nabla : \mathcal{A}^0(M,E) \to \mathcal{A}^2(M,E)$ is linear, so can be considered as an element of $\mathcal{A}^2(M, \mathrm{End}(E))$. Moreover $R^\nabla$ satisfies:*

$$R^\nabla_{X,Y} s = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s.$$

**Proposition 1.98.** *Let $(E,h)$ be a Hermitian holomorphic vector bundle. Then the curvature form of the Chern connection is given locally by*

$$R^D_h := -\partial\bar{\partial} \log \|s\|^2_h,$$

*where $s$ is a local non-vanishing holomorphic section of $E$.*

See Proposition 4.3.8 in [34] for a proof.

**Proposition 1.99** (Cartan's Structural Equation)**.** *In local coordinates, if $A$ is the End$(E)$-valued 1-form such that the connection is given by $\nabla = d + \Theta$, then the curvature form $\nabla$ can be expressed as:*

$$R^\nabla = d\Theta + \Theta \wedge \Theta.$$

*In particular for the trivial conncection $\nabla = d$ and so $R^\nabla = 0$, i.e, the trivial connection is flat.*

*Proof.* By definition

$$\begin{aligned} R^\nabla(s) &= (d+\Theta)\circ(d+\Theta)(s) \\ &= d^2(s) + d(\Theta s) + \Theta ds + \Theta(\Theta s) \\ &= 0 + d\Theta(s) - \Theta ds + \Theta ds + (\Theta\wedge\Theta)(s) \\ &= (d\Theta + \Theta\wedge\Theta)(s). \end{aligned}$$

□

## 1.6 Chern Classes

Let $A$ be an $r \times r$ matrix and let $P_k(A)$ be the homogeneous polynomial with $\deg(P_k) = k$ corresponding to the coefficient of $t^k$ in the characteristic polynomial

$$\det(I + tA) = \sum_{k=0}^{r} P_k(A) t^k.$$

**Definition 1.100** (Chern forms). *Let $(E, h)$ be a complex vector bundle on a smooth manifold $M$ and $\nabla$ is a linear connection on $E$. The Chern forms of $E$ with respect to the connection $\nabla$ is defined by*

$$c_k(E, \nabla) = P_k(\frac{\sqrt{-1}}{2\pi} R^{\nabla}) \in \mathcal{A}^{2k}(M, \textit{End}(E)).$$

**Lemma 1.101** (Chern classes). *The cohomology class $[P_k(\frac{\sqrt{-1}}{2\pi} R^{\nabla})]$ in $H^*(M, \mathbb{C})$ is independent of the choice of connection and it is called $k$th Chern class of $E$ and denoted by $c_k(E)_{\mathbb{R}}$.*

See Lemma 18.2 in [44] for a proof.

The above definition is the standard notion of the Chern classes in complex geometry (Chern-Weil theory). In topology, Chern classes can be defined with integer coefficients, i.e., as elements in $c_k(E) \in H^{2k}(M, \mathbb{Z})$. In this setting, Definition 1.100 corresponds to the torsion free part of the classes defined in $H^*(M, \mathbb{Z})$. An immediate consequence of the definition is following properties of Chern classes.

**Proposition 1.102.** *On a smooth manifold $M$:*

1. *Whitney sum formula: for any two complex vector bundles $E_1$ and $E_2$ on $M$,*

   $$c_k(E_1 \oplus E_2)_{\mathbb{R}} = \sum_{i=0}^{k} c_i(E_1)_{\mathbb{R}} c_{k-i}(E_2)_{\mathbb{R}}.$$

   *In particular, $c_1(E_1 \oplus E_2)_{\mathbb{R}} = c_1(E_1)_{\mathbb{R}} + c_1(E_2)_{\mathbb{R}}$.*

2. *For the dual bundle $E^*$ of a complex vector bundles $E$ on $M$,*

   $$c_k(E^*)_{\mathbb{R}} = (-1)^k c_k(E)_{\mathbb{R}}.$$

3. *Let $N$ be a smooth manifold, $f : M \to N$ a smooth map and $E$ a complex vector bundle on $N$, then*

   $$c_k(f^*E)_{\mathbb{R}} = f^* c_k(E)_{\mathbb{R}}.$$

**Definition 1.103.** *For a complex manifold $M$, the $k$th Chern class of $M$ is defined by the $k$th Chern class of its holomorphic tangent bundle $TM$.*

**Remark 1.104.** *For a complex manifold $X$, the first Chern class of $X$ is the same as the first Chern class of the anti-canonical line bundle $K_X^*$, i.e.,*

$$c_1(X)_{\mathbb{R}} = c_1(K_X^*)_{\mathbb{R}} = -c_1(K_X)_{\mathbb{R}}.$$

**Remark 1.105.** *If $(X, g)$ is a Kähler manifold, the first Chern class of $X$ is given by*

$$c_1(X)_{\mathbb{R}} = [P_1(\frac{\sqrt{-1}}{2\pi}R^D)] = [\frac{\sqrt{-1}}{2\pi}\operatorname{tr}(R^D)] = \frac{1}{2\pi}[\operatorname{Ric}(g)] \in H^2_{\mathrm{dR}}(X, \mathbb{R}) \cap H^{1,1}_{\bar\partial}(X).$$

*In particular, the cohomology class $[\operatorname{Ric}(g)]$ is independent of the choice of Kähler metric.*

Complex line bundles over a complex manifold $X$ are classified topologically by their integral first Chern class, which is an element of the cohomology group $H^2(X, \mathbb{Z})$. Each line bundle is topologically but not holomorphically uniquely determined by its integral first Chern class. Here is the alternative definition of the first Chern class of a line bundle.

**Remark 1.106.** *The exponential short exact sequence on a complex manifold $M$,*

$$0 \to \mathbb{Z} \to \mathcal{O}_M \to \mathcal{O}^*_M \to 0,$$

*gives a connecting cohomology map $\operatorname{Pic}(M) \cong H^1(M, \mathcal{O}^*_M) \to H^2(M, \mathbb{Z})$. For a holomorphic line bundle $\mathcal{L} \in \operatorname{Pic}(M)$, the image of $\mathcal{L}$ under above map is $c_1(\mathcal{L})_{\mathbb{R}} \in H^2(M, \mathbb{Z})$.*

The next remark show that for a compact Kähler manifold, the integral of the scalar curvature is a topological invariant depending to the first Chern class of the manifold.

**Lemma 1.107.** *Let $M$ be a compact Kähler manifold of dimension $n$ with Kähler class $\Omega$. For any $\omega \in \Omega$,*

$$\int_M S(\omega)\omega^n = 2n\pi c_1(M)_{\mathbb{R}} \cup [\omega]^{n-1},$$

*so if $S(\omega)$ is constant, then it equals*

$$S(\omega) = \frac{2n\pi c_1(M)_{\mathbb{R}} \cup [\omega]^{n-1}}{[\omega]^n}.$$

*Proof.* By direct calculation

$$\begin{aligned}\int_M S(\omega)\omega^n &= \int_M \operatorname{tr}_\omega(\operatorname{Ric}(\omega))\omega^n \\ &= \int_M n\operatorname{Ric}(\omega)\omega^{n-1} \\ &= n[\operatorname{Ric}(\omega)] \cup [\omega]^{n-1} \\ &= 2n\pi c_1(M)_{\mathbb{R}} \cup [\omega]^{n-1}.\end{aligned}$$

□

**Example 1.108.** *First Chern Class of $\mathbb{C}^n$:*

$$c_1(\mathbb{C}^n)_{\mathbb{R}} = \frac{1}{2\pi}[\mathrm{Ric}(g_{\mathrm{Euc}})] = 0.$$

**Example 1.109.** *First Chern Class of $\mathbb{CP}^n$:*

$$c_1(\mathbb{CP}^n)_{\mathbb{R}} = \frac{1}{2\pi}[\mathrm{Ric}(g_{\mathrm{FS}})] = \frac{1}{2\pi}[2(n+1)\omega_{\mathrm{FS}}] = (n+1)[\omega_{\mathrm{FS}}].$$

*In particular, the complex projective space $\mathbb{CP}^n$ has a nontrivial first Chern class, since*

$$[\omega_{\mathrm{FS}}] = [\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(1+\|z\|^2)],$$

*is the generator of the cohomology group $H^2(\mathbb{CP}^n,\mathbb{Z})$. Also, for any integer $r$ we get*

$$c_1(\mathcal{O}_{\mathbb{CP}^n}(r))_{\mathbb{R}} = [r\cdot\omega_{\mathit{FS}}] = r[\omega_{\mathit{FS}}].$$

Let $X$ be a compact Kähler manifold. The first Chern class $c_1$ gives a map from holomorphic line bundles to $H^2_{\mathrm{dR}}(X)$. By Hodge theory, one can check that the image of $c_1$ lies in $H^{1,1}_{\bar{\partial}}(X)$. Lefschetz theorem says that the map to $H^2(X,\mathbb{Z})\cap H^{1,1}_{\bar{\partial}}(X)$ is surjective.

**Theorem 1.110** (Lefschetz theorem on $(1,1)$-classes)**.** *Let $X$ be a compact Kähler manifold. Then the map $c_1 : \mathrm{Pic}(X)\to H^2(X,\mathbb{Z})\cap H^{1,1}_{\bar{\partial}}(X)$ is surjective.*

See page 163 in [31] for a proof.

**Theorem 1.111** (Lefschetz)**.** *Let $X$ be an arbitrary Kähler complex manifold. Suppose $\omega$ is a smooth, closed real (1,1)-form on $X$ such that $\omega$ represents an integral cohomology class in $H^2_{\mathrm{dR}}(X,\mathbb{R})$. Then there exists a Hermitian line bundle $(\mathcal{L},h)$ over $X$ such that $\frac{\sqrt{-1}}{2\pi}R^D_{\mathcal{L},h} = \omega$, where $R^D_{\mathcal{L},h}$ is the curvature form of the Hermitian metric on $\mathcal{L}$.*

See page 148 in [31] and [24] for a proof.

**Theorem 1.112** (Calabi-Yau)**.** *Let $(X,\omega)$ be a compact Kähler manifold, and let $\alpha$ be a real $(1,1)$-form representing $c_1(X)_{\mathbb{R}}$. Then there exists a unique Kähler metric $\eta$ on $X$ with $[\eta]=[\omega]$ such that $\mathrm{Ric}(\eta) = 2\pi\alpha$.*

In particular if $c_1(X)_{\mathbb{R}} = 0$, then every Kähler class contains a unique Ricci flat metric. Ricci flat metrics are called Calabi-Yau as well.

## 1.7 Orbifolds in Complex Geometry

In this section, we briefly discuss orbifolds. An orbifold (short for 'orbit-manifold') is a generalization of a manifold. Roughly speaking, an orbifold is a topological space that is locally a finite group quotient of a Euclidean space.

The definition of an orbifold has been provided by Ichirô Satake in the context of automorphic forms in the 1950s under the name "V-manifold" [52], and by William Thurston in the context of the geometry of 3-manifolds in the 1970s [58] when he coined the term "orbifold" following a vote by his students. Formally, an orbifold is defined as follows:

**Definition 1.113** (Orbifold)**.** *An orbifold of dimension $n$ is a Hausdorff topological space $X$, called the underlying space, a covering by a collection of open sets $\{U_i\}$ closed under finite intersections and a finite group $\Gamma_i$ associated to each $U_i$ together with the following data:*

1. *Atlas of Charts: For each $U_i$ there is a homeomorphism $\phi_i : U_i \to \widetilde{U}_i/\Gamma_i$, called an orbifold chart, where $\widetilde{U}_i$ is an open subset of $\mathbb{R}^n$, invariant under a faithful action of a finite group $\Gamma_i$.*
2. *Gluing maps: Whenever $U_i \subseteq U_j$, there is an injective homomorphism $f_{ij} : \Gamma_i \to \Gamma_j$ and a smooth $\Gamma_i$-equivariant[3] gluing map $\widetilde{\phi}_{ij} : \widetilde{U}_i \to \widetilde{U}_j$ such that the gluing maps are compatible with the charts, i.e. $\phi_j \circ \widetilde{\phi}_{ij} = \phi_i$ and the gluing maps are unique up to composition with group elements, i.e. any other possible gluing map from $\widetilde{U}_i$ to $\widetilde{U}_j$ has the form $\gamma \cdot \widetilde{\phi}_{ij}$ for a unique $\gamma$ in $\Gamma_j$. In fact the diagram below commutes.*

$$\begin{array}{ccc} \widetilde{U}_i & \xrightarrow{\widetilde{\phi}_{ij}} & \widetilde{U}_j \\ \downarrow & & \downarrow \\ \widetilde{U}_i/\Gamma_i & \longrightarrow & \widetilde{U}_j/\Gamma_j \\ \downarrow & & \downarrow \\ U_i & \subset & U_j \end{array}$$

**Remark 1.114.** *We regard $\widetilde{\phi}_{ij}$ as being defined only up to composition with elements of $\Gamma_j$, and $f_{ij}$ as being defined up to conjugation by elements of $\Gamma_j$. It is not generally true that $\widetilde{\phi}_{ik} = \widetilde{\phi}_{jk} \circ \widetilde{\phi}_{ij}$ when $U_i \subset U_j \subset U_k$, but there should exist an element $\gamma \in \Gamma_k$ such that $\gamma \widetilde{\phi}_{ik} = \widetilde{\phi}_{jk} \circ \widetilde{\phi}_{ij}$ and $\gamma \cdot f_{ik}(g) \cdot \gamma^{-1} = f_{jk} \circ f_{ij}(g)$.*

**Example 1.115.** *A closed manifold is orbifold, where each group $\Gamma_i$ is the trivial group, so that $\widetilde{U}_i = U_i$.*

[3] i.e., for $\gamma \in \Gamma_i$, $\widetilde{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\widetilde{\phi}_{ij}(x)$ for all $x \in \widetilde{U}_i$.

**Example 1.116.** *A manifold $M$ with a boundary can be given an orbifold structure. In which its boundary becomes a 'mirror'. Any point on the boundary has a neighborhood modelled on $\mathbb{R}^n / \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflection in a hyperplane.*

When $n = 2m$, we say the orbifold is complex of dimension $m$ if each $\widetilde{U}_i$ is seen as a subset of $\mathbb{C}^m$, the gluing maps $\widetilde{\phi}_{ij}$ are holomorphic and the group $\Gamma_i$ acts on $\widetilde{U}_i$ by biholomorphisms.

**Proposition 1.117** (Quotient Orbifolds). *If $M$ is a manifold and $\Gamma$ is a group acting properly discontinuously on $M$, then $M / \Gamma$ has the structure of an orbifold.*

See Proposition 13.2.1. [58] for a proof.

Note that each point $p$ in an orbifold $X$ is associated with a group $\Gamma_p$, well-defined up to isomorphism. In a local coordinate system $U = \widetilde{U}/\Gamma$, $\Gamma_p$ is the isotropy group of any point in $\widetilde{U}$ corresponding to $p$. Alternatively, $\Gamma_p$ may be defined as the smallest group corresponding to some coordinate system containing $p$. In other words, it is the group of transformations that preserve the local geometry of the orbifold at that particular point. This group characterizes the singular behavior around the singularity point and determines the type of singularity at that point.

**Remark 1.118.** *Given an isomorphism class $\mathbf{I}$, we can consider the subset $\Sigma_{\mathbf{I}} = \{p \in X : [\Gamma_p] = \mathbf{I}\}$. This induces a stratification of $X$, $X = \bigsqcup_{\alpha} S_\alpha$, where $S_\alpha$ is a connected component of some $\Sigma_\alpha$. The regular stratum $X_{reg} = \Sigma_{\{\mathrm{Id}\}}$ contains all the nonsingular points. The set $X_{sing} = X \setminus X_{reg} = \{p \in X : [\Gamma_p] \neq \mathrm{Id}\}$ is the singular locus of $X$, and $p \in X_{sing}$ is called an "orbifold point". Clearly, $X$ is a manifold if and only if $X_{sing} = \emptyset$.*

**Example 1.119** (Kummer surface). *Consider the following action of $\mathbb{Z}_2$ on $\mathbb{T}^4$:*

$$(e^{it_1}, e^{it_2}, e^{it_3}, e^{it_4}) \mapsto -(e^{it_1}, e^{it_2}, e^{it_3}, e^{it_4}),$$

*The quotient $\mathbb{T}^4/\mathbb{Z}_2$ is an example of a compact orbifold with sixteen isolated singular points. This orbifold is known as the Kummer surface, and it is a singular $K3$-surface. Notice that the flat metric on $\mathbb{T}^4$ descends to a flat metric on $\mathbb{T}^4/\mathbb{Z}_2$.*

**Example 1.120.** *The hyperbolic plane $\mathbb{H}^2$ being acted upon by the projective group $\mathbb{PSL}(2, \mathbb{Z})$, which consists of $2 \times 2$ matrices with integer entries and determinant 1 taking into account the projective equivalence of*

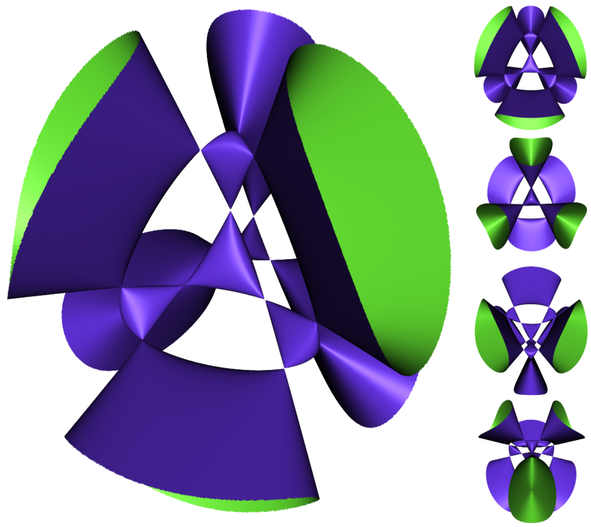


Figure 1.1: Kummer surface (Photo Credit: Claudio Rocchini)

*matrices. This group acts on the upper half-plane in a discontinuous manner via fractional linear transformations. The quotient space* $\mathbb{H}^2/\mathbb{P}\mathrm{SL}(2,\mathbb{Z})$ *gives rise to the modular orbifold, which captures the geometry of the modular group's action on the hyperbolic plane.*

**Definition 1.121.** *An orbifold $X$ is of depth one if, for each connected component $\Sigma$ of $X_{sing}$, the isotropy groups of the points of $\Sigma$ are all isomorphic.*

**Remark 1.122.** *Since orbifolds have only quotient singularities, any smooth object on a manifold has a natural generalization to orbifolds. For instance, a function $f : X \to \mathbb{R}$ on an orbifold is smooth if, in any orbifold chart $(U, \phi, \widetilde{U}, \Gamma)$, its lift to $\widetilde{U}$ is smooth and $\Gamma$-invariant. Similarly, if $X$ is a complex orbifold, a function $f : X \to \mathbb{C}$ is holomorphic if, for any orbifold chart $(U, \phi, \widetilde{U}, \Gamma)$, its lift to $\widetilde{U}$ is holomorphic and $\Gamma$-invariant. We can similarly define smooth forms and tensor fields on an orbifold. This leads to a notion of a Riemannian metric on an orbifold, namely, it is a smooth symmetric 2-tensor that, for each orbifold chart $(U, \phi, \widetilde{U}, \Gamma)$, lifts to a $\Gamma$-invariant Riemannian metric on $\widetilde{U}$. Correspondingly, on a complex orbifold, there is a notion of a Kähler metric.*

**Definition 1.123** (Good Riemannian orbifold)**.** *A good Riemannian orbifold is a triple $(M, g, \Gamma)$ where $(M, g)$ is a Riemannian manifold and $\Gamma$ is a (proper) discontinuous group of isometries $\mathrm{Iso}(M, g)$ acting effectively on $M$. The underlying space of the orbifold is $M/\Gamma$. A bad Riemannian orbifold is a Riemannian orbifold that does not arise as a global quotient.*

**Definition 1.124** (Resolution)**.** *Let $X$ be a singular complex orbifold. A resolution of $X$ is a pair $(\widehat{X}, \pi)$ such* *that $\widehat{X}$ is a complex manifold, the map $\pi : \widehat{X} \to X$ is surjective and*

$$\pi\big|_{\widehat{X} \setminus \pi^{-1}(X_{\text{sing}})} : \widehat{X} \setminus \pi^{-1}(X_{\text{sing}}) \to X \setminus X_{\text{sing}},$$

*is a biholomorphism. The preimage $E := \pi^{-1}(X_{\text{sing}})$ is called the exceptional set of the resolution. Thus, in a resolution we resolve the singularities by replacing each singular point by a submanifold, or more general subvariety. One way to construct resolutions is to use a technique called blowing-up, which we will define in the next section.*

**Example 1.125.** *Consider the complex orbifold $\mathbb{C}^2/\mathbb{Z}_2$ where the group $\mathbb{Z}_2$ acts by reflecting across the coordinate axes. The quotient space $\mathbb{C}^2/\mathbb{Z}_2$ has a singular point at the origin. The line bundle $\mathcal{O}_{\mathbb{CP}^1}(-2)$ is a resolution of $\mathbb{C}^2/\mathbb{Z}_2$, with the exceptional set $E = \pi^{-1}(0) = \mathbb{CP}^1$.*

## 1.8 Blow-up in Complex Geometry

Blowing-up is a process in algebraic and differential geometry that can be used to obtain newer manifolds from known ones and resolve singularities. The blow-up replaces the singular point with a smooth manifold, making it easier to study. Blow-ups play a crucial role in intersection theory, which is a fundamental concept in algebraic and complex geometry. In some cases, blow-ups provide a geometric construction that simplifies the analysis of a complex manifold. They introduce exceptional divisors that can carry valuable geometric and topological information. Before formally defining the blow-up of a complex manifold $X$ at a point $p \in X$, let us consider the blow-up of $\mathbb{C}^n$ at the origin.

**Example 1.126** (Blowing-up origin in $\mathbb{C}^n$). *Consider the $n$-dimensional complex Euclidean space $\mathbb{C}^n$, equipped with coordinates $(z_1, z_2, \dots, z_n)$. The blow-up of $\mathbb{C}^n$ at the origin, denoted as $\mathrm{Bl}_0^{\mathbb{C}^n}$, is constructed by replacing the origin with complex projective space $\mathbb{P}(\mathbb{C}^n) = \mathbb{CP}^{n-1}$. In the Cartesian product $\mathbb{C}^n \times \mathbb{CP}^{n-1}$, consider the tautological subset $\widetilde{\mathbb{CP}^{(n-1)}}$ consisting of pairs $(z, l)$ such that $z \in l$. This tautological subset is infact the total space of the tautological line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ because if $Z = [Z_1; \dots; Z_n]$ are the homogeneous coordinates on $\mathbb{CP}^{n-1}$ we can write*

$$\widetilde{\mathbb{CP}^{(n-1)}} = \{(z, Z) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} : z_i Z_j = z_j Z_i\} = \mathcal{O}_{\mathbb{CP}^{n-1}}(-1).$$

*So we get*

$$\begin{array}{ccccc} \mathrm{Bl}_0^{\mathbb{C}^n} = \mathcal{O}_{\mathbb{CP}^{n-1}}(-1) & \hookrightarrow & \mathbb{CP}^{n-1} \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & \searrow & \downarrow & & \\ & & \mathbb{CP}^{n-1} & & \end{array} \tag{1.2}$$

*Projection on first component,* $\mathrm{Pr}_1 : \mathrm{Bl}_0^{\mathbb{C}^n} \to \mathbb{CP}^{n-1}$ *is onto. Projection on second component* $\mathrm{Pr}_2 : \mathrm{Bl}_0^{\mathbb{C}^n} \to \mathbb{C}^n$ *induces a diffeomorphism except over the zero vector of* $\mathbb{C}^n$*; the preimage of the zero vector is* $\mathbb{CP}^{n-1}$*. The map* $\beta := \mathrm{Pr}_2 : \mathrm{Bl}_0^{\mathbb{C}^n} \to \mathbb{C}^n$ *is the blow-down map and the exceptional divisor of the blow-up,* $E := \beta^{-1}(0)$ *is a copy of* $\mathbb{CP}^{n-1}$*. The map* $\beta$ *collapses the exceptional divisor to the origin. Topologically,* $\mathrm{Bl}_0\,\mathbb{C}^n \setminus E = \mathcal{O}_{\mathbb{CP}^{n-1}}(-1) \setminus \mathbb{CP}^{n-1} \cong \underbrace{\mathbb{C}^n \setminus \{0\}}$ *means that blowing-up consist in gluing* $\mathbb{CP}^{n-1}$ *instead of the origin to* $\mathbb{C}^n \setminus \{0\}$*. Geometrically,* $\mathbb{CP}^{(n-1)}$ *comprises all one-dimensional linear subspaces of* $\mathbb{C}^n$*, but now distinct subspaces have distinct zero vectors.*

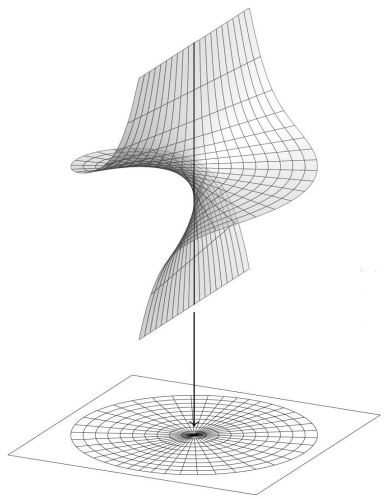


Figure 1.2: Blowing-up origin in $\mathbb{C}^2$ (Photo Credit: Yankı Lekili)

This construction easily generalizes to the blow-up of a point $p$ in a complex manifold $X$:

1. Choose a holomorphic chart $(U, \phi)$ around $p$ on $X$, such that $\phi(U)$ is ball $\mathbb{B}$ in $\mathbb{C}^n$ centred as $0$ and $\phi$ maps $p$ to $0$.

2. Replace $U$ by $\mathrm{Pr}_2^{-1}(\mathbb{B})$ where $\mathrm{Pr}_2 : \mathcal{O}_{\mathbb{CP}^{n-1}}(-1) \to \mathbb{C}^n$ is the projection on second factor in (1.11) on page 48; more precisely, $U \setminus \{p\}$ appears as an open set in $U$ as well as in $X \setminus \{p\}$ and we glue together $\mathrm{Pr}_2^{-1}(\mathbb{B})$ and $X \setminus \{p\}$ along $U \setminus \{p\}$.

The resulting object is denoted by $\mathrm{Bl}_p^X$ and is described by

$$\mathrm{Bl}_p^X = (X \setminus \{p\}) \bigsqcup_{U\setminus\{p\}=\mathrm{Pr}_2^{-1}(\mathbb{B})\setminus \mathrm{Pr}_2^{-1}(0)} \mathrm{Pr}_2^{-1}(\mathbb{B}).$$

This can be easily made into a n-dimensional complex manifold and comes equipped with a natural holomorphic blow-down map $\beta : \mathrm{Bl}_p^X \to X$. The exceptional divisor $E := \beta^{-1}(p)$ is isomorphic to the complex projective space $\mathbb{P}(T_pX)$.

**Example 1.127** (Blowing-up $\mathbb{C}^n$ along $\mathbb{C}^{n-k}$ for $n-k \geq 2$)**.** *With the same process we can define*

$$\mathrm{Bl}_{\mathbb{C}^{n-k}}^{\mathbb{C}^n} := \mathbb{C}^{n-k} \times \mathcal{O}_{\mathbb{CP}^{k-1}}(-1).$$

More generally, as explained in [31] on pages 603-604, this local picture can be used unambiguously to define the blow-up $\mathrm{Bl}_Y^X$ of $X$ along a complex submanifold $Y$ by replacing $Y$ by $E = \mathbb{P}(N_X(Y))$, where $N_X(Y)$ is the normal bundle of $Y$ in $X$. The blow-down $\beta : \mathrm{Bl}_Y^X \to X$ is the proper holomorphic map such that $\beta : \mathrm{Bl}_Y^X \setminus E \to X \setminus Y$ is the identity map. Here we recall some topological and geometric properties of manifolds obtained by blowing-up a complex manifold.

**Proposition 1.128.** *Let $X$ be a complex manifold, and $Y$ a closed, embedded complex submanifold in $X$. Then*

$$\pi_1(\mathrm{Bl}_Y^X) = \pi_1(X).$$

See Lemma 2.2.8 [61] for a proof.

**Proposition 1.129.** *Let $X$ be a complex $n$-manifold, and $Y$ a closed, embedded complex $(n-k)$-submanifold in $X$. Then*

$$b_i(\mathrm{Bl}_Y^X) = b_i(X) + \sum_{j=1}^{k-1} b_{i-2j}(Y).$$

See Corollary 2.2.10 [61] for a proof.

**Proposition 1.130.** *If $X$ is an algebraic variety and $p \in X$, then $\mathrm{Bl}_p^X$ is algebraic.*

See page 192 [31] for a proof.

**Proposition 1.131.** *Suppose that $X$ is a complex manifold of complex dimension $n$, $Y$ a closed complex submanifold in $X$ with of complex dimension $k$, $\mathrm{Bl}_Y^X$ the blow-up of $X$ along $Y$ with exceptional divisor $E$ and $\mathcal{L}_E$ the holomorphic line bundle on $\mathrm{Bl}_Y^X$ associated to $E$. A calculation similar to the adjunction formula shows that*

$$K_{\mathrm{Bl}_Y^X} = \beta^* K_X \otimes \mathcal{L}_E^{n-k-1}.$$

*In particular, the first Chern classes of* $\mathrm{Bl}_p^X$ *and* $X$ *are related by*

$$c_1(\mathrm{Bl}_Y^X) = \beta^* c_1(X) - (n-k-1)c_1(\mathcal{L}_E).$$

See Proposition 6.4.2 [29] for a proof.

## 1.9 Weighted Projective Space

In this section, we discuss the concept of weighted projective space introduced by David Mumford in 1965.

**Definition 1.132** (Compact weighted projective space)**.** *Let* $w_0 \in \mathbb{N}$ *and* $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$*. The compact weighted projective space correponding to the weight vector* $\overrightarrow{w} = (w_0, w)$ *is the quotient*

$$\mathbb{CP}^n_{\overrightarrow{w}} = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*,$$

*with* $\mathbb{C}^*$*-action on* $\mathbb{C}^{n+1}$ *is given by*

$$t.(z_0, z_1, \ldots, z_n) = (t^{w_0} z_0, t^{w_1} z_1, \ldots, t^{w_n} z_n), \forall t \in \mathbb{C}^*.$$

**Definition 1.133** (Non-compact weighted projective space)**.** *Let* $w_0 \in \mathbb{N}$ *and* $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$*. The non-compact weighted projective space correponding to the weight vector* $\overrightarrow{w} = (-w_0, w)$ *is the quotient*

$$\mathbb{CP}^n_{\overrightarrow{w}} = \left(\mathbb{C} \times \mathbb{C}^n \setminus \mathbb{C} \times \{0\}\right)/\mathbb{C}^*,$$

*with* $\mathbb{C}^*$*-action given by*

$$t.(z_0, z_1, \ldots, z_n) = (t^{-w_0} z_0, t^{w_1} z_1, \ldots, t^{w_n} z_n), \forall t \in \mathbb{C}^*.$$

**Remark 1.134.** *The weighted projective space* $\mathbb{CP}^n_{\overrightarrow{w}}$ *(compact or non-compact) has the structure of a complex orbifold, since the* $\mathbb{C}^*$*-action is holomorphic, faithful and orientation preserving.*

**Example 1.135.** *The weighted projective space* $\mathbb{CP}^n_{(1,1,\ldots,1)}$ *is the usual complex projective space* $\mathbb{CP}^n$*.*

**Example 1.136.** *Let* $r \in \mathbb{N}$*, the non-compact weighted projective space* $\mathbb{CP}^n_{(-r,1,\ldots,1)}$ *is the total space of the line bundle* $\mathcal{O}_{\mathbb{CP}^{n-1}}(-r)$*.*

## 1.10 Singularities of the Weighted Projective Space

To describe singularities of the weighted projective space, we begin with the following observation. Assume that $w_0 > 1$, and let $p = (1, 0, \ldots, 0) \in \mathbb{C}^{n+1} \setminus \{0\}$. Under the action of $t \in \mathbb{C}^*$, $(1, 0, \ldots, 0)$ is taken to

$t.p = (t^{w_0}, 0, \ldots, 0)$. So the stablizer of $p$ is given by

$$\mathrm{Stab}(p) = \{t \in \mathbb{C}^* : t^{w_0} = 1\} \cong \mathbb{Z}_{w_0}.$$

With the same idea we can see that for a point $p = [z_0, z_1, \ldots, z_n] \in \mathbb{CP}^n_{(w_0,w)}$, if we set

$$d = \gcd\{w_i : z_i \neq 0, 0 \leq i \leq n\},$$

then we get two cases:

1. Case 1: If $d = 1$, then $p$ is a non-singular point.
2. Case 2: If $d \neq 1$, then $p$ is a singular point. Near the point $p$, the weighted projective space is locally like $\mathbb{C}^{k-1} \times (\mathbb{C}^{n-k+1}/\mathbb{Z}_d)$ where $k = \text{card}\{i : z_i \neq 0, 0 \leq i \leq n\}$ and $\mathbb{Z}_d$ acts on $\mathbb{C}^{n-k+1}$ by

$$e^{\frac{2\pi i}{d}}.(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_{n-k+1}}) = (e^{\frac{2\pi i}{d} w_{i_1}} \xi_{i_1}, e^{\frac{2\pi i}{d} w_{i_2}} \xi_{i_2}, \ldots, e^{\frac{2\pi i}{d} w_{i_{n-k+1}}} \xi_{i_{n-k+1}}),$$

and $\{i_1, \ldots, i_{n-k+1}\} = \{i : z_i = 0\}$. See pages 133-134 [38] for more details.

For the non-compact weighted projective space $\mathbb{CP}^n_{(-w_0,w)}$, the singular points correspond to the singular points of $\mathbb{CP}^{n-1}_w$ in $\mathbb{CP}^n_{(-w_0,w)}$ given by

$$\mathbb{CP}^{n-1}_w = \{[z_0, z_1, \ldots, z_n] \in \mathbb{CP}^n_{(-w_0,w)} : z_0 = 0\} \subset \mathbb{CP}^n_{(-w_0,w)}.$$

**Remark 1.137.** *The above discussion shows that*

*(a) The compact weighted projective space $\mathbb{CP}^n_{(w_0,w)}$ is smooth if and only if $w_0 = 1$ and $w = (1, \ldots, 1)$, i.e, it is the usual weighted projective space $\mathbb{CP}^n$.*

*(b) The non-compact weighted projective space $\mathbb{CP}^n_{(-w_0,w)}$ is smooth if and only if $w = (1, \ldots, 1)$ and $\mathbb{CP}^n_{(-w_0,1,\ldots,1)} \cong \mathcal{O}_{\mathbb{CP}^{n-1}}(-w_0)$.*

**Remark 1.138.** *A non-compact weighted projective space $\mathbb{CP}^n_{(-w_0,w)}$ has only isolated singularities, if and only if*

$$\gcd(w_i, w_j) = 1, \forall i \neq j \in \{0, \ldots, n\}. \tag{1.3}$$

*In this case, the number of singularities is equal to the number of values $w_i$ that are not equal to 1 for $i \in 1, \ldots, n$. If $w_i \neq 1$, the singularity corresponding to the point $[0 : \ldots : 0 : 1 : \ldots : 0] \in \mathbb{CP}^n_{(-w_0,w)}$ and is modeled by the orbifolds $\mathbb{C}^n/\mathbb{Z}_{w_i}$ obtained by the action of the cyclic group of $w_i$-th roots of unity given by*

$$e^{\frac{2\pi i}{w_i}}.(\xi_0, \xi_{i_1}, \ldots, \xi_{i_{n-1}}) = (e^{-\frac{2\pi i}{w_i} w_0} \xi_0, e^{\frac{2\pi i}{w_i} w_{i_1}} \xi_{i_1}, \ldots, e^{\frac{2\pi i}{w_i} w_{i_{n-1}}} \xi_{i_{n-1}}),$$

*with $\{i_1, \ldots, i_{n-1}\} = \{j \in \{1, \ldots, n\} : z_j \neq i\}$.*

**Example 1.139.** *Let $\Gamma_{(-w_0,w)}$ be the cyclic group of $w_0$-roots unity defined by*

$$\Gamma_{(-w_0,w)} = \langle \mathrm{diag}(\xi^{w_1}, \dots, \xi^{w_n}) \rangle \cong \mathbb{Z}_{w_0},$$

*where $\xi = e^{\frac{2\pi i}{w_0}}$. Consider the action of $\Gamma_{(-w_0,w)}$ on $\mathbb{C}^n$ given by*

$$(z_1, \dots, z_n) \mapsto (\xi^{w_1} z_1, \dots, \xi^{w_n} z_n).$$

*The group $\Gamma_{(-w_0,w)}$ is a finite subgroup of $U(n)$, so $X = \mathbb{C}^n / \Gamma_{(-w_0,w)}$ has the structure of an orbifold. It has an isolated singularity at the origin if and only if*

$$\gcd(w_0, w_i) = 1, \forall i \in \{1, \dots, n\}. \tag{1.4}$$

**Definition 1.140.** *Assume that the complex orbifold $X = \mathbb{C}^n / \Gamma_{(-w_0,w)}$ has an isolated singularity at the origin. A blow-up of the origin is given by the non-compact weighted projective space $\mathbb{CP}^n_{(-w_0,w)}$ with the blow-down map $\beta : \mathbb{CP}^n_{(-w_0,w)} \to \mathbb{C}^n / \Gamma_{(-w_0,w)}$ given by*

$$[z_0, z_1, \dots, z_n] \mapsto (z_0^{\frac{w_1}{w_0}} z_1, \dots, z_0^{\frac{w_n}{w_0}} z_n).$$

*The above map is well defined because for any $t \in \mathbb{C}^*$,*

$$\beta([t^{-w_0} z_0, t^{w_1} z_1, \dots, t^{w_n} z_n]) = ((t^{-w_0} z_0)^{\frac{w_1}{w_0}} t^{w_1} z_1, \dots, (t^{-w_0} z_0)^{\frac{w_n}{w_0}} t^{w_n} z_n) = (z_0^{\frac{w_1}{w_0}} z_1, \dots, z_0^{\frac{w_n}{w_0}} z_n).$$

*The exceptional divisor $E = \beta^{-1}(0)$ of this weighted blow-up is naturally identified with the compact weighted projective space $E = \mathbb{CP}^{n-1}_w$.*

**Remark 1.141.** *Note that $\mathbb{C}^n / \Gamma_{(-1,1,\dots,1)} = \mathbb{C}^n$, so*

$$\mathrm{Bl}_0^{\mathbb{C}^n} = \mathbb{CP}^n_{(-1,1,\dots,1)} = \mathcal{O}_{\mathbb{CP}^{n-1}}(-1),$$

*and thus we obtain that this blow-up is consistent with Example 1.126 in the page 38.*

Definition 1.140 shows that for any $w_0 \in \mathbb{N}$, the non compact weighted projective space $\mathbb{CP}^n_{(-w_0,w)}$ is a holomorphic line bundle over the compact weighted projective space $\mathbb{CP}^{n-1}_w$.

**Definition 1.142.** *For $w \in \mathbb{N}^n$, the tautological line bundle of the compact weighted projective space $\mathbb{CP}^{n-1}_w$ is defined by*

$$\mathcal{O}_{\mathbb{CP}^{n-1}_w}(-1) := \mathbb{CP}^n_{(-1,w)},$$

*Similarly for $w_0 \in \mathbb{N}$ we define*

$$\mathcal{O}_{\mathbb{CP}^{n-1}_w}(-w_0) := (\mathcal{O}_{\mathbb{CP}^{n-1}_w}(-1))^{\otimes w_0} = \mathbb{CP}^n_{(-w_0,w)}.$$

Similar properties, as Proposition 1.42, remain valid for the weighted projective space as follows.

**Proposition 1.143.** *For any $w_0 \in \mathbb{N}$ and $w \in \mathbb{N}^n$, the holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}_w}(-w_0)$ has no non-trivial global holomorphic section.*

*Proof.* Let $s \in H^0(\mathbb{CP}^{n-1}_w, \mathcal{O}_{\mathbb{CP}^{n-1}_w}(-w_0))$ be a global holomorphic section. Then the composition with the blow-down map $\beta \circ s : \mathbb{CP}^{n-1}_w \to \mathbb{C}^n / \Gamma_{(-w_0,w)}$ is a holomorphic function defined on the compact weighted projective space $\mathbb{CP}^{n-1}_w$, so it is a constant $\beta \circ s \equiv c$. Clearly, this can only happen if $s = 0$ as an element of $H^0(\mathbb{CP}^{n-1}_w, \mathcal{O}_{\mathbb{CP}^{n-1}_w}(-w_0))$. $\square$

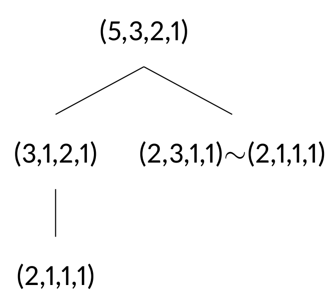
## 1.11 Singularities of Type $\mathcal{I}$

In this section, we introduce the notion of singularities of type $\mathcal{I}$. Singularities of type $\mathcal{I}$ were introduced by Vestislav Apostolov and Yann Rollin in 2016 in [4]. Consider the congruence relation $\sim$ on $\mathbb{N}^{n+1}$ defined by

$$(a_0, a_1, \ldots, a_n) \sim (b_0, b_1, \ldots, b_n) \iff a_0 = b_0, a_i \equiv b_i \mod a_0.$$

We can check that if $(a_0, a) \sim (b_0, b)$ then $\mathbb{C}^n / \Gamma_{(-a_0,a)} \cong \mathbb{C}^n / \Gamma_{(-b_0,b)}$ .

**Example 1.144.** *Consider $\mathbb{C}^3/\Gamma_{(-5,3,2,1)}$, and blow-up the origin by replacing $\mathbb{C}^3/\Gamma_{(-5,3,2,1)}$ with $\mathbb{CP}^3_{(-5,3,2,1)}$. By remark 1.138, this new space is still singular with the isolated singularities at two points $[0:1:0:0]$ and $[0:0:1:0]$. These singularities are locally of the forms $\mathbb{C}^3/\Gamma_{(-3,1,2,1)}$ and $\mathbb{C}^3/\Gamma_{(-2,3,1,1)}$ so we can still blow them up by replacing these by $\mathbb{CP}^3_{(-3,1,2,1)}$ and $\mathbb{CP}^3_{(-2,1,1,1)}$. Since $(2,3,1,1) \sim (2,1,1,1)$, $\mathbb{CP}^3_{(-2,1,1,1)}$ is smooth. However, $\mathbb{CP}^3_{(-3,1,2,1)}$ still has a singularity locally in the form of $\mathbb{C}^3/\Gamma_{(2,1,1,1)}$. By blowing it up and replacing it with $\mathbb{CP}^3_{(-2,1,1,1)}$, we finally obtain a smooth complex manifold. There is a corresponding tree of singularities for $\mathbb{C}^3/\Gamma_{(-5,3,2,1)}$:*




Now we define singularities of type $\mathcal{I}$.

**Definition 1.145** (Singularities of type $\mathcal{I}$)**.** *A singularity of an orbifold $\mathbb{C}^n/\Gamma_{(-w_0,w)}$ with $(-w_0, w)$ as in the* (1.3) *on page 42 is a singularity of type $\mathcal{I}$ if either* *(a)* $(w_0, w) \sim (w_0, 1, \ldots, 1)$.

*or*

*(b)* $(w_0, w) \sim (a_0, a)$ *such that* $\mathbb{CP}^n_{(-a_0,a)}$ *has only isolated singularities of the forms* $\mathbb{C}^n/\Gamma_{(-b_0,b)}$ *of type* $\mathcal{I}$. *That is, after finitely many weighted blow-up we can end with a smooth manifold.*

For singularities of type $\mathcal{I}$, we can represent a tree of singularities as follows. Start with $(a_0, a_1, \ldots, a_n)$, and inductively construct each branch corresponding to $a_i \neq 1$. At each step, a new singularity is obtained by:

$$(a_0, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \to (a_i, a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n),$$

where $x \in \mathbb{N}$ is such that $x \equiv -a_0 \pmod{a_i}$. According to the definition of singularities of type $\mathcal{I}$, each branch is considered complete when it end up to $(w_0, 1, \ldots, 1)$ with corresponding smooth weighted blow-up.

**Example 1.146.** *The singularity of the orbifold* $\mathbb{C}^3/\Gamma_{(-5,3,2,1)}$ *in Example 1.144 is a singularity of type* $\mathcal{I}$ *because we end with* $(2, 1, 1, 1)$ *in each branch.*

**Example 1.147.** *Let* $(w_0, w) = (p, q, 1, \ldots, 1)$, *where* $p$ *and* $q$ *are two positive coprime integers such that* $p > q$. *Then,* $(p, q, 1, \ldots, 1)$ *is of type* $\mathcal{I}$. *A similar inductive procedure shows that by starting with* $(p_0, q_0) = (p, q)$ *and blowing-up at the stage* $k$, *by performing the Euclidean algorithm, we get* $p_k = q_{k-1} < p_{k-1}$ *and* $0 < q_k < p_{k-1}$ *such that* $q_k \equiv -p_{k-1} \pmod{q_{k-1}}$. *Clearly, in each stage,* $p_k$ *and* $q_k$ *are coprime, so we have an isolated singularity. Since* $q_k < q_{k-1}$, *we will eventually obtain a weight vector of the form* $(p_N, 1, 1, \ldots, 1)$.

$$(p, q, 1, \ldots, 1)$$

$$|$$

$$(p_1, q_1, 1, \ldots, 1)$$

$$|$$

$$\ldots$$

$$|$$

$$(p_N, 1, 1, \ldots, 1)$$

Definition 1.139 shows that locally we can glue non-compact weighted projective spaces to resolve partially the isolated singularities of $\mathbb{C}^n/\Gamma_{(-w_0,w)}$. The fact that the singularity is of type $\mathcal{I}$ means that this type of partial resolution can be iterated finitely many times to obtain a smooth manifold. Globally, a complex orbifold with isolated singularities of type $\mathcal{I}$ admits a resolution (which is not necessarily unique) denoted as $\widehat{X}$ of type $\mathcal{I}$. More generally, let $X$ be a compact complex orbifold of depth 1 with singularities of type $\mathcal{I}$ along a connected subset $Y$ with codimension $k$ greater than 2. We can define a type $\mathcal{I}$ resolution of $X$ along $Y$ as follows. Since $X$ has singularities of type $\mathcal{I}$ along $Y$ of codimension $k$, the normal bundle of $Y$ in $X$ is a fiber bundle over $Y$ with fibers of the form $\mathbb{C}^k / \Gamma_{(-w_0,w)}$, where $\Gamma_{(-w_0,w)}$ is a discrete finite subgroup of $U(k)$ as in Definition 1.145. Now, in a local chart

$$\phi : U \to V_1 \times V_2 \subset \mathbb{C}^{n-k} \times (\mathbb{C}^k / \Gamma_{(-w_0,w)}),$$

with $\phi(U \cap Y) = V_1 \times \{0\}$, we can consider the resolution $V_1 \times \widehat{V_2}$ with $\widehat{V_2} = \beta^{-1}(V_2)$, where

$$\beta : \mathbb{CP}^k_{(-w_0,w)} \to \mathbb{C}^k / \Gamma_{(-w_0,w)},$$

is the natural blow-down map of Definition 1.142. That is, we can consider the resolution $\beta_U : \widehat{U} \to U$, inducing a commutative diagram

$$\begin{array}{ccc} \widehat{U} & \xrightarrow{\widehat{\phi}} & V_1 \times \widehat{V_2} \\ \downarrow{\scriptstyle \beta_U} & & \downarrow{\scriptstyle \mathrm{Id} \times \beta} \\ U & \xrightarrow{\phi} & V_1 \times V_2, \end{array}$$

with $\widehat{\phi}$ a biholomorphism. This resolution does not depend on the choice of coordinates. Indeed, if $f : V_1 \times V_2 \to V_1 \times V_2$ is a biholomorphism sending $V_1 \times \{0\}$ onto $V_1 \times \{0\}$, then it lifts to a $\Gamma_{(-w_0,w)}$-equivariant biholomorphism $\widetilde{f} : V_1 \times \widetilde{V_2} \to V_1 \times \widetilde{V_2}$ with $\widetilde{V_2}$ the lift of $V_2$ to $\mathbb{C}^k$ under the quotient map $q : \mathbb{C}^k \to \mathbb{C}^k / \Gamma_{(-w_0,w)}$.

The differential of $\widetilde{f}$ in the $\widetilde{V_2}$ factor induces, when restricted to $V_1 \times \{0\}$, a biholomorphism

$$d\widetilde{f_2} : V_1 \times \mathbb{C}^k \to V_1 \times \mathbb{C}^k,$$

which is linear in the $\mathbb{C}^k$ factor and $\Gamma_{(-w_0,w)}$-equivariant. In paticular, it has a weighted projectivization

$$\mathbb{P}_w(d\widetilde{f_2}) : V_1 \times \mathbb{P}_w(\mathbb{C}^k) \to V_1 \times \mathbb{P}_w(\mathbb{C}^k).$$

One can then easily check that the biholomorphism $f : V_1 \times V_2 \to V_1 \times V_2$ cab be lifted to a biholomorphism

$$\widehat{f} : V_1 \times \widehat{V_2} \to V_1 \times \widehat{V_2},$$

given by $\mathbb{P}_w(d\widetilde{f}_2)$ on $V_1 \times \mathbb{P}_w(\mathbb{C}^k)$ and by $f$ on $V_1 \times (\widehat{V}_2 \setminus \mathbb{P}_w(\mathbb{C}^k)) = V_1 \times (V_2 \setminus \{0\})$.

Clearly, this biholomorphism induces the commutative diagram

$$\begin{array}{ccc} V_1 \times \widehat{V}_2 & \xrightarrow{\widehat{f}} & V_1 \times \widehat{V}_2 \\ \downarrow{\scriptstyle \mathrm{Id} \times \beta} & & \downarrow{\scriptstyle \mathrm{Id} \times \beta} \\ V_1 \times V_2 & \xrightarrow{f} & V_1 \times V_2, \end{array}$$

confirming that the resolution $\widehat{U}$ does not depend on the choice of coordinates. This means that we can consider a partial resolution $\pi : \widehat{X} \to X$ along $Y$ in which an open set $U$ as described above corresponds to the local resolution $\beta_U : \widehat{U} \to U$, and away from $Y$ is simply the identity map.
We say that the partial resolution $\widehat{X}$ is the $(-w_0, w)$-weighted blow-up of $X$ along $Y$. We denote by $E = \pi^{-1}(Y)$ the exceptional divisor of this weighted blow-up. Notice that $\pi : E \to Y$ is a fiber bundle with fibers $\mathbb{CP}_w^{k-1}$. In fact, there is a rank $k$ complex vector bundle $W \to Y$ and a fiberwise $\Gamma_{(-w_0,w)}$-action on $W$ such that $N_X(Y) = W / \Gamma_{(-w_0,w)}$ and $E = \mathbb{P}_w(W)$ is the fiberwise weighted projectivization of $W$. In general, $\widehat{X}$ is not smooth and has orbifold singularities of depth one along suborbifolds corresponding to the isolated singularities of the fibers of $E \to Y$. In particular, these suborbifolds are covers of $Y$. Assuming the initial singularity along $Y$ is of type $\mathcal{I}$, we can perform weighted blow-ups along these suborbifolds. These weighted blow-ups can still have suborbifold singularities of depth one, but by performing additional weighted blow-ups, we can eventually obtain a smooth resolution after finitely many steps.
In other words, when the singularity along $Y$ is of type $\mathcal{I}$, we can find a finite sequence of weighted blow-ups

$$\widehat{X}_l \to \widehat{X}_{l-1} \to \ldots \to \widehat{X}_1 \to X$$

with $\widehat{X}_1 = \widehat{X}$ and $\widehat{X}_l$ smooth.

**Proposition 1.148.** *Let $X$ is a compact complex orbifold of complex dimension $n$. Suppose that $X$ has only depth one singularities of type $\mathcal{I}$ and we denote by $Y$ the singular part of $X$. Assume that the complex codimension $k$ of $Y$ is greater than 2. For any resolution $\widehat{X}$ of $X$ of type $\mathcal{I}$, $\mathfrak{h}(\widehat{X})$ is naturally realized as the Lie subalgebra of $\mathfrak{h}(X)$ consisting of holomorphic vector fields on $X$ tangent to $Y$.*

*Proof.* We proceed with a proof similar to Proposition 6.4.1 in [29]. Firstly, we demonstrate that any (real) holomorphic vector field, denoted as $\widehat{V}$, on $\widehat{X}$ descends to a (real) holomorphic vector field, denoted as $V$, on $X$ which is tangent to $Y$. Indeed, since $\pi : \widehat{X} \setminus E \to X \setminus Y$ is a biholomorphism,

$\widehat{V}$ naturally descends to a holomorphic vector field on $X \setminus Y$. Via the short exact sequence of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TE & \longrightarrow & T\widehat{X} & \longrightarrow & T\widehat{X}/TE & \longrightarrow & 0, \\ & & & & \downarrow & & & & \\ & & & & E, & & & & \end{array}$$

the restriction $\widehat{V}\big|_E$ determines a holomorphic section of the normal bundle

$$T\widehat{X}/TE \cong N_{\widehat{X}}(E).$$

On the other hand, on $E$, the restriction of $\pi$ induces a fiber bundle

$$\begin{array}{c} E \cong \mathbb{P}_w(W) \\ \downarrow \\ Y, \end{array}$$

where $W$ is such that $N_X(Y) = W/\Gamma_{(-w_0,w)}$ for a $\Gamma_{(-w_0,w)}$ of type $\mathcal{I}$ and $\mathbb{P}_w(W)$ is the weighted fiberwise projectivization of $W$. Thus, each fiber of $N_X(Y)$ corresponds to $\mathbb{CP}_w^{k-1}$ with the restriction of $N_{\widehat{X}}(E)$ corresponding to $\mathcal{O}_{\mathbb{CP}_w^{k-1}}(-w_0)$. Using Proposition 1.143, it has no non-trivial global holomorphic section. Consequently, $\widehat{V}\big|_E$ is tangent to $E$. Now via $\pi_* : TE \to \pi^*TY$, $\widehat{V}\big|_E$ induces a section $\pi_*(\widehat{V}\big|_E) \in H^0(E, \pi^*TY)$. On each fiber of $\pi$, $\pi^*TY$ is trivial, so its only holomorphic sections are constant sections. This implies that $V = \pi_*(\widehat{V})$ is a well-defined continuous vector field on $X$, which is tangent to $Y$ and holomorphic on $X \setminus Y$. By Hartogs' theorem, $V$ is holomorphic everywhere on $X$, hence it belongs to the Lie subalgebra,

$$\mathfrak{h}_Y(X) = \{\xi \in \mathfrak{h}(X) : \xi\big|_Y \in H^0(Y, T^{1,0}Y)\}.$$

Furthermore, the resulting map from $\mathfrak{h}(\widehat{X})$ to $\mathfrak{h}_Y(X)$ is a Lie algebra morphism. This map is clearly injective, and we shall now show that it is surjective, establishing that it is a Lie algebra isomorphism.

Indeed, any element $V$ of $\mathfrak{h}_Y(X)$ lifts to a (real) holomorphic vector field, say $\widehat{V}$, on $\widehat{X} \setminus E$, via the isomorphism $\pi : \widehat{X} \setminus E \to X \setminus Y$. We need to check that $\widehat{V}$ extends to all of $\widehat{X}$ as an element of $\mathfrak{h}(\widehat{X})$. Since $\widehat{V}$ is holomorphic on $\widehat{X} \setminus E$, we only have to worry about the behavior of $\widehat{V}$ near points $p$ of the exceptional divisor $E$. Since $p$ has a neighborhood in $\widehat{X}$ isomorphic to $\mathbb{C}^{n-k} \times \mathbb{CP}^k_{(-w_0,w)}$, we can use local holomorphic coordinates

$$(y_1, \ldots, y_{n-k}, Z_1, \ldots, Z_k) \in \mathbb{C}^{n-k} \times \mathbb{C}^k/\Gamma_{(-w_0,w)},$$

near $y = \pi(p)$. Relative to these coordinates, since $V\big|_Y$ is tangent to $Y$, $V$ will be here conveniently regarded as a holomorphic (complex) vector field of type $(1,0)$ in the form of:

$$V = \sum_{i=1}^{n-k} a^i \frac{\partial}{\partial y_i} + \sum_{i,j=1}^{k} b^{ij} Z_j \frac{\partial}{\partial Z_i},$$

where the $a^i$ and $b^{ij}$ are holomorphic functions of $y_1, \ldots, y_{n-k}, Z_1, \ldots, Z_k$. Now, since this is an orbifold chart, $V$ must be $\Gamma_{(-w_0,w)}$-invariant as well, which means that $b^{ij} = 0$ for $i \neq j$ with $w_i \neq w_j$ and $b^{ii}$ does not depend on $(Z_1, \ldots, Z_k)$ whenever $w_i \neq 1$. Correspondingly, near $p$ we can use coordinates $(y_1, \ldots, y_{n-k}, z_1, \ldots, z_k)$ such that

$$\pi(y_1, \ldots, y_{n-k}, z_1, \ldots, z_k) = (y_1, \ldots, y_{n-k}, z_1^{\frac{w_1}{w_0}}, z_1^{\frac{w_2}{w_0}} z_2, \ldots, z_1^{\frac{w_k}{w_0}} z_k) = (y_1, \ldots, y_{n-k}, Z_1, \ldots, Z_k),$$

where we assume without loss of generality that $z_1 \neq 0$ for the weighted projective class corresponding to $p$. Then

$$\pi^*(Z_i \frac{\partial}{\partial Z_i}) = \begin{cases} \frac{w_0}{w_1} z_1 \frac{\partial}{\partial z_1} - \sum_{i=2}^{k} \frac{w_i}{w_1} z_i \frac{\partial}{\partial z_i} & : i = 1 \\ z_i \frac{\partial}{\partial z_i} & : i > 1 \end{cases}$$

and $\pi^*(Z_i \frac{\partial}{\partial Z_j}) = z_i \frac{\partial}{\partial z_j}$ for $i \neq j \neq 1$ with $w_i = w_j = 1$. Moreover, if $w_1 = 1$, then for $i \neq 1$ with $w_i = 1$, $\pi^*(Z_i \frac{\partial}{\partial Z_1}) = w_0 z_1 z_i \frac{\partial}{\partial z_1} - \sum_{j=2}^{k} \frac{w_j}{w_1} z_j z_i \frac{\partial}{\partial z_j}$. This demonstrates that $\widehat{V}$ is well-defined on the whole of $\widehat{X}$ as an element of $\mathfrak{h}(\widehat{X})$. $\square$

# CHAPTER 2

# MANIFOLDS WITH CORNERS AND LIE STRUCTURES AT INFINITY

Many problems in differential geometry and partial differential equations often involve manifolds with boundaries, such as boundary value problems. The category of smooth manifolds alone presents challenges, and even the category of manifolds with boundaries is not sufficiently convenient, as the product of two manifolds with boundaries does not yield a manifold with a boundary. This complexity prompts the introduction of the category of manifolds with corners. Manifolds with corners arise in various ways, as will be shown later. Constructions leading to manifolds with corners include the desingularization of singular varieties (blow-up) and the compactification of non-compact spaces.
Melrose calculus, also known as pseudodifferential operator calculus or boundary value calculus, is a framework that extends the theory of pseudodifferential operators to manifolds with boundaries and corners. Developed by Richard Melrose in the 1980s, it has found applications in microlocal analysis, geometric analysis, etc. The key idea in Melrose calculus is to study operators that behave like pseudodifferential operators near the boundary or corners of a manifold. Pseudodifferential operators are a class of linear operators with symbols that have asymptotic expansions, playing a fundamental role in harmonic analysis and partial differential equations. In Melrose calculus, these operators are generalized to handle boundary value problems and singularities.
We begin this chapter with a brief introduction to manifolds with corners and blow-ups in the sense of Melrose. The definition of manifolds with corners is not universally agreed upon. In this chapter, we follow the approach outlined by Richard Melrose in his book 'Differential Analysis on Manifolds with Corners' [47]. In the second part of this chapter, we introduce Lie structures at infinity based on the series of papers by Ammann-Lauter-Nistor in [2].

## 2.1 Manifolds with Corners

The definition of a manifold with corners below is based on the model spaces

$$\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n \,|\, x_i \geq 0, 1 \leq i \leq k\},$$

which are products of half-lines and lines. The topology on $\mathbb{R}^n_k$ is inherited from $\mathbb{R}^n$. In particular, a subset $\Omega \subset \mathbb{R}^n_k$ is open if there exists an open set $\Omega_0 \subset \mathbb{R}^n$ such that $\Omega = \Omega_0 \cap \mathbb{R}^n_k$. If $\Omega \subset \mathbb{R}^n_k$ is an open subset, we define $C^\infty(\Omega)$ as the set of functions $u : \Omega \to \mathbb{C}$ such that $u$ is smooth in $\Omega^\circ$ with all derivatives bounded on $K \cap \Omega^\circ$ for all subsets $K \Subset \Omega$. Here, $\Omega^\circ = \Omega \cap \mathrm{int}(\mathbb{R}^n_k)$ and $K \Subset \Omega$ means that the closure of $K$ is a compact subset of $\Omega$. Similarly, smooth structures, diffeomorphisms, and partitions of unity are defined in a natural way on $\mathbb{R}^n_k$.

**Definition 2.1** (Smooth structure with corners)**.** *Let $X$ be a Hausdorff topological space. A chart with corners on $X$ is a map $\phi : U \to \mathbb{R}^n_k$, which is a homeomorphism from an open set $U \subseteq X$ onto an open subset of $\mathbb{R}^n_k$, for some $k$. Two charts $(\phi_1, U_1)$ and $(\phi_2, U_2)$ are said to be compatible if either $U_1 \cap U_2 = \emptyset$, or $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is a diffeomorphism between open subsets of $\mathbb{R}^n_{k_1}$ and $\mathbb{R}^n_{k_2}$. An atlas on $X$ is a system of charts $\{(\phi_a, U_a)\}$ for $a \in A$, which are compatible in pairs and cover $X$, i.e., $X = \bigcup_{a \in A} U_a$. A smooth structure with corners on $X$ is a maximal atlas, i.e., an atlas that contains any chart compatible with each element of the atlas.*

**Definition 2.2** ($t$-manifold)**.** *A $t$-manifold is a paracompact Hausdorff space $X$ endowed with some smooth structure with corners on it.*

**Definition 2.3** (Submanifold)**.** *If $X$ is a $t$-manifold, then a submanifold $Y \subseteq X$ is a connected subset with the property that for each $y \in Y$, there exists a coordinate system $(\phi, U)$ around $y$, a linear transformation $G \in \mathrm{GL}(n, \mathbb{R})$, and an open neighborhood $\Omega' \subset \mathbb{R}^n$ of $0$ in terms of which*

$$\phi\big|_U : Y \cap U \to G.(\mathbb{R}^{n'}_{k'} \times \{0\}) \cap \Omega',$$

*for some integers $n'$ and $k' = k'(y)$.*

**Definition 2.4** ($p$-submanifold)**.** *A submanifold $Y$ in a $t$-manifold $X$ is called a $p$-submanifold if, for each $y \in Y$, there exist local coordinates $\phi$ at $y$ within a coordinate neighborhood $\Omega \subset X$, such that*

$$\phi(\Omega \cap Y) = L \cap \phi(\Omega),$$

*where*

$$L = \{x \in \mathbb{R}^n_k : x_{k-j+1} = \ldots = x_k = 0, x_{k+1} = \ldots = x_{k+r} = 0\},$$

*and $j + r$ is the codimension of the submanifold. This implies that $X$ and $Y$ have a common local product decomposition. The 'p' in $p$-submanifold stands for 'product'.*

**Example 2.5.** *The $\mathbb{S}^{n-1}_k := \{x \in \mathbb{R}^n_k : \|x\| = 1\} = \mathbb{S}^{n-1} \cap \mathbb{R}^n_k$ is a $p$-submanifold of the manifold with corners $\mathbb{R}^n_k$.*

We now study the notion of the boundary $\partial X$ for $t$-manifolds.

**Definition 2.6** (Boundary hypersurface)**.** *For a general $t$-manifold set*

$$\partial_l X = \{p \in X : \text{there is a chart } \phi \text{ near } p \text{ with } \phi(p) \in \partial_l \mathbb{R}^n_k\},$$

*where*

$$\partial_l \mathbb{R}^n_k = \{x \in \mathbb{R}^n_k : x_i = 0 \text{ for exactly } l \text{ of the first } k \text{ indices}\}.$$

*Then $X^\circ = \partial_0 X$. More generally, we shall set*

$$\partial^l X = \overline{\partial_l X} = \bigcup_{r \geq l} \partial_r X.$$

*Thus $\partial_l X$ consists precisely of the points in the boundary of $X$ laying in the interior of a corner of codimension $l$, while $\partial^l X$ consists of the points at which the boundary has codimension at least $l$. We also use the notation $\partial X = \partial^1 X$, so $X^\circ = X \setminus \partial X$. A boundary hypersurface of a $t$-manifold $X$ is the closure of a component of $\partial_1 X$; the collection of boundary hypersurfaces will be denoted $M_1(X)$.*

**Definition 2.7** (Manifold with corners)**.** *A manifold with corners is a Hausdorff space with a $C^\infty$ structure with corners (a $t$-manifold) such that each boundary hypersurface is a submanifold in the sense of Definition 2.3.*

**Definition 2.8.** *The cotangent space of a manifold with corners $X$ at $p \in X$ is defined by*

$$T^*_p X = I_p X / (I_p X)^2,$$

*where $I_p X$ is the ideal of smooth functions on $X$ vanishing at $p$:*

$$I_p X = \{f \in C^\infty(X) : f(p) = 0\}.$$

*Therefore, the tangent space at $p$ is defined by the dual of the cotangent bundle, i.e.,*

$$T_p X = (I_p X / (I_p X)^2)^*,$$

*where*

$$(I_p X)^2 = \{f \in C^\infty(X) : \exists k \in \mathbb{N}, g_1, h_1, \ldots, g_k, h_k \in I_p X \quad \text{s.t.} \quad f = \sum_{i=1}^k g_i h_i\}.$$

**Example 2.9.** *Examples and non-examples of manifolds with corners*

*(a) The tetrahedron is a 3-manifold with corners but the square pyramid is not.*

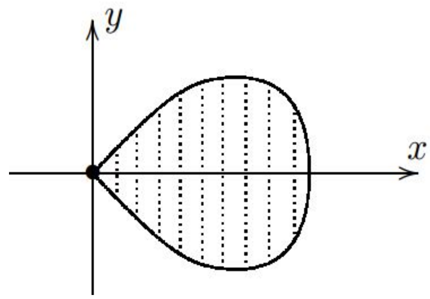



Figure 2.1: The teardrop

(b) *The teardrop* $T = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y^2 \leq x^2 - x^4\}$ *is not a 2-manifold with corners, because its unique boundary hypersurface is not a submanifold.*

**Lemma 2.10.** *In a manifold with corners, each boundary hypersurface* $H$ *has a global defining function in the sense that there exists* $\rho_H \in C^\infty(X)$ *such that* $\rho_H \geq 0$, $H = \rho_H^{-1}(0)$ *is the boundary hypersurface and the differential* $d\rho_H$ *is nowhere zero on* $H$. *Near each point of* $H$, *there are local coordinates with* $\rho_H$ *as the first element.*

**Example 2.11.** *Consider a square in* $\mathbb{R}^2$ *defined by* $[0,1] \times [0,1]$. *This is a manifold with corners. Let us denote its boundary hypersurfaces as* $H_1, H_2, H_3, H_4$, *corresponding to the left, right, bottom, and top sides of the square, respectively. The boundary defining functions can be denoted as* $\rho_{H_1}, \rho_{H_2}, \rho_{H_3}, \rho_{H_4}$. *Specifically:*

$$\rho_{H_1}(x,y) = x \text{ for } H_1,$$

$$\rho_{H_2}(x,y) = 1 - x \text{ for } H_2,$$

$$\rho_{H_3}(x,y) = y \text{ for } H_3,$$

$$\rho_{H_4}(x,y) = 1 - y \text{ for } H_4.$$

*These functions are* $C^\infty$ *and considered as defining functions for the respective boundary hypersurfaces.*

**Definition 2.12.** *Let* $U \subseteq \mathbb{R}^n_k$ *be open. For each* $u = (u_1, \ldots, u_n)$ *in* $U$, *define the boundary depth of* $u$ *in* $U$ *denoted by* $\operatorname{depth}_U(u)$ *as the number of* $u_1, \ldots, u_k$ *which are zero. In other words,* $\operatorname{depth}_U(u)$ *is the number of boundary hypersurfaces of* $U$ *containing* $u$.

**Definition 2.13** (Boundary depth)**.** *Let* $X$ *be an* $n$*-manifold with corners. For* $p \in X$, *choose a local chart* $(U, \phi)$ *around* $p$ *on the manifold* $X$ *with* $\phi(p) = u$ *for* $u \in U \subseteq \mathbb{R}^n_k$, *and define the depth* $\operatorname{depth}_X(x)$ *of* $x$ *in* $X$ *as* $\operatorname{depth}_X(x) = \operatorname{depth}_U(u)$. *This is independent of the choice of* $(U, \phi)$.

**Example 2.14.** *Let $X$ be a manifold with boundary and $p \in X$. If $p \in \operatorname{int}(X)$, then the boundary depth of $p$ is equal to 0 and if $p \in \partial X$, then the boundary depth of $p$ is equal to 1.*

Now, let us define the concept of stratified spaces. These spaces are generalization of manifolds with corners.

**Definition 2.15** (Stratified space)**.** *A stratified space of dimension $n$ is a pair $(X, S)$, where $X$ is a locally compact, separable, metrizable space and $S$ is a stratification, that is, $S = \{S_i\}_{i \in I}$ is a locally finite collection of disjoint locally closed subsets of $X$ and $I$ is a poset such that:*

*(a)* $\bigcup_{i \in I} S_i = X$.

*(b)* $S_i \cap \overline{S_j}$ *is nonempty if and only if* $S_i \subset \overline{S_j}$ *, and this happens if and only if* $i = j$ *or* $i < j$.

*(c) Each $S_i$ is a locally closed smooth submanifold of $\mathbb{R}^n$.*

*The pieces $S_i$ are called strata. The set of strata is itself a poset, with the relation induced from inclusion.*

**Definition 2.16** (Depth of a stratified space)**.** *The depth of a stratified space $(X, S)$ is the largest $k$ such that one can find $k+1$ different strata with $S_1 < S_2 < \ldots < S_k < S_{k+1}$.*

**Example 2.17.** *Any algebraic variety is naturally a stratified space.*

**Example 2.18.** *Consider the closed unit disk $\mathbb{D}^2$ in $\mathbb{R}^2$ and its boundary $\partial\mathbb{D}^2$, which is the unit circle. We can stratify this space as two stratas. The top stratum is the interior of the disk $\mathbb{D}^2$. It is an open subset of $\mathbb{R}^2$ and has dimension 2. And the bottom stratum is the boundary of the disk $\partial\mathbb{D}^2$, which is the unit circle. It is a closed subset of $\mathbb{R}^2$ and has dimension 1. This example illustrates a simple case of a stratified space where each stratum is a subset of the whole space, and they have different dimensions.*

**Example 2.19.** *Any manifold with corners $X$ of dimension $n$ is a stratified space. For each $k = 0, \ldots, n$, define the $k$-th depth stratum of $X$ to be:*

$$S_k(X) = \{x \in X : \operatorname{depth}_X(x) = k\} = \partial_k X.$$

**Definition 2.20** (Manifold with fibered corners)**.** *We say that $(M, \phi)$ is a manifold with fibered corners if there is a partial order on the boundary hypersurfaces such that:*

*(a) Any subset $I$ of boundary hypersurfaces such that $\bigcap_{i \in I} H_i \neq \emptyset$ is totally ordered.*

*(b) If $H_i < H_j$ , then $H_i \cap H_j \neq \emptyset$, $\phi_i|_{H_i \cap H_j} : H_i \cap H_j \to S_i$ is a surjective submersion and $S_{ji} := \phi_j(H_i \cap H_j)$ is one of the boundary hypersurfaces of the manifold with corners $S_j$ . Moreover, there is a surjective submersion $\phi_{ji} : S_{ji} \to S_i$ such that $\phi_{ji} \circ \phi_j = \phi_i$ on $H_i \cap H_j$.*

*(c) The boundary hypersurfaces of $S_j$ are given by the $S_{ji}$ for $H_i < H_j$ .*

**Example 2.21.** *As explained in Remark 1.118, an orbifold is naturally a stratified space. In fact, it is not hard to see that an orbifold is naturally a smoothly stratified space, with the corresponding manifold with fibered corners obtained by blowing-up the strata in an order compatible with the partial order. Notice, in particular, that in terms of Definition 1.121, an orbifold is of depth 1 if and only if it is of depth 1 as a smoothly stratified space.*

### 2.1.1 Blow-up in Melrose Sense

Why consider blow-up at all? If we work on category of smooth manifolds, there isn't a compelling rationale for initiating any form of blow-up. Nonetheless, there are three interconnected scenarios where the process of blow-up can prove highly beneficial. These instances involve attempting to 'resolve' the following:

(a) A singular function, e.g., $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

(b) A singular space, e.g., $C = \{(t, x, y) \mid t^2 = x^2 + y^2, t \geq 0\}$.

(c) Degenerate vector fields, e.g., the span of $z_i \dfrac{\partial}{\partial z_i}$, $i = 1, 2, 3$, on $\mathbb{R}^3$.

In Melrose blow-up (real blow-up) as introduced in [33], the idea is simply to work in polar coordinates around the singular point. That is, we lift everything up to a manifold with a boundary by using the polar map. Before defining the Melrose blow-up, we need to recall the definition of sphere bundle.

**Definition 2.22** (Sphere bundle)**.** *Let $E \to X$ be a smooth vector bundle. The sphere bundle of $E$, denoted by $S(E)$ is a fiber bundle whose fiber is an $n$-sphere and is defined as the set of (positive) rays in the bundle $E$, that is,*

$$S(E) = (E \setminus X) / \mathbb{R}^+.$$

*If we fix a smooth metric on $E$, then the fiber of $S(E)$ over a point $p$ is the set of all unit vectors in*

*$E_p$, the fiber over $p$ in $E$. When the vector bundle is the tangent bundle $TX$, the unit sphere bundle is known as the unit tangent bundle.*

Blowing-up the origin in $\mathbb{R}^2$ simply amounts to the introduction of polar coordinates. We define $\mathbb{R}^2$ blown up at $\{0\}$ to be

$$[\mathbb{R}^2, \{0\}] = \mathbb{S}^1 \times [0, \infty)_r,$$

together with the associated blow-down map $\beta : \mathbb{S}^1 \times [0, \infty)_r \to \mathbb{R}^2$ defined by $\beta(\omega, r) = r\omega$. This is a diffeomorphism from $[\mathbb{R}^2, \{0\}]$ onto $\mathbb{R}^2 \setminus \{0\}$ and has rank 1 at the boundary $\partial[\mathbb{R}^2, \{0\}] = \mathbb{S}^1 \times \{0\}$, which projects to $\{0\}$.

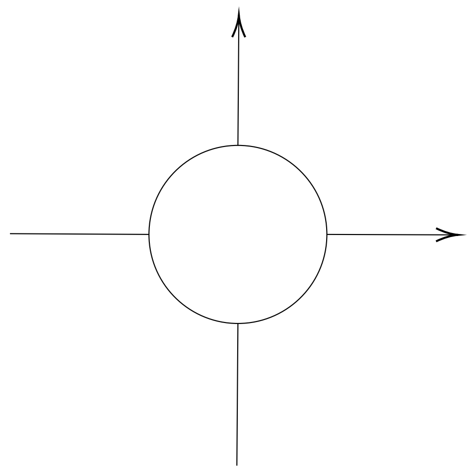


Figure 2.2: Blowing-up the origin in $\mathbb{R}^2$

We can generalize the above idea and get that

$$[\mathbb{R}^n, \{0\}] = \mathbb{S}^{n-1} \times [0, \infty),$$
$$[\mathbb{C}^n, \{0\}] = \mathbb{S}^{2n-1} \times [0, \infty).$$

Another example is blowing-up the origin in $\mathbb{R} \times [0, \infty)_\varepsilon$. Again, by using polar coordinates, we define

$$[\mathbb{R} \times [0, \infty)_\varepsilon; \{0\}] = \mathbb{S}^1_+ \times [0, \infty)_r,$$

where $\mathbb{S}^n_+ = \{(x_0, \dots, x_n) \in \mathbb{S}^n : x_n \geq 0\}$ with the blow-down map $\beta : \mathbb{S}^1_+ \times [0, \infty)_r \to \mathbb{R} \times [0, \infty)_\varepsilon$ defined by $\beta(\omega, r) = r\omega$.

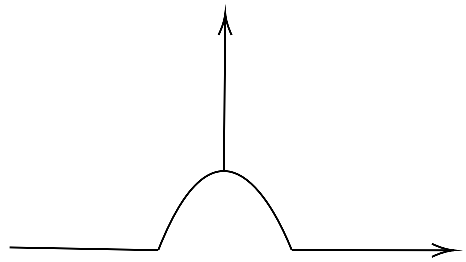


Figure 2.3: Blowing-up the origin in $\mathbb{R} \times [0, \infty)_\varepsilon$

We can generalize the above idea and get that

$$[\mathbb{R}^n \times [0, \infty), \{0\}] = \mathbb{S}^n_+ \times [0, \infty),$$

and even more

$$[\mathbb{R}^n_k, \{0\}] = \mathbb{S}^{n-1}_k \times [0, \infty).$$

**Remark 2.23.** *One can check that the action of* $\mathrm{GL}(n)$ *on* $\mathbb{R}^n$ *lifts to a smooth action of* $\mathrm{GL}(n)$ *on* $[\mathbb{R}^n, \{0\}]$*. This means that the Lie algebra,* $\mathfrak{gl}(n)$*, lifts to* $[\mathbb{R}^n, \{0\}]$*. Since the exponentials of linear vector fields are linear transformations, this implies that for each* $i$ *and* $j$*, there are smooth vector fields* $V_{ij}$ *on* $[\mathbb{R}^n, \{0\}]$ *such that*

$$\beta_* V_{ij} = x_i \partial_{x_i}.$$

*This shows that any smooth vector field on* $\mathbb{R}^n$ *which vanishes at* $0$ *lifts to a smooth vector field on* $[\mathbb{R}^n, \{0\}]$*:*

$$\beta_*^{-1}(a_{ij}(x) x_i \partial_{x_i}) = a_{ij}(r\theta) V_{ij}.$$

In general, for a vector space $V$, the blow-up of $V$ at $0$ is defined as a set by

$$[V, \{0\}] = ((V \setminus \{0\})/\mathbb{R}_+) \bigsqcup (V \setminus \{0\}).$$

Thus, the blow-up of $V$ at $\{0\}$ is the disjoint union of the projective sphere in $V$ and the complement of $\{0\}$. The choice of a basis in $V$ gives a linear isomorphism $V \to \mathbb{R}^n$, which allows us to identify $[V, \{0\}]$ and $[\mathbb{R}^n; \{0\}]$. To show that the smooth structure, as a manifold with boundary, of $[V, \{0\}]$ is well-defined, we therefore need to check that the action of GL$(n)$ on $\mathbb{R}^n$ lifts to a smooth action of GL$(n)$ on $[\mathbb{R}^n, \{0\}]$.

If $E$ is a vector bundle over a manifold with corners $Y$, then we identify $Y$ as the zero section of $E$ and define $E$ blown up along $X$ to be

$$[E, Y] = \bigcup_{y \in Y} [E_y, \{0\}],$$

with the blow-down map $\beta : [E, Y] \to V$ that is, we simply blow-up the origin of each fiber. The blow-up $[E, Y]$ has a natural $C^\infty$ structure as a manifold with corners. Now we can define the blow-up in Melrose sense formally.

**Definition 2.24** (Blow-up in Melrose sense along a submanifold)**.** *Let $X$ be a smooth manifold with corners and $Y \subset X$ be a closed $p$-submanifold. The blow-up of $X$ along $Y$, denoted $[X, Y]$, is a manifold with corners, given as a point set by*

$$[X, Y] = S(N_X(Y)) \bigsqcup (X \setminus Y),$$

*where $S(N_X(Y))$ represents the inward-pointing part of the spherical normal bundle $N_X(Y)$, i.e, the inward-pointing normal space $N_{X,y}(Y)$ at a point $y \in Y$ is defined as the quotient*

$$N_{X,y}(Y) = T_y X \,/ T_y Y,$$

*and the spherical normal bundle is then given by*

$$S(N_X(Y)) = (N_{X,y}(Y) \setminus \{0\}) \,/ \mathbb{R}^+.$$

*The blow-up $[X, Y]$ has a natural $C^\infty$ structure as a manifold with corners. There is a unique smooth map $[X, Y] \to X$ extending the identity on $X \setminus Y$ called the blow-down map.*

**Example 2.25.** *Blowing-up $\mathbb{R}^n$ along $\mathbb{R}^{n-k}$ in Melrose sense:*

$$\begin{aligned}[\mathbb{R}^n, \mathbb{R}^{n-k}] &= [\mathbb{R}^{n-k} \times \mathbb{R}^k, \mathbb{R}^{n-k} \times \{0\}] \\ &= \mathbb{R}^{n-k} \times [\mathbb{R}^k, \{0\}] \\ &= \mathbb{R}^{n-k} \times S^+(N_{\mathbb{R}^n}(\mathbb{R}^{n-k})) \bigsqcup (\mathbb{R}^k \setminus \{0\}) \\ &= \mathbb{S}^{k-1} \times [0, \infty)_r \times \mathbb{R}^{n-k}.\end{aligned}$$

*By $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we get that*

$$[\mathbb{C}^n, \mathbb{C}^{n-k}] = \mathbb{S}^{2k-1} \times [0, \infty)_r \times \mathbb{C}^{n-k}.$$

*We also have that*

$$[\mathbb{R}^n \times [0, \infty)_\varepsilon, \mathbb{R}^{n-k} \times \{0\}] = \mathbb{S}^k_+ \times [0, \infty)_r \times \mathbb{R}^{n-k},$$

$$[\mathbb{C}^n \times [0, \infty)_\varepsilon, \mathbb{C}^{n-k} \times \{0\}] = \mathbb{S}^{2k}_+ \times [0, \infty)_r \times \mathbb{C}^{n-k}.$$

**Definition 2.26** (Lifting submanifolds)**.** *If $Z \subset X$ is a closed subset of a manifold with corners, and $Y \subset X$ is a closed $p$-submanifold, we shall define the lift of $Z$ to $[X, Y]$ under the blow-down map $\beta : [X, Y] \to X$ in two distinct cases. First, if $Z \subset Y$, then*

$$\beta(Z) = \beta^{-1}(Z).$$

*Secondly, if $Z = \overline{Z \setminus Y}$, then*

$$\beta(Z) = \overline{\beta^{-1}(Z \setminus Y)}.$$

## 2.2 Lie Structures at Infinity

In this section, we introduce the concept of structural Lie algebras of vector fields, which is then employed to define manifolds with a Lie structure at infinity. However, before delving into this, we need to revisit the concept of finitely generated projective $C^\infty(M)$-modules and the Serre-Swan theorem. This theorem serves as a bridge between the geometric notion of vector bundles and the algebraic concept of finitely generated projective $C^\infty(M)$-modules. This equivalence allows us to study and comprehend both vector bundles and finitely generated projective $C^\infty(M)$-modules using a unified framework.

**Definition 2.27** (Geometric fiber)**.** *Let $M$ be a manifold with corners and let $V$ be a $C^\infty(M)$-module with module structure $C^\infty(M) \times V \ni (f, v) \to fv \in V$. For $p \in M$, the ideal of smooth functions on $M$ vanishing at $p$, i.e,*

$$I_pM = \{f \in C^\infty(M) : f(p) = 0\},$$

*is a complex subspace of $V$ and $V/((I_pM)V)$ is called the geometric fiber of $V$ at $p$. In general, geometric fibers are vector spaces with different dimensions.*

Recall from algebra that $S \subset V$ is called a basis for a $C^\infty(M)$-module $V$, if any $v \in V$ could be written uniquely as $v = \sum_{s \in S} f_s s$ such that $f_s \in C^\infty(M)$ and $\{s \in S : f_s \neq 0\}$ is a finite set. A module is called free, if it has a basis. For free modules, geometric fibers have same dimension and this dimension is equal to the cardinality of the basis. A $C^\infty(M)$-module $V$ is called finitely generated projective, if there exist a $C^\infty(M)$-module $W$ such that $V \oplus W$ is free with finite basis. This is equivalent with the concept of locally free $C^\infty(M)$-modules.

**Theorem 2.28** (Serre-Swan). *Let $V$ be a finitely generated projective $C^\infty(M)$-module. Then there exists a natural smooth vector bundle, $E \to M$, and a natural map $\iota : E \to V$ such that*

$$V = \iota_* \Gamma(M, E),$$

*and with the fiber of $E$ above $p \in M$ canonically identified with $V/((I_p M)V)$. Conversely, for any finite rank smooth vector bundle $E \to M$, $\Gamma(M, E)$ is a finitely generated projective $C^\infty(M)$-module.*

We refer to [39] for more details about the Serre-Swan theorem.

**Definition 2.29** (Structural Lie algebra of vector fields). *A structural Lie algebra of vector fields on a manifold $M$ (possibly with corners) is a subspace, $\mathcal{V} \subset \mathfrak{X}(M)$, of the real vector space of vector fields on $M$ with the following properties:*

*(a) $\mathcal{V}$ is closed under Lie brackets.*

*(b) $\mathcal{V}$ is a finitely generated projective $C^\infty(M)$-module.*

*(c) the vector fields in $\mathcal{V}$ are tangent to all faces in $M$.*

**Example 2.30** (Lie algebra of $b$-vector fields). *Let $M$ be a manifold with corners, and*

$$\begin{aligned}\mathcal{V}_b(M) &= \{X \in \mathfrak{X}(M) : X \text{ is tangent to all faces of } M\} \\ &= \{X \in \mathfrak{X}(M) : X\rho_H = a_H \rho_H, a_H \in C^\infty(X), \forall H \in \partial M\},\end{aligned}$$

*where $\rho_H$ is a boundary defining function of the hypersurface $H$. Then $\mathcal{V}_b(M)$ is a structural Lie algebra of vector fields. This is the fundamental object in the theory of Melrose's b-calculus. A vector field $X \in \mathcal{V}_b(M)$ is called a $b$-vector field $X$. In local coordinates near a point $p \in \partial X$ any $b$-vector field $X$ is of the form*

$$X = \sum_{i=1}^{k} a_i(x,y) x_i \partial_{x_i} + \sum_{i=1}^{n-k} b_i(x,y) \partial_{y_i},$$

*where $x_1, \ldots, x_k$ are boundary defining functions, $y \in \mathbb{R}^{n-k}$, $a_i$ and $b_i$ are smooth functions. This shows that the Lie algebra of $b$-vector fields is generated in a neighborhood $U$ of $p$ by $x_j \partial_{x_j}$ and $\partial_{y_j}$ as a $C^\infty(M)$-module. Any structural Lie algebra of vector fields on $M$ is contained in $\mathcal{V}_b(M)$.*

**Example 2.31** (Lie algebra of scattering vector fields). *Let $M$ be a compact manifold with boundary, and let $x : M \to \mathbb{R}_+$ be a boundary defining function. Then the Lie algebra $\mathcal{V}_{\mathrm{SC}}(M) := x\mathcal{V}_b(M)$*

*does not depend on the choice of $x$, and the vector fields in $\mathcal{V}_{\mathrm{SC}}(M)$ are called scattering vector fields. In a local coordinate $(x, y_1, \dots, y_{n-1})$ near a point $p$ any scattering vector field $X$ is of the form*

$$X = a(x,y)x^2\partial_x + \sum_{i=1}^{n-1} x b_i(x,y)\partial_{y_i},$$

*where $a$ and $b_i$ are smooth functions. In fact the Lie algebra of scattering vector fields is generated in a neighborhood $U$ of $p$ by $x^2\partial_x$ and $x\partial_{y_i}$ as a $C^\infty(M)$-module.*

**Example 2.32** (Lie algebra of edge vector fields)**.** *Let $M$ be a manifold with boundary $\partial M$, which is the total space of a fibration $\pi : \partial M \to B$ of smooth manifolds. We let*

$$\mathcal{V}_e(M) = \{X \in \mathfrak{X}(M) : X \text{ is tangent to all fibers of } \pi \text{ at the boundary}\}$$

*be the space of edge vector fields. Clearly, $\mathcal{V}_e(M)$ is closed under the Lie bracket. This is the fundamental object in the theory of Mazzeo's edge calculus. If $(x, y, z)$ are coordinates in a local product decomposition near the boundary, where $x$ corresponds to the boundary-defining function, $y$ corresponds to a set of variables on the base $B$ lifted to $\partial M$ through $\pi$, and $z$ is a set of variables in the fibers of $\pi$, then edge vector fields are generated by $x\partial_x$, $x\partial_y$, and $\partial_z$. In other words, any $X \in \mathcal{V}_e(M)$ can be expressed locally as*

$$X = a(x,y,z)x\partial_x + \sum_{i=1}^{b} b_i(x,y,z)x\partial_{y_i} + \sum_{i=1}^{f} c_j(x,y,z)\partial_{z_i},$$

*where $a, b_i, c_j \in C^\infty(M)$.*

**Example 2.33.** *As a particular case of edge vector fields, the fibration $\mathrm{id} : \partial M \to \partial M$ for a manifold with boundary $M$ yields the Lie algebra of $0-$vector fields in the $0$-calculus of Mazzeo-Melrose in [46], i.e,*

$$\mathcal{V}_0(M) = \{X \in \mathfrak{X}(M) : X\big|_{\partial M} = 0\} = x\mathfrak{X}(M),$$

*where $x$ is a boundary-defining function.*

**Proposition 2.34.** *If $\mathcal{V}$ is a structural Lie algebra of vector fields, then $\mathcal{V}$ is a finitely generated projective $C^\infty(M)$-module. So there exists a vector bundle $E$ such that $\mathcal{V} = \Gamma(M, E)$ and a natural vector bundle map $\varrho_{\mathcal{V}} : E \to TM$ such that the induced map $\varrho_\Gamma : \mathcal{V} \to \mathfrak{X}(M)$ identifies with the inclusion map.*

See Proposition 2.12 in [2] for a proof.

Now, let us recall the definition of a Lie algebroid. In a general sense, a Lie algebroid can be viewed as the multi-object version of a Lie algebra.

**Definition 2.35** (Lie algebroid)**.** *A Lie algebroid $\mathcal{A}$ over a manifold $M$ is a vector bundle $\mathcal{A}$ over $M$, together with a Lie algebra structure on the space $\Gamma(\mathcal{A})$ of smooth sections of $\mathcal{A}$ and a bundle map $\varrho : \mathcal{A} \to TM$, extended to a map $\varrho_\Gamma : \Gamma(\mathcal{A}) \to \Gamma(TM)$ between sections of these bundles, such that the right Leibniz rule is also satisfied*

$$[X, fY] = f[X, Y] + (\varrho_\Gamma(X)f)Y, \quad \forall X, Y \in \Gamma(A),\ f \in C^\infty(M).$$

*The map $\varrho_\Gamma$ is called the anchor of $\mathcal{A}$.*

**Remark 2.36.** *By the antisymmetry of the bracket, the left Leibniz rule is also satisfied:*

$$[fX, Y] = f[X, Y] - (\varrho_\Gamma(X)f)Y, \quad \forall X, Y \in \Gamma(A),\ f \in C^\infty(M).$$

**Remark 2.37.** *By direct calculation as Proposition 1.21 in [3] one can check that the anchor map is a morphism of Lie algebras. In other words,*

$$\varrho_\Gamma([X, Y]) = [\varrho_\Gamma(X), \varrho_\Gamma(Y)], \quad \forall X, Y \in \Gamma(A),$$

*where on the left, we have the Lie algebroid bracket, and on the right, we have the Lie algebra bracket of vector fields.*

**Example 2.38.** *Examples of Lie algebroid*

*(a) All Lie algebras are Lie algebroids. In fact, a Lie algebroid over a one-point set, with the zero anchor, is a Lie algebra.*

*(b) Any bundle of Lie algebras is a Lie algebroid with zero anchor and Lie bracket defined pointwise.*

*(c) The tangent bundle $TM$ of a manifold $M$, with as bracket the Lie bracket of vector fields and with as anchor the identity of $TM$, is a Lie algebroid over $M$ which is called tangent Lie algebroid.*

*(d) Given the action of a Lie algebra $\mathfrak{g}$ on a manifold $M$ that is, a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{X}(M)$, the action algebroid is the trivial vector bundle $\mathfrak{g} \times M \to M$, with the anchor given by the Lie algebra action and brackets uniquely determined by the bracket of $\mathfrak{g}$ on constant sections $M \to \mathfrak{g}$ and by the Leibniz identity.*

**Definition 2.39** (Lie structure at infinity)**.** *A Lie structure at infinity on a manifold $M$ is a pair $(M, \mathcal{V})$, where $M$ is a compact manifold with corners and $\mathcal{V}$ is a structural Lie algebra of vector fields on $M$ such that its anchor $\varrho_\mathcal{V}$ : ${}^\mathcal{V}TM \to TM$ is an isomorphism on $M^\circ$, i.e,${}^\mathcal{V}TM|_{TM^\circ} \cong TM^\circ$.*

**Remark 2.40.** *If $M^\circ$ is compact without boundary, then it follows from the Definition 2.39 that $M = M^\circ$ and ${}^{\mathcal{V}}TM = TM$, so a Lie structure at infinity on $M^\circ$ gives no additional information on $M^\circ$. The interesting cases are thus the ones when $M^\circ$ is noncompact.*

**Definition 2.41** (Riemannian manifold with a Lie structure at infinity)**.** *Let $(M, \mathcal{V})$ be a Lie structure at infinity for a manifold with corners $M$. Let $\varrho_{\mathcal{V}} : {}^{\mathcal{V}}TM \to TM$ be the associated anchor and $g$ a Riemannian metric on ${}^{\mathcal{V}}TM$, that is, a smooth positive definite symmetric 2-tensor $g$ on ${}^{\mathcal{V}}TM$. In this case, $(M^\circ, (\varrho_{\mathcal{V}}^{-1})^*(g\big|_{M^\circ}))$ is called a Riemannian manifold with a Lie structure at infinity.*

Riemannian manifold with a Lie structure at infinity have some nice geometric property, for instance the following proposition shows that the volume of any noncompact Riemannian manifold with a Lie structure at infinity is infinite.

**Proposition 2.42.** *Let $M^\circ$ be a Riemannian manifold with Lie structure $(M, \mathcal{V}, g)$ at infinity. Let $f \geq 0$ be a smooth function on $M$. If $\displaystyle\int_{M^\circ} f \,\mathrm{dvol}_g < \infty$, then $f$ vanishes on each boundary hyperface of $M$. In particular, the volume of any noncompact Riemannian manifold with a Lie structure at infinity is infinite.*

See Proposition 4.1.in [2] for a proof.

**Proposition 2.43.** *Let $M^\circ$ be a Riemannian manifold with a Lie structure $(M, \mathcal{V}, g)$ at infinity. Then $M^\circ$ is complete in the induced metric $g$.*

See Corollary 4.9. in [2] for a proof.

**Proposition 2.44.** *Let $M^\circ$ be a connected Riemannian manifold with a Lie structure $(M, \mathcal{V}, g)$ at infinity. Then $(M^\circ, g)$ is of bounded geometry.*

See Corollary 4.3 in [2] and Theorem 5.2. in [11] for a proof.

**Definition 2.45** (Melrose $b$-metric)**.** *Let $M$ be a compact Riemannian manifold with boundary, equipped with a Lie structure at infinity $(M, \mathcal{V}_b, g_b)$. The Riemannian metric $g_b$ is referred to as the $b$-metric.*

**Example 2.46** (Manifold with asymptotically cylindrical end)**.** *A manifold with cylindrical ends is a Riemannian manifold $(M, g)$ for which there exists a compact subset $K$ (topologic part) such that outside*

*$K$, $M$ resembles a cylinder with the product metric. This can be expressed through the identification:*

$$M \setminus K \cong N \times (0, \infty)_r,$$

*where $N$ is a closed manifold with $\dim N = \dim M - 1$ and a cylindrical metric on $N \times (0, \infty)_r$, i.e,*

$$g\big|_{M\setminus K} = g_{\text{cyl}} = g_N + dr^2,$$

*where $g_N$ is a metric on $N$ and $r \in (0, \infty)$ is a coordinate for $(0, \infty)$. Let $(M, \mathcal{V}_b)$ be a compact Riemannian manifold with boundary $M$, equipped with a Lie structure at infinity. By using a tubular neighborhood of $N = \partial M$ in $M$ to make*

$$M^\circ \setminus K \cong \partial M \times (0, \infty)_r.$$

*So, we have the cylindrical end with the cylindrical metric*

$$g\big|_{M\setminus K} = g_{\partial M} + dr^2.$$

*By attaching $\partial M$ at infinity, we obtain a compactification $M := M^\circ \cup \partial M$, which is a compact manifold with boundary. Introducing the change of variable $x = e^{-r}$ where $r$ is the coordinate for $(0, \infty)_r$, yields a defining function for $\partial M$, so we can write this cylindrical metric as*

$$g\big|_{M\setminus K} = g_{\partial M} + \frac{dx^2}{x^2} \in \Gamma(S^2({}^bT^*M)),$$

*which is compatible with the Lie structure at infinity $\mathcal{V}_b$, i.e, it is a $b-$metric.*
*More generally, we say $(M, g)$ has asymptotically cylindrical end if $g \to g_{\partial M} + \dfrac{dx^2}{x^2}$ when $x \to 0$ in the follwoing sense: there exist $\gamma > 0$ such that*

$$g - (g_{\partial M} + \frac{dx^2}{x^2}) \in x^\gamma C_b^\infty(M, S^2({}^bT^*M)) = x^\gamma C_b^\infty(M) \otimes_{C^\infty(M)} \Gamma(S^2({}^bT^*M)),$$

*where*

$$C_b^\infty(M^\circ) = \{f \in C^\infty(M^\circ) : \forall k \in \mathbb{N}_0, \{V_1, \ldots, V_k\} \subset \mathcal{V}_b(M), \sup_{M^\circ} |V_1 \ldots V_k f| < \infty\},$$

*and*

$$x^\gamma C_b^\infty(M^\circ) = \{f \in C^\infty(M^\circ) : \frac{f}{x^\gamma} \in C_b^\infty(M^\circ)\}.$$

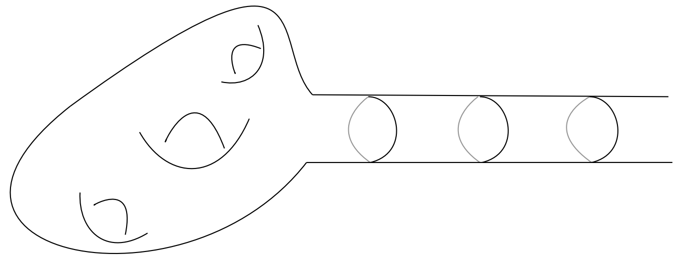


Figure 2.4: A manifold with a cylindrical end

**Definition 2.47** (Melrose $\mathrm{SC}$-metric)**.** *A Riemannian metric compatible with a Lie structure at infinity $(M, \mathcal{V}_{\mathrm{SC}})$, where $M$ is a compact manifold with boundary, is called scattering metric ($\mathrm{SC}$-metric for short).*

**Example 2.48.** *Let $M$ be a compact manifold with boundary. A metric of the form*

$$g_{\mathrm{SC}} = \frac{g_{\partial M}}{x^2} + \frac{dx^2}{x^4},$$

*close to the boundary $\partial M$, is an example of $\mathrm{SC}$-metric, where $x$ is a defining function for the boundary, and $g_{\partial M}$ is the Riemannian metric $g$ restricted to the boundary. Notice that a $\mathrm{SC}$-metric is always of the form $g_{\mathrm{SC}} = \frac{g_b}{x^2}$ for some $b$-metric.*

**Example 2.49.** *A simple example in the Euclidean case is the radial compactification of $\mathbb{R}^n$ with the boundary being the sphere $\mathbb{S}^{n-1}$. This compactification is given by the stereographic projection $\mathrm{SP}$ defined by*

$$\mathrm{SP} : \mathbb{R}^n \to \mathbb{S}^n_+ := \{z = (z_0, ..., z_n) \in \mathbb{S}^n : \; z_0 \geq 0\},$$

$$\mathrm{SP}(x) = \frac{1}{\sqrt{1+|z|^2}} (1, z_1, ..., z_n),$$

*where $\mathbb{S}^n_+$ is a compact manifold with boundary. $\mathrm{SP}$ identifies $\mathbb{R}^n$ with the interior of the upper half-sphere $\mathbb{S}^n_+$. The Euclidean metric is a scattering metric on $\mathbb{R}^n$ given by*

$$g_{\mathbb{R}^n} = dr^2 + r^2 g_{\mathbb{S}^{n-1}} = \frac{dx^2}{x^4} + \frac{g_{\mathbb{S}^{n-1}}}{x^2},$$

*where $r = |z|$ and $x = \frac{1}{r}$ is a defining function for the boundary i.e. $\partial\mathbb{S}^n_+ = \mathbb{S}^{n-1} = \{x = 0\}$.*

**Example 2.50** (Manifold with asymptotically conical end)**.** *A manifold with conical ends is a Riemannian manifold $(M, g)$ for which there exists a compact subset $K$ (topologic part) such that outside $K$,*

$(M, g)$ *is a cone. This can be expressed through the identification:*

$$M \setminus K \cong N \times (R, \infty)_r, \quad \text{for some } R > 0,$$

*where* $N$ *is a closed manifold with* $\dim N = \dim M - 1$ *and a concic metric on* $N \times (R, \infty)_r$*, i.e,*

$$g\big|_{M\setminus K} = g_{\textit{cone}} = r^2 g_N + dr^2, r \geq R,$$

*where* $g_N$ *is a metric on* $N$ *and* $r \in (0, \infty)$ *is a coordinate for* $(0, \infty)$*. If* $M$ *is a compact manifold with boundary, by using a tubular neighborhood of* $N = \partial M$ *in* $M$*, we can find an identification*

$$M^\circ \setminus K \cong \partial M \times (R, \infty)_r,$$

*so that a conical metric* $g$ *on* $M^\circ$ *with* $g\big|_{M^\circ\setminus K} = r^2 g_{\partial M} + dr^2$ *is a* $\mathrm{SC}$*-metric. Indeed, attaching this cone to* $\partial M$ *at infinity, we obtain a compactification* $M := M^\circ \cup \partial M$*, which is a compact manifold with boundary. Introducing the change of variable* $x = \dfrac{1}{r}$ *where* $r$ *is the coordinate for* $(0, \infty)_r$*, yields a defining function for* $\partial M$*, so we can write this conical metric as*

$$g\big|_{M\setminus K} = \frac{g_{\partial M}}{x^2} + \frac{dx^2}{x^4} \in \Gamma(S^2({}^{\mathrm{SC}}T^*M)),$$

*which is compatible with the Lie structure at infinity* $\mathcal{V}_{\mathrm{SC}}$*, i.e, it is a* $\mathrm{SC}$ *−metric.*

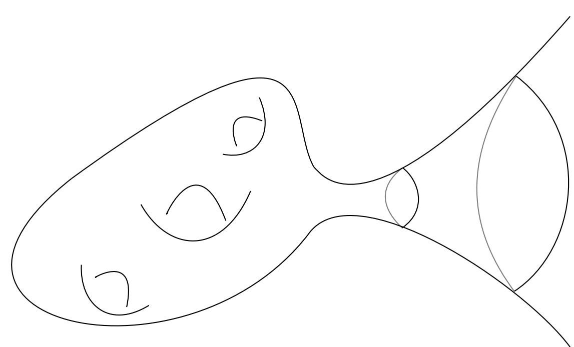


Figure 2.5: A manifold with a conical end

**Definition 2.51** (Asymptotically Conical Metric)**.** *Let* $(L, g)$ *be a compact Riemannian manifold. On* $C = (0, \infty) \times L$*, consider the conic metric*

$$g_{\textit{cone}} = dr^2 + r^2 g,$$

*where $r$ is the coordinate on $(0,\infty)$. A Riemannian manifold $(M,g)$ is called asymptotically conical (AC for short), asymptotic to $g_{\text{cone}}$, if there exists a diffeomorphism $\pi : M\setminus K \to (R,\infty)\times L$, for some $R>0$ and $K\subset M$ compact, there is a positive constants $c$ and $\mu$ such that for any $k\geq 0$,*

$$|\nabla^k(\pi_*(g)-g_{\text{cone}})|_{g_{\text{cone}}} \leq \frac{c}{r^{\mu+k}}.$$

*Here, $\nabla$ denotes the Levi-Civita connection of $g_{\text{cone}}$.*

Now, we define asymptotically locally Euclidean metrics, which are important examples of asymptotically conical metrics. In fact when $g_{\text{cone}}$ is a quotient of the euclidean flat space by a finite subgroup of the orthogonal matrices which acts freely on the unit sphere, the corresponding AC metrics are called asymptotically locally euclidean or ALE for short.

**Definition 2.52** (Asymptotically Locally Euclidean metric). *Let $\Gamma$ be a finite subgroup of $U(n)$ acting freely on $\mathbb{C}^n\setminus 0$, so $\mathbb{C}^n/\Gamma$ has an isolated quotient singularity at $0$, and the Euclidean metric is $\Gamma$-invariant. Thus, $(\mathbb{C}^n/\Gamma, g_{\mathrm{Euc}})$ is a Riemannian cone. Let $M$ be a non-compact complex manifold with end asymptotic to the cone $\mathbb{C}^n/\Gamma$ at infinity (e.g. a resolution of $\mathbb{C}^n/\Gamma$), i.e., there is a compact subset $K\subset M$ and a map $\pi : M\setminus K\to \mathbb{C}^n/\Gamma$ that is a diffeomorphism between $M\setminus K$ and $\{z\in\mathbb{C}^n/\Gamma : d_{\mathrm{Euc}}(z,0)>R\}$ for a fixed positive constant $R$. A Riemannian (Kähler ) metric $g$ on $M$ is called asymptotically locally Euclidean (ALE-for short) if $\pi_*(g)$ is asymptotic to $g_{\mathrm{Euc}}$ at infinity, i.e., there is a positive constant $c$ such that for any $k\geq 0$,*

$$|\nabla^k(\pi_*(g)-g_{\mathrm{Euc}})| \leq \frac{c}{r^{n+k}},$$

*where $\nabla$ is the Levi-Civita connection of $g_{\mathrm{Euc}}$ on $\mathbb{C}^n/\Gamma$.*

**Example 2.53** (Burns and Simanca ALE scalar-flat Kähler metrics on $\mathcal{O}_{\mathbb{CP}^{m-1}}(-1)$). *In 1991 Burns (case $m=2$) and Simanca [55] (case $m>1$) constructed a cscK metric on $\mathcal{O}_{\mathbb{CP}^{m-1}}(-1)$. They showed that the Kähler potential of this scalar flat Kähler metric on $\mathcal{O}_{\mathbb{CP}^{m-1}}(-1)$ is radially symmetric and of the form*

$$H_{BS} = \|Z\|^2 + \gamma(\|Z\|)\log\|Z\|^2 + \|Z\|^{4-2m} + \psi(\|Z\|^2),$$

*where $\gamma:\mathbb{R}\to\mathbb{R}$ is the cut-off function such that $\gamma(t)=1$ for $t<1$, $\gamma(t)=0$ for $t>2$ and*

$$|\nabla^k\psi(t)| \leq \frac{c}{t^{m+k-2}},$$

*for all $k\geq 0$. Here $\nabla$ is the Levi-Civita connection of $g_{FS}$ on $\mathcal{O}_{\mathbb{CP}^{m-1}}(-1)$.*

More generally, for any natural number $r$, as discussed in section 2 of [4], the line bundle $\mathcal{O}_{\mathbb{CP}^{m-1}}(-r)$ admits an ALE scalar-flat Kähler metric as follows.

**Example 2.54** (ALE scalar-flat Kähler metric on $\mathcal{O}_{\mathbb{CP}^{m-1}}(-r)$)**.** *The Burns-Simanca metric is generalized by Eguchi-Hanson [27] ($m = 2$, $r = 2$), LeBrun [40] ($m = 2$, $r > 2$), Pedersen-Poon [49] and Rollin-Singer [51] ($m > 2$, $r > 2$) . In summary, the Burns-Eguchi-Hanson-LeBrun-Pedersen-Poon-Simanca metric on $\mathcal{O}_{\mathbb{CP}^{m-1}}(-r)$ has Kähler potential:*

$$H = \frac{1}{2}\|Z\|^2 + A\|Z\|^{4-2m} + O(\|Z\|^{3-2m}),$$

*when $m \geq 3$ and*

$$H = \frac{1}{2}\|Z\|^2 + A\log\|Z\| + O(\|Z\|^{-2}),$$

*when $m = 2$, where $A$ is some constant. Recall that $T = O(t^s)$ if $|T| \leq ct^s$ for some $c > 0$.*

**Remark 2.55.** *There is a generalization of asymptotically locally Euclidean metrics (ALE-metrics for short) to quasi asymptotically locally Euclidean metrics (QALE-metrics for short) by D. Joyce in [36] when the action of $\Gamma$ is not necessarily free on $\mathbb{C}^n \setminus \{0\}$.*

**Remark 2.56.** *There is a generalization of AC-metrics and QALE-metrics to quasi-asymptotically conical metrics (QAC-metrics for short) by Degeratu and Mazzeo [23] that we will discuss and use in Chapter 4 for our main construction.*

As a summary, we have these relations between these special metrics

$$\begin{array}{ccccccc} \text{AE} \subset & \text{ALE} & \subset & \text{AC} & \subset \mathrm{SC} \\ & \cap & & \cap & \\ & \text{QALE} & \subset & \text{QAC} & \end{array}$$

We finish this chapter by defining the edge-metric in Mazzeo's sense.

**Definition 2.57** (Mazzeo edge metric)**.** *A Riemannian metric compatible with a Lie structure at infinity $(M, \mathcal{V}_e)$, where $M$ is a compact manifold with fibered boundary is called and edge metric. An edge metrics $g_e$ close to the boundary $\partial M$ is an element of $S^2({}^eT^*M)$ which locally generated by*

$$\{\frac{dx^2}{x^2}, \frac{dy_i^2}{x^2}, \frac{dx\otimes dy_i}{x^2}, dz_j^2, \frac{dx\otimes dz_j}{x}, \frac{dy_i\otimes dz_j}{x}\},$$

*in terms of the local coordinates of example 2.32.*

**Example 2.58** ($0-$metrics)**.** *An interesting class of a edge metrics is the class of $0$-metrics, i.e., metrics corresponding to the Lie structure at infinity $(M, \mathcal{V}_0)$ as discussed in Example 2.33. A $0$-metric $g_0$ close to the boundary $\partial M$ is of the form*

$$g_0 = \frac{dx^2}{x^2} + \frac{dy_i^2}{x^2}.$$

*In fact, if $(M, g)$ is a compact Riemann manifold with boundary, and $x$ is a defining function for the boundary, then the metric in the interior of $M$,*

$$g_0 = \frac{g}{x^2},$$

*is complete and is an example of $0-$metric. In particular, the hyperbolic space is of this type. The sectional curvature of $g_0$ approaches $-|dx|_g^2$ at the boundary, so $g_0$ has negative curvature outside a compact set. For more information, see Lemma 2.5 in [46].*

# CHAPTER 3
# CONSTANT SCALAR CURVATURE KÄHLER METRICS

In this chapter we focus on constant scalar curvature Kähler (cscK) metrics. A special case of a constant scalar curvature Kähler metric is a Kähler-Einstein (KE) metric, which has been the main focus of Kähler geometry since the inception of the celebrated Calabi conjecture on the existence of canonical Kähler metrics in the 1950s: In every Kähler Class of every compact Kähler manifolds, there must exists one best, canonical Kähler metrics.

In fact, Calabi proposed the following conjectures for an compact Kähler manifold $(X, \omega_X)$:

**Conjecture 1:** If $\mathrm{Aut}(X) = 1$, then there exists a unique cscK metric on $X$ in $[\omega_X]$.

**Conjecture 2:** There exists an extremal Kähler metric on $X$ in $[\omega_X]$, unique up to $\mathrm{Aut}(X)$.

Calabi's vision, now six decades later, has been the inspiration for fundamental work in Kähler geometry up to the present day. From Yau's celebrated theorem [64], based on Calabi's $C^3$ estimate for the Monge-Ampère equation in 1958 [13], for which he received the Fields Medal in 1976, to the conjecture of Yau-Tian-Donaldson in the Kähler-Einstein Fano case that was finally solved in 2012 by Chen, Donaldson, and Sun [18, 19, 20] and Tian [59].

We begin this chapter with the scalar curvature function and the definition of extremal metrics. Then, we briefly look at Kähler-Einstein metrics and Conjecture 1. Following that, we study cscK metrics and discuss classic results by Matsushima-Lichnerowicz and Arezzo-Pacard. Finally, we wrap up this chapter by constructing new examples of cscK orbifolds with singularities of type $\mathcal{I}$.

## 3.1 Scalar Curvature Function

**Lemma 3.1.** *Let $(M, g)$ be a Kähler manifold and $D$ denote its Levi-Civita connection. For a real 1-form $\alpha$ if we denote by $D^-\alpha$ the J-anti-invariant part of the covariant derivative $D\alpha$, then we have*

$$D^-\alpha = -\frac{1}{2} g(J(\mathcal{L}_{\alpha^\sharp} J)\bullet, \bullet) = -\frac{1}{2}\omega(J(\mathcal{L}_{\alpha^\sharp} J)\bullet, \bullet).$$

See Lemma 1.22.2. [29] for a proof.

**Definition 3.2** (Lichnerowicz operator)**.** *Let $(M, g)$ be a Kähler manifold and $D$ the Levi-Civita connection. If we set $\mathcal{D} = D^- d$ and denote by $\mathcal{D}^* = (D^- d)^*$ its formal adjoint, then the fourth-order* *operator $\mathcal{D}^*\mathcal{D} : C^\infty(M,\mathbb{C}) \to C^\infty(M,\mathbb{C})$ is called the Lichnerowicz operator. The Lichnerowicz operator $\mathcal{D}^*\mathcal{D}$ is a formally self-adjoint, semipositive differential operator acting on (real) functions. For instance for complex-valued functions $\phi$, $\psi$ defined on compact Kähler manifold $(M,g)$ we have*

$$\int_M (\mathcal{D}^*\mathcal{D}\phi)\overline{\psi}\omega^n = \int_M \phi\overline{\mathcal{D}^*\mathcal{D}\psi}\omega^n.$$

**Lemma 3.3** (Lichnerowicz operator)**.** *Suppose that $(M,g)$ is a Kähler manifold and $u$ is a smooth complex-valued function defined on $M$. Then*

$$\begin{aligned}\mathcal{D}^*\mathcal{D}u &= \frac{1}{2}\Delta_g^2 u + \mathrm{Ric}_g . \nabla^2 u + \frac{1}{2}\nabla S(\omega).\nabla u \\ &= \frac{1}{2}\Delta_g^2 u + R^{\bar{j}i}\nabla_i\nabla_{\bar{j}}u + \frac{1}{2}g^{i\bar{j}}\nabla_i S(\omega).\nabla_{\bar{j}}u.\end{aligned}$$

See Lemma 1.22.5. [29] for a proof.

**Lemma 3.4.** *Let $M$ be a Kähler manifold with Kähler metric $g$ and corresponding Kähler form $\omega$. Assume that $\omega + \sqrt{-1}\partial\bar{\partial}u$ is a small perturbation of $\omega$ by a function $u \in C^4(M)$ with $\|u\|_{C^4(M)} < c$, for a sufficiently small $c > 0$. Then we can linearize the scalar curvature operator in the following way:*

$$S(\omega + \sqrt{-1}\partial\bar{\partial}u) = \sum_{k=0}^{+\infty} \frac{d^k}{dt^k}|_{t=0} S(\omega + t\sqrt{-1}\partial\bar{\partial}u) = S(\omega) + L_\omega(u) + Q_\omega(\nabla^2 u),$$

*where $L_\omega(u) = \frac{d}{dt}|_{t=0}S(\omega + t\sqrt{-1}\partial\bar{\partial}u)$ is the linear part and*

$$Q_\omega(\nabla^2 u) = \sum_{k=2}^{+\infty} \frac{d^k}{dt^k}|_{t=0} S(\omega + t\sqrt{-1}\partial\bar{\partial}u),$$

*is a second-order non-linear differential operator that collects all the non-linear terms. Moreover, the linearization of the scalar curvature operator $L_\omega(u)$ can be expressed as:*

$$L_\omega(u) = -(\frac{1}{2}\Delta_g^2 u + \langle \mathrm{Ric}_g, \sqrt{-1}\partial\bar{\partial}u\rangle_g) = -\frac{1}{2}\Delta_g^2 u - R^{\bar{j}i}\partial_i\partial_{\bar{j}}u = \frac{1}{2}\nabla S(\omega).\nabla u - \mathcal{D}^*\mathcal{D}u.$$

*Also, the non-linear part $Q_\omega$ could decomposes with finite sums as follows:*

$$\begin{aligned}Q_\omega(\nabla^2 u) = &\sum_q B_{q,4,2}(\nabla^4 u, \nabla^2 u)C_{q,4,2}(\nabla^2 u) \\ &+ \sum_q B_{q,3,3}(\nabla^3 u, \nabla^3 u)C_{q,3,3}(\nabla^2 u) \\ &+ |z|\sum_q B_{q,3,2}(\nabla^3 u, \nabla^2 u)C_{q,3,2}(\nabla^2 u) \\ &+ \sum_q B_{q,2,2}(\nabla^2 u, \nabla^2 u)C_{q,2,2}(\nabla^2 u),\end{aligned}$$

*where $B^i$s are bilinear forms and $C^i$s are smooth functions.*

See Lemma 2.158 [10], Equation (31) in [5], Lemma 1.2 and Lemma 1.3 in [41] for a proof.

**Lemma 3.5.** *Let $M$ be a Kähler manifold of real dimension $n$ with Kähler metric $g$ and corresponding Kähler form $\omega$. The scalar curvature of a conformally changed metric $\omega' = e^{2f}\omega$ can be computed by:*

$$S(\omega') = e^{-2f}(S(\omega) + 2(n-1)\Delta_\omega f - (n-1)(n-2)\|\nabla f\|^2_\omega).$$

See Section 1J in [10] for a proof.

## 3.2 Extremal Metrics

Extremal metrics were defined by Calabi [14] in 1982 as follows:

**Definition 3.6** (Extremal Metrics)**.** *Suppose that $M$ is a compact Kähler manifold. An extremal Kähler metric on $M$ in the class $\Omega \in H^2_{\mathrm{dR}}(M, \mathbb{R})$ is a critical point of the functional*

$$\mathrm{Cal}(\omega) = \int_M S(\omega)^2 \omega^n,$$

*for $\omega \in \Omega$, where $S(\omega)$ is the scalar curvature of $\omega$. This functional is called the Calabi energy functional.*

**Theorem 3.7.** *A Kähler metric $\omega$ on compact Kähler manifolds $M$ is extremal if and only if $S(\omega)$ is a Killing potential, i.e, one of the following equivalent conditions holds:*

*(a)* $\nabla^{1,0}S(\omega)$ *is a holomorphic vector field.*

*(b)* $\mathcal{D}^*\mathcal{D}S(\omega) = 0$. *(Here, $\mathcal{D}^*\mathcal{D}$ is the Lichnerowicz operator defined in Definition 3.2 on page 70.)*

See Lemma 1.23.2 in [29] or Theorem 4.2. in [57] for a proof.

**Example 3.8.** *The most important examples of extremal metrics are cscK metrics. In particular Kähler-Einstein metrics have constant scalar curvature, so they are examples of extremal metrics, i.e,*

*KE $\subset$ cscK $\subset$ Extremal.*

**Remark 3.9.** *Suppose that $M$ is a compact Kähler manifold.*

*(a) If $\mathfrak{h}(M) = 0$, i.e, $M$ admits no non-trivial holomorphic vector fields, then every extremal Kähler metric must have constant scalar curvature.*

*(b) If the Kähler class $\Omega$ is proportional to $c_1(M)$, then any constant scalar curvature metric in $\Omega$ is Kähler-Einstein.*

See Lemma 2.2.3 in [28] for a proof.

Now we discuss the **Conjecture 1**. The existence of Kähler–Einstein metrics for compact Kähler manifolds depends on the sign of the first Chern class of the Kähler manifold.

**Theorem 3.10** (Yau). *Let $M$ be a compact Kähler manifold with $c_1(M) = 0$. Then, every Kähler class contains a unique Ricci flat metric. These types of manifolds are called Calabi-Yau. Calabi–Yau manifolds are complex manifolds that generalize K3 surfaces to higher dimensions.*

This is just a special case of the theorem 1.112 on page 34. The case when the first Chern class is negative is proved independently in 1978 by Thierry Aubin [8] and Shing-Tung Yau [64] as follows:

**Theorem 3.11** (Aubin-Yau). *Let $M$ be a compact Kähler manifold with $c_1(M) < 0$. Then, there is a unique Kahler metric $\omega \in -2\pi c_1(M)$ such that $\mathrm{Ric}(\omega) = -\omega$.*

When the first Chern class is positive, existence of Kähler-Einstein metrics remained a well-known open problem for many years. In this case, there are a non-trivial obstructions to existence. In 2012, Xiuxiong Chen, Simon Donaldson, and Song Sun [18],[19], [20] as well as Tian [59] proved that in this case existence is equivalent to an algebro-geometric property called K-stability.

## 3.3 Constant Scalar Curvature Kähler Metrics

Now we state Matsushima - Lichnerowicz theorem. This classical theorem gives us obstructions to the existence of cscK metric based on the structure of the Lie algebra of holomorphic vector fields.

**Theorem 3.12** (Matsushima - Lichnerowicz). *Let $(M, J, g)$ be a cscK manifold. Then, the Lie algebra $\mathfrak{h}(M)$ of holomorphic vector fields decomposes as a direct sum:*

$$\mathfrak{h}(M) = \mathfrak{h}_0(M) \oplus \mathfrak{a}(M),$$

*where $\mathfrak{a}(M) \subset \mathfrak{h}(M)$ is the abelian subalgebra of parallel holomorphic vector fields and $\mathfrak{h}_0(M)$ is the subalgebra of holomorphic vector fields with zeros. Furthermore, $\mathfrak{h}_0(M)$ is the complexification of the killing fields with zeros, i.e,*

$$\mathfrak{h}_0(M) = (\mathfrak{k}(M, g) / \mathfrak{a}(M)) \otimes_{\mathbb{R}} \mathbb{C},$$

*where $\mathfrak{k}(M, g)$ denotes the Lie algebra of real Killing vector fields on $(M, g)$. In particular, $\mathfrak{h}(M)$ is a reductive Lie algebra, i.e, it is the direct sum of an abelian and a semisimple Lie algebra.*

See [43] and [45] for a proof.

**Corollary 3.13.** *Let $(M, J, g)$ be a cscK manifold. Then the identity component of $\mathrm{Iso}(M, g, J)$ is the maximal compact subgroup of the identity component $\mathrm{Aut}(M, J)$.*

**Corollary 3.14.** *The theorem of Lichnerowicz and Matsushima implies that a compact Kähler manifold $(M, J)$ whose identity component $\mathrm{Aut}_0(M, J)$ of the automorphism group is not reductive does not admit any cscK metric.*

**Example 3.15.** *Let $n > 1$, then the projective space $\mathbb{CP}^n$ blown-up at one or two points does not admit any cscK metric. See [10] page 331 for more details.*

**Remark 3.16.** *There is a general version of the Matsushima-Lichnerowicz theorem for extremal metrics proved by Calabi (Theorem 2.3.6 in [28]). In particular, a compact complex manifold $(M, J)$ for which the connected group of automorphisms is non-trivial but has no connected compact subgroup apart from $\{\mathrm{Id}\}$ cannot have any extremal Kähler metric. Examples of Kähler compact complex surfaces satisfying these hypotheses, hence admitting no extremal Kähler metric, were first given by M. Levine [42]. As a consequence, the answer of* ***Conjecture 2*** *is negative in general. A Kähler manifold is called a* ***Calabi dream manifold*** *if every Kähler class on it admits an extremal metric. All compact Riemann surfaces, complex projective spaces $\mathbb{CP}^n$, Hirzebruch surfaces $\mathbb{F}_k \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$, and all compact Calabi-Yau manifolds [64] are Calabi dream manifolds.*

**Proposition 3.17** (Calabi). *Let $g$ be an extremal Kähler metric on a compact complex manifold $(M, J)$. Let $\mathrm{Iso}_0(M, g)$ denote the identity component of the isometry group of $(M, g)$, and let $\mathrm{Aut}_0(M, J)$ denote the identity component of the biholomorphism group of $(M, J)$. Then $\mathrm{Iso}_0(M, g)$ is a maximal compact subgroup of $\mathrm{Aut}_0(M, J)$.*

See Theorem 3.5.1 [29] for a proof.

**Proposition 3.18.** *Let $(M, g)$ be a compact cscK manifold and let $L_\omega(u) = -\mathcal{D}^*\mathcal{D}u$ be the linearization of the scalar curvature operator in Lemma 3.4, then*

$$\dim_{\mathbb{R}}(\ker(L_\omega)) = \dim_{\mathbb{C}}(\mathfrak{h}_0(M)) + 1.$$

*In particular, when the identity component of the biholomorphism group $\mathrm{Aut}(M, J)$ is discrete, $\ker(L_\omega)$ consists only of constant functions.*

See Proposition 1 in [41] for a proof.

Now, we recall some notable results of Arezzo-Pacard [5] and [6] that provide a vast collection of cscK manifolds.

**Theorem 3.19** (Arezzo-Pacard)**.** *Let $(M, \omega)$ be a constant scalar curvature compact Kähler manifold or Kähler orbifold of complex dimension $m$ with isolated singularities. Assume that there is no nonzero holomorphic vector field vanishing somewhere on $M$. Then, given finitely many smooth points $p_1, \ldots, p_n$ in $M$ and positive numbers $a_1, \ldots, a_n > 0$, there exists $\varepsilon_0 > 0$ such that the blow-up of $M$ at $p_1, \ldots, p_n$ carries constant scalar curvature Kähler forms*

$$\omega_\varepsilon \in \pi^*[\omega] - \epsilon^2 \left( a_1^{\frac{1}{m-1}}[E_1] + \ldots + a_n^{\frac{1}{m-1}}[E_n] \right),$$

*where $[E_i]$ are the Poincaré duals of the $(2m-2)$-homology classes of the exceptional divisors of the blow-up at $p_i$, and $\varepsilon \in (0, \varepsilon_0)$. Moreover, as $\varepsilon$ tends to $0$, the sequence of metrics $(g_\varepsilon)_\varepsilon$ converges to $g$ (in smooth topology) on compact subsets away from the exceptional divisors.*

See Theorem 1.1 in [5] for a proof.

**Theorem 3.20** (Arezzo-Pacard)**.** *Assume that $(M, J, g, \omega)$ is a constant scalar curvature compact Kähler manifold. There exists $n_g \geq 1$ such that for all $n \geq n_g$, there exists a nonempty open subset*

$$V_n \subset M^n_\Delta := \{(p_1, \ldots, p_n) \in M^n \mid p_a \neq p_b \text{ for all } a \neq b\},$$

*such that for all $(p_1, \ldots, p_n) \in V_n$, the blow-up of $M$ at $p_1, \ldots, p_n$ carries a family of constant scalar curvature Kähler metrics $(g_\varepsilon)_\varepsilon$ converging to $g$ (in smooth topology) on compact subsets away from the exceptional divisors, as the parameter $\varepsilon$ tends to $0$.*

See Theorem 1.2 in [6] for a proof.

**Theorem 3.21** (Arezzo-Pacard). *Assume that $(M, J, g, \omega)$ is a compact Kähler manifold of complex dimension $m$ with constant scalar curvature and that $(p_1, \dots, p_n) \in M^n_\Delta$ are chosen so that:*

*(a)* $\xi(p_1), \dots, \xi(p_n)$ *span $\mathfrak{h}^*$, where $\mathfrak{h}$, the space of Killing vector fields with zeros.*

*(b) there exist $a_1, \dots, a_n > 0$ such that $\sum_{i=1}^{n} a_i \xi(p_i) = 0 \in \mathfrak{h}^*$.*

*Then, there exist $c > 0$, $\varepsilon_0 > 0$, and for all $\varepsilon \in (0, \varepsilon_0)$, there exists on the blow-up of $M$ at $p_1, \dots, p_n$ a constant scalar curvature Kähler metric $g_\varepsilon$ which is associated to the Kähler form*

$$\omega_\varepsilon \in [\omega] - \varepsilon^2 \left( a_{1,\varepsilon}^{\frac{1}{m-1}} [E_1] + \dots + a_{n,\varepsilon}^{\frac{1}{m-1}} [E_n] \right),$$

*where the $[E_i]$ are the Poincaré duals of the $(2m-2)$-homology classes of the exceptional divisors of the blow-up at $p_i$ and where*

$$|a_{i,\varepsilon} - a_i| \leq c\varepsilon^{\frac{2}{2m+1}}.$$

*Finally, the sequence of metrics $(g_\varepsilon)_\varepsilon$ converges to $g$ (in smooth topology) on compact subsets, away from the exceptional divisors.*

See Theorem 1.3 in [6] for a proof.

**Theorem 3.22** (Kronheimer-Joyce). *Let $(M, \omega)$ be a nondegenerate compact $m$-dimensional constant scalar curvature Kähler orbifold with $m = 2$ or $3$ and isolated singularities. Let $p_1, \dots, p_n \in M$ be any set of points with a neighborhood biholomorphic to a neighborhood of the origin in $\mathbb{C}^m/\Gamma_i$, where $\Gamma_i$ is a finite subgroup of $SU(m)$. Let further $N_i$ be a Kähler crepant resolution of $\mathbb{C}^m/\Gamma_i$ (which always exists). Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there exists a constant scalar curvature Kähler form $\omega_\varepsilon$ on $M \bigsqcup_{p_1,\varepsilon} N_1 \bigsqcup_{p_2,\varepsilon} \cdots \bigsqcup_{p_n,\varepsilon} N_n$.*

See Corollary 8.2 in [5] for a proof.

**Theorem 3.23** (Apostolov-Rollin). *For any $k \geq 2$, the orbifold $\mathbb{CP}^k_{(-w_0,w)}$ admits a scalar-flat Kähler ALE metric $g_{\mathrm{ALE}}$ with quotient singularity at infinity $\mathbb{C}^n/\Gamma_{(-w_0,w)}$ and a Kähler potential $H$ for the Kähler form $\omega_{\mathrm{ALE}}$ written as*

$$H = \frac{1}{2}\|Z\|^2 + A\|Z\|^{4-2k} + O(\|Z\|^{3-2k}), \tag{3.1}$$

*when $k \geq 3$ and*

$$H = \frac{1}{2}\|Z\|^2 + A\log\|Z\| + O(\|Z\|^{-2}),$$

*when $k = 2$, where $A$ is a real constant and $\|Z\|^2$ is the square norm function on $\mathbb{C}^k$. Furthermore, the constant $A = 0$ iff the metric $g_{\mathrm{ALE}}$ is Ricci-flat.*

See Proposition 17 in [4] for a proof.

**Theorem 3.24** (Arezzo-Pacard)**.** *Any compact complex surface of general type admits cscK metrics.*

We finish this chapter with a concrete example that we construct. We will demonstrate how to find a constant scalar curvature Kähler metric on it using our theorem at the end of the thesis.

## 3.4 An example of orbifold with singularities of type $\mathcal{I}$

Let $r$ be a natural number and consider the cyclic subgroup of $U(1) \leq U(k)$ given by

$$\Gamma_{(-r,1,\dots,1)} = \langle \xi \,\mathrm{Id}_k \rangle \cong \mathbb{Z}_r,$$

where $\xi = e^{\frac{2\pi i}{r}}$ is the primitive $r$-th rooth of unity. Then for $l \geq 2$,

$$\Gamma := \langle \mathrm{diag}(\mathrm{Id}_l, \gamma) : \gamma \in \Gamma_{(-r,1,\dots,1)} \rangle \subset U(l) \times U(k) \subset U(l+k),$$

is a finite subgroup of the group of isometry $\mathrm{Iso}(\mathbb{CP}^{l+k-1}, g_{\mathrm{FS}}) \cong U(l+k)/U(1)$ acting on $\mathbb{CP}^{l+k-1}$ via the standard action on $\mathbb{C}^{l+k}$, where $g_{\mathrm{FS}}$ is the Fubini-Study mertic.

**Lemma 3.25.** *The orbifold $M = \mathbb{CP}^{l+k-1}/\Gamma$ has two disjoint singular strata of type $\mathcal{I}$ at*

$$S_1 = \{[z_0 : \ldots : z_{l-1} : 0 : \ldots : 0] \in \mathbb{CP}^{l+k-1}/\Gamma : [z_0 : \ldots : z_{l-1}] \in \mathbb{CP}^{l-1}\} \cong \mathbb{CP}^{l-1}/\Gamma,$$
$$S_2 = \{[0 : \ldots : 0 : z_l : \ldots : z_{l+k-1}] \in \mathbb{CP}^{l+k-1}/\Gamma : [z_l : \ldots : z_{l+k-1}] \in \mathbb{CP}^{k-1}\} \cong \mathbb{CP}^{k-1}/\Gamma.$$

*Also,*

$$N_M(S_1) = (\underbrace{\mathcal{O}_{\mathbb{CP}^{l-1}}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{CP}^{l-1}}(1)}_{k-\text{ times}})/\Gamma$$

*and*

$$N_M(S_2) = (\underbrace{\mathcal{O}_{\mathbb{CP}^{k-1}}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{CP}^{k-1}}(1)}_{l-\text{ times}})/\Gamma.$$

*Proof.* The set of points of $\mathbb{CP}^{l+k-1}$ fixed by the action of $\Gamma$ are $[z_0 : \ldots : z_{l+k-1}] \in \mathbb{CP}^{l+k-1}$ such that

$$\operatorname{diag}(\mathrm{Id}_l, \xi^s \mathrm{Id}_k)[z_0 : \ldots : z_{l+k-1}] = [z_0 : \ldots : z_{l+k-1}], 0 \leq s < r,$$

so this action fixes the two disjoint submanifolds

$$S_1 = \{[z_0 : \ldots : z_{l-1} : 0 : \ldots : 0] \in \mathbb{CP}^{l+k-1}/\Gamma : [z_0 : \ldots : z_{l-1}] \in \mathbb{CP}^{l-1}\} \cong \mathbb{CP}^{l-1},$$
$$S_2 = \{[0 : \ldots : 0 : z_l : \ldots : z_{l+k-1}] \in \mathbb{CP}^{l+k-1}/\Gamma : [z_l : \ldots : z_{l+k-1}] \in \mathbb{CP}^{k-1}\} \cong \mathbb{CP}^{k-1}/\Gamma_{(-r,1,\ldots,1)}.$$

To identify the normal bundles note that $S_1$ could be considered as the intesection of $k$ hyperplanes $D_1 \cap \ldots \cap D_k$, so that

$$N_M(S_1) = (N_M(D_1)|_{S_1} \oplus \ldots \oplus N_M(D_k)|_{S_1})/\Gamma = (\mathcal{O}_{\mathbb{CP}^{l-1}}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{CP}^{l-1}}(1))/\Gamma. \quad (3.2)$$

Similarly, for $S_2$, we can write it as the intesection of $l$ hyperplanes $D'_1 \cap D'_2 \cap \ldots \cap D'_l$, so we have that

$$N_M(S_2) = (N_M(D'_1)|_{S_2} \oplus \ldots \oplus N_M(D'_l)|_{S_2})/\Gamma = (\mathcal{O}_{\mathbb{CP}^{k-1}}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{CP}^{k-1}}(1))/\Gamma. \quad (3.3)$$

$\square$

Note that $\mathrm{Aut}(\mathbb{CP}^{l+k-1}) = \mathbb{P}\mathrm{GL}(l+k, \mathbb{C})$. For $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in M_{l+k}(\mathbb{C})$ and $\gamma = (\mathrm{Id}_l, \xi^q \mathrm{Id}_k) \in \Gamma$, $U\gamma = \left(\begin{array}{c|c} A & \xi^q B \\ \hline C & \xi^q D \end{array}\right)$ and $\gamma U = \left(\begin{array}{c|c} A & B \\ \hline \xi^q C & \xi^q D \end{array}\right)$, so $U\gamma = \gamma U$ and $\xi^q \neq 1$ imply that $B = 0$ and $C = 0$, so the orbifold $M = \mathbb{CP}^{l+k-1}/\Gamma$ has group of automorphism $\mathbb{P}(\mathrm{GL}(l, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C}))$, which is still quite large. To obtain an example with discrete automorphism group, we will use Theorem 1.4 in [6] by blowing-up $\mathbb{CP}^{l+k-1}$ at a sufficient number of points $\{p_1, p_2, \ldots, p_n\}$. Let $\mathfrak{h} = \mathrm{Lie}(\mathbb{P}\mathrm{GL}(l+k))$ be the Lie algebra of killing vector fields with zeros on $\mathbb{CP}^{l+k-1}$ and denote by $\mathfrak{h}^\Gamma$ the Lie subalgebra of $\mathfrak{h}$ consisting of $\Gamma$-invariant vector fields. Also denote the corresponding momentum maps by $\mu : \mathbb{CP}^{l+k-1} \to \mathfrak{h}^*$ and $\mu^\Gamma : \mathbb{CP}^{l+k-1} \to \mathfrak{h}^{\Gamma *}$. If $\iota : \mathfrak{h}^\Gamma \hookrightarrow \mathfrak{h}$ is the inclusion map and $\iota^* : \mathfrak{h}^* \to \mathfrak{h}^{\Gamma *}$ the dual map, notice that $\mu^\Gamma = \iota^* \circ \mu$.

Using Lemma 6.3 in [6], there exists $n_g \geq \dim \mathfrak{h} = (l + k - 1)^2 - 1$ such that for all $n \geq n_g$, there exists a nonempty open set $V_n \subset \{(p_1, p_2, \ldots, p_n) \in (\mathbb{CP}^{l+k-1})^n : p_a \neq p_b \quad \forall a \neq b\}$ such that, for all $(p_1, p_2, \ldots, p_n) \in V_n$, $\{\mu(p_1), \ldots, \mu(p_n)\}$ spans $\mathfrak{h}^*$ and there exist $a_1, a_2, \ldots, a_n > 0$ such that $\sum_{i=1}^{n} a_i \mu(p_i) = 0 \in \mathfrak{h}^*$. In particular, since $\mu^\Gamma = \iota^* \circ \mu$, this means that:

(a) The set $\{\mu^{\Gamma}(p_1), \ldots, \mu^{\Gamma}(p_n)\}$ spans $\mathfrak{h}^{\Gamma *}$.

(b) There exist positive integers $a_1, a_2, \ldots, a_n$ with $a_i = a_j$ if $p_j = \sigma(p_{j'})$ for some $\sigma \in \Gamma$ and $\sum_{i=1}^{n} a_i \mu^{\Gamma}(p_i) = 0 \in \mathfrak{h}^{\Gamma *}$.

Without loss of generality we can choose $p_1, p_2, \ldots, p_n \in \mathbb{CP}^{l+k-1} \setminus (S_1 \cup S_2)$, since $V_n$ is open and $\mathbb{CP}^{l+k-1} \setminus (S_1 \cup S_2)$ is open and dense. Note that if we add a point $p_{n+1}$, conditions (1) and (2) remain valid because of Lemma 6.2 in [6]. Thus, by adding points if necessary, we can assume that the set $\{p_1, \ldots, p_n\}$ is $\Gamma$-invariant. Using Theorem 1.4 in [6] shows that there exist $c > 0$, $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a $\Gamma$-invariant constant scalar curvature Kähler metric $\widetilde{g}_{\varepsilon}$ on $\mathrm{Bl}^{\mathbb{CP}^{l+k-1}}_{\{p_1,\ldots,p_n\}}$. This metric induces a constant scalar curvature Kähler metric $g_{\varepsilon}$ on $\mathrm{Bl}^{\mathbb{CP}^{l+k-1}}_{\{p_1,\ldots,p_n\}} / \Gamma$ in the sense of Remark 1.122. Consider the quotient map $q : \mathbb{CP}^{l+k-1} \to \mathbb{CP}^{l+k-1} / \Gamma$ and choose $q_1, \ldots, q_m \in \mathbb{CP}^{l+k-1}$ so that $q(\{p_1, \ldots, p_n\}) = \{q_1, \ldots, q_m\}$ where $m = \dfrac{n}{|\Gamma|}$. The natural identification,

$$\mathrm{Bl}^{\mathbb{CP}^{l+k-1}}_{\{p_1,\ldots,p_n\}} \Big/ \Gamma \cong \mathrm{Bl}^{\mathbb{CP}^{l+k-1}/\Gamma}_{\{q_1,\ldots,q_m\}},$$

then gives us a constant scalar curvature Kähler metric on the orbifold $X = \mathrm{Bl}^{\mathbb{CP}^{l+k-1}/\Gamma}_{\{q_1,\ldots,q_m\}}$ in the sense of Remark 1.122.

Using the Proposition 1.148 on page 47 the Lie algebra of holomorphic vector fields of $\mathrm{Bl}^{\mathbb{CP}^{l+k-1}/\Gamma}_{\{q_1,\ldots,q_m\}}$ is realized as

$$\{v \in \mathrm{Lie}(\mathbb{P}(\mathrm{GL}(l) \times \mathrm{GL}(k))) : v(q_1) = \ldots = v(q_m) = 0\},$$

so the identity component of the automorphism group of $\mathrm{Aut}(\ \mathrm{Bl}^{\mathbb{CP}^{l+k-1}/\Gamma}_{\{q_1,\ldots,q_m\}}\ )$ is the subgroup of elements of $\mathbb{P}(\mathrm{GL}(l) \times \mathrm{GL}(k))$ which fix $\{q_1, \ldots, q_m\}$. Now, a fixed point of $f \in \mathbb{P}(\mathrm{GL}(l) \times \mathrm{GL}(k))$ is the same as an eigenvector of a choice of representative $\widetilde{f} \in \mathrm{GL}(l) \times \mathrm{GL}(k)$, so $f(q) = q$ means $\widetilde{f}(p) = \lambda p$ for a representative $p$.

Suppose that we pick $n > l + k$ and the first $l + k$ points $p_1, \ldots, p_{l+k}$ in $\mathbb{CP}^{l+k-1}$ represented by points $\tilde{p}_1, \ldots, \tilde{p}_{l+k} \in \mathbb{C}^{l+k}$ forming a basis of $\mathbb{C}^{l+k}$ and with $q(p_i) \neq q(p_j)$ for $i \neq j$ with $i, j \leq l+k$, where $q : \mathbb{CP}^{l+k-1} \to \mathbb{CP}^{l+k-1} / \Gamma$ is the quotient map. Then with respect to this basis, the lift $\widetilde{f} \in \mathrm{GL}(l) \times \mathrm{GL}(k)$ of an element $f \in \mathbb{P}(\mathrm{GL}(l) \times \mathrm{GL}(k))$ such that $f(p_i) = p_i$ for $i \in \{1, \ldots, k+l\}$ will be diagonal. Then if we take $p_{l+k+1}$ such that its representative is $\tilde{p}_{l+k+1} = \sum_{i=1}^{l+k} \tilde{p}_i$, we see that the only element $f \in \mathbb{P}(\mathrm{GL}(l) \times \mathrm{GL}(k))$ fixing $p_1, \ldots, p_{l+k+1}$ is the identity (i.e, $\widetilde{f}$ is a multiple of the identity). Perturbing the representatives $\tilde{p}_1, \dots, \tilde{p}_{l+k}$ if necessary, we can assume that $q(p_{l+k+1}) \notin S_1 \cup S_2$. We have proven the following theorem:

**Theorem 3.26.** *Let $r$ be a natural number and consider the cyclic subgroup of $U(1) \leq U(k)$ given by*

$$\Gamma_{(-r,1,\dots,1)} = \langle \xi \,\mathrm{Id}_k \rangle \cong \mathbb{Z}_r,$$

*where $\xi = e^{\frac{2\pi i}{r}}$ is the primitive $r$-th rooth of unity. Then for $l \geq 2$,*

$$\Gamma := \langle \mathrm{diag}(\mathrm{Id}_l, \gamma) : \gamma \in \Gamma_{(-r,1,\dots,1)} \rangle \subset U(l+k),$$

*is a finite subgroup of the group of isometry $\mathrm{Iso}(\mathbb{CP}^{l+k-1}, g_{\mathrm{FS}})$ acting on $\mathbb{CP}^{l+k-1}$ via the standard action on $\mathbb{C}^{l+k}$, where $g_{\mathrm{FS}}$ is the Fubini-Study mertic. Then the orbifold $M = \mathbb{CP}^{l+k-1}/\Gamma$ has two disjoint singularities of type $\mathcal{I}$ at*

$$S_1 = \{[z_0 : \dots : z_{l-1} : 0 : \dots : 0] \in \mathbb{CP}^{l+k-1} : [z_0 : \dots : z_{l-1}] \in \mathbb{CP}^{l-1}\} \cong \mathbb{CP}^{l-1},$$
$$S_2 = \{[0 : \dots : 0 : z_l : \dots : z_{l+k-1}] \in \mathbb{CP}^{l+k-1} : [z_l :: z_{l+k-1}] \in \mathbb{CP}^{k-1}\} \cong \mathbb{CP}^{k-1}/\Gamma_{(-r,1,\dots,1)}.$$

*Also, we can choose $p_1, p_2, \dots, p_{r_m} \in \mathbb{CP}^{l+k-1} \setminus (S_1 \cup S_2)$ with $m \geq l+k+1$ such that there exists a constant scalar curvature Kähler metric on the orbifold $X = \mathrm{Bl}^{\mathbb{CP}^{l+k-1}/\Gamma}_{\{q_1,\dots,q_m\}}$ ( in the sense of Remark 1.122) where $q_1, \dots, q_m \in \mathbb{CP}^{l+k-1}/\Gamma$ are such that $q(\{p_1, \dots, p_{r_m}\}) = \{q_1, \dots, q_m\}$ where $q : \mathbb{CP}^{l+k-1} \to \mathbb{CP}^{l+k-1}/\Gamma$ is the quotient map. Furthermore, the identity component of the automorphism group of $\mathrm{Aut}(\mathrm{Bl}^{\mathbb{CP}^{l+k-1}/\Gamma}_{\{q_1,\dots,q_m\}})$ is trivial.*

# CHAPTER 4

# GLUING TECHNIQUE

Let $X$ be an orbifold of depth one in the sense of Definition 1.121, with singularities of type $\mathcal{I}$ along a connected suborbifold $Y$ of complex codimension $k$ i.e., we can find a finite subgroup $\Gamma_{(-w_0,w)}$ of $U(k)$ acting freely on $\mathbb{C}^k \setminus \{0\}$ such that any point $p \in Y$ has a local orbifold uniformization chart of the form $\mathbb{C}^{n-k} \times (\mathbb{C}^k/\Gamma_{(-w_0,w)})$ and $\Gamma_{(-w_0,w)}$ is of type $\mathcal{I}$ in the sense of Definition 1.145. Suppose that $X$ admits a cscK metric in the sense of Remark 1.122. Let $\pi : \widehat{X} \to X$ be a partial resolution of $X$ by performing a $(-w_0, w)$-weighted blow-up along $Y$ as described on page 47.

The goal of this section will be to construct a Kähler metric on the resolution $\widehat{X}$ which is close to the cscK metric $\omega_X$ on $X$. To do so, we will follow the approch of [21] and introduce an auxiliary space on which this construction will take place. We follow the following steps:

(a) Step 1: Consider first the orbifold with boundary $X \times [0, \infty)_\varepsilon$ and blow-up the submanifold $Y \times \{0\}$ in the sense of Melrose to obtain the orbifold with corners $\mathcal{X} := [X \times [0, \infty)_\varepsilon, Y \times \{0\}]$ where $\varepsilon$ is the parameter of deformation. Let $\beta : \mathcal{X} \to X \times [0, \infty)_\varepsilon$ be the corresponding blow-down map.

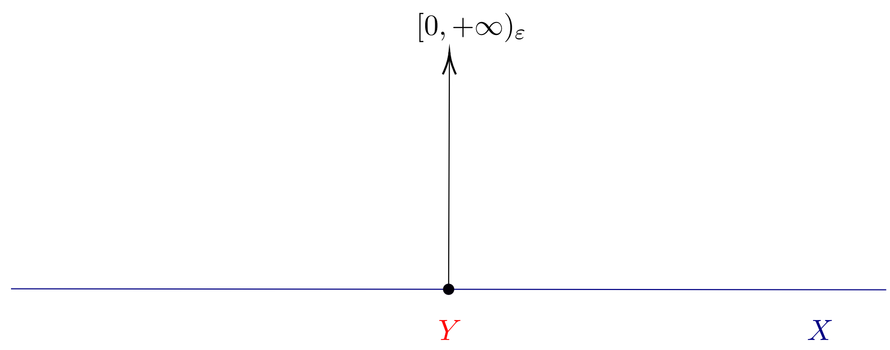



Figure 4.1: Orbifold with corner $X \times [0, \infty)_\varepsilon$

In local coordinates near $Y \times \{0\}$, this means that we replace $\mathbb{C}^{n-k} \times \mathbb{C}^k /\Gamma \times [0, \infty)$ by $\mathbb{C}^{n-k} \times \mathbb{S}^{2k}_+ /\Gamma \times [0, +\infty)$, where $\mathbb{S}^{2k}_+$ is the half sphere $\mathbb{S}^{2k}_+ = \{(z, \varepsilon) \in \mathbb{C}^k \times [0, \infty) : |z|^2 + \varepsilon^2 = 1\}$ and the local blow-down map

$$\mathbb{C}^{n-k} \times \mathbb{S}^{2k}_+ \Big/ \Gamma \times [0, +\infty) \to \mathbb{C}^{n-k} \times \mathbb{C}^k /\Gamma \times [0, \infty),$$

is given by $(w, (z, \varepsilon), r) \mapsto (w, rz, r\varepsilon)$.

Let $H_1$ be the boundary hypersurface of $\mathcal{X}$ obtained by the blow-up of $Y \times \{0\}$ and let $H_2$ be the boundary hypersurface correponding to the lift of $X \times \{0\}$ in $X \times [0, \infty)_\varepsilon$. In our metric model $H_1 = \overline{N_X(Y)} = N_X(Y) \sqcup S(N_X(Y))$ is the radial compactification of the normal bundle $N_X(Y)$ of $Y$ and $H_2 = [X, Y]$. Also note that $H_1 \cap H_2 = \partial H_1 = \partial H_2 = S(N_X(Y))$ and

$$\widehat{H}_1^\circ \cong \mathcal{O}_{E/Y}(-w_0), \tag{4.1}$$

where $E = \mathbb{P}_w(W)$ is the weighted projectivization of some vector bundle $W \to Y$ of rank $k$ such that $N_X(Y) = W/\Gamma_{(-w_0, w)}$.

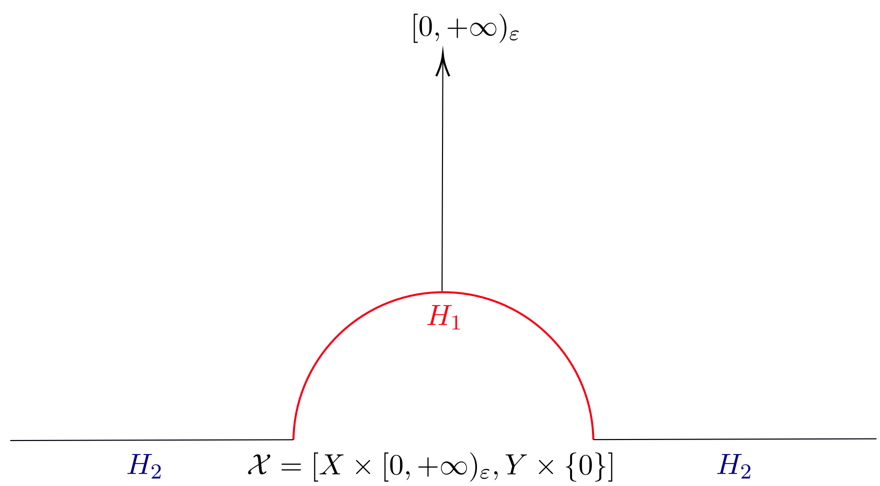



Figure 4.2: Blowing-up the orbifold $X \times [0, \infty)_\varepsilon$ along $Y \times \{0\}$

(b) Step 2: The partial resolution $\pi : \widehat{X} \to X$ naturally induces a partial resolution

$$\widehat{\mathcal{X}} \xrightarrow{\pi} \mathcal{X} = [X \times [0, \infty)_\varepsilon; \overline{Y} \times \{0\}] \xrightarrow{\beta} X \times [0, \infty)_\varepsilon.$$

As an orbifold, $X$ is automatically a stratified space with two strata $\Sigma_1 = Y$ and $\Sigma_2 = X \setminus Y$. The orbiofld $\mathcal{X}$ has corners with boundary hypersurfaces $H_1 = \overline{N_X(Y)}$ and $H_2 = [X, Y]$ coresponding to the stratas $\Sigma_1$ and $\Sigma_2$. The boundary hypersurfaces are naturally equiped with a fiber bundle structure

$$\begin{array}{ccc} \overline{V_1} = \overline{\mathbb{C}^k/\Gamma} & \longrightarrow & H_1 = \overline{N_X(Y)}, \\ & & \downarrow{\scriptstyle \varphi_1} \\ & & S_1 = Y \end{array} \qquad \begin{array}{ccc} V_2 = \{\mathrm{pt}\} & \longrightarrow & H_2 = [X, Y], \\ & & \downarrow{\scriptstyle \varphi_2 = \mathrm{id}} \\ & & S_2 = [X, Y] \end{array}$$

where $\overline{V_1}$ and $V_2$ are the fibers.

This is a particular case of Lemma 4.3 in [21], namely the orbifold $\mathcal{X}$ is in fact an orbifold with fibered corners $(\mathcal{X}, \varphi)$ where $\varphi_1 : H_1 = \overline{N_X(Y)} \to Y$ and $\varphi_2 = \mathrm{id} : H_2 = [X, Y] \to H_2$ are the fiber bundle maps. Indeed, the partial order in this case is just the order $H_1 < H_2$ on the set $\{H_1, H_2\}$ of boundary hypersurfaces and $H_1 \cap H_2 = \partial H_1 = \partial H_2 = \partial(\overline{N_X(Y)})$, $\varphi_1\big|_{H_1 \cap H_2} :$ $H_1 \cap H_2 \to S_1$ is clearly a surjective submersion, $S_{21} := \varphi_2(H_1 \cap H_2) = H_1 \cap H_2 = \partial H_2$ is the boundary of $H_2 = S_2$ and $\varphi_{21} := \varphi_1\big|_{H_1 \cap H_2}$ is such that $\varphi_{21} \circ \varphi_2 = \varphi_1 \circ \mathrm{id} = \varphi_1$.

Now we consider the Lie algebra of of $b$**-vector fields** in the sense of Example 2.30 and Remark 1.122 on the orbifold with fibered corners $\mathcal{X}$, that is, smooth vector fields on $\mathcal{X}$ which are tangent to all boundary hypersurfaces:

$$\begin{aligned}\mathcal{V}_b(\mathcal{X}) :&= \{\xi \in \mathfrak{X}(\mathcal{X}) : \xi \text{ tangent to } H_1 \text{ and } H_2\} \\ &= \{\xi \in \mathfrak{X}(\mathcal{X}) : \xi x_1 \in x_1 \mathcal{C}^\infty(\mathcal{X}) \text{ and } \xi x_2 \in x_2 \mathcal{C}^\infty(\mathcal{X})\},\end{aligned}$$

where $x_1$ and $x_2$ are boundary defining functions for $H_1$ and $H_2$, i.e, $x_i \geq 0$, $x_i^{-1}(0) = H_i$ and $dx_i \neq 0$ on $H_i$. Choose $x_1$ and $x_2$ so that $\varepsilon = x_1.x_2$ is the corresponding total boundary function. In the local coordinates $(x_1, x_2, u_i)$, vector fields in $\mathcal{V}_b(\mathcal{X})$ are of the form

$$\xi = a x_1 \frac{\partial}{\partial x_1} + b x_2 \frac{\partial}{\partial x_2} + \sum_i c_i u_i \frac{\partial}{\partial u_i},$$

where $a, b, c_i$ are smooth functions.

Also we consider the Lie subalgebra $\mathcal{V}_{\mathrm{QAC}}(\mathcal{X})$ of **quasi asymptotically conical vector fields** or $\mathrm{QAC}$**-vector fields** on $\mathcal{X}$ originally introduced by Conlon, Degeratu and Rochon in [21]. By definition (see Definitions 1.11 and 1.14 in [21]), these are the $b$-vector fields $\xi \in \mathcal{V}_b(\mathcal{X})$ such that:

(a) $\mathrm{QAC}\,1$: $\xi\big|_{H_i}$ is tangent to the fibers of $\varphi_i : H_i \to S_i$ for all $i$,

(b) $\mathrm{QAC}\,2$: $\xi\varepsilon \in \varepsilon^2 C^\infty(\mathcal{X})$.

The Lie algebra $\mathcal{V}_{\mathrm{QAC}}(\mathcal{X})$ is a finitely generated projective $\mathcal{C}^\infty(\mathcal{X})$-module, so by the Serre-Swan theorem (Theorem 2.28), there exists a natural smooth vector bundle, ${}^\varphi T\mathcal{X} \to \mathcal{X}$, and a natural map $\iota_\varphi : {}^\varphi T\mathcal{X} \to T\mathcal{X}$ such that

$$\mathcal{V}_{\mathrm{QAC}}(\mathcal{X}) = (\iota_\varphi)_* \Gamma(\mathcal{X}, {}^\varphi T\mathcal{X}).$$

The above vector bundle ${}^{\varphi}T\mathcal{X}$, originally introduced in [21] is called the $\mathrm{QAC}$**-tangent bundle** over $\mathcal{X}$. The $\mathrm{QAC}$**-cotangent bundle** ${}^{\varphi}T^{*}\mathcal{X}$ over $\mathcal{X}$ is the vector bundle dual to ${}^{\varphi}T\mathcal{X}$.
Using the function $\varepsilon = \beta^{*}\operatorname{pr}_2 \in \mathcal{C}^{\infty}(\mathcal{X})$, we can define the Lie subalgebra of $\mathcal{V}_{\mathrm{QAC}}(\mathcal{X})$, corresponding to $\mathrm{QAC}$-vector fields tangent to the level sets of $\varepsilon$:

$$\mathcal{V}_{\mathrm{QAC},\varepsilon}(\mathcal{X}) := \{\xi \in \mathcal{V}_{\mathrm{QAC}}(\mathcal{X}) : \xi\varepsilon \equiv 0\}.$$

Again by the Serre-Swan theorem (Theorem 2.28 above), there exist a natural vector bundle $\mathcal{E} \to \mathcal{X}$ and a natural map $\iota_{\varepsilon} : \mathcal{E} \to T\mathcal{X}$ such that there is a canonical identification

$$\mathcal{V}_{\mathrm{QAC},\varepsilon}(\mathcal{X}) = (\iota_{\varepsilon})_{*}\Gamma(\mathcal{X}, \mathcal{E}).$$

In fact, $\mathcal{E}$ is naturally a vector subbundle of ${}^{\varphi}T\mathcal{X}$, which induces a natural map $\iota_{\varepsilon}^{*} : {}^{\varphi}T^{*}\mathcal{X} \to \mathcal{E}^{*}$. This means that a smooth $\mathrm{QAC}$-metric (i.e. a bundle metric for ${}^{\varphi}T\mathcal{X}$) naturally restricts to define an element of $\Gamma(\mathcal{X}, \mathcal{E}^{*} \otimes \mathcal{E}^{*})$. Now we look at the pull-back of the orbifold cscK-metric $g_X$ on $X$ to $g_{\mathcal{X}} := \beta^{*}\operatorname{pr}_1^{*} g_X$ on $\mathcal{X}$, where $\operatorname{pr}_1 : X \times [0, +\infty) \to X$ is the projection on the first factor.

**Lemma 4.1.** *The pull-back $g_{\varepsilon} := \iota_{\varepsilon}^{*}\beta^{*}\operatorname{pr}_1^{*} g_X$ is such that $\dfrac{g_{\varepsilon}}{\varepsilon^2} \in \Gamma(\mathcal{X}, \mathcal{E}^{*} \otimes \mathcal{E}^{*})$.*

*Proof.* This is a special case of Lemma 4.5 in [21] but we will provide a direct proof for the convenience of the reader. It is sufficient to show that $\beta^{*}\operatorname{pr}_1^{*} s \in \Gamma(\mathcal{X}, {}^{\varphi}T^{*}\mathcal{X})$ for $s \in \Gamma(X, T^{*}X)$. Let us choose $(y, z)$ as a local coordinate of $X$, where $y$ is a coordinate of $Y \subseteq X$ and $z = (z_1, \ldots, z_k) \in \mathbb{C}^k/\Gamma$ is normal to $Y$, i.e, $Y = \{z = 0\}$. Write $z = (r, \omega) \in \mathbb{R}^{+} \times \mathbb{S}^{2k-1}$ in the spherical coordinates. The boundary defining function of $H_1$ and $H_2$ can be choosen to be $x_1 = \sqrt{r^2 + \varepsilon^2}$ and $x_2 = \dfrac{\varepsilon}{x_1} = \dfrac{\varepsilon}{\sqrt{r^2+\varepsilon^2}}$. The bundle ${}^{\varphi}T^{*}\mathcal{X}$ over $\mathcal{X}$ is then generated by $\{\dfrac{\mathrm{d}\varepsilon}{\varepsilon^2}, \dfrac{\mathrm{d}x_2}{x_2^2}, \dfrac{\mathrm{d}y}{\varepsilon}, \dfrac{\mathrm{d}\omega}{x_2}\}$, while $\mathcal{E}^{*}$ is generated by $\{\dfrac{\mathrm{d}x_2}{x_2^2}, \dfrac{\mathrm{d}y}{\varepsilon}, \dfrac{d\omega}{x_2}\}$. We need to check that $\dfrac{r\mathrm{d}\omega}{\varepsilon}$ and $\dfrac{\mathrm{d}r}{\varepsilon}$ are sections of ${}^{\varphi}T^{*}\mathcal{X}$. Now

$$\frac{r\mathrm{d}\omega}{\varepsilon} = \frac{r}{\sqrt{r^2+\varepsilon^2}}\frac{\sqrt{r^2+\varepsilon^2}\mathrm{d}\omega}{\varepsilon} = \frac{r}{\sqrt{r^2+\varepsilon^2}}\frac{\mathrm{d}\omega}{x_2}.$$

Since the coefficient $\dfrac{r}{\sqrt{r^2+\varepsilon^2}} = \sqrt{1-x_2^2}$ is a smooth function on $\mathcal{X}$, this shows that $\dfrac{r\mathrm{d}\omega}{\varepsilon}$ is a smooth section of $\mathcal{E}^*$. On the other hand:

$$\begin{aligned}\frac{\mathrm{d}r}{\varepsilon} &= \frac{1}{\varepsilon}\mathrm{d}(\sqrt{x_1^2-\varepsilon^2}) \\ &= \frac{1}{\varepsilon}\mathrm{d}(\varepsilon\sqrt{\frac{x_1^2}{\varepsilon^2}-1}) \\ &= \frac{1}{\varepsilon}\mathrm{d}(\varepsilon\sqrt{\frac{1}{x_2^2}-1}) \\ &= \frac{\mathrm{d}\varepsilon}{\varepsilon}\sqrt{\frac{1}{x_2^2}-1}+\frac{\varepsilon}{\varepsilon}\mathrm{d}(\sqrt{\frac{1}{x_2^2}-1}) \\ &= \frac{\mathrm{d}\varepsilon}{\varepsilon}\sqrt{\frac{1}{x_2^2}-1}-\frac{\mathrm{d}x_2}{x_2^2\sqrt{1-x_2^2}}.\end{aligned}$$

Since $\frac{\mathrm{d}\varepsilon}{\varepsilon}\sqrt{\frac{1}{x_2^2}-1} = \frac{\sqrt{1-x_2^2}}{x_2}\frac{\mathrm{d}\varepsilon}{\varepsilon} = x_1\sqrt{1-x_2^2}\frac{\mathrm{d}\varepsilon}{\varepsilon^2}$ is a smooth section of ${}^{\varphi}T^*\mathcal{X}$ over $\mathcal{X}$ and $\frac{\mathrm{d}x_2}{x_2^2\sqrt{1-x_2^2}}$ is a section of $\mathcal{E}^*$ over $\mathcal{X}$, this shows that $\frac{\mathrm{d}r}{\varepsilon}$ is a smooth section of ${}^{\varphi}T^*\mathcal{X}$.

□

Now, we describe the restrictions of $\frac{g_\varepsilon}{\varepsilon^2}$ to $H_1$ and $H_2$. First describe the restriction of $\mathcal{E}$ to $H_1$ and $H_2$.

On $H_1$, restricting the boundary defining function of $\mathcal{X}$ to $H_1$ gives us a Lie algebra of $\mathrm{QAC}$-vector fields and a corresponding $\mathrm{QAC}$-tangent bundle ${}^{\varphi}T(H_1\,/S_1)$ in the fibers of $\varphi_1 : H_1 \to S_1$. Since these fibers are manifolds with boundary, this is in each fiber, the Lie algebra of scattering vector field in the Melrose sense. So there is a natural map $\mathcal{E}\big|_{H_1} \to {}^{\mathrm{SC}}T(H_1\,/S_1)$ and a short exact sequence

$$0 \to N_1\mathcal{E} \to \mathcal{E}\big|_{H_1} \to {}^{\mathrm{SC}}T(H_1\,/S_1) \to 0,$$

where $N_1\mathcal{E} := \ker(\mathcal{E}\big|_{H_1} \to {}^{\mathrm{SC}}T(H_1\,/S_1))$.
Since there is a natural inclusion ${}^{\mathrm{SC}}T(H_1\Big/S_1) \subset \mathcal{E}\big|_{H_1}$, the above short exact sequence splits. There is another natural short exact sequence involving ${}^{\mathrm{SC}}T(H_1\,/S_1)$,

$$0 \to {}^{\mathrm{SC}}T(H_1\Big/S_1) \to {}^{\mathrm{SC}}TH_1 \to x_2\varphi_1^*(TS_1) \to 0,$$

where $TS_1$ is the tangent bundle of $S_1 = Y$, so

$$x_2\varphi_1^*(TS_1) = {}^{\mathrm{SC}}TH_1\Big/{}^{\mathrm{SC}}T(H_1\,/S_1). \quad (4.2)$$

Multiplication by the boundary defining function $x_1$ induces the identification

$$ {}^{\mathrm{SC}}TH_1 \Big/ {}^{\mathrm{SC}}T(H_1 \Big/ S_1) = \ker\left(\mathcal{E}\big|_{H_1} \to {}^{\mathrm{SC}}T(H_1 / S_1)\right). \quad (4.3) $$

In particular, we see from (4.2) and (4.3) that there is a natural identification

$$ \ker\left(\mathcal{E}\big|_{H_1} \to {}^{\mathrm{SC}}T(H_1 / S_1)\right) \cong \varepsilon\varphi_1^*(TS_1). $$

Hence, we have a canonical decomposition

$$ \mathcal{E}\big|_{H_1} = \varepsilon\varphi_1^*(TS_1) \oplus {}^{\mathrm{SC}}T(H_1 / S_1). $$

By [21], the family of metric $\frac{g_\varepsilon}{\varepsilon^2}$ splits accordingly

$$ \frac{g_\varepsilon}{\varepsilon^2}\Big|_{H_1} = \frac{\varphi_1^* g_{S_1}}{\varepsilon^2} + g_{\varphi_1}, $$

where $g_{\varphi_1}$ and $\frac{\varphi_1^* g_{S_1}}{\varepsilon^2}$ are the metrics induced by $\frac{g_\varepsilon}{\varepsilon^2}$ in the fiber of $\varphi_1 : H_1 \to S_1$ and the bundle $N_1\mathcal{E}$.

On the resolution $\widehat{\mathcal{X}}$, the boundary hypersurface $H_1$ is replaced by a boundary hypersurface $\widehat{H}_1$ that is a resolution of $H_1$, and the fiber bundle

$$ \begin{array}{ccc} \overline{V_1} = \overline{\mathbb{C}^k / \Gamma} & \longrightarrow & H_1 = \overline{N_X(Y)}\,, \\ & & \big\downarrow \varphi_1 \\ & & S_1 = Y \end{array} $$

is replaced by

$$ \begin{array}{ccc} \overline{Z_1} = \widehat{\overline{\mathbb{C}^k / \Gamma}} & \longrightarrow & \widehat{H}_1 = \widehat{\overline{N_X(Y)}}\,, \\ & & \big\downarrow \widehat{\varphi_1} \\ & & \widehat{S_1} = \widehat{Y} \end{array} $$

where $\overline{Z_1}$ is a resolution of $\overline{V_1}$. The function $\varepsilon$ on $\mathcal{X}$ naturally extends to a smooth function on $\widehat{\mathcal{X}}$, also denoted by $\varepsilon$. Similarly, the boundary defining functions $x_1$ can be chosen to lift to a smooth boundary defining function on $\widehat{\mathcal{X}}$, yielding a natural Lie algebra $\mathcal{V}_{\mathrm{QAC}}(\widehat{\mathcal{X}})$. We can define a Lie subalgebra by

$$ \mathcal{V}_{\mathrm{QAC},\varepsilon}(\widehat{\mathcal{X}}) := \{\xi \in \mathcal{V}_{\mathrm{QAC}}(\widehat{\mathcal{X}}) : \xi\varepsilon \equiv 0\}. $$

There is a corresponding vector bundle $\widehat{\mathcal{E}} \to \widehat{\mathcal{X}}$ and a natural map $\iota_\varepsilon : \widehat{\mathcal{E}} \to T\widehat{\mathcal{X}}$ yielding a canonical identification

$$\mathcal{V}_{\mathrm{QAC},\varepsilon}(\mathcal{X}) = (\iota_\varepsilon)_*\Gamma(\widehat{\mathcal{X}}, \widehat{\mathcal{E}}).$$

The following theorem constructs a family of Kähler metrics on $\widehat{X}$ which are approximately cscK.

**Theorem 4.2.** *Let $(X, \omega_X)$ be a compact cscK complex orbifold with singularities of type $\mathcal{I}$ along a subset $Y$, i.e, the normal bundle of $Y$ in $X$ has fibers of the form $W/\Gamma$ where $W$ is a vector bundle of rank $k$ on $X$ and $\Gamma$ is a finite subgroup of $U(k)$ of type $\mathcal{I}$. Then, there exists a smooth closed $(1,1)$-form $\omega_{\widehat{\varphi_1}}$ on $\widehat{H}_1$ restricting on each fiber of $\widehat{\varphi_1}$ to the Kähler form of a scalar flat $\mathrm{ALE}$-metric asymptotic to $\omega_{\varphi_1}$. Moreover, for $\mu > 0$ small, there is $\widehat{\omega}_\varepsilon \in \Gamma(\widehat{X}, \widehat{\mathcal{E}^*} \wedge \widehat{\mathcal{E}^*})$ such that $\frac{\widehat{\omega}_\varepsilon}{\varepsilon^2}\big|_{\widehat{H}_1} = \omega_{\widehat{\varphi_1}} + \frac{\varphi_1^*\omega_{S_1}}{\varepsilon^2}$, $\widehat{\omega}_\varepsilon\big|_{H_2} = \omega_X\big|_{H_2}$ and which yields a positive definite closed $(1,1)-$form on*

$$\widehat{X}_c = \{p \in \widehat{\mathcal{X}} : \varepsilon(p) = c\} \cong \widehat{X},$$

*for each $0 < c < \mu$.*

*Proof.* Let $\Gamma = \langle\gamma\rangle$. Since the unitary matrix $\gamma$ is diagonalizable, the eigenspaces of $\gamma$ in each fiber of $W \to Y$ induce an $\omega_X$-orthogonal decomposition

$$W = \bigoplus_{i=1}^{l} W_i, \quad \text{where} \quad \dim W_i = k_i, \quad \sum_{i=1}^{l} k_i = k.$$

Now we can consider the action of $\bigtimes_{i=1}^{l} U(k_i)$ on $W$. Let $e = (e_1, \ldots, e_k)$ be an orthonormal basis for smooth sections of $W$ on an open set $U \subset Y$, compatible with the decomposition of $W$. This gives us a trivialization $W|_U \cong \mathbb{C}^k \times U$, and in this trivialization, $\gamma$ acts diagonally on $\mathbb{C}^k$ by

$$\gamma = \mathrm{diag}(e^{\frac{iw_1}{w_0}}, \ldots, e^{\frac{iw_k}{w_0}}).$$

Let $W|_V \cong \mathbb{C}^k \times V$ be another such trivialization for an open subset $V \subset Y$ with orthonormal basis $e' = (e'_1, \ldots, e'_k)$. Then $e' = fe$ for a smooth function $f : U \cap V \to \bigtimes_{i=1}^{l} U(k_i)$. Note that $\widehat{H}_1 = \widehat{N_X(Y)}$ and $\widehat{H}_1^\circ$ is the total space of a vector bundle with fibers isomorphic to $\mathbb{CP}^k_{(-w_0,w)} = \mathcal{O}_{\mathbb{CP}^{k-1}_w}(-w_0)$. We get a natural line bundle:

$$\varpi : \widehat{H}_1^\circ \to E, \quad \text{where} \quad E = \mathbb{P}_w(W) \quad \text{is the weighted fiberwise projectivization of } W.$$

As discussed in Theorem 3.23 on page 76, there exists a scalar-flat Kähler metric $g_{\mathrm{ALE}}$ on $\mathbb{CP}^k_{(-w_0,w)}$ modelled on $\mathbb{C}^k / \Gamma_{(-w_0,w)}$ at infinity with a Kähler potential $H$ for the Kähler form $\omega_{\mathrm{ALE}}$ written as

$$H = \frac{1}{2}\|Z\|^2 + A\|Z\|^{4-2k} + O(\|Z\|^{3-2k}), \tag{4.4}$$

when $k \geq 3$ and

$$H = \frac{1}{2}\|Z\|^2 + A\log\|Z\| + O(\|Z\|^{-2}),$$

when $k = 2$, where $A$ is a constant. Since the metric $g_{\mathrm{ALE}}$ and the potential $H$ are invariant under the action of $\bigtimes_{i=1}^{l} U(k_i)$, we can use the above trivialization to obtain a well-defined fiberwise potential

$$H_N : N_X(Y) \to \mathbb{R},$$

corresponding to $H$ in each fibers of $N_X(Y)$ for any choice of trivialization as described above. Set $\omega_{\widehat{\varphi_1}} = 2\sqrt{-1}\partial\bar{\partial}(H_N)$ on $N_X(Y) \setminus Y \cong \widehat{H}_1^\circ \setminus E$. On each fiber of $\widehat{H}_1^\circ \to Y$, this closed $(1,1)$-form extends to an $\mathrm{ALE}$ scalar flat metric. Globally, this extends to a smooth $(1,1)$-form on $\widehat{H}_1^\circ$ that we will also denoted by $\omega_{\widehat{\varphi_1}}$. Since $\omega_{\widehat{\varphi_1}}$ is closed on the complement of $E$, it is closed everywhere by continuity. Hence, $\omega_{\widehat{\varphi_1}}$ is the desired closed $(1,1)$-form. Finally we define

$$\widehat{\omega}_\varepsilon = \omega_X + \varepsilon^2\sqrt{-1}\partial\bar{\partial}[\gamma_1(\frac{d}{r_\varepsilon})f(\frac{d}{\varepsilon})],$$

where $\gamma_1 : \mathbb{R} \to \mathbb{R}$ is a cut-off function such that $\gamma_1(t) = 1$ for $t < 1$ and $\gamma_1(t) = 0$ for $t > 2$, $d = d_{g_X} \circ \mathrm{Pr}_1 \circ \beta \circ \pi : \widehat{\mathcal{X}} \to [0, +\infty)$, where $d_{g_X}$ is the distance from $Y$ on $X$ with respect to the cscK metric $g_X$ (we can use $d = r$ in terms of the coordinates used in the proof of Lemma 4.1), $r_\varepsilon = \varepsilon^{\frac{2k}{2k+1}}$ and

$$f = A\|Z\|^{4-2k} + O(\|Z\|^{3-2k}),$$

is a function defined on the complement of the exceptional divisor such that

$$\omega_{\widehat{\varphi_1}} = \omega_{\varphi_1} + \sqrt{-1}\partial\bar{\partial}f.$$

By construction, we obtain three different regions on $\widehat{X}$ as follows:

(a) Near the exceptional divisor, on $\Omega_1 = \{x \in \widehat{X} : d(x) < r_\varepsilon\}$ we have $\gamma_1(\frac{d}{r_\varepsilon}) = 1$ so we get

$$\widehat{\omega}_\varepsilon = \omega_X + \varepsilon^2\sqrt{-1}\partial\bar{\partial}f(\frac{d}{\varepsilon}).$$

(b) On the intermediate region $\Omega_2 = \{x \in \widehat{X} : r_\varepsilon < d(x) < 2r_\varepsilon\}$ we get

$$\widehat{\omega}_\varepsilon = \omega_X + \varepsilon^2\sqrt{-1}\partial\bar{\partial}[\gamma_1(\frac{d}{r_\varepsilon})f(\frac{d}{\varepsilon})].$$

(c) Far from the exceptional divisor, $\Omega_3 = \{x \in \widehat{X} : 2r_\varepsilon < d(x)\}$ we have $\gamma_1(\frac{d}{r_\varepsilon}) = 0$, so we get

$$\widehat{\omega}_\varepsilon = \omega_X.$$

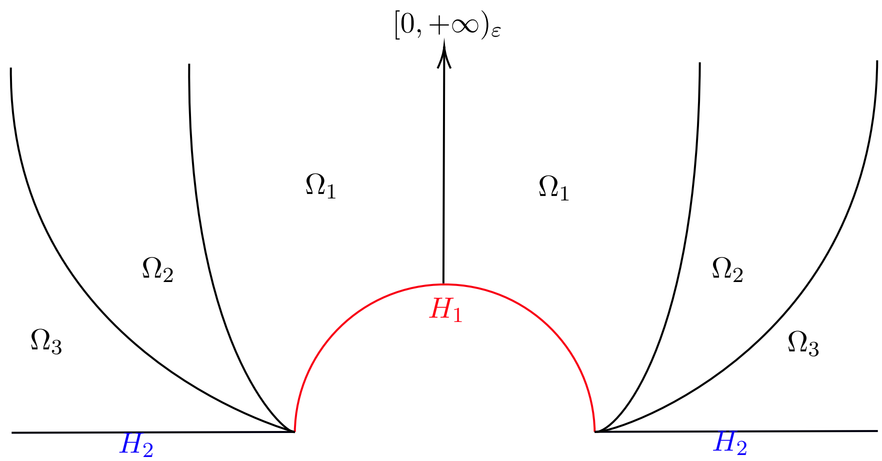



Figure 4.3: Three different regions on $\widehat{X}$

Far from the exceptional divisor, $\widehat{\omega}_\varepsilon$ is well-defined on the complement of the exceptional divisor. Moreover, by construction, $\frac{\widehat{\omega}_\varepsilon}{\varepsilon^2}$ extends to a metric on the bundle $\widehat{\mathcal{E}}$ for small $\varepsilon$. In fact on $\widehat{H}_1$ we get

$$\frac{\widehat{\omega}_\varepsilon}{\varepsilon^2}|_{H_1} = \omega_{\widehat{\varphi_1}} + \frac{\varphi_1^*\omega_{S_1}}{\varepsilon^2},$$

and on $\widehat{H}_2$ we get same restriction as $\frac{\omega_X}{\varepsilon^2}$. Since $\frac{\widehat{\omega}_\varepsilon}{\varepsilon^2}$ is positive definite on both $\widehat{\mathcal{E}}\big|_{H_1}$ and $\widehat{\mathcal{E}}\big|_{H_2}$, so is $\frac{\widehat{\omega}_\varepsilon}{\varepsilon^2}$ near $\widehat{H}_1$ and $\widehat{H}_2$. Consequently, it remains positive definite for small $\varepsilon > 0$, which yields a positive definite closed $(1,1)-$form on the level sets $\widehat{X}_c = \{p \in \widehat{\mathcal{X}} : \varepsilon(p) = c\} \cong \widehat{X}$ as claimed. □

**Remark 4.3.** *Since the cohomology in degree 2 of $\mathbb{CP}^k_{(-w_0,w)}$ is generated by the divisor $[\mathbb{CP}^{k-1}_w]$, notice that by reparametrizing $\varepsilon$, if necessary, we can assume without loss of generality in Theorem 4.2 that $[\widehat{\omega}_\varepsilon] = [\omega_X] - \varepsilon^2[E]$.*

# CHAPTER 5
# LINEAR ANALYSIS

The Kähler metrics $\widehat{\omega}_\varepsilon$ provided by Theorem 4.2 are not necessarily cscK, but since their asymptotic models on $\widehat{H}_1$ and $H_2$ are, they will be almost cscK for small $\varepsilon$. We can therefore hope to solve the nonlinear equation

$$S(\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}u) = R,$$

perturbatively, for some well-chosen constant $R$.

To do this, the purpose of this section is to first study the linearization of the scalar curvature of the metric $\widehat{\omega}_\varepsilon$ perturbed by a potential $u$:

$$S(\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}u) = S(\widehat{\omega}_\varepsilon) + L_{\widehat{\omega}_\varepsilon}(u) + Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u), \tag{5.1}$$

where $L_{\widehat{\omega}_\varepsilon}$ is linear part and $Q_{\widehat{\omega}_\varepsilon}$ is nonlinear part. By the Proposition 3.4 we get,

$$L_{\widehat{\omega}_\varepsilon}(u) = -(\frac{1}{2}\Delta^2_{\widehat{g}_\varepsilon} + \mathrm{Ric}_{\widehat{g}_\varepsilon}.\nabla^2_{\widehat{g}_\varepsilon})u.$$

In terms of Lichnerowicz operator $\mathcal{D}^*\mathcal{D}$, we can write

$$L_{\widehat{\omega}_\varepsilon}(u) = \frac{1}{2}\nabla S(\widehat{\omega}_\varepsilon).\nabla u - \mathcal{D}^*\mathcal{D}u.$$

Let us define $\widetilde{L}_\varepsilon : C^\infty(\widehat{X})_0 \times \mathbb{R} \to C^\infty(\widehat{X})$ by

$$\widetilde{L}_\varepsilon(u, R) = L_{\widehat{\omega}_\varepsilon}(u) - R, \tag{5.2}$$

where $C^\infty(\widehat{X})_0$ is the space of smooth functions $u$ that have zero integral with respect to the metric $\widehat{g}_\varepsilon$.

**Definition 5.1** (Hölder space). *Let $(M, g)$ is a smooth Riemannian manifold, $k \in \mathbb{N}_0$ and $\alpha \in (0, 1]$. Then the Hölder space $\mathcal{C}^{k,\alpha}_g(M)$ consists of functions $f : M \to \mathbb{R}$ such that*

$$\|f\|_{\mathcal{C}^{k,\alpha}_g} := \|f\|_{g,k} + [\nabla^k f]_{g,\alpha} < \infty,$$

*where $\|f\|_{g,k} = \sum_{i=0}^{k} \sup_{p\in M} \|\nabla^i f(p)\|_g$ with $\|\cdot\|_g$ the pointwise norm induced by the metric $g$ and $[\nabla^k f]_{g,\alpha}$ is the Hölder semi norm defined by*

$$[\nabla^k f]_{g,\alpha} = \sup_{\gamma(0)\neq\gamma(1)} \{\frac{\|P_\gamma(\nabla^k f(\gamma(0))) - \nabla^k f(\gamma(1))\|_g}{l(\gamma)^\alpha} : \gamma \text{ a smooth curve on } M\},$$

*and $P_\gamma$ is parallel transport along $\gamma$.*

**Definition 5.2** (Weighted Hölder space). *Let $(M,g)$ is a smooth Riemannian manifold, $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$. The weighted Hölder space $\rho\mathcal{C}^{k,\alpha}_g(M)$ for a weight function $\rho \in \mathcal{C}^\infty(M,\mathbb{R}^+)$ consists of functions $f : M \to \mathbb{R}$ such that*

$$\|f\|_{\rho\mathcal{C}^{k,\alpha}_g} := \|\frac{f}{\rho}\|_{\mathcal{C}^{k,\alpha}_g} < \infty.$$

**Remark 5.3.** *(a) If $\rho = \sqrt{d^2+\varepsilon^2}$ is a boundary defining function for $\widehat{H}_1$ in $\widehat{\mathcal{X}}$, then the restriction of $\widehat{g}_\varepsilon$ to $\widehat{H}_1$ induces a family of fiberwise b-metric in the fibers of $\varphi_1 : \widehat{H}_1 \to \widehat{S}_1$.*

*(b) The restriction $\frac{\widehat{g}_\varepsilon}{\rho^2}|_{H_2} = \frac{g_X}{\rho^2}|_{H_2} = \frac{g_X}{d^2}$ is an edge metric in the sense of Mazzeo.*

On a fiber $\overline{Z_1}$ of $\widehat{\varphi}_1 : \widehat{H}_1 \to \widehat{S}_1$, the metric $\frac{\widehat{g}_\varepsilon}{\varepsilon^2}$ induced by restriction is a scalar flat Kähler ALE metric $g_{\widehat{\varphi}_1}$. It is convenient to study the mapping properties of its operator $L_{\widehat{\omega}_\varepsilon}$ in terms of the weighted Hölder space induced by the $b$-metric $g_{\widehat{\varphi}_{1,b}}$ obtained by restriction of $\frac{\widehat{g}_\varepsilon}{\rho^2}$ to $\overline{Z_1}$.

**Lemma 5.4.** *If $\delta < 0$ and $k > 2$, then the linear operator*

$$L_{\omega_{\widehat{\varphi}_1}} : (\frac{\rho}{\varepsilon})^\delta \mathcal{C}^{4,\alpha}_{g_{\widehat{\varphi}_{1,b}}}(Z_1) \to (\frac{\rho}{\varepsilon})^{\delta-4} \mathcal{C}^{0,\alpha}_{g_{\widehat{\varphi}_{1,b}}}(Z_1),$$

*has trivial kernel, where $Z_1$ is the interior of $\overline{Z_1}$.*

*Proof.* We will proceed as the proof of Proposition 8.9 in [57]. Since the scalar curvature of $g_{\widehat{\varphi}_1}$ is zero,

$$L_{\omega_{\widehat{\varphi_1}}}(u) = -\mathcal{D}^*\mathcal{D}u.$$

Also by Proposition 17 of [4], the Kähler potential of $\omega_{\widehat{\varphi}_1}$ as $\|Z\| \to \infty$ is

$$\frac{1}{2}\|Z\|^2 + A\|Z\|^{4-2k} + O(\|Z\|^{3-2k}),$$

where $Z$ denotes the Euclidean coordinates on $(\mathbb{C}^k / \Gamma_{(-w_0,w)}) \setminus \{0\}$ identified with complement of the exceptional divisor in $Z_1$. Suppose $u \in (\frac{\rho}{\varepsilon})^\delta \mathcal{C}^{4,\alpha}_{g_{\widehat{\varphi}_{1,b}}}(Z_1)$ and $\mathcal{D}^*\mathcal{D}u = 0$. Consider a smooth cutoff function $\gamma$ such that it vanishes on $\pi^{-1}(B_1(0))$ and is 1 outside $\pi^{-1}(B_2(0))$, where $B_r(0)$ is the ball with radius $r$ centred origin in $\mathbb{C}^k / \Gamma_{(-w_0,w)}$, so $\gamma u$ is a smooth function on $\mathbb{C}^k / \Gamma_{(-w_0,w)}$. Now we try to compare the Lichnerowicz operator $\mathcal{D}^*\mathcal{D}$ with Euclidean operator $\Delta^2_{\mathrm{Euc}}$, note that $\mathcal{D}^*_{\mathrm{Euc}}\mathcal{D}_{\mathrm{Euc}} = \frac{1}{2}\Delta^2_{\mathrm{Euc}}$, but

$$\mathcal{D}^*\mathcal{D}(u) = \frac{1}{2}\Delta^2_{g_{\widehat{\varphi}_1}}(u) + R^{\bar{k}j}\partial_j\partial_k u + \frac{1}{2}\nabla S(\omega_{\mathrm{ALE}}) \cdot \nabla u,$$

so $\mathcal{D}^*\mathcal{D} - \frac{1}{2}\Delta^2_{\mathrm{Euc}}$ is in $O(\|Z\|^{2-2k})$ as a scattering operator as $\|Z\| \to \infty$.

Since $u \in (\frac{\rho}{\varepsilon})^\delta \mathcal{C}^{4,\alpha}_{g_{\widehat{\varphi}_{1,b}}}(Z_1)$, this means that $\Delta^2_{\mathrm{Euc}}(\gamma u) \in (\frac{\rho}{\varepsilon})^{\delta-2-2k}\mathcal{C}^{0,\alpha}_{g_{H_1}}((\mathbb{C}^k \setminus \{0\})/\Gamma_{(-w_0,w)})$. Applying Theorem 8.3 in [57] on the orbifold cover $\mathbb{C}^k$ of $\mathbb{C}^k/\Gamma_{(-w_0,w)}$, we conclude that there exist a function $v \in (\frac{\rho}{\varepsilon})^{\delta+2-2k}\mathcal{C}^{4,\alpha}_{g_{\widehat{\varphi}_1,b}}((\mathbb{C}^k\setminus\{0\})/\Gamma_{(-w_0,w)})$ such that $\Delta^2_{\mathrm{Euc}}(v) = \Delta^2_{\mathrm{Euc}}(\gamma u)$. Hence $v-\gamma u$ is a biharmonic function that decays at infinity. Since there is no indicial roots in $(4-2k, 0)$ we have $v - \gamma u \in (\frac{\rho}{\varepsilon})^{4-2k}\mathcal{C}^{4,\alpha}_{g_{\widehat{\varphi}_{1,b}}}((\mathbb{C}^k \setminus B_2(0))/\Gamma_{(-w_0,w)})$ and this implies that $\gamma u \in (\frac{\rho}{\varepsilon})^{4-2k}\mathcal{C}^{4,\alpha}_{g_{\widehat{\varphi}_{1,b}}}((\mathbb{C}^k \setminus B_2(0))/\Gamma_{(-w_0,w)})$ so $u \in (\frac{\rho}{\varepsilon})^{4-2k}\mathcal{C}^{4,\alpha}_{g_{\widehat{\varphi}_{1,b}}}(Z_1)$. Since we assume $k > 2$, this decay allows us to integrate by parts

$$\int_{Z_1} \|\mathcal{D}u\|^2 \, \mathrm{d}\omega_{\widehat{\varphi}_1} = \int_{Z_1} u\mathcal{D}^*\mathcal{D}u \mathrm{d}\omega_{\widehat{\varphi}_1} = 0,$$

and then $\mathcal{D}u = 0$, so $\nabla^{1,0}u$ is a holomorphic vector field on $Z_1 = \widehat{\mathbb{C}^k/\Gamma_{(-w_0,w)}}$. The resolution $\pi : Z_1 \to \mathbb{C}^k/\Gamma_{(-w_0,w)}$ implies a biholomorphism $Z_1 \setminus \pi^{-1}(0) \cong (\mathbb{C}^k \setminus \{0\})/\Gamma_{(-w_0,w)}$, so we can represent the holomorphic vector field $\nabla^{1,0}u|_{Z_1\setminus\pi^{-1}(0)}$ as $\sum_{i=1}^k a_i \frac{\partial}{\partial z_i}$ where $a_i$ are $\Gamma_{(-w_0,w)}$-invariant function on $\mathbb{C}^k\setminus\{0\}$. By applying the Hartogs theorem, for each $a_i$ there exist a holomorphe extension $\widetilde{a}_i$ on $\mathbb{C}^k$. Because of the boundedness of $\widetilde{a}_i$, Liouville's theorem implies that $\widetilde{a}_i$ is a constant function, hence its decay at infinity implies that $\widetilde{a}_i = 0$. So $\nabla^{1,0}u \equiv 0$ on $Z_1 \setminus \pi^{-1}(0)$. Now notice that $Z_1\setminus\pi^{-1}(0)$ is a dense subset of $Z_1$, so by continuity $\nabla^{1,0}u \equiv 0$ on $Z_1$. Finally, $\nabla u = \nabla^{1,0}u + \nabla^{0,1}u = \nabla^{1,0}u + \overline{\nabla^{1,0}u} = 0$, so $u$ is a constant function on $Z_1$ that decays at infinity, implying that $u = 0$. □

**Lemma 5.5.** *If $\delta < 0$ and $L_0(u) = 0$ for $u \in (\frac{\rho}{\varepsilon})^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(Z_1 \times \mathbb{C}^{n-k})$, then $u = 0$. Here $L_0$ denotes the Lichnerowicz operator on the product space.*

*Proof.* It suffices to replace $\mathrm{Bl}_0^{\mathbb{C}^k}$ by $\mathbb{CP}^k_{(-w_0,w)}$ in Lemma 11 of [53] and use Lemma 5.4 instead of Proposition 8.9 in [57] in the proof of Lemma 11 in [53]. □

**Lemma 5.6.** *If $4-2k < \delta < 0$ and $\Delta^2_{\mathrm{Euc}}u = 0$ for $u \in (1+\frac{d^2}{\varepsilon^2})^{\frac{\delta}{2}} C^{4,\alpha}_{\frac{g_X}{\rho^2}}(((\mathbb{C}^k/\Gamma_{(-w_0,w)}) \setminus \{0\}) \times \mathbb{C}^{n-k})$, then $u = 0$.*

*Proof.* It suffices to pull-back to $(\mathbb{C}^k \setminus \{0\}) \times \mathbb{C}^{n-k}$ and use Lemma 12 in [53]. □

**Lemma 5.7.** *Suppose that $X$ has no non trivial holomorphic vector fields. If $4-2k < \delta < 0$, then the linear operator*

$$\widetilde{L}_{\omega_0} : \rho^\delta \mathcal{C}^{4,\alpha}_{\frac{g_X}{\rho^2}}(H_2)_0 \times \mathbb{R} \to \rho^{\delta-4}\mathcal{C}^{0,\alpha}_{\frac{g_X}{\rho^2}}(H_2),$$

*defined by* $\widetilde{L}_{\omega_0}(u,R) = L_{\omega_0}(u) - R$ *where* $\omega_0 = \omega_X$ *and* $\rho^\delta \mathcal{C}^{4,\alpha}_{\frac{g_X}{\rho^2}}(H_2)_0$ *is the subspace of functions in* $\rho^\delta \mathcal{C}^{4,\alpha}_{\frac{g_X}{\rho^2}}(H_2)$ *that are* $L^2$*-orthogonal to the constant functions, has trivial kernel.*

*Proof.* On $H_2$,

$$\widetilde{L}_{\omega_0}(u,R) = \frac{1}{2}\nabla S(\omega_0).\nabla u - \mathcal{D}^*\mathcal{D}u - R = -\mathcal{D}^*\mathcal{D}u - R.$$

Let $u \in \rho^\delta \mathcal{C}^{4,\alpha}_{\frac{g_X}{\rho^2}}(H_2)_0$ and $R \in \mathbb{R}$ such that $\widetilde{L}_{\omega_\varepsilon}(u,R) = 0$, so $\mathcal{D}^*\mathcal{D}u + R = 0$. Both $\mathcal{D}^*\mathcal{D}u$ and $u$ belongs to $\rho^{\delta-4}\mathcal{C}^{0,\alpha}_{\frac{g_X}{\rho^2}}(H_2)$ and $\delta - 4 > -2k$, so they are integrable on $X$. This ensures that for a test function $\varphi \in \mathcal{C}^\infty(X)$,

$$\int_X \varphi \mathcal{D}^*\mathcal{D}u \mathrm{d}\omega_X = \int_X (\mathcal{D}^*\mathcal{D}\varphi) u \mathrm{d}\omega_X,$$

that is, $\mathcal{D}^*\mathcal{D}u + R = 0$ in the sense of distribution on $X$. Elliptic regularity implies that $u$ is a smooth function in the orbifold sense on $X$. Hence, on $X$,

$$\begin{aligned}
\|\mathcal{D}^*\mathcal{D}u\|^2_{L^2} &= \langle \mathcal{D}^*\mathcal{D}u, \mathcal{D}^*\mathcal{D}u\rangle_{L^2} \\
&= \langle -R, \mathcal{D}^*\mathcal{D}u\rangle_{L^2} \\
&= \langle -\mathcal{D}R, \mathcal{D}u\rangle_{L^2} \\
&= \langle 0, \mathcal{D}u\rangle_{L^2} = 0,
\end{aligned}$$

so $\mathcal{D}^*\mathcal{D}u = 0$ and so $R = 0$. Since the kernel of $\mathcal{D}^*\mathcal{D}$ on $X$ consists of constant functions (we assume $X$ has no non trivial holomorphic vector fields), this shows that $u = 0$.

□

**Proposition 5.8.** *Assume* $X$ *has no non trivial holomorphic vector fields. For* $4 - 2k < \delta < 0$ *and* $\varepsilon > 0$ *small enough, the operator* (5.2)

$$\widetilde{L}_\varepsilon : \rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0 \times \mathbb{R} \to \rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X}),$$

*where* $\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0$ *is the subspace of functions in* $\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})$ *that are* $L^2$*-orthogonal to the constant functions, is invertible and its inverse* $P_\varepsilon := \widetilde{L}_\varepsilon^{-1}$ *is bounded by constant independent of* $\varepsilon$.

*Proof.* We will closely follow the proof proposition 9 in [53]. By the Schauder estimates[1], there is a

[1] Schauder interior estimates theorem: Consider the elliptic second order partial differential equation

$$Lu(x) = a^{i,j}(x)D^2_{ij}u(x) + b^i(x)D_iu(x) + c(x)u(x) = f(x), \tag{5.3}$$

uniform constant $C$ independent of $\varepsilon$, such that

$$\|u\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}} + |R| \leq C(\|u\|_{\rho^\delta C^{0}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}} + |R| + \|\widetilde{L}_\varepsilon(u,R)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}}). \tag{5.4}$$

We want to show that there exists a uniform constant $\widetilde{C}$ such that

$$\|u\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}} + |R| \leq \widetilde{C}\|\widetilde{L}_\varepsilon(u,R)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}}. \tag{5.5}$$

To prove (5.5), for a contradiction suppose that there is a sequence $\varepsilon_i \to 0$ with $u_i$ and $R_i$ such that $\|u_i\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_{\varepsilon_i}}{\rho^2}}} + |R_i| > i\|\widetilde{L}_{\varepsilon_i}(u_i,R_i)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_{\varepsilon_i}}{\rho^2}}}$ for each $i$. In particular, by (5.4),

$$i\|\widetilde{L}_{\varepsilon_i}(u_i,R_i)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_{\varepsilon_i}}{\rho^2}}} < \|u_i\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_{\varepsilon_i}}{\rho^2}}} + |R_i| \leq C(\|u_i\|_{\rho^\delta C^{0}_{\frac{\widehat{g}_{\varepsilon_i}}{\rho^2}}} + |R_i| + \|\widetilde{L}_{\varepsilon_i}(u_i,R_i)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_{\varepsilon_i}}{\rho^2}}}),$$

so that

$$(\frac{i}{C} - 1)\|\widetilde{L}_{\varepsilon_i}(u_i,R_i)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_{\varepsilon_i}}{\rho^2}}} < \|u_i\|_{\rho^\delta C^{0,\alpha}_{\frac{\widehat{g}_{\varepsilon_i}}{\rho^2}}} + |R_i|.$$

Without loss of generality, by multiplying $u_i$ and $R_i$ by a constant $\lambda_i$, we can suppose that $\|u_i\|_{\rho^\delta C^{0}_{\frac{g_{\varepsilon_i}}{\rho^2}}} + |R_i| = 1$, so $\|\widetilde{L}_\varepsilon(u_i,R_i)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{g_{\varepsilon_i}}{\rho^2}}} < \dfrac{1}{\frac{i}{C} - 1}$. This shows that $\|\widetilde{L}_\varepsilon(u_i,R_i)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{g_\varepsilon}{\rho^2}}} \to 0$ as $i \to \infty$. Moreover, the Schauder estimates (5.4) show that $u_i : X \setminus Y \to \mathbb{R}$ is uniformly bounded and uniformly equicontinuous in $\rho^{\delta-4} C^{4,\alpha}_{\frac{g_\varepsilon}{\rho^2}}$, so by the Arzelà-Ascoli theorem[23] there exists a convergent subsequence $\{u_{i_j}\}$ converging a function $u \in \rho^\delta C^{4,\alpha}_{\frac{g_X}{\rho^2}}(X \setminus Y)$ with convergence in $\rho^\delta C^{4}_{\frac{g_X}{\rho^2}}(K)$ on each compact subset $K \subset X \setminus Y$.[4] Also, $\{R_i\}$ is a bounded numerical sequence, so by the Bolzano-

---

on the domain $\Omega$, where the source term satisfies $f \in C^\alpha(\Omega)$. If there exists a constant $\lambda > 0$ such that the $a_{i,j}$ are strictly elliptic, $a^{i,j}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for all $x \in \Omega, \xi \in \mathbb{R}^n$ and the relevant norms coefficients are all bounded by another constant $\Lambda$ Then the weighted $C^{2,\alpha}$ norm of a bounded solution $u \in C^{2,\alpha}(\Omega)$ is controlled by the supremum of $u$ and the Holder norm of $f$, i.e, there exist a constant $C = C_{n,\alpha,\lambda,\Lambda} < \infty$ such that

$$\|u\|^*_{2,\alpha;\Omega} \leq C(\|u\|_{0;\Omega} + \|f\|^{(2)}_{0,\alpha;\Omega}).$$

See Theorem 5.5 in section 8.5 in [30] for a proof.

[2] Arzelà-Ascoli Theorem: Every bounded equicontinuous sequence of functions in $C^0([a,b],\mathbb{R})$ has a uniformly convergent subsequence. See Theorem 14 on page 224 [50] for a proof.

[3] Classically, Arzelà-Ascoli theorem could apply for compact sets, here we can cover $X \setminus Y$ by compact sets and inductively choosing subsequences.

[4] In general, if $\{u_i\} \subset C^{k,\alpha}$ and $\|u_i - u\|_{C^k} \to 0$, then $u \in C^{k,\alpha}$.

Weierstrass theorem, we can assume that the subsequence $\{R_{i_j}\}$ converges to some $R$, so that

$$\widetilde{L}_{\omega_X}(u,R) = \lim_{j\to\infty} \widetilde{L}_{\omega_{\varepsilon_{i_j}}}(u_{i_j}, R_{i_j}) = 0.$$

Lemma 5.4 implies that $u = 0$ and $R = 0$, so $R_{i_j} \to 0$. Since $\nabla S(\omega_\varepsilon) \to \nabla S(\omega_X) = 0$ when $\varepsilon \to 0$ and

$$\|\widetilde{L}_\varepsilon(u_{i_j}, R_{i_j})\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{g_{\varepsilon_{i_j}}}{\rho^2}}} = \|L_{\omega_{\varepsilon_{i_j}}}(u_{i_j}) - \frac{1}{2}\nabla S(\omega_{\varepsilon_{i_j}}).\nabla u_{i_j} - R_{i_j}\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{g_{\varepsilon_{i_j}}}{\rho^2}}},$$

we see that

$$\lim_{j\to\infty} \|L_{\omega_{\varepsilon_{i_j}}}(u_{i_j})\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{g_{\varepsilon_{i_j}}}{\rho^2}}} = 0.$$

On the other hand, $\sup_{x\in\widehat{X}} |\frac{u_{i_j}(x)}{\rho^\delta(x)}| \leq \|\frac{u_{i_j}}{\rho^\delta}\|_{C^0_{\frac{g_{\varepsilon_{i_j}}}{\rho^2}}} \leq 1$, so $\frac{u_{i_j}(x)}{\rho^\delta(x)}$ is bounded by 1 on the compact manifold $\widehat{X}$ for each $j$. In particular, it achieves a maximum, say at $q_j \in \widehat{X}$. In particular $\frac{u_{i_j}(q_j)}{\rho^\delta(q_j)} \leq 1$ and $\frac{u_{i_j}(q_j)}{\rho^\delta(q_j)} \to 1$ as $j \to \infty$, since $R_{i_j} \to 0$ as $j \to \infty$. On any compact subset of $X \setminus Y$, $u_{i_j}(q_j) \to 0$, so we must have $\rho(q_j) \to 0$. Taking a subsequence if needed, we can therefore assume that $q_j \to q \in \widehat{H}_1$. There are two possibilities, either $q \in \widehat{H}_1 \setminus (\widehat{H}_1 \cap \widehat{H}_2)$, or else $q \in \widehat{H}_1 \cap \widehat{H}_2$. If $q \in \widehat{H}_1 \setminus (\widehat{H}_1 \cap \widehat{H}_2)$, then $\frac{u_{i_j}(x)}{\rho^\delta(q_j)}$ converges to a function that satisfies Lemma 5.5, so $\frac{u_{i_j}(x)}{\rho^\delta(q_j)} \to 0$, in contradiction $\frac{u_{i_j}(q_j)}{\rho^\delta(q_j)} \to 1$. Otherwise, if $q \in \widehat{H}_1 \cap \widehat{H}_2$, then $\frac{u_{i_j}(x)}{\rho^\delta(q_j)}$ converges to a function that satisfies Lemma 5.6, $\frac{u_{i_j}(q_j)}{\rho^\delta(q_j)} \to 0$, again yielding to a contradiction with $\frac{u_{i_j}(q_j)}{\rho^\delta(q_j)} \to 1$. Consequently the inequality (5.5) holds. The inequality (5.5) shows that the kernel of $\widetilde{L}_\varepsilon$ is trivial and since it has index zero, it is invertible. Also $\widetilde{L}_\varepsilon$ is bijective bounded linear operator from one Banach space to another, so $\widetilde{L}_\varepsilon$ has bounded inverse. □

# CHAPTER 6

# NONLINEAR ANALYSIS AND THE MAIN THEOREM

If a perturbed metric $\widetilde{\omega}_\varepsilon = \widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}u$ has a constant scalar curvature $S(\widetilde{\omega}_\varepsilon) = R_\varepsilon$ for all $\varepsilon$, then $R_\varepsilon$ can be determined from the Kähler class $[\widehat{\omega}_\varepsilon]$ of $\widehat{\omega}_\varepsilon$. We try to find an approximate for $R$.

**Proposition 6.1.** *Let $X$ be a compact complex orbifold with singularities of type $\mathcal{I}$ along a subset $Y$ with codimension $k$ greater than 2, i.e., the normal bundle of $Y$ in $X$ has fibers of the form $\mathbb{C}^k \big/ \Gamma_{(-w_0,w)}$ for some weight $w$. Then the first Chern class of the $(-w_0, w)-$weighted blow-up $\widehat{X}$ of $X$ along $Y$ is*

$$c_1(\widehat{X}) = \pi^* c_1(X) - (\frac{1}{w_0}\sum_{i=1}^{k} w_i - 1)[E]|_E,$$

*where $[E]$ is the Poincaré class of the exceptional divisor.*

*Proof.* Away from $Y$, we have the canonical identification of canonical bundles $K_{\widehat{X}} = K_X$. Let $N_{\widehat{X}}(E)$ be the normal bundle of $E$ in $\widehat{X}$. By the adjunction formula

$$K_{\widehat{X}}|_E = K_E \otimes N^*_{\widehat{X}}(E) = K_E \otimes [-E]|_E.$$

On the other hand, if $V = \ker(\pi|_{E*})$ is the vertical tangent bundle of $\pi|_E : E \to Y$, then

$$T^*E \cong V^* \oplus \pi^* TY. \tag{6.1}$$

Since $\pi : E \to Y$ is a projective bundle with fiber $\mathbb{CP}^{k-1}_w$, $E = \mathbb{P}_w(W)$ is the weighted projectivization of some vector bundle $W \to Y$ of rank $k$ such that $N_X(Y) = W \big/ \Gamma_{(-w_0,w)}$. Now the canonical bundle of the weighted projective space is given by

$$K_{\mathbb{CP}^{k-1}_w} = \mathcal{O}_{\mathbb{CP}^{k-1}_w}(-\sum_{i=1}^{k} w_i),$$

see, for instance, [25] or 6.7.2 in [38]. Keeping in mind the decomposition of $W$ in the proof of Theorem 4.2, and using equation (6.1), this means that

$$\begin{aligned}
K_E &= \wedge^{k-1}(V^*) \otimes \pi^*(K_Y) \\
&= \pi^*(K_Y) \otimes \pi^*(\wedge^k(W^*)) \otimes \mathcal{O}_{E/Y}(-\sum_{i=1}^{k} w_i) \\
&= \pi^*(K_X)|_E \otimes \mathcal{O}_{E/Y}(-\sum_{i=1}^{k} w_i).
\end{aligned}$$

Finally, we have that

$$\begin{aligned} K_{\widehat{X}}|_E &= K_E \otimes N^*_{\widehat{X}}(E) \\ &= \pi^*(K_X)|_E \otimes \mathcal{O}_{E/Y}(-\sum_{i=1}^{k} w_i) \otimes \mathcal{O}_{E/Y}(w_0) \\ &= \pi^*(K_X)|_E \otimes \mathcal{O}_{E/Y}(-\sum_{i=1}^{k} w_i + w_0) = \pi^*(K_X)|_E \otimes (N_{\widehat{X}}(E))^{\frac{1}{w_0}\sum_{i=1}^{k} w_i - 1}. \end{aligned}$$

Globally on $\widehat{X}$, $N_{\widehat{X}}(E) = [E]$ is trivial on $\widehat{X} \setminus E$, hence

$$K_{\widehat{X}} = \pi^*(K_X) \otimes (N_{\widehat{X}}(E))^{\frac{1}{w_0}\sum_{i=1}^{k} w_i - 1}.$$

Since $c_1(X) = -c_1(K_X)$ and $c_1(\widehat{X}) = -c_1(K_{\widehat{X}})$, we finally obtain

$$c_1(\widehat{X}) = \pi^* c_1(X) - (\frac{1}{w_0}\sum_{i=1}^{k} w_i - 1)[E]|_E.$$

□

**Proposition 6.2.** *Let $\widehat{\omega}_\varepsilon$ be the family of Kähler metric of Theorem 4.2. Assume the singularity of type $\mathcal{I}$ is $\mathbb{C}^k / \Gamma_{(-w_0,w)}$. If there is a constant scalar curvature metric $\widetilde{\omega}_\varepsilon$ in the Kähler class $[\widehat{\omega}_\varepsilon]$, the scalar curvature of $\widetilde{\omega}_\varepsilon$ can be represented by*

$$S(\widetilde{\omega}_\varepsilon) = S(\omega_X) + \lambda \varepsilon^{2k-2} + R_\varepsilon,$$

*where $|R_\varepsilon| \le c\varepsilon^{2k}$ for some constant $c > 0$ independent of $\varepsilon$ and $\lambda$ is a topological constant depending on the Kähler class $[\widehat{\omega}_\varepsilon]$ and first Chern class of $\widehat{X}$.*

*Proof.* In this sense,

$$\begin{aligned} S(\widetilde{\omega}_\varepsilon)\operatorname{Vol}_{\widehat{\omega}_\varepsilon}(\widehat{X}) &= \int_{\widehat{X}} S(\widetilde{\omega}_\varepsilon)\tilde{\omega}_\varepsilon^n \\ &= \int_{\widehat{X}} (S(\widetilde{\omega}_\varepsilon)\tilde{\omega}_\varepsilon) \wedge \tilde{\omega}_\varepsilon^{n-1} \\ &= \int_{\widehat{X}} 2n\rho \wedge \widetilde{\omega}_\varepsilon^{n-1} \\ &= \int_{\widehat{X}} 2n(2\pi c_1(\widehat{X})) \wedge \widetilde{\omega}_\varepsilon^{n-1} \\ &= 4n\pi \int_{\widehat{X}} c_1(\widehat{X}) \cup [\widetilde{\omega}_\varepsilon]^{n-1} \\ &= 4n\pi \int_{\widehat{X}} c_1(\widehat{X}) \cup [\omega_\varepsilon]^{n-1}, \end{aligned}$$

so that

$$S(\tilde{\omega}_\varepsilon) = \frac{4n\pi \displaystyle\int_{\widehat{X}} c_1(\widehat{X}) \cup [\omega_\varepsilon]^{n-1}}{\mathrm{Vol}_{\widehat{\omega}_\varepsilon}(\widehat{X})} = \frac{4\pi n}{\displaystyle\int_{\widehat{X}} [\omega_\varepsilon]^n} \int_{\widehat{X}} c_1(\widehat{X}) \cup [\omega_\varepsilon]^{n-1}.$$

We set $C_\varepsilon = \dfrac{4\pi n}{\displaystyle\int_{\widehat{X}} [\widehat{\omega}_\varepsilon]^n}$ and $C = \dfrac{4\pi n}{\displaystyle\int_{\widehat{X}} [\omega_X]^n}$ as well, then by Remark 4.3,

$$\begin{aligned}
\int_{\widehat{X}} [\widehat{\omega}_\varepsilon]^n &= \int_X ([\omega_X] - \varepsilon^2[E])^n \\
&= \int_X [\omega_X]^n + \int_X \sum_{i=1}^{n} \binom{n}{i} (-\varepsilon^2[E])^i [\omega_X]^{n-i} \\
&= \int_X [\omega_X]^n + \sum_{i=1}^{n} \binom{n}{i} (-\varepsilon^2)^i \int_E [E]^{i-1} [\omega_X]^{n-i} \\
&= \int_X [\omega_X]^n + \sum_{i=k}^{n} \binom{n}{i} (-\varepsilon^2)^i \int_E [E]^{i-1} [\omega_X]^{n-i} \\
&= \int_X [\omega_X]^n + O(\varepsilon^{2k}),
\end{aligned}$$

since we must have that $i-1 \geq k-1$ for the second integral to be non-zero. Indeed, the only vertical contribution of $[E]^i[\omega_X]^{n-i}$ with respect to the fiber bundle $\pi : E \to Y$ is coming from $[E]^i$.

Hence

$$C_\varepsilon = \frac{4\pi n}{\displaystyle\int_{\widehat{X}} [\widehat{\omega}_\varepsilon]^n} = \frac{4\pi n}{\displaystyle\int_{\widehat{X}} [\omega_X]^n + O(\varepsilon^{2k})} = \frac{4\pi n}{\displaystyle\int_{\widehat{X}} [\omega_X]^n} + O(\varepsilon^{2k}) = C + O(\varepsilon^{2k}).$$

From the Lemma 6.1, $c_1(\widehat{X}) = \pi^* c_1(X) - (\dfrac{1}{w_0} \displaystyle\sum_{i=1}^{k} w_i - 1)[E]$ and by the Remark 4.3 again, $[\widehat{\omega}_\varepsilon] = [\omega_X] - \varepsilon^2[E]$, so

$$
\begin{aligned}
S(\widetilde{\omega}_\varepsilon) &= C_\varepsilon \int_{\widehat{X}} (\pi^* c_1(X) - (\frac{1}{w_0}\sum_{i=1}^{k} w_i - 1)[E]) \cup ([\omega_X] - \varepsilon^2[E])^{n-1} \\
&= C_\varepsilon \int_X c_1(X) \cup [\omega_X]^{n-1} \\
&+ C_\varepsilon \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^i \varepsilon^{2i} \int_{\widehat{X}} \pi^* c_1(X) \cup [\omega_X]^{n-1-i} \cup [E]^i \\
&- C_\varepsilon \sum_{i=0}^{n-1} (\frac{1}{w_0}\sum_{i=1}^{k} w_i - 1) \binom{n-1}{i} (-1)^i \varepsilon^{2i} \int_X [\omega_X]^{n-1-i} \cup [E]^{i+1} \\
&= S(\omega_X) + C_\varepsilon \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^i \varepsilon^{2i} \int_E \pi^* c_1(X) \cup [\omega_X]^{n-1-i} \cup [E]^{i-1} \\
&+ C_\varepsilon \sum_{i=0}^{n-1} (\frac{1}{w_0}\sum_{i=1}^{k} w_i - 1) \binom{n-1}{i} (-1)^i \varepsilon^{2i} \int_E [\omega_X]^{n-1-i} \cup [E]^i + O(\varepsilon^{2k}).
\end{aligned}
$$

Since $\pi^* c_1(X)$ and $\pi^*[\omega_X]^{n-i-1}$ are basic with respect to the bundle map $\pi : E \to Y$, in the first sum, we must have that $i - 1 \geq k - 1$ for the integral to be non-zero, while in the second sum, $i \geq k - 1$ for the integral to be non-zero. Hence we see that $S(\widetilde{\omega}_\varepsilon) = S(\omega_X) + \lambda\varepsilon^{2k-2} + R_\varepsilon$ with constant coefficient $\lambda = C(\frac{1}{w_0}\sum_{i=1}^{k} w_i - 1)\binom{n-1}{k-1}(-1)^{k-1}\int_E [\omega_X]^{n-k} \cup [E]^{k-1}$ and $R_\varepsilon$ as claimed.

□

Now we can find a better approximation of $u$ by looking at solution of $\mathcal{D}^*\mathcal{D}\Gamma = \lambda$ on $X \setminus Y$ for $\lambda$ defined in Proposition 6.2. To do so, we will consider the function

$$\Lambda(x) = \int_Y G(x, y)\mathrm{d}y,$$

where $G(x, y)$ is the Green function of the Lichnerowicz operator $\mathcal{D}^*\mathcal{D}$. The operator $G$ is associated to the Green function of $\mathcal{D}^*\mathcal{D}$ and by definition

$$\mathcal{D}^*\mathcal{D}G = \mathbf{Id} - \mathbf{P},$$

where $\mathbf{P}$ is the projection on constant functions, i.e, $\mathbf{P}(f) = \int_X \frac{f(y)}{\mathrm{Vol}(X)}$ . In terms of Schwartz kernels,

$$\mathcal{D}^*_x\mathcal{D}_x G(x, y) = \delta(x - y) - \frac{1}{\mathrm{Vol}(X)}, \quad \text{on X.}$$

In the distributional sense, we thus have that

$$\mathcal{D}^*_x\mathcal{D}_x \Lambda = \delta_Y - \frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(X)}, \quad \text{on X,}$$

where $\delta_Y$ is the current of integration along $Y$. Let us consider the function

$$\Gamma(x) = -\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(Y)}\lambda\Lambda(x).$$

To determine the asymptotic behaviour of $\Lambda(x)$ and $\Gamma(x)$ near $Y$, notice that in local coordinates near $Y$,

$$\begin{aligned} G(x,y) &= \mathcal{F}^{-1}(\sigma_4((\mathcal{D}^*\mathcal{D})^{-1}))(x-y) + O(|x-y|^{5-2n}) \\ &= \mathcal{F}^{-1}(|\xi|^{-4})(x-y) + O(|x-y|^{5-2n}) \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y).\xi}|\xi|^{-4}\mathrm{d}\xi + O(|x-y|^{5-2n}) \\ &= \frac{c}{|x-y|^{2n-4}} + O(|x-y|^{5-2n}), \end{aligned}$$

where $c$ is some positive constant and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on $\mathbb{R}^n$. Let $x_s$ be the projection of $x$ on $Y$ and $d = |x - x_s|$ be the distance to $Y$ in local coordinates. Then $|x-y|^2 = d^2 + |y-x_s|^2$, so setting $y_s = y - x_s$, we get that

$$\int_Y \frac{\mathrm{d}y_1 \dots \mathrm{d}y_{2n-2k}}{|x-y|^{2n-4}} = \int_Y \frac{\mathrm{d}y_{s_1} \dots \mathrm{d}y_{s_{2n-2k}}}{(d^2+|y_s|^2)^{n-2}}.$$

In polar coordinates in a ball of radius 1, this yields

$$\begin{aligned} \int_{B_1(0)} \frac{\mathrm{d}y_{s_1} \dots \mathrm{d}y_{s_{2n-2k}}}{(d^2+|y_s|^2)^{n-2}} &= \int_{\mathbb{S}^{2n-2k-1}}\int_0^1 \frac{r^{2n-2k-1}}{(d^2+r^2)^{n-2}}\mathrm{d}r\mathrm{d}\omega \\ &= \mathrm{Vol}(\mathbb{S}^{2n-2k-1})\int_0^1 \frac{r^{2n-2k-1}}{(d^2+r^2)^{n-2}}\mathrm{d}r \\ &= \frac{2\pi^{n-k}}{\Gamma(n-k)}\int_0^1 \frac{r^{2n-2k-1}}{(d^2+r^2)^{n-2}}\mathrm{d}r. \end{aligned}$$

By substituting $r = Rd$, we get

$$\int_0^1 \frac{r^{2n-2k-1}}{(d^2+r^2)^{n-2}}\mathrm{d}r = \int_0^{\frac{1}{d}} \frac{(Rd)^{2n-2k-1}d}{(d^2+(Rd)^2)^{n-2}}\mathrm{d}R = \frac{1}{d^{2k-4}}\int_0^{\frac{1}{d}} \frac{R^{2n-2k-1}}{(1+R^2)^{n-2}}\mathrm{d}R.$$

Note that the intregral $\int_0^{\frac{1}{d}} \frac{R^{2n-2k-1}}{(1+R^2)^{n-2}}\mathrm{d}R$ converges when $d \to 0$ by the Riemann criterion because $2(n-2)-(2n-2k-1) = 2k-3 > 1$ since $k > 2$. Moreover the integral can be computed explicitly

$$\begin{aligned} \int_0^{+\infty} \frac{R^{2n-2k-1}}{(1+R^2)^{n-2}}\mathrm{d}R &= \int_0^{\frac{\pi}{2}} \sin^{2n-2k-1}\theta\cos^{2k-5}\theta\mathrm{d}\theta, \text{ posing } R = \tan\theta \\ &= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{n-k-1}(\cos^2\theta)^{k-2}\sin\theta\cos\theta\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^1 t^{n-k-1}(1-t)^{k-3}\mathrm{d}t = \frac{\Gamma(n-k)\Gamma(k-2)}{2\Gamma(n-2)}. \end{aligned}$$

Hence,

$$\Lambda(x) \approx c(\frac{2\pi^{n-k}}{\Gamma(n-k)})(\frac{\Gamma(n-k)\Gamma(k-2)}{2\Gamma(n-2)})(\frac{1}{d})^{2k-4}+O((\frac{1}{d})^{2k-5}) = c'(\frac{1}{d})^{2k-4}+O((\frac{1}{d})^{2k-5}), \quad (6.2)$$

for $c'$ another constant.

Recalling that $r_\varepsilon = \varepsilon^{\frac{2k}{2k+1}}$, we can use the function $\Gamma$ to define a new metric

$$\widetilde{\omega}_\varepsilon := \widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}(\varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma),$$

where $\gamma_2 : \mathbb{R} \to \mathbb{R}$ is a cutoff function such that $\gamma_2(t) = 0$ for $t < 1$ and $\gamma_2(t) = 1$ for $t > 2$. On the support of $\gamma_2(\frac{d}{r_\varepsilon})$, $\rho \leq Cd$ for some constant and $d \geq r_\varepsilon$, so $\rho^2(\frac{1}{d})^{2k-4} \leq C^2(\frac{1}{d})^{2k-6} \leq C^2 r_\varepsilon^{6-2k}$. Hence if we denote $\Omega = \{x \in \widehat{X} : r_\varepsilon \leq d(x)\}$, then using (6.2),

$$\begin{aligned}
\|\varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma\|_{\rho^2 C^{l,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} &\leq c\varepsilon^{2k-2}\|\Gamma\|_{\rho^2 C^{l,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega)} \\
&\leq c'\varepsilon^{2k-2} r_\varepsilon^{6-2k} \\
&\leq c'\varepsilon^{\frac{5(2k)-2}{2k+1}} \\
&\leq c'\varepsilon^{\frac{4(2k)+2k-2}{2k+1}} \leq c'\varepsilon^4, \text{since } k > 2,
\end{aligned}$$

which tends to zero as $\varepsilon \to 0$. So $\widetilde{\omega}_\varepsilon = \widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}(\varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma)$ is a small perturbation of $\widehat{\omega}_\varepsilon$ and so $L_{\widetilde{\omega}_\varepsilon}$ is a small perturbation of $L_{\widehat{\omega}_\varepsilon}$ for sufficiently small $\varepsilon$. Now we would like to solve the non-linear equation

$$S(\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}u) = R, \quad (6.3)$$

with $u$ and $R$ of the form

$$\begin{aligned}
u &= \varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma + v, \\
R &= S(\omega_X) + \lambda\varepsilon^{2k-2} + R_\varepsilon.
\end{aligned}$$

So the goal is to find $v$. By replacing $u$ in the left side of (6.3), we get from (5.1) page 90 that

$$\begin{aligned}
S(\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}u) &= S(\widehat{\omega}_\varepsilon) + L_{\widehat{\omega}_\varepsilon}(u) + Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u) \\
&= S(\widehat{\omega}_\varepsilon) + \varepsilon^{2k-2}L_{\widehat{\omega}_\varepsilon}(\gamma_2(\frac{d}{r_\varepsilon})\Gamma) + L_{\widehat{\omega}_\varepsilon}(v) + Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u),
\end{aligned}$$

so solving (6.3) means to solve

$$L_{\widehat{\omega}_\varepsilon}(v) - R_\varepsilon = S(\omega_X) - S(\widehat{\omega}_\varepsilon) + \lambda\varepsilon^{2k-2} - \varepsilon^{2k-2}L_{\widehat{\omega}_\varepsilon}(\gamma_2(\frac{d}{r_\varepsilon})\Gamma) - Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u).$$

Now we define $F_\varepsilon : \rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0 \times \mathbb{R} \to \rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X}) \times \mathbb{R}$ by

$$F_\varepsilon(v, R) := S(\omega_X) - S(\widehat{\omega}_\varepsilon) + \lambda \varepsilon^{2k-2} - \varepsilon^{2k-2} L_{\widehat{\omega}_\varepsilon}(\gamma_2(\frac{d}{r_\varepsilon})\Gamma) - Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u).$$

**Remark 6.3.** *Note that the function $F_\varepsilon$ does not depends on $R$, but to use Banach fixed point theorem, we consider $F_\varepsilon$ as a function of $v$ and $R$.*

**Lemma 6.4.** *Suppose $\delta > 0$. Then there exists constants $c_0, c > 0$ such that if $\|\varphi\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} < c_0$, then*

$$\|Q_{\widehat{\omega}_\varepsilon}(\nabla^2 \varphi)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \le c \|\varphi\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \|\varphi\|_{\rho^2 C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}.$$

*Proof.* From Lemma 3.4 we have the following decomposition with finite sums:

$$\begin{aligned} Q_{\widehat{\omega}_\varepsilon}(\nabla^2 \varphi) = & \sum_q B_{q,4,2}(\nabla^4 \varphi, \nabla^2 \varphi) C_{q,4,2}(\nabla^2 \varphi) \\ & + \sum_q B_{q,3,3}(\nabla^3 \varphi, \nabla^3 \varphi) C_{q,3,3}(\nabla^2 \varphi) \\ & + |z| \sum_q B_{q,3,2}(\nabla^3 \varphi, \nabla^2 \varphi) C_{q,3,2}(\nabla^2 \varphi) \\ & + \sum_q B_{q,2,2}(\nabla^2 \varphi, \nabla^2 \varphi) C_{q,2,2}(\nabla^2 \varphi), \end{aligned}$$

where $B^i$s are bilinear forms and $C^i$s are smooth functions. Each term of the above decomposition is controlled by the $\|\varphi\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \|\varphi\|_{\rho^2 C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}$, for example

$$\begin{aligned} \|B_{q,3,3}(\nabla^3 \varphi, \nabla^3 \varphi)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} & \le \|\rho^{4-\delta} B_{q,3,3}(\nabla^3 \varphi, \nabla^3 \varphi)\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \\ & \le \|B_{q,3,3}\|_{\mathrm{op}} \|\rho^{3-\delta} \nabla^3 \varphi\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \|\rho \nabla^3 \varphi\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \\ & = \|B_{q,3,3}\|_{\mathrm{op}} \|\nabla^3 \varphi\|_{\rho^{\delta-3} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \|\nabla^3 \varphi\|_{\rho^{-1} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \\ & \le c \|\varphi\|_{\rho^\delta C^{3,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \|\varphi\|_{\rho^2 C^{3,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \\ & \le c \|\varphi\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \|\varphi\|_{\rho^2 C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}, \end{aligned}$$

using the fact that the embedding $\rho^\delta C^{k,\alpha}_g \to \rho^{\delta'} C^{k',\alpha'}_g$ is compact for $k' + \alpha' < k + \alpha$ and $\delta' < \delta$ and also that, $\|\nabla^i f\|_{\rho^\delta C^{k,\alpha}_g} \le c \|f\|_{\rho^{\delta+i} C^{k+i,\alpha}_g}$. □

**Proposition 6.5.** *For $4-2k<\delta<0$ very close to $4-2k$, there is a constant $c$ independent of $\varepsilon$ such that*

$$\|F_\varepsilon(0,0)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \leq c r_\varepsilon^{3-\delta}.$$

*Proof.* We consider four possible regions:

(a) On $\Omega_1=\{x\in\widehat{X}: d(x)<\varepsilon\}$ we have $\gamma_2(\frac{d}{r_\varepsilon})=0$ so we get

$$\begin{aligned} F_\varepsilon(0,0) &= S(\omega_X)-S(\widehat{\omega}_\varepsilon)+\lambda\varepsilon^{2k-2}-\varepsilon^{2k-2}L_{\widehat{\omega}_\varepsilon}(\gamma_2(\frac{d}{r_\varepsilon})\Gamma)-Q_{\widehat{\omega}_\varepsilon}(\nabla^2(\varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma)) \\ &= S(\omega_X)-S(\widehat{\omega}_\varepsilon)+\lambda\varepsilon^{2k-2}. \end{aligned} \tag{6.4}$$

Furthermore, in this region,

$$\rho=\sqrt{\varepsilon^2+d^2}\leq\sqrt{\varepsilon^2+\varepsilon^2}=\sqrt{2}\varepsilon.$$

By Lemma 3.5, the scalar curvature of the conformally changed metric $\omega'=e^{2f}\omega$ is equal to

$$S(\omega')=e^{-2f}(S(\omega)+2(2n-1)\Delta_\omega f-(2n-1)(2n-2)\|\nabla f\|^2_\omega).$$

Using this formula for $\omega'=\varepsilon^{-2}\widehat{\omega}_\varepsilon$ and $f=-\ln\varepsilon$ constant, we get $S(\varepsilon^{-2}\widehat{\omega}_\varepsilon)=\varepsilon^2S(\widehat{\omega}_\varepsilon)$ or $S(\widehat{\omega}_\varepsilon)=\varepsilon^{-2}S(\varepsilon^{-2}\widehat{\omega}_\varepsilon)$. Since $\varepsilon^{-2}\widehat{\omega}_\varepsilon$ tends to a scalar flat $\mathrm{ALE}$ metric in the fibers of $\widehat{\varphi_1}:\widehat{H}_1\to Y$, by Theorem 4.2 on page 87, $S(\widehat{\omega}_\varepsilon)=\varepsilon^{-2}O(\varepsilon)=O(\varepsilon^{-1})$. Hence

$$\begin{aligned} \|F_\varepsilon(0,0)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} &= \|S(\omega_X)-S(\widehat{\omega}_\varepsilon)+\lambda\varepsilon^{2k-2}\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \\ &= \|\rho^{4-\delta}(S(\omega_X)-S(\widehat{\omega}_\varepsilon)+\lambda\varepsilon^{2k-2})\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \\ &\leq (\sqrt{2}\varepsilon)^{4-\delta}(\|S(\omega_X)\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}+\|S(\widehat{\omega}_\varepsilon)\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}+\|\lambda\varepsilon^{2k-2}\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}) \\ &\leq (\sqrt{2}\varepsilon)^{4-\delta}(\|S(\omega_X)\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}+\|\varepsilon^{-2}S(\varepsilon^{-2}\widehat{\omega}_\varepsilon)\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}+\|\lambda\varepsilon^{2k-2}\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}) \\ &\leq (\sqrt{2}\varepsilon)^{4-\delta}(c_1+\varepsilon^{-2}c_2\varepsilon+\varepsilon^{2k-2}c_3) \\ &\leq c(\sqrt{2}\varepsilon)^{3-\delta}, \end{aligned}$$

where $c_1$, $c_2$, $c_3$ and $c$ are constants independent of $\varepsilon$.

(b) On $\Omega_2 = \{x \in \widehat{X} : \varepsilon < d(x) < r_\varepsilon\}$ we have $\gamma_2(\frac{d}{r_\varepsilon}) = 0$,

$$\varepsilon \leq \rho = \sqrt{\varepsilon^2 + d^2} \leq \sqrt{\varepsilon^2 + r_\varepsilon^2} \leq \sqrt{r_\varepsilon^2 + r_\varepsilon^2} = \sqrt{2} r_\varepsilon,$$

and (6.4) still holds. The terms $S(\omega_X)$ and $\lambda\varepsilon^{2k-2}$ can be estimated as in (a). By (4.4) on page 88 and the fact that $\omega_{\widehat{\varphi_1}}$ is scalar flat, we have that

$$S(\varepsilon^{-2}\widehat{\omega}_\varepsilon) = O(\varepsilon(\frac{d}{\varepsilon})^{-2k}),$$

$$S(\widehat{\omega}_\varepsilon) = \varepsilon^{-2} S(\varepsilon^{-2}\widehat{\omega}_\varepsilon) = O(\varepsilon^{-1}(\frac{d}{\varepsilon})^{-2k}) = O(\frac{1}{d}).$$

This means that

$$\|S(\widehat{\omega}_\varepsilon)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \leq C \| \frac{\rho^{\delta-4}}{d} \|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \leq C(\sqrt{2} r_\varepsilon)^{3-\delta},$$

for some constant $C$.

(c) On $\Omega_3 = \{x \in \widehat{X} : r_\varepsilon < d(x) < 2r_\varepsilon\}$ we have

$$\begin{aligned} F_\varepsilon(0,0) &= S(\omega_X) - S(\widehat{\omega}_\varepsilon) + \lambda\varepsilon^{2k-2} - \varepsilon^{2k-2} L_{\widehat{\omega}_\varepsilon}(\gamma_2(\frac{d}{r_\varepsilon})\Gamma) - Q_{\widehat{\omega}_\varepsilon}(\nabla^2(\varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma)) \\ &= S(\omega_X) + \lambda\varepsilon^{2k-2} - (S(\widehat{\omega}_\varepsilon) + \varepsilon^{2k-2} L_{\widehat{\omega}_\varepsilon}(\gamma_2(\frac{d}{r_\varepsilon})\Gamma) + Q_{\widehat{\omega}_\varepsilon}(\nabla^2(\varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma))) \\ &= S(\omega_X) + \lambda\varepsilon^{2k-2} - S(\widetilde{\omega}_\varepsilon). \end{aligned}$$

As in the previous case, $\|S(\omega_X)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \leq c r_\varepsilon^{3-\delta}$ and $\|\lambda\varepsilon^{2k-2}\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \leq c r_\varepsilon^{3-\delta}$, so we just need to control $\|S(\widetilde{\omega}_\varepsilon)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)}$. To check this, let us write $\widetilde{\omega}_\varepsilon$ as

$$\widetilde{\omega}_\varepsilon = \omega_{\mathrm{Euc}} + \sqrt{-1}\partial\bar{\partial} H,$$

where $\omega_{\mathrm{Euc}} = \sqrt{-1}\partial\bar{\partial}(|z|^2 + |w|^2)$ and

$$\begin{aligned} H &= \phi_1(z,w) + A\varepsilon^{2k-2}|z|^{4-2k}(1 + \phi_2(z,w))^{4-2k} + \varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\widetilde{\Gamma} + O(\varepsilon^{2k-1}|z|^{3-2k}) \\ &= A\varepsilon^{2k-2}|z|^{4-2k} + \widetilde{H}, \end{aligned}$$

where $\phi_1$ and $\phi_2$ are smooth functions and $A$ is a constant. Note that

$$\nabla^2 H = O(r_\varepsilon + \varepsilon^{2k-2} r_\varepsilon^{2-2k} + \varepsilon^{2k} r_\varepsilon^{-2k}) = O(\varepsilon^{2k-2} r_\varepsilon^{2-2k}) \to 0 \text{ as } \varepsilon \to 0. \tag{6.5}$$

Now

$$\begin{aligned}\|S(\widetilde{\omega}_\varepsilon)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)} &\leq \|S(\widetilde{\omega}_\varepsilon) - L_{\omega_{\mathrm{Euc}}}(H)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)} + \|L_{\omega_{\mathrm{Euc}}}(H)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)}\\ &= \|Q_{\omega_{\mathrm{Euc}}}(\nabla^2 H)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)} + \|\Delta^2_{\mathrm{Euc}} H\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)},\end{aligned}$$

and with the same procedure in the Proposition 13 in [53], we will show that each term is $O(r_\varepsilon^{3-\delta})$. From (6.5) on page 104 we get

$$\begin{aligned}\|Q_{\omega_{\mathrm{Euc}}}(\nabla^2 H)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)} &= \|\sum_q B_{q,4,2}(\nabla^4 H, \nabla^2 H) C_{q,4,2}(\nabla^2 H)\\ &+ \sum_q B_{q,3,3}(\nabla^3 H, \nabla^3 H) C_{q,3,3}(\nabla^2 H)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)}\\ &\leq c_1 r_\varepsilon^{4-\delta} \|\nabla^4 H\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)} \|\nabla^2 H\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_2)}\\ &+ c_2 r_\varepsilon^{4-\delta} \|\nabla^3 H\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)} \|\nabla^3 H\|_{C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)}\\ &\leq c_1 r_\varepsilon^{4-\delta} (\varepsilon^{2k-2} r_\varepsilon^{-2k})(\varepsilon^{2k-2} r_\varepsilon^{2-2k})\\ &+ c_2 r_\varepsilon^{4-\delta} (\varepsilon^{2k-2} r_\varepsilon^{1-2k})(\varepsilon^{2k-2} r_\varepsilon^{1-2k})\\ &\leq c\varepsilon^{4k-4} r_\varepsilon^{6-4k-\delta} \leq c r_\varepsilon^{3-\delta}.\end{aligned}$$

For the $L_{\omega_{\mathrm{Euc}}}(H)$ note that $L_{\mathrm{Euc}} = -\Delta^2_{\mathrm{Euc}}$ and $\Delta^2_{\mathrm{Euc}}(|Z|^{4-2k}) = 0$, so

$$\begin{aligned}\|\Delta^2_{\mathrm{Euc}} H\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)} &= \|\Delta^2_{\mathrm{Euc}}(A\varepsilon^{2k-2}|Z|^{4-2k} + \widetilde{H})\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)}\\ &= \|\Delta^2_{\mathrm{Euc}} \widetilde{H}\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)}\\ &\leq C r_\varepsilon^{4-\delta}(1 + \varepsilon^{2k-2} r_\varepsilon^{1-2k})\\ &\leq c r_\varepsilon^{3-\delta}.\end{aligned}$$

(d) On $\Omega_4 = \{x \in \widehat{X} : 2r_\varepsilon < d(x)\}$ we have $\gamma_2(\frac{d}{r_\varepsilon}) = 1$, $\widehat{\omega}_\varepsilon = \omega_X$, $L_{\widehat{\omega}_\varepsilon}(\Gamma) = \lambda$ and

$$\begin{aligned}F_\varepsilon(0,0) &= S(\omega_X) - S(\omega_X) + \lambda\varepsilon^{2k-2} - \varepsilon^{2k-2} L_{\widehat{\omega}_\varepsilon}(\Gamma) - Q_{\widehat{\omega}_\varepsilon}(\nabla^2(\varepsilon^{2k-2}\Gamma))\\ &= -Q_{\widehat{\omega}_\varepsilon}(\varepsilon^{2k-2}\nabla^2\Gamma).\end{aligned}$$

By the approximation of $\Gamma$ and the assumption $4-2k < \delta < 0$ we get $\|\Gamma\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_4)} \leq \rho^{-\delta} d^{4-2k} \leq (\sqrt{2})^{-\delta} r_\varepsilon^{4-2k-\delta}$, because $\rho \leq \sqrt{2}\varepsilon$ and $d \geq 2r_\varepsilon$. Similarly $\|\Gamma\|_{\rho^2 C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_4)} \leq$

$\rho^{-2}r_\varepsilon^{4-2k} \leq \varepsilon^{-2}r_\varepsilon^{4-2k} \leq r_\varepsilon^{2-2k-\frac{1}{k}}$ since $\rho \geq \varepsilon$.

Therefore Lemma 6.4 implies that

$$\begin{aligned}
\|F_\varepsilon(0,0)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_4)} &= \|-Q_{\widehat{\omega}_\varepsilon}(\varepsilon^{2k-2}\nabla^2\Gamma)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_4)} \\
&\leq c\varepsilon^{4k-4}\|\Gamma\|_{\rho^{\delta}C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_4)}\|\Gamma\|_{\rho^{2}C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\Omega_3)} \\
&\leq c'\varepsilon^{4k-4}r_\varepsilon^{4-2k-\delta}r_\varepsilon^{4-2k-2-\frac{1}{k}} \\
&\leq c'\varepsilon^{4k-4}r_\varepsilon^{6-4k-\delta-\frac{1}{k}} \\
&\leq c'(r_\varepsilon^{1+\frac{1}{2k}})^{4k-4}r_\varepsilon^{6-4k-\delta-\frac{1}{k}} \\
&\leq c'r_\varepsilon^{2-\delta+2-\frac{3}{k}} \\
&\leq c'r_\varepsilon^{3-\delta}.
\end{aligned}$$

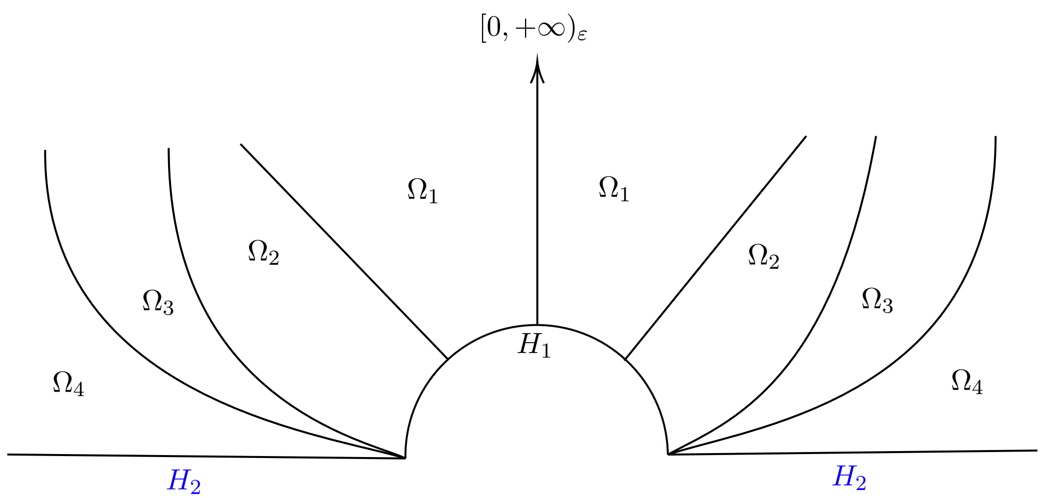



Figure 6.1: Four different regions on $\widehat{X}$

Finally, to prove existence of a solution for the non-linear equation

$$S(\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}u) = R,$$

for $\varepsilon > 0$ small enough, we show that there exist $v_\varepsilon \in \rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0$ such that

$$F_\varepsilon(v_\varepsilon, R_\varepsilon) = L_{\widehat{\omega}_\varepsilon}(v_\varepsilon) - R_\varepsilon.$$

Define $\mathcal{N}_\varepsilon : \rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0 \times \mathbb{R} \to \rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0 \times \mathbb{R}$ by $\mathcal{N}_\varepsilon(v,R) = P_\varepsilon F_\varepsilon(v,R)$, where $P_\varepsilon := \widetilde{L}_\varepsilon^{-1} : \rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X}) \to \rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0 \times \mathbb{R}$ is as Proposition 5.8 on page 93. If we show that

$\mathcal{N}_\varepsilon$ is a contraction, then by the Banach fixed point theorem, there exist unique $(v_\varepsilon, R)$, such that $\mathcal{N}_\varepsilon(v_\varepsilon, R) = (v_\varepsilon, R)$ or equivalently $F_\varepsilon(v_\varepsilon, R) = \widetilde{L}_\varepsilon(v_\varepsilon, R)$. Since $\widetilde{L}_\varepsilon(v_\varepsilon, R) = L_{\widehat{\omega}_\varepsilon}(v_\varepsilon) - R$, then $F(v_\varepsilon) + R = L_{\omega_\varepsilon}(v_\varepsilon)$. Now we are going to show that $\mathcal{N}_\varepsilon$ is a contraction on a suitable domain. By Proposition 6.2, we must have that $R = R_\varepsilon$.

**Lemma 6.6.** *There exist constants $c_0, \varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,*

$$\|\mathcal{N}_\varepsilon(v_1, R_1) - \mathcal{N}_\varepsilon(v_2, R_2)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \le \frac{1}{2}\|v_1 - v_2\|_{\rho^\delta C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})},$$

*for $(v_i, R)$ such that $\|v_i\|_{\rho^2 C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} < c_0$.*

*Proof.* The proof is essentially the same as Lemma 23 in [56]. Since $P_\varepsilon$ is bounded independently of $\varepsilon$, we just need to control

$$\|F_\varepsilon(v_1, R_1) - F_\varepsilon(v_2, R_2)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} = \| - Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u_1) + Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u_2)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})}.$$

By the mean value theorem there exist $t \in [0, 1]$ such that for $X = (1 - t)u_1 + tu_2$,

$$S(\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}u_1) - S(\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}u_2) = L_{\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}X}(u_1 - u_2).$$

Hence, this means that

$$Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u_1) - Q_{\widehat{\omega}_\varepsilon}(\nabla^2 u_2) = (L_{\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}X} - L_{\widehat{\omega}_\varepsilon})(u_1 - u_2).$$

The linear operator $L_{\widehat{\omega}_\varepsilon}$ is bounded independently of $\varepsilon$, so

$$\begin{aligned}
\|(L_{\widehat{\omega}_\varepsilon + \sqrt{-1}\partial\bar{\partial}X} - L_{\widehat{\omega}_\varepsilon})(u_1 - u_2)\|_{\rho^{\delta-4}C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} &\le C(\|u_1\|_{\rho^2 C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} + \|u_2\|_{\rho^2 C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})})\|u_1 - u_2\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \\
&\le 2c'C\|u_1 - u_2\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \\
&\le 2c'C\|v_1 - v_2\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})},
\end{aligned}$$

where the constant $c'$ can be chosen as small as we want, provided $c_0$ and $\varepsilon$ are sufficiently small, since $u_i = \varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma + v_i$ and when $\varepsilon \to 0$

$$\|\varepsilon^{2k-2}\gamma_2(\frac{d}{r_\varepsilon})\Gamma\|_{\rho^2 C^{k,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \le c(\frac{\varepsilon}{r_\varepsilon})^{2k-2} = o(1).$$

By Properties 5.8, the result follows. □

Now we define open set

$$\mathcal{U}_\varepsilon = \{v \in \rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0 : \|v\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \le (1 + 2c)Cr_\varepsilon^{3-\delta}\},$$

where $C$ is the independent bound of $P_\varepsilon$ and $c$ is the constant in Properties 6.5.

**Proposition 6.7.** *Suppose $\delta < 0$ is sufficiently close to $4-2k$. Then for $\varepsilon > 0$ sufficiently small, the map $\mathcal{N}_\varepsilon : \mathcal{U}_\varepsilon \to \mathcal{U}_\varepsilon$ is a contraction and therefore has a fixed point $v_\varepsilon$.*

*Proof.* Note that if $(v, R) \in \mathcal{U}_\varepsilon$, then we have

$$\|v\|_{\rho^2 C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \leq \varepsilon^{\delta-2}\|v\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})} \leq (1+2c)C\varepsilon^{\delta-2}r_\varepsilon^{3-\delta} \leq c_0,$$

for sufficiently small $\varepsilon$, so Lemma 6.6 applies to $\mathcal{U}_\varepsilon$. It remains to check that $\mathcal{N}_\varepsilon(\mathcal{U}_\varepsilon) \subseteq \mathcal{U}_\varepsilon$. To do this, for any $v \in \mathcal{U}_\varepsilon$, Proposition 6.5 and Lemma 6.6 implies that:

$$\begin{aligned}\|\mathcal{N}_\varepsilon(v,R)\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0} &\leq \|\mathcal{N}_\varepsilon(v,R) - \mathcal{N}_\varepsilon(0,0)\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0} + \|\mathcal{N}_\varepsilon(0,0)\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0} \\ &\leq \frac{1}{2}\|(v,R)\|_{\rho^\delta C^{4,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0} + C\|F_\varepsilon(0,0)\|_{\rho^{\delta-4} C^{0,\alpha}_{\frac{\widehat{g}_\varepsilon}{\rho^2}}(\widehat{X})_0} \\ &\leq \frac{1}{2}((1+2c)Cr_\varepsilon^{3-\delta}) + Cc(r_\varepsilon^{3-\delta}) \leq (1+2c)Cr_\varepsilon^{3-\delta}.\end{aligned}$$

□

The above proposition completes the proof of our main theorem.

**Theorem 6.8.** *Suppose that $X$ is a compact cscK orbifold with no holomorphic vector fields, and such that the set of singular points $Y$ of $X$ is of complex co-dimension $> 2$. Suppose, furthermore, that any point $p \in Y$ has a local orbifold uniformization chart of the form $\mathbb{C}^{n-k} \times (\mathbb{C}^k / \Gamma_{(-w_0,w)})$ where $\Gamma_{(-w_0,w)}$ is a finite linear group of type $\mathcal{I}$. Then on the $(-w_0, w)$-weighted blow-up $\widehat{X}$ of $X$ along $Y$, the class $[\omega_X] - \varepsilon^2[E]$ admits a cscK metric for $\varepsilon > 0$ sufficiently small, where $E = \pi^{-1}(Y)$ is the exceptional divisor of the partial resolution $\pi : \widehat{X} \to X$.*

Unless the singularity of type $\mathcal{I}$ is of the form $(-r, 1, \dots, 1)$ for some $r \in \mathbb{N}$, $\widehat{X}$ also has a singularity of type $\mathcal{I}$ along a suborbifold of complex codimension $k$. However, as described on page 47, since the singularity is of type $\mathcal{I}$, we can find a sequence of weighted blow-ups

$$\widehat{X}_l \to \widehat{X}_{l-1} \to \dots \to \widehat{X}_1 \to X,$$

with $\widehat{X}_1 = \widehat{X}$ and $\widehat{X}_l$ smooth. Thanks to Proposition 1.148, we can apply Theorem 6.8 iteratively to each $\widehat{X}_i$ to obtain on $\widehat{X}_l$ a cscK metric, which establishes Corollary C in the introduction.

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