UNIVERSITÉ DU QUÉBEC À MONTRÉAL

UN THÉORÈME D'ISOMORPHISME POUR CERTAINS OPÉRATEURS DIFFÉRENTIELS ELLIPTIQUES LINÉAIRES SUR DES VARIÉTÉS QUASI-ASYMPTOTIQUEMENT CONIQUES

THÈSE

PRÉSENTÉE

COMME EXIGENCE PARTIELLE

DU DOCTORAT EN MATHÉMATIQUES

PAR

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OCTOBRE 2024

UNIVERSITÉ DU QUÉBEC À MONTRÉAL

AN ISOMORPHISM THEOREM FOR SOME LINEAR ELLIPTIC DIFFERENTIAL OPERATORS ON QUASI-ASYMPTOTICALLY CONICAL MANIFOLDS

THESIS

PRESENTED

AS PARTIAL REQUIREMENT

TO THE PH.D IN MATHEMATICS

BY

MOHAMED NOUIDHA

OCTOBER 2024

ACKNOWLEDGMENTS

First and foremost, I give thanks to God for all the beautiful things in my life; this thesis is certainly one of them. I also acknowledge that any negative aspects are due to my own actions.

It is customary to thank one's research director, in my case it couldn't be more appropriate. I would like to thank Frédéric Rochon for introducing me the subject of this thesis. His help, insightful remarks and patience were crucial to the success of this project. The rigor with which he apprehends mathematics is something to look up to. I will be always in his debt.

I also want to thank the university's deanship for allowing me to undertake this project and for considering the challenging period I experienced.

Finally, I want to thank the members of CIRGET for fostering a great environment for research in mathematics.

CONTENTS

RÉSUMÉ

Dans cette thèse nous étudions un opérateur différentiel elliptique linéaire de la forme $\mathcal{P} = \Delta + V - \lambda$ sur une variété quasi-asymptotiquement conique (**QAC**) (M, g), où g est une métrique polyhomogène, et V est un $b-$champ de vecteur non borné par rapport à la métrique g.

Au chapitre 4 nous étudions un opérateur elliptique plus général (à coefficients non bornés) de la forme $\mathcal{A} = \Delta + V + r - \lambda$, où r est une fonction bornée supérieurement, et nous prouvons une estimation de Schauder globale pour l'opérateur $\mathcal{A}$ sur une variété Riemannienne non compacte.

Puis au chapitre 3, nous développons des espaces de Hölder à poids qui prennent en compte le comportement asymptotique de l'opérateur $\mathcal{P}$ sur les variétés **QAC**, et démontrons un théorème d'isomorphisme sur les espaces à poids définis en utilisant le résultat prouvé au chapitre 4.

Le chapitre 2 contient quelques résultats sur les tenseurs polyhomogènes. Faute de référence, nous avons décidé d'ajouter des preuves à certains résultats qui sont nécessaires pour le chapitre 3. Par exemple, nous montrons que l'opérateur de Hodge-Laplace d'une métrique QAC polyhomogène est de la forme $\Delta = x_{max}^2 \mathrm{P}_{\mathcal{V}_{Qb}}$, où $\mathrm{P}_{\mathcal{V}_{Qb}}$ est un polynôme du second ordre de champs de vecteurs Qb avec des coefficients polyhomogènes et sans terme d'ordre 0.

Mots clés: Opérateurs différentiels elliptiques, Espaces de Hölder à poids, Variétés Riemanniennes non compactes, Estimées de Schauder, Variétés quasi-asymptotiquement coniques, Metriques polyhomogènes.

ABSTRACT

In this thesis we study a linear elliptic differential operator of the form $\mathcal{P} = \Delta + V - \lambda$ on a quasi-asymptotically conical (**QAC**) manifold (M, g) where g is a polyhomogeneous metric and V is a $b-$vector field that is unbounded with respect to the metric g.

In chapter 4 we study a more general elliptic operator (with unbounded coefficients) of the form $\mathcal{A} = \Delta + V + r - \lambda$, where r is a function bounded above, and prove a global Schauder estimate for the operator $\mathcal{A}$ on a non compact Riemannian manifold.

Then in chapter 3, we develop weighted Hölder spaces that take into account the asymptotic behavior of the operator $\mathcal{P}$ on **QAC** manifolds, and prove an isomorphism theorem on the defined weighted spaces using the result proved in chapter 4.

Chapter 2 contains some results on polyhomogeneous tensors. For a lack of reference, we decided to add proofs to some results that are needed in chapter 3. For example, we show that the Hodge-Laplace operator of a polyhomogeneous $QAC-$metric is of the form $\Delta = x_{max}^2 \,\mathrm{P}_{\mathcal{V}_{Qb}}$, where $\mathrm{P}_{\mathcal{V}_{Qb}}$ is a polynomial of second order of $Qb-$vector fields with polyhomogeneous coefficients and without a term of order 0.

Keywords: Elliptic differential operators, Weighted Hölder spaces, Non-compact Riemannian manifolds, Schauder estimates, Quasi-asymptotically conical manifolds, Polyhomogeneous metrics.

INTRODUCTION

Let (M, g) be a complete Riemannian manifold, and $\mathcal{P}$ a linear elliptic operator of second order defined by $\mathcal{P} = \Delta + V - \lambda$, where V is a smooth vector field on M. When M is compact, the mapping properties of $\mathcal{P}$ are relatively well understood (see for instance theorem 1.4.4). This ceases to be true on non compact manifolds, where we need **weighted** Hölder spaces adapted to the asymptotic behavior of Δ and V at infinity. One of the issues with the operator $\mathcal{P}$ is that the vector V is potentially unbounded. For instance, in (Chaljub-Simon and Choquet-Bruhat, 1979) they consider a similar operator on **AE** manifolds but they require that the coefficients of order 1 and 0 be decreasing at infinity.

The theory of elliptic operators on weighted Hölder spaces was introduced by (Nirenberg and Walker, 1973) and studied extensively by (Lockhart and Mc Owen, 1985) and (McOwen, 1979). It was also used by (Chaljub-Simon and Choquet-Bruhat, 1979) to study regularity of linear elliptic operator of second order on **asymptotically euclidean** (**AE**) manifolds, the work of whom was adapted by Joyce to study **asymptotically locally euclidean** (**ALE**) manifolds, and **quasi-asymptotically locally euclidean** (**QALE**) manifolds.
More recently, (Degeratu and Mazzeo, 2017) proved Fredholm results of generalized Laplace-type operators for weighted Sobolev and Hölder spaces on **quasi-asymptotically conical** (**QAC**) manifolds. We also mention the work of Conlon, Degeratu and Rochon (Conlon *and al.*, 2019), where such a result is used to solve a complex Monge-Ampère equation on weighted Hölder spaces in order to build Ricci-flat $QAC-$metrics.

Before we state our results, we would like to explain the motivation behind the choice of the differential operator $\mathcal{P}$, which is related to the existence of Kähler Ricci solitons on non compact manifolds.

0.1 Ricci solitons

0.1.1 Riemannian Ricci soliton

On a Riemannian manifold (X, g_0), the Ricci flow is a heat-like equation of the form:

$$\begin{aligned} &\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)), \\ &g(0) = g_0, \end{aligned} \tag{1}$$

where $\mathrm{Ric}(g(t))$ denotes the Ricci curvature of the metric $g(t)$. A **Ricci soliton** is a self-similar solution of this equation. More precisely, $(M, g(t))$ is called a Ricci solition if

$$g(t) = \sigma(t)\phi_t^* g(0),$$

where $\phi_t : M \to M$ is a time dependent family of diffeomorphisms of M, and $\sigma(t)$ a time dependent scale factor, satisfying $\phi_0 = \mathrm{Id}$ and $\sigma(0) = 1$. If we plug $g(t)$ in equation (1) (and evaluate at $t = 0$), we obtain:

$$-2\phi_t^* \operatorname{Ric}(g_0) = \sigma'(t)\phi_t^* g_0 - \sigma(t)\phi_t^* \mathcal{L}_X g_0,$$
$$\operatorname{Ric}(g_0) - \frac{1}{2}\mathcal{L}_X g_0 + \frac{\sigma'(0)}{2} g_0 = 0,$$

where $X = -\frac{d}{dt}_{|t=0}\phi_t$. Letting $\lambda = \frac{\sigma'(0)}{2}$, the soliton is called shrinking, steady or expanding if $\lambda < 0, \lambda = 0$ or $\lambda > 0$ respectively.

We also could define a **Ricci soliton** as a triple (M, g, X) where (M, g) is a Riemannian manifold and X is a smooth vector field satisfying

$$\operatorname{Ric}(g) - \frac{1}{2}\mathcal{L}_X g + \lambda g = 0, \tag{2}$$

for $\lambda \in \{-1, 0, 1\}$. When $X = \nabla^g f$, we say that (M, g, X) is a **gradient Ricci soliton**, and the above equation translates to

$$\operatorname{Ric}(g) - \operatorname{Hess}(f) + \lambda g = 0. \tag{3}$$

0.1.2 Kähler Ricci soliton

Suppose now that (M, J, g) is a Kähler manifold. If (M, g) is a Ricci soliton and the vector field X is real holomorphic, then the soliton equation can be rewritten as:

$$\rho_\omega - \frac{1}{2}\mathcal{L}_X \omega + \lambda\omega = 0, \tag{4}$$

$\omega(.,.) = g(J.,.)$ being the Kähler form, and $\rho_\omega(.,.) = \operatorname{Ric}(J.,.)$ the Ricci form. We say that (M, g, X) is a **Kähler Ricci soliton**.

0.2 Motivation

0.2.1 Asymptotically conical Kähler-Ricci soliton

Let $\pi : M \to C$ be an equivariant (with respect to the real holomorphic torus action generated by the Reeb vector field) crepant resolution of a Kähler cone (C, J_C, g_C), and let X denote the holomorphic lift of the radial vector field on the cone. If we impose certain topological conditions on M (see propositions 3.1 and

3.2 of (Conlon and Deruelle, 2016)) we could define an asymptotically conical Kähler metric g on M and a function $F \in C^{\infty}(M)$ such that

$$\mathcal{L}_{JX}\omega_g = 0,$$
$$\mathcal{L}_{JX}F = 0,$$

(ω_g being the Kähler form of g) and such that F satisfies the following equation

$$\rho_{\omega_g} - \frac{1}{2}\mathcal{L}_X\omega_g + \omega_g = i\partial\overline{\partial}F. \tag{5}$$

Furthermore, the function F can be chosen such that it decays at infinity together with its derivatives. The metric g which is sometimes referred to as a **background metric** can also be chosen such that $g = \pi^*(g_C + \mathrm{Ric(g_C)})$ outside of a compact set. Suppose now that $\omega_\phi = \omega_g + i\partial\overline{\partial}\phi$ is a Kähler metric that solve the soliton equation (4). Then combining equations (5) and (4) we obtain that

$$i\partial\overline{\partial}\left(\log\frac{\omega_\phi^n}{\omega_g^n} + \frac{1}{2}X\phi - \lambda\phi\right) = i\partial\overline{\partial}F.$$

Hence, if ϕ satisfies the following complex Monge-Ampère equation

$$\log\frac{\omega_\phi^n}{\omega_g^n} + \frac{1}{2}X\phi - \lambda\phi = F, \tag{6}$$

then ω_ϕ is automatically a solution of equation (4).

Then we ask the following question: does there exist a smooth function ϕ that solves the complex Monge-Ampère equation (6)?

It turns out that if we choose our weighted spaces carefully, we should be able to solve it.

To show the existence of a solution we usually use the continuity method. The openness follows from the fact that the linearization of the previous operator is exactly the operator $\mathcal{P}$ which is an isomorphism. The closedness is the more difficult part, since we deal with an operator with unbounded coefficients on a non-compact manifold.

As examples of Asymptotically conical Kähler-Ricci solitons, Conlon and Deruelle show that given any negative line bundle L over a projective manifold D, the total space of $L^{\otimes p}$ admits an asymptotically conical expanding gradient Kähler-Ricci soliton for any p such that $c_1\left(K_D \bigotimes (L^*)^{\otimes p}\right) > 0$. This is actually the particular case of a more general construction described in Corollary B of (Conlon and Deruelle, 2016).

0.2.2 QAC Kähler-Ricci soliton

Let L be holomorphic line bundle over a compact complex orbifold D. The total space of L has singularities going off to infinity. In this case, the crepant resolution of L will introduce some topology at infinity, which make it harder to build QAC solitons using the same technique as in the AC case. For instance, if we take the canonical bundle K_X over a complex orbifold X with isolated singularities, then, the canonical bundle over the crepant resolution of X is a crepand resolution of K_X.

Conlon, Degeratu and Rochon solve this problem by using a natural compactification of L into an **orbifold with fibred corners** $\widetilde{L}$. Then, given a background metric of the right type (in their case a Calabi-Yau conic orbifold metric) they proceed by gluing suitable local models near each singularity.

We think that it is possible to proceed in the same manner in order to build QAC Kähler-Ricci soliton. In fact we could use the same technique as in the AC case to solve the soliton equation away from the singular set, and use the technique in (Conlon *and al.*, 2019) to glue suitable models near each singularity.

In this setting, the radial vector field with respect to the (background) conique metric on $L\backslash D$ is a $b-$vector field on $\widetilde{L}$.

This work focuses on solving one of the needed steps in order to build examples of QAC Kähler-Ricci soliton.

0.3 Main results

In chapter 4 we prove a version of Lunardi's theorem (Lunardi, 1998) which is a global Schauder estimate that is essential in the proof of our main result. We modify and expand the proof in (Deruelle, 2015) to obtain a more general version of the theorem. See chapter 3 (section 3.2) for the definition of the functional spaces mentioned below.

Theorem 0.3.1 (Lunardi) *Let (M^n, g) be a complete Riemannian manifold with positive injectivity radius, and V be a smooth vector field on M. Let $\mathcal{A}$ be an elliptic differential operator acting on tensors over M such that:*

$$\mathcal{A} = \underbrace{\Delta + \nabla_V}_{\Delta_V} + r(x)\ ,\ r \in C^3(M).$$

Suppose that $sup_{x\in M} r(x) = r_0 < \infty$ and that there exists a positive constant C such that $\sum_{i=1}^{3} ||\nabla^i r|| < C$. Assume also that there exists a positive constant K such:

$$||Rm(g)||_{C^3(M,E)} + ||Rm(g) * V||_{C^3(M,E)} + ||\nabla V||_{C^2(M,E)} \leq K,$$

where $Rm(g) * V = Rm(g)(V, ., ., .)$. Assume also that there exists a function $\phi \in C^2(M)$ and a constant $\lambda_0 \geq r_0$ such that:

$$\lim_{x \to \infty} \phi(x) = +\infty,\ sup_{x \in M} \left(\mathcal{A}(\phi)(x) - \lambda_0 \phi\left(x\right)\right) < \infty.$$

Then:

1. *For any $\lambda > r_0$, there exists a positive constant C such that for any $H \in C^0\left(M, E\right)$, there exists a unique tensor $h \in D^2_{\mathcal{A}}\left(M, E\right)$, satisfying:*

$$\mathcal{A}(h) - \lambda h = H,\ ||h||_{D^2_{\mathcal{A}}(M,E)} \leq C ||H||_{C^0(M,E)}.$$

   Moreover $D^2_{\mathcal{A}}\left(M, E\right)$ is continuously embedded in $C^\theta\left(M, E\right)$ for any $\theta \in (0, 2)$, i.e. there exists a positive constant $C(\theta)$ such that for any $h \in D^2_{\mathcal{A}}\left(M, E\right)$,

$$||h||_{C^\theta(M,E)} \leq C(\theta) ||h||^{\frac{\theta}{2}}_{D^2_{\mathcal{A}}(M,E)} ||h||^{1-\frac{\theta}{2}}_{C^0(M,E)}.$$

2. *For any $\lambda > r_0$, there exists a positive constant C such that for any $H \in C^{0,\theta}\left(M, E\right)$, $\theta \in \left(0, 1\right)$, there exists a unique tensor $h \in C^{2,\theta}\left(M, E\right)$ satisfying:*

$$\mathcal{A}(h) - \lambda h = H,\ ||h||_{C^{2,\theta}(M,E)} \leq C ||H||_{C^{0,\theta}(M,E)}.$$

Then in chapter 3 we prove the main results. Given a $QAC-$manifold (X, g) such that g is a polyhomogeneous metric, and a $b-$vector field V on X (we will denote by $M = \mathring{X}$), we obtain the following Schauder estimates for the linear elliptic operator $\mathcal{P}_\alpha$ (described in equation (3.7) and remark $3.3.4$):

Theorem 0.3.2 *Let $\mathcal{C}^{k;j,\theta}(M, E)$ be the functional space defined by:*

$$\mathcal{C}^{k;j,\theta}(M, E) = \left\{ h \in C^{k+j+\lfloor\theta\rfloor, \theta-\lfloor\theta\rfloor}_{loc}(M, E) \mid x^{-i}_{max} \nabla^i h \in C^{j+\lfloor\theta\rfloor, \theta-\lfloor\theta\rfloor}(M, E),\ \ \forall i = 0, \ldots, k \right\}.$$

such that $\theta \in (0, 2)$, and endowed with the norm:

$$||h||_{\mathcal{C}^{k;j,\theta}(M,E)} = \sum_{i=0}^{k} ||x^{-i}_{max} \nabla^i h||_{C^{j+\lfloor\theta\rfloor, \theta-\lfloor\theta\rfloor}(M,E)}$$

Suppose also that:

$$||Rm(g) * V||_{C^0(M,E)} + ||\nabla V||_{C^0(M,E)} < \infty$$

Then, for any constant $\lambda \in \mathbb{R}$ such that:

$$\lambda > \max\left(\sup_M V\ln(v^\alpha x_{max}^k), \sup_M V\ln(v^\alpha x_{max}^{k-1})\right)$$

we have that:

- *There exists a positive constant C such that for any $H \in C_{Qb}^{k,\theta}(M,E)$ there exists a unique $h \in D_{\mathcal{P}_\alpha}^{k+2,\theta}(M,E)$ satisfying:*

  $$\mathcal{P}_\alpha(h) - \lambda h = H,\ ||h||_{D_{\mathcal{P}_\alpha}^{k+2,\theta}(M,E)} \le C||H||_{C_{Qb}^{k,\theta}(M,E)};\ \theta \in [0,1)$$

  i.e. the operator

  $$\mathcal{P}_\alpha - \lambda : D_{\mathcal{P}_\alpha}^{k+2,\theta}(M,E) \to C_{Qb}^{k,\theta}(M,E)$$

  is an isomorphism of Banach spaces. Moreover, $D_{\mathcal{P}_\alpha}^{k+2}(M,E)$ embeds continuously in $\mathcal{C}^{k;0,\theta}(M,E)$ for any $\theta \in (0,2)$, i.e there exists a positive constant C such that for any $h \in D_{\mathcal{P}_\alpha}^{k+2}(M,E)$,

  $$||h||_{\mathcal{C}^{k;0,\theta}(M,E)} \le C||h||^{\frac{\theta}{2}}_{D_{\mathcal{P}_\alpha}^{k+2}(M,E)}||h||^{1-\frac{\theta}{2}}_{C_{Qb}^{k}(M,E)}$$

- *There exists a positive constant C such that, for $\theta \in (0,1)$*

  $$||h||_{\mathcal{C}^{k;2,\theta}(M,E)} \le C||H||_{C_{Qb}^{k,\theta}(M,E)}$$

As a consequence of the previous theorem, we prove the following result:

Theorem 0.3.3 (Isomorphism theorem) *Le (X,g) be a $QAC-$manifold such that g is polyhomogeneous. We will denote by $M = \mathring{X}$. Let V be a $b-$vector field on X such that:*

$$||Rm(g) * V||_{C^0(M,E)} + ||\nabla V||_{C^0(M,E)} < \infty$$

Then, the operator $\Delta_V - \lambda : D_{\Delta_V,\alpha}^{2+k,\theta}(M,E) \to C_{Qb,\alpha}^{k,\theta}(M,E)$ is an isomorphism of Banach spaces, for any $\theta \in (0,1)$ and any constant $\lambda \in \mathbb{R}$ such that:

$$\lambda > \max\left(\sup_M V\ln(x^\alpha x_{max}^k), \sup_M V\ln(x^\alpha x_{max}^{k-1})\right)$$

Note that the $b-$vector V is unbounded with respect to the $QAC-$metric g, which was taken into consideration when defining the weighted Hölder spaces used in this result.
It is also worth mentioning that this result generalizes a previous result of Deruelle (Deruelle, 2015) on asymptotically conical manifolds that was used to build expanding Kähler Ricci solitons in (Conlon and Deruelle, 2016) and in the analogous problem of constructing **QAC-**expanders, proves openness.

0.4 Future projects

We hope that this result will allow us to build quasi-asymptotically conical Kähler Ricci expanding solitons. That will be the logical sequel to this result.

CHAPTER 1

ANALYSIS ON RIEMANNIAN MANIFOLDS

We will try to summarize in this chapter the material needed in the following chapters, the main source being (Jost, 2008) and the three first chapters of (Joyce, 2000).

Let (M, g) be a non compact, complete Riemannian manifold of dimension n with positive injectivity radius. We will denote by dV the volume element induced by the metric g. When using coordinate notation, we implicitly refer to some local frame $(e_1, \ldots, e_n)$ on TM, and its dual frame $(e_1^*, \ldots, e_n^*)$ on T^*M.

1.1 Vector bundles

Definition 1.1.1 *Let E a be a vector bundle over M. A connection ∇ on E is a linear map $\nabla : \Gamma(E) \to \Gamma(E) \bigotimes \Gamma(T^*M)$ satisfying:*

- $\nabla(fs) = f\nabla s + s \otimes df$*, for all* $s \in \Gamma(E)$ *and* $s \in C^\infty(M)$*,*
- $\nabla_{\alpha v+w} s = \alpha \nabla_v s + \nabla_w s$*, for all* $v, w \in \Gamma(TM)$*,* $s \in \Gamma(E)$ *and* $\alpha \in C^\infty(M)$*.*

Remark 1.1.2 *To prevent any confusion we use $\Gamma(E)$ to denote the space of **smooth** global sections of E, i.e elements of $C^\infty(X, E)$.*

Proposition 1.1.3 *Let E be a vector bundle over M endowed with a connection ∇. Then, there exists a unique section $R(\nabla) \in \Gamma(\mathrm{End}(E)) \bigotimes \Lambda^2 T^*M)$ called the curvature that satisfies the following equation:*

$$R(\nabla)(X, Y)e = [\nabla_X, \nabla_Y] e - \nabla_{[X,Y]} e\,, \text{ for all } X, Y \in \Gamma(TM) \text{ and } e \in \Gamma(E)\,. \tag{1.1}$$

Remark 1.1.4 *Given two vector bundles E and F over M, endowed with connections ∇^E and ∇^F respectively, we can define a connection ∇ for each of the following vector bundles:*

- $E \bigoplus F$: $\nabla(e + f) = \nabla^E e + \nabla^F f$;

- $E \bigotimes F$: $\nabla(e \otimes f) = \nabla^E e \otimes f + e \otimes \nabla^F f$;
- E^*: $(\nabla L)(e) = d(L(e)) - L(\nabla^E e)$.

Definition 1.1.5 *Let $(E, \langle ., . \rangle)$ be a vector bundle over g endowed with a bundle metric. A connection ∇ on E is compatible with the bundle metric (or just **metric**) if*

$$X \langle S, T \rangle = \langle \nabla_X S, T \rangle + \langle S, \nabla_X T \rangle \ , \forall S, T \in \Gamma(E) \ , \forall X \in \Gamma(TM).$$

Theorem 1.1.6 (Fundamental Theorem of Riemannian Geometry) *There exists a unique torsion free (i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$), compatible connection on TM (equipped with the bundle metric g), defined by the following **Koszul formula**:*

$$2 \langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle . \quad (1.2)$$

This connection is called the Levi-Civita connection of the metric g.

1.1.1 Tensor bundles

Using the musical isomorphisms, a Riemannian metric g on a manifold M induces an Euclidean metric g^{-1} on the vector bundle T^*M. Locally, we will use notation g_{ij} and g^{ij} to refer to $g(e_i, e_j)$ and $g^{-1}(e_i^*, e_j^*)$ respectively. The Euclidean metrics g and g^{-1} induce a bundle metric $\langle ., . \rangle$ on each tensor bundle of the form $E = TM^{\otimes^r} \bigotimes T^*M^{\otimes^s}$ in the following manner:
Let $T, S \in \Gamma(E)$ be two sections of E, then

$$\langle T, S \rangle = g_{i_1 p_1} \dots g_{i_r p_r} g^{j_1 q_1} \dots g^{j_s q_s} \, T^{i_1 \dots i_r}_{j_1 \dots j_s} S^{p_1 \dots p_r}_{q_1 \dots q_s} , \quad (1.3)$$

where

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} \, e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^*_{j_1} \otimes \dots \otimes e^*_{j_s},$$
$$S = S^{p_1 \dots p_r}_{q_1 \dots q_s} \, e_{p_1} \otimes \dots \otimes e_{p_r} \otimes e^*_{q_1} \otimes \dots \otimes e^*_{q_s}.$$

This allows us to define the norm of a tensor T as $|T| = \langle T, T \rangle^{\frac{1}{2}}$, which is a continuous function on M. We will use this notation when defining function spaces.

The bundle of $k-$forms $\Lambda^k T^*M$ is a sub-bundle of $T^*M^{\otimes^k}$ locally spanned by sections of the form

$$e^*_{i_1} \wedge \cdots \wedge e^*_{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) e^*_{i_{\sigma(1)}} \otimes \cdots \otimes e^*_{i_{\sigma(k)}}.$$

Using this identification, we can compute product and norms of $k-$forms. We can also define the formal adjoint d^* of the exterior derivative $d : \Gamma\left(\Lambda^k T^*M\right) \to \Gamma\left(\Lambda^{k+1} T^*M\right)$, in order to define the **de Rham Laplacian** $\Delta = dd^* + d^*d$.

The Levi-Civita connection ∇ of g can be extended to a connection on a vector bundle of the form $TM^{\otimes^r} \bigotimes T^*M^{\otimes^s}$ (using remark 1.1.4). In particular, the induced connection is compatible with the bundle metric previously introduced.

1.2 Functional spaces

Let $(E, \langle .,. \rangle, \nabla)$ be a vector bundle over M endowed with a bundle metric and a compatible connection. The space of continuous sections of E that have k continuous bounded derivatives we denote by $C^k(M,E)$, and admits a norm:

$$||s||_{C^k(M,E)} = \sum_{i=0}^{k} \sup_M |\nabla^i s|, \tag{1.4}$$

making it a Banach space.

1.2.1 Hölder spaces

We define the **Hölder** space of continuous sections of E that have $k+\alpha$ ($\alpha \in (0,1)$) continuous bounded derivatives to be:

$$C^{k,\alpha}(M,E) = \left\{ s \in C^k(M,E) \mid ||s||_{C^{k,\alpha}(M,E)} = ||s||_{C^k(M,E)} + \left[\nabla^k s\right]_\alpha < \infty \right\}, \tag{1.5}$$

where the **Hölder** semi-norm $\left[\nabla^k s\right]_\alpha$ is defined by

$$[T]_\alpha = \sup_{x \in M} \sup_{\substack{y \in M \\ 0 < d(x,y) < \delta}} \frac{|T(x) - P^*_{x,y} T(y)|}{d(x,y)^\alpha},$$

$P_{x,y}$ being the parallel transport along the unique minimizing geodesic from x to y, and δ is the injectivity radius of g. Notice that $C^{k,\alpha}(M,E)$ is a Banach space with the norm $||.||_{C^{k,\alpha}(M,E)}$. When E is a trivial line bundle with the trivial connection, we denote those spaces by $C^k(M)$ and $C^{k,\alpha}(M)$ respectively.

Now, we list some useful results regarding Hölder spaces.

Proposition 1.2.1 *For $\alpha \in (0,1)$, $u \in C^{k,\alpha}(M)$, and $T \in C^{k,\alpha}(M,E)$, $uT \in C^{k,\alpha}(M,E)$. More precisely, there exists a positive constant $C > 0$ such that*

$$||uT||_{C^{k,\alpha}(M,E)} \leq C \left(\sum_{p=0}^{k} ||u||_{C^p(M)} ||T||_{C^{k-p,\alpha}(M,E)} + ||u||_{C^{p,\alpha}(M)} ||T||_{C^{k-p}(M,E)} \right)$$

Proof. Notice that $u(x)T(x) - u(y)P^*_{x,y}T(y) = u(x)\left(T(x) - P^*_{x,y}T(y)\right) + P^*_{x,y}T(y)\left(u(x) - u(y)\right)$. Consequently, we obtain that:

$$[uT]_\alpha \leq ||u||_{C^0(M)} [T]_\alpha + ||T||_{C^0(M,E)} [u]_\alpha$$

which then implies that

$$\begin{aligned} ||uT||_{C^{0,\alpha}(M,E)} &\leq ||u||_{C^0(M)} ||T||_{C^0(M,E)} + [uT]_\alpha \\ &\leq ||u||_{C^0(M)} ||T||_{C^0(M,E)} + ||u||_{C^0(M)} [T]_\alpha + ||T||_{C^0(M,E)} [u]_\alpha \\ &\leq ||u||_{C^{0,\alpha}(M)} ||T||_{C^0(M,E)} + ||u||_{C^0(M)} ||T||_{C^{0,\alpha}(M,E)}. \end{aligned}$$

For $k = 1$ we use the Leibniz rule to obtain that $\nabla(uT)(x) - \nabla(uT)(y) = \nabla u(x) \otimes (T(x) - T(y)) + T(y)(\nabla u(x) - \nabla u(y)) + \nabla T(x)(u(x) - u(y)) + u(y)(\nabla T(x) - \nabla T(y))$. This implies that:

$$[\nabla(uT)]_\alpha \leq ||\nabla u||_{C^0(M)} [T]_\alpha + ||T||_{C^0(M,E)} [\nabla u]_\alpha + ||\nabla T||_{C^0(M,E)} [u]_\alpha + ||u||_{C^0(M)} [\nabla T]_\alpha \tag{1.6}$$

Then we use the fact that $||uT||_{C^1(M,E)} \leq ||u||_{C^0(M)} ||T||_{C^1(M,E)} + ||\nabla u||_{C^0(M)} ||T||_{C^0(M,E)}$ combined with equation (1.6) to prove the case $k = 1$. We proceed by induction to prove the result for $k > 1$. □

Remark 1.2.2 *For any non-negative real value θ, we will denote by $C^\theta(M,E)$ the Hölder space $C^{\lfloor\theta\rfloor,\theta-\lfloor\theta\rfloor}(M,E)$.*

Theorem 1.2.3 (The Mean Value Theorem) *Let V and W be two normed vector spaces, $\Omega \subset V$ a convex subset of V and $f \in C^1(\Omega, W)$. Let $a, b \in \Omega$, and suppose that there exists a positive constant M such that*

$$||f'(ta + (t-1)b)|| \leq M \ , \ \text{ for all } t \in [0,1] \, .$$

Then we have:

$$||f(a) - f(b)|| \leq M||a - b||.$$

As a consequence of the previous theorem, we have

Proposition 1.2.4 *Let E be a vector bundle over M and $T \in C^1(M, E)$. Then there exist a positive constant C depending only on the constant δ used in the definition of Hölder spaces, such that:*

$$||T||_{C^{0,\theta}(M,E)} \leq C||T||_{C^1(M,E)}, \forall\theta \in (0, 1) \tag{1.7}$$

1.2.2 Sobolev Spaces

Let $p \geq 1$, we define the **Lebesgue space** $L^p(M, E)$ as the set of locally integrable sections (elements of $L^1_{loc}(M, E)$) of E such that the norm

$$||s||_{L^p(M,E)} = \left(\int_M |s|^p dV\right)^{\frac{1}{p}},$$

is finite. Given a non-negative integer k, we define the **Sobolev space**

$$W^{k,p}(M, E) = \left\{s \in L^p(M, E) \mid \nabla^i s \in L^p(M, E)\right\},$$

with the norm

$$||s||_{W^{k,p}(M,E)} = \left(\sum_{i=0}^{k} \int_M |\nabla^i s|^p dV\right)^{\frac{1}{p}}.$$

Notice that $W^{k,p}(M, E)$ is a Banach space with the norm $||.||_{W^{k,p}(M,E)}$. Note also that the derivatives in the previous definition are meant in the weak sense. The local Sobolev space $W^{k,p}_{loc}(M, E)$ consists of all locally integrable sections s of E whose restriction to any pre-compact $Q \Subset M$ lies on $W^{k,p}(Q, E)$.

$$W^{k,p}_{loc}(M, E) = \left\{s \in L^1_{loc}(M, E) \mid \forall Q \Subset M \ : \ s_{|_Q} \in W^{k,p}(Q, E)\right\}.$$

1.3 Linear differential operators

Let $\left(E, \langle.,.\rangle_E, \nabla^E\right)$ and $\left(F, \langle.,.\rangle_F, \nabla^F\right)$ be two vector bundles over M of ranks s and t respectively. Before we proceed to the definition, let us recall that for any $x \in M$ there exists a neighborhood $U \subset M$ of x, $(e_i)_{1\leq i\leq s} \in \Gamma\left(E_{|_U}\right)$, and $(f_i)_{1\leq i\leq t} \in \Gamma\left(F_{|_U}\right)$ such that $(e_i(y))_{1\leq i\leq s}$ and $(f_i(y))_{1\leq i\leq t}$ are basis of E_y and F_y respectively for any $y \in U$. $(e_i)_{1\leq i\leq s}$ and $(f_i)_{1\leq i\leq t}$ are called local frames. These frames can be used to locally trivialize E and F respectively.

Definition 1.3.1 *A linear map $P : \Gamma(E) \to \Gamma(F)$ is called a smooth linear differential operator of order l if in local trivializing neighborhood it has the following form:*

$$Pe = \sum_{i=1}^{l} P_i e \tag{1.8}$$

where $s \in \Gamma(E)$, and such that P_i is an $t \times s$ matrix whose components are of the form:

$$\sum_{|\beta|=i} a_\beta(x)\nabla^\beta$$

β being a multi-index with indices ranging from $1 \ldots n$, and a_β are locally defined smooth functions.
Let $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$ and $\sigma_\xi(P,x)$ the $t \times s$ matrix obtained from P_l by replacing ∇^β by ξ^β and evaluated at x. $\sigma(P,x)$ is called the ***principal symbol*** *of P. P is a linear elliptic differential operator if $\sigma_\xi(P,x)$ is an isomorphism for any $\xi \neq 0$.*

Remark 1.3.2 *Note that if a linear differential operator $P : \Gamma(E) \to \Gamma(F)$ is elliptic, then $\mathrm{rank}(E) = \mathrm{rank}(F)$.*

Definition 1.3.3 *The formal adjoint of a linear differential operator $P : \Gamma(E) \to \Gamma(F)$ is a smooth linear operator $P^* : \Gamma(F) \to \Gamma(E)$ satisfying:*

$$\int_M \langle Pe, f\rangle_F \ \mathrm{dV} = \int_M \langle e, P^* f\rangle_F \ \mathrm{dV} \text{ for all } e \in C_c^\infty(M,E), \text{ and } f \in C_c^\infty(M,F). \tag{1.9}$$

P is elliptic if and only if P^* is. The formal adjoint P^* depends on the choice of the bundles metrics on E and F as well as the Riemannian metric g on M.

1.3.1 Linear elliptic differential operators

Theorem 1.3.4 (Schauder estimates) *Let E and F be vector bundles over M, $\Omega \Subset M$ be a bounded domain, K a compact subset of Ω, and $L : \Gamma(E) \to \Gamma(F)$ a linear elliptic differential operator of order k with coefficients in $C^{k,\theta}(\Omega)$, where $k \geq 0$ and $\theta \in (0,1)$.*
Then, there exists a positive constant $C(K, \Omega, g, \theta, l,$ coefficients of L$)$ such that for any $u \in C^{k+l,\theta}(U,E)$ we have that:

$$||u||_{C^{k+l,\theta}(K,E)} \leq C\left(||L(u)||_{C^{l,\theta}(\Omega,F)} + ||u||_{C^0(\Omega,E)}\right). \tag{1.10}$$

Remark 1.3.5 *Let $f \in C^2(M)$, and assume that f attains its maximum (minimum) at a point $p \in M$. Then*

$$\Delta f(p) \leq 0 \ (\Delta f(p) \geq 0) \ , \text{ and } df(p) = 0$$

As a consequence, let $\mathcal{A} = \Delta + X$ where X is a smooth vector field, and $f \in C^2(M)$. If f attains its maximum (minimum) at a point $p \in M$, then $\mathcal{A}f(p) \leq 0$ ($\mathcal{A}f(p) \geq 0$).

1.4 Fredholm operators

Definition 1.4.1 *Let V and W be two Banach spaces and $P \in \mathcal{L}(V, W)$, $\mathcal{L}(V, W)$ being the set of continuous linear maps from V to W. Then, P is a* ***Fredholm*** *operator if:*

- $\mathrm{dim}(\mathrm{ker(P)})$ *is finite.*
- $\mathrm{ran(P)}$ *is a closed sub-space of W with finite codimension. Actually, finite codimension implies closedness.*

In this case, we define the index of P by the equation:

$$\mathrm{index}\, P = \mathrm{dim}(\mathrm{ker(L)}) - \mathrm{dim}(\mathrm{coker(L)})\,. \tag{1.11}$$

In particular, an isomorphism between V and W is Fredholm of index zero.

Definition 1.4.2 *Let V and W be two Banach spaces and $P \in \mathcal{L}(V, W)$. Then, P is a* ***compact*** *operator if the image under P of any bounded sequence in V contains a convergent sub-sequence in W.*

Remark 1.4.3 *The index of a Fredholm operator does not change under perturbation by a compact operator. In other words, if P and K are a Fredholm operator and a compact operator respectively, then, $P + K$ is also Fredholm with the same index as P.*

The following theorem is partly a consequence of theorem 1.3.4 , which in particular states that linear elliptic operators over a bounded domain are Fredholm.

Theorem 1.4.4 (Theorem 1.5.4 (Joyce, 2000)) *Let $k > 0$ and $l \geq k$ be integers, and $\theta \in (0, 1)$. Suppose that E and F are vector bundles over a compact manifold M, equipped with bundle metrics. Suppose also that $P : \Gamma(E) \to \Gamma(F)$ is a linear elliptic operator of order k with $C^{l,\theta}$ coefficients. Then*

- *P^* is elliptic with $C^{l-k,\theta}$ coefficients, and both $kerP$, $kerP^*$ are finite sub-spaces of $C^{k+l,\theta}(M, E)$ and $C^{l,\theta}(M, F)$ respectively.*

- *If $f \in C^{l,\theta}(M, F)$ then there exists $u \in C^{k+l,\theta}(M, E)$ such that $Pu = f$ if and only if $f \perp kerP^*$ (f is in the subspace $F/kerP^*$), and if one requires that $u \perp kerP$ then u is unique.*

The notation $\perp$ refers to the L^2 inner product defined in equation (1.9). The previous theorem is a statement of the Fredholm alternative.

CHAPTER 2
QUASI-ASYMPTOTICALLY CONICAL MANIFOLDS

Manifolds with fibred corners are a powerful tool to encode the asymptotic behavior of Riemannian metrics in term of the Lie algebra of vector fields. This chapter is a rather bare-bones introduction to the subject, the main source of which are the work of Richard Melrose, (Albin *and al.*, 2012), (Debord *and al.*, 2015), (Conlon *and al.*, 2019) and (Kottke and Rochon, 2021).

2.1 Stratified spaces

Definition 2.1.1 *A **smoothly stratified** space X of dimension n is a metrizable, locally compact, second countable space which decomposes into a locally finite union of locally closed **strata** $\mathcal{S} = \{S_\alpha\}$, where each S_α is a smooth manifold of dimension dim $S_\alpha \leq n$. The set of strata $\mathcal{S}$ obeys the following properties:*

(i) *Each strata S is endowed with a tubular neighborhood T_S and a radial function in the tubular neighborhood $\rho_S : T_S \to [0,1)$ such that $\rho_S^{-1}(0) = S$, together with a continuous retraction $\pi_S : T_S \to S$.*

(ii) *If $S_\alpha, S_\beta \in \mathcal{S}$, then $T_{S_\alpha} \cap S_\beta \neq \emptyset \Leftrightarrow S_\alpha \cap \overline{S_\beta} \neq \emptyset \Leftrightarrow S_\alpha \subset \overline{S_\beta}$. In this case we write $S_\alpha \leq S_\beta$. If moreover $S_\alpha \neq S_\beta$ then we write $S_\alpha < S_\beta$. This induces a partial order on the set of strata $\mathcal{S}$.*

(iii) *The retraction $\pi_S : T_S \to S$ is a locally trivial fibration with fibre the cone $C(L_S)$ over some compact stratified space L_S.*

(iv) *If we let X_i be the union of strata of dimension less than or equal to i, then we obtain a filtration $\emptyset \subset X_1 \subset \cdots \subset X_n = X$, X_{n-1} being the singular set and $X \backslash X_{n-1}$ the regular set.*

Remark 2.1.2 *Although we don't specify any restriction on the codimension of the singular set, in some cases (complex algebraic varieties) the singular set is at least of real codimension 2.*

Definition 2.1.3 *Let $(X, \mathcal{S})$ be a smoothly stratified space. The **depth** of X is the largest k such that $S_1 < S_2 < \cdots < S_{k+1}$ is a totally ordered chain in $\mathcal{S}$.*
*The **relative depth** of a stratum S is the largest k such that $S < S_1 < \cdots < S_k$ is a totally ordered chain in $\mathcal{S}$. The **relative depth** of a point $x \in X$ is the relative depth of the unique stratum that contains it.*

Example 2.1.4 *Orbifolds are a good example of stratified spaces. Indeed, let X be a complex orbifold. A point $x \in X$ is singular if in an orbifold chart, its local isotropy subgroup H_x is non trivial, and regular otherwise. We will denote by $\mathcal{S} = \{S_{\Sigma_\alpha}\}$ the canonical stratification of X such that each S_{Σ_α} is the union of singular points with isotropy subgroups in the isomorphism class Σ_α, S_{Σ_e} being the regular stratum.*

2.2 Manifold with corners

Let n be a positive integer and k an integer such that $0 \leq k \leq n$. Let us define $\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k}$. we define the set

$$\partial_l \mathbb{R}^n_k = \{x \in \mathbb{R}^n_k \mid x_i = 0 \text{ for exactly } l \text{ of the first } k \text{ indices } \}\,. \tag{2.1}$$

An open subset of $\mathbb{R}^n_k$ is a set $\Omega = \widetilde{\Omega} \cap \mathbb{R}^n_k$ for some open set $\widetilde{\Omega} \subset \mathbb{R}^n$. We will denote by $\partial_l \Omega = \widetilde{\Omega} \cap \partial_l \mathbb{R}^n_k$ the boundary of codimension l of Ω. Given two open sets Ω_1 and Ω_2 of $\mathbb{R}^n_k$, a map $\phi : \Omega_1 \to \Omega_2$ is a diffeomorphism if there exists two open sets $\widetilde{\Omega_1}$ and $\widetilde{\Omega_2}$ such that $\Omega_i = \widetilde{\Omega_i} \cap \mathbb{R}^n_k$ for $i = 1, 2$ and ϕ extends to a diffeomorphism $\phi : \widetilde{\Omega_1} \to \widetilde{\Omega_2}$ in the usual sense. Such a diffeomorphism restricts to a bijective map between boundaries of the same codimension.

Definition 2.2.1 *Let X be a paracompact Hausdorff topological space. A chart with corners (U, ϕ) on X is a homeomorphism $\phi : U \to V \subset \mathbb{R}^n_{k_\phi}$ for some integer k_ϕ, such that U and V are open sets of X and $\mathbb{R}^n_{k_\phi}$ respectively. Two charts with corners (U, ϕ) and (W, ψ) are compatible if $U \cap W = \emptyset$ or*

$$\psi \circ \phi^{-1} : \phi(U \cap W) \to \psi(U \cap W)$$

is a diffeomorphism in the sense described earlier. A maximal set of compatible charts that covers X is called a $\mathcal{C}^\infty$ structure with corners on X of dimension n. A $t-$manifold is a pair $(X, \mathcal{F} = C^\infty(X))$ such that $C^\infty(X)$ is the set of smooth functions on X induced by some $\mathcal{C}^\infty$ structure with corners.
We denote by $\partial_l X$ the set of boundaries of X of codimension l, defined by:

$$\partial_l X = \{p \in X \mid \text{charts around } p \text{ maps } p \text{ to } \partial_l \mathbb{R}^n_k\}\,.$$

The boundary hypersurfaces of X are the closure of connected components of $\partial_1 X$, the set of which will be denoted by $M_1(X)$.

Definition 2.2.2 (Melrose) *A manifold with corners X of dimension n and **depth** at most k, is a $t-$manifold of dimension n such that the boundary hypersurfaces of X (corners of codimension 1) are embedded submanifolds (with corners) and such that $\partial_l X = \emptyset$ for any integer $l > k$.*

Some definitions of manifold with corners drops the second part of the previous definition. For instance, Joyce's definition (see definition 2.2 of (Joyce, 2016)) doesn't impose any requirements on hypersurfaces. For example, the **teardrop** $T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y^2 \leq x^2 - x^4\}$ fits the definition of Joyce of a manifold with corners of dimension 2, but does not qualify as such according to definition 2.2.2 because ∂T self intersects.

Remark 2.2.3 *We will assume that each boundary hypersurface H_i of X has a defining function $x_i \in \mathcal{C}^\infty(X)$ such that:*

(1) $H_i = x_i^{-1}(0)$*;*

(2) x_i *is positive on* $X \backslash H_i$*;*

(3) dx_i *is nowhere vanishing on* H_i*;*

(4) *Each point* $p \in H_i$ *has a local coordinate system with* x_i *as one of its elements.*

Example 2.2.4 *As an example of a manifold with corners, we can take the product $X = X_1 \times X_2$ of two connected manifolds with boundaries. In this case, X has two hypersurfaces $H_1 = \partial X_1 \times X_2$ and $H_2 = \partial X_2 \times X_1$, the corner (of codimension 2) being $\partial X_1 \times \partial X_2$.*

2.3 QFB-metric

2.3.1 Iterated fibration structure

The notion of iterated fibration structure was introduced by Melrose in the context of the resolution (blowup) of smoothly stratified spaces. In fact, there is a one to one correspondence between smoothly stratified spaces and manifolds with fibred corners; see for instance propositions 2.5 and 2.6 in (Albin *and al.*, 2012).

Definition 2.3.1 (Melrose) *Let X be a manifold with corners and $(H_i)_{1 \leq i \leq l}$ the list of boundary hypersurfaces of X. An* ***iterated fibration structure*** *on X is a collection of fibrations $\pi = (\pi_i)_{1 \leq i \leq l}$ such that:*

(i) *Each $\pi_i : H_i \to S_i$ is a fiber bundle with fiber F_i where both F_i and S_i are manifolds with corners.*

(ii) *If $H_{ij} = H_i \cap H_j \neq \emptyset$ then dim $\mathsf{F}_i \neq$ dim F_j.*

(iii) *We write $H_i < H_j$ if $H_{ij} \neq \emptyset$ and dim $\mathsf{F}_i <$ dim F_j. In this case, $\pi_i : H_{ij} \to S_i$ is a surjective submersion.*

(iv) *The boundary hypersurfaces of S_j are exactly the $S_{ij} = \pi_j(H_{ij})$ with $H_i < H_j$. Moreover, there exists a surjective submersion $\pi_{ij} : S_{ij} \to S_i$ such that when restricted to H_{ij} we have $\pi_{ij} \circ \pi_j = \pi_i$.*

The iterated fibration structure induces a partial order on the set of boundary hypersurfaces. Thus, we define the **relative depth** of a boundary hypersurface H as the largest k such that $H < H_1 < H_2 < \cdots < H_{k-1}$ for some $k-1$ hypersurfaces H_i. Notice that if H_i and H_j are respectively minimal and maximal hypersurfaces , then both F_j and S_i are closed manifolds.

Let $H_1, \ldots, H_l$ be an exhaustive list of boundary hypersurfaces, and let $x_1, \ldots, x_l$ be the corresponding boundary defining functions. In what follows, we will denote by $v = \prod_{k=1}^{l} x_k$ a total boundary defining function.

Definition 2.3.2 *A **tube system** for a hypersurface H is a triplet $(\mathcal{N}_H, r_h, x_h)$ with $\mathcal{N}_H$ an open neighborhood of H in X, $r_h : \mathcal{N}_H \to H$ a smooth retraction, and $(r_h, x_h) : \mathcal{N}_H \to H \times [0, \infty)$ a diffeomorphism onto its image.*

Definition 2.3.3 *A **manifold with fibred corners** is a manifold with corners endowed with an iterated fibration structure (X, π). We say that the boundary defining functions are **compatible** with the iterated fibration structure, if for each boundary hypersurfaces $H_i < H_j$, the restriction of x_i to H_j is constant along the fibers of $\pi_j : H_j \to S_j$.*

We will always assume that the boundary defining functions are compatible with the iterated fibration structure in sense of definition 2.3.3, and such that x_i is identically equal to 1 outside of a tubular neighborhood of H_i.

Definition 2.3.4 *An **iterated fibred tube system** of a manifold with fibred corners X, is a family of tube* *systems* $(\mathcal{N}_i, r_i, x_i)$ *for* $H_i \in M_1(X)$ *such that for any hypersurfaces* $H_i < H_j$ *we have:*

$$r_j\left(\mathcal{N}_i \cap \mathcal{N}_j\right) \subset \mathcal{N}_i\ ,\ x_i \circ r_j = x_i\ ,\ \pi_i \circ r_i \circ r_j = \pi_i \circ r_i \text{ on } \mathcal{N}_i \cap \mathcal{N}_j \tag{2.2}$$

and the restriction of x_i *to* H_j *is constant along the fibers of* $\pi_j : H_j \to S_j$*.*

The existence of an iterated fibred tube system on a manifold with fibred corners is proved in lemma 1.4 of (Debord *and al.*, 2015).

2.3.2 Quasi fibred boundary metrics

We will review the notion of Quasi fibred boundary metrics introduced in (Conlon *and al.*, 2019). Let (X, π) be a manifold with fibred corners. We denote by:

$$\mathcal{V}_b = \left\{\xi \in C^\infty(X; TX) \mid \xi \text{ is tangent to the hypersurfaces of } X\right\}, \tag{2.3}$$

the Lie algebra of **b-vector fields**. This intrinsic definition is equivalent the following one

$$\mathcal{V}_b = \left\{\xi \in C^\infty(X; TX) \mid \xi\left(x_i\right) \in x_i \mathcal{C}^\infty\left(X\right)\right\}, \tag{2.4}$$

which is easier to use.

Definition 2.3.5 *A **quasi fibred boundary vector field** (or **QFB-vector field**) is a b-vector field* ξ *such that:*

(i) $\xi|_{H_i}$ *is tangent to the fibers of* π_i*;*

(ii) $\xi\left(v\right) \in v^2\mathcal{C}^\infty\left(X\right)$*, where* v *is a total boundary defining function.*

*These conditions are clearly still satisfied for the Lie bracket of two such vector fields. Thus, the set of **quasi fibred boundary** vector fields is a Lie algebra, which will be denoted by* $\mathcal{V}_{QFB}(X)$*.*

Remark 2.3.6 *The definition of **QFB-vector fields** depends on the choice of a total boundary defining function* $v \in C^\infty(X)$ *(see lemma 1.1 of (Kottke and Rochon, 2021)).*

Using definition 2.3.1 we can give an explicit description of QFB-vector fields. Indeed, let $H_1 < H_2 < \cdots < H_k$ be a totally ordered chain and $(x_1, y_1, x_2, y_2, \cdots, x_k, y_k, z)$ a local coordinate system around a point $p \in H_1 \cap H_2 \cap \cdots \cap H_k$ that straightens out the fibrations $\pi_i : H_i \to S_i$ such that:

- x_i is a boundary defining function of H_i;
- $y_i = \left(y_i^1, \cdots, y_i^{k_i}\right)$ for $i \in \{1, \cdots, k\}$ and $z = (z_1, \cdots, z_q)$;
- Each fibration π_i corresponds to the map

$$(x_1, y_1, \cdots, \hat{x}_i, y_i, \cdots, x_k, y_k, z) \mapsto (x_1, y_1, \cdots, x_{i-1}, y_{i-1}, y_i)\,. \tag{2.5}$$

Using equation (2.5) we see that $(x_{i+1}, y_{i+1}, \cdots, z)$ are coordinates on the fibers of π_i. Thus, the space of b-vector fields tangent to the fibers of the fibrations π_i is locally spanned by:

$$\frac{\partial}{\partial z_j},\ \frac{\partial}{\partial y_j^l},\ x_j\frac{\partial}{\partial x_j} \text{ for } j > i. \tag{2.6}$$

Now, using the second part of definition 2.3.5 we deduce that QFB-vector fields are spanned by:

$$\frac{\partial}{\partial z_j},\ v_i\frac{\partial}{\partial y_i^j},\ v_1x_1\frac{\partial}{\partial x_1},\ v_{i+1}\left(x_{i+1}\frac{\partial}{\partial x_{i+1}} - x_i\frac{\partial}{\partial x_i}\right), i = 1 \dots k-1,, \tag{2.7}$$

where $v_i = \prod\limits_{j=i}^{k} x_j$.

Remark 2.3.7 *$\mathcal{V}_{QFB}(X)$ is called a* ***structural Lie algebra*** *in the sense of definition 1.4 in* $(Ammann$and al.$, 2004)$. *In particular, structural Lie algebras are finitely generated protective $C^\infty(X)$-modules. Thus, using the* ***Serre-Swan*** *theorem, there exists a smooth vector bundle (the $QFB-$tangent bundle) ${}^\pi TX \to X$ such that $\mathcal{V}_{QFB}(X) \simeq \Gamma({}^\pi TX)$. This vector bundle is actually a* ***boundary tangential Lie algebroid*** *(see definition 1.14 of (Ammann* and al.*, 2004)). The same thing goes for $\mathcal{V}_b(X)$ and $\mathcal{V}(X)$ (definition 2.3.11 below).*

We will denote by $i_\pi : {}^\pi TX \to TX$ the natural bundle map that restricts to an isomorphism over $\mathring{X}$, such that:

$$\mathcal{V}_{QFB}(X) = i_{\pi*}\mathcal{C}^\infty(X; {}^\pi TX)\,. \tag{2.8}$$

The **QFB-cotangent bundle** ${}^{\pi}T^*X$ is then defined as the vector bundle dual to the QFB-tangent bundle ${}^{\pi}TX$, and is locally spanned by:

$$dz_j,\ \frac{dv_i}{v_i^2},\ \frac{dy_i^j}{v_i}. \tag{2.9}$$

Definition 2.3.8 *A* ***quasi fibred boundary metric*** *(or* ***QFB-metric****) is a positive-definite tensor $g_\pi \in \mathcal{C}^\infty(\mathring{X}; \mathrm{Sym}^2\left({}^{\pi}T^*X\right))$ that restricts to a Riemannian metric on $\mathring{X}$ via the map $i_\pi : {}^{\pi}TX \to TX$. We say that g_π is a smooth $QFB-$metric if it is smooth up to the boundary. We say that (X, g_π) is a $QFB-$manifold.*

Definition 2.3.9 *If a manifold with fibred corners (X,π) is such that $H_i = S_i$ and $\pi_i = Id$ for each maximal boundary hypersurface H_i, then a QFB-vector field is called* ***quasi-asymptotically conical vector field*** *(or* ***QAC-vector field****) and in the same manner, a QFB-metric is called a* ***quasi-asymptotically conical metric*** *(or* ***QAC-metric****). If g_{QAC} is a $QAC-$metric on X, then (X, g_{QAC}) is called a* ***QAC-manifold****.*

2.3.3 Examples of QAC manifolds

2.3.3.1 Asymptotically Conical manifolds

As we said, manifolds with fibred corners can be used to encode asymptotic conditions on certain complete manifolds. Let us for instance consider a non-compact Riemannian manifolds (M, g) of dimension $n+1$, and a compact subset $K \subset M$. Suppose that $M \backslash K$ is diffeomorphic to the non-compact ends of the Riemannian cone $\left((1,\infty) \times Y, g_c = dr^2 + r^2 h\right)$ with (Y, h) a compact Riemannian manifold (called the link of the cone). Suppose also that under such an identification we have that:

$$||\nabla^k \left(g - g_c\right)|| = O(r^{-\epsilon - k}) \text{ for all } k \in \mathbb{N}_0\text{, and some } \epsilon > 0.$$

Then, (M, g) is called an **Asymptotically Conical** (or AC) manifold. In particular, (M, g) is called **Asymptotically Euclidean** (or **AE**) manifold if (Y, h) is $\mathbb{S}^n$ equipped with the round metric, and **Asymptotically Locally Euclidean** (or **ALE**) if $Y = \mathbb{S}^n \backslash \Gamma$ where the finite subgroup $\Gamma \subset O(n)$ acts freely on $\mathbb{S}^n$. So **AC** manifolds can be seen as a generalization of **AE** and **ALE** manifolds.

Let (C, g) be a manifold with boundary, and g a $QAC-$metric on it. Then in a neighborhood of the boundary, the **QAC-cotangent bundle** ${}^{\pi}T^*C$ is generated locally by

$$\frac{d\rho}{\rho^2},\ \frac{dy^j}{\rho},$$

with ρ is a boundary defining function, and such that the metric g is of the form

$$g = \underbrace{\frac{d\rho^2}{\rho^4} + \frac{h}{\rho^2}}_{g_0} + \eta,$$

where η is some mixed terms tensor, and h is a Riemannian metric on ∂C. If we suppose that $||\eta||_{g_0} = O(\rho^\epsilon)$ (for some $\epsilon > 0$), then it becomes clear that $AC-$metrics are a particular case of $QAC-$metrics on manifold with boundary. In fact, in manifold with boundary, QAC-metrics corresponds the **scattering metrics** of(Melrose, 1995).

2.3.3.2 Quasi- Asymptotically Conical manifolds

Let us start with the case of **Quasi-Asymptotically Locally Euclidean** (or **QALE**) manifolds. These were introduced by Joyce (Joyce, 2001b) to study the existence of **Kähler metrics** on the resolution of $\mathbb{C}^n \backslash \Gamma$ where $\Gamma \subset U(n)$ is a finite subgroup that does not act freely on $\mathbb{C}^n \backslash \{0\}$. In this case, fixed points are subspaces of $\mathbb{C}^n$ with potentially different isotropy subgroups of Γ. The main source of examples of $QALE-$metrics are crepant resolutions of $\mathbb{C}^n \backslash \Gamma$ (see theorem 3.3 of (Joyce, 2001b)). Although, (Carron, 2011) showed that the Nakajima metric (Nakajima, 1999) is a $QALE-$metric in the sens of Joyce.
Mazzeo gave a description of these singular sets in terms of **iterated cone-edge** spaces (Mazzeo, 2006), a sub class of stratified spaces and together with Degeratu (Degeratu and Mazzeo, 2017) introduced **Quasi-Asymptotically Conical manifolds** as resolution blow-ups of these manifolds (into a manifold with fibred corners). An alternative description of these metrics was given in section 1 of (Conlon *and al.*, 2019). In some sense, $QAC-$manifolds generalize $QALE-$manifolds the way $AC-$manifolds generalize $ALE-$manifolds. In their work (Conlon *and al.*, 2019), Conlon, Degeratu and Rochon built $Calabi-Yau\, QAC-$metrics that are neither $QALE-$metrics nor Cartesian products of $AC-$metrics. Concrete examples of such metrics can be built using the following theorem

Theorem 2.3.10 (Corollary 4.10 of (Conlon *and al.*, 2019)) *Let (D, g_D) be a Kähler-Einstein Fano orbifold with isolated singularities of complex codimension at least two with each locally admitting a Kähler crepant resolution, then D admits a Kähler crepant resolution $\widehat{D}$ and the $QAC-$compactification $\widehat{X}_{QAC}$ of $K_{\widehat{D}}$ admits a Kähler $QAC-$metric asymptotic to g_C (a quasi-regular Calabi-Yau cone metric on $K \backslash D$) with rate δ for any $\delta > 0$.*

2.3.4 Qb manifolds

Definition 2.3.11 *Let (X,π) be a QAC-manifold and x_{max} the product of boundary defining functions of the maximal hypersurfaces of X. A smooth* ***quasi b-metric*** *(**Qb-metric**) is a metric g_{Qb} of the form:*

$$g_{Qb} = x_{max}^2 g_{QAC}. \tag{2.10}$$

for some smooth QAC-metric g_{QAC}. The Lie algebra of Qb-vector fields is defined as

$$\mathcal{V}_{Qb} = \{\xi \in C^{\infty}(X, TX) \mid \sup_{\mathring{X}} g_{Qb}(\xi,\xi) < \infty\}.$$

or equivalently as b-vector fields such that for each i

- *$\xi|_{H_i}$ is tangent to the fibers of π_i if H_i is not a maximal hypersurface;*
- *$\xi v \in \frac{v^2}{x_{max}} C^{\infty}(X)$.*

Remark 2.3.12 *From the previous definition, it is easy to see that $\mathcal{V}_{QAC}(X) = x_{max}\mathcal{V}_{Qb}(X)$.*

Proposition 2.3.13 *Given two QAC-vector fields $\tilde{V}$ and $\tilde{W}$ we have that $\left[\tilde{V},\tilde{W}\right] \in x_{max}\mathcal{V}_{QAC}(X)$.*
In addition, $X(f) \in x_{max}C^{\infty}(X)$ for any $QAC-$vector field X and function $f \in C^{\infty}(X)$.
Proof. Using remark 2.3.12, any QAC-vector field $\tilde{V}$ is of the form $x_{max}V$ for some some Qb-vector field V. Then, $\left[\tilde{V},\tilde{W}\right] = [x_{max}V, x_{max}W]$ for some Qb-vector fields V and W. A straightforward computation shows that:

$$[x_{max}V, x_{max}W] = x_{max}^2\left([V,W] + \frac{V(x_{max})}{x_{max}}W - \frac{W(x_{max})}{x_{max}}V\right).$$

By definition 2.3.11, both $\frac{W(x_{max})}{x_{max}}$ and $\frac{W(x_{max})}{x_{max}}$ are in $C^{\infty}(X)$, which implies that $[x_{max}V, x_{max}W] \in x_{max}\mathcal{V}_{QAC}(X)$. The second assertion follows from the fact that $\xi(f) \in C^{\infty}(X)$ for any $Qb-$vector field ξ.
□

2.4 Polyhomogeneity

Note that when defining a $QAC-$metric we only require that the given tensor is defined on $\mathring{X}$. In some cases, such tensors can be extended smoothly up to the boundary, but most of the time requiring a metric to be smooth up to the boundary is restrictive. In this section we introduce a class of tensors that, while not smooth up to the boundary, have a Taylor like asymptotic expansion near the boundary. These are called polyhomogeneous tensors.

Definition 2.4.1 *An index set K is a subset of $\mathbb{C} \times \mathbb{N}_0$ such that:*

$$(z,k) \in K \ , \ |(z,k)| \to \infty \implies \text{Re } z \to \infty, \tag{2.11}$$

$$(z,k) \in K \ , p \in \mathbb{N} \implies (z+p,k) \in K, \tag{2.12}$$

$$(z,k) \in K \implies (z,p) \in K \ \forall \ 0 \le p \le k. \tag{2.13}$$

An index set K is a non-negative index set if it also satisfies the following conditions:

$$\mathbb{N}_0 \times \{0\} \subset K, \tag{2.14}$$

$$(z,k) \in K \implies Imz = 0 \text{ and } Rez \ge 0, \tag{2.15}$$

$$(0,k) \in K \implies k = 0. \tag{2.16}$$

Given two index sets G and K, we define the index set $G + K$ as follows:

$$G + K = \{(z_1 + z_2, k_1 + k_2) \mid (z_1,k_1) \in G, (z_2,k_2) \in K\}. \tag{2.17}$$

Note that, if K and G are non-negative index sets, then $K \cup G \subset K + G$. In particular, given a non-negative index set H, we define the index set

$$H_\infty = \sum_{i=1}^{\infty} H = \bigcup_{i=1}^{\infty} \sum_{j=1}^{i} H. \tag{2.18}$$

It is easy to see that $H_\infty + H = H_\infty$. Before we define polyhomogeneous functions on a manifold with corners, let's start with the simpler case of a manifold with boundary.

Definition 2.4.2 *Let M be a compact manifold with boundary, and ρ the boundary defining function of ∂M. The set of polyhomogeneous functions on M with respect to an index set K, denoted by $\mathcal{A}^K_{phg}(M)$, is the the set of functions $f \in C^\infty(\mathring{M})$ such that:*

$$f \sim \sum_{(z,k)\in K} a_{(z,k)} \rho^z (\log \rho)^k \ , \ a_{(z,k)} \in C^\infty(\partial M), \tag{2.19}$$

where $\sim$ means that for any $N \in \mathbb{N}$ we have that:

$$f - \sum_{\substack{(z,k)\in K \\ Rez \le N}} a_{(z,k)} \rho^z (\log \rho)^k \in \dot{C}^N(M), \tag{2.20}$$

where $\dot{C}^N(M)$ is the set of N differentiable functions on M that restrict to zero on ∂M together with all their derivatives up to order N. We define

$$\dot{C}^\infty(M) = \bigcap_{N \in \mathbb{N}} \dot{C}^N(M).$$

Remark 2.4.3 *Note that if $K = \mathbb{N}_0 \times \{0\}$ then $\mathcal{A}^K_{phg}(M) = C^\infty(M)$. It is also easy to see that $\mathcal{A}^\emptyset_{phg}(M) = \dot{C}^\infty(M)$. Another important remark is that the multiplicative inverse of a positive polyhomogeneous function f that is bounded away from zero, is also polyhomogeneous. This is a direct consequence of theorems $B.1$ and $B.6$ of (Sher, 2023).*

Now we are ready to define polyhomogeneous functions on a manifold with corners X.

Definition 2.4.4 *An index family $\mathcal{K}$ on a manifold with corner X, is an assignment of an index set $\mathcal{K}(H)$ to each hypersurface $H \in M_1(X)$. If F is a boundary surface of X, then we will denote by $\mathcal{K}_{|_F}$ the family index that assigns $\mathcal{K}(H)$ to the boundary hypersurface $F \cap H$ of F (such that $H \in M_1(X)$). $\mathcal{K}$ is a non-negative family index if $\mathcal{K}(H)$ is a non-negative index set for each $H \in M_1(X)$.*
Given two family indices $\mathcal{G}$ and $\mathcal{K}$, we define family indices $\mathcal{G} + \mathcal{K}$ and $\mathcal{G}_\infty$ as follows:

$$\mathcal{G}_\infty(H) = (\mathcal{G}(H))_\infty,$$
$$(\mathcal{G} + \mathcal{K})(H) = \mathcal{G}(H) + \mathcal{K}(H) \text{ for each } H \in M_1(X).$$

We will denote by $\mathcal{A}^{\mathcal{K}}_{phg}(X)$ the space of polyhomogeneous functions on X with index family $\mathcal{K}$.

Definition 2.4.5 *$\mathcal{A}^{\mathcal{K}}_{phg}(X)$ is the set of functions $f \in C^\infty(\mathring{X})$ such that near each boundary hypersurface H:*

$$f \sim \sum_{(z,k) \in \mathcal{K}(H)} a_{(z,k)} \rho_H^z (\log \rho_H)^k \ , \ a_{(z,k)} \in \mathcal{A}^{\mathcal{K}_{|_H}}_{phg}(H), \tag{2.21}$$

ρ_H being the defining function of H.

Note that in the previous definition, the $a_{(z,k)}$ coefficients are well defined since the induction will ends when reaching a corner of maximal codimension which consists of a closed manifold.

Definition 2.4.6 *Let (X, π) be a QAC-manifold, and E a vector bundle over X. The set of polyhomogeneous sections of E with family index $\mathcal{K}$ is defined by:*

$$\mathcal{A}^{\mathcal{K}}_{phg}(X, E) = \mathcal{A}^{\mathcal{K}}_{phg}(X) \bigotimes_{C^\infty(X)} \Gamma(E)\,. \tag{2.22}$$

Definition 2.4.7 *We define a* ***polyhomogeneous QFB-metric*** *as an euclidean metric $g \in \mathcal{A}^{\mathcal{K}}_{phg}(X, \mathrm{Sym}^2({}^{\pi}T^*X))$ such that $\mathcal{K}$ is a non-negative family index. We define* ***polyhomogeneous QAC-metric*** *in the same manner. As a direct consequence of this definition, we have that $||\xi||_g$ is uniformly bounded on $\mathring{X}$ for any $QFB-$vector field ξ.*

Proposition 2.4.8 *The inverse of a* ***polyhomogeneous QFB-metric*** *g is also a* ***polyhomogeneous QFB-metric****. Thus, g induces a* ***polyhomogeneous euclidean metric*** *on the vector bundles $E = {}^{\pi}TX^{\otimes^r} \otimes {}^{\pi}T^*X^{\otimes^s}$.*
Proof. Use remark 2.4.3 □

Example 2.4.9 *Using equations (2.6) and (2.7) we can see that $\mathcal{V}_b \subset \mathcal{A}^{\mathcal{G}}_{phg}(X, {}^{\pi}TX)$ such that $\mathcal{G}$ is an index family satisfying $\mathcal{G}(H) \subset \mathbb{Z}_{\{\geq -1\}} \times \{0\}$ for every $H \in M_1(X)$.*

2.5 Some results on **polyhomogeneous QAC-metrics**

Some of the work done in $(Ammann$and al.$, 2004)$ can actually be extended to **polyhomogeneous QAC-metric**. For instance, let $(X, \mathcal{V}_{QAC}, g_{QAC})$ be a **QAC-manifold** such that g_{QAC} is polyhomogeneous with respect to a non-negative family index $\mathcal{G}$. We will denote by $\mathrm{Diff}_{\mathcal{V}_{QAC}}(X)$ the algebra of differential operators generated by vectors in $\mathcal{V}_{QAC}$ and with coefficients in $C^\infty(X)$.

Definition 2.5.1 *Given two vector bundles E_1 and E_2 over X, we will denote by*

$$\mathrm{Diff}_{\mathcal{V}_{QAC}}(X, E_1, E_2) = \mathrm{Diff}_{\mathcal{V}_{QAC}}(X) \bigotimes_{C^\infty(X)} \Gamma(E_1^* \otimes E_2) \tag{2.23}$$

the algebra of differential operators taking sections of E_1 to sections of E_2 generated by vectors in $\mathcal{V}_{QAC}$ and with coefficients in $C^\infty(X)$. To clarify this definition, let $\mathcal{U}$ be a local trivializing neighborhood of both E_1 and E_2. Then, $\Gamma(E_{i_{|\mathcal{U}}}) \simeq C^\infty(\mathcal{U}) \bigotimes \mathbb{C}^{N_i}$ for $i = 1, 2$. So locally, the elements of $\mathrm{Diff}_{\mathcal{V}_{QAC}}(X, E_1, E_2)$

are a linear combination of the composition of operators of the form $X \otimes A$, such that X in $\mathcal{V}_{QAC}(X)$, and A a smooth family of linear mappings in $\mathcal{L}\left(\mathbb{C}^{N_1}, \mathbb{C}^{N_2}\right)$. Note that this definition doesn't depend on the local trivialization since two local trivializations only differ by a smooth family of linear mappings $\in \mathcal{L}\left(\mathbb{C}^{N_1}, \mathbb{C}^{N_2}\right)$.

In the same manner, we define

$$\operatorname{Diff}_{\mathcal{V}_{QAC},\mathcal{G}}(X, E_1, E_2) = \mathcal{A}^{\mathcal{G}}_{phg}(X) \bigotimes_{C^\infty(X)} \operatorname{Diff}_{\mathcal{V}_{QAC}}(X, E_1, E_2), \tag{2.24}$$

the algebra of differential operators taking sections of E_1 to sections of E_2 generated by vectors in $\mathcal{V}_{QAC}$ and with polyhomogeneous coefficients in $\mathcal{A}^{\mathcal{G}}_{phg}(X)$. When $E_1 = E_2$ we will use the simpler notations $\operatorname{Diff}_{\mathcal{V}_{QAC}}(X, E)$ and $\operatorname{Diff}_{\mathcal{V}_{QAC},\mathcal{G}}(X, E)$.

Proposition 2.5.2 *Let ∇ be the Levi-Civita connection of g_{QAC}. Then, ∇ can be extended to a differential operator in $x_{max} \operatorname{Diff}_{\mathcal{V}_{QAC},\mathcal{G}}(X, {}^{\pi}TX, {}^{\pi}TX^* \otimes {}^{\pi}TX)$. Consequently, the Riemannian curvature tensor R is an element of $x^2_{max}\mathcal{A}^{\mathcal{G}}_{phg}(X, \Lambda^2 ({}^{\pi}T^*X) \bigotimes End({}^{\pi}TX))$.*

Proof. Let X, Y and Z be three QAC-vector fields. Then, using the Koszul identity, we have that

$$2\langle \nabla_X Y, Z\rangle = X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) + \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle - \langle [X, Z], Y\rangle. \tag{2.25}$$

Then, using proposition 2.3.13 we deduce that each term in the right side of equation (2.25) is in $x_{max}\mathcal{A}^{\mathcal{G}}_{phg}(X)$. Thus, $\langle \nabla_X Y, Z\rangle \in x_{max}\mathcal{A}^{\mathcal{G}}_{phg}(X)$, which implies that $\nabla_X Y \in x_{max}\mathcal{A}^{\mathcal{G}}_{phg}(X, {}^{\pi}TX)$. Since $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, the remaining statements are direct consequences of the previous one. $\square$

As a consequence of proposition 2.5.2 , we have the following corollary.

Corollary 2.5.3 *Let $k \in \mathbb{N}$. Then $\nabla^k R \in x^{2+k}_{max}\mathcal{A}^{\mathcal{G}_\infty}_{phg}(X, {}^{\pi}T^*X^{\otimes^k} \otimes \Lambda^2 ({}^{\pi}T^*X) \bigotimes End({}^{\pi}TX))$. More generally, if $T \in \mathcal{A}^{\mathcal{K}}_{phg}\left(X, {}^{\pi}TX^{\otimes^r} \otimes {}^{\pi}T^*X^{\otimes^s}\right)$, then $\nabla^k T \in x^k_{max}\mathcal{A}^{\mathcal{K}+\mathcal{G}_\infty}_{phg}\left(X, {}^{\pi}T^*X^{\otimes^k} \otimes {}^{\pi}TX^{\otimes^r} \otimes {}^{\pi}T^*X^{\otimes^s}\right)$.*

Proposition 2.5.4 *Let $T \in \mathcal{A}^{\mathcal{K}}_{phg}\left(X, {}^{\pi}TX^{\otimes^r} \otimes {}^{\pi}T^*X^{\otimes^s}\right)$ be a polyhomogeneous tensor with respect to some index family $\mathcal{K}$ such that $\mathcal{K}(H) \subset \mathbb{Z} \times \mathbb{N}_0$ for any $H \in M_1(X)$. Then, $||T||_{g_{QAC}} < \infty$ if and only if $\mathcal{K}$ can be chosen to be a non-negative index family.*

Proof. Let us choose a local frame near the boundary of X that diagonalizes the metric g_{QAC}. Then, we *have that:*

$$||T||^2_{g_{QAC}} = g_{QAC,i_1i_1}\cdots g_{QAC,i_ri_r}g^{j_1j_1}_{QAC}\cdots g^{j_sj_s}_{QAC}{T^{i_1\ldots i_r}_{j_1\ldots j_s}}^2.$$

The coefficients $g_{QAC,i_1i_1}\cdots g_{QAC,i_ri_r}g^{j_1j_1}_{QAC}\cdots g^{j_sj_s}_{QAC}$ are positive and bounded on $\mathring{X}$. Thus, $||T||_{g_{QAC}} < \infty$ implies that the coefficients $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ are also uniformly bounded on $\mathring{X}$. Consequently, $\mathcal{K}$ can be chosen to be a non-negative index family. To prove the opposite direction we only use that polyhomogeneous functions with respect to a non-negative index family are bounded. $\square$

As a consequence of proposition 2.5.4 we have the following important corollary that is going to be essential in the proof of the isomorphism theorem in the next chapter.

Corollary 2.5.5 *Let V be a $b-$vector field such that:*

$$||Rm(g_{QAC}) * V||_{C^0(\mathring{X})} + ||\nabla V||_{C^0(\mathring{X})} < \infty \tag{2.26}$$

Then both $(Rm(g_{QAC}) * V)$ and ∇V are polyhomogeneous tensors with respect to non-negative family indices. Consequently, we have that:

$$||x^{-(k+1)}_{max}\nabla^k \left(Rm(g_{QAC}) * V\right)||_{C^0(\mathring{X})} + ||x^{-k}_{max}\nabla^{k+1}V||_{C^0(\mathring{X})} < \infty \tag{2.27}$$

for any positive integer k. In particular, equation (2.26) is satisfied for any $QAC-$vector field V.
Proof. Let us note that both $Rm(g_{QAC}) * V$ and ∇V are polyhomogeneous tensors. Then using proposition 2.5.4 , the equation (2.26) implies that both $(Rm(g_{QAC}) * V)$ and ∇V are polyhomogeneous tensors with respect to a non-negative family indices. Equation (2.27) is a combination of propositions 2.5.3 and 2.5.4. The last assertion follows from the fact that $QAC-$vector fields are smooth up to the boundary. $\square$

2.5.1 Adjoints of differential operators

Let $(E, \langle,\rangle_E)$ be a hermitian vector bundle over $\mathring{X}$, and $C^\infty_{\mathcal{C}}(\mathring{X}, E)$ the space of compactly supported sections of E. We consider the hermitian inner product on $C^\infty_{\mathcal{C}}(\mathring{X}, E)$ defined by:

$$\langle \alpha, \beta\rangle := \int_{\mathring{X}} \langle \alpha, \beta\rangle_E \; dV \tag{2.28}$$

such that dV is the volume element induced by g_{QAC} over $\mathring{X}$.

Lemma 2.5.6 *Let $D \in \mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{G}}(X,E)$ be a differential operator and $D^{\#}$ its formal adjoint. Then, $D^{\#} \in \mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{K}}(X,E)$ for some non-negative index family $\mathcal{K}$.*

Proof. As we noted in definition 2.5.1, elements of $\mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{G}}(X,E)$ are locally linear combination of the composition of operators of the form $X \otimes A$ such that $X \in \mathcal{V}_{QAC}(X)$ and A a smooth family of endomorphisms of $\mathbb{C}^N$. We know that the adjoint of an endomorphism is again an endomorphism. It only remains to prove that the adjoint of an operator of the form $X \otimes 1$ is in $\mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{K}}(X,E)$.

Let $\alpha, \beta \in C^{\infty}_{\mathcal{C}}(\mathring{X},E)$, and U a trivializing neighborhood of E with respect to a unitary frame. Then

$$\alpha_{|U} = f_i \otimes z_i, \tag{2.29}$$

$$\beta_{|U} = h_j \otimes w_i\ ,\ f_i, h_j \in C^{\infty}(U), \text{and } z_i, w_j \in \mathbb{C}^N. \tag{2.30}$$

For the sake of simplicity we will suppose that $\alpha_{|U} = f \otimes z$ and $\beta_{|U} = h \otimes w$. Then

$$\int_U \langle X \otimes 1\alpha, \beta\rangle_E \ \mathrm{dV} = \int_U \left\langle \alpha, X^{\#} \otimes 1\beta \right\rangle_{\mathbb{C}^N} \ \mathrm{dV}$$
$$= \int_U X(f)\overline{h}\,\langle z,w\rangle_{\mathbb{C}^N} \ \mathrm{dV} = \int_U f\overline{X^{\#}(h)}\,\langle z,w\rangle_{\mathbb{C}^N} \ \mathrm{dV}$$

where $\langle,\rangle_{\mathbb{C}^N}$ is the standard hermitian product on $\mathbb{C}^N$. So $X^{\#}$ is such that

$$\int_{\mathring{X}} X(f)\overline{h} \ \mathrm{dV} = \int_{\mathring{X}} f\overline{X^{\#}(h)} \ \mathrm{dV}$$

for any $f, h \in C^{\infty}_{\mathcal{C}}(\mathring{X})$. Since $\int_{\mathring{X}} X(f)\overline{h} \ \mathrm{dV} = \int_{\mathring{X}} X(f\overline{h})\,dV - \int_{\mathring{X}} f\overline{X(h)} \ \mathrm{dV}$ and that $0 = \int_{\mathring{X}} \mathrm{div}(f\overline{h}X) \ \mathrm{dV} = \int_{\mathring{X}} f\overline{h}\,\mathrm{div}(X) \ \mathrm{dV} + \int_{\mathring{X}} X(f\overline{h}) \ \mathrm{dV}$, we obtain :

$$\int_{\mathring{X}} X(f)\overline{h} \ \mathrm{dV} = -\int_{\mathring{X}} f\overline{h}\,\mathrm{div}(X) \ \mathrm{dV} - \int_{\mathring{X}} f\overline{X(h)} \ \mathrm{dV} = \int_{\mathring{X}} f\overline{-(\mathrm{divX} + X(h))} \ \mathrm{dV}$$

This implies that $X^{\#} = -\mathrm{divX} - X$. We recall that $\mathrm{divX} = \mathrm{tr}_{g_{QAC}} \nabla X$, which implies that $\mathrm{divX} \in x_{max}\mathcal{A}^{\mathcal{K}}_{phg}(X)$ (by proposition 2.5.2) for some non-negative index family $\mathcal{K}$. $\square$

Corollary 2.5.7 *Let E_1 and E_2 be two Hermitian vector bundles over X and $D \in \mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{G}}(X,E_1,E_2)$. Then the formal adjoint of D is in $\mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{K}}(X,E_2,E_1)$ for some non-negative index family.*

Proof. Let define $E = E_1 \bigoplus E_2$ and use the natural matrix (block) notation to describe $\mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{G}}(X,E_1,E_2)$ as a subset of $\mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{G}}(X,E)$, then apply lemma 2.5.6. $\square$

Proposition 2.5.8 *The exterior derivative d is a differential operator in $\mathrm{Diff}_{\mathcal{V}_{QAC}}(X, \Lambda^{p\pi}T^*X, \Lambda^{p+1\pi}T^*X)$. In particular, $d \in x_{max}\,\mathrm{Diff}_{\mathcal{V}_{Qb}}(X, \Lambda^{p\pi}T^*X, \Lambda^{p+1\pi}T^*X)$.*

Proof. Let $\omega \in \Gamma\left(\Lambda^{p\pi}T^*X\right)$, and $\{X_i\}_{1\leq i\leq n}$ a local frame of $QAC-$vector fields. Then, we have :

$$(d\omega)\left(X_{i_0},\ldots,X_{i_p}\right)=\sum_{j=0}^{p}(-1)^j\,X_{i_j}\left(\omega\left(X_{i_0},\ldots,\widehat{X_{i_j}},\ldots,X_{i_p}\right)\right)-$$
$$\sum_{0\leq s<t\leq p}^{p}(-1)^{s+t}\,\omega\left(\left[X_{i_s},X_{i_t}\right],X_{i_0},\ldots,\widehat{X_{i_s}},\ldots,\widehat{X_{i_t}},\ldots,X_{i_p}\right)$$

Since $[X_i,X_j]=c_{ij}^s X_s$ where $c_{ij}^s\in x_{max}C^\infty(X)$, we deduce that $d\in\mathrm{Diff}_{\mathcal{V}_{QAC}}(X,\Lambda^{p\pi}T^*X,\Lambda^{p+1\pi}T^*X)$. Actually we can see that $d\in x_{max}\,\mathrm{Diff}_{\mathcal{V}_{Qb}}(X,\Lambda^{p\pi}T^*X,\Lambda^{p+1\pi}T^*X)$ $\square$

Corollary 2.5.9 *The Hodge-Laplace operator on $\left(\mathring{X},g_{QAC}\right)$ defined by $\Delta_{g_{QAC}}=(d+d^*)^2$ is a differential operator in $\mathrm{Diff}_{\mathcal{V}_{QAC},\mathcal{K}}(X,\Lambda^{p\pi}T^*X)$ for some non-negative index family $\mathcal{K}$. In particular, $\Delta_{g_{QAC}}\in x_{max}^2\,\mathrm{Diff}_{\mathcal{V}_{Qb},\mathcal{K}}(X,\Lambda^{p\pi}T^*X)$.*

Proof. The proof is a combination of corollary 2.5.7 and proposition 2.5.8. $\square$

CHAPTER 3

THE ISOMORPHISM THEOREM

3.1 Introduction

Let (X, g_{QAC}) be a QAC-manifold such that g_{QAC} is a polyhomogeneous metric. We will denote by $(H_i)_{1\leq i\leq k}$ the boundary hypersurfaces of X, and $(x_i)_{1\leq i\leq k}$ their respective defining functions. We define x_{max} as the product of boundary defining functions of maximal hypersurfaces, and $v = \prod_{i=1}^{k} x_i$.
The Riemannian manifold $(M = X\backslash\partial X, g_{QAC})$ is a complete manifold of bounded geometry and positive injectivity radius (proposition 1.3 in (Conlon *and al.*, 2019)).
Let $E = \left(T^*M^{\otimes^r} \otimes TM^{\otimes^s}\right)$ be a tensor bundle over M, and $\mathcal{A}$ the elliptic operator defined by:

$$\mathcal{A} = \underbrace{\Delta + \nabla_V}_{\Delta_V} - \lambda \tag{3.1}$$

acting on sections of E, such that:

- V is a b-vector field on X.
- λ is a positive constant.
- Δ and ∇ are the Laplacian and the Levi-Civita connection of g_{QAC} respectively.

In this chapter, we will prove that $\mathcal{A} : D^{k+2,\theta}_{\Delta_V,f}(M,E) \to C^{k,\theta}_{Qb,f}(M,E)$ is an isomorphism of Banach spaces for some positive function f to be defined later and such that $D^{k+2,\theta}_{\Delta_V,f}(M,E)$ and $C^{k,\theta}_{Qb,f}(M,E)$ are as defined below.

3.2 Function spaces

In the following functional spaces, we will consider the norm with respect to g_{QAC} and the euclidean structure on E. Covariant derivatives will be taken with respect to the Levi-Civita connection of g_{QAC} and the connection on E. Motivated by the work of (Siepmann, 2013) and (Deruelle, 2015), we define the following weighted holder spaces:

- $C^{k,\theta}_{Qb}(M,E) := \left\{ h \in C^{k,\theta}_{loc}(M,E) \mid ||h||_{C^{k,\theta}_{Qb}(M,E)} < \infty \right\}$, where

$$||h||_{C^{k}_{Qb}(M,E)} := \sum_{i=0}^{k} \sup_{M} |x_{max}^{-i} \nabla^i h|,$$

$$||h||_{C^{k,\theta}_{Qb}(M,E)} := ||h||_{C^{k}_{Qb}(M,E)} + \left[x_{max}^{-k} \nabla^k h \right]_{\theta},$$

and

$$[T]_{\theta} := \sup_{x \in M} \sup_{y \in B(x,\delta) \setminus \{x\}} \frac{|T(x) - P^{*}_{x,y} T(y)|}{d(x,y)^{\theta}}, \tag{3.2}$$

$P_{x,y}$ being the parallel transport along the unique minimizing geodesic from x to y, and δ the injectivity radius of g_{QAC}.

Remark 3.2.1 *The space $C^{k,\theta}_{Qb}(M,E)$ defined here is different from the one considered in (Conlon* and al., *2019), since in (3.2) it is the distance of the QAC-metric which is used instead of the distance of the Qb-metric.*

Given an elliptic differential operator $\mathcal{P}$ acting on sections of E, we also define the following spaces:

- $D^{2+k}_{\mathcal{P}}(M,E) := \left\{ h \in \bigcap_{p \geq 1} W^{2+k,p}_{loc}(M,E) \mid h \in C^{k}_{Qb}(M,E)\ ;\ \mathcal{P}(h) \in C^{k}_{Qb}(M,E) \right\}$, with the norm

$$||h||_{D^{2+k}_{\mathcal{P}}(M,E)} := ||h||_{C^{k}_{Qb}(M,E)} + ||\mathcal{P}(h)||_{C^{k}_{Qb}(M,E)}.$$

- $D^{2+k,\theta}_{\mathcal{P}}(M,E) := \left\{ h \in C^{k+2,\theta}_{loc}(M,E) \mid h \in C^{k,\theta}_{Qb}(M,E)\ ;\ \mathcal{P}(h) \in C^{k,\theta}_{Qb}(M,E) \right\}$, with the norm

$$||h||_{D^{2+k,\theta}_{\mathcal{P}}(M,E)} := ||h||_{C^{k,\theta}_{Qb}(M,E)} + ||\mathcal{P}(h)||_{C^{k,\theta}_{Qb}(M,E)}.$$

The following weighted spaces are defined using a positive function f to be defined later:

- $C^{k,\theta}_{Qb,f}(M,E) := f^{-1} C^{k,\theta}_{Qb}(M,E)$ with the norm $||h||_{C^{k,\theta}_{Qb,f}(M,E)} := ||fh||_{C^{k,\theta}_{Qb}(M,E)}$.
- $D^{k+2,\theta}_{\mathcal{P},f}(M,E) := f^{-1} D^{k+2,\theta}_{\mathcal{P}}(M,E)$ with the norm $||h||_{D^{k+2,\theta}_{\mathcal{P},f}(M,E)} := ||fh||_{D^{k+2,\theta}_{\mathcal{P}}(M,E)}$.

Remark 3.2.2 *It is worth mentioning that weighted holder spaces were introduced to study the behavior of the Laplacian on non compact manifolds. For instance, (Chaljub-Simon and Choquet-Bruhat, 1979) introduced the following spaces:*

$$C^{k,\theta}_{\beta,\rho}(M,E) := \left\{ h \in C^{k,\theta}_{loc}(M,E) \mid ||h||_{C^{k,\theta}_{\beta,\rho}(M,E)} := ||h||_{C^{k}_{\beta,\rho}(M,E)} + \left[\nabla^k h \right]_{\theta,\beta-k-\theta,\rho} < \infty \right\},$$

such that:

$$||h||_{C^k_{\beta,\rho}(M,E)} = \sum_{i=0}^{k} ||\rho^{i-\beta}\nabla^j h||_{C^0(M,E)},$$

$$[T]_{\theta,\gamma,\rho} := \sup_{x\in M} \sup_{y\in B(x,\delta)\setminus\{x\}} \inf(\rho(x),\rho(y))^{-\gamma}\frac{|T(x)-P^*_{x,y}T(y)|}{d(x,y)^\theta},$$

to study linear elliptic operators on asymptotically euclidean manifolds (ρ being some distance function). These spaces were adapted by Joyce in (Joyce, 2001a) and (Joyce, 2001b) to study asymptotically locally euclidean manifolds and quasi-asymptotically euclidean manifolds. Note that $C^{k,\theta}_{0,x^{-1}_{max}}(M,E) \subset C^{k,\theta}_{Qb}(M,E)$.

Remark 3.2.3 *It is also important to note that the space $C^{k,\theta}_{Qb}(M,E)$ as defined here is not equal to the interpolation space $\left(C^k_{Qb}(M,E), C^{k+1}_{Qb}(M,E)\right)_{\theta,\infty}$ which can be identified as follows:*

$$\left(C^k_{Qb}(M,E), C^{k+1}_{Qb}(M,E)\right)_{\theta,\infty} = \left\{h \in C^k_{Qb}(M,E) \mid \left[x^{-k}_{max}\nabla^k h\right]_{Qb,\theta} < +\infty\right\}$$

such that:

$$[T]_{Qb,\theta} = \sup_{x\in M} \sup_{y\in B(x,\frac{\delta}{x_{max}})\setminus\{x\}} \min\left\{x^{-\theta}_{max}(x), x^{-\theta}_{max}(y)\right\}\frac{|T(x)-P^*_{x,y}T(y)|}{d(x,y)^\theta}$$

where δ is a positive constant depending on the lower bound of $\inf_{x\in M} inj(x, g_{QAC})x_{max}(x)$.

Proposition 3.2.4 *Let $\mathcal{P}$ be an elliptic differential operator and λ a constant such that $\mathcal{P}-\lambda : D^{k+2,\theta}_{\mathcal{P}}(M,E) \to C^{k,\theta}_{Qb}(M,E)$ is an isomorphism. Then, the space $D^{2+k,\theta}_{\mathcal{P}}(M,E)$ is a Banach space.*

Proof. We are going to use the fact that $C^{k,\theta}_{Qb}(M,E)$ is a Banach space. Let $(h_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $D^{2+k,\theta}_{\mathcal{P}}(M,E)$. Then, there exists $h \in C^{k,\theta}_{Qb}(M,E)$ such that h_k converges to h in $C^{k,\theta}_{Qb}(M,E)$. Since $((\mathcal{P}-\lambda)(h_k))$ is also a Cauchy sequence in $C^{k,\theta}_{Qb}(M,E)$, there exists $\tilde{h} \in C^{k,\theta}_{Qb}(M,E)$ such that $(\mathcal{P}-\lambda)(h_k)$ converges to $(\mathcal{P}-\lambda)(\tilde{h})$. Note also that $(\mathcal{P}-\lambda)(h_k)$ converges to $(\mathcal{P}-\lambda)(h)$ in the $C^{k,\theta}_{loc}(M,E)$ topology which implies that $(\mathcal{P}-\lambda)(\tilde{h}) = (\mathcal{P}-\lambda)(h)$. Consequently, $h \in D^{2+k,\theta}_{\mathcal{P}}(M,E)$. $\square$

3.3 Lunardi's theorem on QAC manifolds

In order to prove the isomorphism theorem, we are going to use the following theorem:

Theorem 3.3.1 (Lunardi) *Let (M^n, g) be a complete Riemannian manifold with positive injectivity radius, and V be a smooth vector field on M. Let $\mathcal{P}$ be an elliptic differential operator acting on tensors over M such that:*

$$\mathcal{P} = \underbrace{\Delta + \nabla_V}_{\Delta_V} + r(x) \ , \ r \in C^3(M). \tag{3.3}$$

Suppose that $sup_{x\in M} r(x) = r_0 < \infty$ and that there exists a positive constant C such that $\sum_{i=1}^{3} ||\nabla^i r|| < C$. Assume also that there exists a positive constant K such:

$$||Rm(g)||_{C^3(M,E)} + ||Rm(g) * V||_{C^3(M,E)} + ||\nabla V||_{C^2(M,E)} \leq K, \tag{3.4}$$

where $Rm(g) * V = Rm(g)(V, ., ., .)$. Assume also that there exists a function $\phi \in C^2(M)$ and a constant $\lambda_0 \geq r_0$ such that:

$$\lim_{x\to\infty} \phi(x) = +\infty, \ sup_{x\in M} \left(\mathcal{P}(\phi)(x) - \lambda_0 \phi\left(x\right)\right) < \infty. \tag{3.5}$$

Then:

1. *For any $\lambda > r_0$, there exists a positive constant C such that for any $H \in C^0\left(M, E\right)$, there exists a unique tensor $h \in D^2_{\mathcal{P}}\left(M, E\right)$, satisfying:*

   $$\mathcal{P}(h) - \lambda h = H, \ ||h||_{D^2_{\mathcal{P}}(M,E)} \leq C||H||_{C^0(M,E)}.$$

   Moreover $D^2_{\mathcal{P}}\left(M, E\right)$is continuously embedded in $C^\theta\left(M, E\right)$ for any $\theta \in (0, 2)$, i.e there exists a positive constant $C(\theta)$ such that for any $h \in D^2_{\mathcal{P}}\left(M, E\right)$,

   $$||h||_{C^\theta(M,E)} \leq C(\theta)||h||^{\frac{\theta}{2}}_{D^2_{\mathcal{P}}(M,E)}||h||^{1-\frac{\theta}{2}}_{C^0(M,E)}.$$

2. *For any $\lambda > r_0$, there exists a positive constant C such that for any $H \in C^{0,\theta}\left(M, E\right)$, $\theta \in \left(0, 1\right)$, there exists a unique tensor $h \in C^{2,\theta}\left(M, E\right)$ satisfying:*

   $$\mathcal{P}(h) - \lambda h = H, \ ||h||_{C^{2,\theta}(M,E)} \leq C||H||_{C^{0,\theta}(M,E)}.$$

We will prove theorem 3.3.1 in the next chapter.

Remark 3.3.2 *Note that in order to satisfy condition (3.4) for the $b-$vector field V, it is sufficient to have*

$$||Rm(g) * V||_{C^0(M,E)} + ||\nabla V||_{C^0(M,E)} < \infty. \tag{3.6}$$

This is a direct consequence of corollaries 2.5.3 and 2.5.5. Note also that condition (3.6) is satisfied if the vector field V is a $QAC-$vector field.

Let's start by showing that condition (3.5) is easily satisfied.

Proposition 3.3.3 *There exists a smooth function $\phi : M \to \mathbb{R}_+$ such that:*

$$\lim_{p\to\infty} \phi(p) = +\infty,\ \sup_{p\in M} (\Delta_X)(\phi) < \infty,$$

for any b-vector field X.

Proof. Let us set $\phi(p) = -\ln(v(p))$. Using the definition of a b-vector field and the fact that the Laplacian of g_{QAC} can be expressed as a polynomial of degree at most 2 (without terms of order 0) in QAC-vector fields (Ammann and al., *2004) we see that both $\Delta\phi$ and $\nabla_X\phi$ are bounded on M.* $\square$

Let $\mathcal{A}_f$ be the differential operator defined by $\mathcal{A}_f(h) = f\mathcal{A}(f^{-1}h)$. Then, since:

$$\begin{aligned} f\Delta(f^{-1}h) &= f\left(f^{-1}\Delta h + h\Delta f^{-1} + 2 < \nabla f^{-1}, \nabla h >\right) \\ &= \Delta h + hf\Delta f^{-1} - 2 < \nabla \ln(f), \nabla h > \\ &= \Delta h + h\left(||\nabla \ln(f)||^2 - \Delta \ln(f)\right) - 2 < \nabla \ln(f), \nabla h >, \end{aligned}$$

and

$$f\nabla_V(f^{-1}h) = f\left(f^{-1}\nabla_V h + h\nabla_V f^{-1}\right) f = \nabla_V h - h\nabla_V \ln(f),$$

we have that:

$$\mathcal{A}_f(h) = \left(\underbrace{\Delta + \nabla_{V-2\nabla\ln(f)} - V\ln(f)}_{\mathcal{P}_f} - \lambda\right)(h) + \underbrace{\left(||\nabla \ln(f)||^2 - \Delta \ln(f)\right)}_{\mathcal{K}_f} h. \tag{3.7}$$

Remark 3.3.4 *The operator $\mathcal{A} : D^{k+2,\theta}_{\Delta_V,f}(M,E) \to C^{k,\theta}_{Qb,f}(M,E)$ is an isomorphism of Banach spaces if and only if $\mathcal{A}_f : D^{k+2,\theta}_{\Delta_V}(M,E) \to C^{k,\theta}_{Qb}(M,E)$ is. In what follows, we are going to set $f = v^{-\alpha}$ for some positive real value α.*

We are also going to use the following notation:

- $r_\alpha := \alpha V \ln(v)$;

- $V_\alpha := V + 2\alpha\nabla\ln(v)$;

- $\mathcal{P}_\alpha := \Delta_{V_\alpha} + r_\alpha$;

- $\mathcal{K}_\alpha := \alpha^2||\nabla\ln(v)||^2 - \alpha\Delta\ln(v)$;

- $C^{k,\theta}_{Qb,\alpha}(M,E) := C^{k,\theta}_{Qb,f}(M,E)$;

- $D^{k+2,\theta}_{\Delta_V,\alpha}(M,E) := D^{k+2,\theta}_{\Delta_V,f}(M,E)$.

Proposition 3.3.5 *Given a function f as defined in the previous remark, we have that:*

(i) *r_α is bounded on M.*

(ii) *$\nabla\ln(v)$ is a QAC-vector field. Thus, both $||Rm(g)*\nabla\ln(v)||_{C^k(M,E)}$ and $||\nabla\nabla\ln(v)||_{C^{k-1}(M,E)}$ are bounded for any integer $k\geq 1$.*

(iii) $\lim\limits_{x\to\infty}||\nabla^i\mathcal{K}_\alpha|| = 0$ *for $i\geq 0$.*

Proof. (i) *: This follows from the proof of proposition 3.3.3.*

(ii) : $d\ln(f) = -\alpha v\frac{dv}{v^2}$ *which is clearly a QAC-covector (see equation (2.9)) that tends to 0 near the boundary. The rest follows from the fact that both $Rm(g_{QAC})$ and $\nabla\ln(f)$ are tensors over the QAC-vector bundle over X, thus bounded with respect to the QAC-metric together with its derivatives.*

(iii) *: Since the Laplacian is a polynomial on $QAC-$vector fields without a constant term, we have that $\Delta\ln(v)\in vC^\infty(X)$. Taking covariant derivatives will increase the decay towards maximal hypersurfaces.*

□

Proposition 3.3.6 *For any $h\in C^{k,\theta}_{Qb}(M,E)$ we have that $\mathcal{K}_\alpha h\in C^{k,\theta}_{Qb}(M,E)$, we also have that*

$$\Delta_V(h)\in C^{k,\theta}_{Qb}(M,E)\Longleftrightarrow \mathcal{P}_\alpha(h)\in C^{k,\theta}_{Qb}(M,E) \tag{3.8}$$

As a consequence, we have that $D^{k+2,\theta}_{\mathcal{P}_\alpha+\mathcal{K}_\alpha}(M,E) = D^{k+2,\theta}_{\mathcal{P}_\alpha}(M,E) = D^{k+2,\theta}_{\Delta_V}(M,E)$.

Proof. $\mathcal{K}_\alpha h\in C^{k,\theta}_{Qb}(M,E)$ *follows from (iii) of proposition 3.3.5 and proposition 1.2.1. Since we have that*

$$|||\mathcal{P}_\alpha(h)+\mathcal{K}_\alpha h||_{C^{k,\theta}_{Qb}(M,E)} \leq |||\mathcal{P}_\alpha(h)||_{C^{k,\theta}_{Qb}(M,E)} + ||\mathcal{K}_\alpha h||_{C^{k,\theta}_{Qb}(M,E)},$$

and

$$|\ ||\mathcal{P}_\alpha(h)||_{C^{k,\theta}_{Qb}(M,E)} - ||\mathcal{K}_\alpha h||_{C^{k,\theta}_{Qb}(M,E)}| \leq |||\mathcal{P}_\alpha(h)+\mathcal{K}_\alpha h||_{C^{k,\theta}_{Qb}(M,E)},$$

which implies that $(\mathcal{P}_\alpha+\mathcal{K}_\alpha)(h) \in C^{k,\theta}_{Qb}(M,E) \iff \mathcal{P}_\alpha(h) \in C^{k,\theta}_{Qb}(M,E)$*. We proceed in the same manner using* (i) *and* (iii) *of proposition 3.3.5 to prove* (3.8). □

Remark 3.3.7 *Going back to remark 3.3.4, in order to prove that* $\mathcal{A}_f : D^{k+2,\theta}_{\Delta_V}(M,E) \to C^{k,\theta}_{Qb}(M,E)$ *is an isomorphism, it's suffice to prove that* $\mathcal{P}_\alpha - \lambda : D^{k+2,\theta}_{\mathcal{P}_\alpha}(M,E) \to C^{k,\theta}_{Qb}(M,E)$ *is an isomorphism and that* $\mathcal{K}_\alpha$ *is a compact operator.*

3.4 The isomorphism theorem

Theorem 3.4.1 (Isomorphism theorem) *Let* $\mathcal{C}^{k;j,\theta}(M,E)$ *be the functional space defined by:*

$$\mathcal{C}^{k;j,\theta}(M,E) = \left\{ h \in C^{k+j+\lfloor\theta\rfloor,\theta-\lfloor\theta\rfloor}_{loc}(M,E) \mid x^{-i}_{max}\nabla^i h \in C^{j+\lfloor\theta\rfloor,\theta-\lfloor\theta\rfloor}(M,E),\ \ \forall i=0,\ldots,k \right\}.$$

such that $\theta \in (0,2)$*, and endowed with the norm:*

$$||h||_{\mathcal{C}^{k;j,\theta}(M,E)} = \sum_{i=0}^{k} ||x^{-i}_{max}\nabla^i h||_{C^{j+\lfloor\theta\rfloor,\theta-\lfloor\theta\rfloor}(M,E)}$$

Suppose also that:

$$||Rm(g)*V||_{C^0(M,E)} + ||\nabla V||_{C^0(M,E)} < \infty \tag{3.9}$$

Then, for any constant $\lambda \in \mathbb{R}$ *such that:*

$$\lambda > \max\left(\sup_M V\ln(v^\alpha x^k_{max}), \sup_M V\ln(v^\alpha x^{k-1}_{max})\right) \tag{3.10}$$

we have that:

- *There exists a positive constant* C *such that for any* $H \in C^{k,\theta}_{Qb}(M,E)$ *there exists a unique* $h \in D^{k+2,\theta}_{\mathcal{P}_\alpha}(M,E)$ *satisfying:*

$$\mathcal{P}_\alpha(h) - \lambda h = H,\ ||h||_{D^{k+2,\theta}_{\mathcal{P}_\alpha}(M,E)} \leq C||H||_{C^{k,\theta}_{Qb}(M,E)};\ \theta \in [0,1) \tag{3.11}$$

i.e. the operator

$$\mathcal{P}_\alpha - \lambda : D^{k+2,\theta}_{\mathcal{P}_\alpha}(M,E) \to C^{k,\theta}_{Qb}(M,E) \tag{3.12}$$

is an isomorphism of Banach spaces. Moreover, $D^{k+2}_{\mathcal{P}_\alpha}(M,E)$ embeds continuously in $\mathcal{C}^{k;0,\theta}(M,E)$ for any $\theta \in (0,2)$, i.e there exists a positive constant C such that for any $h \in D^{k+2}_{\mathcal{P}_\alpha}(M,E)$,

$$||h||_{\mathcal{C}^{k;0,\theta}(M,E)} \leq C||h||^{\frac{\theta}{2}}_{D^{k+2}_{\mathcal{P}_\alpha}(M,E)}||h||^{1-\frac{\theta}{2}}_{C^k_{Qb}(M,E)}$$

- *There exists a positive constant C such that, for $\theta \in (0,1)$*

$$||h||_{\mathcal{C}^{k;2,\theta}(M,E)} \leq C||H||_{C^{k,\theta}_{Qb}(M,E)} \tag{3.13}$$

In order to prove the previous theorem, we are going to proceed by induction on k. Let us consider the case $k=0$.

Theorem 3.4.2 (Isomorphism theorem (k=0)) *Suppose that:*

$$||Rm(g) * V||_{C^0(M,E)} + ||\nabla V||_{C^0(M,E)} < \infty. \tag{3.14}$$

Then, for any constant $\lambda \in \mathbb{R}$ such that:

$$\lambda > \sup_M \left(V \ln(v^\alpha)\right), \tag{3.15}$$

we have that:

- *There exists a positive constant C such that, for any $H \in C^0(M,E)$, there exists a unique tensor $h \in D^2_{\mathcal{P}_\alpha}(M,E)$ satisfying*

  $$\mathcal{P}_\alpha(h) - \lambda h = H, \ ||h||_{D^2_{\mathcal{P}_\alpha}} \leq C||H||_{C^0(M,E)}.$$

  Moreover, $D^2_{\mathcal{P}_\alpha}(M,E)$ is continuously embedded in $C^\theta(M,E) := C^{\lfloor\theta\rfloor,\theta-\lfloor\theta\rfloor}(M,E)$ for any $\theta \in (0,2)$, i.e there exists a positive constant C such that for any $h \in D^2_{\mathcal{P}_\alpha}(M,E)$,

  $$||h||_{C^\theta(M,E)} \leq C||h||^{\frac{\theta}{2}}_{D^2_{\mathcal{P}_\alpha}(M,E)}||h||^{1-\frac{\theta}{2}}_{C^0(M,E)}.$$

- *There exists a positive constant C such that, for any $H \in C^{0,\theta}(M,E)$, with $\theta \in (0,1)$, there exists a unique tensor $h \in C^{2,\theta}(M,E)$ satisfying*

  $$\mathcal{P}_\alpha(h) - \lambda h = H, \ ||h||_{C^{2,\theta}(M,E)} \leq C||H||_{C^{0,\theta}(M,E)}.$$

  Moreover, the operator $\mathcal{A} : D^{2,\theta}_{\Delta_V,\alpha}(M,E) \to C^{0,\theta}_{Qb,\alpha}(M,E)$ is an isomorphism of Banach spaces.

Proof. *Condition 3.14 together with remark 3.3.2 implies condition (3.4) of theorem 3.3.1. We also have that condition 3.5 is satisfied by proposition 3.3.3. This proves the first point and the first part of the second point. In order to prove that $\mathcal{A}$ is an isomorphism of Banach spaces we are going to use the fact that the index of a Fredholm operator remains unchanged under a perturbation by a compact operator. Thus, if $\mathcal{P}_\alpha - \lambda$ is an isomorphism and $\mathcal{K}_\alpha$ is compact, then $\mathcal{P}_\alpha + \mathcal{K}_\alpha - \lambda$ is of index 0. Then, injectivity of $\mathcal{A}$ implies surjectivity.*

(1) $\mathcal{A} : D^{2,\theta}_{\Delta_V,\alpha}(M,E) \to C^{0,\theta}_\alpha(M,E)$ *is injective.*
Let $h \in D^{2,\theta}_{\Delta_V,\alpha}(M,E)$ be such that $\mathcal{A}(h) = 0$ and $h_k = h - \frac{\phi}{k}$ with ϕ the smooth function of proposition 3.3.3. Then, $\sup\limits_{p\in M} h_k = h_k(p_k)$ for some $p_k \in M$. Moreover $\lim_{k\to\infty} \sup\limits_{p\in M} h_k = \sup\limits_{p\in M} h$.
Since $\mathcal{A}(h_k) = -\frac{\mathcal{A}(\phi)}{k}$, we have that $\mathcal{A}(h_k) \geq -\frac{\sup\limits_{p\in M}\mathcal{A}(\phi)}{k}$. Evaluating the last inequality at point p_k we get $(\lambda - \sup\limits_{p\in M} V\ln(v^\alpha))h_k(p_k) \leq \frac{\sup\limits_{p\in M}\mathcal{A}(\phi)}{k}$. By taking the limit $k \to \infty$ we find that $\sup\limits_{p\in M} h \leq 0$. By applying the same method to $-h$ we deduce that $h = 0$. Hence, $\mathcal{A}$ is injective.

(2) $\mathcal{K}_\alpha : D^{2,\theta}_{\mathcal{P}_\alpha}(M,E) \to C^{0,\theta}(M,E)$ *is a compact operator.*
Let $(h_k)_{k\in\mathbb{N}}$ be a bounded sequence in $D^{2,\theta}_{\mathcal{P}_\alpha}(M,E)$ and $(U_k)_{k\in\mathbb{N}}$ be a sequence of precompact open sets of M such that $\overline{U_k} \subset U_{k+1}$ and $M = \cup_k U_k$. By Schauder estimates 1.3.4, the sequence $(h_k)_k$ is bounded in $C^{2,\theta}(U_i, E|_{U_i})$ for any i.
Since $C^{2,\theta}(\overline{U_i}, E|_{\overline{U_i}})$ is compactly embedded into $C^2(\overline{U_i}, E|_{\overline{U_i}})$, there exists a sub-sequence $(h^i_k)_k$ that converges uniformly in $C^2(\overline{U_i}, E|_{\overline{U_i}})$. Let $(g_k)_k$ be a sub-sequence such that $g_k = h^k_k$. Then, $(g_k)_k$ converges in the topology of $C^2_{loc}(M,E)$ to $h \in D^{2,\theta}_{\mathcal{P}_\alpha}(M,E)$ (since it converges uniformly on every compact of M).
Before we finish the proof, we need the following lemma:

Lemma 3.4.3 *Let $h \in C^{0,\theta}(M,E)$ and $f \in C^1(M)$. Then, for any compact set $K \subset M$ there exists a precompact set Q containing K and a positive constant C such that:*

$$||fh||_{C^{0,\theta}(M,E)} \leq C\left(||f||_{C^1(Q)}||h||_{C^{0,\theta}(Q,E|_Q)} + ||f||_{C^1(M\setminus K)}||h||_{C^{0,\theta}(M\setminus K, E|_{M\setminus K})}\right).$$

Proof. *Let Q be a precompact set containing K such that $\forall x \in K$ we have that $B(x,\delta) \subset Q$ (δ being the injectivity radius). Then:*

$$\begin{aligned}||fh||_{C^{0,\theta}(M,E)} &\leq \left(||fh||_{C^{0,\theta}(Q,E|_Q)} + ||fh||_{C^{0,\theta}(M\setminus K, E|_{M\setminus K})}\right)\\ &\leq \left(||f||_{C^{0,\theta}(Q)}||h||_{C^{0,\theta}(Q,E|_Q)} + ||f||_{C^{0,\theta}(M\setminus K)}||h||_{C^{0,\theta}(M\setminus K, E|_{M\setminus K})}\right).\end{aligned}$$

Now using a local version of the mean value theorem, it is easy to see that there exists a positive constant C (that depends only on the injectivity radius) such that:

$$||f||_{C^{0,\theta}(Q)} \leq C||f||_{C^1(Q)},$$
$$||f||_{C^{0,\theta}(M\setminus K)} \leq C||f||_{C^1(M\setminus K)}.$$

Consequently, we have that

$$||fh||_{C^{0,\theta}(M,E)} \leq C\left(||f||_{C^1(Q)}||h||_{C^{0,\theta}(Q,E|_Q)} + ||f||_{C^1(M\setminus K)}||h||_{C^{0,\theta}(M\setminus K,E|_{M\setminus K})}\right).$$

□

Since $\lim_{p\to\infty}|\nabla^i\mathcal{K}_\alpha|(p) = 0$ for $i = 0, 1$ (by proposition 3.3.5); it follows that for all $\epsilon > 0$ there exists a compact set $K \subset M$ such that $||\mathcal{K}_\alpha||_{C^1(M\setminus K)} < \epsilon$.
Using the previous lemma, we have that

$$||\mathcal{K}_\alpha\left(g_n - h\right)||_{C^{0,\theta}(M,E)} \leq C(||\mathcal{K}_\alpha||_{C^1(Q)}||\left(g_n - h\right)||_{C^{0,\theta}(Q,E|_Q)} +$$
$$||\mathcal{K}_\alpha||_{C^1(M\setminus K)}||\left(g_n - h\right)||_{C^{0,\theta}(M\setminus K,E|_{M\setminus K})}) < C\epsilon$$

for some precompact set Q containing K and n large enough. This proves that $(\mathcal{K}_\alpha g_n)_{n\in\mathbb{N}}$ converges to $\mathcal{K}_\alpha h$ in the $C^{0,\theta}(M,E)$ topology, which proves that $\mathcal{K}_\alpha$ is a compact operator. □

Proposition 3.4.4 *Let f be a $C^2_{loc}(M)$ function such that:*

$$\mathcal{T}(f) = (\Delta_{V+2\alpha\nabla\ln(v)} + V\ln(v^\alpha x^k_{max}) - \lambda)(f) \geq 0 \text{ and } f = O(x^{-1}_{max}), \tag{3.16}$$

λ being a constant such that:

$$\lambda > \max\left(\sup_M V\ln(v^\alpha x^k_{max}), \sup_M V\ln(v^\alpha x^{k-1}_{max})\right). \tag{3.17}$$

Then, $\sup_M f \leq 0$.
Proof. First of all, let us note that f is only potentially unbounded near the maximal hypersurfaces (since $f = O(x^{-1}_{max})$). If f is bounded above then we will use the exhaustion function of proposition 3.3.3 to prove the proposition. Otherwise, f is unbounded from above near maximal hypersurfaces, and we will use x^θ_{max} with $\theta < -1$ as a barrier function.
Before we proceed, let us note that inequality 3.16 implies that:

$$\Delta_{V+2\alpha\nabla\ln(v)}(f) \geq (\lambda - V\ln(v^\alpha x^k_{max}))f. \tag{3.18}$$

In addition, given a function f_s that attains its supremum at a point $p_s \in M$ we have that:

$$\Delta_{V+2\alpha\nabla\ln(v)}(f_s)(p_s) \leq 0. \tag{3.19}$$

***First case**: f is bounded above on M:*

Let us define $f_s = f - \frac{\phi}{s}$, ϕ being the function in proposition 3.3.3. Since f is bounded above, f_s attains its supremum at some point $p_s \in M$. Using inequality (3.18) we deduce that:

$$\Delta_{V+2\alpha\nabla\ln(v)}(f_s) = \Delta_{V+2\alpha\nabla\ln(v)}(f) - \frac{1}{s}\Delta_{V+2\alpha\nabla\ln(v)}(\phi) \geq (\lambda - V\ln(v^\alpha x^k_{max}))f \\ - \frac{1}{s}\Delta_{V+2\alpha\nabla\ln(v)}(\phi).$$

Combining this inequality with the fact that

$$(\lambda - V\ln(v^\alpha x^k_{max}))f - \frac{1}{s}\Delta_{V+2\alpha\nabla\ln(x)}(\phi) = (\lambda - V\ln(v^\alpha x^k_{max}))f_s - \frac{1}{s}\mathcal{T}(\phi).$$

we obtain that:

$$\Delta_{V+2\alpha\nabla\ln(v)}(f_s) \geq (\lambda - V\ln(v^\alpha x^k_{max}))f_s - \frac{1}{s}\mathcal{T}(\phi).$$

Finally, evaluating this inequality at p_s and using inequality (3.19) we obtain that:

$$(\lambda - V\ln(v^\alpha x^k_{max}))f_s(p_s) \leq \frac{1}{s}\mathcal{T}(\phi)(p_s).$$

By letting $s \to \infty$ and using the fact that $\mathcal{T}(\phi)$ is bounded above (proposition 3.3.3), we obtain that $\sup_M f \leq 0$.

***Second case**: f is unbounded above near the maximal hypersurfaces:*

Let $\theta \in (-2,-1)$ be a constant such that (see lemma 3.4.5 below for a proof of existence)

$$\lambda > \max\left(\sup_M V\ln(v^\alpha x^k_{max}), \sup_M V\ln(v^\alpha x^{k+\theta}_{max})\right) \tag{3.20}$$

and let us set $f_s = f - \frac{x^\theta_{max}}{s}$. Since f is unbounded above near maximal hypersurfaces, there exists $s_0 \in \mathbb{N}_0$ such that for $s \geq s_0$ there exists a point $p_s \in M$ such that $\sup_{p\in M} f_s(p) = f_s(p_s)$.

Using inequality (3.18), we have that:

$$\begin{aligned}\Delta_{V+2\alpha\nabla\ln(v)}(f_s) &= \Delta_{V+2\alpha\nabla\ln(v)}(f) - \frac{1}{s}\Delta_{V+2\alpha\nabla\ln(v)}(x^\theta_{max}) \\ &\geq \left(\lambda - V\ln(v^\alpha x^k_{max})\right) f - \frac{1}{s}\Delta_{V+2\alpha\nabla\ln(v)}(x^\theta_{max}) \\ &\geq \left(\lambda - V\ln(v^\alpha x^k_{max})\right) f_s - \frac{1}{s}\mathcal{P}(x^\theta_{max}).\end{aligned}$$

When evaluating the previous inequality at p_s, we have that:

$$\left(\lambda - V\ln(v^\alpha x_{max}^k)\right)(f_s)(p_s) \leq \frac{1}{s}\mathcal{T}(x_{max}^\theta)(p_s).$$

A simple computation shows that

$$\begin{aligned}\Delta_{V+2\alpha\nabla\ln(v)}(x_{max}^\theta) &= \Delta x_{max}^\theta + V(x_{max}^\theta) + 2\alpha\nabla\ln(v)(x_{max}^\theta)\\ &= \Delta x_{max}^\theta + x_{max}^\theta V\ln(x_{max}^\theta) + 2\alpha\theta v x_{max}^{\theta+1}\left\langle \frac{dv}{v^2}, \frac{dx_{max}}{x_{max}^2}\right\rangle\\ &\leq C + x_{max}^\theta V\ln(x_{max}^\theta)\end{aligned}$$

since both Δx_{max}^θ and $v x_{max}^{\theta+1}\left\langle \frac{dv}{v^2}, \frac{dx_{max}}{x_{max}^2}\right\rangle$ are bounded by corollary 2.5.9.
Consequently, using inequality (3.20) we deduce that

$$\begin{aligned}\left(\lambda - V\ln(v^\alpha x_{max}^k)\right)(f_s)(p_s) &\leq \frac{1}{s}\left(x_{max}^\theta\left(V\ln(x_{max}^\theta) + V\ln(v^\alpha x_{max}^k) - \lambda\right) + C\right)\\ &\leq \frac{1}{s}\left(x_{max}^\theta\left(V\ln(v^\alpha x_{max}^{k+\theta}) - \lambda\right) + C\right)\\ &\leq \frac{C}{s}.\end{aligned}$$

*This implies that when $s \to \infty$ we have $\sup_{p\in M} f(p) \leq 0$ which contradicts the hypothesis of the **second case**.*
$\square$

Lemma 3.4.5 *Let $\lambda \in \mathbb{R}$ be such that*

$$\lambda > \max\left(\sup_M V\ln(v^\alpha x_{max}^k), \sup_M V\ln(v^\alpha x_{max}^{k-1})\right). \tag{3.21}$$

Then, there exists $\theta \in (-2,-1)$ such that

$$\lambda > \max\left(\sup_M V\ln(v^\alpha x_{max}^k), \sup_M V\ln(v^\alpha x_{max}^{k+\theta})\right). \tag{3.22}$$

Proof. Since V is a b-vector field, we have that $V\ln(x_{max}) \in C^\infty(X)$, thus is bounded on M.

Consequently, the difference $V\ln(x_{max}^{k-1}) - V\ln(x_{max}^{k+\theta}) = V\ln(v_{max}^{-(1+\theta)})$ can be made arbitrary close to zero by choosing the constant $\theta \in (-2,-1)$ close enough to -1. This implies that $\sup_M V\ln(v^\alpha x_{max}^{k+\theta})$ can be made arbitrarily close to $\sup_M V\ln(v^\alpha x_{max}^{k-1})$ by a choice of a constant θ as described previously, which then preserves inequality (3.22). $\square$

Corollary 3.4.6 *Let h be a tensor such that $h \in \cap_{p\geq 1} W_{loc}^{2,p}(M,E)$ and*

$$\mathcal{T}(h) = (\Delta_{V+2\alpha\nabla\ln(v)} + V\ln(v^\alpha x_{max}^k) - \lambda)(h) = 0 \text{ and } h = O(x_{max}^{-1}),$$

such that $\lambda > \max\left(\sup_M V\ln(v^\alpha x^k_{max}), \sup_M V\ln(v^\alpha x^{k-1}_{max})\right)$.

Then, $h \equiv 0$.

Proof. Let us define $f = ||h||^2$. *Then we have :*

$$\mathcal{T}(f) = 2\left\langle \Delta_{V+2\alpha\nabla\ln(v)} h, h\right\rangle - \left(\lambda - V\ln(v^\alpha x^k_{max})\right) f + 2||\nabla h||^2.$$

Since $\mathcal{T}(h) = 0$ *we have that* $\Delta_{V+2\alpha\nabla\ln(v)} h = \left(\lambda - V\ln(v^\alpha x^k_{max})\right) h$.

Consequently,

$$\mathcal{T}(f) = \left(\lambda - V\ln(v^\alpha x^k_{max})\right) f + 2||\nabla h||^2$$

which implies that

$$\mathcal{T}(f) \geq 0.$$

By applying proposition 3.4.4 we get that $\sup_{p\in M} f \leq 0$. *Thus ,*$f \equiv 0$. $\square$

3.4.1 Proof of theorem 3.4.1

Proof. Uniqueness follows from theorem 3.4.2. *In order to prove the existence of a solution, we are going to proceed by induction on* k. *The case* $k = 0$ *is exactly theorem* 3.4.2. *Let* k *be a positive integer and* $H \in C^{k,\theta}_{Qb}(M,E)$. *Using the induction hypothesis and the fact that* $H \in C^{k-1,\theta}_{Qb}(M,E)$, *there exists* $h \in D^{2+k-1}_{\mathcal{P}_\alpha}(M,E)$ *such that* $\mathcal{P}_\alpha(h) - \lambda h = H$.

Let us define $h_i = x^{-i}_{max}\nabla^i h$ *for* $i = 0,\ldots,k$. *We want to prove that* $h \in D^{2+k}_{\mathcal{P}_\alpha}(M,E)$. *This amounts to proving that* $h_k \in D^2_{\mathcal{P}_\alpha}(M,E)$. *In order to do that, we are going to compute the evolution equation of* h_k. *Let us recall that* $\mathcal{P}_\alpha = \Delta + \nabla_{\underbrace{V - 2\nabla\ln(v^{-\alpha})}_{V_\alpha}} \underbrace{- V\ln(v^{-\alpha})}_{r_\alpha}$, *so that*

$$\mathcal{P}_\alpha(h_k) = \Delta_{V_\alpha}(h_k) + r_\alpha h_k$$

In order to compute $\mathcal{P}_\alpha(h_k)$ *we are going to use lemma 4.3.2 below. Consequently, we have that:*

$$\begin{aligned}
&\nabla_{V_\alpha}(h_k) = V_\alpha(x^{-k}_{max})\nabla^k h + x^{-k}_{max}\nabla_{V_\alpha}\nabla^k h \\
&= (-V\ln(x^k_{max}) + a)h_k + x^{-k}_{max}\nabla^k\nabla_{V_\alpha}h + x^{-k}_{max}\sum_{j=0}^{k-1}\nabla^{k-j}V_\alpha * \nabla^{j+1}h + \nabla^{k-1-j}\left(Rm(g_{QAC}) * V_\alpha\right) * \nabla^j h \\
&= (-V\ln(x^k_{max}) + a)h_k + x^{-k}_{max}\nabla^k\nabla_{V_\alpha}h + \sum_{j=0}^{k-1} x^{-k+j+1}_{max}\nabla^{k-j}V_\alpha * h_{j+1} + x^{-k+j}_{max}\nabla^{k-1-j}\left(Rm(g_{QAC}) * V_\alpha\right) * h_j
\end{aligned} \tag{3.23}$$

with $a \in x_{max}C^{\infty}(X)$. We also have that:

$$\Delta(h_k) = \Delta(x_{max}^{-k})\nabla^k h + bx_{max}^{-k}\nabla^{k+1}h + x_{max}^{-k}\nabla^k \Delta h + x_{max}^{-k}\sum_{j=0}^{k}\nabla^{k-j}Rm(g_{QAC}) * \nabla^j h$$
$$= bx_{max}h_{k+1} + (Rm(g_{QAC}) + c) * h_k + x_{max}^{-k}\nabla^k\Delta h + \sum_{j=0}^{k-1} x_{max}^{-k+j}\nabla^{k-j}Rm(g_{QAC}) * h_j \quad (3.24)$$

with $b, c \in x_{max}C^{\infty}(X)$.

Now, using equations (3.23) and (3.24) we deduce that

$$(\mathcal{P}_\alpha + V\ln(x_{max}^k))(h_k) - \lambda h_k = bx_{max}h_{k+1} + H_k + (a + c + Rm(g_{QAC})) * h_k$$
$$+ \sum_{j=0}^{k-1} x_{max}^{-k+j+1}\nabla^{k-j}V_\alpha * h_{j+1} + x_{max}^{-k+j}\nabla^{k-1-j}\left(Rm(g_{QAC}) * V_\alpha\right) * h_j$$
$$+ \sum_{j=0}^{k-1} x_{max}^{-k+j}\nabla^{k-j}Rm(g_{QAC}) * h_j$$

such that $H_k = x_{max}^{-k}\nabla^k H$. Thus

$$||(\mathcal{P}_\alpha + V\ln(x_{max}^k))(h_k) - \lambda h_k||_{C^{0,\theta}(M,E)} \leq ||bx_{max}h_{k+1}||_{C^{0,\theta}(M,E)} + ||H_k||_{C^{0,\theta}(M,E)}$$
$$+ ||\left(a + c + Rm(g_{QAC})\right) * h_k||_{C^{0,\theta}(M,E)}$$
$$+ \sum_{j=0}^{k-1}||x_{max}^{-k+j+1}\nabla^{k-j}V_\alpha * h_{j+1}||_{C^{0,\theta}(M,E)} + \quad (3.25)$$
$$||x_{max}^{-k+j}\nabla^{k-1-j}\left(Rm(g_{QAC}) * V_\alpha\right) * h_j||_{C^{0,\theta}(M,E)}$$
$$+ \sum_{j=0}^{k-1}||x_{max}^{-k+j}\nabla^{k-j}Rm(g_{QAC}) * h_j||_{C^{0,\theta}(M,E)}.$$

From the induction hypothesis, there exists a positive constant C such that:

$$||h||_{D^{2+k-1}_{\mathcal{P}_\alpha}(M,E)} \leq C||H||_{C^{k-1}_{Qb}(M,E)} \quad (3.26)$$
$$||h||_{\mathcal{C}^{k-1;0,\theta}(M,E)} \leq C||h||^{\frac{\theta}{2}}_{D^{2+k-1}_{\mathcal{P}_\alpha}(M,E)}||h||^{1-\frac{\theta}{2}}_{C^{k-1}_{Qb}(M,E)} \text{ for any } \theta \in (0,2). \quad (3.27)$$

From inequality 3.13 and the induction hypothesis we have that

$$||x_{max}^{-(k-1)}\nabla^{k-1}h||_{C^{2,\theta}(M,E)} \leq C||H||_{C^{k-1}_{Qb}(M,E)}, \quad (3.28)$$

which then implies that

$$||x_{max}^{-(k-1)}\nabla^{k}h||_{C^{0,\theta}(M,E)} \leq C||H||_{C^{k,\theta}_{Qb}(M,E)},$$
$$||x_{max}^{-(k-1)}\nabla^{k+1}h||_{C^{0,\theta}(M,E)} \leq C||H||_{C^{k,\theta}_{Qb}(M,E)}.$$

Using the previous inequalities we get that :

$$
\begin{aligned}
||bx_{max}h_{k+1}||_{C^{0,\theta}(M,E)} &\leq \tilde{C}||\frac{b}{x_{max}}||_{C^1(M,E)}||x_{max}^{-(k-1)}\nabla^{k+1}||_{C^{0,\theta}(M,E)},\\
&\leq B||H||_{C^{k,\theta}_{Qb}(M,E)},
\end{aligned}
$$

and

$$
\begin{aligned}
||\left(a+c+Rm(g_{QAC})\right)*h_k||_{C^{0,\theta}(M,E)} &\leq \tilde{C}||\frac{a+c+Rm(g_{QAC})}{x_{max}}||_{C^1(M,E)}||x_{max}^{-(k-1)}\nabla^{k}||_{C^{0,\theta}(M,E)},\\
&\leq B||H||_{C^{k,\theta}_{Qb}(M,E)}.
\end{aligned}
$$

for some positive constant B. We proceed in the same manner for the other terms in (3.25) using the fact that $||x_{max}^{-k+j+1}\nabla^{k-j}V_\alpha||_{C^1(M,E)}$, $||x_{max}^{-k+j}\nabla^{k-1-j}\left(Rm(g_{QAC})*V_\alpha\right)||_{C^1(M,E)}$ *and* $||x_{max}^{-k+j}\nabla^{k-j}Rm(g_{QAC})||_{C^1(M,E)}$ *are bounded (corollary 2.5.5).*

Thus, there exists a positive constant B such that:

$$
||(\mathcal{P}_\alpha + V\ln(x_{max}^k))(h_k) - \lambda h_k||_{C^{0,\theta}(M,E)} \leq B||H||_{C^{k,\theta}_{Qb}(M,E)}.
$$

Therefore, by theorem 3.3.1, there exists a solution $\tilde{h}_k \in D^{2,\theta}_{\mathcal{P}_\alpha}(M,E)$ satisfying

$$
\begin{aligned}
(\mathcal{P}_\alpha + V\ln(x_{max}^k))(\tilde{h}_k) - \lambda\tilde{h}_k =& bx_{max}h_{k+1} + H_k + (a+c+Rm(g_{QAC}))*h_k\\
&+\sum_{j=0}^{k-1} x_{max}^{-k+j+1}\nabla^{k-j}V_\alpha * h_{j+1} + x_{max}^{-k+j}\nabla^{k-1-j}\left(Rm(g_{QAC})*V_\alpha\right)*h_j\\
&+\sum_{j=0}^{k-1} x_{max}^{-k+j}\nabla^{k-j}Rm(g_{QAC})*h_j,
\end{aligned}
$$

and such that

$$
||\tilde{h}_k||_{C^{2,\theta}(M,E)} \leq C||H||_{C^{k,\theta}_{Qb}(M,E)} \text{ with } \theta\in(0,1).
$$

It remains to prove that $\tilde{h}_k \equiv h_k$. First of all, as $H \in C^{k,\theta}_{loc}(M,E)$ we have that $h \in C^{k+2,\theta}_{loc}(M,E)$ (by elliptic regularity). As a consequence, the difference $T = \tilde{h}_k - h_k$ satisfies:

$$
T \in \cap_{p\geq 1}W^{2,p}_{loc}(M,E)\,;\ \left((\mathcal{P}_\alpha + V\ln(x_{max}^k)) - \lambda\right)(T) = 0.
$$

Near the maximal hypersurfaces, we only have that $x_{max}^{-(k-1)}\nabla^k h$ is bounded, so we deduce that $x_{max}T$ is bounded ($T = O(x_{max}^{-1})$). By corollary 3.4.6 we have that $T\equiv 0$.

Consequently, $h_k \in D^{2,\theta}_{\mathcal{P}_\alpha}(M,E)$ and

$$
||h_k||_{C^{2,\theta}(M,E)} \leq C||H||_{C^{k,\theta}_{Qb}(M,E)} \text{ with } \theta\in(0,1).
$$

We also have that

$$||h_k||_{C^\theta(M,E)} \leq C||h||^{\frac{\theta}{2}}_{D^2_{\mathcal{P}_\alpha}(M,E)}||h||^{1-\frac{\theta}{2}}_{C^0(M,E)} \text{ with } \theta \in (0,2).$$

□

As a consequence of theorem 3.4.1 together with remark 3.3.4, we have the following result:

Corollary 3.4.7 *Suppose that:*

$$||Rm(g) * V||_{C^0(M,E)} + ||\nabla V||_{C^0(M,E)} < \infty \tag{3.29}$$

Then, the operator $\mathcal{A} : D^{2+k,\theta}_{\Delta_V,\alpha}(M,E) \to C^{k,\theta}_{Qb,\alpha}(M,E)$ *is an isomorphism of Banach spaces, for any* $\theta \in (0,1)$ *and any constant* $\lambda \in \mathbb{R}$ *such that:*

$$\lambda > \max\left(\sup_M V \ln(x^\alpha x^k_{max}), \sup_M V \ln(x^\alpha x^{k-1}_{max})\right)$$

Corollary 3.4.8 *The spaces* $D^{2+k,\theta}_{\Delta_V}(M,E)$ *and* $D^{2+k,\theta}_{\Delta_V,\alpha}(M,E)$ *are Banach spaces.*

Proof. We use proposition 3.2.4 and the previous corollary. □

CHAPTER 4

LUNARDI'S THEOREM

4.1 Introduction

In this chapter we study a class of linear elliptic operators of the form $\Delta + \nabla_V + r$ with unbounded coefficients. Such operators where studied by Alessandra Lunardi in (Lunardi, 1998) on $\mathbb{R}^n$ then a version was proven by (Deruelle, 2015) in the context of Riemannian manifold and such that $r \equiv 0$.

Theorem 4.1.1 (Lunardi) *Let (M^n, g) be a complete Riemannian manifold with positive injectivity radius, and V be a smooth vector field on M. Let $\mathcal{A}$ be an elliptic differential operator acting on tensors over M such that:*

$$\mathcal{A} = \underbrace{\Delta + \nabla_V}_{\Delta_V} + r(x)\ ,\ r \in C^3(M). \tag{4.1}$$

Suppose that $sup_{x\in M} r(x) = r_0 < \infty$ and that there exists a positive constant C such that $\sum_{i=1}^{3} ||\nabla^i r|| < C$. Assume also that there exists a positive constant K such:

$$||Rm(g)||_{C^3(M,E)} + ||Rm(g) * V||_{C^3(M,E)} + ||\nabla V||_{C^2(M,E)} \leq K, \tag{4.2}$$

where $Rm(g) * V = Rm(g)(V, ., ., .)$. Assume also that there exists a function $\phi \in C^2(M)$ and a constant $\lambda_0 \geq r_0$ such that:

$$\lim_{x\to\infty} \phi(x) = +\infty,\ sup_{x\in M} \left(\mathcal{A}(\phi)(x) - \lambda_0 \phi\left(x\right)\right) < \infty. \tag{4.3}$$

Then:

1. *For any $\lambda > r_0$, there exists a positive constant C such that for any $H \in C^0\left(M, E\right)$, there exists a unique tensor $h \in D^2_{\mathcal{A}}\left(M, E\right)$, satisfying:*

   $$\mathcal{A}(h) - \lambda h = H,\ ||h||_{D^2_{\mathcal{A}}(M,E)} \leq C||H||_{C^0(M,E)}.$$

   Moreover $D^2_{\mathcal{A}}\left(M, E\right)$ is continuously embedded in $C^\theta\left(M, E\right)$ for any $\theta \in (0, 2)$, i.e. there exists a positive constant $C(\theta)$ such that for any $h \in D^2_{\mathcal{A}}\left(M, E\right)$,

   $$||h||_{C^\theta(M,E)} \leq C(\theta)||h||^{\frac{\theta}{2}}_{D^2_{\mathcal{A}}(M,E)}||h||^{1-\frac{\theta}{2}}_{C^0(M,E)}.$$

2. *For any $\lambda > r_0$, there exists a positive constant C such that for any $H \in C^{0,\theta}(M,E)$, $\theta \in (0,1)$, there exists a unique tensor $h \in C^{2,\theta}(M,E)$ satisfying:*

$$\mathcal{A}(h) - \lambda h = H,\ ||h||_{C^{2,\theta}(M,E)} \leq C||H||_{C^{0,\theta}(M,E)}.$$

4.2 Uniqueness of the solution

Proposition 4.2.1 (Injectivity) *Let $h \in \cap_{p\geq 1} W^{2,p}_{loc}(M,E)$ be a bounded tensor and $\lambda > r_0$ a constant such that $\mathcal{A}(h) - \lambda h = 0$. Then $h \equiv 0$.*

Proof. Let us define $h_\epsilon = \sqrt{||h||^2 + \epsilon^2}$ (for some positive constant ϵ). Then:

$$\mathcal{A}(h_\epsilon) - \lambda h_\epsilon = \frac{1}{h_\epsilon}\left(\langle \Delta_V h, h\rangle - (\lambda - r)\, h_\epsilon^2 + ||\nabla h||^2 - \frac{||\nabla ||h||^2||^2}{4h_\epsilon^2}\right).$$

Since $\Delta_V h = (\lambda - r)\, h$ and $||\nabla ||h||^2||^2 = 4|\langle \nabla h, h\rangle|^2 \leq 4||\nabla h||^2\ ||h||^2$, we have that:

$$\mathcal{A}(h_\epsilon) - \lambda h_\epsilon = \frac{1}{h_\epsilon}\left(-(\lambda - r)\,\epsilon^2 + ||\nabla h||^2 - \frac{|\langle \nabla h, h\rangle|^2}{h_\epsilon^2}\right) \geq \frac{\epsilon^2}{h_\epsilon}\left(-(\lambda - r) + \frac{||\nabla h||^2}{h_\epsilon^2}\right) \geq -(\lambda - r)\,\epsilon.$$

Let us define $h_{\epsilon,k} = h_\epsilon - \frac{\phi}{k}$ for an integer $k \geq 1$. Then, $\lim_{k\to\infty} sup_M h_{\epsilon,k} = sup_M h_\epsilon$, and we also have:

$$\mathcal{A}(h_{\epsilon,k}) - \lambda h_{\epsilon,k} \geq -(\lambda - r)\epsilon - \frac{sup_M\,(\mathcal{A}(\phi) - \lambda\phi)}{k}.$$

Since ϕ is an exhaustion function (that can be chosen to be positive), $h_{\epsilon,k}$ attains its maximum in a point $x_k \in M$. When evaluating previous inequality at x_k, we get that:

$$sup_M h_{\epsilon,k} \leq \epsilon + \frac{sup_M\,(\mathcal{A}(\phi) - \lambda\phi)}{k\,(\lambda - r_0)}.$$

By letting $k \to \infty$ and $\epsilon \to 0$ we get $sup_M h_\epsilon \leq 0$, and consequently $h \equiv 0$. □

4.3 Existence of the solution

In order to study the existence and the regularity of the solution of the equation

$$\mathcal{A}(h) - \lambda h = H, \tag{4.4}$$

we need to study the semigroup $T(t)$ associated to the following Cauchy problem:

$$\begin{cases} u_t(t,x) = \mathcal{A}(u)(t,x), \\ u(0,x) = u_0(x). \end{cases} \tag{4.5}$$

Using the interpolation procedure in $(Lunardi, 1996)$ we will be able to characterize the domain of the generator of $T(t)$ and provide an optimal description of the regularity of the solution of (4.4).
As a first step, we are going to find an estimate of $||T(t)||_{\mathcal{L}(C^\alpha(M,E),C^\theta(M,E))}$ such that $0 \leq \alpha \leq \theta \leq 3$. In order to do that, we will be using the following version of the maximum principle:

Proposition 4.3.1 *Let $(z(t,))_{t\in[0,T]}$ be a classic bounded solution of the Cauchy problem*

$$\begin{cases} z_t(t,x) - \mathcal{A}(z)(t,x) = g(t,x), \\ z(0,x) = z_0(x), \end{cases} \tag{4.6}$$

and $\lambda_0 \geq r_0$. Then

1. *If $\sup\limits_M z > 0$, and if $g(t,x) \leq 0$ for all $t \in [0,T]$ and $x \in M$, then*
$$\sup_M z \leq e^{\lambda_0 t} \sup_M z_0. \tag{4.7}$$
2. *If $\inf\limits_M z < 0$, and if $g(t,x) \geq 0$ for all $t \in [0,T]$ and $x \in M$, then*
$$\inf_M z \geq e^{\lambda_0 t} \inf_M z_0. \tag{4.8}$$
3. *In particular, if $g \equiv 0$, then*
$$||z||_\infty \leq e^{\lambda_0 t}\, ||z_0||_\infty. \tag{4.9}$$

Proof. In order to prove inequality 4.7, we define $v(t,x) = e^{-\lambda t}\, z(t,x)$ for $\lambda > \lambda_0$. Then,

$$\begin{cases} v_t(t,x) - \mathcal{A}(v)(t,x) + \lambda v = e^{-\lambda t} g(t,x) \\ v(0,x) = z_0(x) \end{cases}$$

We also define $v_k(t,x) = v(t,x) - \frac{\phi(x)}{k}$. For k large enough, v_k admits a positive maximum (since $sup_M z > 0$) at (t_k, x_k). If $t_k \equiv 0$ for all k, then

$$sup_{[0,T]\times M}\, v_k \leq sup_M z_0 - inf_M\, \frac{\phi}{k}.$$

Consequently

$$sup_{[0,T]\times M}\, e^{-\lambda t}\, z \leq sup_M z_0,$$

hence inequality 4.7*. Now, suppose that* $t_k > 0$*. Since* $\partial_t v(t_k, x_k) \geq 0$ *(because* $\partial_t v(t_k, x_k) = \partial_t v_k(t_k, x_k) \geq 0$*), we have that:*

$$\mathcal{A}(v)(t_k, x_k) - \lambda v(t_k, x_k) \geq 0.$$

Adding $-\frac{(\mathcal{A}(\phi)-\lambda\phi)(t_k,x_k)}{k}$ *to both sides of the previous inequality, we find that:*

$$(\lambda - r_0)\, v_k(t_k, x_k) \leq \frac{(\mathcal{A}(\phi) - \lambda\phi)\,(t_k, x_k)}{k},$$

which is impossible for k *large enough.*

Inequality 4.8 *can be proved by replacing* z *with* $-z$ *in inequality* 4.7 *and using the fact that* $-sup_M\,(-z_0) \geq inf_M\, z_0$*. The last inequality is a combination of the previous ones.* □

Before we proceed with the next theorem, we will need the following technical lemma.

Lemma 4.3.2 *Let* (M, g) *be a Riemannian manifold and* ∇ *and* Δ *the Levi-Civita connection and the Laplacian respectively associated to* g*. Then, we have that:*

$$\begin{aligned} \left[\nabla^k, \Delta\right] &= \sum_{j=0}^{k} \nabla^{k-j} Rm(g) * \nabla^j, \\ \left[\nabla^k, \nabla_V\right] &= \sum_{j=0}^{k-1} \nabla^{k-j} V * \nabla^{j+1} + \nabla^{k-1-j}(Rm(g) * V) * \nabla^j, \end{aligned} \tag{4.10}$$

where $Rm(g)$ *is the Riemannian curvature tensor, and* $Rm(g) * V = Rm(g)(V, ., ., .)$*.*

Proof. We will proceed by induction on k *to prove equality* (4.10)*. Let us prove the result for* $k = 1$ *using normal coordinates. By definition of the curvature tensor , we have that*

$$\nabla_i \nabla_V h = \nabla_V \nabla_i h + (Rm(g) * V) * h + \nabla_{\nabla V} h = \nabla_V \nabla_i h + (Rm(g) * V) * h + \nabla V * \nabla h,$$

where $(Rm(g) * V) * h$ *corresponds to the action of the curvature tensor on tensors over* M*. This proves the second relation for* $k = 1$*. Regarding the first the equality, a simple computation shows that:*

$$\nabla \nabla^2_{i,j} h = \nabla^2_{i,j} \nabla h + \nabla_i\,(Rm(g) * h) + Rm(g) * \nabla_j h,$$

which proves the first relation of (4.10) *for* $k = 1$*.*

Now suppose that the result is true for $k \geq 1$. Then we have that:

$$
\begin{aligned}
\nabla^{k+1}\nabla_V h &= \nabla\left(\nabla_V\nabla^k h + \sum_{j=0}^{k-1}\nabla^{k-j}V * \nabla^{j+1}h + \nabla^{k-1-j}(Rm(g)*V)*\nabla^j h\right) \\
&= \nabla\nabla_V\nabla^k h + \sum_{j=0}^{k-1}\nabla^{(k+1)-j}V*\nabla^{j+1}h + \nabla^{(k+1)-(j+1)}V*\nabla^{(j+1)+1}h \\
&+ \nabla^{(k+1)-(j+1)}(Rm(g)*V)*\nabla^j h + \nabla^{(k+1)-(j+2)}(Rm(g)*V)*\nabla^{(j+1)}h \\
&= \nabla_V\nabla^{k+1}h + (Rm(g)*V)*\nabla^k h + \nabla V*\nabla^{k+1}h \\
&+ \sum_{j=0}^{k}\nabla^{(k+1)-j}V*\nabla^{j+1}h + \nabla^{(k+1)-(j+1)}(Rm(g)*V)*\nabla^j h \\
&= \sum_{j=0}^{k}\nabla^{(k+1)-j}V*\nabla^{j+1}h + \nabla^{(k+1)-(j+1)}(Rm(g)*V)*\nabla^j h
\end{aligned}
$$

This proves the second part of equality (4.10). We proceed in the same manner to prove the first one. □

Theorem 4.3.3 *Let $(u(t,.))_{t\in[0,T)}$ be a bounded solution of the Cauchy problem (4.5) with initial condition $u_0 \in C^\infty(M,E)$. Assume that there exists an integer $1 \leq k \leq 3$ and a positive constant $K(k)$ such that:*

$$
||Rm(g)||_{C^k(M,E)} + ||Rm(g)*V||_{C^k(M,E)} + ||\nabla V||_{C^{k-1}(M,E)} \leq K(k), \tag{4.11}
$$

where $Rm(g)*V = Rm(g)(V,.,.,.)$.

Then, for any $T>0$, there exists a constant $\omega = \omega(n,k,\lambda_0)$ such that :

$$
||u(t)||^2_{C^0(M,E)} + \sum_{i=1}^{k}\frac{(\alpha t)^i}{i}||\nabla^i u(t)||^2_{C^0(M,E)} \leq e^{\omega t}||u_0||^2_{C^0(M,E)} \;\; \forall t\in[0,T], \tag{4.12}
$$

α being a positive constant that will be defined later in order to obtain the right estimates.

Proof. We are going to derive the evolution of the heat equation of the following function:

$$
s(t,x) = ||u||^2(t,x) + \sum_{i=1}^{i=k}\frac{(\alpha t)^i}{i}||\nabla^i u||^2(t,x),
$$

and then apply the maximum principle for some values of α. We compute that

$$
s_t = 2\langle u_t, u\rangle + \alpha\sum_{i=1}^{k}(\alpha t)^{i-1}||\nabla^i u||^2 + 2\sum_{i=1}^{k}\frac{(\alpha t)^{i-1}}{i}\left\langle \nabla^i u_t, \nabla^i u\right\rangle.
$$

Since u is a solution of (4.5), we obtain:

$$
s_t = 2\left(\langle \Delta_V u, u\rangle + r||u||^2\right) + \alpha\sum_{i=1}^{k}(\alpha t)^{i-1}||\nabla^i u||^2 + 2\sum_{i=1}^{k}\frac{(\alpha t)^i}{i}\left\langle \nabla^i\Delta_V u + \nabla^i r u, \nabla^i u\right\rangle.
$$

On the other hand, we have that:

$$\mathcal{A}(s) = 2\left(||\nabla u||^2 + \langle \Delta_V u, u\rangle + \sum_{i=1}^{k} \frac{(\alpha t)^i}{i}\left(||\nabla^{i+1}u||^2 + \left\langle \Delta_V \nabla^i u, \nabla^i u\right\rangle\right)\right) + rs.$$

Thus,

$$s_t - \mathcal{A}(s) = 2\sum_{i=1}^{k} \frac{(\alpha t)^i}{i}\left\langle \left[\nabla^i, \Delta_V\right] u, \nabla^i u\right\rangle + (\alpha - 2)\,||\nabla u||^2 - \frac{2\,(\alpha t)^k}{k}||\nabla^{k+1}u||^2 +$$
$$\sum_{i=1}^{k-1} (\alpha t)^i \left(\alpha - \frac{2}{i}\right)||\nabla^{i+1}u||^2 - r\left(s - 2||u||^2\right) + 2\sum_{i=1}^{k} \frac{(\alpha t)^i}{i}\left\langle \nabla^i r u, \nabla^i u\right\rangle .$$

If we choose $\alpha \leq \frac{2}{k-1}$ if $k \geq 2$, and $\alpha \leq 2$ if $k = 1$, we obtain the following inequality:

$$s_t - \mathcal{A}(s) \leq 2\sum_{i=1}^{k} \frac{(\alpha t)^i}{i}\left\langle \left[\nabla^i, \Delta_V\right] u, \nabla^i u\right\rangle - r\left(s - ||u||^2 - 2\sum_{i=1}^{k} \frac{(\alpha t)^i}{i}||\nabla^i u||^2\right) +$$
$$2\sum_{i=1}^{k} \frac{(\alpha t)^i}{i}\sum_{j=1}^{i} \frac{i!}{(i-j)!j!}\left\langle \nabla^j r \nabla^{i-j}u, \nabla^i u\right\rangle .$$

Now, using the fact that:

$$-r\left(s - ||u||^2 - 2\sum_{i=1}^{k} \frac{(\alpha t)^i}{i}||\nabla^i u||^2\right) = rs \leq r_0 s,$$

and that there exists a positive constant C_1 (depending on T and using the fact that $\alpha \leq 2$) such that:

$$2\sum_{i=1}^{k} \frac{(\alpha t)^i}{i}\sum_{j=1}^{i} \frac{i!}{(i-j)!j!}\left\langle \nabla^j r \nabla^{i-j}u, \nabla^i u\right\rangle \leq C_1 \sum_{i=1}^{k}\sum_{j=1}^{i} ||\nabla^{i-j}u||\;||\nabla^i u|| \leq kC_1 s,$$

we obtain that:

$$s_t - \mathcal{A}(s) \leq 2\sum_{i=1}^{k} \frac{(\alpha t)^i}{i}\left\langle \left[\nabla^i, \Delta_V\right] u, \nabla^i u\right\rangle + (r_0 + kC_1)s.$$

On the other hand, by lemma 4.3.2 we have:

$$\left[\nabla^i, \Delta\right] h = \sum_{j=0}^{i} \nabla^j h * \nabla^{i-j} Rm(g),$$

and

$$\left[\nabla^i, \nabla_V\right] h = \sum_{j=0}^{i-1} \nabla^{i-j} V * \nabla^{j+1} h + \nabla^{i-1-j}(Rm(g) * V) * \nabla^j h.$$

Using condition 4.11 we deduce that there exists a positive constant C_2 such that:

$$||\left[\nabla^i, \Delta_V\right] h|| \leq C_2 \sum_{j=0}^{i} ||\nabla^j h|| \text{ for } i \leq k.$$

Consequently, there exists a positive constant $\tilde{C}$ such that:

$$\sum_{i=1}^{k} \frac{(\alpha t)^i}{i} \left\langle \left[\nabla^i, \Delta_V\right] u, \nabla^i u \right\rangle \leq C_2 \sum_{i=1}^{k} \frac{(\alpha t)^i}{i} \sum_{j=0}^{i} ||\nabla^j h|| \, ||\nabla^i h|| \leq \tilde{C} s$$

Finally, using the fact that $s(0,t) = ||u_0||^2$ and by applying proposition 4.3.1 to the operator $\tilde{\mathcal{A}} = \mathcal{A} + \left(2\tilde{C} + r_0 k C_1\right) I$ ($u_t - \tilde{\mathcal{A}}(u) \leq 0$), we deduce that:

$$s(t,x) \leq e^{\omega t} ||u_0||^2_{C^0(M,E)} \tag{4.13}$$

such that $\omega = r_0 + 2\tilde{C} + r_0 k C_1$. □

Remark 4.3.4 *The initial condition of the Cauchy problem (4.5) doesn't need to be smooth. In fact, using estimates (4.12) and the fact that $C^\infty(M,E)$ is dense in $C^0(M,E)$ (in the strong topology), we can see that u_0 can be chosen in $C^0(M,E)$.*

Remark 4.3.5 *The Cauchy problem 4.5 defines a semigroup of linear operators $T(t)$ that acts on $C^0(M,E)$, such that*

$$\left(T(t)u_0\right)(x) = u(t,x) \quad t \geq 0, \;\; x \in M, \;\; u_0 \in C^0(M,E).$$

The estimates of theorem $4.3.3$ implies that for all integers $0 \leq s \leq k \leq 3$ and $t \in (0,1]$, we have that:

$$||T(t)||_{\mathcal{L}(C^s(M,E),C^k(M,E))} \leq \frac{Ce^{\omega t}}{t^{\frac{k-s}{2}}}, \tag{4.14}$$

such that C is a positive constant independent of t. Moreover, using the maximum principle we get the following estimate:

$$||T(t)u_0||_{C^0(M,E)} \leq e^{\lambda_0 t} \, ||u_0||_{C^0(M,E)}. \tag{4.15}$$

In order to prove the existence of a solution to the Cauchy problem (4.5), we are going to approximate the problem using a sequence of elliptic operators with bounded coefficients. These operators have a unique solution on $M \times (0,\infty)$ (using the theory of parabolic equations with bounded coefficients).

Theorem 4.3.6 (Lunardi) *For any $u_0 \in C^0(M,E)$, there exists a unique (bounded) solution $(u(t))_{t\in[0,\infty)}$ of the Cauchy problem (4.5).*

Proof. Unicity is a direct consequence of the maximum principle (proposition $4.3.1$). In order to prove the *existence we will proceed as follows. Let $F \in C^{\infty}(M)$ such that:*

$$lim_{x\to\infty}F(x) = \infty,\ F(x) \leq c(1 + d_p(x)),\ \ \forall x \in M,\ \ ||\nabla F|| + ||\nabla^2 F|| \leq c, \tag{4.16}$$

where d_p is the distance function with respect to a fixed point $p \in M$. The existence of such a function is proved in theorem 3.6 in $(Shi, 1997)$ and uses the fact that the curvature is bounded.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\psi(t) = 1$ if $0 \leq t \leq 1$ and $\psi(t) = 0$ if $t \geq 2$. We define $\psi_s(x) = \psi(\frac{F(x)}{s})$, $V_s = \psi_s V$ and $r_s = \psi_s r$.

Note that V_s is bounded on M and that condition 4.11 continues to be satisfied by V_s. Consequently, the following Cauchy problem

$$\begin{cases} \partial_t u_s(t,x) = \mathcal{A}_s(u_s)(t,x) \\ u_s(0,x) = u_0(x) \end{cases}$$

has a unique (bounded) solution $(u_s(t))_{t\in(0,\infty)}$, where $\mathcal{A}_s = \Delta_{V_s} + r_s$.

Using equation (4.16) and the fact that ψ is a compactly supported function, there exists a positive constant C independent of s such that $||\psi_s||_{C^2(M,E)} \leq C$. Thus, inequality (4.11) is satisfied for the vector fields V_s with a constant $K(k)$ independent of s. The same thing applies to inequality (4.3) with the operator $\mathcal{A}_s$.

Consequently, the estimates of theorem $4.3.3$ applies to u_s, such that ω is a positive constant independent of s. In particular, $||u_s||_{C^k(M,E)}$ is uniformly bounded.

Using Arzela-Ascoli, there exists a subsequence $(u_{k_i})_{k_i\in\mathbb{N}}$ and a tensor $u \in C^{\infty}(M,E)$ such that u_{k_i} converges uniformly to u on any compact subset of $(0,\infty) \times M$. □

4.4 Interpolation spaces

Before we proceed with that last step of the proof, let us recall some results on interpolation spaces (see $(Lunardi, 2018)$ for more details).

Definition 4.4.1 *Let X and Y be two Banach spaces such that Y is continuously embedded into X. An intermediate space between X and Y is a Banach space E such that $Y \subset E \subset X$ with continuous inclusions.*

Definition 4.4.2 *The interpolation space $(X,Y)_{\theta,\infty}$ is an intermediate space between X and Y defined by:*

$$(X,Y)_{\theta,\infty} = \left\{x \in X \mid ||x||_{\theta,\infty} = sup_{t\in(0,1)} t^{-\theta} K\left(t,x,X,Y\right) < \infty\right\}, \tag{4.17}$$

such that:

$$K(t,x,X,Y) = inf\left\{||a||_X + t||b||_Y \mid x = a+b,\ (a,b) \in X \times Y\right\}. \tag{4.18}$$

We can see that $\left((X,Y)_{\theta,\infty}, ||\,.\,||_{\theta,\infty}\right)$ *is a Banach space.*

Theorem 4.4.3 (Interpolation theorem) *Let Y_1 and Y_2 be two Banach space continuously embedded into respectively the Banach spaces X_1 and X_2. If $T \in \mathcal{L}(X_1,X_2) \cap \mathcal{L}(Y_1,Y_2)$, then $T \in \mathcal{L}\left((X_1,Y_1)_{\theta,\infty}, (X_2,Y_2)_{\theta,\infty}\right)$ for any $\theta \in (0,1)$. Moreover*

$$||T||_{\mathcal{L}\left((X_1,Y_1)_{\theta,\infty},(X_2,Y_2)_{\theta,\infty}\right)} \leq \left(||T||_{\mathcal{L}(X_1,X_2)}\right)^{1-\theta}\left(||T||_{\mathcal{L}(Y_1,Y_2)}\right)^{\theta}. \tag{4.19}$$

Definition 4.4.4 *Let $\theta \in [0,1]$ and E be an intermediate space between X and Y. Then, we say that :*

(i) *E belongs to the class $J_\theta(X,Y)$ if there exists a constant $c > 0$ such that:*

$$||x||_E \leq c||x||_X^{1-\theta}||x||_Y^{\theta}\ , \forall x \in Y. \tag{4.20}$$

(ii) *E belongs to the class $K_\theta(X,Y)$ if there exists a constant $c > 0$ such that:*

$$K\left(t,x,X,Y\right) \leq ct^{\theta}||x||_E,\ \forall x \in E,\ \forall t > 0. \tag{4.21}$$

The last inequality implies that a Banach space E is of class $K_\theta(X,Y)$ if and only if E is embedded continuously into $(X,Y)_{\theta,\infty}$. We also have that $(X,Y)_{\theta,\infty} \in J_\theta\left(X,Y\right) \cap K_\theta\left(X,Y\right)$.

Theorem 4.4.5 (Reiteration theorem) *Let $0 \leq \theta_0 < \theta_1 \leq 1$ and $\theta \in (0,1)$. If $\omega = (1-\theta)\,\theta_0 + \theta\theta_1$, then:*

(i) *If E_i is of class $K_{\theta_i}(X,Y)$ $(i = 0,1)$, then*

$$(E_0,E_1)_{\theta,\infty} \subset (X,Y)_{\omega,\infty}. \tag{4.22}$$

(ii) *If E_i is of class $J_{\theta_i}(X,Y)$ $(i = 0,1)$, then*

$$(X,Y)_{\omega,\infty} \subset (E_0,E_1)_{\theta,\infty}. \tag{4.23}$$

Consequently, if $E_i \in K_{\theta_i}(X,Y) \cap J_{\theta_i}(X,Y)$, then

$$(E_0, E_1)_{(\theta,\infty)} = (X,Y)_{(\omega,\infty)} . \tag{4.24}$$

Proposition 4.4.6 (proposition 2.8 (Deruelle, 2015)) *Let (M,g) be a complete Riemannian manifold with positive injectivity radius and bounded curvature together with its covariant derivatives. Then*

(i) *For $\theta \in (0,1)$ and $k \in \mathbb{N}$,*

$$\left(C^k (M,E), C^{k+1}(M,E)\right)_{\theta,\infty} = C^{k,\theta}(M,E) . \tag{4.25}$$

(ii) *Let $0 \le \theta_1 \le \theta_2$ and $0 \le \theta \le 1$. Then, if $\omega = (1-\theta)\theta_1 + \theta\theta_2$ is not an integer,*

$$\left(C^{\theta_1}(M,E), C^{\theta_2}(M,E)\right)_{\theta,\infty} = C^{\omega}(M,E) , \tag{4.26}$$

such that

$$C^{\omega}(M,E) := C^{\lfloor\omega\rfloor, \omega - \lfloor\omega\rfloor}$$

$\lfloor\omega\rfloor$ *being the integer part of ω.*

4.5 Regularity of the solution

Corollary 4.5.1 *Using the previous notation (of C^ω), we have that*

$$||T(t)||_{\mathcal{L}(C^\theta((M,E)),C^\alpha(M,E))} \le \frac{Ce^{\omega t}}{t^{\frac{\alpha-\theta}{2}}} , \; 0 \le \theta \le \alpha \le 3 \tag{4.27}$$

Proof. If θ and α are both integers then the previous inequality is a direct consequence of estimate 4.14. On the other hand, if α is an integer and θ is not an integer, then:

$$||T(t)||_{\mathcal{L}(C^\theta((M,E)),C^\alpha(M,E))} \le ||T(t)||_{\mathcal{L}\left(C^{\lfloor\theta\rfloor}((M,E)),C^\alpha(M,E)\right)}$$

If α is not an integer, let $0 \le k_1 \le k_2 \le 3$ be two integers and $s \in (0,1)$ such that $\alpha = (1-s)k_1 + sk_2$. Using proposition $4.4.6$,we have that $C^\alpha(M,E) = \left(C^{k_1}(M,E), C^{k_2}(M,E)\right)_{s,\infty}$ and $C^{\lfloor\theta\rfloor}(M,E) = \left(C^{\lfloor\theta\rfloor}(M,E), C^{\lfloor\theta\rfloor}(M,E)\right)_{s,\infty}$. Now, using theorem $4.4.3$, we deduce that

$$||T||_{\mathcal{L}(C^\theta(M,E),C^\alpha(M,E))} \le \left(||T||_{\mathcal{L}(C^{\lfloor\theta\rfloor}(M,E),C^{k_1}(M,E))}\right)^{1-s} \left(||T||_{\mathcal{L}(C^{\lfloor\theta\rfloor}(M,E),C^{k_2}(M,E))}\right)^{s} = \frac{Ce^{\omega t}}{t^{\frac{\alpha-\lfloor\theta\rfloor}{2}}}$$

Since we could restrict ourselves to $t \in (0,1)$ (using the semigroup law), this concludes the proof. □

In section 4 of $(Lunardi, 1998)$, the author notices that even though the semigroup $T(t)$ is not strongly continuous in $C^0(M,E)$, we could still define a realization of the operator $\mathcal{A}$ in $C^0(M,E)$. Let λ>λ_0 and R be the following linear operator

$$(R(\lambda)u)(x) = \int_0^\infty e^{-\lambda t}(T(t)u)(x)\ dt,\ \ x \in M$$

R is well defined (using estimate 4.15). Moreover, $||R(\lambda)||_{\mathcal{L}(C^0(M,E))} \leq \frac{1}{\lambda-\lambda_0}$.

Since $R(\lambda)(u)(x)$ is the Laplace transform of the tensor $t \mapsto T(t)(u)(x)$, R is injective. Thus, there exists a closed linear operator $A : D(A) \to C^0(M,E)$ (infinitesimal generator of $\mathcal{A}$), such that $R(\lambda)$ is the resolvent of$A$, and $D(A) = Image(R(\lambda))$. By proposition 4.1 of $(Lunardi, 1998)$, we have that

Proposition 4.5.2 (Lunardi)

$$D(A) = D^2_{\mathcal{A}}(M,E)$$

$$Ah = \mathcal{A}h,\ \forall h \in D(A)$$

Moreover, for any $\theta \in (0,2)$, there exists a positive constant C such that

$$||h||_{C^\theta(M,E)} \leq C||h||^{1-\frac{\theta}{2}}_{C^0(M,E)}||h||^{\frac{\theta}{2}}_{D(A)},\ \forall h \in D(A), \tag{4.28}$$

where $||h||_{D(A)} = ||h||_{C^0(M,E)} + ||\mathcal{A}(h)||_{C^0(M,E)}$.

Proof. [Proof of theorem $4.1.1$] Equation (4.28) shows that $D^2_{\mathcal{A}}(M,E)$ is continuously embedded into $C^\theta(M,E)$ for all $\theta \in (0,2)$ which proves the first part of theorem $4.1.1$.

To prove the second part of theorem $4.1.1$ we proceed as follows. Let $H \in C^\theta(M,E)$ ($\theta \in (0,1)$) and $\lambda > \lambda_0$. Then

$$h(x) := \int_0^\infty e^{-\lambda t}(T(t)H)(x)\ dt, \tag{4.29}$$

is well defined. Moreover, since $h \in D(A)$ (because $(\lambda - \mathcal{A})(h) = H$), we have that $h \in C^\theta(M,E)$ (using equation (4.28)),

$$||h||_{C^\theta(M,E)} \leq C||h||^{1-\frac{\theta}{2}}_{C^0(M,E)}||h||^{\frac{\theta}{2}}_{D(A)} \leq \frac{C}{(\lambda-\lambda_0)^{1-\frac{\theta}{2}}}||H||_{C^\theta(M,E)}.$$

It remains to prove that $h \in C^{2,\theta}(M,E)$. By proposition $4.4.6$ we have that

$C^{2,\theta}(M,E) = \left(C^\alpha(M,E), C^{2,\alpha}(M,E)\right)_{\gamma,\infty}$ for $\gamma = 1 - \frac{(\alpha-\theta)}{2}$ and $\alpha \in (\theta, 1)$.

Let $\eta > \omega$ (ω of corollary $4.5.1$). Then, h satisfies $(\eta - \mathcal{A})h = H + (\eta - \lambda)h = \tilde{H}$. The latter satisfies:

$$||\tilde{H}||_{C^\theta(M,E)} \leq \left(1 + \frac{C|\eta-\lambda|}{(\lambda-\lambda_0)^{1-\frac{\theta}{2}}}\right)||H||_{C^\theta(M,E)}.$$

Since $\eta > \lambda_0$, we have that

$$h\left(x\right) = \int_0^{\infty} e^{-\eta t} \left(T\left(t\right) \tilde{H}\right)\left(x\right)\ dt.$$

For every $\epsilon > 0$, set

$$a(x) = \int_0^{\epsilon} e^{-\eta t} \left(T\left(t\right) \tilde{H}\right)\left(x\right)\ dt\ ;\ b(x) = \int_{\epsilon}^{\infty} e^{-\eta t} \left(T\left(t\right) \tilde{H}\right)\left(x\right)\ dt.$$

Then, $h(x) = a(x) + b(x)$. Using estimate 4.27, there exists positive constants C_1 and C_2 such that

$$||a||_{C^{\alpha}(M,E)} \leq C_1 \epsilon^{\gamma} ||\tilde{H}||_{C^{\theta}(M,E)}$$
$$||b||_{C^{2,\alpha}(M,E)} \leq C_2 \epsilon^{\gamma-1} ||\tilde{H}||_{C^{\theta}(M,E)}$$

Consequently (using the definition of $||h||_{\gamma,\infty}$), we obtain that:

$$||h||_{\gamma,\infty} \leq sup_{\epsilon\in(0,1)} \epsilon^{-\gamma} \left(||a||_{C^{\alpha}(M,E)} + \epsilon ||b||_{C^{2,\alpha}(M,E)}\right) = \left(C_1 + C_2\right) ||\tilde{H}||_{C^{\theta}(M,E))}$$
$$\leq \left(C_1 + C_2\right) \left(1 + \frac{C|\eta - \lambda|}{\left(\lambda - \lambda_0\right)^{1-\frac{\theta}{2}}}\right) ||H||_{C^{\theta}(M,E)}.$$

Since $||h||_{\gamma,\infty} = ||h||_{C^{2,\theta}(M,E)}$ this concludes the proof. □

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