UNIVERSITÉ DU QUÉBEC À MONTRÉAL - UNIVERSITÉ TOULOUSE III

UNE CORRESPONDANCE DE YAU–TIAN–DONALDSON SUR UNE CLASSE DE FIBRATIONS TORIQUES

THÈSE

PRÉSENTÉE

COMME EXIGENCE PARTIELLE

DU DOCTORAT EN MATHÉMATIQUES

PAR

SIMON JUBERT

JUILLET 2023

UNIVERSITÉ DU QUÉBEC À MONTRÉAL - UNIVERSITÉ TOULOUSE III

A YAU–TIAN–DONALDSON CORRESPONDENCE ON A CLASS OF TORIC FIBRATIONS

DISSERTATION

PRESENTED

AS PARTIAL REQUIREMENT

OF THE DOCTORATE IN MATHEMATICS

BY

SIMON JUBERT

JULY 2023

REMERCIEMENT

Mes sincères remerciements vont à mes encadrants Vestislav Apostolov et Éveline Legendre pour leur générosité dans le partage de leurs idées et leur aide constante pendant ce doctorat. Je tiens à exprimer ma reconnaissance à Éveline Legendre pour son soutien et ses encouragements depuis mon premier cours de géométrie différentielle en master 1, et à Vestislav Apostolov pour son accueil chaleureux à l'Université du Québec à Montréal.

Je tiens également à remercier l'ensemble du personnel du CIRGET à Montréal et de l'Institut mathématiques de Toulouse pour les conditions favorables à l'apprentissage des mathématiques qu'ils ont su créer. Je suis également reconnaissant aux membres du laboratoire Jean Leray à Nantes pour m'avoir accueilli dans une ambiance à la fois conviviale et professionnelle durant ces 3 mois d'automne 2022.

Ma gratitude va vers toutes les personnes rencontrées en conférence ou en séminaire pour les moments agréables passés ensemble, mathématiques ou non.

je salue mes associations de musique, MoonTrip à Toulouse et Vagues Records à Montréal qui m‘ont permis de passer tant de moments hors du temps, bien souvent nécessaires. Je salue également mes partenaires de badminton, toulousains et montréalais.

Pour conclure, mes remerciements profonds vont à mes parents qui m'ont toujours soutenus et encouragés dans mes études. Je remercie tendrement celle qui fût à mes côtés au quotidien, Lola.

CONTENTS

LIST OF FIGURES

RÉSUMÉ

Dans cette thèse, nous montrons une correspondance du type Yau–Tian–Donaldson sur une large classe de fibrations toriques, introduites par Apostolov–Calderbank–Gauduchon–Tonnesen-Friedman et appelées fibrés principaux toriques semisimples. Nous démontrons l'équivalence entre l'existence d'une métrique kählérienne extrémale sur l'espace total et une notion d'uniforme K-stabilité pondérée du polytope de Delzant correspondant à la fibre torique. Pour cela, nous utilisons qu'une métrique kählérienne extrémale sur l'espace total compatible avec la structure de fibré, correspond à une métrique avec courbure scalaire constante pondérée (au sens de Lahdili) sur la fibre torique associée. Comme application, nous démontrons que le fibré en plans projectifs $\mathbb{P}(L_1 \oplus L_2 \oplus L_3)$ au dessus d'une courbe elliptique admet une métrique kählérienne extrémale dans chaque classe de Kähler.

Dans une seconde partie, en travail commun avec Delcroix, nous obtenons plusieurs conditions assurant l'uniforme K-stabilité pondérée d'un polytope compact convexe simple. Nos critères peuvent être vérifiées en pratique. En vertu de la correspondance de Yau–Tian–Donaldson mentionnée ci-dessus, nous obtenons l'existence d'une métrique kählérienne extrémale sur l'espace total d'un fibré semisimple principal torique dès qu'une des conditions suffisantes sur la fibre est satisfaite. Nous montrons en particulier qu'une de classe fibrés principaux semisimples toriques Y dont la première classe de Chern $c_1(Y)$ est positive admet une métrique kählérienne extrémale dans $c_1(Y)$ si sa fonction affine extrémale est majorée par $2(\dim(Y) + 1)$. Nous appliquons ensuite la condition suffisante pour exhiber de nouveaux exemples de basse dimension de métriques kählériennes extrémales.

Finalement, dans un travail en collaboration avec Apostolov et Lahdili, nous introduisons une généralisation des fibrés principaux toriques semisimples où la fibre n'est pas nécessairement torique. Nous supposons à la place que la fibre est une variété kählérienne munie d'une action isométrique hamiltonienne d'un tore. Pour de tels fibrés, appelés fibrés principaux semisimples, nous démontrons que l'existence d'une métrique kählérienne extrémale sur l'espace total est équivalente à l'existence d'une métrique à courbure scalaire constante pondérée sur la fibre ainsi qu'à une notion de propreté de l'énergie de Mabuchi pondérée de la fibre.

ABSTRACT

In this thesis, we establish a Yau–Tian–Donaldson type correspondence on a large class of toric fibrations, introduced by Apostolov–Calderbank–Gauduchon–Tonnesen-Friedman and called semisimple principal toric fibrations. We establish the equivalence between the existence of an extremal Kähler metric on the total space of such fibration and a suitable notion of weighted uniform K-stability of the Delzant polytope of the toric fiber. To this end, we use that an extremal Kähler metric on the total space compatible with the bundle structure corresponds to a weighted constant scalar curvature Kähler metric (in the sense of Lahdili) on the toric fiber. As an application, we show that the projective plane bundle $\mathbb{P}(L_1 \oplus L_2 \oplus L_3)$ over an elliptic curve, admits an extremal metric in every Kähler class.

In a second part of the thesis, which is a joint work with Delcroix, we prove various sufficient conditions for a simple compact convex polytope to be weighted uniform K-stable. Our conditions can be checked effectively. By virtue of the Yau–Tian–Donaldson correspondence mentioned above, we obtain the existence of an extremal Kähler metric on the total space of a semisimple principal toric fibration as soon as one of the sufficient conditions on the fiber is satisfied. We show in particular that a certain class of a semisimple principal toric fibrations with positive first Chern class $c_1(Y)$ admits an extremal Kähler metric in $c_1(Y)$ as soon as their affine extremal function is bounded above by $2(\dim(Y) + 1)$. We apply this to exhibit new low dimensional examples of extremal Kähler metrics.

Finally, in a collaboration with Apostolov and Lahdili, we introduce a generalization of semisimple principal toric fibrations, in which the fiber is not necessarily a toric variety. Instead, we assume that the fiber is a Kähler manifold with an isometric hamiltonian action of a torus. For such fibrations, referred to as semisimple principal fibrations, we show that the existence of an extremal Kähler metric on the total space is equivalent to the existence of a weighted cscK metric on the fiber and also to a properness condition of the weighted Mabuchi energy of the fiber.

INTRODUCTION

0.1 Motivations

Un problème central en géométrie kählérienne, proposé par Calabi [20] dans les années 1980, est de trouver un représentant canonique, appelé métrique kählérienne extrémale, dans une classe de Kähler donnée. Les métriques kählériennes extrémales incluent en particulier les métriques kählériennes à courbure scalaire constante (cscK) et donc les métriques Kähler–Einstein. Par définition, une métrique kählérienne extrémale est une métrique kählérienne telle que le champ de vecteurs hamiltonien défini par la courbure scalaire est un champ de vecteurs (réel) holomorphe, c.-à-d. son flot préserve la structure complexe. Dans son article fondateur [20], Calabi a démontré que les surfaces complexes de Hirzebruch n'admettent pas de métriques à cscK, mais admettent des métriques kählériennes extrémales.

Bien que le problème de Calabi soit encore ouvert en général, certains cas particuliers sont établis. C'est le cas des surfaces de Riemann pour lesquelles le théorème d'uniformisation [52, 70] implique qu'elles admettent une métrique à cscK dans chaque classe de Kähler. Le théorème de Yau, conjecturé par Calabi dans les années 1950 et démontré par Yau [80] à la fin des années 1970, affirme que, dans le cas où $c_1(X) = \{0\}$, chaque classe de Kähler sur X admet une métrique cscK (qui est aussi Ricci plate). Si la classe de Chern de la variété est négative, Aubin et Yau ont montré [13, 14, 80] qu'il existe une unique métrique Kähler–Einstein dans $-c_1(X)$.

Pour une variété complexe X, Matsushima et Lichnerowicz ont montré [64, 67] que le groupe d'automorphismes complexes $\mathrm{Aut}(X)$ doit etre réductif pour qu'il existe une métrique cscK. L'existence d'une métrique kählérienne extrémale est également obstruée en termes de $\mathrm{Aut}(X)$. En effet, Calabi a prouvé [21] qu'un certain sous-groupe de $\mathrm{Aut}(X)$ doit être réductif pour qu'il existe une métrique kählérienne extrémale sur X. Cette obstruction est non triviale comme l'a observé Levin [57] en donnant le premier exemple de variété kählérienne compacte n'admettant aucune métrique kählérienne extrémale. D'autre part, Tian a donné des exemples [77] de variétés sans automorphisme qui ne possèdent pas de métrique Kähler–Einstein. Ses travaux (voir par exemple [74, 75, 76, 77, 78]) ont contribué à une conjecture générale, appelée *conjecture de Yau–Tian–Donaldson* (YTD), qui prédit que l'existence d'une métrique kählérienne extrémale dans une classe de Kähler α est équivalente à une certaine notion de *stabilité* de (X, α). Cette conjecture est toujours ouverte mais elle a été confirmée dans plusieurs situations. C'est le cas des variétés de Fano par le théorème de Chen–Donaldson–Sun [27, 28, 29] (voir également [17, 63, 78]). Le cas des surfaces toriques a également été conclu par le théorème de Donaldson [39] et en dimension quelconque plus récemment par le théorème de Chen–Cheng [25, 26], suites aux efforts de nombreux mathématiciens [1, 22, 25, 26, 37, 48, 61, 81]. Dans le contexte des variétés toriques, la notion de stabilité est exprimée en termes du polytope de Delzant correspondant [37, 48]. Pour certaines autres classes de variétés admettant un large groupe de symétries, telles que les variétés sphériques, la conjecture de Yau–Tian–Donaldson est également établie (voir [36]).

Dans ce manuscrit, nous nous intéresserons à la conjecture de Yau–Tian–Donaldson pour certains fibrés principaux toriques, introduits par Apostolov–Calderbank–Gauduchon–Tonnesen-Friedman [5] et dénommés fibrés principaux toriques semisimples. Ces fibrés sont obtenus via une généralisation de la construction des $\mathbb{P}^1$-fibrés de Calabi [20]. Nous en rappelons briévement les élèments principaux et donneront plus de détails en section 1.2. Ils sont définis à partir d'un fibré principal Q en tores $\mathbb{T}$ et d'une variété kählérienne torique $(X, J_X, \omega_X, \mathbb{T})$. On suppose que la base B du fibré principal $\pi : Q \to B$ est un produit de variétés kählériennes $(B, J_B, \omega_B) := \prod_{a=1}^{k}(B_a, J_a, \omega_a)$ à cscK et que la première classe de Chern de Q satisfait

$$c_1(Q) = \sum_{a=1}^{k} p_a \otimes \pi^*[\omega_a], \tag{1}$$

où les p_a sont des éléments du réseau $\Lambda \subset \mathfrak{t}$ du tore $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$ (où $\mathfrak{t}$ désigne l'algèbre de Lie de $\mathbb{T}$). L'espace total du fibré principal torique semisimple associé est défini par

$$Y := Q \times_{\mathbb{T}} X. \tag{2}$$

Ensuite, nous munissons Y d'une structure complexe J_Y construite à partir d'une connexion $\theta \in \Omega^1(Q) \otimes \mathfrak{t}$ dont le courbure est égale à

$$d\theta = \sum_{a=1}^{k} p_a \otimes \pi^*(\omega_a),$$

de la structure complexe J_B de B et la structure complexe J_X de X. De plus, on peut montrer que toute métrique kählérienne ω_X sur X définie une métrique kählérienne ω_Y sur Y tel que son tiré en arrière sur $X \times Q$ par l'application quotient $\pi_Y : X \times Q \to Y$ est la 2-forme basique

$$\pi_Y^*(\omega_Y) = \omega_X + \langle m, \theta\rangle + \sum_{a=1}^{k} \big(\langle p_a, m\rangle + c_a\big)\pi^*(\omega_a), \tag{3}$$

où m désigne l'application moment de l'action de $\mathbb{T}$ sur (X, ω_X), $\langle \cdot, \cdot \rangle$ est la contraction naturelle entre $\mathfrak{t}$ et $\mathfrak{t}^*$ et les constantes réelles c_a, choisies de manière que $\langle p_a, m\rangle + c_a > 0$, déterminent la classe de Kähler $[\omega_Y]$. De telles métriques sont appelées *métriques compatibles* et les classes de Kähler correspondantes sont appelées *classes de Kähler compatibles.* Ci-dessous, deux examples de fibrés semisimple princpaux toriques pour lesquels la fibre est l'espace projectif $\mathbb{P}^k$.

1. La projectivisation

$$Y = \mathbb{P}(L_0 \oplus \cdots \oplus L_k) \to \Sigma,$$

   d'une somme directe de fibrés en droites holomorphes $L_i \to \Sigma$ au-dessus d'une surface de Riemann Σ.

2. La projectiviation

$$Y = \mathbb{P}(L_0 \oplus \cdots \oplus L_k) \to B_1 \times \cdots \times B_k,$$

   où (B_i, L_i) sont des variétés polarisées à cscK, sont des exemples de fibrés principaux toriques semisimples.

Les fibrés principaux toriques semisimples définis ci-dessus correspondent aux fibrés rigides semisimples dans le sens de [5] lorsque la base est un produit global de variétés kählériennes à cscK et quand il n'y a pas de *blow-down.* Dans [7], la conjecture suivante est énoncée

Conjecture 1. *Soit $(Y, J_Y, \mathbb{T})$ un fibré rigide semisimple torique muni d'une métrique compatible ω_Y. Soit $(X, J_X, \omega_X, \mathbb{T})$ la fibre torique dont le polytope de Delzant associé est P_X. Alors, les énoncés suivants sont équivalents.*

1. *Il existe une métrique kählérienne extrémale dans $[\omega_Y]$.*

2. *Il existe une métrique kählérienne extrémale compatible dans* $[\omega_Y]$.

3. *Le polytope* P *respecte une certaine condition de stabilité pondérée.*

Dans le 3ème énoncé, la notion de stabilité est une version pondérée de la notion de K-stabilité torique introduite par Donaldson [37]. En présence de blow-down, la notion de métrique compatible doit être raffinée par rapport à celle introduite ci-dessus (3), voir [5, 7].

0.2 Résultats principaux

Nous obtenons la résolution de la conjecture 1 dans un article accepté dans Annales de l'institut Fourier (voir [50])

Théorème 0.2.1. *Pour un fibré principal torique semisimple, la conjecture* 1 *est vraie.*

Dans le théorème 0.2.1, la notion de K-stabilité uniforme pondérée est définie à l'aide de fonctions convexes continues sur le polytope et lisses sur l'intérieur, et par rapport à la norme L^1 sur le polytope, voir la définition 1.6.8. Par continuité et densité $\mathcal{C}^0$, cette notion est équivalente à la stabilité uniforme pondérée sur P définie en termes de fonctions convexes affines par morceaux à coefficients rationnels (et relativement à la norme L^1). De plus, par [68], on peut voir que cette notion de stabilité est équivalente à celle proposée initialement dans la conjecture 1 (voir [7, Conjecture 2]) pour l'énoncé précis).

On scinde le théorème 0.2.1 en deux énoncés : les théorèmes A et B ci-dessous. Le théorème A correspond à l'équivalence (1) $\Leftrightarrow$ (2) et le théorème B à l'équivalence (2) $\Leftrightarrow$ (3) dans la conjecture 1.

Théorème A. *Soit* $(Y, J_Y, \omega_Y, \mathbb{T})$ *un fibré principal torique semisimple dont la fibre est la variété torique* $(X, J_X, \omega_X, \mathbb{T})$*. Les énoncés suivants sont équivalents.*

1. *Il existe une métrique kählérienne extrémale dans* $[\omega_Y]$*.*
2. *Il existe une métrique kählérienne extrémale* compatible *dans* $[\omega_Y]$*.*
3. *Il existe une métrique à courbure scalaire constante pondérée dans* $[\omega_X]$*.*

Dans la 3ème assertion, la notion de métrique kählérienne à courbure scalaire constante pondérée est au sens de Lahdili (voir [53]) et les poids sont définis par les données topologiques de la fibration, voir (1.15). L'équivalence entre (2) et (3) est établie dans [5] et rappelée dans le corollaire 1.2.4. L'idée principale pour obtenir (1) $\Rightarrow$ (2) est d'utiliser que l'existence d'une métrique kählérienne extrémale implique une certaine notion de propreté de la fonctionnelle de Mabuchi relative [25, 26, 47] (voir la section 1.4). Nous montrons ensuite que, sous l'hypothèse de propreté de l'énergie de Mabuchi relative, le chemin de continuité de Chen [24] peut être résolu dans le sous-espace des métriques compatibles dans $[\omega_Y]$.

Le théorème B ci-dessous correspond à l'énoncé (1) $\Leftrightarrow$ (3) dans la conjecture 1. Il caractérise l'existence de métriques kählériennes extrémales sur l'espace total Y par une certaine condition de stabilité du polytope de Delzant P_X de la fibre torique X.

Théorème B. *Soit* $(Y, J_Y, \omega_Y, \mathbb{T})$ *un fibré torique principale semisimple dont la fibre est la variété torique* $(X, J_X, \omega_X, \mathbb{T})$ *avec polytope de Delzant* P_X*. Alors, il existe une métrique kählérienne à courbure scalaire constante pondérée dans* $[\omega_X]$ *si et seulement si* P_X *est pondérément uniformément K-stable relativement aux poids définis dans* (1.15)*. En particulier, cette condition est nécessaire et suffisante pour que* $[\omega_Y]$ *admette une métrique kählérienne extrémale.*

La stratégie de la démonstration consiste à exprimer les métriques kählériennes extrémales compatibles en termes de métriques à courbure scalaire constante pondérée sur la fibre via le théorème A. On utilise ensuite le formalisme d'Abreu–Guillemin et une adaptation au cas pondérée de résultats de Chen–Li–Sheng [22], Donaldson [37] et Zhou–Zhu [81] pour établir l'équivalence sur $(X, J_X, \omega_X, \mathbb{T})$ entre l'existence d'une métrique extremal sur Y et la stabilité pondérée uniforme de P_X. Une direction, à savoir que « l'existence d'une métrique extrémale compatible sur Y implique que le polytope de Delzant est pondérément uniformément K-stable », découle d'un résultat de [22], démontré par Li–Lian–Sheng [60]. Pour la direction réciproque, nous adaptons des arguments de Donaldson et Zhou–Zhu [37, 81] au cas pondéré pour montrer que la stabilité uniforme pondérée du polytope implique une certaine notion de propreté de l'énergie de Mabuchi pondérée sur X, voir la proposition 1.6.9. On montre alors que la dernière condition correspond à la propreté de l'énergie de Mabuchi de Y restreinte à l'espace des métriques compatibles, et on conclut en utilisant [25, 26, 47].

Nous nous intéressons ensuite à une certaine classe de métriques presque kählériennes (c.-a-d. compatible avec une forme symplectique mais dont la structure presque complexe compatible n'est pas necéssairement intégrable) sur une variété torique $(X, \omega_X, \mathbb{T})$. Elles sont, par définition, des métriques presque kählériennes telles que la distribution orthogonale aux orbites de $\mathbb{T}$ est involutive (voir [56]) et sont appelées *métriques presque kählériennes toriques involutives.* L'idée d'étudier de telles métriques provient de Donaldson [37] (voir [5] pour le cas pondéré), où il a été conjecturé que l'existence d'une métrique presque kählérienne involutive torique à courbure scalaire constante est équivalente à l'existence d'une métrique kählérienne torique à courbure scalaire constante.

Proposition 0.2.1. *Soit* $(X, \omega_X, \mathbb{T})$ *une variété torique symplectique associée à un polytope de Delzant* P_X. *Alors, pour les poids définis en* (1.15)*, les énoncés* *suivants sont équivalents.*

1. *Il existe une métrique kählérienne torique à courbure scalaire constante pondérée sur* $(X, \omega_X, \mathbb{T})$.
2. *Il existe une métrique presque kählérienne torique involutive à courbure scalaire constante pondérée sur* $(X, \omega_X, \mathbb{T})$.
3. P_X *est pondérément uniformément K-stable.*

Nous appliquons ensuite la proposition 0.2.1 pour démontrer l'existence de métriques kählériennes extrémales sur la projectivisation $Y = \mathbb{P}(L_0 \oplus L_1 \oplus L_2) \to \Sigma_{\mathbf{g}}$ de la somme directe des fibrés en droites L_i au-dessus d'une courbe complexe $\Sigma_{\mathbf{g}}$ de genre $\mathbf{g} = 0, 1$. En effet, dans [7, Proposition 4], les auteurs ont construit explicitement des métriques presque kählériennes toriques involutives à courbure scalaire constante pondérée sur $\mathbb{P}^2$, pour des poids correspondant à la variété Y. En combinant cette construction avec la proposition 0.2.1 on en déduit :

Corollaire 0.2.2. *Soit* $Y = \mathbb{P}(L_0 \oplus L_1 \oplus L_2) \longrightarrow \Sigma_{\mathbf{g}}$ *un fibré projectif au-dessus d'une courbe complexe* $\Sigma_{\mathbf{g}}$ *de genre* $\mathbf{g}$, $\mathbf{g} = 0, 1$. *Alors* Y *est une variété rêve de Calabi, c.-a-d.* Y *admet une métrique kählérienne extrémale dans chaque classe de Kähler. De plus, les métriques kählériennes extrémales sont compatibles, c.-a-d. de la fome* (3).

Lorsque $\mathbf{g} = 0$, la partie existence du corollaire 0.2.2 a été démontrée dans [55]. Dans ce cas, nous obtenons de plus que les métriques kählériennes extrémales sont compatibles, i.e. de la forme (3).

Dans un travail en collaboration avec Delcroix [35] accepté pour publication dans les Annales Henry Lebesgue, nous avons obtenus des conditions suffisantes pour qu'un polytope compact convexe simple P_X dans $\mathbb{R}^\ell$ soit uniformément pondérément K-stable au sens de Lahdili [53] pour des poids arbitraires, voir le théorème 2.1.4. La preuve de ce résultat est basée sur une généralisation au cas pondéré de la méthode de Zhou–Zhu [81]. Nous nous sommes ensuite concentré sur le cas des poids correspondants à un fibré principal torique semisimple Y. En vertu du theorème B ci-dessus, nous obtenons l'existence d'une métrique kählérienne extrémale sur Y dès que cette condition est satisfaite. Lorsqu'on suppose que la fibre X est une variété torique de Fano, la condition s'exprime de la manière suivante

Théorème C. *Soit* (Y, J_Y) *un fibré principal torique semisimple dont la fibre* (X, J_X) *est une variété torique de Fano. On considère une classe de Kähler* $[\omega_Y]$ *définie à partir de la classe* $[\omega_X] := t2\pi c_1(X)$ *sur* (X, J_X) *via* (3). *On suppose que pour tout* a, $2\dim(B_a)c_a \geq ts_a$ *(voir* (3) *au-dessus) et que à chaque sommet* x *du polytope* P_X *de* $(X, [\omega_X])$,

$$2(\dim(Y)+1) + \sum_{a=1}^{k} \frac{ts_a - 2\dim(B_a)c_a}{\langle p_a, x\rangle + c_a} - tl_{\mathrm{ext}}(x) \geq 0, \tag{4}$$

où $l_{\mathrm{ext}} \in \mathrm{Aff}(\mathfrak{t}^*)$ *est la fonction affine extremale associée à* $[\omega_Y]$. *Alors il existe une métrique kählérienne extrémale dans* $[\omega_Y]$.

Dans l'énoncé ci-dessus, s_a désigne la courbure scalaire (constante) de la métrique ω_a sur B_a où $B := \prod_{a=1}^k B_a$ est la base de la fibration. Pour une variété kählérienne compacte Y, la fonction affine extrémale est définie sur le dual de l'algèbre de Lie d'un tore maximal du groupe d'automorphismes réduit $\mathrm{Aut}_{\mathrm{red}}(Y)$. Pour des fibrés principaux semisimples toriques Y, le tore $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(Y)$ n'est pas maximal en général dû à la contribution de la base (B, J_B). En revanche, pour ces fibrations, il est montré que la fonction affine extrémale d'une classe compatible est une fonction affine sur le dual de l'algébre de Lie $\mathfrak{t}$ de $\mathbb{T}$, voir [7, section 3.4] ou la section 1.2.3. Un des aspects notables du théorème C est que la condition (4) doit être vérifiée sur un nombre fini de points, les sommets du polytope moment.

Supposons de plus que l'espace total Y est Fano et que la base B est un produit de variétés Kähler–Einstein à cscK positive $\prod_{a=1}^{k}(B_a, J_a, \omega_a)$. Il s'ensuit que X est également de Fano et que $c_1(Y)$ est une classe de Kähler compatible, voir l'annexe C. Ainsi la condition suffisante s'écrit simplement :

Corollaire 0.2.3 (du théorème C)**.** *Soit Y un fibré principal semisimple de Fano au dessus d'un produit de variétés Kähler–Einstein $\prod_{a=1}^{k}(B_a, J_a, \omega_a)$ à cscK positive. Alors Y admet une métrique kählérienne extrémale dans la première classe de Chern $2\pi c_1(Y)$ si*

$$l_{\mathrm{ext}} \leq 2\left(\dim(Y) + 1\right),$$

où $l_{\mathrm{ext}} \in \mathrm{Aff}(\mathfrak{t}^*)$ est la fonction affine-extremale associée.

Nous appliquons ensuite le théorème C pour démontrer l'existence de nouvelles métriques kählériennes extrémales. Pour cela, nous avons écrit un programme Python (voir l'annexe B) permettant de vérifier (4) dans le cas où la base ne possède qu'un facteur et que la fibre est $\mathbb{P}^1$ ou $\mathbb{P}^2$. Nous en déduisons

Proposition 0.2.4. *Soit B une variété Kähler–Einstein dont le fibré anticanonique vérifie $-K_B = L^{\mathrm{Ind}(B)}$ pour un certain fibré holomorphe L, où $\mathrm{Ind}(B)$ est l'indice de Fano de B. Soit $\pi : \mathbb{P}(E) \to B$ la projectivisation de $E := \mathbb{C} \oplus L^{-p_1} \oplus L^{-p_2} \to B$, où $1 \leq p_1 \leq p_2$ sont entiers. Alors, il existe une métrique kählérienne extrémale dans la classe de Kähler $c_1(O_E(3)) + \lambda\pi^* c_1(L)$, pour tout $\lambda \geq 7p_2$.*

Nous nous sommes ensuite interessés à l'existence de v-solitons. On considère une variété complexe X de Fano muni de l'action hamiltonienne d'un tore $\mathbb{T}$ et on fixe le polytope moment P_X canoniquement normalisé par rapport à $(2\pi c_1(X), \mathbb{T})$, voir (3.31). Par définition (voir [18, 62]), un v-soliton est une métrique kählérienne $\omega \in 2\pi c_1(X)$ telle que

$$Ric(\omega) - \omega = \frac{1}{2} dd^c \log(\mathrm{v}(m_\omega)),$$

où $Ric(\omega)$ est la forme de Ricci de ω, v est une fonction lisse strictement positive sur P_X et m_ω est l'application moment canoniquement normalisée de l'action de $\mathbb{T}$ sur (X, ω). Il est connu [9] qu'un v-soliton sur un fibré principal semisimple torique Y correspond à une métrique à courbure scalaire constante pondérée sur la fibre X pour des poids bien choisis. Pour un poids v correspondant à un v-soliton sur Y, on montre que notre condition suffisante est satisfaite dès que l'invariant de Futaki pondéré $\mathcal{F}_\mathrm{v}$ sur X s'annule. Nous obtenons ainsi

Proposition 0.2.5. *Soit* $(Y, J_Y, \omega_Y, \mathbb{T})$ *un fibré principal torique semisimple de Fano comme dans le corollaire 0.2.3 dont la fibre torique est* $(X, J_X, \omega_X, \mathbb{T})$ *avec polytope de Delzant* P_X*. Soit* $\mathcal{F}_\mathrm{v} : \mathrm{Aff}(P_X) \to \mathbb{R}$ *l'invariant de Futaki pondéré pour les poids correspondant à un* v*-soliton sur* Y*. Si* $\mathcal{F}_\mathrm{v}$ *s'annule, alors* P_X *est pondérément uniformément K-stable.*

Dans l'énoné ci-dessus, $\mathrm{Aff}(P_X)$ désigne l'ensemble des fonctions afffines sur P_X. De plus, dans la proposition 1.6.9, nous démontrons que la stabilité uniforme pondérée de P_X implique une certaine notion de propreté de l'énergie de Mabuchi pondérée $\mathcal{M}_\mathrm{v}$ correspondante. Nous en déduisons

Corollary 0.2.6. *Soit* $(Y, J_Y, \omega_Y, \mathbb{T})$ *comme dans la proposition 0.2.5. Si* $\mathcal{F}_\mathrm{v}$ *s'annule, alors l'energie de Mabuchi pondérée* $\mathcal{M}_\mathrm{v}$ *de la fibre est propre, c.-a-d. satisfait* (1.64).

Il est démontré dans [9], en utilisant des arguments de K-stabilité de Han–Li [62], qu'un fibré principal semisimple torique de Fano Y admet un v-soliton si l'invariant de Futaki pondérée de la fibre est nul. En combinant [62, Theorem 3.5] avec le corollaire 0.2.6 nous obtenons une nouvelle preuve complétement analytique de ce résultat.

Finalement, dans un travail en collaboration avec Apostolov et Lahdili [9] accepté pour publication dans Géometry & Tppology, nous avons introduit une généralisation des fibrés principaux toriques semisimples où la fibre X n'est pas nécessairement torique. Nous supposons à la place que $\mathbb{T}$ s'injecte dans le groupe réduit d'automorphismes $\mathrm{Aut}_{\mathrm{red}}(X)$ de X. On peut ainsi définir un fibré lisse Y, appelé *fibré principal semisimple*, comme dans (2). Similairement au cas où les fibres sont toriques, on peut définir une structure complexe J_Y et la notion de métriques compatibles (3) s'étend à ce contexte. Nous référons à la section 3.1 pour plus de détails. On démontre ainsi

Théorème D. *Soit* $(Y, J_Y, \omega_Y, \mathbb{T})$ *un fibré principal semisimple dont la fibre est une variété kählérienne compacte* $(X, J_X, \omega_X, \mathbb{T})$ *telle que* $\mathbb{T}$ *est un tore maximal dans* $\mathrm{Aut}_{\mathrm{red}}(X)$. *Alors les énoncés suivants sont équivalents.*

1. *Il existe une métrique kählérienne extrémale dans* $[\omega_Y]$.

2. *Il existe une métrique kählérienne extrémale* compatible *dans* $[\omega_Y]$.

3. *Il existe une métrique à courbure scalaire constante pondérée dans* $[\omega_X]$.

4. *La fonctionnelle de Mabuchi pondérée de la fibre est propre relativement à* $\mathbb{T}^{\mathbb{C}}$.

La notion de propreté dans l'affirmation 4 est introduite dans la définition 3.2.1 et s'exprime en termes de la distance d_1 de Darvas (voir (1.20)) relativement aux orbites de l'action du tore complexifié $\mathbb{T}^{\mathbb{C}}$ sur l'espace des potentiels kählériens normalisés de la fibre (X, ω_X).

Dans l'énoncé ci dessus, l'équivalence (2) $\Leftrightarrow$ (3) découle des lemmes 3.15 et 3.1.10 et consiste à montrer, comme dans le cas où X est torique, que la courbure scalaire d'une métrique compatible s'exprime en termes de la courbure scalaire pondérée de la fibre X et que le champ de vecteurs extrémal est tangent aux fibres. Les preuves des implications (1) $\Rightarrow$ (3) et (4) $\Rightarrow$ (3) sont similaires et reposent sur les résultats de Chen–Cheng et He [25, 26, 47], mais nécessitent une modification par rapport au cas où X est torique. Pour la fermeture du chemin de continuité, nous ne pouvons plus utiliser la caractérisation des fibrés principaux semisimples par le fait que l'action du tore $\mathbb{T}$ est rigide et semisimple, voir [4, 5]. Nous utilisons à la place que toute métrique kählérienne compatible est invariante par l'action d'un tore maximal $\mathbb{K}_Y$ dans le groupe d'automorphismes réduit $\mathrm{Aut}_{\mathrm{red}}(Y)$ contenant $\mathbb{T}$, voir le lemme 3.1.10. Il s'ensuit des propriétés de $\mathbb{K}_Y$ (voir le lemme 3.1.10) que tout élément γ de $\mathbb{K}_Y$ envoie une fibre X_b au-dessus de $b \in B$ sur une fibre $X_{\gamma(b)}$ au-dessus d'un point $\gamma(b) \in B$. Nous pouvons ensuite conclure en utilisant que la fonction affine extrémale est une fonction sur le dual de l'algèbre de Lie $\mathfrak{t}$ de $\mathbb{T}$, et la convergence lisse d'une suite de métriques compatibles (obtenue par la fermeture du chemin de continuité) vers une métrique kählérienne extrémale (par [25, 26, 47]).

Dans le cadre du théorème D, la preuve que l'existence d'une métrique à courbure scalaire constante pondérée implique une certaine notion de stabilité pondérée a été obtenue dans [9] pour des poids généraux au sens de [53] (la notion de stabilité pondérée est une généralisation de celle définie par [33, 48, 61, 73], voir [9] pour la définition précise). Soulignons que la réciproque, à savoir qu'une certaine notion de stabilité pondérée implique l'existence d'une métrique à courbure scalaire constante pondérée, a été obtenue dans le théorème B dans le cas où

X est torique. Dans ce cas, la notion de stabilité pondérée s'exprime en termes du polytope de la fibre, et cette dernière implique une certaine notion de propreté de l'énergie de Mabuchi pondéree correspondante (voir la proposition 1.6.9). Dans le cadre général d'une variété kählérienne compacte, ce type de résultat n'est pas connu à ce jour, même dans le cas non pondéré.

0.3 Organisation du manuscrit

Dans le chapitre 1, nous démontrons les théorèmes A et B, la proposition 0.2.1 et le corollaire 0.2.2. La section 1.1 est une brève introduction à la notion de métriques kählériennes à courbure scalaire constante pondérée selon [53]. Dans la section 1.2, nous rappelons la construction et certains résultats clés établis dans [5, 7] concernant les fibrés principaux toriques semisimples. Dans la section 1.2.5, nous introduisons les distances pondérées, fonctionnelles pondérées et opérateurs différentiels pondérés. La section 1.4 présente une brève introduction aux résultats d'existence de métriques kählériennes extrémales [25, 26, 47]. Nous expliquons pourquoi les arguments fonctionnent identiquement lorsque la notion de propreté est relative à un tore maximal du groupe d'automorphismes réduit (et non seulement d'un sous groupe compact connexe maximal). Dans la section 1.5, le résultat principal du chapitre, le théorème A, est énoncé et démontré. Dans la section 1.6, nous rapellons certains réultats de géométrie torique kählérienne et donnons la preuve du théorème B. Nous concluons le chapitre 1 par la section 1.7, où nous démontrons le corollaire 0.2.2.

Dans le chapitre 2 nous complétons les preuves du théorème C, du corollaire 0.2.3 et des propositions 0.2.4 and 0.2.5. Dans la section 2.1, nous prouvons le théorème 2.1.4 qui donne une condition suffisante pour qu'un polytope soit uniformément pondérément K-stable pour des poids quelconques. Dans la section

2.2, nous exposons également les preuves du théorème C, du corollaire 0.2.3, de la proposition 0.2.5 et du corollaire 0.2.6. Nous présentons finalement, dans la section 2.3, plusieurs applications de la condition suffisante. Nous démontrons en particulier la proposition 0.2.4.

Le chapitre 3 est dédié à l'introduction des fibrés principaux semisimples et à la preuve du théorème D. Dans la section 3.1, nous construisons les fibrés principaux semisimples et nous généralisons les résultats clés établis dans le cas où les fibres sont toriques. Dans la section 3.2, nous prouvons le théorème D.

En annexe A, nous présentons un programme Python permettant de calculer la condition suffisante lorsque la base du fibré principal torique semisimple ne possède qu'un seul facteur et que la fibre est la droite projective $\mathbb{P}^1$ ou le plan projectif $\mathbb{P}^2$. L'annexe B est un compléement de preuve de la proposition 0.2.4. Nous ajoutons en annexe C, la preuve du lemme C.0.1. Il donne un critère pour qu'une certaine classe de fibrations toriques semisimples soient Fano. Finalement, nous ajoutant en annexe D une preuve des décompositions en espaces verticaux et horizontaux des opérateurs différentiels associés au chemin de continuité de Chen sur un fibré principal semisimple.

INTRODUCTION

0.4 Motivations

A central problem in Kähler geometry, proposed by Calabi in the 1980's [20], is to find a canonical representative, called extremal, in a given Kähler class. Extremal Kähler metrics include, in particular, the much studied constant scalar curvature Kähler metrics (cscK) and Kähler–Einstein metrics. By definition, an extremal Kähler metric is a Kähler metric such that the hamiltonian vector field generated by the scalar curvature is (real) holomorphic, i.e. its flow preserves the complex structure. Calabi shows in its semind article [20] that the Hirzebruch complexes surfaces do not admit any cscK metric but do admit extremal Kahler metrics.

Although the Calabi problem is still open in general, some particular cases have been established. The Uniformization Theorem [52, 70] implies that any Riemann surface admits a cscK metric in each Kähler class. Yau's Theorem [80] states that, when $c_1(X) = \{0\}$, each Kähler class admits a cscK metric (which is also Ricci flat). If the first Chern class of the manifold is negative, Aubin and Yau proved [13, 14, 80] that $-c_1(X)$ has a unique Kähler–Einstein metric.

Matsushima and Lichnerowicz showed [64, 67] that the group of complex automorphism $\mathrm{Aut}(X)$ of a compact complex manifold X must be reductive in order for X to admit a cscK metric. The existence of an extremal metric is also obstructed in terms of $\mathrm{Aut}(X)$. Indeed, Calabi proved [21] that a certain subgroup of $\mathrm{Aut}(X)$ must be reductive in order for X to admit an extremal metric.

This obstruction is not trivial, as observed by Levin [57] giving the first example of compact Kähler manifold that does not admit any extremal Kähler metric. On the other hand, Tian found examples [77] of Kähler manifolds without automorphism, which do not admit any Kähler–Einstein metric. His work (see e.g. [74, 75, 76, 77, 78]) contributed to a formulation of a general conjecture, called *the Yau–Tian–Donaldson conjecture* (YTD), which predicts that the existence of an extremal metric in a Kähler class α is equivalent to a certain notion of *stability* of (X, α). Even though this conjecture is still open in general, some particular cases are established. For example, the case of Fano manifolds is proved by Chen–Donaldson–Sung [27, 28, 29] (see also [17, 63, 78]). The case of toric surface was resolved by Donaldson [39] and by Chen–Cheng's [25, 26] for arbitrary dimension, building on works of many mathematicians [1, 25, 26, 22, 37, 48, 61, 81]. In the context of toric manifolds, the notion of stability is expressed in terms of the corresponding Delzant polytope [37, 48]. The Yau–Tian–Donaldson conjecture is also established for other classes of manifolds admitting a large group of symmetry, such as spherical manifolds, see [36].

In this thesis, we are interested in the Yau–Tian–Donaldson conjecture on a certain class of toric fibrations, introduced by Apostolov–Calderbank–Gauduchon–Tonessen-Friedman [5] and called *semisimple principal toric fibrations*. They are obtained from a generalization of the construction of extremal Kähler metric on $\mathbb{P}^1$-bundle of Calabi [20]. The semisimple principal toric fibrations are defined from a principal bundle Q with structure group a torus $\mathbb{T}$ and a Kähler compact toric manifold $(X, J_X, \omega_X, \mathbb{T})$. We assume that the base B of the principal $\mathbb{T}$-bundle $\pi : Q \to B$ is a product of cscK compact manifolds $(B, J_B, \omega_B) := \prod_{a=1}^{k}(B_a, J_a, \omega_a)$ and that the first Chern class of Q satisfies

$$c_1(Q) = \sum_{a=1}^{k} p_a \otimes \pi^*[\omega_a], \tag{5}$$

where the p_a are elements of the lattice $\Lambda \subset \mathfrak{t}$ of the torus $\mathbb{T}$ (where $\mathfrak{t}$ denotes the Lie algebra of $\mathbb{T}$). As a smooth manifold, the total space of the semisimple principal toric fibration is defined by

$$Y := Q \times_{\mathbb{T}} X. \tag{6}$$

Then, we endow Y with complex structure J_Y defined from a connection $\theta \in \Omega^1(Q) \otimes \mathfrak{t}$ with curvature

$$d\theta = \sum_{a=1}^{k} p_a \otimes \pi^*(\omega_a), \tag{7}$$

the lifted complex structure J_B of B and the complex structure J_X of X. Furthermore, one can show that any Kähler metric ω_X on X defines a Kähler metric ω_Y on Y such that its pullback to $X \times Q$ by the quotient map $\pi_Y : X \times Q \to Y$ is the basic 2-form

$$\pi_Y^*(\omega_Y) = \omega_X + \langle m, \theta \rangle + \sum_{a=1}^{k} \big(\langle p_a, m \rangle + c_a\big)\pi^*(\omega_a), \tag{8}$$

where m denotes the moment map of the $\mathbb{T}$-action on (X, J_X, ω_X), $\langle \cdot, \cdot \rangle$ is the natural contraction of $\mathfrak{t}^*$ and $\mathfrak{t}$, and the constant c_a defined the Kähler class $[\omega_Y]$ in such way that $\langle p_a, m \rangle + c_a > 0$. Such metrics are called *compatible Kähler metrics* and the correpsonding Kähler classes $[\omega_Y]$ are called *compatible Kähler class*. We refer to Section 1.2.1 (see also Section 3.1 for the case when X is not necessarily toric) for more details on this construction. We give there two examples of semisimple principal toric fibrations whose fiber is the projective space $\mathbb{P}^k$.

1. The projectivization

$$Y = \mathbb{P}(L_0 \oplus \cdots \oplus L_k) \to \Sigma,$$

   of the direct sum of holomorphic line bundles $L_i \to \Sigma$ over a Riemann surface Σ.

2. The projectivization

$$Y = \mathbb{P}(L_0 \oplus \cdots \oplus L_k) \to B_1 \times \cdots \times B_k,$$

   where (B_i, L_i) are polarized cscK manifolds.

Semisimple principal toric fibrations as defined above correspond to semisimple rigid toric fibrations in the sense of [5] when the basis is a global product of cscK Kähler manifolds and when there is no *blow-down*. In [7], the authors made the following conjecture:

Conjecture 2. *Let $(Y, J_Y, \omega_Y, \mathbb{T})$ be a semisimple rigid toric fibration and P its associated Delzant polytope. Suppose $[\omega_Y]$ is a compatible Kähler class. Then the following statements are equivalent:*

1. *$[\omega_Y]$ admits an extremal Kähler metric.*
2. *$[\omega_Y]$ admits a compatible extremal Kähler metric.*
3. *The Delzant polytope P satisfies a certain notion of weighted stability.*

In the third assertion, the notion of *weighted stability* is a weighted version of the notion of K-stability introduced in [37]. In the presence of blow-down, the notion of compatble Kähler metric needs to be refined compared to the one introduced above, see [5, 7].

0.4.1 Main results

We obtain the resolution of Conjecture 2 for semisimple principal toric fibrations in an article published in Annales de l'Institut Fourier, see [50].

Theorem 0.4.1. *For Y a semisimple principal toric fibration, Conjecture 2 is true.*

In the above statement, in order to define *uniform weighted K-stability* (see Definition 1.6.8), we use normalized continuous convex functions which are smooth in the interior of P and the usual L^1-norm of P. By C^0 density and continuity, this is equivalent to the uniform (weighted) K-stability of P, defined in terms of normalized convex piece-wise linear functions with rational coefficients and the L^1-norm. Moreover, by [68], this stability notion is equivalent to the one use in Conjecture 2 (see [7, Conjecture 2] for the precise statement).

We split Theorem 0.4.1 in two statements: Theorems A and B below. Theorem A corresponds to the statement "(1) $\Leftrightarrow$ (2)" in Conjecture 2.

Theorem A. *Let $(Y, J_Y, \omega_Y, \mathbb{T})$ be a semisimple principal toric fibration whose fiber is the toric Kähler manifold $(X, J_X, \omega_X, \mathbb{T})$. Then, the following statements are equivalent.*

1. *There exists an extremal Kähler metric in $[\omega_Y]$.*
2. *There exists a compatible extremal Kähler metric in $[\omega_Y]$.*

3. *There exists a weighted cscK metric in* $[\omega_X]$ *for the weights defined in* (1.15) *below.*

In the third assertion, the notion of weighted cscK metric is in the sense of [53], see Section 1.1 for a precise definition. The equivalence (2) $\Leftrightarrow$ (3), established in [5], is recalled in Section 1.2. The main idea behind the proof of (1) $\Rightarrow$ (2) is to use that the existence of an extremal Kähler metric implies a certain properness condition [25, 26, 47] for the corresponding relative Mabuchi functional (see Theorem 1.4.2 below for a precise statement). Then, under the hypothesis that the relative Mabuchi is proper, we show that the continuity path of [24] can be solved in the subspace of *compatible* Kähler metrics in $[\omega_Y]$.

Theorem B below corresponds to the statement "(1) $\Leftrightarrow$ (3)" in Conjecture 2 and provides a *criterion* for verifying the equivalent conditions of Theorem A, expressed in terms of the Delzant polytope P of the fiber and data depending of the topology of Y and the compatible Kähler class.

Theorem B. *Let* $(Y, J_Y, \omega_Y, \mathbb{T})$ *be a semisimple principal toric fibration with fiber* $(X, J_X, \omega_X, \mathbb{T})$ *and denote by* P_X *its associated Delzant polytope. Then, there exists a weighted cscK metric in* $[\omega_X]$ *if and only if* P_X *is weighted uniformly* K*-stable, for the weights defined in* (1.15)*. In particular, the latter condition is necessary and sufficient for* $[\omega_Y]$ *to admit an extremal Kähler metric.*

The strategy of the proof consists in considering the compatible extremal Kähler metrics on the total space $(Y, J_Y, \mathbb{T})$ as weighted cscK metrics on the corresponding toric fiber $(X, J_X, \mathbb{T})$ via Theorem A. We then use the Abreu–Guillemin formalism and a weighted adaptation of the results in Chen–Li–Sheng [22], Donaldson [37] and Zhou–Zhu [81] to establish the equivalence on $(X, J_X, \omega_X, \mathbb{T})$. In one direction, namely showing that the existence of an compatible extremal Kähler metric implies that the polytope P_X is weighted uniformly K-stable, the argument follows from a straightforward modifcation of the result of Chen–Li–Sheng [22], which was proved by Li–Lian–Sheng [60]. To show the other direction, we build on arguments of Donaldson [37] and Zhou–Zhu [81] to obtain in Proposition 1.6.9 that the uniform weighted K-stability of the polytope implies a certain notion of properness of the weighted Mabuchi energy on X. We then show that the later implies the properness of the Mabuchi energy of Y, and we finally conclude by involing again [25, 26, 47].

Finally, we are interested in a certain class of almost Kähler metrics (i.e. metrics compatible with a symplectic form whose almost complex structures are not necessarily integrable) on a toric manifold $(X, \omega_X, \mathbb{T})$. They are, by definition, $\mathbb{T}$-equivariant almost Kähler metrics such that the orthogonal distribution to the $\mathbb{T}$-orbits is involutive (see [56]) and we will refer to such metrics as *involutive toric almost Kähler metrics.* The idea of studying such metrics comes from Donaldson [37] (see [5] for the weighted case), where it is conjectured that the existence of a toric involutive constant scalar curvature almost Kähler metric is equivalent to the existence of a cscK metric.

Proposition 0.4.2. *Let $(X, \omega_X, \mathbb{T})$ be a toric manifold associated to a Delzant polytope P_X. Then, for the weights defined in* (1.15)*, the following statements are equivalent.*

1. *There exists a toric weighted cscK metric on $(X, \omega_X, \mathbb{T})$.*
2. *There exists an involutive toric weighted csc almost Kähler metric on $(X, \omega_X, \mathbb{T})$.*
3. *P is weighted uniformly K-stable.*

As an application of the above result, we study the existence of extremal

Kähler metrics on the projectivisation $\mathbb{P}(L_0 \oplus L_1 \oplus L_2)$ of a direct sum of line bundles $L_i \to \Sigma_{\mathbf{g}}$ over a compact complex curve $\Sigma_{\mathbf{g}}$ of genus $\mathbf{g} = 0, 1$. In [7, Proposition 4], the authors established the existence of involutive toric weighted constant scalar curvature almost Kähler metrics on $\mathbb{P}(L_0 \oplus L_1 \oplus L_2)$. Combining with Proposition 0.4.2, we deduce the following:

Corollary 0.4.3. *Let $Y = \mathbb{P}(L_0 \oplus L_1 \oplus L_2) \longrightarrow \Sigma_{\mathbf{g}}$ be a projective $\mathbb{P}^2$-bundle over a complex curve $\Sigma_{\mathbf{g}}$ of genus $\mathbf{g}$, $\mathbf{g} = 0, 1$. Then Y is a Calabi dream manifold, i.e. Y admits an extremal Kähler metric in each Kähler class. Furthermore, the extremal Kähler metrics are compatibles.*

When $\mathbf{g} = 0$, the existence part of Corollary 0.2.2 was already obtained in [55]. We prove in addition that these extremal metrics are compatibles, i.e. satsisfy (8).

In a work in collaboration with Delcroix [35] accepted in Annales Henry Lebesgue, we obtained a sufficient condition for the weighted uniform K-stability of a simple convex compact polytope in the sense of Lahdili [53] to hold for arbitrary weights, see Theorem 2.1.4. The proof of this result is based on a generalization to the weighted case of a method of Zhou–Zhu [81]. Next, we considered the weights corresponding to a semisimple principal fibration. By virtue of Theorem B above, we obtain the existence of an extremal Kähler metric on the total space as soon as our criterion is satisfied. When the fiber X is a Fano manifold, the condition is expressed as follows

Theorem C. *Let (Y, J_Y) be a semisimple principal toric fibration whose fiber is a toric Fano manifold (X, J_X). We consider a compatible Kähler class $[\omega_Y]$ on Y defined from a Kähler class $[\omega_X] = t2\pi c_1(X)$ on X via (8). We suppose that for all a, $2\dim(B_a)c_a \geq ts_a$ (see (8) above) and that at each vertex x of the polytope P_X of X*

$$2(\dim(Y)+1)+\sum_{a=1}^{k}\frac{ts_a-2\dim(B_a)c_a}{\langle p_a,x\rangle+c_a}-tl_{\mathrm{ext}}(x)\geq 0, \tag{9}$$

where $l_{\mathrm{ext}} \in \mathrm{Aff}(\mathfrak{t}^*)$ *is the affine extremal function of* $[\omega_Y]$*. Then there exists an extremal Kähler metric in* $[\omega_Y]$*.*

The constants s_a in the preceding statement are the constant scalar curvatures of the Kähler metrics ω_a. In the general context of a compact Kähler manifold, the affine extremal function is an affine function on the dual Lie algebra of a maximal torus in the reduced automorphism group. Because of the contribution of the base (B, J_B), for a semisimple principal toric fibration, the torus $\mathbb{T}$ is not, in general, maximal in the reduced automorphism group. However, it is shown that the affine extremal function of a compatible Kähler class is in fact affine-linear on the dual Lie algebra $\mathfrak{t}^*$ of $\mathbb{T}$, see [7, Section 3.4] or Section 1.2.3. One of the important aspects of Theorem C is that the condition (9) needs to be checked only at a finite number of points, the vertices of P_X.

Furthermore, suppose that the total space Y is Fano and that the base B is a product of positive Kähler–Einstein manifolds $\prod_{a=1}^{k}(B_a, J_a, \omega_a)$. Then X is also Fano and $c_1(Y)$ is a compatibe Kähler class, see Appendix C. Then the condition (9) is simplies

Corollary 0.4.4 (of Theorem C)**.** *Let* Y *be Fano semisimple principal toric fibration* Y *over a product of positive Kähler–Einstein manifolds* $\prod_{a=1}^{k}(B_a, J_a, \omega_a)$*. Then* Y *admits an extremal Kähler metric in its first Chern class* $c_1(Y)$ *if its extremal function* l_{ext} *satisfies:*

$$\sup l_{\mathrm{ext}} \leq 2(\dim(Y)+1).$$

We apply Theorem C to show the existence of new extremal Kähler metrics. For that, we wrote a Python programme (see appendix B) allowing us to check (9) in the case when the base has only one factor and the fibers are $\mathbb{P}^1$ of $\mathbb{P}^2$. We deduce

Proposition 0.4.5. *Let B be a Kähler-Einstein Fano threefold whose anticanonical line bundle satisfies $-K_B = L^{\mathrm{Ind}(B)}$ for some holomorphic line bundle L, where $\mathrm{Ind}(B)$ is the Fano index of B. Let $\pi : \mathbb{P}(E) \to B$ the projectivisation of $E := \mathbb{C} \oplus L^{-p_1} \oplus L^{-p_2} \to B$, where $1 \leq p_1 \leq p_2$ are integers. Then there exists an extremal metric in the Kähler class $c_1(O_E(3)) + \lambda\pi^* c_1(L)$, for every $\lambda \geq 7p_2$.*

Next, we are interested in the existence of v-solitons. Consider a compact complex Fano manifold X endowed with a hamiltonian action of a torus $\mathbb{T}$ and we fix the canonically normalized moment polytope P_X associated to $(2\pi c_1(X), \mathbb{T})$, see (3.31). A v-soliton is a Kähler metric $\omega \in c_1(X)$ such that

$$Ric(\omega) - \omega = \frac{1}{2} dd^c \log(\mathrm{v}(m_\omega)), \tag{10}$$

where $Ric(\omega)$ is the Ricci form of ω, v is a positive smooth function on P_X and m_ω is the moment map of ω. It is known [9] that a v-soliton on a semisimple principal fibration Y corresponds to a weighted cscK metric on the fiber X for suitable weight functions. For a weight v corresponding to v-soliton on Y, we show that our sufficient condition is strictly satisfied as soon as the weighted Futaki invariant of the fiber $\mathcal{F}_{\mathrm{v}}$ vanishes. We get

Proposition 0.4.6. *Let $(Y, J_Y, \omega_Y, \mathbb{T})$ be a semisimple principal toric fibration as in Corollary 0.4.4 whose fiber is $(X, J_X, \omega_X, \mathbb{T})$ with associated Delzant polytope P_X. We let $\mathcal{F}_{\mathrm{v}} : \mathrm{Aff}(P_X) \to \mathbb{R}$ be the weighted Futaki invariant corresponding to a* v*-soliton problem on Y. If $\mathcal{F}_{\mathrm{v}}$ vanishes, then P_X is weighted uniformly K-stable.*

In the above statement, $\mathrm{Aff}(P_X)$ denote the space of affine functions on P_X. Moreover, we show that the weighted uniform K-stability of P_X implies a certain properness condition of the corresponding weighted Mabuchi energy $\mathcal{M}_{\mathrm{v}}$ of the fiber X, see Proposition 1.6.9.

Corollary 0.4.7. *Let* $(Y, J_Y, \omega_Y, \mathbb{T})$ *be as in Proposition 0.4.6. If the weighted Futaki invariant* $\mathcal{F}_{\mathrm{v}}$ *vanishes, then the weighted Mabuchi energy is proper, i.e. satisfies* (1.64).

It is shown in [9], using the K-stability arguments of Han–Li [62], that a Fano semisimple principal toric fibration Y admits a v-soliton if the weighted Futaki invariant of the fiber, for weights corresponding to v-soliton on the total space, vanishes. Combining [62, Theorem 3.5] with Proposition 0.4.6, we obtain an analytic proof of this result.

Finlay, in a work in collaboration with Apostolov and Lahdili [9] accepted in Geometry & Topology, we introduce a generalization of semisimple principal toric fibrations where the fiber X is not necessarily toric. Instead, we assume that $\mathbb{T}$ is embedded in the reduced automorphism group $\mathrm{Aut}_{\mathrm{red}}(X)$ of X. We can then define a smooth fibration Y as in (6) with fiber X, called *semisimple principal fibration*. Similarly to the case with toric fibers, we define a complex structure J_Y on Y and a compatible Kähler metrics via (8). We then show

Theorem D. *Let* $(Y, J_Y, \omega_Y, \mathbb{T})$ *be a semisimple principal fibration with fiber* $(X, J_X, \omega_X, \mathbb{T})$ *such that* $\mathbb{T}$ *is maximal in* $\mathrm{Aut}_{\mathrm{red}}(X)$*. Then, the following conditions are equivalent.*

1. *There exists an extremal Kähler metric in* $[\omega_Y]$*.*

2. *There exists a compatible extremal Kähler metric in* $[\omega_Y]$*.*

3. *There exists a weighted cscK metric in* $[\omega_X]$.

4. *The weighted Mabuchi energy* $\mathcal{M}_{\mathrm{v,w}}$ *of the fiber is* $\mathbb{T}^{\mathbb{C}}$*-relatively proper.*

In the above statement the weights are defined by the topological data of the fibration Y. The properness in the fourth statement is introduced in Definition 3.2.1 and is expressed in terms of Darvas d_1-distance (see (1.20)) relative to the orbits of the complexified torus $\mathbb{T}^{\mathbb{C}}$ on the space of normalized Kähler potentials of the fiber $(X, [\omega_X])$.

The equivalence (2) $\Leftrightarrow$ (3) in Theorem D follows from Lemmas 3.15 and 3.1.10 and consists of proving, as in the case when X is toric, that the scalar curvature of a compatible Kähler metric on Y is expressed in terms of the weighted scalar curvature of the fiber X, and that the extremal vector field is tangent to the fibers. The proofs of the statements (1) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) of Theorem D are similar to the ones give in the case of toric fibrations and rely on the results of Cheng–Cheng and He [25, 26, 47], but the convergence arguments need to be modified. More precisely, we cannot use the characterization of semisimple principal fibrations as Kähler manifolds endowed with a rigid and semisimple action of a torus $\mathbb{T}$ (see [4, 5]). We use instead that any compatible Kähler metric is invariant under the action of a maximal torus $\mathbb{K}_Y$ in the reduced automorphism group $\mathrm{Aut}_{\mathrm{red}}(Y)$ containing $\mathbb{T}$, see Lemma 3.1.10. Using the properties of $\mathbb{K}_Y$ (see Lemma 3.1.10), any element γ of $\mathbb{K}_Y$ sends a fiber X_b over $b \in B$ to a fiber $X_{\gamma(b)}$ over a point $\gamma(b) \in B$. We can then conclude that X_b admits a weighted cscK metric by using that the affine extremal function is a function on the dual of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ and the smooth convergence of a sequence of compatible Kähler metrics to an extremal metric (see [25, 26, 47]).

In the context of Theorem D, the proof that the existence of a weighted cscK metric implies a weighted K-stability condition is obtained in [9] for arbitrary weights (the notion of weighted stability is a generalization of the one defined by [33, 48, 61, 73], see [9] for a precise definition). We would like to emphasise that when X is toric, Theorem B gives the converse, namely that the relevant notion of weighted K-stability implies the properness of the corresponding weighted Mabuchi energy (see Proposition 1.6.9). In the general case of Kähler manifold, similar result is not known as yet, even in the unweighted case.

0.5 Outline of the manuscript

In Chapter 1, we establish Theorems A and B, Proposition 0.4.2 and Corollary 0.4.3. Section 1.1 is a brief summary of the notion of weighted (v, w)-scalar curvature introduced by Lahdili [53]. In Section 1.2, we recall the construction and key results of semisimple principal toric fibrations established in [5, 7]. In Section 1.3, we introduce weighted distances, weighted functionals, and weighted differential operators. Section 1.4 gives an account on the existence results for extremal Kähler metrics obtained by Chen–Cheng and He [25, 26, 47]. We explain why their argument works equally when the properness is relative to a maximal torus of the reduced group of automorphism (rather than a connected maximal compact subgroup). In Section 1.5, our main result, Theorem A, is stated and proved. In Section 7, after introducing some fundamental notions of toric Kähler geometry, we prove Theorem B. In Section 1.7, we show Corollary 0.4.3.

In Chapter 2 we give the proofs of Theorem C, Corollary 0.4.4, and of Propositions 0.4.5 and 0.4.6. In Section 2.1, we establish Theorem 2.1.4, which give a sufficient condition for the weighted uniform K-stability of a labelled polytope to hold. In Section 2.2, we give the proofs of Theorem C, of Proposition 0.4.6 and of Corollaries 0.4.4 and 0.4.7. Finally, we present in Section 2.3 several applications of Theorem C. In particular, we establish Proposition 0.4.5.

Chapter 3 is dedicated to the introduction of semisimple principal fibrations and to the proof of Theorem D. In Section 3.1, we introduce the semisimple principal fibrations and generalize the main results established when the fiber X is toric. In Section 3.2, we prove Theorem D.

We include in Appendix A a Python program allowing us to test the sufficient condition when the base B has only one factor and the fiber is $\mathbb{P}^1$ or $\mathbb{P}^2$. In Appendix B, we give a complement of proof of Proposition 0.4.5. In Appendix C, we show Lemma C.0.1 which gives a criterion for a certain class of semimple principal toric fibrations to be Fano. Finally, in Appendix D we show the decomposition with respect to the vertical and horizontal distributions of the operators associated to the continuity path of Chen for a semisimple principal fibration.

CHAPTER I

A YAU–TIAN–DONALDSON CORRESPONDENCE ON SEMISIMPLE PRINCIPALE TORIC FIBRATIONS (BASED ON [50])

1.1 Lahdili's v-scalar curvature

In this section, we review briefly the notion of *weighted* v*-scalar curvature* introduced by Lahdili in [53]. Consider a smooth compact Kähler manifold (X, J, ω). We denote by $\mathrm{Aut}_{\mathrm{red}}(X)$ the reduced group of automorphisms whose Lie algebra $\mathfrak{h}_{\mathrm{red}}$ is given by the ideal of real holomorphic vector fields with zeros, see [42]. Let $\mathbb{T}$ be an ℓ-dimensional real torus in $\mathrm{Aut}_{\mathrm{red}}(X)$ with Lie algebra $\mathfrak{t}$. Suppose ω_0 is a $\mathbb{T}$-invariant Kähler form and consider the set of smooth $\mathbb{T}$-invariant Kähler potentials $\mathcal{K}(X,\omega_0)^{\mathbb{T}}$ relative to ω_0. For $\varphi \in \mathcal{K}(X,\omega_0)^{\mathbb{T}}$ we denote by $\omega_\varphi = \omega_0 + dd^c\varphi$ the corresponding Kähler metric. It is well known that the $\mathbb{T}$-action on X is ω_φ-Hamiltonian (see [42]) and we let $m_\varphi : X \longrightarrow \mathfrak{t}^*$ denote a ω_φ-moment map of $\mathbb{T}$. It is also known [12, 45, 53] that $P_\varphi := m_\varphi(X)$ is a convex polytope in $\mathfrak{t}^*$ and we can normalize m_φ by

$$m_\varphi = m_0 + d^c\varphi, \tag{1.1}$$

in such a way that $P = P_\varphi$ is φ-independent, see [53, Lemma 1].

Definition 1.1.1. *For* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ *we define the (weighted)* v*-scalar curvature* *of the Kähler metric* ω_φ*,* $\varphi \in \mathcal{K}(X,\omega_0)^{\mathbb{T}}$*, to be*

$$Scal_{\mathrm{v}}(\omega_\varphi) := \mathrm{v}(m_\varphi)Scal(\omega_\varphi) + 2\Delta_{\omega_\varphi}\big(\mathrm{v}(m_\varphi)\big) + \mathrm{Tr}\big(G_\varphi \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\varphi)\big),$$

where Δ_{ω_φ} *is the Riemannian Laplacian associated to* $g_\varphi := \omega_\varphi(\cdot, J\cdot)$*,* $\mathrm{Hess}(\mathrm{v})$ *is the Hessian of* v *viewed as bilinear form on* $\mathfrak{t}^*$ *whereas* G_φ *is the bilinear form with smooth coefficients on* $\mathfrak{t}$*, given by the restriction of the Riemannian metric* g_φ *on fundamental vector fields and* $Scal(\omega_\varphi)$ *is the scalar curvature of* (X, J, ω_φ)*.*

In a basis $\boldsymbol{\xi} = (\xi_i)_{i=1,\cdots,\ell}$ of $\mathfrak{t}$ we have

$$\mathrm{Tr}\big(G_\varphi \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\varphi)\big) = \sum_{1\le i,j\le \ell} \mathrm{v}_{,ij}(m_\varphi) g_\varphi(\xi_i, \xi_j)$$

where $\mathrm{v}_{,ij}$ stands for the partial derivatives of v in the dual basis of $\boldsymbol{\xi}$.

Definition 1.1.2. *Let* (X, J, ω_0) *be a compact Kähler manifold,* $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(X)$ *a real torus with normalized moment image* $P \subset \mathfrak{t}^*$ *associated to* $[\omega_0]$*, and* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$*,* $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$*. A* (v,w)*-cscK metric is a* $\mathbb{T}$*-invariant Kähler metric satisfying*

$$Scal_{\mathrm{v}}(\omega_\varphi) = \mathrm{w}(m_\varphi). \tag{1.2}$$

The motivation for studying (1.2) is that many natural geometric problems in Kähler geometry correspond to (1.2) for suitable choices of v and w. For example, if $\mathbb{T}$ is a maximal torus in $\mathrm{Aut}_{\mathrm{red}}(X)$, $\mathrm{v} \equiv 1$ and l is an affine function on $\mathfrak{t}^*$, the $(1,l)$-cscK metrics are extremal Kähler metrics in the sense of Calabi. Another example, which will be of main interest to this thesis, is the existence theory of extremal Kähler metrics on a class of fibrations, which can be reduced to the study of (v, w)-cscK on the fiber for suitable choices of v and w. Weighted Kähler metrics have been extensively studied and related to a notion of (v, w)-weighted K-stability, see for example [8, 11, 49, 53].

1.2 A class of toric fibrations

1.2.1 Semisimple principal toric fibrations

Let $\mathbb{T}$ be an ℓ-dimensional torus. We denote by $\mathfrak{t}$ its Lie algebra and by $\Lambda \subset \mathfrak{t}$ the lattice of generators of circle subgroups, so that $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$. Consider $\pi_B : Q \longrightarrow (B, J_B)$ a principal $\mathbb{T}$-bundle over a $2n$-dimensional product of cscK Hodge manifolds $(B, J_B, \omega_B) = \prod_{a=1}^{k}(B_a, J_a, \omega_a)$. Let $\theta \in \Omega^1(Q) \otimes \mathfrak{t}$ be a connection 1-form on Q with curvature

$$d\theta = \sum_{a=1}^{k} \pi_B^* \omega_a \otimes p_a \quad p_a \in \Lambda \subset \mathfrak{t}. \tag{1.3}$$

The connection 1-form θ gives rise to a *horizontal* distribution $\mathcal{H} := ann(\theta)$ and the tangent space splits as

$$TQ = \mathcal{H} \oplus \mathfrak{t},$$

where, by definition, $\mathcal{H}_b \overset{(\pi_B)_*}{\cong} T_bB$. The complex structure J_B acts on vector fields in $\mathcal{H}$ via the unique horizontal lift from TB.

Let $(X, J_X, \omega, \mathbb{T})$ be a 2ℓ-dimensional compact toric Kähler manifold with associated compact Delzant polytope P [36]. We will consider various actions of $\mathbb{T}$ in this paper. In order to avoid confusion, we specify the space on which $\mathbb{T}$ acts as a subscript, e.g. $\mathbb{T}_Q$ acts on Q. The interior P^0 is the set of regular values of the moment map $m_\omega : X \longrightarrow P \subset \mathfrak{t}^*$ of $(X, \omega, \mathbb{T})$ and $X^0 := m_\omega^{-1}(P^0)$ is the open dense subset of points with regular $\mathbb{T}_X$-orbits. Introducing angular coordinates $t : X^0 \rightarrow \mathfrak{t}/2\pi\Lambda$ with respect to the Kähler structure (J_X, ω) (see e.g. [2]), we identify

$$X^0 \cong \mathbb{T} \times P^0 \quad \text{and} \quad T_x X^0 \cong \mathfrak{t} \oplus \mathfrak{t}^*. \tag{1.4}$$

for all $x \in X^0$. Notice that the first diffemorphism is $\mathbb{T}$-equivariant.

We consider the $2m = 2(\ell + n)$ dimensional smooth manifold

$$Y^0 := Q \times_{\mathbb{T}} X^0,$$

where the $\mathbb{T}_{Q \times X^0}$-action is given by

$$\gamma \cdot (q, x) = (\gamma \cdot q, \gamma^{-1} \cdot x), \; q \in Q, x \in X^0, \text{ and } \gamma \in \mathbb{T}.$$

Using (1.4) we identify

$$Y^0 \cong Q \times P^0. \tag{1.5}$$

We will still denote by $\pi_B : Y^0 \longrightarrow B$ the projection. At the level of tangent space we get

$$TY^0 = \mathcal{H} \oplus \mathcal{V}, \tag{1.6}$$

where, for all $b \in B$, $\mathcal{V}_b := \ker(\pi_B)_* \cong \mathfrak{t} \oplus \mathfrak{t}^*$ is the *vertical space*. Since X^0 compactifies as X, the smooth manifold Y^0 compactifies as a fiber bundle over B with fiber $X_b \cong X$:

$$Y := \overline{Y^0} = Q \times_{\mathbb{T}} X.$$

By construction, Y^0 is an open dense subset of Y consisting of points with regular $\mathbb{T}_Y$-orbits.

One can show that the almost complex structure $J_Y := J_B \oplus J_X$ on Y^0 is integrable and extends to Y (as J_X extends to X). In other words, Y is a compactification of the principal $(\mathbb{C}^*)^\ell$-bundle $\pi_B : (Y^0, J_Y) \longrightarrow (B, J_B)$.

1.2.2 Compatible Kähler metrics

Following [7], we introduce a family of Kähler metrics *compatible with the bundle structure*. In moment-angular coordinates (m_ω, t), the Kähler form ω of $(X, J_X, \mathbb{T})$ is writen on X^0 as

$$\omega = \langle dm_\omega \wedge dt \rangle, \tag{1.7}$$

where the angle bracket denotes the pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$. By (1.5), we can equivalently define a 1-form θ on $Y^0 = Q \times_{\mathbb{T}} X^0$ which satisfies $\theta(\xi^Y) = \xi$ and $\theta(J_Y \xi^Y) = 0$, where ξ^Y is the fundamental vector field defined by $\xi \in \mathfrak{t}$. Then, $\langle dm_\omega \wedge \theta \rangle$ is well defined on Y^0 and restricts to $\langle dm_\omega \wedge dt \rangle$ on each fibers. Thus, we get a 2-form on Y

$$\omega := \langle dm_\omega \wedge \theta \rangle. \tag{1.8}$$

We choose the real constants c_a, $1 \leq a \leq k$, such that the affine-linear functions $\langle p_a, x\rangle + c_a$ are positive on P, where, we recall the elements $p_a \in \Lambda$ are defined by (1.3). We then define the 2-form on Y^0

$$\tilde{\omega} = \sum_{a=1}^{k} \left(\langle p_a, m_\omega\rangle + c_a\right)\omega_a + \langle dm_\omega \wedge \theta\rangle, \tag{1.9}$$

which extends to a smooth Kähler form on (Y, J_Y) since ω does on (X, J_X). In the sequel, we fix the metrics ω_a, the 1-form θ and the constants c_a, noting that $p_a \in \mathfrak{t}$ are topological constants associed to Y. The Kähler manifold $(Y, J_Y, \tilde{\omega}, \mathbb{T})$ is then a fiber bundle over B obtained from the principal $\mathbb{T}$-bundle Q, with fiber the Kähler toric manifold $(X, J_X, \omega, \mathbb{T})$. Following [7], we define:

Definition 1.2.1. *The Kähler manifold $(Y, J_Y, \tilde{\omega}, \mathbb{T})$ constructed above is referred to as a semisimple principal toric fibration and the Kähler metric given explicitly on Y^0 by (1.9) is referred to as a compatible Kähler metric. The corresponding Kähler classes on (Y, J_Y) are called compatible Kähler classes and, in the above set up, are parametrized by the real constant c_a satisfyng $\langle p_a, x\rangle + c_a > 0$.*

Remark 1.2.2. *Let $(Y, J_Y, \tilde{g}, \tilde{\omega})$ be a compact Kähler $2m$-manifold endowed with an effective isometric hamiltonian action of an ℓ-torus $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(Y)$ and moment map $m : Y \to \mathfrak{t}^*$. Following [7], we say the action is rigid if for all x in Y, $\tilde{g}(\xi_1^Y, \xi_2^Y)$ depends only on $m(x)$, where ξ_i are fundamental vector fields of the $\mathbb{T}$-action on Y, $i = 1, 2$. The $\mathbb{T}$-action is said to be semisimple rigid if moreover, for any regular value x_0 of the moment map, the derivative with respect to x of the family $\omega_B(x)$ of Kähler forms induced on the complex quotient (B, J_B) of (Y, J_Y) are parallel and diagonalizable with respect to $\omega_B(x_0)$. By the results of [4, 5, 7], semisimple principal toric fibrations $(Y, J_Y, \tilde{\omega}, \mathbb{T})$ correspond to Kähler manifolds with a semisimple rigid torus action, such that the Kähler quotient B is a global product of cscK manifolds, and there are no blow-downs.*

The volume form of a compatible Kähler metric (1.9) satisfies

$$\tilde{\omega}^{[\ell+n]} = \omega_B^{[n]} \wedge \mathrm{v}(m_\omega)\omega^{[\ell]} = \bigwedge_{a=1}^{k} \omega_a^{[n_a]} \wedge \mathrm{v}(m_\omega)\omega^{[\ell]} \tag{1.10}$$

where $\mathrm{v}(m_\omega) := \prod_{a=1}^{k} \big(\langle p_a, m_\omega\rangle + c_a\big)^{n_a}$, n_a is the complex dimension of B_a and $\omega^{[i]} := \frac{\omega^i}{i!}$ for $1 \leq i \leq m$. It follows from [5] and [53, Sect. 6] that the scalar curvature of a compatible metric is given by

$$Scal(\tilde{\omega}) = \sum_{a=1}^{k} \frac{s_a}{\langle p_a, m_\omega\rangle + c_a} + \frac{1}{\mathrm{v}(m_\omega)} Scal_{\mathrm{v}}(\omega), \tag{1.11}$$

where s_a is the constant scalar curvature of (B_a, J_a, ω_a) and $Scal_{\mathrm{v}}(\omega)$ is the v-weighted scalar curvature of $(X, J_X, \omega, \mathbb{T})$, see Definition 1.1.1.

1.2.3 The extremal vector field

We now recall the definition of the extremal vector field on a general compact Kähler manifold Y. To this end, we fix a maximal compact torus $\mathbb{K} \subset \mathrm{Aut}_{\mathrm{red}}(Y)$ and a Kähler class $[\tilde{\omega}_0]$. Given any $\mathbb{K}$-invariant Kähler metric $\tilde{\omega} \in [\tilde{\omega}_0]$, we consider the $L^2_{\tilde{\omega}}$-orthogonal projection

$$\Pi_{\tilde{\omega}} : L^2_{\tilde{\omega}} \longrightarrow \mathcal{P}^{\mathbb{K}}_{\tilde{\omega}}, \tag{1.12}$$

where $\mathcal{P}^{\mathbb{K}}_{\tilde{\omega}}$ is the space of $\tilde{\omega}$-Killing potentials, which is the space of function $f \in \mathcal{C}^\infty(Y)^{\mathbb{K}}$ such that the hamiltonian vector field $X := \tilde{\omega}^{-1}(df)$ is holomorphic. Futaki and Mabuchi [41] showed that $\Pi_{\tilde{\omega}}\big(Scal(\tilde{\omega})\big)$ does not depend on the chosen $\mathbb{K}$-invariant Kähler metric $\tilde{\omega}$ in $[\tilde{\omega}_0]$. Therefore, with respect to a normalized moment map $m_{\tilde{\omega}} : Y \longrightarrow Lie(\mathbb{K})^*$, see (1.1), one can write

$$\Pi_{\tilde{\omega}}\big(Scal(\tilde{\omega})\big) = \langle \xi_{\rm ext}, m_{\tilde{\omega}}\rangle + c_{\rm ext} =: \ell_{\rm ext}(m_{\tilde{\omega}}) \tag{1.13}$$

where $\xi_{\rm ext} \in Lie(\mathbb{K})$, $c_{\rm ext} \in \mathbb{R}$ and $\ell_{\rm ext} \in {\rm Aff}\big(Lie(\mathbb{K})^*\big)$, see [53, Lemma 1] for more details.

Assume now that $(Y, J_Y, \tilde{\omega}, \mathbb{T})$ is a semisimple principal toric fibration. Then by [7, Proposition 1], the extremal vector field is tangent to the fibers, i.e. $\xi_{\rm ext} \in \mathfrak{t}$.

Proposition 1.2.3. *Let (Y, J_Y) be a semisimple principal toric fibration and $\mathbb{K}_B$ be a maximal torus in the isometry group of $g_B := \sum_{a=1}^{k} g_a$, where g_a is the Riemannian metric of ω_a. Denote by $\mathfrak{k}_B$ the Lie algebra of $\mathbb{K}_B$. Any compatible Kähler metric $\tilde{\omega}$ on (Y, J_Y) is invariant by the action of a maximal torus $\mathbb{K}_Y \subset {\rm Aut}_{\rm red}(Y)$ whose Lie algebra fits in the exact sequence of Lie algebras*

$$\{0\} \longrightarrow \mathfrak{t} \longrightarrow \mathfrak{k}_Y \longrightarrow \mathfrak{k}_B \rightarrow \{0\}, \tag{1.14}$$

where $\mathfrak{k}_Y$ the Lie algebra of $\mathbb{K}_Y$. Moreover, the extremal vector field $\xi_{\rm ext}$ belongs in the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$.

As shown in [7], we get from (1.11) and (1.13):

Corollary 1.2.4. *A compatible Kähler metric $\tilde{\omega}$ on (Y, J_Y) is extremal if and only if the corresponding toric Kähler metric ω on (X, J_X) is (v, w)-cscK in the sense of Definition 1.2, where the weights are given by*

$$\begin{aligned} \mathrm{v}(x) &= \prod_{a=1}^{k} \big(\langle p_a, x\rangle + c_a\big)^{n_a} \\ \mathrm{w}(x) &= \mathrm{v}(x)\left(l_{\rm ext}(x) - \sum_{a=1}^{k} \frac{s_a}{\langle p_a, x\rangle + c_a}\right), \end{aligned} \tag{1.15}$$

where $l_{\mathrm{ext}} \in \mathrm{Aff}(P)$ *is defined in* (1.13).

1.2.4 The space of invariant functions

Any $\mathbb{T}_Y$-invariant smooth function on Y pulls back to a $\mathbb{T}_Q \times \mathbb{T}_X$-invariant function on $Q \times X$, and therefore descends to a $\mathbb{T}_X$-invariant smooth function on $B \times X$ (see Section 1.2.1). This gives rise to an isomorphism of Fréchet spaces

$$C^\infty(Y)^{\mathbb{T}} \cong C^\infty(B \times X)^{\mathbb{T}_X}, \tag{1.16}$$

which we shall tacitly use throughout the paper. Moreover, by (1.5) we get

$$C^\infty(Y^0)^{\mathbb{T}} \cong C^\infty(B \times P^0)$$

Given $f \in \mathcal{C}^\infty(Y)^{\mathbb{T}}$, for any $b \in B$, we denote by $f_b \in \mathcal{C}^\infty(X)^{\mathbb{T}}$ the induced smooth function on X with respect to the identification (1.16). Similarly, for any $x \in X$, we denote by $f_x \in \mathcal{C}^\infty(B)$ the induced smooth function on B. It follows that on $\mathcal{C}^\infty(Y)^{\mathbb{T}}$, the differential operator d_Y splits as $d_Y = d_B + d_X$, where d_B and d_X, is the exterior derivative on B and X respectively. We get

$$\mathcal{C}^\infty(X)^{\mathbb{T}} \cong \{f \in \mathcal{C}^\infty(Y)^{\mathbb{T}} \mid d_B f_x = 0 \ \forall x \in X\}, \tag{1.17}$$

showing that $\mathcal{C}^\infty(X)^{\mathbb{T}}$ is closed in $\mathcal{C}^\infty(Y)^{\mathbb{T}}$ for the Fréchet topology.

1.2.5 The space of compatible Kähler potentials

We fix a reference compatible Kähler metric $\tilde{\omega}_0$ on (Y, J_Y), its corresponding Kähler metric ω_0 on (X, J_X) and let (v, w) be the weights given by (1.15). We denote by $\mathcal{K}(X,\omega_0)^{\mathbb{T}}$ the space of smooth Kähler potentials on X relative to ω_0 and by $\omega_\varphi = \omega_0 + d_X d_X^c \varphi$ be the corresponding Kähler metric on (X, J_X). Similarly, $\mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$ denotes the space of smooth $\mathbb{T}$-invariant Kähler potentials on (Y, J_Y) relative to $\tilde{\omega}_0$ and $\tilde{\omega}_\varphi = \tilde{\omega}_0 + d_Y d_Y^c \varphi$ is the corresponding Kähler metric. The following Lemma is established in [7, Lemma 7].

Lemma 1.2.5. *Let $\omega_\varphi = \omega_0 + d_X d_X^c \varphi$ be a $\mathbb{T}$-invariant Kähler metric on (X, J_X) and m_φ be the moment map which satisfies the normalization* (1.1)*. Then, the compatible Kähler metric $\tilde{\omega}_\varphi$ induced by ω_φ on Y is given by $\tilde{\omega}_\varphi = \tilde{\omega}_0 + d_Y d_Y^c \varphi$, where φ is seen as a smooth function on Y via* (1.16)*.*

It follows that $\mathcal{K}(X,\omega_0)^{\mathbb{T}}$ parametrizes the compatible Kähler metric on (Y, J) given explicitly by (1.9) and will be referred to as *the space of compatible Kähler potentials*. Proposition 1.2.3 and Lemma 1.2.5 give

Corollary 1.2.6. *Let $\mathbb{K}$ be as in Lemma 1.2.3. Then, there is an embedding of Frechet spaces $\mathcal{K}(X,\omega_0)^{\mathbb{T}} \hookrightarrow \mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{K}}$.*

1.3 Weighted distance, functionals and operators

1.3.1 Weighted distance

Thanks to the work of Mabuchi [65, 66], it is known, that $\mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$ is an infinite dimensional formal Riemannian manifold with respect to the Mabuchi Riemann metric:

$$\langle \dot{\varphi}_0, \dot{\varphi}_1 \rangle_\varphi = \int_Y \dot{\varphi}_0 \dot{\varphi}_1 \tilde{\omega}_\varphi^{[n+\ell]} \quad \forall \dot{\varphi}_0, \dot{\varphi}_1 \in T_\varphi \mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}},$$

where $T_\varphi \mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}} \cong \mathcal{C}^\infty(X)^{\mathbb{T}}$ denotes the tangent space of the Fréchet space $\mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$ at φ. Furthermore, a path $(\varphi_t)_{t\in[0,1]} \in \mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$ connecting two points is a smooth geodesic for the Mabuchi metric if and only if (see [66])

$$\ddot{\varphi}_t - |d\dot{\varphi}_t|^2_{\varphi_t} = 0. \tag{1.18}$$

In [43], Guan showed the existence of a smooth geodesic between two Kähler potentials on a toric manifold. This argument yields the geodesic connectedness $\mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$. The following result is proven in Lemma 3.1.7 in the more general context of semisimple principal fiber bundles.

Lemma 1.3.1. *Let $\varphi \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}$ and $f \in T_{\varphi}\mathcal{K}(X, \omega_0)^{\mathbb{T}}$, also viewed as an element of $T_{\varphi}\mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$. Then*

$$|df|^2_{\tilde{\omega}_{\varphi}} = |df|^2_{\omega_{\varphi}}.$$

In particular, $\mathcal{K}(X, \omega_0)^{\mathbb{T}}$ is a totally geodesic submanifold of $\mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ with respect to the Mabuchi metric.

Remark 1.3.2. *On a general compact Kähler manifold (X, J, ω_0), Darvas [32] introduced the distance d_1*

$$d_1(\varphi_0, \varphi_1) := \inf_{\varphi_t} \int_0^1 \int_X |\dot{\varphi}_t| \omega^{[\ell]}_{\varphi_t}, \tag{1.19}$$

where $\omega^{[\ell]}_{\varphi_t}$ is the volume form associated to the metric $\omega_{\varphi_t} = \omega_0 + dd^c\varphi_t$ and the infimum is taken over the space of smooth curves $\{\varphi_t\}_{t\in[0,1]} \subset \mathcal{K}(X, \omega_0)^{\mathbb{T}}$ joining φ_0 to φ_1. In the above formula, $\dot{\varphi}_t$ is the variation of φ_t with respect to t. It is shown in [32] that $d_1(\varphi_0, \varphi_1)$ equals to the length of the unique (weak) $C^{1,\bar{1}}$ geodesic [23] joining φ_0 and φ_1.

Lemma 1.3.3. *Up to a positive multiplicative constant, the distance d_1 restricts to $\mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ as the distance $d_{1,\mathrm{v}}$, defined by*

$$d^X_{1,\mathrm{v}}(\varphi_0,\varphi_1) := \inf_{\varphi_t} \int_0^1 \int_X |\dot{\varphi}_t| \mathrm{v}(m_t)\omega_t^{[\ell]}. \tag{1.20}$$

Proof. This is a direct consequence of (1.10) and Lemma 1.3.1. □

1.3.2 Weighted functionals

We consider the Mabuchi energy on $\mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$ relative to $\mathbb{T}$, characterized by its variation

$$d_\varphi \mathcal{M}^{\mathbb{T}}(\dot{\varphi}) = -\int_Y \dot{\varphi}\big(Scal(\tilde{\omega}_\varphi) - \Pi_{\tilde{\omega}_\varphi}\big(Scal(\tilde{\omega}_\varphi)\big)\big)\tilde{\omega}_\varphi^{[n+\ell]}, \quad \mathcal{M}^{\mathbb{T}}(0) = 0, \tag{1.21}$$

where $\Pi_{\tilde{\omega}_\varphi}$ is introduced in (1.12). When restricted to $\mathcal{K}(X,\omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$, this functional reduces to the weighted Mabuchi functional introduced in [53]. Indeed, we have the following Lemma, established in [7], which follows directly from (1.11) and (1.15).

Lemma 1.3.4. *The restriction of the relative Mabuchi energy $\mathcal{M}^{\mathbb{T}}$ on Y to the subspace $\mathcal{K}(X,\omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$ is equal (up to a positive multiplicative constant) to the weighted Mabuchi energy on X, defined by*

$$d_\varphi \mathcal{M}_{\mathrm{v},\mathrm{w}}(\dot{\varphi}) := -\int_X \big(Scal_{\mathrm{v}}(\omega_\varphi) - \mathrm{w}(m_\varphi)\big)\dot{\varphi}\omega_\varphi^{[\ell]}, \quad \mathcal{M}_{\mathrm{v},\mathrm{w}}(0) = 0, \tag{1.22}$$

where weight functions (v,w) are given by (1.15), and $\varphi \in \mathcal{K}(X,\omega_0)^{\mathbb{T}}$, and $\dot{\varphi} \in T_\varphi\mathcal{K}(X,\omega_0)^{\mathbb{T}}$. In particular, the compatible extremal Kähler metrics in $[\tilde{\omega}_0]$ are critical points of $\mathcal{M}_{\mathrm{v},\mathrm{w}} : \mathcal{K}(X,\omega)^{\mathbb{T}} \longrightarrow \mathbb{R}$, i.e. its corresponding Kähler metric ω on X satisfies $Scal_{\mathrm{v}}(\omega) = \mathrm{w}(m_\omega)$.

The Aubin–Mabuchi functional $\mathcal{I} : \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}} \longrightarrow \mathbb{R}$ is defined by

$$d_\varphi \mathcal{I}(\dot{\varphi}) = \int_Y \dot{\varphi}\tilde{\omega}_\varphi^{[m]}, \quad \mathcal{I}(0) = 0,$$

for any $\dot{\varphi} \in T_\varphi \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$. By (1.10), its restriction to $\mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ is equal (up to a positive multiplicative constant) to

$$d_\varphi \mathcal{I}_{\mathrm{v}}(\dot{\varphi}) := \int_X \dot{\varphi}\mathrm{v}(m_\varphi)\omega_\varphi^{[\ell]}, \ \mathcal{I}_{\mathrm{v}}(0) = 0, \dot{\varphi} \in T_\varphi \mathcal{K}(X, \tilde{\omega}_0)^{\mathbb{T}} \tag{1.23}$$

We define the space of $\mathcal{I}$-normalized relative Kähler potentials as

$$\mathring{\mathcal{K}}(Y, \tilde{\omega}_0)^{\mathbb{T}} := \mathcal{I}^{-1}(0) \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}. \tag{1.24}$$

It is well known, see e.g. [42, Chapter 4], that this space is totally geodesic in $\mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$. Similarly, we define

$$\mathring{\mathcal{K}}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}} := \mathcal{I}_{\mathrm{v}}^{-1}(0) \subset \mathcal{K}(X, \omega_0)^{\mathbb{T}}. \tag{1.25}$$

It follows from (1.23) that we also have $\mathring{\mathcal{K}}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}} \subset \mathring{\mathcal{K}}(Y, \tilde{\omega}_0)^{\mathbb{T}}$.

1.3.3 Weighted differential operators

Following [7], we introduce the v-Laplacian of (X, J_X, ω) acting on smooth functions by

$$\Delta^X_{\omega,\mathrm{v}} f := \frac{1}{\mathrm{v}(m_\omega)} \delta\big(\mathrm{v}(m_\omega) d_X f\big),$$

where δ is the formal adjoint of the differential d_X with respect to the Riemann metric g_ω. This definition immediately implies that $\Delta^X_{\omega,\mathrm{v}}$ is self-adjoint with respect to $\mathrm{v}(m_\omega)\omega^{[\ell]}$. Moreover, it follows from the computations in [7, Lemma 8] that $\Delta^X_{\omega,\mathrm{v}}$ can be alternatively expressed as

$$\Delta^X_{\omega,\mathrm{v}} f = \Delta^X_\omega f - \sum_{a=1}^{k} \frac{n_a d^c_X f(p^X_a)}{\langle p_a, m_\omega\rangle + c_a} \tag{1.26}$$

for any $f \in \mathcal{C}^\infty(X)^{\mathbb{T}}$, where Δ^X_ω is the Laplacian with respect to ω and p^X_a is the fundamental vector field on X defined by $p_a \in \mathfrak{t}$. As in [53], we introduce the v-weighted Lichnerowicz operator of (X, J_X, ω) defined on smooth functions $f \in \mathcal{C}^\infty(X)$ to be

$$\mathbb{L}^X_{\omega,\mathrm{v}} f := \frac{\delta\delta\big(\mathrm{v}(m_\omega)(D^- d_X f)\big)}{\mathrm{v}(m_\omega)}, \tag{1.27}$$

where D is the Levi-Civita connection of ω, $D^- d_X$ denotes the $(2,0)+(0,2)$ part of Dd_X and $\delta : \otimes^p T^*X \longrightarrow \otimes^{p-1} T^*X$ is defined in any local orthogonal frame $\{e_1, \dots, e_{2n}\}$ by

$$\delta\psi := -\sum_{i=1}^{2n} D_{e_i}(e_i, \cdot)\psi$$

The operator $\delta\delta$ is the formal adjoint of $D^- d_X$ with respect to $\omega^{[\ell]}$. Hence, the v-weighted Lichnerowicz operator is self-adjoint with respect to the volume form $\mathrm{v}(m_\omega)\omega^{[\ell]}$. Let $\tilde\omega$ be the compatible Kähler metric on (Y, J_Y) corresponding to ω. For any $x \in X$, we denote by $\omega_B(x)$ the Kähler form on (B, J_B) induced by $\tilde\omega$:

$$\omega_B(x) := \sum_{a=1}^{k} \big(\langle p_a, m_\omega(x)\rangle + c_a\big)\omega_a.$$

The following is etablished in the proof of [7, Lemma 8].

Proposition 1.3.5. *Let f be a $\mathbb{T}$-invariant smooth function on Y, seen as a $\mathbb{T}_X$-invariant function on $X \times B$ via (1.16). We denote by $\Delta_{\tilde{\omega}}$ the Laplacian of $(Y, J_Y, \tilde{\omega})$ and by Δ_x^B, respectively $\mathbb{L}_x^B$, the Laplacian, respectively the Lichnerowicz operator, of $(B, J_B, \omega_B(x))$. We then have*

$$\Delta_{\tilde{\omega}}^Y f = \Delta_{\omega,\mathrm{v}}^X f_b + \Delta_x^B f_x$$

Furthermore, the corresponding Licherowicz's operators $\mathbb{L}_{\tilde{\omega}}^Y$, $\mathbb{L}_{\omega,\mathrm{v}}^X$ and $\mathbb{L}_x^B$ are related by

$$\mathbb{L}_{\tilde{\omega}}^Y f = \mathbb{L}_{\omega,\mathrm{v}}^X f_s + \mathbb{L}_x^B f_x + \Delta_x^B \big(\Delta_{\omega,\mathrm{v}}^X f_b\big)_x + \Delta_{\omega,\mathrm{v}}^X \big(\Delta_x^B f_x\big)_b + \sum_{a=1}^{k} Q_a(x) \Delta_a f_x \tag{1.28}$$

where Δ_a stands for the Laplacian with respect to (B_a, J_a, ω_a) and $Q_a(x)$ is a $\mathbb{T}$-invariant smooth function on X.

We fix a compatible Kähler metric

$$\tilde{\chi} = \sum_{a=1}^{k} (\langle p_a, m_\chi \rangle + c_{a,\alpha}) \omega_a + \chi$$

corresponding to a Kähler metric χ on (X, J_X), where m_χ is a moment map with respect to χ and $c_{a,\alpha}$ are constants depending on $\alpha := [\tilde{\chi}]$ such that $\langle p_a, m_\chi \rangle + c_{a,\alpha} > 0$. Hashimoto introduced in [46] the operator $\mathbb{H}_{\tilde{\omega}}^{\tilde{\chi}} : \mathcal{C}^\infty(Y)^{\mathbb{T}} \longrightarrow \mathcal{C}^\infty(Y)^{\mathbb{T}}$:

$$\mathbb{H}_{\tilde{\omega}}^{\tilde{\chi}} f := g_{\tilde{\omega}}\big(\tilde{\chi}, dd^c f\big) + g_{\tilde{\omega}}\big(d\Lambda_{\tilde{\omega}} \tilde{\chi}, df\big). \tag{1.29}$$

According to [46, Lemma 1], $\mathbb{H}_{\tilde{\omega}}^{\tilde{\chi}}$ is a second order self-adjoint elliptic differential operator with respect to $\tilde{\omega}^{[n]}$. Furthermore, the kernel of $\mathbb{H}_{\tilde{\omega}}^{\tilde{\chi}}$ is the space of constant functions. We define the v-weighted *Hashimoto operator* $\mathbb{H}_{\omega,\mathrm{v}}^{\chi} : \mathcal{C}^{\infty}(X)^{\mathbb{T}} \to \mathcal{C}^{\infty}(X)^{\mathbb{T}}$ by

$$\mathbb{H}_{\omega,\mathrm{v}}^{\chi} f := g_{\omega}\big(\chi, d_X d_X^c f\big) + g_{\omega}\big(d_X \Lambda_{\omega}\chi, d_X f\big) + \frac{1}{\mathrm{v}(m_{\omega})} g_{\omega}\big(\chi, d_X \mathrm{v}(m_{\omega}) \wedge d_X^c f\big). \quad (1.30)$$

Proposition 1.3.6. *Let f be a $\mathbb{T}$-invariant smooth function on Y, seen as a $\mathbb{T}_X$-invariant function on $X \times B$ via (1.16). The Hashimoto operator admits the following decomposition*

$$\mathbb{H}_{\tilde{\omega}}^{\tilde{\chi}} f = \mathbb{H}_{\omega,\mathrm{v}}^{\chi} f_b + \sum_{a=1}^{k} R_a(x) \Delta_a f_x,$$

where $R_a(x)$ are smooth functions on X depending on χ and α.

Proof. For simplicity, we denote by m the moment map of ω and

$$q(m) := \sum_{a=1}^{k} \frac{n_a\big(\langle p_a, m_{\chi}\rangle + c_{a,\alpha}\big)}{\langle p_a, m\rangle + c_a}.$$

Let $K \in \mathcal{C}^{\infty}(X)^{\mathbb{T}} \otimes \mathfrak{t}^*$ be the generator of the $\mathbb{T}_X$-action. By definition, $d_X^c f(K)$ is a smooth $\mathbb{T}_X$-invariant $\mathfrak{t}^*$-valued function on X and induces a smooth $\mathbb{T}_Y$-invariant $\mathfrak{t}^*$-valued function on Y via (1.16). It is shown in the proof of [7, Lemma 8] that on Y^0

$$\begin{aligned} d_Y d_Y^c f =& \langle d_X (d_X^c f_b(K))_b \wedge \theta\rangle + \langle d_B (d_X^c f_b(K))_x \wedge \theta\rangle \\ &+ \sum_{a=1}^{k} d_X^c f_b(p_a^X)\omega_a + d_B d_B^c f_x + \langle d_B^c (d_X^c f_b(K))_x, J_Y \theta\rangle. \end{aligned} \quad (1.31)$$

First, we recall the general identity

$$g_{\tilde{\omega}}(d_Y d_Y^c f, \tilde{\chi})\tilde{\omega}^{[n]} = -d_Y d_Y^c f \wedge \tilde{\chi} \wedge \tilde{\omega}^{[\ell+n-2]} - \Delta_{\tilde{\omega}} f \Lambda_{\tilde{\omega}}(\tilde{\chi})\tilde{\omega}^{[n+\ell]}. \tag{1.32}$$

From the expressions of $\tilde{\chi}$ and $\tilde{\omega}$, we can see that

$$\Big(\langle d_B(d_X^c f_b(K))_x \wedge \theta\rangle + \langle d_B^c(d_X^c f_b(K))_x, J\theta\rangle\Big) \wedge \tilde{\chi} \wedge \tilde{\omega}^{[n+\ell-2]} = 0.$$

A straightforward computation gives

$$\Lambda_{\tilde{\omega}}(\tilde{\chi}) = \Lambda_{\omega}(\chi) + \sum_{a=1}^{k} \frac{n_a\big(\langle p_a, m_\chi\rangle + c_{a,\alpha}\big)}{\langle p_a, m\rangle + c_a}. \tag{1.33}$$

From Proposition 1.3.5, (1.33) and (1.32) we have

$$g_{\tilde{\omega}}(d_Y d_Y^c f_b, \tilde{\chi}) = g_{\omega}(d_X d_X^c f_b, \chi) + \sum_{a=1}^{k} \frac{n_a d_X f_s(p_a^X)(\langle p_a, m_\chi\rangle + c_{a,\alpha})}{(\langle p_a, m\rangle + c_a)^2}. \tag{1.34}$$

Using (1.33) we get

$$g_{\tilde{\omega}}\big(d_Y \Lambda_{\tilde{\omega}}(\tilde{\chi}), df_b\big) = g_{\omega}\big(d_X \Lambda_{\omega}(\chi), d_X f_b\big) + g_{\omega}\big(d_X q(m), d_X^c f_b\big). \tag{1.35}$$

To summarize, we have shown

$$\begin{aligned} \mathbb{H}_{\tilde{\omega}}^{\tilde{\chi}} f_s = & g_{\omega}(d_X d_X^c f_s, \chi) + g_{\omega}\big(d_X \Lambda_{\omega}(\chi), d_X f_s\big) \\ & + g_{\omega}\big(d_X q(m), d_X^c f_s\big) + \sum_{a=1}^{k} \frac{n_a d_X f_s(p_a^X)(\langle p_a, m_\chi\rangle + c_{a,\alpha})}{(\langle p_a, m\rangle + c_a)^2}. \end{aligned} \tag{1.36}$$

Using (1.15) we have

$$\frac{1}{\mathrm{v}(m)} g_\omega\big(\chi, d_X \mathrm{v}(m) \wedge d^c_X f_s\big) = g_\omega\big(d_X q(m), d^c_X f_s\big) + \sum_{a=1}^{k} \frac{n_a d_X f_s(p_a^X)(\langle p_a, m_\chi\rangle + c_{a,\alpha})}{(\langle p_a, m\rangle + c_a)^2}. \tag{1.37}$$

From (1.36) and (1.37) we get

$$\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}} f_s = \mathbb{H}^{\chi}_{\omega,\mathrm{v}} f_s.$$

The term $\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}} f_x$ is obtained via similar computation.

□

Remark 1.3.7. *Proposition 1.3.5 implies in particular that the restriction of $\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}}$ to the Frechet subspace $\mathcal{C}^\infty(X)^{\mathbb{T}} \subset \mathcal{C}^\infty(Y)^{\mathbb{T}}$ coincides with $\mathbb{H}^{\chi}_{\omega,\mathrm{v}}$. It follows that $\mathbb{H}^{\chi}_{\omega,\mathrm{v}}$ is a self-adjoint second order elliptic operator with respect to $\mathrm{v}(m_\omega)\omega^{[\ell]}$.*

1.4 An analytic criterion for the existence of extremal Kähler metrics

In this section we recall the general existence results for extremal Kähler metrics in a given Kähler class, proved by Chen–Cheng [25, 26] in the constant scalar curvature case and extended by He [47] to the extremal case.

We fix a compact Kähler manifold (X, J), a maximal compact connected subgroup $\mathbb{G}$ of $\mathrm{Aut}_{\mathrm{red}}(X)$ and a $\mathbb{G}$-invariant Kähler metric ω_0. Let $\xi_{\mathrm{ext}} \in Lie(\mathbb{G})$ denotes the corresponding extremal vector field, see Section 1.2.3. Since the extremal vector field ξ_{ext} is in the center of the Lie algebra of $\mathbb{G}$, $\exp(t\xi)$ generates a torus $\mathbb{T}_{\mathrm{ext}}$ in the center of $\mathbb{G}$. As in [47], we consider the space of $\mathbb{T}_{\mathrm{ext}}$-invariant

Kähler potentials $\mathcal{K}(X,\omega_0)^{\mathbb{T}_{\text{ext}}}$. The group $\mathbb{G}^{\mathbb{C}}$ acts on $\mathring{\mathcal{K}}(X,\omega_0)^{\mathbb{T}_{\text{ext}}}$ (as $\mathbb{T}_{\text{ext}}$ is in the center of $\mathbb{G}^{\mathbb{C}}$) via the natural action on Kähler metrics in $[\omega_0]$ and the normalization (1.24). We introduce the distance $d_{1,\mathbb{G}^{\mathbb{C}}}$ relative to $\mathbb{G}^{\mathbb{C}}$ by

$$d_{1,\mathbb{G}^{\mathbb{C}}}(\varphi_1,\varphi_2) := \inf_{\gamma\in\mathbb{G}^{\mathbb{C}}} d_1(\varphi_1,\gamma\cdot\varphi_2), \tag{1.38}$$

where d_1 is defined in (1.19). Let $\mathcal{M}^{\mathbb{T}_{\text{ext}}}$ be the Mabuchi energy relative to $\mathbb{T}_{\text{ext}}$, see (1.21). We recall the following definition from [33]:

Definition 1.4.1. *The relative Mabuchi energy $\mathcal{M}^{\mathbb{T}_{\text{ext}}}$ is said to be proper with respect to $d_{1,\mathbb{G}^{\mathbb{C}}}$ if*

- *$\mathcal{M}^{\mathbb{T}_{\text{ext}}}$ is bounded from below on $\mathcal{K}(X,\omega_0)^{\mathbb{T}_{\text{ext}}}$;*
- *for any sequence of normalized potentials $\varphi_i \in \mathring{\mathcal{K}}(X,\omega_0)^{\mathbb{T}_{\text{ext}}}$, $d_{1,\mathbb{G}^{\mathbb{C}}}(0,\varphi_i)\to\infty$ implies that $\mathcal{M}^{\mathbb{T}_{\text{ext}}}(\varphi_i)\to\infty$.*

Theorem 1.4.2. *The relative Mabuchi energy $\mathcal{M}^{\mathbb{T}_{\text{ext}}}$ restricted to $\mathcal{K}(X,\omega_0)^{\mathbb{G}} \subset \mathcal{K}(X,\omega_0)^{\mathbb{T}_{\text{ext}}}$ is $d_{1,\mathbb{G}^{\mathbb{C}}}$-proper if and only if there exists an extremal Kähler metric in $(X,J,[\omega_0])$ with extremal vector field ξ_{ext}. Moreover, the same assertion holds if we remplace $\mathbb{T}_{\text{ext}}$ and $\mathbb{G}$ by a maximal torus $\mathbb{K}\subset \mathrm{Aut}_{\text{red}}(X)$, and $\mathbb{G}^{\mathbb{C}}$ by the complexification $\mathbb{K}^{\mathbb{C}}$ of $\mathbb{K}$.*

Proof. The first assertion is established in [47, Theorem 3.1]. We can directly modify the argument to obtain the second. Indeed, in the one direction, suppose that $\mathcal{M}^{\mathbb{K}}$ is $\mathbb{K}^{\mathbb{C}}$-proper, in the sense that $\mathcal{M}^{\mathbb{K}}$ is bounded from below on $\mathcal{K}(M,\omega_0)^{\mathbb{K}}$ and for any sequence $\varphi_i\in\mathring{\mathcal{K}}(X,\omega_0)^{\mathbb{K}}$, $d_{1,\mathbb{K}^{\mathbb{C}}}(0,\varphi_i)\to\infty$ implies that $\mathcal{M}^{\mathbb{K}}(\varphi_i)\to\infty$. Since $\mathbb{K}\subset\mathbb{G}$ is a maximal torus it must contain the center of $\mathbb{G}$, i.e. $\mathbb{T}_{\text{ext}}\subset\mathbb{K}$. Hence, $\mathcal{M}^{\mathbb{K}}|_{\mathcal{K}(X,\omega_0)^{\mathbb{G}}} = \mathcal{M}^{\mathbb{T}_{\text{ext}}}|_{\mathcal{K}(X,\omega_0)^{\mathbb{G}}}$, where $\mathcal{K}(X,\omega_0)^{\mathbb{G}}\subset\mathcal{K}(X,\omega_0)^{\mathbb{K}}$ is the subspace of $\mathbb{G}$-invariant ω_0-relative Kähler potentials. As any $\mathbb{K}^{\mathbb{C}}$-orbit of an element of $\mathcal{K}(X,\omega_0)^{\mathbb{G}}$ belongs to its $\mathbb{G}^{\mathbb{C}}$-orbit, the $d_{1,\mathbb{K}^{\mathbb{C}}}$-properness of $\mathcal{M}^{\mathbb{K}}$ implies that $\mathcal{M}^{\mathbb{T}_{\mathrm{ext}}}$ is $d_{1,\mathbb{G}^{\mathbb{C}}}$-proper when restricted to the subspace $\mathcal{K}(X,\omega_0)^{\mathbb{K}}$. By [47, Theorem 3.1], this implies the existence of a $\mathbb{G}$-invariant (and hence $\mathbb{K}$-invariant) extremal Kähler metric in $[\omega_0]$.

Conversely, suppose that $[\omega_0]$ admits a $\mathbb{K}$-invariant extremal Kähler metric. Then the proof of [47, Theorem 3.7] yields the $\mathbb{K}^{\mathbb{C}}$-properness of $\mathcal{M}^{\mathbb{K}}$, should one have the uniqueness of the $\mathbb{K}$-invariant extremal Kähler metrics modulo $\mathbb{K}^{\mathbb{C}}$. Generalizing the result of Berman-Berndtsson [16] and Chen-Paun-Zeng [30], Lahdili showed, in the more general context of (v,w)-weighted metrics [54, Theorem 2, Remark 2], that the $\mathbb{K}$-invariant extremal metrics are unique modulo the action of $\mathbb{K}^{\mathbb{C}}$.

□

1.5 An analytic criterion in the case of semisimple principal toric fibrations

This section is devoted to prove of the following result (where we use notation of Section 1.2).

Theorem 1.5.1. *Let $(Y, J_Y, \tilde{\omega}_0, \mathbb{T})$ be a semisimple principal toric fibration with Kähler toric fiber $(X, J_X, \omega_0, \mathbb{T})$ and let (v,w) be the corresponding weight functions defined in (1.15). Then, the following statements are equivalent.*

1. *There exists an extremal Kähler metric in $(Y, J_Y, [\tilde{\omega}_0], \mathbb{T})$.*
2. *There exists a compatible extremal Kähler metric in $(Y, J_Y, [\tilde{\omega}_0], \mathbb{T})$.*
3. *There exists a (v,w)-cscK metric in $(X, J_X, [\omega_0], \mathbb{T})$.*

The statement (2) ⇔ (3) is established in Corollary 1.2.4 whereas the statement (2) ⇒ (1) is clear. We focus on (1) ⇒ (2).

We follow the argument of He [47] by restricting the continuity path of Chen [24] to compatible Kähler metrics. We consider the continuity path for $\varphi \in \mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ given by

$$t\big(Scal_{\mathrm{v}}(\omega_\varphi) - \mathrm{w}(m_\varphi)\big) = (1-t)\big(\Lambda_{\omega_\varphi,\mathrm{v}}(\chi) - n - \ell\big), \quad t \in [0,1], \tag{1.39}$$

for some Kähler metric $\chi \in [\omega_0]$ that we will choose below in (1.44). In the above formula

$$\Lambda_{\omega_\varphi,\mathrm{v}}(\chi) := \Lambda_{\omega_\varphi}(\chi) + \sum_{a=1}^{k} \frac{n_a\big(\langle p_a, m_\chi\rangle + c_a\big)}{\langle p_a, m_\varphi\rangle + c_a} \tag{1.40}$$

is a smooth function on X equal to $\Lambda_{\tilde{\omega}_\varphi}(\tilde{\chi})$. By definition, a solution φ_t at $t = 1$ corresponds to a compatible extremal metric on (Y, J_Y) or equivalently to a (v, w)-cscK on (X, J_X). For $t_1 \in (0, 1]$, we define

$$S_{t_1} := \{t \in (0, t_1] \mid (1.39) \text{ has a solution } \varphi_t \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}\}. \tag{1.41}$$

We need to show that S_1 is open, closed and non empty.

1.5.1 Openness

Proposition 1.5.2. *S_1 is open and non empty.*

For a compatible Kähler form $\tilde{\omega}$ on (Y, J_Y) corresponding to a Kähler metric ω on (X, J_X), we denote by $C^\infty(Y, \tilde{\omega})^{\mathbb{T}}$ the space of $\mathbb{T}_Y$-invariant smooth functions with zero mean value with respect to $\tilde{\omega}^{[m]}$ and by $\mathcal{C}^\infty_{\mathrm{v}}(X, \omega)^{\mathbb{T}} \subset C^\infty(Y, \tilde{\omega})^{\mathbb{T}}$ the space of $\mathbb{T}_X$-invariant smooth functions with zero mean with respect to $\mathrm{v}(m_\omega)\omega^{[\ell]}$. The following is an adaptation of [47, Lemma 3.2].

Lemma 1.5.3. *S_1 is non empty.*

Proof. Let ω be a Kähler metric on (X, J_X) and $\tilde{\omega}$ its associate compatible Kähler metric on (Y, J_Y) via (1.9). Since $\Delta^X_{\omega,\mathrm{v}}$ is self-adjoint with respect to $\mathrm{v}(m_\omega)\omega^{[\ell]}$, it follows from the proof of Proposition 1.5.4 below that

$$\Delta^X_{\omega,\mathrm{v}} : \mathcal{C}^\infty_{\mathrm{v}}(X,\omega)^{\mathbb{T}} \longrightarrow \mathcal{C}^\infty_{\mathrm{v}}(X,\omega)^{\mathbb{T}} \tag{1.42}$$

is an isomorphism. Denote by $f \in C^\infty(Y, \tilde{\omega})^{\mathbb{T}}$ the unique solution of

$$\Delta^Y_{\tilde{\omega}} f = Scal_{\mathrm{v}}(\omega) - \mathrm{w}(m_\omega). \tag{1.43}$$

By (1.42), $f \in \mathcal{C}^\infty_{\mathrm{v}}(X,\omega)^{\mathbb{T}}$. Now we choose

$$\tilde{\chi} := \tilde{\omega} - d_Y d^c_Y \frac{f}{r}. \tag{1.44}$$

Since f is a $\mathbb{T}_X$-invariant smooth function on X by Lemma 1.2.5, $\tilde{\chi}$ is both Kähler and compatible for r sufficiently large. We denote by χ the corresponding Kähler metric on (X, J_X). Then

$$\begin{aligned}\Delta^Y_{\tilde{\omega}} f =& r\Delta^Y_{\tilde{\omega}} f\frac{1}{r} = -r\Lambda_{\tilde{\omega}} d_Y d^c_Y \frac{f}{r}\\ =& r\Lambda_{\tilde{\omega}}\left(\tilde{\omega} - d_Y d^c_Y\frac{f}{r} - \tilde{\omega}\right)\\ =& r\big(\Lambda_{\omega,\mathrm{v}}(\chi) - n - \ell\big).\end{aligned}$$

Now let $r := t_0^{-1} - 1$, for $t_0 \in (0,1)$ sufficiently small. Then (ω, t_0) is a solution of (1.39).

□

Now we show that S_1 is open. We fix a solution (ω_{t_0}, t_0) of (1.39) given by Lemma 1.5.3. Let $\tilde{\omega}_{t_0} = \tilde{\omega}_0 + d_Y d_Y^c \varphi_{t_0}$ be the corresponding compatible Kähler metric on (Y, J_Y), with $\varphi_{t_0} \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}$. Let $\pi : \mathcal{C}^\infty(Y)^{\mathbb{T}} \longrightarrow \mathcal{C}^\infty(Y, \tilde{\omega}_{t_0})^{\mathbb{T}}$ be the projection to the space $\mathcal{C}^\infty(Y, \tilde{\omega}_{t_0})^{\mathbb{T}}$ of $\mathbb{T}$-invariant smooth function on Y with zero mean with respect to $\tilde{\omega}_{t_0}$

$$\pi(f) := f - \frac{1}{\int_Y \tilde{\omega}_{t_0}^{[n]}} \int_Y f \tilde{\omega}_{t_0}^{[n]}.$$

We consider

$$R : \mathring{\mathcal{K}}(Y, \tilde{\omega}_0)^{\mathbb{T}} \times [0,1] \longrightarrow \mathcal{C}^\infty(Y)^{\mathbb{T}},$$

defined by

$$R(\varphi, t) := t\big(Scal(\tilde{\omega}_\varphi) - \Pi_{\tilde{\omega}_\varphi}(Scal(\tilde{\omega}_\varphi))\big) - (1-t)(\Lambda_{\tilde{\omega}_\varphi}(\tilde{\chi}) - n - \ell).$$

The linearization of $\pi \circ R$ at (φ_{t_0}, t_0) is given by

$$\begin{aligned} &D(\pi \circ R)(\varphi_{t_0}, t_0)[f, s] = \\ &\pi\bigg(\mathcal{L}_{\tilde{\omega}_{t_0}} f + s\bigg(Scal(\tilde{\omega}_{t_0}) - \Pi_{\tilde{\omega}_{t_0}}\big(Scal(\tilde{\omega}_{t_0})\big) + \Lambda_{\tilde{\omega}_{t_0}}(\tilde{\chi}) - n - \ell\bigg)\bigg), \end{aligned} \tag{1.45}$$

where

$$\mathcal{L}_{\tilde{\omega}_{t_0}} = -2t_0\mathbb{L}^Y_{\tilde{\omega}_{t_0}} + (1-t_0)\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}_{t_0}}.$$

Above we used the notation

$$\begin{aligned}\mathbb{L}^Y_{\tilde{\omega}_{t_0}} f :&= \delta\delta D^- d_Y f \\ &= \frac{1}{2}(\Delta^Y_{\tilde{\omega_{t_0}}})^2 f + g_{\tilde{\omega}_{t_0}}\big(d_Y d^c_Y f, Ric(\varphi_{t_0})\big) + \frac{1}{2} g_{\tilde{\omega}_{t_0}}\big(d_Y f, d_Y Scal(\varphi_{t_0})\big),\end{aligned}$$

where $D^- d_Y$ and δ is introduced in (1.27) and $\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}_{t_0}}$ is introduced in (1.29). Since $\mathcal{L}_{\tilde{\omega}_{t_0}}$ is a self-adjoint operator with respect to $\tilde{\omega}^{[n+\ell]}_{t_0}$ we get

$$D(\pi\circ R)(\varphi_{t_0}, t_0)[f,s] = \mathcal{L}_{\tilde{\omega}_{t_0}} f.$$

By Propositions 1.3.5 and 1.3.6, the restriction of $\mathcal{L}_{\tilde{\omega}_{t_0}}$ to $\mathcal{C}^\infty(X)^{\mathbb{T}}$ is equal to

$$\mathcal{L}^X_{\omega_{t_0},\mathrm{v}} := -2t\mathbb{L}^X_{\omega_{t_0},\mathrm{v}} + (1-t)\mathbb{H}^{\chi}_{\omega_{t_0},\mathrm{v}}. \tag{1.46}$$

By Propositions 1.3.5 and 1.3.6 we obtain

$$\begin{aligned}\mathcal{L}^Y_{\tilde{\omega}_{t_0}} f =& \mathcal{L}^X_{\omega_{t_0},\mathrm{v}} f_b + t_0\mathbb{L}^B_x f_x + t_0\Delta^S_x\big(\Delta^X_{\omega_{t_0},\mathrm{v}} f_b\big)_x \\ &+ t_0\Delta^X_{\omega_{t_0},\mathrm{v}}\big(\Delta^B_x f_x\big)_b + \sum_{a=1}^k U_a(x)\Delta_a f_x\end{aligned} \tag{1.47}$$

for all $f \in \mathcal{C}^\infty(Y)^{\mathbb{T}}$, where $U_a(x)$ is a smooth function on X. By [47, Lemma 3.1] the operator $\mathcal{L}^Y_{\tilde{\omega}_{t_0}}$ extends to an isomorphism between the Hölder spaces

$$\mathcal{L}^Y_{\tilde{\omega}_{t_0}} : \mathcal{C}^{4,\alpha}(Y, \tilde{\omega}_{t_0})^{\mathbb{T}} \longrightarrow \mathcal{C}^{0,\alpha}(Y, \tilde{\omega}_{t_0})^{\mathbb{T}}, \tag{1.48}$$

where $\mathcal{C}^{4,\alpha}(Y, \tilde{\omega}_{t_0})^{\mathbb{T}}$ is the space of $\mathbb{T}_Y$-invariant functions with regularity $(4, \alpha)$ with zero mean with respect to $\tilde{\omega}^{[n]}_{t_0}$ and similarly for $\mathcal{C}^{0,\alpha}(Y, \tilde{\omega}_{t_0})^{\mathbb{T}}$. By (1.46), the restriction of the operator $\mathcal{L}^Y_{\tilde{\omega}_{t_0}}$ to the space $\mathcal{C}^{4,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}}$ is equal to $\mathcal{L}^X_{\omega_{t_0},\mathrm{v}}$, where $\mathcal{C}^{4,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}}$ is the space of $\mathbb{T}_X$-invariant functions of regularity $(4, \alpha)$ with zero mean with respect to $\mathrm{v}(m_{t_0})\omega^{[\ell]}_{t_0}$.

Proposition 1.5.4. *The operator* $\mathcal{L}^X_{\omega_{t_0},\mathrm{v}} : \mathcal{C}^{4,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}} \longrightarrow \mathcal{C}^{0,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}}$ *is an isomorphism.*

Proof. Since $\mathcal{L}^X_{\omega_{t_0},\mathrm{v}}$ is the restriction of an injective operator, it is enough to prove its surjectivity. We proceed analogously to the proof of [7, Lemma 8].

We denote by $L^2_{0,\mathrm{v}}(X)^{\mathbb{T}}$ the completion for the L^2-norm of $\mathcal{C}^{0,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}}$. We argue by contradiction. Assume that

$$\mathcal{L}^X_{\omega_{t_0},\mathrm{v}} : \mathcal{C}^{4,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}} \to \mathcal{C}^{0,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}}$$

is not surjective. Then, there exists $\phi \in L^2_{0,\mathrm{v}}(X)^{\mathbb{T}}$ satisfying

$$\int_X \mathcal{L}^X_{\omega_{t_0},\mathrm{v}}(f)\phi \mathrm{v}(m_{t_0})\omega^{[\ell]}_{t_0} = 0 \tag{1.49}$$

for all $f \in \mathcal{C}^{4,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}}$. We claim that

$$\int_Y \mathcal{L}^Y_{\tilde{\omega}_{t_0}}(f)\phi\tilde{\omega}^{[n+\ell]}_{t_0} = 0 \tag{1.50}$$

for all functions $f \in \mathcal{C}^{4,\alpha}(Y, \tilde{\omega}_{t_0})^{\mathbb{T}}$. Now, (1.50) contradicts the surjectivity of $\mathcal{L}^{Y}_{\tilde{\omega}_{t_0}} : \mathcal{C}^{4,\alpha}(Y, \tilde{\omega}_{t_0})^{\mathbb{T}} \longrightarrow \mathcal{C}^{0,\alpha}(Y, \tilde{\omega}_{t_0})^{\mathbb{T}}$ established in [47, Lemma 3.1]. To obtain (1.50), we argue similarly to the proof of [7, Lemma 8] using that the image of $\mathcal{L}^{X}_{\omega_{t_0},\mathrm{v}}$ is L^2-orthogonal to the subspace of constant functions with respect to $\mathrm{v}(m_{t_0})\omega_{t_0}^{[\ell]}$.

□

By the Implicit Function Theorem applied to the differential operator $\mathcal{L}^{X}_{\omega_{t_0},\mathrm{v}} : \mathcal{C}^{4,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}} \longrightarrow \mathcal{C}^{0,\alpha}_{\mathrm{v}}(X, \omega_{t_0})^{\mathbb{T}}$, there exists an open interval $U \subset (0,1)$ containing t_0 such that for any $t \in U$, there exists a compatible potential φ_t of regularity $\mathcal{C}^{4,\alpha}$ such that (t, φ_t) is a solution of (1.39). By a well-known bootstrapping argument, any solution of (1.39) of regularity $\mathcal{C}^{4,\alpha}$ is smooth. This concludes the proof of Proposition 1.5.2.

1.5.2 Closedness

Proposition 1.5.5. *S_1 is closed*

Proof. By assumption, there exists an extremal Kähler metric in $[\tilde{\omega}_0]$. By Theorem 1.4.2, the relative Mabuchi energy $\mathcal{M}^{\mathbb{K}}$ is $d_{1,\mathbb{K}^{\mathbb{C}}}$-proper. Let $\{\varphi_i\}_{i\in\mathbb{N}} \subset \mathcal{K}(X, \omega_0)^{\mathbb{T}}$ be a sequence of solutions of (1.39) given by Proposition 1.5.2 with $t_i \to t_1 < 1$. By Corollary 1.2.6, $\{\varphi_i\}_{i\in\mathbb{N}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{K}}$. Consequently, the same argument as in [47, Lemma 3.3] shows the existence of a smooth limit $\varphi_{t_1} \in \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{K}}$. Moreover, it follows from (1.14) and (1.16) that $\mathcal{K}(X, \omega_0)^{\mathbb{T}}$ is closed in $\mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{K}}$ in the Fréchet topology. Then, φ_{t_1} belongs to $\mathcal{K}(X, \omega)^{\mathbb{T}}$. In particular $\tilde{\omega}_{\varphi_{t_1}}$ is a compatible Kähler metric.

Let $\tilde{\varphi}_{t_i} \in \mathring{\mathcal{K}}(X, \omega_0)^{\mathbb{T}}$ (see (1.25)) be the solution of (1.39) at t_i for t_i increasing to 1. By Theorem 1.4.2, $\mathcal{M}^{\mathbb{K}}$ is $d_{1,\mathbb{K}^{\mathbb{C}}}$-proper. Then, by Corollary 1.2.6, we get a upper bound with respect to $d_{1,\mathbb{K}^{\mathbb{C}}}$, that is

$$\sup_{i\in\mathbb{N}} d_{1,\mathbb{K}^{\mathbb{C}}}(0,\tilde{\varphi}_{t_i}) < \infty.$$

By definition of $d_{1,\mathbb{K}^{\mathbb{C}}}$, there exists $\gamma_i \in \mathbb{K}^{\mathbb{C}}$ such that $\varphi_{t_i} := \gamma_i \cdot \tilde{\varphi}_{t_i}$ satisfy

$$\sup_{i\in\mathbb{N}} d_1(0,\varphi_{t_i}) < \infty.$$

The Kähler metric $\tilde{\omega}_{\varphi_{t_i}}$ is not compatible in general since the connection form θ and the base Kähler metrics ω_a may change after the action of γ_i. However, by Proposition 1.2.3, the $\mathbb{K}^{\mathbb{C}}$-action commutes with the $\mathbb{T}_Y$-action. Then, for each t_i, the $\mathbb{T}_Y$-action is still rigid and semisimple (see Remark 1.2.2). According to [5], $\tilde{\omega}_{\varphi_{t_i}}$ is then given by the generalized Calabi ansatz with a fixed stable quotient $B = \prod_{a=1}^k B_a$ with respect to the complexified action $\mathbb{T}_Y^{\mathbb{C}}$. Thus, there exists a connection 1-form θ_{t_i} with curvature

$$d\theta_{t_i} = \sum_{a=1}^k \pi_B^*(\omega_{a,t_i}) \otimes p_{a,t_i} \quad p_{a,t_i} \in \Lambda$$

such that $\tilde{\omega}_{\varphi_{t_i}}$ is given by

$$\tilde{\omega}_{\varphi_{t_i}} = \sum_{a=1}^k (\langle p_{a,t_i}, m_{\varphi_{t_i}}\rangle + c_{a,t_i})\pi_B^*(\omega_{a,t_i}) + \langle dm_{\varphi_{t_i}}, \theta_{t_i}\rangle.$$

In the above formula, ω_{a,t_i} are cscK metrics on B_a belonging to the Kähler class $[\omega_a]$ of the initial cscK metric. Since $\tilde{\omega}_{\varphi_{t_i}} \in [\tilde{\omega}_0]$, $c_{a,t_i} = c_a$ and $p_{a,t_i} = p_a$. By [47, Theorem 3.5], $\tilde{\omega}_{t_i}$ converges smoothly to an extremal metric ω_{φ_1}. Furthermore, by Propositon 1.2.3, the extremal vector field ξ_{ext} of $[\tilde{\omega}_0]$ relatif to $\mathbb{K}$ is in the Lie algebra $\mathfrak{t}$ of $\mathbb{T}_Y$. Then, by Corollary 1.2.4 and the smooth convergence of $\tilde{\omega}_{\varphi_{t_i}}$ to $\tilde{\omega}_{\varphi_1}$, we get

$$\langle m_{\varphi_1}, \xi_{ext}\rangle + c_{ext} = \sum_{a=1}^{k} \frac{Scal(\omega_{a,1})}{\langle p_a, m_{\varphi_1}\rangle + c_a} + \frac{1}{\mathrm{v}(m_{\varphi_1})} Scal_{\mathrm{v}}(\omega_{\varphi_1}), \tag{1.51}$$

where ω_{φ_1} is the Kähler metric on (X, J_X) corresponding to $\tilde{\omega}_{\varphi_1}$. Taking the exterior differential d_{B_a} on B_a in (1.16) we get $d_{B_a} Scal(\omega_{a,1}) = 0$ for all $1 \leq a \leq k$, i.e. $\omega_{a,1}$ has constant scalar curvature. Yet, $[\omega_{a,1}] = [\omega_a]$, showing that $Scal(\omega_{a,1}) = s_a$. By the definition of $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$, we get

$$Scal_{\mathrm{v}}(\omega_{\varphi_1}) = \mathrm{w}(m_{\varphi_1}).$$

□

Corollary 1.5.6. *In a compatible Kähler class, the extremal Kähler metrics are given by the generalized Calabi ansatz of [5]. Equivalently, in a compatible Kähler class, the extremal metrics are induced by* (v, w)*-cscK metrics on* (X, J_X) *via* (1.9) *for a suitable connection* 1*-form* θ *and suitable Kähler metrics* ω_a*.*

Proof. Suppose there exists an extremal metric ω_1 in $[\tilde{\omega}]$. By a result of Calabi [21], ω_1 is invariant by some maximal torus $\mathbb{K} \subset \mathrm{Aut}_{\mathrm{red}}(Y)$. Conjugating if necessary, we can assume that $\mathbb{T}_Y \subset \mathbb{K}$. By Theorem 1.5.1, there exists a compatible extremal metric ω_2 in $[\tilde{\omega}]$. By Lemma 1.2.6, ω_2 is $\mathbb{K}$-invariant. Then, by uniqueness of the $\mathbb{K}$-invrariant extremal Kähler metrics modulo $\mathbb{K}^{\mathbb{C}}$ [16, 30, 53], there exists $\gamma \in \mathbb{K}^{\mathbb{C}}$ such that $\omega_1 = \gamma^* \omega_2$. Since $\mathbb{T}_Y \subset \mathbb{K}$, the action of $\mathbb{T}_Y$ on (Y, J_Y, ω_1) is still rigid and semisimple, see Remark 1.2.2. Thus, according to [5], ω_1 is given by the generalized Calabi ansatz.

□

1.6 Toric weighted K-stability

1.6.1 Complex and symplectic points of view

We briefly recall the well-known correspondence between symplectic and Kähler potentials of toric Kähler manifolds [4, 5, 37, 44]. We use the notation and the conventions of [3], which differ in places from those used in [1, 37].

Let $(X, \omega_0, \mathbb{T})$ be a toric symplectic manifold classified by its *labelled integral Delzant polytope* $(P, \mathbf{L})$ [4, 36], where $\mathbf{L} = (L_j)_{j=1,\ldots,d}$ is the collection defining affine-linear functions for

$$P := \{x \in \mathfrak{t}^* \mid L_j(x) \geq 0, \forall j = 1, \ldots, d\},$$

with dL_j being primitive elements of the lattice Λ of circle subgroups of $\mathbb{T}$. Choose a $\mathbb{T}$-invariant ω_0-compatible Kähler structure (g, J) on $(X, \omega_0, \mathbb{T})$ and denote $m_0 : X \longrightarrow \mathfrak{t}^*$ the moment map of $(X, \omega_0, \mathbb{T})$ and by $t_J : X^0 \longrightarrow \mathfrak{t}/2\pi\Lambda$ the angular coordinates (defined modulo an additive constant) depending on the complex structure J (see [5, Remark 3]). The coordinates (m_0, t_J) are symplectic, i.e. ω_0 is given by (1.7). The Kähler structure (g, J) is defined on X^0 by a smooth strictly convex function u on P^0 via

$$g = \langle dm_0, \mathbf{G}, dm_0 \rangle + \langle dt_J, \mathbf{H}, dt_J \rangle \quad \text{and} \quad Jdm_0 = \langle \mathbf{H}, dt_J \rangle, \tag{1.52}$$

where $\mathbf{G} := \mathrm{Hess}(u)$ is a positive definite $S^2\mathfrak{t}$-valued function and $\mathbf{H} := \mathbf{G}^{-1}$ a is $S^2\mathfrak{t}^*$-valued function on P^0 (thus $\mathbf{H} : \mathfrak{t} \longrightarrow \mathfrak{t}^*$ and $\mathbf{G} : \mathfrak{t}^* \longrightarrow \mathfrak{t}$ at each point of P^0) and $\langle \cdot, \cdot, \cdot \rangle$ denote the point wise contraction $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^*$ or its dual. It is shown in [5, Lemma 3] that for two $\mathbb{T}$-invariant Kähler structures on $(X, \omega_0, \mathbb{T})$, given on X^0 by (1.52) with the same matrix $\mathbf{H}$, there exists a $\mathbb{T}$-equivariant Kähler isometry between them.

Conversely, any smooth strictly convex function u on P^0 defines a $\mathbb{T}$-invariant ω_0-compatible Kähler structure on X^0 via (1.52). The following Proposition established in [5] gives a criterion for the Kähler metric on X^0 to extend to X.

Proposition 1.6.1. *Let $(X, \omega_0, \mathbb{T})$ be a compact toric symplectic 2ℓ-manifold with moment map $m_\omega : X \to P$ and u be a smooth strictly convex function on P^0. Then the positive definite $S^2\mathfrak{t}^*$-valued function $\mathbf{H} := \mathrm{Hess}(u)^{-1}$ on P^0 comes from a $\mathbb{T}$-invariant, ω-compatible Kähler metric g via* (1.52) *if and only if $\mathbf{H}$ satisfies the following conditions:*

- [smoothness] $\mathbf{H}$ *is the restriction to P^0 of a smooth $S^2\mathfrak{t}^*$-valued function on P;*
- [boundary values] *for any point y on the codimension one face $F_j \subset P$ with inward normal u_j, we have*

$$\mathbf{H}_y(u_j, \cdot) = 0 \text{ and } (d\mathbf{H})_y(u_j, u_j) = 2u_j, \tag{1.53}$$

  where the differential $d\mathbf{H}$ is viewed as a smooth $S^2\mathfrak{t}^* \otimes \mathfrak{t}$-valued function on P;
- [positivity] *for any point y in the interior of a face $F \subseteq P$, $\mathbf{H}_y(\cdot, \cdot)$ is positive definite when viewed as a smooth function with values in $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$, where $\mathfrak{t}_F \subset \mathfrak{t}$ is the vector subspace spanned by the inward normals u_j in $\mathfrak{t}$ to the codimension one faces F.*

Definition 1.6.2. *Let $\mathcal{S}(P, \mathbf{L})$ be the space of smooth strictly convex functions on the interior of P^0 such that $\mathbf{H} = \mathrm{Hess}(u)^{-1}$ satisfies the conditions of Proposition* 1.6.1.

Remark 1.6.3. *In Proposition 1.6.1 and Definition 1.6.2, one uses as a model metric the Guillemin Kähler metric* (g_0, J_0) *[44], given by* (1.52) *for the symplectic potential*

$$u_0 := \frac{1}{2}\sum_{j=1}^{d} L_j \log L_j,$$

where $L_j, j = 1, \ldots, d$ *are the affine-linear functions defining the polytope. This introduces a discrepancy of a factor* $1/2$ *with respect to the normalization used in [37], which in turn will result in some obvious modification of the formula for the (weighted) Futaki invariant in Section* 1.6.3.

Thus, there exists a bijection between $\mathbb{T}$-equivariant isometry classes of $\mathbb{T}$-invariant ω_0-compatible Kähler structures and smooth $S^2\mathfrak{t}^*$-valued functions $\mathbf{H} = \mathrm{Hess}(u)^{-1}$, where $u \in \mathcal{S}(P, \mathbf{L})$.

We fix the Guillemin Kähler structure J_0 on $(X, \omega_0, \mathbb{T})$. Consider another ω_0-compatible Kähler structure J_u defined by a symplectic potential $u \in S(P, \mathbf{L})$ via (1.52). Donaldson shows [38] that there is a biholomorphism $\Phi_u : (X, J_u) \cong (X, J_0)$. Let $\omega_u := \Phi_u^*(\omega_0)$ and ϕ_u and ϕ_{u_0} be the Legendre transforms of u and u_0, respectively. By [44], we have that

$$\omega_u = \omega_0 + dd^c_{J_0}\varphi_u, \qquad \varphi_u(y_0) := \phi_u(y_0) - \phi_{u_0}(y_0), \tag{1.54}$$

where $y_0 = \nabla u_0$ are the pluriharmonic coordinates with respect to J_0.

Conversely, using the dual Legendre transform, any $\mathbb{T}$-invariant Kähler potential $\varphi \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}$ gives rise to a symplectic potential $u \in \mathcal{S}(P, \mathbf{L})$ through (1.54). A key point of this correspondence is that a path $\varphi_{u_t} \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}$ corresponding $u_t \in \mathcal{S}(P, \mathbf{L})$ satisfies (see [43])

$$\frac{d}{dt}u_t = -\frac{d}{dt}\varphi_t. \tag{1.55}$$

1.6.2 Generalized Abreu's equation

Thanks to Abreu [1], the scalar curvature $Scal(u)$ associated to a symplectic potential $u \in \mathcal{S}(P, \mathbf{L})$ is expressed by

$$Scal(u) = \sum_{i,j=1}^{\ell} -(H^u_{ij})_{,ij}, \tag{1.56}$$

where the partial derivatives of the inverse Hessian $(H^u_{ij}) = \mathrm{Hess}(u)^{-1}$ of u are taken in a fixed basis $\boldsymbol{\xi}^*$ of $\mathbf{t}^*$.

From [53, Sect. 6] and the computation in [11, Sect. 3], the v-scalar curvature associated to a symplectic potential $u \in \mathcal{S}(P, \mathbf{L})$ and a positive weight function v is given by

$$Scal_{\mathrm{v}}(u) = -\sum_{i,j=1}^{\ell} (\mathrm{v}H^u_{ij})_{,ij}. \tag{1.57}$$

Let $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ and $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$. According to Definition 1.1.2, a Kähler structure (J_u, g_u) on $(X, \omega_0, \mathbb{T})$ with respective symplectic potential $u \in \mathcal{S}(P, \mathbf{L})$, is (v, w)-cscK if and only if

$$-\sum_{i,j=1}^{\ell} (\mathrm{v}H^u_{ij})_{,ij} = \mathrm{w}. \tag{1.58}$$

This formula generalizes the expression (1.56) and is referred to as *the generalized Abreu equation.* This equation has been studied for example in [7, 58, 59, 60].

1.6.3 Weighted Donaldson–Futaki invariant

Following [37, 53, 58], for $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ and $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$, we introduce the (v, w)-Donaldson–Futaki invariant

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) := 2\int_{\partial P} f \mathrm{v} d\sigma - \int_P f \mathrm{w} dx, \tag{1.59}$$

for all continuous functions f on P, where $d\sigma$ is the induced measure on each face $F_j \subset \partial P$ by letting $dL_j \wedge d\sigma = -dx$, where dx is the Lesbegue measure on P. We shall consider on this chapter weight functions which satify the following

Convention 1.6.4. *The weights* $\mathrm{v} > 0$ *and* $\mathrm{w} \in C^\infty(P, \mathbb{R})$ *satisfy*

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) = 0, \tag{1.60}$$

for all f *affine-linear on* P*.*

Integration by parts in (1.59) (see e.g. [37]) yields that (1.60) is a necessary condition for the existence of (v, w)-cscK metric on $(X, \omega_0, \mathbb{T})$.

Remark 1.6.5. *Notice that in the case of semisimple principal toric fibrations, the weights given by* (1.15) *satisfy the condition* (1.60)*.*

1.6.4 Weighted Mabuchi energy

The volume form $\omega_0^{[\ell]}$ on X is pushed forward to the measure $(2\pi)^\ell dx$ via the moment map m_0. Seen as functional on $\mathcal{S}(P, \mathbf{L})$ via (1.55), up to multiplication by $(2\pi)^\ell$, the weighted Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ defined in (1.22) is given by

$$d_u\mathcal{M}_{\mathrm{v,w}}(\dot{u}) = \int_P \Big(-\sum_{i,j=1}^{\ell} \big(\mathrm{v}H^u_{ij}\big)_{,ij} - \mathrm{w}\Big)\dot{u}dx.$$

From [11, Lemma 6] (see also Lemma 1.6.6 below) we get

$$d_u\mathcal{M}_{\mathrm{v,w}}(\dot{u}) = \mathcal{F}_{\mathrm{v,w}}(\dot{u}) - \int_P \sum_{i,j=1}^{\ell} \mathrm{v}H^u_{ij}\dot{u}_{,ij}dx,$$

where $\mathcal{F}_{\mathrm{v,w}}$ is the weighted Donaldson–Futaki invariant defined in (1.59). Using that $d\log\det\mathbf{H} = \mathrm{tr}\mathbf{H}^{-1}d\mathbf{H}$, we get (up to a positive multiplicative constant)

$$\mathcal{M}_{\mathrm{v,w}}(u) = \mathcal{F}_{\mathrm{v,w}}(u) - \int_P \log\det\mathrm{Hess}(u)\mathrm{Hess}(u_0)^{-1}\mathrm{v}dx.$$

We denote by $\mathcal{CV}^\infty(P)$ the set of continuous convex functions on P which are smooth in the interior P^0. Using the same argument as in [37, Lemma 3.3.5], and since v is smooth and P is compact, we get:

Lemma 1.6.6. *Let* $\mathbf{H}$ *be a smooth* $S^2\mathfrak{t}^*$*-valued function on* P *which satisfies the boundary conditions* (1.53) *of Proposition 1.6.1 (but not necessarily the positivity condition). For any* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ *and* $f \in \mathcal{CV}^\infty(P)$*:*

$$\int_P \sum_{i,j=1}^{\ell} \big(\mathrm{v}H_{ij}\big) f_{,ij}dx = \int_P \Big(\sum_{i,j=1}^{\ell} \big(\mathrm{v}H_{ij}\big)_{,ij}\Big) f dx + 2\int_{\partial P} f\mathrm{v}d\sigma. \tag{1.61}$$

In particular, $\int_P \sum_{i,j=1}^{\ell} \big(\mathrm{v}H_{ij}\big) f_{,ij}dx < \infty$.

The following result and proof are generalizations of [37, Proposition 3.3.4].

Proposition 1.6.7. *Let* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ *and* $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$*. The Mabuchi energy* $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ *extends to the set* $\mathcal{CV}^\infty(P)$ *as functional with values in* $(-\infty, +\infty]$*. Moreover, if there exists* $u \in \mathcal{S}(P, \mathbf{L})$ *corresponding to a* (v, w)*-cscK metric, i.e. which satisfies* (1.58)*, then* u *realizes the minimum of* $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ *on* $\mathcal{CV}^\infty(P)$*.*

Proof. The linear term $\mathcal{F}_{\mathrm{v},\mathrm{w}}$ is well-defined on $\mathcal{CV}^\infty(P)$. We then focus on the non-linear term of $\mathcal{M}_{\mathrm{v},\mathrm{w}}$. Let $u \in \mathcal{S}(P, \mathbf{L})$ and $h \in \mathcal{CV}^\infty(P)$. Suppose $\det \mathrm{Hess}(h) \neq 0$. By convexity of the functional $-\log\det$ on the space of positive definite matrices, we get:

$$-\log\det \mathrm{Hess}(h) + \log\det \mathrm{Hess}(u) \geq -\mathrm{Tr}\big(\mathrm{Hess}(u)^{-1}\mathrm{Hess}(f)\big),$$

where $f = h - u$. Turning this around and multiplying by v, we obtain:

$$\mathrm{v}\log\det \mathrm{Hess}(h) \leq \mathrm{v}\log\det \mathrm{Hess}(u) + \mathrm{v}\mathrm{Tr}\big(\mathrm{Hess}(u)^{-1}\mathrm{Hess}(f)\big).$$

By linearity of (1.61), the equality still holds when we replace f by a difference of two functions in $\mathcal{CV}^\infty(P)$. In particular, this shows that $\mathrm{v}\mathrm{Tr}\big(\mathrm{Hess}(u)^{-1}\mathrm{Hess}(f)\big)$ is integrable on P and hence, by the previous inequality, $\mathrm{v}\log\det \mathrm{Hess}(h)$ is integrable too. Now if the determinant of $\mathrm{Hess}(h)$ is equal to 0, we define the value of $\mathcal{M}_{\mathrm{v},\mathrm{w}}(h)$ to be $+\infty$. Then $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ is well-defined on $\mathcal{CV}^\infty(P)$. Suppose u satisfies (1.58). If $\det \mathrm{Hess}(f) = 0$, then we trivially get $\mathcal{M}_{\mathrm{v},\mathrm{w}}(u) \leq \mathcal{M}_{\mathrm{v},\mathrm{w}}(f)$. Now, suppose that $\det \mathrm{Hess}(f) \neq 0$. The function $g(t) := \mathcal{M}_{\mathrm{v},\mathrm{w}}(u + tf)$ is therefore a convex function which is differentiable at $t = 0$ with

$$g'(0) = -\int_P \left(\sum_{i,j=1}^{\ell} (\mathrm{v}H^u_{ij})_{,ij} - \mathrm{w}\right) f dx = 0,$$

by hypothesis on u. Then $\mathcal{M}_{\mathrm{v,w}}(u) \leq \mathcal{M}_{\mathrm{v,w}}(f)$ by the convexity of g.

□

1.6.5 Properness and (v, w)-uniform K-stability

Following [37, 71] (see also [3, Chapter 3.6]), we fix $x_0 \in P^0$ and consider the following normalization

$$\mathcal{CV}^\infty_*(P) := \{f \in \mathcal{CV}^\infty(P) \mid f(x) \geq f(x_0) = 0, \forall x \in P\}. \tag{1.62}$$

Then, any $f \in \mathcal{CV}^\infty(P)$ can be written uniquely as $f = f^* + f_0$, where f_0 is affine-linear and $f^* \in \mathcal{CV}^\infty_*(P)$.

Definition 1.6.8. *A Delzant polytope $(P, \mathbf{L})$ is (v, w)-uniformly K-stable if there exists $\lambda > 0$ such that*

$$\mathcal{F}_{\mathrm{v,w}}(f) \geq \lambda \|f^*\|_1 \tag{1.63}$$

for all $f \in \mathcal{CV}^\infty(P)$, where $\|\cdot\|_1$ denotes the L^1-norm on P.

Proposition 1.6.9. *Suppose $(P, \mathbf{L})$ is (v, w)-uniformly K-stable. Then there exists $C > 0$ and $D \in \mathbb{R}$ such that*

$$\mathcal{M}_{\mathrm{v,w}}(u) \geq C\|u^*\|_1 + D \tag{1.64}$$

for all $u \in \mathcal{S}(P, \mathbf{L})$.

Proof. This result in the case when $\mathrm{v} = 1$ is due to [37, 81]. The proof is an adaptation of the proof in [3].

Let $u_0 \in \mathcal{S}(P, \mathbf{L})$ be the Guillemin Kähler potential. We consider $\mathcal{F}_{\mathrm{v},\mathrm{w}_0}$ given by (1.59) where $\mathrm{w}_0 := Scal_{\mathrm{v}}(u_0)$. For any $f \in \mathcal{CV}^{\infty}_{*}(P)$ there exists $C > 0$ such that

$$|\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) - \mathcal{F}_{\mathrm{v},\mathrm{w}_0}(f)| \leq 2C\|f\|_1.$$

Since $(P, \mathbf{L})$ is (v, w)-uniformly K-stable we get

$$|\mathcal{F}_{\mathrm{v},\mathrm{w}_0}(f) - \mathcal{F}_{\mathrm{v},\mathrm{w}}(f)| \leq C_1\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) - C\|f\|_1,$$

where C_1 is a positive constant depending (1.63). We deduce that

$$\mathcal{F}_{\mathrm{v},\mathrm{w}_0}(f) \leq \tilde{C}\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) - C\|f\|_1, \tag{1.65}$$

where $\tilde{C} := C_1 + 1$. By Proposition 1.6.7, the Mabuchi energy extends to $\mathcal{CV}^{\infty}_{*}(P)$. Then, by (1.65) and (1.60), for any $u \in \mathcal{S}(P, \mathbf{L})$,

$$\begin{aligned}
\mathcal{M}_{\mathrm{v},\mathrm{w}}(u) &= \mathcal{F}_{\mathrm{v},\mathrm{w}}(u^*) - \int_P \mathrm{v} \log\det \mathrm{Hess}(u^*)\mathrm{Hess}(u_0)^{-1} dx \\
&\geq \tilde{C}\mathcal{F}_{\mathrm{v},\mathrm{w}_0}(u^*) + C\|u^*\|_1 - \int_P \mathrm{v} \log\det \mathrm{Hess}(u^*)\mathrm{Hess}(u_0)^{-1} dx \\
&= \mathcal{M}_{\mathrm{v},\mathrm{w}_0}(\tilde{C}u^*) + \int_P \mathrm{v} \log\det \mathrm{Hess}(\tilde{C}u^*)\mathrm{Hess}(u^*)^{-1} dx + C\|u^*\|_1 \\
&= \mathcal{M}_{\mathrm{v},\mathrm{w}_0}(\tilde{C}u^*) + n \log \tilde{C} \int_P \mathrm{v} dx + C\|u^*\|_1.
\end{aligned}$$

The Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}_0}$ reaches its minimum at the potential $u_0 \in \mathcal{S}(P, \mathbf{L})$, as u_0 is solution of

$$Scal_{\mathrm{v}}(u_0) = \mathrm{w}_0.$$

In particular, $\mathcal{M}_{\mathrm{v},\mathrm{w}_0}$ is bounded from below on $\mathcal{CV}^{\infty}(P)$ according to Proposition 1.6.7. Letting $D := \inf_{\mathcal{CV}^{\infty}} \mathcal{M}_{\mathrm{v},\mathrm{w}_0} + n \log \tilde{C} \int_P \mathrm{v} dx$ we get the result.

□

1.6.6 Existence of a (v, w)-cscK metric is equivalent to (v, w)-uniform K-stability

The following result is established in [60, Theorem 2.1] and in [22] in the case $\mathrm{v} = 1$. The arguments adapts to the weighted case in a straightforward way.

Proposition 1.6.10. *Suppose there exists an* (v, w)*-cscK metric in* $(X, [\omega_0], \mathbb{T})$*, i.e.* (1.58) *admits a solution* $u \in \mathcal{S}(P, \mathbf{L})$*. Then* P *is* (v, w)*-uniformly K-stable.*

We now focus on the converse. We consider the space of normalized Kähler potentials (1.25) and normalized symplectic potentials

$$\mathring{\mathcal{S}}_{\mathrm{v}}(P, \mathbf{L}) := \{u \in \mathcal{S}(P, \mathbf{L}) \mid \int_P u \mathrm{v} dx = \int_P u_0 \mathrm{v} dx\}. \tag{1.66}$$

Lemma 1.6.11. *For any symplectic potential* $\mathring{u}_t \in \mathring{\mathcal{S}}_{\mathrm{v}}(P, \mathbf{L})$*, the corresponding Kähler potential* $\varphi_t = \varphi_{\mathring{u}_t}$ *obtained via* (1.54) *belongs to the space of normalized Kähler potential* $\mathring{\mathcal{K}}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}}$ *defined in* (1.25)*. Conversely, any path in* $\mathring{\mathcal{K}}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}}$ *comes from a path* $\mathring{u}_t$ *in* $\mathring{\mathcal{S}}_{\mathrm{v}}(P, \mathbf{L})$*.*

Proof. By definition of $\mathcal{I}_{\mathrm{v}}$ (1.23), a path $\varphi_t \in \mathcal{K}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}}$ starting from 0 belongs to $\mathring{\mathcal{K}}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}}$ if and only if

$$\int_X \dot{\varphi}_t \mathrm{v}(m_{\varphi_t}) \omega_t^{[\ell]} = 0, \tag{1.67}$$

for all $\dot{\varphi}_t \in T_{\varphi_t}\mathcal{K}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}}$. By pushing-forward the measure $\omega_t^{[\ell]}$ via $m_{\omega_{\varphi_t}}$ and using (1.55) we get that (1.67) is equivalent to

$$\int_P \dot{u}_t \mathrm{v} dx = 0,$$

where u_t is the path corresponding to φ_t via (1.54). The conclusion follows from the convexity of $\mathcal{S}(P, \mathbf{L})$. □

Theorem 1.6.12. *Let $(Y, J_Y, \tilde{\omega}_0, \mathbb{T})$ be a semisimple principal toric fibration with toric Kähler fiber $(X, J_X, \omega_0, \mathbb{T})$. Let (v, w) be the weights defined in (1.15) and denote by P the Delzant polytope associated to $(X, \omega_0, \mathbb{T})$. Then there exists a (v, w)-weighted cscK metric in $[\omega_0]$ if and only if P is (v, w)-uniformly K-stable. In particular, the latter condition is necessary and sufficient for $[\tilde{\omega}_0]$ to admit an extremal Kähler metric.*

Proof. Suppose there exists a (v, w)-cscK metric in $[\omega_0]$. By Proposition 1.6.10, P is (v, w)-uniformly K-stable.

Conversely, suppose that P is (v, w)-uniformly K-stable. We are going to show that there are uniform positive constants $\tilde{A}$ and $\tilde{B}$ such that

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}(\varphi) \geq \tilde{A} \inf_{\gamma \in \mathbb{T}^{\mathbb{C}}} d_{1,\mathrm{v}}^X(0, \gamma \cdot \varphi) - \tilde{B}, \tag{1.68}$$

where $d_{1,\mathrm{v}}^X$ is defined in Lemma 1.3.3 and $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ is the weighted Mabuchi energy of the Kähler toric fiber $(X, J_X, [\omega_0], \mathbb{T})$, see (1.22). For all $\varphi \in \mathring{\mathcal{K}}(X,\omega_0)^{\mathbb{T}}$, there exists $\gamma \in \mathbb{T}^{\mathbb{C}}$ such that the symplectic potential $u_{\gamma\cdot\varphi}$ corresponding to $\gamma\cdot\varphi$ satisfies $d_{x_0}u_{\gamma\cdot\varphi} = 0$. By Lemma 1.6.11 and the inequality in [3, (66)], we have

$$d_{1,\mathbb{T}^{\mathbb{C}}}^X(0, \gamma\cdot\varphi) \leq A\int_P |u_\varphi^* - u_0^*|dx \leq A\|u_\varphi^*\|_1 + B, \qquad (1.69)$$

for some uniform constants $A > 0$ and $B > 0$, where $d_{1,\mathbb{T}^{\mathbb{C}}}^X$ is the d_1 distance relative to $\mathbb{T}^{\mathbb{C}}$ (see (1.38)) on $\mathcal{K}(X,\omega_0)^{\mathbb{T}}$, $u_\varphi \in S(P,\mathbf{L})$ is the symplectic potential corresponding to φ and u_φ^* is its normalization in $S(P,\mathbf{L}) \cap \mathcal{CV}_*^\infty(P)$, see (1.62). Since $\mathrm{v} > 0$ on P, we have for the weighted distance $d_{1,\mathrm{v}}^X \leq C d_1^X$. Then (1.68) follows from (1.69) and Proposition 1.6.9.

Let $\mathbb{K}$ be a maximal torus in $\mathrm{Aut}_{\mathrm{red}}(Y)$ containing $\mathbb{T}_Y$ and satisfying (1.14). By Lemma 1.3.4 and our choice of normalization (1.25), the Mabuchi energy $\mathcal{M}^{\mathbb{K}}$ restricted to $\mathring{\mathcal{K}}_{\mathrm{v}}(X,\omega_0)^{\mathbb{T}}$ is $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ (up to a positive multiplicative constant). We denote by $d_{1,\mathbb{K}^{\mathbb{C}}}$ the d_1 distance relative to $\mathbb{K}^{\mathbb{C}}$ on $\mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{K}}$ (see (1.38)). Since any $\mathbb{T}_Y^{\mathbb{C}}$-orbit lies in a $\mathbb{K}^{\mathbb{C}}$-orbit, by (1.68) and Lemma 1.3.3, $\mathcal{M}^{\mathbb{K}}$ is $d_{1,\mathbb{K}^{\mathbb{C}}}$-proper on $\mathring{\mathcal{K}}_{\mathrm{v}}(X,\omega_0)^{\mathbb{T}}$ in the sense of Definition 1.4.1.

In the proof of (1) $\Rightarrow$ (2) in Theorem 1.5.1, we have used the $d_{1,\mathbb{K}^{\mathbb{C}}}$-properness of the Mabuchi energy $\mathcal{M}^{\mathbb{K}}$ only on sequences included in $\mathcal{K}(X,\omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{K}}$. This allows us to obtain the existence of a (v,w)-cscK metric by the same argument.

The last assertion follows from Theorem 1.5.1. $\square$

1.7 Applications

1.7.1 Almost Kähler metrics

An almost Kähler metric is a Riemannian metric defined by a symplectic form and a compatible almost complex structure which is not necessarily integrable. As observed in [37], for fixed angular coordinates dt_0 with respect to a reference Kähler structure J_0, one can use (1.52) to define $\mathbb{T}$-invariant *almost Kähler* metrics on X, as soon as $\mathbf{H}$ satisfies the smoothness, boundary value and positivity conditions of Proposition 1.6.1 (even if the inverse matrix $\mathbf{G} := \mathbf{H}^{-1}$ is not the Hessian of a smooth function). We shall refer to such almost Kähler metrics as *involutive.* One can further use involutive almost Kähler metrics on X to build a compatible metric $\tilde{g}_{\mathbf{H}}$ on Y by the formula

$$\tilde{g}_{\mathbf{H}} = \sum_{a=1}^{k} \big(\langle p_a, m\rangle + c_a\big) g_a + \langle dm, \mathbf{G}, dm\rangle + \langle \theta, \mathbf{H}, \theta\rangle.$$

It is shown in [7] that $\tilde{g}_{\mathbf{H}}$ is an extremal almost Kähler on Y in the sense of [56] (i.e. its hermitian scalar curvature is a Killing potential) if and only if $\mathbf{H}$ satisfies the equation

$$-\sum_{i,j} (\mathrm{v} H_{ij})_{,ij} = \mathrm{w}, \tag{1.70}$$

for (v, w) defined in (1.15). We shall more generally consider involutive almost Kähler metrics on X satisfying the equation (1.70) for weight functions $\mathrm{v} > 0$ and w. For such almost Kähler metrics we say that $(X, J_{\mathbf{H}}, , g_{\mathbf{H}}, \omega, \mathbb{T})$ is an *involutive* (v, w)-*csc almost Kähler metric.*

The motivation for considering involutive (v, w)-csc almost Kähler metrics come from the fact that (1.70) is a linear indeterminate PDE for the smooth coefficients of $\mathbf{H}$ (which would therefore admit infinitely many solutions if we drop the positivity assumption of $\mathbf{H}$), which in some special cases is easier to solve explicitly, as demonstrated in [7]. On the other hand, it was observed in [37] (see [7] for the weighted case) that the existence of a (v, w)-csc almost Kähler metric implies that $\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) \geq 0$ with equality iff $f^* = 0$, and it was conjectured that the existence of a (v, w)-csc almost Kähler metric is equivalent to the existence of a (v, w)-cscK metric on $(X, \omega, \mathbb{T})$. Legendre [55] observed that the existence of an involutive extremal almost Kähler metric implies the stronger yet uniform stability of P via the result of [25, 26] and [47], and thus confirmed the conjecture in the case where $\mathrm{v} = 1$ and $\mathrm{w} = l_{\mathrm{ext}}$, see Section 1.2.3. Our additional observation is that arguments similar to the ones in the proof of Proposition 1.6.10 show the following.

Proposition 1.7.1. *Let $(X, \omega, \mathbb{T})$ be a toric manifold associated to Delzant polytope P. Let $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ and $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$ be such that (1.60) is satisfied . Suppose there exists an involutive (v, w)-csc almost Kähler metric on $(X, \omega, \mathbb{T})$, i.e. there exists $\mathbf{H}$ satisfying the smoothness, boundary value and positivity conditions of Proposition 1.6.1 and equation (1.70). Then P is (v, w)-uniformly K-stable.*

Combining this result (for the special weights (v, w) associated to a semisimple principal torus bundle via (1.15)) with Theorem 1.6.12, we deduce:

Proposition 1.7.2. *Let $(X, \omega, \mathbb{T})$ be a toric manifold associated to Delzant polytope P. Let (v, w) be the weights defined in (1.15). Then the following statements are equivalent.*

1. *There exists a (v, w)-cscK metric on $(X, \omega, \mathbb{T})$.*

2. *There exists an involutive* (v,w)*-csc almost Kähler metric on* $(X,\omega,\mathbb{T})$.

3. P *is* (v,w)*-uniformly K-stable in the sense of Definition 1.6.8.*

1.7.2 Proof of Corollary 0.2.2

Let $\Sigma_{\mathbf{g}}$ be a compact complex curve of genus $\mathbf{g}$ and $L_i \longrightarrow \mathbf{g}$ a holomorphic line bundle, $i = 0,1,2$. We consider $(Y, J_Y) := \mathbb{P}(L_0 \oplus L_1 \oplus L_2) \to \Sigma_{\mathbf{g}}$. Since (Y, J_Y) is invariant by tensoring $L_0 \oplus L_1 \oplus L_2$ with a line bundle, we can suppose without loss of generality that $(Y, J_Y) = \mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2)$, where $\mathcal{O} \longrightarrow S$ is the trivial line bundle and $\boldsymbol{p}_i := -deg\big(L_i\big) \geq 0$, $i = 1,2$. Suppose $\boldsymbol{p}_1 = \boldsymbol{p}_2 = 0$. Then by a result of Fujiki [40] (see also [7, Remark 2]) (Y, J) admits an extremal metric in every Kähler class. If $\boldsymbol{p}_2 = \boldsymbol{p}_1 > 0$ or $\boldsymbol{p}_2 > \boldsymbol{p}_1 = 0$ and $\mathbf{g} = 0,1$, there exists an extremal metric in every Kähler class by [6, Theorem 6]. We then suppose $\boldsymbol{p}_2 > \boldsymbol{p}_1 > 0$.

Suppose $\mathbf{g} = 0$, i.e. $(B, J_B) = \mathbb{CP}^1$. Then Y is a toric variety. Using the existence of an extremal almost Kähler metric of involutive type compatible with any Kähler metric on Y (see [7, Proposition 4]) and the Yau–Tian–Donaldson correspondence on toric manifold, it is shown in [55] that there exists an extremal Kähler metric in every Kähler class of $\mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2) \longrightarrow \mathbb{CP}^1$. Observe that by applying Proposition 1.7.2 and Theorem 1.5.1 we obtain that these extremal metrics are compatible, i.e. are of the form of (1.9).

Now suppose that $\Sigma_{\mathbf{g}}$ in an elliptic curve, i.e. $\mathbf{g} = 1$. The complex manifold (Y, J) is not toric. However, it is shown in [7] that (Y, J_Y) is a semisimple principal toric fibration. By the Leray–Hirsch Theorem $H^2(Y, \mathbb{R})$ is of dimension 2. In particular, up to scaling, any Kähler class on (Y, J_Y) is compatible. It is shown in [7, Proposition 4] that Y admits an extremal almost Kähler metric in any compatible Kähler class. Using Proposition 1.7.2 and Theorem 1.6.12, we conclude that there exists an extremal Kähler metric in every Kähler class. Furthermore, by Theorem 1.5.1, the extremal Kähler metrics are of the form of (1.9), i.e. are given by the generalized Calabi ansatz.

Remark 1.7.3. *If* $\mathbf{g} \geq 2$, *it is shown in [7, Theorem 2] that there exists an extremal Kähler metric in sufficiently small compatible Kähler classes. On the other hand, by [7, Proposition 2], if* $\mathbf{g} > 2$ *and* $\boldsymbol{p}_1, \boldsymbol{p}_2$ *satisfying* $2(\mathbf{g}-1) > \boldsymbol{p}_1 + \boldsymbol{p}_2$*, there is no extremal Kähler metric in sufficiently big Kähler classes.*

CHAPTER II

AN EFFECTIVE WEIGHTED K-STABILITY CONDITION FOR POLYTOPES AND SEMISIMPLE PRINCIPAL TORIC FIBRATIONS (BASED ON [35])

2.1 Weighted K-stability of labelled polytopes: a sufficient condition

2.1.1 Weighted K-stability of labelled polytopes

Let V be an affine space of dimension ℓ, equipped with a fixed Lebesgue measure dx. A *labelled polytope* in V is a pair $(P, \mathbf{L})$ where P is a (compact, convex, simple) polytope in V and $\mathbf{L} = (L_j)_{j=1}^d$ is a minimal set of defining affine functions for P, that is,

$$P = \{x \in V \mid \forall j, \quad L_j(x) \geq 0\}$$

where d is the number of facets (codimension one faces) of P. We denote by $F_j := \{x \in P \mid L_j(x) = 0\}$ the facet of P defined by L_j.

Definition 2.1.1. *The* labelled boundary measure $d\sigma$ *is the measure on* ∂P *whose restriction to the facet* F_j *is defined by* $dL_j \wedge d\sigma = -dx$.

Note that the labelled boundary measure $d\sigma$ depends on the choice of labelling $\mathbf{L} = (L_j)$. For example, for any tuple (r_j) of positive real numbers, the $(L'_j) =$

(r_jL_j) is another labelling of P. The associated labelled boundary measure $d\sigma'$ satisfies $d\sigma' = \frac{1}{r_j}d\sigma$ on F_j.

Following [48, 68], we introduce

$$\|f\|_J = \inf_{l\in \mathrm{Aff}(V)} \int_P \Big(f + l - \inf_P(f+l)\Big)\, dx,$$

for any conitinuous functions f on P, where $\mathrm{Aff}(V)$ denotes the space of affine functions on V. Recall that a labelled simple compact convex polytope $(P, \mathbf{L})$ (non-necessarily Delzant) is (v, w)-uniformly K-stable if it satisfies the condition of Definition 1.6.8. The latter depends on the choice of a point $x_0 \in P^0$. We recall that $\mathcal{CV}^\infty(P)$ denotes the set of continuous convex functions on P which are smooth in the interior P^0. Also, for $f \in \mathcal{C}^\infty(P)$, f^* denotes the projection of f to the space of normalized functions in $\mathcal{C}^\infty(P)$ such that $f(x) \geq f(x_0) = 0$ for all $x \in P$.

Lemma 2.1.2. *The labelled polytope* $(P, \boldsymbol{L})$ *is* (v, w)*-uniformly K-stable in the sens of Definition 1.6.8 if and only if there exists* $\lambda > 0$ *such that for all* $f \in \mathcal{CV}^\infty(P)$*,*

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) \geq \lambda \|f\|_J \tag{2.1}$$

where $\mathcal{F}_{\mathrm{v},\mathrm{w}}$ *is introduced in* (1.59)*.*

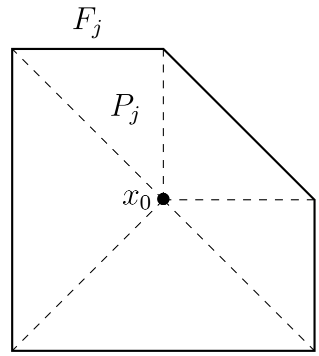
Proof. From [68, Proposition 5.4.1 (3)], there exists a constant $C_1 > 0$ such that for all continuous convex functions on P,

$$\|f\|_J \leq \|f^*\|_{L^1} \leq C_1 \|f\|_J$$

The conclusion follows immediately. □

Remark 2.1.3. *The condition of Definition 1.6.8 is the one that we will effectively use in the sequel. Lemma 2.1.2 show that the notion of uniform K-stability*

Figure 2.1 The cone decomposition



introduced in Definition 1.6.8 is independent of the choice of x_0*. In the more familiar unweighted case, the equivalence between various notions of K-stability of polytopes was shown by Nitta and Saito [68].*

2.1.2 A sufficient condition

Recall that F_j denotes the facet of P defined by $L_j = 0$. For each j, let P_j be the cone with base F_j and vertex x_0 as illustrated in Figure 2.1.2. For a function $f \in \mathcal{C}^\infty(P, \mathbb{R})$, we denote by $d_x f$ its differential at $x \in P$. The following is the main technical result of this chapter. It is inspired by the proof by Zhou and Zhu [81] of a coercivity criterion for the modified Mabuchi functional on toric manifolds.

Theorem 2.1.4. *Let* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ *and let* $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$*. Assume that* $\mathcal{F}_{\mathrm{v},\mathrm{w}}$ *satisfies* (1.60) *and that for all* $j = 1, \ldots, d$ *and for all* $x \in P_j$,

$$\frac{1}{L_j(x_0)} \left(\mathrm{v}(x)(\ell + 1) + d_x \mathrm{v}(x - x_0) \right) - \frac{\mathrm{w}(x)}{2} \geq 0. \tag{2.2}$$

Then $(P, \boldsymbol{L})$ *is* (v, w)*-uniformly K-stable.*

Proof. Since $L_j(x) = 0$ for $x \in F_j$, we have $L_j(x_0) = d_x L_j(x_0 - x)$ for all $x \in F_j$.

In particular,

$$\int_{F_j} f \mathrm{v} d\sigma = \int_{F_j} f \mathrm{v} \frac{-d_x L_j(x-x_0)}{L_j(x_0)} d\sigma.$$

For each facet F of ∂P_j different from F_j, and $x \in F$, the interior product $\iota_{x-x_0}(dx)$ vanishes since it vanishes on the affine space spanned by F. If we further use that $-dL_j \wedge d\sigma = dx$ on F_j, we obtain

$$\int_{F_j} f \mathrm{v} d\sigma = \frac{1}{L_j(x_0)} \int_{\partial P_j} f \mathrm{v} \iota_{x-x_0}(dx).$$

Hence by Stokes theorem we obtain

$$\int_{F_j} f \mathrm{v} d\sigma = \frac{1}{L_j(x_0)} \int_{P_j} \left(\mathrm{v} d_x f(x-x_0) + \ell f \mathrm{v} + f d_x \mathrm{v}(x-x_0) \right) dx.$$

Summing the previous identities over j we get

$$\begin{aligned} \mathcal{F}_{\mathrm{v},\mathrm{w}}(f) = & \sum_{j=1}^{d} \frac{2}{L_j(x_0)} \int_{P_j} \left(d_x f(x-x_0) - f \right) \mathrm{v} dx \\ & + \sum_{j=1}^{d} \int_{P_j} \left(\frac{2}{L_j(x_0)} \left((\ell+1)\mathrm{v} + d_x \mathrm{v}(x-x_0) \right) - \mathrm{w} \right) f dx. \end{aligned} \tag{2.3}$$

Assume condition (2.2) is satisfied and $(P, \mathbf{L})$ is not (v, w)-uniformly K-stable. We will show a stronger condition than condition (1.63). Namely, assume that condition (2.2) is satisfied and that there exists a sequence of $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{CV}^\infty(P)$ such that

$$\lim_{k\to\infty} \mathcal{F}_{\mathrm{v},\mathrm{w}}(f_k^*) = 0 \qquad \text{and} \qquad \int_{\partial P} \mathrm{v} f_k^* \, d\sigma = 1, \tag{2.4}$$

$\forall k \in \mathbb{N}$. Recall from [37, Lemma 5.1.3] (see [68, Proposition 5.1.2] for a detailed proof and explicit constant C) that there exists a positive constant $C > 0$ such that for all $f \in \mathcal{CV}^\infty(P)$,

$$\int_{\partial P} f^* \, d\sigma \geq C \|f^*\|_{L^1}.$$

As a consequence, since $\mathrm{v} > 0$ on P, there exists a constant $C' > 0$ such that for all $f \in \mathcal{CV}^\infty(P)$,

$$\int_{\partial P} \mathrm{v} f^* \, d\sigma \geq C' \|f^*\|_{L^1}.$$

In particular, the sequence $\{f_k^*\}_{k\in\mathbb{N}}$ has bounded L^1 norm. By [37, Corollary 5.2.5], $\{f_k^*\}_{k\in\mathbb{N}}$ converges (up to a sub-sequence, still denoted by $\{f_k^*\}_{k\in\mathbb{N}}$) locally uniformly in P^0 to a convex function f_∞^* which still satisfies $\inf f_\infty^* = f_\infty^*(x_0) = 0$. Since in addition all f_k^* are smooth and convex we have $d_x f_k^*(x - x_0) - f_k^*(x) \geq 0$. Then, since condition (2.2) is assumed to hold, all terms of the sum in (2.3) are non-negative. Evaluating (2.3) at f_k^* and passing to the limit reveals that $\lim_{k\to\infty} d_x f_k^*(x - x_0) - f_k^*(x) = 0$ almost everywhere in P^0, showing that f_∞^* is affine on P^0. Using again $\inf f_\infty^* = f_\infty^*(x_0) = 0$, we conclude that f_∞^* is the zero function on P^0.

The local uniform convergence of $\{f_k^*\}$ to the zero function shows that

$$\lim_{k\to\infty} \int_P f_k^* \mathrm{w}\, dx = 0$$

that is,

$$\lim_{k\to\infty} 2\int_{\partial P} f_k^* \mathrm{v}\, d\sigma - \mathcal{F}_{\mathrm{v},\mathrm{w}}(f_k^*) = 0$$

which is in contradiction with condition (2.4).

From this contradiction it follows that there exists a constant $m > 0$ such that for all $f \in \mathcal{CV}^\infty(P)$,

$$\begin{aligned}\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) &\geq m \int_{\partial P} \mathrm{v} f^*\, d\sigma \\ &\geq mC' \|f^*\|_{L^1}\end{aligned}$$

which concludes the proof. □

Remark 2.1.5. *We stress that the property of* (v, w)*-uniform K-stability is independent of the choice of* $x_0 \in P^0$ *by Lemma 2.1.2, but the condition* (2.2) depends *on that choice. It is possible and useful in practice to vary* x_0 *according to the data of the problem.*

Remark 2.1.6. *Condition* (2.2) *depends continuously on the labelled polytope, the weights* v *and* w*, and the choice of* x_0*.*

2.1.3 Monotone polytopes

Let us recall the terminology of monotone polytopes, used in [55].

Definition 2.1.7. *A labelled polytope* $(P, \boldsymbol{L})$ *is* monotone *if there exists an* $x_0 \in P^0$ *such that* $L_1(x_0) = L_2(x_0) = \cdots = L_d(x_0)$.

There is thus a natural choice of x_0 in this case and our sufficient condition indeed becomes much simpler.

Corollary 2.1.8. *Let* $(P, \boldsymbol{L})$ *be a monotone labelled polytope satisfying* $L_1(x_0) = L_2(x_0) = \cdots = L_d(x_0) = t$. *Let* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R})$ *such that* v *is positive on* P *and let* $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$. *Assume that* $\mathcal{F}_{\mathrm{v},\mathrm{w}}$ *satisfies the condition* (1.60) *and that for all* $x \in P$,

$$\frac{1}{t}\left(\mathrm{v}(x)(\ell+1) + d_x\mathrm{v}(x - x_0)\right) - \frac{\mathrm{w}(x)}{2} \geq 0. \tag{2.5}$$

Then $(P, \boldsymbol{L})$ *is* (v, w)*-uniformly K-stable.*

The inequalities involved form a finite set of conditions to check, contrary to the definition of (v, w)-uniform K-stability. It is furthermore easy to implement the condition in a computer program, *via* formal or numerical computations depending on the data $(P, \mathbf{L}, \mathrm{v}, \mathrm{w})$. The same is true for the more general Theorem 2.1.4, but the decomposition in cones makes it a bit more tedious.

2.2 Geometric applications

2.2.1 Weighted cscK toric manifolds

The results from Section 2.1 are motivated by the study of existence of weighted cscK metrics on toric manifolds shown in Theorem 1.6.12.

We consider a toric Kähler manifold $(X, \omega, \mathbb{T})$ with labelled Delzant polytope $(P, \mathbf{L})$ as introduced in Section 1.6.

Remark 2.2.1. *We focus here on smooth manifolds, but let us mention that the cases of orbifolds would also be natural settings to consider. In these situations, the labelling could be more general (see [1]), thus justifying our choice to allow arbitrary labellings in the previous section.*

In general, no Yau–Tian–Donaldson correspondence is proved so far for the existence of weighted cscK metrics on toric manifolds (see (1.58)). However, by analogy with the unweighted cscK case, there is a known candidate for the corresponding K-stability condition, namely the (v, w)-uniform K-stability of $(P, \mathbf{L})$ introduced in Definition 1.6.8. In fact, the direction showing that existence of weighted cscK metrics implies weighted K-stability is proved by Li–Lian–Sheng [60].

Theorem 2.2.2 ([60, Theorem 2.1])**.** *If ω is a (v, w)-cscK metric, then $(P, \mathbf{L})$ is (v, w)-uniformly K-stable.*

The converse direction is in general much harder, but is known for special choices of weights.

- If v and w are constants, this is the uniform YTD conjecture for cscK metrics on toric manifolds. If v is constant and w is affine-linear, this is the uniform YTD conjecture for extremal metrics on toric manifolds. Both these conjectures were proved recently [22, 37, 48, 61, 68] thanks to the breakthrough of Chen–Cheng [23, 26], its adaptation by He to the extremal setting [47], and earlier works, notably [37, 81].

- If only v is constant, the converse of Theorem 2.2.2 is know for all $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$ by [60].

- For v-solitons on Fano toric manifolds, which correspond to choosing an arbitrary weight $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ and $\mathrm{w}(x) = 2(\ell \mathrm{v}(x) + d_x\mathrm{v}(x))$ (see [9, Proposition 1]), it was proved in [15] that the converse of Theorem 2.2.2 holds for general weight v, and much earlier in [79] for the weight corresponding to Kähler-Ricci solitons. We note that [62] established a general uniform Yau–Tian–Donaldson correspondance for v-solitons on a Fano manifold. We refer to Section 2.2.5 for a discussion of v-solitons on semisimple principal toric fibrations.

- The converse of Theorem 2.2.2 is established in Theorem 1.6.12 for weights corresponding to the extremal Kähler problem on semisimple principal toric fibrations.

2.2.2 The general statement

We use the notations of Section 1.2 for semisimple principal toric fibrations and the notations of Section 2.1 for the cone decomposition of the polytope $P = \bigcup_j P_j$.

Corollary 2.2.3 (of Theorem 2.1.4). *The semisimple principal toric fibration (Y, ω_Y) admits an extremal Kähler metric in $[\omega_Y]$ if there exists an $x_0 \in P^0$ and corresponding cone decomposition $P = \bigcup_j P_j$ such that for all j and for all $x \in P_j$*

$$\frac{1}{L_j(x_0)}\left(\ell + 1 + \sum_{a=1}^{k} \frac{n_a p_a(x - x_0)}{p_a(x) + c_a}\right) - \frac{1}{2}\left(l_{\mathrm{ext}}(x) - \sum_{a=1}^{k} \frac{s_a}{p_a(x) + c_a}\right) \geq 0.$$

Proof. By Theorem 1.5.1, the sufficient condition of Theorem 2.1.4 translates to a sufficient condition for the existence of extremal Kähler metrics on Y. To obtain the statement above, it suffices to note that for the weight v in (1.15), we have

$$d_x\mathrm{v}(y) = \left(\sum_{a=1}^{k} \frac{n_a p_a(y)}{p_a(x) + c_a}\right) \mathrm{v}(x).$$

so that we can factor by $\mathrm{v}(x)$ in the condition in Theorem 2.1.4, which is positive everywhere. □

2.2.3 Fibrations with Fano fiber

We now turn to the semisimple principal toric fibrations with Fano fiber. We will use Corollary 2.1.8. We assume furthermore that the Kähler class $[\omega_X]$ is a multiple of the anticanonical class $2\pi c_1(X)$. This implies that the labelled polytope $(P, \mathbf{L})$ monotone, with a preferred point x_0 such that $L_1(x_0) = \cdots = L_d(x_0) = t$ (see [44]). By adding a constant if necessary, we may further assume that $x_0 = 0$, and we let $t = \frac{[\omega]}{2\pi c_1(X)}$.

Corollary 2.2.4. *The semisimple principal toric fibration $(Y, [\omega_Y])$ with Fano toric fiber admits an extremal Kähler metric in $[\omega_Y]$ if $\forall x \in P$,*

$$2(\ell + \sum_a n_a) + 2 + \sum_a \frac{ts_a - 2n_a c_a}{p_a(x) + c_a} - tl_{\mathrm{ext}}(x) \geq 0, \ \forall x \in P, \tag{2.6}$$

where $n_a := \dim_{\mathbb{C}}(B_a)$.

Proof. Since all $L_j(0)$ are equal to t, the condition from Corollary 2.2.3 further simplifies to

$$2\ell + 2 + \sum_{a=1}^{k} \frac{2n_a p_a(x) + ts_a}{p_a(x) + c_a} - tl_{\mathrm{ext}}(x) \geq 0 \ \forall x \in P \tag{2.7}$$

as in Corollary 2.1.8. Writing $2n_a p_a(x) = 2n_a(p_a(x) + c_a) - 2n_a c_a$ yields the statement. □

While simple enough, and tractable with numerical optimization techniques, (2.7) is a polynomial inequality in several variables of degree equal to the complex dimension of the base B plus one. It is this difficult to check in pratice, but there is a further ramification that allows us to get a simpler condition which can be checked by a finite number of evaluations of polynomial functions.

Corollary 2.2.5. *In the hypothesis of Corollary 2.2.4, suppose furthermore that for all a,* $c_a \geq \frac{ts_a}{2n_a}$*. Then the semisimple principal toric fibration* $(Y, [\omega_Y])$ *admits an extremal Kähler metric in* $[\omega_Y]$ *if inequation* (2.6) *is satisfied at every vertex of* P*.*

Proof. If f is a positive affine function, $\frac{1}{f}$ is also convex. Hence under the condition in the statement, the function $\frac{ts_a - 2n_a c_a}{p_a(x)+c_a}$ is concave on P. Condition (2.6) thus amounts to check the non-negativity of a concave function on a convex polytope: it is thus enough to check the non-negativity on vertices. □

Remark 2.2.6. *We can formulate a similar statement for the general case of toric fibrations, by working with the cone decomposition (see Figure 2.1.2). In this case the conditions to impose are: for all* j*, for all* a*,* $L_j(x_0)s_a - 2n_a(p_a(x_0) + c_a) \leq 0$ *and condition* (2.2.3) *is satisfied at all vertices of* P_j*, that is, some vertices of* P *and* x_0*.*

2.2.4 Extremal metrics in the anticanonical class

An important special case is when the semisimple principal toric fibrations are themselves Fano. We assume that X is Fano, $\omega \in 2\pi c_1(X)$ and (B_a, ω_a) are positive Hodge Kähler–Einstein manifolds, see Lemma C.0.1.

Corollary 2.2.7. *A Fano semisimple principal toric fibration* Y *defined as in Lemma C.0.1 over a product of positive Kähler–Einstein manifolds* $\prod_{a=1}^{k}(B_a, \omega_a)$ *admits an extremal Kähler metric in* $2\pi c_1(Y)$ *if its extremal function* l_{ext} *satisfies*

$$\sup l_{\text{ext}} \leq 2(\dim(Y) + 1). \tag{2.8}$$

Proof. When $[\omega_X] = 2\pi c_1(X)$, by Lemma C.0.1, for all a, $s_a = 2n_a c_a$. The condition from Corollary 2.2.4 becomes

$$2\dim(Y) + 2 - l_{\mathrm{ext}} \geq 0 \text{ on } P.$$

□

Of course, as in Corollary 2.2.5, it is enough to check this condition on the vertices of the polytope. Furthermore, if l_{ext} is constant, it is equal to $2\dim(Y)$ since the class is the anticanonical one. As a consequence, the condition is always satisfied:

$$2\dim(Y) + 2 - l_{\mathrm{ext}} = 2 > 0. \tag{2.9}$$

In particular, we obtain

Proposition 2.2.8. *Let (Y, J_Y, ω_Y) be a Fano semisimple principal toric fibration defined as in Lemma C.0.1 with vanishing Futaki invariant. Then there exists a Kähler–Einstein metric in $2\pi c_1(Y)$.*

Proposition 2.2.8 also follows from the fact that any Fano semisimple toric fibration given by the construction of Lemma C.0.1 admits a Kähler Ricci soliton [9] (see [34, 69, 79] for related results). This latter result, which is obtained by different methods in [9], can be in turn derived from our new proof of Corollary 2.2.11 given below.

2.2.5 Weighted solitons on Fano semisimple principal toric fibrations

Recall that a v-soliton is a Kähler metric ω satisfying (10). On a Fano semisimple principal toric fibration Y with fiber X, a Kähler metric $\omega_Y \in 2\pi c_1(Y)$ is a v-soliton if and only if its corresponding metric $\omega_X \in 2\pi c_1(X)$ is $(\mathrm{v}\mathrm{v}_0, \tilde{\mathrm{v}})$-cscK (see [9, Lemma 2.2, Lemma 5.11]) for the weights v_0 defined in (1.15) and

$\tilde{\mathrm{v}} := 2(\ell \mathrm{v}_0(x)\mathrm{v}(x) + d_x(\mathrm{v}_0\mathrm{v})(x))$. Since the polytope P is monotone, with $x_0 = 0$ and $t = 1$, the condition (2.5) becomes $\mathrm{v} \geq 0$ on the polytope, which is obviously satisfied. We deduce

Proposition 2.2.9. *Let $(Y, J_Y, \omega_Y, \mathbb{T})$ be a Fano semisimple principal toric fibration defined as in Lemma C.0.1 with associate Delzant polytope P. If the weighted Futaki invariant $\mathcal{F}_{\mathrm{vv}_0,\tilde{\mathrm{v}}}$ of X vanishes, then P is $(\mathrm{vv}_0, \tilde{\mathrm{v}})$ uniformly K-stable.*

By virtue of Proposition 1.6.8, we deduce

Corollary 2.2.10. *Let $(Y, J_Y, \omega_Y, \mathbb{T})$ be as in Proposition 2.2.9. If $\mathcal{F}_{\mathrm{vv}_0,\tilde{\mathrm{v}}}$ vanishes, then the weighted Mabuchi energy $\mathcal{M}_{\mathrm{vv}_0,\tilde{\mathrm{v}}}$ is proper, i.e. satisfies (1.64).*

Thanks to [62, Theorem 3.5], there exists a weighted vv_0-soliton in $2\pi c_1(X)$ as soon as the weighted Mabuchi energy is proper relatively to $\mathbb{T}^{\mathbb{C}}$. We thus get

Corollary 2.2.11. *Let $(Y, J_Y, \omega_Y, \mathbb{T})$ be as in Proposition 2.2.9. If $\mathcal{F}_{\mathrm{vv}_0,\tilde{\mathrm{v}}}$ vanishes, there exists a v-soliton in $2\pi c_1(Y)$.*

Corollary 2.2.11 was obtained in [9, Theorem 3], by applying K-stability arguments from [62]. The proof proposed above allows to remain fully on the analytical side.

2.3 Examples

2.3.1 Projective bundles as semisimple principal toric fibrations

We now provide an effective way of constructing examples of semisimple principal toric fibrations from a direct sum of holomorphic complex line bundles over a product cscK manifolds.

Let $(B, J_B, \omega_B) := \prod_{a=1}^{k}(B_a, J_a, \omega_a)$ be a product of compact complex manifolds (B_a, J_a) endowed with cscK metrics ω_a such that the Kähler classes $[\omega_a]$ are primitive elements of $H^2(B_a, \mathbb{Z})$. We consider holomorphic line bundles $\pi : L_i \longrightarrow B$, $i = 1, \ldots, \ell$. Let h_i be the hermitian metric on L_i whose curvature is

$$\omega_{h_i} = -\sum_{a=1}^{k} p_{ai}\pi^*\omega_a,$$

and $S_i \subset L_i$ the principle $\mathbb{S}^1$-bundle over B of vectors of unit norm of L_i with respect to h_i. Letting $r_i := ||\cdot||_{h_i}$ be the corresponding fiberwise norm function, the 1-form $d^c \log r_i$ restricted to S_i gives rise to a connection 1-form θ_i such that

$$d\theta_i = \sum_{a=1}^{k} p_{ai}\pi^*\omega_a.$$

We then consider the principle $\mathbb{T}$-bundle $Q := S_1 \times_B \cdots \times_B S_\ell \to B$ endowed with the connection 1-form $\theta = \sum_{i=1}^{\ell} \theta_i$ satisfying

$$d\theta = \sum_{a=1}^{k} \pi^*\omega_a \otimes p_a, \tag{2.10}$$

where $p_a = \sum_{i=1}^{\ell} p_{ai}\xi_i$ in the basis $(\xi_1, \ldots \xi_\ell)$ of the circle subgroups of $\mathbb{S}^1_i \subset \mathbb{T}$. It is not hard to see that the semisimple principal toric fibration Y build from Q and the toric fiber $\mathbb{P}^\ell$ is biholomorphic to $\mathbb{P}(\mathcal{O} \oplus L_1 \oplus \cdots \oplus L_\ell) \to B$.

Suppose $\omega_{\mathbb{P}^\ell}$ belongs to $2\pi c_1(\mathbb{P}^\ell)$ and denote by P the ℓ-simplex associated to $(\mathbb{P}^\ell, 2\pi c_1(\mathbb{P}^\ell), \mathbb{T}^\ell)$ via the Delzant correspondence [36]. By (2.10), any compatible Kähler metric on Y is of the form

$$\omega_Y = \sum_{a=1}^{k} \left(\sum_{i=1}^{\ell} \langle p_a, x \rangle + c_a \right) \pi^*(\omega_a) + \omega_{\mathbb{P}^\ell}, \tag{2.11}$$

with

$$c_a > -\sum_{i=1}^{\ell} p_{ai}, \tag{2.12}$$

In the above formula, by abuse of notation, $\omega_{\mathbb{P}^d}$ denotes both the Kähler metric on $\mathbb{P}^\ell$ and its induced 2-form on Y as in (1.8). The tuples (c_a) satisfy (2.12) and parametrize the compatible Kähler classes.

Furthermore, suppose that (B, J_B, ω_B) is a product of positive Kähler–Einstein manifolds $(B, J_B, \omega_B) := \prod_{a=1}^{k}(B_a, J_a, \omega_a)$. By Lemma 3.1.13, if we choose c_a equal to the Fano index I_a of B_a, the corresponding compatible Kähler form ω_Y defined in (2.11) belongs to the first Chern class $2\pi c_1(Y)$. In particular, if

$$I_a > -\sum_{i=1}^{\ell} p_{ai}, \tag{2.13}$$

Y is a Fano manifold with compatible first Chern class.

2.3.2 Projective line bundles over a negative Kähler–Einstein 3-fold

Consider B a three-dimensional canonically polarized manifold, equipped with its Kähler–Einstein metric in $-2\pi c_1(B)$, whose scalar curvature is thus equal to -6. We consider the sufficient condition for existence of extremal Kähler metrics in admissible Kähler classes on the $\mathbb{P}^1$-bundles $\mathbb{P}(\mathcal{O}_B \oplus K_B^m)$. Up to rescaling and symmetry, this amounts to check the (v, w)-uniform K-stability of the polytope $[-1, 1] \subset \mathbb{R}$ with respect to the weights

$$\mathrm{v}(x) = (px + c)^3 \qquad \text{and} \qquad \mathrm{w}(x) = \left(l_{\mathrm{ext}}(x) - \frac{-6}{px + c}\right)(px + c)^3$$

where $p \in \mathbb{Q}$, $c \in \mathbb{R}$ and $c > p > 0$. Our sufficient condition allows to obtain the following families of extremal Kähler classes. We only show an example with very rough estimates to illustrate the results, but of course one could get more examples by using more precise estimates in the proof, and further classes by using the sufficient condition in Theorem 2.1.4 in its full generality.

Proposition 2.3.1. *In the above setting, if* $c \geq 15p$, *then* $[-1,1]$ *is* (v,w)*-uniformly K-stable. The corresponding Kähler classes on the* $\mathbb{P}^1$*-bundles* $\mathbb{P}(\mathcal{O} \oplus K_B^m)$ *admit extremal Kähler metrics.*

Proof. Using Program A.1 in the appendix or straightforward but tedious computations, we obtain that the sufficient condition reads as

$$\begin{aligned} &75c^7 - 300c^6 - 65c^5p^2 + 160c^4p^2 - 15c^3p^4 - 180c^2p^4 - 27cp^6 + 48p^6 \\ &> |-75c^6p + 5c^4p^3 + 80c^3p^3 - 105c^2p^5 + 15p^7| \end{aligned} \tag{2.14}$$

Without attempting to give an optimal result, we may as well check that the

$$\text{RHS of (2.14)} > 75c^6p + 5c^4p^3 + 80c^3p^3 + 105c^2p^5 + 15p^7$$

since c and p are positive. Writing $c = \alpha p$ for some $\alpha > 1$ and simplifying by p^6, we get a linear inequality in p

$$pA + B \geq 0 \tag{2.15}$$

where

$$\begin{aligned} A &= 75\alpha^7 - 75\alpha^6 - 65\alpha^5 - 5\alpha^4 - 15\alpha^3 - 105\alpha^2 - 27\alpha - 15 \\ B &= -300\alpha^6 + 160\alpha^4 - 80\alpha^3 - 180\alpha^2 + 48 \end{aligned}$$

Since $\alpha > 1$, the coefficient A is larger than $(75\alpha - 307)\alpha^6$ and in particular, it is non-negative for $\alpha \geq \frac{307}{75}$. Using the same lower bound for the leading coefficient, inequality (2.15) is certainly satisfied at $p = 1$ if

$$(75\alpha - 307)\alpha^6 - 300\alpha^6 - 160\alpha^4 - 80\alpha^3 - 180\alpha^2 - 48 \geq 0$$

Using again $\alpha > 1$ and very rough estimates, this is implied by the inequality

$$(75\alpha - 1075)\alpha^6 \geq 0$$

The latter is satisfied at least for $\alpha \geq 15$, and since $15 \geq \frac{307}{75}$, we obtain that if $\alpha \geq 15$, the sufficient condition is satisfied for all $p \geq 1$. $\square$

2.3.3 Projective plane bundles over a positive Kähler–Einstein 3-fold

We consider the 2-dimensional projective space $(\mathbb{P}^2, \mathbb{T}^2, 2\pi c_1(\mathbb{P}^2))$. Identifying the lattice Λ of $\mathbb{T}^2$ with $\mathbb{Z}^2$, its labelled moment polytope $(P, \mathbf{L})$ in $\mathbb{R}^2$ is

$$P = \{(x_1, x_2) =: x \in \mathbb{R}^2 \mid x_1 + 1 \geq 0,\ x_2 + 1 \geq 0,\ - x_1 - x_2 + 1 \geq 0\}, \quad (2.16)$$

Let (B, ω_B) be a Kähler–Einstein Fano threefold with $\alpha_B := [\omega_B]$ a primitive element of $H^2(B, \mathbb{Z})$. Let $L_i \longrightarrow B$ be an holomorphic line bundle of degree $-p_i$ proportional to the anticanonical line bundle $-K_B$, i.e. $-p_i\alpha_B = 2\pi c_1(L_i)$. We consider a semisimple principal toric fibration $\pi : Y := \mathbb{P}(L_0 \oplus L_1 \oplus L_2) \longrightarrow B$ with one factor (B, ω_B) on the base. Since (Y, J_Y) is invariant by tensoring $E := L_0 \oplus L_1 \oplus L_2$ with a line bundle, we can suppose without loss of generality that $L_0 = \mathcal{O}$ is the trivial line bundle and $p_i \geq 0$, $i = 1, 2$. When $p_1 = p_2 > 0$ or $p_2 > p_1 = 0$, it is known [6, Proposition 11] that there exists an extremal metric in every compatible Kähler classes. We then suppose $p_2 \geq p_1 > 0$.

Since α_B is primitive, the constant scalar curvature s_B of ω_B is $s_B = 6I$, where I is the Fano index of B. By a Theorem of Kobayashi–Ochiai [51], the only possible Fano indices I are 1, 2, 3 or 4. Consequently, to cover all Kähler–Einstein Fano 3-fold, we only need to check our sufficient condition for $s_B = 6, 12, 18, 24$.

The compatible Kähler classes are parametrized by the constants c and are of the form (see [10])

$$\alpha_c := 2\pi c_1(O_E(3)) + c\pi^*(\alpha_B). \tag{2.17}$$

In the case when B is the quadric Q_3 or the projective space $\mathbb{P}^3$ (i.e. if $I = 3$ or $I = 4$ respectively, see [51]), Leray–Hirsch Theorem shows that $H^2(Y, \mathbb{R}) \cong \mathbb{R}^2$. It follows that, up to scaling, all Kähler classes are compatible, i.e. are of the form of (2.17). By [7, Theorem 4] there exists a constant c_0 such that for all $c \geq c_0$ the class α_c is extremal. The following Proposition gives a precise value for c, depending on p_1 and p_2, such that α_c admits an extremal metric.

Proposition 2.3.2. *Let $\pi : \mathbb{P}(E) \to B$ be the projectivization of $E := \mathbb{C} \oplus L^{-p_1} \oplus L^{-p_2} \to B$, where $1 \leq p_1 \leq p_2$ are integer and $L^{Ind(B)} = -K_B$. Then there exists an extremal metric in α_c for $c \geq 7p_2$.*

Proof. Since the arguments are identical for each Fano index I, we give the proof only for $I = 4$.

By Corollary 2.2.5, for $c \geq 4$, it is sufficient to check (2.6) evaluated at each vertex $v_1 := (-1, 2)$, $v_2 := (-1, -1)$, $v_3 := (2, -1)$ of the polytope P.

Using Program A.2 in Appendix A, we find that the LHS of (2.6) evaluated at v_1 is a rational fraction in the variables c, p_1, p_2:

$$\text{LHS of (2.6)} = \frac{P(c, p_1, p_2)}{Q(c, p_1, p_2)}.$$

We give the explicit expression of the polynomials P and Q in Appendix B. Suppose now $c \geq 7p_2$ and $p_2 \geq p_1 \geq 1$. Then we can find two polynomials

$$R(c) := 12250c^{10} - 73500c^9 - 295470c^8 + 1296540c^7 - 3657150c^6 + 3776220c^5 - 6537672c^4 + 5624964c^3 - 6193584c^2 + 85920232c - 1889568$$

and

$$S(c) := 6125c^{10} + 18375c^9 + 6615c^8 + 19845c^7 + 127575c^6 + 382725c^5 + 17496c^4 + 52488c^3 - 288684c^2 - 866052c$$

such that R satisfies $0 < R(c) \leq P(c, p_1, p_2)$ and S satisfies $0 < S(c)$ and $S(c) \geq Q(c, p_1, p_2)$. This implies that

$$\text{LHS of (2.6)} = \frac{P(c, p_1, p_2)}{Q(c, p_1, p_2)} \geq \frac{R(c)}{S(c)} \geq 0.$$

We proceed analogously for the verticies v_2 and v_3. We conclude the proof by using Corollary 2.2.5.

□

Remark 2.3.3. *In Proposition 2.3.2, we obtained a lower bound on c depending only on p_1 and p_2. For given values of p_1 and p_2 it is possible to obtain a sharper result. Indeed, suppose p_1 and p_2 are fixed. Then, the LHS of* (2.6) *is a rational fraction F depending only on the variable c. We then only need to look for constant c_0 such that F is non-negative for $c \geq c_0$. For example, if $B = \mathbb{P}^3$, respectively $B = Q_3$, $p_1 = 1$ and $p_2 = 2$,* (2.6) *show the existence of an extremal metric in α_c for $c \geq 7.09$, respectively $c \geq 9.08$. We refer to Appendix A for further examples of application of the sufficient condition on simple principal $\mathbb{P}^2$-fibrations.*

CHAPTER III

THE CALABI PROBLEM ON SEMISIMPLE PRINCIPAL FIBRATIONS (BASED ON A PART OF [9])

3.1 Semisimple principal fibrations

Let (X, ω) be a compact Kähler 2ℓ-manifold, endowed with a hamiltonian isometric action of an r-dimensional torus $\mathbb{T}$. We use lower script to denote the space on which $\mathbb{T}$ acts, as in Chapter 1. Let $\mathfrak{t}$ be the Lie algebra of $\mathbb{T}$ and $\Lambda \subset \mathfrak{t}$ the lattice of generators of circle groups in $\mathbb{T}$ (i.e. $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$). We denote by $m_\omega : X \to P \subset \mathfrak{t}^*$ the normalized $\mathbb{T}_X$-moment map of ω, i.e. whose image is a fixed compact convex polytope $P \subset \mathfrak{t}^*$.

Let $B = B_1 \times \cdots \times B_k$ be a $2n$-dimensional cscK manifold, where each $(B_a, J_a, \omega_a), a = 1, \ldots, k$ is a compact cscK Hodge Kähler $2n_a$-manifold (i.e. $\frac{1}{2\pi}[\omega_a] \in H^2(B_a, \mathbb{Z})$), and $\pi_B : Q \to B$ a principal $\mathbb{T}$-bundle endowed with a connection 1-form $\theta \in \Omega^1(Q, \mathfrak{t})$ with curvature

$$d\theta = \sum_{a=1}^{k} (\pi_B^* \omega_a) \otimes p_a, \qquad p_a \in \Lambda. \tag{3.1}$$

Remark 3.1.1. *The principle $\mathbb{T}$-bundle Q can be described in terms of r complex line bundles over B as follows. Fixing a lattice basis $\{\xi_1, \ldots, \xi_r\}$ of $\mathfrak{t}$, and writing $p_a = \sum_{i=1}^{r} p_{ai}\xi_i, , p_{ai} \in \mathbb{Z}, a = 1, \ldots k$, 3.1 yields that Q is the (fiber-wise) product* *of r principle $U(1)$-bundles $Q_i \to B$, where each Q_i is associated to a complex line bundle L_i^* on B with Chern class $2\pi c_1(L_i^*) = -\sum_{a=1}^{r} p_{ai}\pi_B^*[\omega_a]$, i.e. we have*

$$2\pi c_1(P) := -2\pi \sum_{i=1}^{r} c_1(L_i^*) \otimes \xi_i = \sum_{a=1}^{k} \pi_B^*[\omega_a] \otimes p_a.$$

Fixing a connection 1-form θ on Q as in (3.1) amounts to introducing a hermitian metric h_i^* on each L_i^*, with curvature $-\sum_{a=1}^{r} p_{ai}\pi_B^*(\omega_{B_a})$, and identifying $Q_i \subset L_i^*$ with the corresponding unitary $\mathbb{S}^1$-bundle.

Let $\mathcal{D} = \mathrm{ann}(\theta) \subset TQ$ be the horizontal distribution defined by θ, leading to a splitting

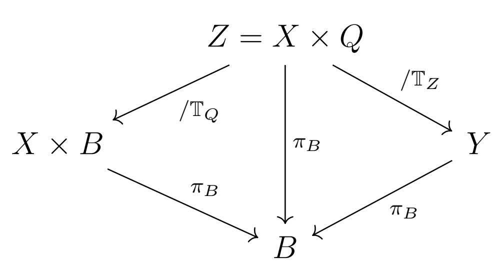
$$TQ = \mathcal{D} \oplus \mathfrak{t}_Q,$$

where $\mathfrak{t}_Q$ denotes the Lie algebra of $\mathbb{T}_Q$ inside $C^\infty(Q, TQ)$, corresponding to the $\mathbb{T}$-action $\mathbb{T}_Q$ on Q. The lift J_B of the complex structure of B gives rise to a smooth section of TY which preserve $\mathcal{D}$.

We further let $Z := X \times Q$ and consider the induced $\mathbb{T}$-action, denoted $\mathbb{T}_Z$, generated by $(-\xi_i^X + \xi_i^Q)$ for any basis of Λ as above. We thus define

$$Y := Z/\mathbb{T}_Z.$$

It follows that Y is a $2(m+n)$-dimensional smooth manifold, and $\pi_Y : Z = X \times Q \to Y$ is a principal $\mathbb{T}$-bundle over Y whereas $\pi_B : Q \to B$ defines a fibration $\pi_B : Y \to B$ with smooth fibres X, as summarized in the diagram below.



The $\mathbb{T}_X$-action on the factor X in $Z = X \times Q$ descends to a $\mathbb{T}$-action on Y, denoted $\mathbb{T}_Y$, which preserves each fibre (and thus coincides with the action of $\mathbb{T}_X$). Notice that the 1-form θ also defines a connection 1-form on Z with horizontal distribution $\mathcal{H}$:

$$T(X \times Q) = \mathcal{H} \oplus \mathfrak{t}_Z, \qquad \mathcal{H} = TX \oplus \mathcal{D} = \mathrm{ann}(\theta), \tag{3.2}$$

giving rise to a section $J = J_X \oplus J_B$ of TZ preserving $(\mathcal{H},)$, which is clearly invariant under the $\mathbb{T}_Z$-action, and therefore defines a $\mathbb{T}_Y$-invariant complex structure J_Y on Y.

We now consider Kähler metrics on Y, compatible with the fibre-bundle construction of the above form. We denote by ω a $\mathbb{T}$-invariant Kähler structure in the class α on X, and by $\tilde{\omega}$ the resulting Kähler structure on Y, which is defined in terms of a basic 2-form on $Z = X \times Q$, depending on k real constants $c_a \in \mathbb{R}$ (which will be fixed) such that for each $a = 1, \dots, k$, the affine linear function $\langle p_a, x\rangle + c_a$ on $\mathfrak{t}^*$ is strictly positive on the moment image P:

$$\begin{aligned} \tilde{\omega} &:= \omega + \sum_{a=1}^{k} (\langle p_a, m_\omega \rangle + c_a)\pi_B^*\omega_a + \langle dm_\omega \wedge \theta\rangle \\ &= \omega + \sum_{a=1}^{k} c_a(\pi_B^*\omega_a) + d\left(\langle m_\omega, \theta\rangle\right). \end{aligned} \tag{3.3}$$

In the above expression, $\langle \cdot, \cdot \rangle$ stands for the natural pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$: thus $\langle p_a, m_\omega\rangle$ is a smooth function, $\langle m_\omega, \theta\rangle$ is a 1-form, and $\langle dm_\omega \wedge \theta\rangle$ is a 2-form on Z. One can directly check from the above expression that $\tilde{\omega}$ is closed, $\mathbb{T}_Z$-basic, and is positive definite on $(\mathcal{H}, J_X \oplus J_B)$, so it is the pullback of a Kähler form on Y. We shall tacitly identify in the sequel the Kähler form on Y with its pullback (3.3) on $Z = X \times Q$. Notice that $\tilde{\omega}$ is $\mathbb{T}_Y$-invariant and m_ω, seen as a smooth $\mathbb{T}_Z$-invariant function on Z, is the P-normalized moment map.

Definition 3.1.2. *The Kähler manifold* $(Y, \mathbb{T}_Y)$ *constructed as above will be called a* semisimple $(X, \mathbb{T})$-principal fibration *associated to the Kähler manifold* $(X, \mathbb{T})$ *and the product cscK manifold* $B = B_1 \times \cdots \times B_k$. *The* $\mathbb{T}_Y$*-invariant Kähler metric* $\tilde{\omega}$ *on* Y *constructed from a* $\mathbb{T}_X$*-invariant Kähler metric* ω *on* X *(and fixed cscK metrics* ω_a *on* B_a*) will be called* bundle-compatible.

Remark 3.1.3. *In the case when* $(X, \mathbb{T}, \omega)$ *is a toric Kähler manifold, a semisimple* $(X, \mathbb{T})$*-principal fibration endowed with a bundle-compatible Kähler metric is exactly the semisimple principal toric fibration studied in Chapter 1.*

3.1.1 The space of bundle-compatible functions

The above bundle construction gives rise to a natural embedding of the space $C^\infty(X)^{\mathbb{T}}$ of $\mathbb{T}_X$-invariant smooth functions on X to the space $C^\infty(Y)^{\mathbb{T}}$ of $\mathbb{T}_Y$-invariant smooth functions on Y: for any $\varphi \in C^\infty(X)^{\mathbb{T}}$ we consider the induced function on $Z = X \times Q$, which is clearly $\mathbb{T}_Z$-invariant, and thus descends to a smooth $\mathbb{T}_Y$-invariant function on Y. We shall tacitly identify φ and its image in $C^\infty(Y)^{\mathbb{T}}$, i.e. we shall consider

$$C^\infty(X)^{\mathbb{T}} \subset C^\infty(Y)^{\mathbb{T}}.$$

Notice that the above embedding is closed in the Fréchet topology, as we can identify a smooth $\mathbb{T}_X$-invariant function on X with a smooth $\mathbb{T}_Y$-invariant function φ on Y, which has the property

$$d_Q(\pi_Y^* \varphi) = 0$$

on $Z = X \times Q$.

More generally, for any $\mathbb{T}_Y$-invariant smooth function $\psi \in C^\infty(Y)^{\mathbb{T}}$ its lift $\pi_Y^* \psi$ to $Z = X \times Q$ a smooth function which is both $\mathbb{T}_Z$ and $\mathbb{T}_X$-invariant, or equivalently $\mathbb{T}_X$ and $\mathbb{T}_Q$ invariant. It thus follows that $\pi_Y^*\psi$ can be equivalently viewed as a $\mathbb{T}_X$-invariant smooth function on $X \times B$, i.e. we have an identification

$$C^\infty(Y)^{\mathbb{T}} \cong C^\infty(X \times B)^{\mathbb{T}}. \tag{3.4}$$

In particular, for any fixed point $x \in X$, we shall denote by $\psi_x \in C^\infty(B)$ the induced smooth function on B, and for any fixed point $b \in B$ by $\psi_b \in C^\infty(X)^{\mathbb{T}}$ the induced function on X. We thus have the identification

$$C^\infty(X)^{\mathbb{T}} \cong \{\psi \in C^\infty(Y)^{\mathbb{T}} \,|\, d_B\psi_x = 0 \,\forall x \in X\}.$$

3.1.2 The space of bundle-compatible Kähler metrics

We shall next use the construction of (3.3) in order to identify the space $\mathcal{K}(X,\omega_0)^{\mathbb{T}}$ of $\mathbb{T}_X$-invariant ω_0-relative Kähler potentials on X as a subset of the space $\mathcal{K}(Y,\tilde\omega_0)^{\mathbb{T}}$ of $\mathbb{T}_Y$-invariant $\tilde\omega_0$-relative Kähler potentials on Y.

Lemma 3.1.4. *Let $\omega_\varphi = \omega_0 + d_X d_X^c\varphi$ be an $\mathbb{T}_X$-invariant Kähler form on X in the Kähler class $\alpha = [\omega_0]$, where $\varphi \in \mathcal{K}(X,\omega_0)^{\mathbb{T}}$ is a $\mathbb{T}_X$-invariant smooth function on X. Denote by m_φ the moment map of $\mathbb{T}_X$ with respect to ω_φ, satisfying the normalization $m_\varphi(X) = P$, and by $\tilde\omega_\varphi$ the induced Kähler metric on Y, via* (3.3)*. Then,*

$$\tilde\omega_\varphi = \tilde\omega_0 + d_Y d_Y^c\varphi,$$

where φ stands for the induced smooth function on Y.

Proof. Recall that $m_\varphi = m_0 + d_X^c\varphi$ (see (1.1)). By (3.3), the pullback of $\tilde\omega_\varphi$ to

$Z = X \times Q$ is

$$\begin{aligned}\tilde{\omega}_\varphi &= \omega_\varphi + \sum_{a=1}^{k} c_a(\pi_B^*\omega_a) + d_Z\langle m_\varphi, \theta\rangle \\ &= \omega_0 + \sum_{a=1}^{k} c_a(\pi_B^*\omega_a) + d_X d_X^c\varphi + d_Z\langle m_\varphi, \theta\rangle \\ &= \tilde{\omega}_0 + d_Z d_X^c\varphi + d_Z\big(\langle d_X^c\varphi, \theta\rangle\big),\end{aligned}$$

so it is enough to check that

$$d_Y^c\varphi = d_X^c\varphi + \langle d_X^c\varphi, \theta\rangle, \tag{3.5}$$

for any $\mathbb{T}_X$-invariant smooth function φ on X. To this end, let us choose a basis $\{\xi_1, \ldots, \xi_r\}$ of $\mathfrak{t}$, with dual basis $\{\xi^1, \ldots, \xi^r\}$ of $\mathfrak{t}^*$, and we write $d_X^c\varphi = \sum_{j=1}^{r}(d_X^c\varphi)(\xi_j^X)\xi^j$ and $\theta = \sum_{j=1}^{r}\theta_j\xi_j$ for 1-forms θ_j on Z such that θ_j is zero on $\mathcal{H}$ and $\theta_j(\xi_i^Q) = \theta_j(-\xi_i^X + \xi_i^Q) = \delta_{ij}$. Thus, (3.5) is equivalent to

$$d_Y^c\varphi = d_X^c\varphi + \sum_{j=1}^{r}(d_X^c\varphi)(\xi_j^X)\theta_j.$$

Evaluating the RHS of the above equality on the generators $(-\xi_j^X + \xi_j^Q)$ of $\mathfrak{t}_Z$, we see that it is a π_Y-basic 1-form on Z, and thus is the pullback of a 1-form on Y via π_Y. The claim follows easily. □

Thus, Lemma 3.1.4 defines an embedding $\mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ and we have also identified in Section 3.1.1 a natural embedding of the space of $\mathbb{T}_X$-invariant functions on X into the space of $\mathbb{T}_Y$-invariant functions on Y, through their pullbacks to $Z = X \times Q$.

Letting $\theta := \sum_{j=1}^{r}\theta_j \otimes \xi_j^Q$ be the decomposition of the connection 1-form θ on Q in a basis $\{\xi_1, \ldots, \xi_r\}$ of the lattice $\Lambda \subset \mathfrak{t}$, and $\theta^{\wedge r} := \theta_1 \wedge \cdots \wedge \theta_r$, it follows from (3.3) and Lemma 3.1.4 that for any $\varphi \in \mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$, the measure

$\tilde{\omega}_\varphi^{[m+n]}$ on Y is the push-forward of the measure on Z:

$$\frac{1}{(2\pi)^r}\tilde{\omega}_\varphi^{[m+n]} \wedge \theta^{\wedge r} = \frac{1}{(2\pi)^r}\left(\mathrm{v}(m_\varphi)\omega_\varphi^{[m]} \wedge \bigwedge_{a=1}^{k} \pi_B^*\omega_a^{[n_a]}\right) \wedge \theta^{\wedge r}, \tag{3.6}$$

where

$$\mathrm{v}(x) := \prod_{a=1}^{k}(\langle p_a, x\rangle + c_a)^{n_a}, \qquad n_a = \dim_{\mathbb{C}}(B_a) \tag{3.7}$$

is a positive polynomial on P, determined by the semisimple $(X, \mathbb{T})$-principal fibration Y and the given bundle-compatible Kähler class on it. It thus follows that any $\mathbb{T}_X$-invariant integrable function f on X defines an integrable $\mathbb{T}_Y$-invariant function on Y and, for any $\varphi \in \mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$, we have

$$\int_Y f\tilde{\omega}_\varphi^{[n+m]} = \mathrm{Vol}(B, \omega_B)\int_X \mathrm{v}(m_\varphi) f\omega_\varphi^{[m]}. \tag{3.8}$$

Corollary 3.1.5. *There exists an embedding $\mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ such that, for any smooth curve $\psi_t \in \mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$, we have*

$$L_1^Y(\psi_t) = \mathrm{Vol}(B, \omega_B)L_{1,\mathrm{v}}^X(\psi_t),$$

where $\mathrm{v}(x)$ is the positive weight function on P defined in (3.7), $L_{1,\mathrm{v}}^X$ is the $\mathrm{v}(x)$-weighted length function on $\mathcal{K}(X, \omega_0)^{\mathbb{T}}$ given by

$$L_{1,\mathrm{v}}^X(\psi_t) := \int_0^1 \left(\int_X |\dot{\psi}_t|\mathrm{v}(m_{\psi_t})\omega_{\psi_t}^{[m]}\right) dt,$$

and L_1^Y is the length function on $\mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ corresponding to the weight $\mathrm{v} = 1$. In particular, for any compatbile Kähler potentials $\varphi_0, \varphi_1 \in \mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$, $d_1^Y(\varphi_0, \varphi_1) = \mathrm{Vol}(B, \omega_B)d_{1,\mathrm{v}}^X(\varphi_0, \varphi_1)$, where $d_{1,\mathrm{v}}^X$ is the induced distance via the length functional $L_{1,\mathrm{v}}^X$.

Proof. A direct consequence of (3.8). □

Lemma 3.1.6. *For any weight $\mathrm{v} > 0$, there exists an uniform constant $C = C(X, \omega_0, \mathrm{v}) > 0$ such that*

$$\frac{1}{C}d_1(\varphi_0, \varphi_1) \leq d_{1,\mathrm{v}}(\varphi_0, \varphi_1) \leq Cd_1(\varphi_0, \varphi_1), \qquad \forall\varphi_0, \varphi_1 \in \mathcal{K}_{\mathbb{T}}(X, \omega_0), \tag{3.9}$$

where $d_1 := d_{1,1}$ is the distance introduced in [32]. In particular, $d_{1,\mathrm{v}}$ is a distance on $\mathcal{K}_{\mathbb{T}}(X,\omega_0)$ which is equivalent to d_1.

Proof. The relation (3.9) follows from the fact that $v(x)$ is positive and uniformly bounded on P. This yields that $d_{1,\mathrm{v}}$ is a distance, as d_1 is a distance according to [32]. □

Lemma 3.1.7. *Let φ be a smooth $\mathbb{T}_X$-invariant function on X, also considered as a smooth $\mathbb{T}_Y$-invariant function on Y, and ω be an $\mathbb{T}_X$-invariant Kähler metric on X with $\tilde{\omega}$ the corresponding $\mathbb{T}_Y$-invariant Kähler metric on Y given by* (3.3)*. Then*

$$||d_X\varphi||^2_{\omega} = ||d_Y\varphi||^2_{\tilde{\omega}}.$$

Proof. We use that

$$||d_X\varphi||^2_{\omega} = \frac{d_X\varphi \wedge d^c_X\varphi \wedge \omega^{[m-1]}}{\omega^{[m]}} = \frac{d_X\varphi \wedge d^c_X\varphi \wedge \omega^{[m-1]} \wedge \mathrm{v}(m_\omega)(\pi_B^*\omega_B)^{[n]} \wedge \theta^{\wedge r}}{\omega^{[m]} \wedge \mathrm{v}(m_\omega)(\pi_B^*\omega_B)^{[n]} \wedge \theta^{\wedge r}}$$
$$||d_Y\varphi||^2_{\tilde{\omega}} = \frac{d_Y\varphi \wedge d^c_Y\varphi \wedge \tilde{\omega}^{[m+n-1]}}{\tilde{\omega}^{[m+n]}} = \frac{d_Y\varphi \wedge d^c_Y\varphi \wedge \tilde{\omega}^{[m+n-1]} \wedge \theta^{\wedge r}}{\tilde{\omega}^{[m+n]} \wedge \theta^{\wedge r}}$$

(where the RHS are written on $X \times Q$) together with $d_X\varphi = d_Y\varphi$ and (3.6). □

Proposition 3.1.8. *The embedding in Corollary 3.1.5 is totally geodesic with respect to the weak $C^{1,\bar{1}}$ geodesics.*

Proof. Let $\varphi_0, \varphi_1 \in \mathcal{K}(X,\omega_0)^{\mathbb{T}}$. If φ_0 and φ_1 can be connected with a *smooth* geodesic φ_t, i.e. with a smooth curve in $\mathcal{K}(X,\omega_0)^{\mathbb{T}}$ such that

$$\ddot{\varphi} = ||d_X\dot{\varphi}||^2_{\omega_\varphi}, \tag{3.10}$$

then, by Lemma 3.1.7, it follows that φ_t is also a smooth geodesic in $\mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$ connecting $\varphi_0, \varphi_1 \in \mathcal{K}(Y,\tilde{\omega}_0)^{\mathbb{T}}$.

In general, by the results in [23], φ_0, φ_1 can be connected only with a weak $C^{1,\bar{1}}$-geodesic in $\mathcal{K}^{1,\bar{1}}(X,\omega_0)^{\mathbb{T}}$, where $\mathcal{K}^{1,\bar{1}}(X,\omega_0)$ stand for the space of $C^1(X)$ functions φ on X such that $\omega_0 + dd^c\varphi \geq 0$ and has bounded coefficients as a $(1,1)$-current. More precisely, letting $\Sigma := \{1 < z < e\} \subset \mathbb{C}$, it is shown in [23] that there exists a unique weak solution (i.e. a positive $(1,1)$-current in the sense of Bedford–Taylor) of the homogeneous Monge–Ampère equation

$$\begin{aligned} &(\pi_X^*\omega_0 + d_X d_X^c \Phi)^{m+1} = 0, && \pi_X^*\omega_0 + d_X d_X^c \Phi \geq 0,\ \Phi \in C^{1,\alpha}(X \times \overline{\Sigma}), \\ &\Phi(x,1) = \varphi_0(x), && \Phi(x,e) = \varphi_1(x). \end{aligned} \tag{3.11}$$

It was later shown in [31] that Φ is actually of regularity $C^{1,1}(X \times \overline{\Sigma})$. Note that, by the uniqueness, Φ is $\mathbb{T}$-invariant as soon as φ_0 and φ_1 are. The link with (3.10) is (see [72]) that if Φ were actually smooth, we can recover the smooth geodesic φ_t joining φ_0 and φ_1 by letting $t := \log|z|$ and $\varphi_t(x) := \Phi(x, \log|z|)$. In the general case, the curve φ_t of (weak) ω_0-relative pluri-subharmonic potentials (of regularity $C^{1,1}(X \times [0,1])$) is referred to as the *weak* $C^{1,\bar{1}}$-geodesic joining φ_0 and φ_1.

We are thus going to check that any weak $C^{1,\bar{1}}$-geodesic on X (invariant under $\mathbb{T}_X$) defines, via Lemma 3.1.4, a $C^{1,\bar{1}}$-geodesic on Y. To this end, we need to show that Φ satisfies

$$(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Phi)^{m+n+1} = 0, \qquad \pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Phi \geq 0, \tag{3.12}$$

the regularity statements being automatically satisfied on Y.

By the results in [23] and [19], Φ can be approximated as $\varepsilon \to 0$, both it the weak sense of currents and in $C^{1,\alpha}(X \times \overline{\Sigma})$ (for a fixed $\alpha \in (0,1)$), by smooth functions $\Psi^\varepsilon(x,z)$ on $X \times \overline{\Sigma}$ which solve

$$\begin{aligned} &(\pi_X^*\omega_0 + d_X d_X^c \Psi^\varepsilon)^{[m+1]} = \varepsilon \left((\pi_X^*\omega_0)^{[m]} \wedge (dx \wedge dy) \right),\ \varepsilon > 0, \\ &\pi_X^*\omega_0 + d_X d_X^c \Psi^\varepsilon > 0,\ \Psi^\varepsilon(x,1) = \varphi_0,\ \ \Psi^\varepsilon(x,e) = \varphi_1. \end{aligned} \tag{3.13}$$

By the uniqueness of the smooth solution of (3.13) (and using that both φ_0, φ_1 are $\mathbb{T}_X$-invariant), we have that $\Psi^\varepsilon(x,z)$ is a $\mathbb{T}_X$-invariant smooth function on X for any $z \in \overline{\Sigma}$; furthermore, the positivity condition on the second line yields that $\Psi^\varepsilon(x,z) \in \mathcal{K}(X,\omega_0)^{\mathbb{T}}$ for any $z \in \overline{\Sigma}$. We can then also see $\Psi^\varepsilon(x,z)$, via its pull-back to $X \times Q \times \overline{\Sigma}$, as a $\mathbb{T}_Y$-invariant smooth function on $Y \times \overline{\Sigma}$; the arguments in the proof of Lemma 3.1.4 yield that $\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon > 0$ on $Y \times \overline{\Sigma}$. Furthermore, by the same proof, we have the following equality of volume forms on $X \times Q \times \overline{\Sigma}$:

$$\begin{aligned}(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} \wedge \theta^{\wedge r} &= \mathrm{v}(m_{\Psi^\varepsilon})(\pi_X^*\omega_0 + dd^c\Psi^\varepsilon)^{[m+1]} \wedge (\pi_B^*\omega_B)^{[n]} \wedge \theta^{\wedge r} \\ &= \varepsilon \mathrm{v}(m_{\Psi^\varepsilon})(\pi_X^*\omega_0)^{[m+1]} \wedge (\pi_B^*\omega_B)^{[n]} \wedge \theta^{\wedge r},\end{aligned} \tag{3.14}$$

where, we recall, $\mathrm{v}(x) := \prod_{a=1}^k (\langle p_a, x\rangle + c_a)^{n_a}$, $\theta^{\wedge r} := \theta_1 \wedge \cdots \wedge \theta_r$ (for $\theta = \sum_{i=1}^r \theta_i \otimes \xi_i^Q$ with respect to a basis $\{\xi_1, \ldots, \xi_r\}$ of $\Lambda \subset \mathfrak{t}$), and, for any fixed $z \in \overline{\Sigma}$, m_{Ψ^ε} denotes the normalized $\mathbb{T}_X$-moment map (1.1) of $\omega_0 + d_X d_X^c \Psi^\varepsilon$. Notice that, as v is uniformly bounded on P by positive constants, it follows by (3.14) that

$$\lim_{\varepsilon \to 0} \left((\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} \wedge \theta^{\wedge r} \right) = 0,$$

weakly (as measures on $Z \times \overline{\Sigma}$). The push-forward measure of the measure $(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} \wedge \theta^{\wedge r}$ to Y is the measure $(\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]}$, so we obtain on Y:

$$\lim_{\varepsilon \to 0} \left((\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} \right) = 0.$$

Furthermore, using the $C^{1,\alpha}$-convergence of Ψ^ε to Φ, we get the weak convergences (of positive $(1,1)$-currents):

$$\begin{aligned}\lim_{\varepsilon \to 0} (\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon) &= \pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Phi \geq 0; \\ 0 = \lim_{\varepsilon \to 0} (\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Psi^\varepsilon)^{[m+n+1]} &= (\pi_Y^*\tilde{\omega}_0 + d_Y d_Y^c \Phi)^{[m+n+1]}.\end{aligned}$$

Thus, (3.12) follows. □

3.1.3 The scalar curvature and the extremal vector field

Lemma 3.1.9. *Let ω, $\tilde{\omega}$ be $\mathbb{T}$-invariant Kähler metrics respectively on X and Y, given by* (3.3)*, and suppose (B_a, ω_a) has constant scalar curvature $Scal(\omega_a) = s_a$. Then, the scalar curvature $Scal(\tilde{\omega})$, considered as smooth function on $X \times Q$, is given by*

$$Scal(\tilde{\omega}) = \frac{1}{\mathrm{v}(m_\omega)} Scal_{\mathrm{v}}(\omega) + \sum_{a=1}^{k} \frac{s_a}{\langle p_a, m_\omega \rangle + c_a} \tag{3.15}$$

with $\mathrm{v}(x) = \prod_{a=1}^{k} (\langle p_a, x\rangle + c_a)^{n_a}$.

Proof. We apply the arguments in the proof [4, Prop. 7] to both $(X, \mathbb{T}_X)$ and $(Y, \mathbb{T}_Y)$ to compute the corresponding scalar curvatures, and compare the results.

On X, we consider the open dense subset $\mathring{X} \subset X$ of regular orbits for the $\mathbb{T}_X$-action. Consider the point-wise ω-orthogonal and $\mathbb{T}$-invariant decomposition

$$T\mathring{X} = H \oplus \mathfrak{t}_X \oplus J\mathfrak{t}_X,$$

and write the Kähler structure (g, J, ω) on X as

$$g = g_H + \sum_{i,j=1}^{r} H_{ij} \left(\eta_i \otimes \eta_j + J\eta_i \otimes J\eta_j\right),$$
$$\omega = \omega_H + \sum_{i,j=1}^{r} H_{ij} \eta_i \wedge J\eta_j,$$

where, for a fixed basis $\{\xi_1, \ldots, \xi_r\}$ of $\mathfrak{t}$, the 1-forms η_j on $\mathring{X}$ are defined by $(\eta_j)_{|H} = 0$, $\eta_j(\xi_i^X) = \delta_{ij}$; $\eta_j(J\xi_i^X) = 0$ and $H_{ij} = g(\xi_i^X, \xi_j^X)$.

The distribution $(\mathfrak{t}_X)_{\mathbb{C}} := \mathfrak{t}_X \oplus J\mathfrak{t}_X$ is integrable and defines a holomorphic foliation $\mathcal{F}$. We consider an open subset $U \subset \mathring{X}$ on which $\mathcal{F}$ is trivial. We denote by S the space of leaves of $\mathcal{F}$ in U (then S can be seen as an open subset of $\mathbb{C}^{m-r}$). We next fix a volume form Vol_S on S in some holomorphic coordinates, and write point-wisely

$$\omega_H^{[\ell-r]} = F\pi_S^*(\mathrm{Vol}_S), \tag{3.16}$$

for some positive (locally defined) smooth function F on U (where both $\omega_H^{[\ell-1]}$ and $\pi_S^*(\mathrm{Vol}_S)$ are seen as sections of $\bigwedge^{\ell-1} H^*$). According to [4, Prop. 7], we have that

$$\kappa := -\frac{1}{2}\left(\log(F) + \log\det(H_{ij})\right) \tag{3.17}$$

is a (local) Ricci potential of ω, i.e. $Ric(\omega) = d_X d_X^c \kappa$, and thus

$$Scal(\omega) = -2\frac{d_X d_X^c \kappa \wedge \omega^{[\ell-1]}}{\omega^{[\ell]}}.$$

By the same argument as above we get that on $\mathring{Z} = \mathring{X} \times B$

$$\tilde{\omega} = \omega_H + \sum_{i,j=1}^{r} H_{ij}\eta_i \wedge J\eta_j + \sum_{a=1}^{k}(\langle \mu_\omega, p_a\rangle + c_a)\pi_B^*\omega_a$$

We now take a local holomorphic volume form on $S \times U_B$ of the form $\mathrm{Vol}_S \wedge \mathrm{Vol}_{B_1} \wedge \cdots \wedge \mathrm{Vol}_{B_k}$, where $U_B \subset B$ is a open subset trivialazing the anticanonical bundle of B. Using (3.3), we see that a Ricci potential on Y (when pulled back to $X \times Q$) is written as

$$\tilde{\kappa} = \sum_{a=1}^{k} \kappa_a - \frac{1}{2}\left(\log(\tilde{F}) + \log\det(H_{ij})\right),$$

where $\kappa_a := -\frac{1}{2}\log\left(\frac{\omega_a^{[n_a]}}{\mathrm{Vol}_{B_a}}\right)$ is a Ricci potential of (B_a, ω_a) and

$$\tilde{F} = \mathrm{v}(m_\omega)F.$$

Thus, we obtain

$$\tilde{\kappa} = \sum_{a=1}^{k} \kappa_a + \kappa - \frac{1}{2}\log \mathrm{v}(m_\omega), \tag{3.18}$$

as functions on $X \times Q$. Introducing a basis $(\xi_i)_i$ of Λ and writing the connection 1-form $\theta \in \Omega^1(Q, \mathfrak{t})$ as $\theta = \sum_{j=1}^r \theta_j \otimes \xi_j^Q$ (where the 1-forms θ_j on Q are such that θ_j is zero on $\mathcal{D}$ and $\theta_j(\xi_i^Q) = \delta_{ij}$), we compute for the scalar curvature of $\tilde{\omega}$

$$\begin{aligned} Scal(\tilde{\omega}) &= -\frac{d_Y d_Y^c \tilde{\kappa} \wedge \tilde{\omega}^{[\ell+n-1]}}{\tilde{\omega}^{[\ell+n]}} \text{ (on } Y) \\ &= -\frac{d_Y d_Y^c \tilde{\kappa} \wedge \tilde{\omega}^{[\ell+n-1]} \wedge \theta^{\wedge r}}{\tilde{\omega}^{[\ell+n]} \wedge \theta^{\wedge r}} \text{ (on } X \times Q). \end{aligned} \tag{3.19}$$

By (3.5) and (3.18), the pullback of $d_Y d_Y^c \tilde{\kappa}$ to $X \times Q$ is given by,

$$\begin{aligned} d_Y d_Y^c \tilde{\kappa} =& d_Y d_Y^c \left(\kappa - \frac{1}{2} \log \mathrm{v}(m_\omega) \right) + \sum_{a=1}^k d_Y d_Y^c \kappa_a \\ =& d_X d_X^c \left(\kappa - \frac{1}{2} \log \mathrm{v}(m_\omega) \right) + \sum_{j=1}^r d_X \left(d_X^c \left(\kappa - \frac{1}{2} \log \mathrm{v}(m_\omega) \right) (\xi_j^X) \right) \wedge \theta_j \\ &+ \sum_{j=1}^r d_X^c \left(\kappa - \frac{1}{2} \log \mathrm{v}(m_\omega) \right) (\xi_j) d_Q \theta_j + \sum_{a=1}^k d_{B_a} d_{B_a}^c \kappa_a \\ =& d_X d_X^c \kappa - \frac{1}{2} d_X d_X^c \left(\log \mathrm{v}(m_\omega) \right) \\ &+ \sum_{j=1}^r d_X \left(d_X^c \left(\kappa - \frac{1}{2} \log \mathrm{v}(m_\omega) \right) (\xi_j^X) \right) \wedge \theta_j \\ &+ \sum_{a=1}^k d_X^c \left(\kappa - \frac{1}{2} \log \mathrm{v}(m_\omega) \right) (p_a)(\pi_B^* \omega_a) + \sum_{a=1}^k d_{B_a} d_{B_a}^c \kappa_a, \end{aligned} \tag{3.20}$$

where in the last equality we used (3.1) and we have denoted by p_a the induced vector field on X by the element $p_a \in \mathfrak{t}$. We shall compute the term $d_X^c \kappa(p_a)$ on S : using (3.17) we get

$$d_X^c \kappa(p_a) = \frac{1}{2} \left(\frac{\mathcal{L}_{Jp_a} F}{F} + \mathrm{Tr} \left(H_{ij}^{-1} (\mathcal{L}_{Jp_a} H_{ij}) \right) \right). \tag{3.21}$$

Taking the wedge product of both sides of (3.16) with

$$\left(\sum_{i,j=1}^r H_{ij} \eta_i \wedge J\eta_j \right)^{[r]} = \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J\eta_j),$$

gives

$$\omega^{[\ell]} = F\pi_S^*\mathrm{Vol}_S \wedge \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J\eta_j).$$

Applying the Lie derivative $\mathcal{L}_{Jp_a}$ to the above equality yields

$$\begin{aligned}(\Delta_\omega m_\omega^{p_a})\,\omega^{[\ell]} = (\mathcal{L}_{J\xi_a}F)\pi_S^*\mathrm{Vol}_S \wedge \det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J\eta_j) \\ + F\pi_S^*\mathrm{Vol}_S \wedge \mathcal{L}_{Jp_a}\left(\det(H_{ij}) \bigwedge_{j=1}^r \eta_j \wedge J\eta_j\right)\end{aligned}$$

$$\begin{aligned}F\pi_S^*\mathrm{Vol}_S \wedge \mathcal{L}_{Jp_a}\left(\det(H_{ij}) \bigwedge_{j=1}^r (\eta_j \wedge J\eta_j)\right) \\ = \left(\mathrm{Tr}\left(H_{ij}^{-1}(\mathcal{L}_{Jp_a}H_{ij})\right)\right)F\pi_S^*\mathrm{Vol}_S \wedge (\det(H_{ij})) \bigwedge_{j=1}^r \eta_j \wedge J\eta_j,\end{aligned}$$

where we used that $\mathcal{L}_{Jp_a}\eta_j$ is a basic form (since $(\mathcal{L}_{Jp_a}\eta_j)(\xi_i) = -\eta_j([Jp_a, \xi_i]) = 0$). We thus get $\Delta_\omega m_\omega^{p_a} = \frac{\mathcal{L}_{Jp_a}F}{F} + \mathrm{Tr}\left(H_{ij}^{-1}(\mathcal{L}_{Jp_a}H_{ij})\right)$, or equivalently, in terms of (3.21)

$$d_X^c\kappa(p_a) = \frac{1}{2}\left(\Delta_\omega m_\omega^{p_a}\right). \tag{3.22}$$

Using the above equation in (3.20), we continue the computation

$$\begin{aligned}d_Y d_Y^c\tilde{\kappa} =& d_X d_X^c\kappa - \frac{1}{2}dd_X^c\left(\log \mathrm{v}(m_\omega)\right) + \sum_{j=1}^r d_X\left(d_X^c\left(\kappa - \frac{1}{2}\log \mathrm{v}(m_\omega)\right)(\xi_j^X)\right) \wedge \theta_j \\ &+ \frac{1}{2}\sum_{a=1}^k\left(\Delta_\omega m_\omega^{p_a} + \frac{(\mathcal{L}_{Jp_a}(\mathrm{v}(m_\omega))}{\mathrm{v}(m_\omega)}\right)(\pi_B^*\omega_a) + \sum_{a=1}^k d_{B_a}d_{B_a}^c\kappa_a.\end{aligned} \tag{3.23}$$

Recall that (3.6) on Z we have $\tilde{\omega}^{[\ell+n]} \wedge \theta^{\wedge r} = \mathrm{v}(m_\omega)\omega^{[\ell]} \wedge \bigwedge_{a=1}^k \pi_B^*\omega_a^{[n_a]} \wedge \theta^{\wedge r}$.

Similarly, by (3.3),

$$\tilde{\omega}^{[\ell+n-1]} \wedge \theta^{\wedge r} = \sum_{b=1}^{k} \left(\frac{\mathrm{v}(m_\omega)}{(\langle m_\omega, p_b\rangle + c_b)} \omega^{[\ell]} \wedge (\pi_B^*\omega_b)^{[n_b-1]} \wedge \bigwedge_{\substack{a=1\\a\neq b}}^{k} (\pi_B^*\omega_a)^{[n_a]} \wedge \theta^{\wedge r} \right) + \mathrm{v}(m_\omega)\omega^{[\ell-1]} \wedge \bigwedge_{a=1}^{k} (\pi_B^*\omega_a)^{[n_a]} \wedge \theta^{\wedge r}. \tag{3.24}$$

Using (3.19), (3.23), (3.6) and (3.24), we obtain

$$\begin{aligned} Scal(\tilde{\omega}) =& Scal(\omega) + \Delta_\omega(\log \mathrm{v}(m_\omega)) \\ &+ \sum_{a=1}^{k} \left(\frac{n_a}{(\langle m_\omega, p_a\rangle + c_a)} \left[\Delta_\omega m_\omega^{p_a} + \frac{\mathcal{L}_{Jp_a}\mathrm{v}(m_\omega)}{\mathrm{v}(m_\omega)} \right] + \frac{s_a}{(\langle m_\omega, p_a\rangle + c_a)} \right) \\ =& Scal(\omega) + \sum_{a=1}^{k} n_a \Delta_\omega(\log(\langle m_\omega, p_a\rangle + c_a)) \\ &+ \sum_{a=1}^{k} \frac{n_a}{(\langle m_\omega, p_a\rangle + c_a)} \left(\Delta_\omega(\langle m_\omega, p_a\rangle) + \frac{\mathcal{L}_{Jp_a}(\mathrm{v}(m_\omega))}{\mathrm{v}(m_\omega)} \right) \\ &+ \sum_{a=1}^{k} \frac{n_a}{(\langle m_\omega, p_a\rangle + c_a)} \frac{s_a}{(\langle m_\omega, p_a\rangle + c_a)} \\ =& Scal(\omega) - \sum_{a,b=1}^{k} \frac{n_a n_b g(p_a, p_b)}{(\langle m_\omega, p_a\rangle + c_a)(\langle m_\omega, p_b\rangle + c_b)} \\ &+ \sum_{a=1}^{k} \left(\frac{2n_a \Delta_\omega(\langle m_\omega, p_a\rangle)}{(\langle m_\omega, p_a\rangle + c_a)} + \frac{n_a |\xi_a|^2_{g_\omega}}{(\langle m_\omega, p_a\rangle + c_a)^2} + \frac{s_a}{(\langle m_\omega, p_a\rangle + c_a)} \right). \end{aligned} \tag{3.25}$$

On the other hand, using a basis (ξ_i) of $\mathfrak{t}$ with a dual basis (ξ^i) of $\mathfrak{t}^*$, we compute

$$\begin{aligned} Scal_{\mathrm{v}}(\omega) :=& \mathrm{v}(m_\omega) Scal(\omega) + 2\sum_{i=1}^{r} \mathrm{v}_{,i}(m_\omega) \Delta_\omega(\langle m_\omega, \xi_i\rangle) - \sum_{i,j=1}^{r} \mathrm{v}_{,ij}(m_\omega) g_\omega(\xi_i, \xi_j) \\ =& \mathrm{v}(m_\omega) Scal(\omega) + 2\sum_{i=1}^{r} \Delta_\omega(\langle m_\omega, \xi_i\rangle) \sum_{a=1}^{k} \frac{n_a \xi^i(p_a)\mathrm{v}(m_\omega)}{\langle m_\omega, p_a\rangle + c_a} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{r} g_\omega(\xi_i,\xi_j)\left(\sum_{a=1}^{k}\frac{n_a\xi^i(p_a)\xi^j(p_a)\mathrm{v}(m_\omega)}{(\langle m_\omega,p_a\rangle+c_a)^2}\right)\\
&-\sum_{i,j=1}^{r} g_\omega(\xi_i,\xi_j)\left(\sum_{a,b=1}^{k}\frac{n_an_b\xi^i(p_a)\xi^j(p_a)\mathrm{v}(m_\omega)}{(\langle m_\omega,p_a\rangle+c_a)(\langle m_\omega,p_b\rangle+c_b)}\right)\\
=&\mathrm{v}(m_\omega)Scal(\omega)+2\Delta_\omega(\langle m_\omega,p_a\rangle)\sum_{a=1}^{k}\frac{n_a\mathrm{v}(m_\omega)}{(\langle m_\omega,p_a\rangle+c_a)}\\
&+\left[\sum_{a=1}^{k}\frac{n_a|p_a|^2_{g_\omega}\mathrm{v}(m_\omega)}{(\langle m_\omega,p_a\rangle+c_a)^2}-\sum_{a,b=1}^{k}\frac{n_an_bg_\omega(p_a,p_b)\mathrm{v}(m_\omega)}{(\langle m_\omega,p_a\rangle+c_a)(\langle m_\omega,p_b\rangle+c_b)}\right].
\end{aligned}
$$

Comparing the above expression with (3.25), we obtain

$$
Scal(\tilde\omega)=\frac{1}{\mathrm{v}(m_\omega)}Scal_{\mathrm{v}}(\omega)+\left(\sum_{a=1}^{k}\frac{s_a}{\langle m_\omega,p_a\rangle+c_a}\right). \tag{3.26}
$$

□

The following is a straightforward extension of [7, Lemma 5].

Lemma 3.1.10. *Suppose Y is a semisimple principal $(X,\mathbb{T})$-fibration over B, such that $\mathbb{T}$ is a maximal torus in the reduced group of automorphisms $\mathrm{Aut}_r(X)$. Let $\tilde\omega$ be a bundle-compatible Kähler metric on Y corresponding to a $\mathbb{T}$-invariant Kähler metric ω on X, and $\mathbb{K}_B\subset\mathrm{Aut}_r(B)$ be a maximal compact torus in the reduced group of automorphisms of B which (without loss by Lichnerowicz–Matsushima theorem) belongs to the isometry group of ω_B. Then $\tilde\omega$ is invariant under the action of a maximal torus $\mathbb{K}_Y\subset\mathrm{Aut}_r(Y)$, and we have an exact sequence of Lie algebra*

$$
\{0\}\to\mathfrak{t}_Y\to\mathfrak{k}_Y\to\mathfrak{k}_B\to\{\,0\},
$$

where $\mathfrak{k}_B$ and $\mathfrak{k}_Y$ denote respectively the Lie algebras of $\mathbb{K}_B$ and $\mathbb{K}_Y$. Furthermore, the affine extremal function $\ell_{\rm ext}$ of a compatible Kähler class $(Y, J_Y, [\omega_Y], \mathbb{K}_Y)$ is an affine function on P.

Proof. The proof of the above result is not materially different than the proof of [7, Lemma 5] (where $(X, \mathbb{T})$ is toric). We only give a sketch. A Killing potential f for a Killing vector field $K \in \mathfrak{k}_B := \mathrm{Lie}(\mathbb{K}_B)$ is of the form $f = \sum_{a=1}^k f_a$, where f_a is a Killing potential of (B_a, ω_a). Letting $\tilde{K}$ be the horizontal lift of K to Q (using the $\mathfrak{t}_Q$-valued connection 1-form θ), one can check that the vector field on Q

$$\hat{K} = \tilde{K} + \sum_{a=1}^{k} f_a \xi^Q_{p_a}$$

is a CR vector field on $(Q, \mathcal{D}, J_B)$, hence also on $(Z, \mathcal{H}, J_B \oplus J_X)$. Furthermore, a direct verification in (3.3) reveals that

$$\imath_{\hat{K}}\tilde{\omega} = -d\left(\sum_{a=1}^{k}(\langle p_a, m_\omega\rangle + c_a)f_a\right) \tag{3.27}$$

showing that $\hat{K}$ also preserves $\tilde{\omega}$. We thus obtain a lift $\hat{\mathfrak{k}}_B$ of the Lie algebra $\mathfrak{k}_B = \mathrm{Lie}(T_B)$ to Z, which clearly commutes with the action $\mathbb{T}_Z$, and preserves both the CR structure of $(Z, \mathcal{H})$ and the 2-form $\tilde{\omega}$. The Lie algebra $\mathfrak{k}_Y$ of $\mathbb{K}_Y$ is then induced by $\mathfrak{t}_X \oplus \hat{\mathfrak{k}}_B \subset TZ$, which descend to an abelian Lie algebra of Killing fields on Y. The maximality of $\mathbb{K}_Y \subset \mathrm{Aut}_{\rm red}(Y)$ and the exactness of the sequence follow from the maximality of each $\mathbb{K}_B \subset \mathrm{Aut}_{\rm red}(B)$ and $\mathbb{T} \subset \mathrm{Aut}_{\rm red}(X)$, and the fact that (recall that Y is a locally trivial X-fibre bundle and therefore the fibres have trivial normal bundle) any holomorphic vector field on Y projects under π_B to a holomorphic vector field on B.

For the second claim in Lemma 3.1.10, we recall that, by definition, the affine-extremal function is the unique affine function such that the Futaki invariant vanishes

$$\mathbf{F}(l) := \int_Y l(m_{\tilde{\omega}})(Scal(\tilde{\omega}) - l_{ext}(m_{\tilde{\omega}}))\tilde{\omega}^{[m]} = 0, \tag{3.28}$$

for any $l \in \mathrm{Aff}(\mathfrak{k}^*)$. By (3.27) the Killing potentials of all lifted Killing vector fields $\hat{K}$ from B are of the form $\sum_{a=1}^k (\langle p_a, m_\omega\rangle + c_a) f_a$. Thus, by Lemma 3.1.9 and using (3.6), the integral condition (3.28) holds for any such Killing potential, as soon as we normalize $\int_{B_a} f_a \omega_a^{n_a} = 0$. Moreover, by the exactness of the sequence established above and the previous comment, (3.28) is equivalent to

$$\int_X l(m_\omega)(Scal_{\mathrm{v}}(\omega) - l_{ext}(m_\omega))\omega^{[\ell]} = 0, \tag{3.29}$$

for any $l \in \mathrm{Aff}(\mathfrak{t}^*)$, showing that l_{ext} is an affine-linear function on $\mathfrak{t}^*$.

□

We deduce the two following straightforward corollaries.

Corollary 3.1.11. *A compatible Kähler metric ω_Y is extremal if and only if its corresponding metric ω_X on X is* (v, w)*-cscK for the weights*

$$\begin{aligned} \mathrm{v}(x) &= \prod_{a=1}^k \left(\langle p_a, x\rangle + c_a\right)^{n_a} \\ \mathrm{w}(x) &= \mathrm{v}(x)\left(l_{\mathrm{ext}}(x) - \sum_{a=1}^k \frac{s_a}{\langle p_a, x\rangle + c_a}\right), \end{aligned} \tag{3.30}$$

Proof. It is a direct consequence of (3.15) and Lemma 3.1.10. □

Corollary 3.1.12. *Fix $\mathbb{T}_Y$ and $\mathbb{K}_Y$ as in Lemma 3.1.10. The restriction of the $\mathbb{K}_Y$-relative Mabuchi energy $\mathcal{M}^Y$ on Y to the subspace $\mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{K}}$ is equal to $C\mathcal{M}^X_{\mathrm{v},\mathrm{w}}$, where* v, w *are given in (3.30) and* $C = \mathrm{Vol}(B, \omega_B)$.

Proof. It is a direct corollary of Lemmas 3.1.9 and 3.1.10 and of Definitions 1.21 and 1.22 . □

3.1.4 Fano semisimple principal fibrations

In this subsection, we specialize to the case when each (B_a, ω_a) is a Hodge Kähler–Einstein manifold with positive scalar curvature $s_a = 2n_a k_a$, $k_a \in \mathbb{N}$. Equivalently, $2\pi c_1(B_a) = k_a[\omega_a]$ for a positive integer k_a and an integral Kähler class $\frac{1}{2\pi}[\omega_a]$. Notice that k_a must be a positive divisor of the Fano index $\mathrm{Ind}(B_a)$ of B_a, which yields the a priori bound $1 \leq k_a \leq \mathrm{Ind}(B_a)$. We also assume that $(X, \mathbb{T})$ is Fano with canonically normalized moment polytope, i.e. for any $\mathbb{T}$-invariant Kähler form $\omega \in 2\pi c_1(X)$ we normalize m_ω such that

$$m_{Ric(\omega)} = m_\omega + d^c_X h, \tag{3.31}$$

where h is a $\mathbb{T}$-invariant smooth function on X such that $Ric(\omega) = \omega + d_X d^c_X h$, and $m_{Ric(\omega)} := \frac{1}{2}\Delta_\omega m_\omega$ is the "moment map" of $Ric(\omega)$.

We then have

Lemma 3.1.13. *In the setting above, if the affine linear functions $(\langle p_a, x\rangle + k_a) > 0$ on P, the bundle-compatible Kähler metric $\tilde{\omega}$ on Y corresponding to the constants $c_a = k_a$ belongs to deRham class $2\pi c_1(Y)$.*

Proof. By using (3.22) and rearranging the terms in (3.20), we have the following relation (written on Z):

$$\begin{aligned} Ric(\tilde{\omega}) =& Ric(\omega) + \sum_{a=1}^{k} \left(\langle p_a, m_{Ric(\omega)}\rangle + c_a\right)\omega_a + \langle dm_{Ric(\omega)} \wedge \theta\rangle \\ &+ \sum_{a=1}^{k} (Ric(\omega_a) - c_a\omega_a) - \frac{1}{2} d_Y d^c_Y \log \mathrm{v}(m_\omega), \end{aligned} \tag{3.32}$$

where $Ric(\tilde{\omega})$, $Ric(\omega)$ and $Ric(\omega_a)$ respectively denote the Ricci forms of $(Y, \tilde{\omega})$, (X, ω) and (B_a, ω_a), pulled back to Z, and $m_{Ric(\omega)} := d^c_X \kappa$ is the "moment map" with respect to the Ricci form $Ric(\omega)$. As in (3.22), we have $m_{Ric(\omega)} = \frac{1}{2}\Delta_\omega m_\omega$. A closer look at the proof of Lemma 3.1.4 and the relation (3.32) (with $c_a = \frac{s_a}{2n_a} = k_a$) show that

$$Ric(\tilde{\omega}) - \tilde{\omega} = \frac{1}{2} d_Y d^c_Y \tilde{h}, \qquad \tilde{h} := h - \log \mathrm{v}(m_\omega).$$

.

□

Remark 3.1.14. *Lemma 3.1.13 provides a useful way to construct semisimple* $(X, \mathbb{T})$*-principal Fano fibrations. Indeed, for given positive Hodge Kähler–Einstein manifolds* (B_a, J_a, ω_a) *as above, with corresponding integer constants* k_a*, and a given Fano manifold* $(X, \mathbb{T})$ *with associated canonical polytope* P*, one can try to find all the possible principal* $\mathbb{T}$*-bundles* Q *over* $B = \prod_{a=1}^k B_a$*, for which the corresponding semisimple* $(X, \mathbb{T})$*-principal fibration is Fano. Such principal* $\mathbb{T}$*-bundles* Q *are in correspondence with the choice of lattice elements* $p_a \in \Lambda \subset \mathfrak{t}$ *and Lemma 3.1.13 tels us that for a set of elements* p_a *to determine a Fano semisimple* $(X, \mathbb{T})$*-principal fibration* Y*, it is sufficient to check that they satisfy*

$$\langle p_a, x\rangle + k_a > 0 \text{ on } P.$$

We refer to Section 2.3 for explicit constructions of Fano semisimple principal toric fibration with $\mathbb{P}^1$ *and* $\mathbb{P}^2$ *fiber.*

3.2 An analytic criterion for semisimple principal fibrations

This section is devoted to the proof of Theorem 3.2.2 below. We first introduced the relevant notion of properness for the weighted Mabuchi energy.

Definition 3.2.1. *The weighted Mabuchi energy* $\mathcal{M}_{\mathrm{v,w}} : \mathcal{K}(X, \omega_0)^{\mathbb{T}} \to \mathbb{R}$ *is said* $d_{1,\mathbb{T}^{\mathbb{C}}}$*-proper if*

- $\mathcal{M}_{\mathrm{v,w}}$ *is bounded from below on* $\mathcal{K}(X, \omega_0)^{\mathbb{T}}$*;*
- *for any sequence* $\varphi_i \in \mathring{\mathcal{K}}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}}$, $d_{1,\mathbb{T}^{\mathbb{C}}}(0, \varphi_i) \to \infty$ *implies that* $\mathcal{M}_{\mathrm{v,w}}(\varphi_i) \to \infty$,

where the relative Darvas distance $d_{1,\mathbb{T}^{\mathbb{C}}}$ *is defined in* (1.19) *and the space of normalized relative potential* $\mathring{\mathcal{K}}_{\mathrm{v}}(X, \omega_0)^{\mathbb{T}}$ *is defined in* (1.25)*.*

It is worth noting that, according to Lemma 3.1.6, the notion of properness defined above with the distance d_1 is equivalent to the one defined with weighted distance $d_{1,\mathrm{v}}$. We conclude the section with the main theorem of this chapter.

Theorem 3.2.2. *Suppose* Y *is a semisimple principal* $(X, \mathbb{T})$*-fibration, with a Kähler metric* ω_Y *induced by a* $\mathbb{T}$*-invariant Kähler metric* ω_X *on* X*. We suppose, moreover, that* $\mathbb{T}$ *is a maximal torus in the reduced group of automorphisms* $\mathrm{Aut}_{\mathrm{red}}(X)$*. Then, the following conditions are equivalent*

1. Y *admits an extremal Kähler metric in the Kähler class* $[\omega_Y]$*.*
2. Y *admits a compatible extremal Kähler metric in the Kähler class* $[\omega_Y]$*.*
3. X *admits a* $\mathbb{T}$*-invariant* (v, w)*-cscK metric in the Kähler class* $[\omega_X]$*, with weights* (3.30)*.*
4. *The weighted Mabuchi energy* $\mathcal{M}_{\mathrm{v,w}}$ *of* $(X, [\omega_X], \mathbb{T})$ *is* $d_{1,\mathbb{T}^{\mathbb{C}}}$*-proper for the weights* (v, w) *defined in* (3.30)*.*

Proof. The implication (3) ⇒ (4) is established in [9, Theorem 1] and is admit there. The equivalence (2) ⇔ (3) follows from Lemmas 3.1.9 and 3.1.10. We shall prove below (4) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). The arguments are similar to those used in the proof of Theorem 1.5.1 when $(X, \mathbb{T})$ is toric. We will highlight only the differences. Let $\mathbb{K}_Y$ be a maximal torus in $\mathrm{Aut}_{\mathrm{red}}(Y)$ that contains $\mathbb{T}_Y$ as defined in Lemma 3.1.10. We fix a $\mathbb{K}_Y$-invariant compatible Kähler form $\tilde{\omega}_0$ on Y corresponding to a $\mathbb{T}$-invariant Kähler form ω_0 on X, see Lemma 3.1.4.

Proof of (4) $\Rightarrow$ (3). As in the proof of Theorem 1.5.1, we consider the continuity path for $\varphi \in \mathcal{K}(X, \omega_0)^{\mathbb{T}} \subset \mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ given by

$$t\big(Scal_{\mathrm{v}}(\omega_\varphi) - \mathrm{w}(m_\varphi)\big) = (1-t)\big(\Lambda_{\omega_\varphi, \mathrm{v}}(\chi) - n - \ell\big), \quad t \in [0,1], \tag{3.33}$$

for some Kähler metric $\chi \in [\omega_0]$ that we will choose latter and $\Lambda_{\omega_\varphi,\mathrm{v}}(\chi)$ is defined in (1.40). We consider

$$S_{t_1} := \{t \in (0, t_1] \mid (3.33) \text{ has a solution } \varphi_t \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}\}. \tag{3.34}$$

We need to show that S_1 is open, closed and non empty in $(0, 1]$.

Step 1: Openness. The arguments to prove that S_1 is non-empty are identical to those used to prove Proposition 1.5.3: we can choose a suitable χ compatible with the bundle construction i.e. of the form of (1.9) such that there exists a solution φ_{t_0} for some $t_0 > 0$. The openness of the continuity path follows by applying the inverse function theorem and using the decomposition of the operators established in Appendix D.0.3.

Step 2: Closedness. Assuming (4) in Theorem 3.2.2, by *Step 1* and similar arguments to those used in the proof of Proposition 1.5.5, we can find compatible

Kähler metrics $\tilde{\omega}_{\varphi_{t_i}}$ which are solutions of (3.33) at t_i, and elements $\gamma_i \in \mathbb{T}_Y^{\mathbb{C}} \subset \mathbb{K}_Y^{\mathbb{C}}$ such that $\gamma_i^*(\tilde{\omega}_{\varphi_{t_i}}) = \tilde{\omega}_{\varphi_{t_i}} + d_Y d_Y^c(\gamma \cdot \varphi_{t_i})$ converges to an extremal Kähler metric $\tilde{\omega}_{\varphi_1}$, where $\gamma \cdot \varphi_{t_i}$ is the normalized Kähler potential in $\mathring{\mathcal{K}}(Y, \tilde{\omega}_0)^{\mathbb{T}}$ (see (1.24)) corresponding to $\gamma_i^*(\tilde{\omega}_{\varphi_{t_i}})$. Since each $\tilde{\omega}_{\varphi_{t_i}}$ are compatible (i.e. of the form of (1.9)), by Lemma 3.1.4, $\varphi_{t_i} \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}$. Moreover, the $\mathbb{T}^{\mathbb{C}}$-action preserves $\mathcal{K}(X, \omega_0)^{\mathbb{T}}$, showing that $\gamma_i \cdot \varphi_{t_i} \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}$. On the other hand, by (3.1.1), $\mathcal{K}(X, \omega_0)^{\mathbb{T}}$ is closed in $\mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{T}}$, showing that $\varphi_1 \in \mathcal{K}(X, \omega_0)^{\mathbb{T}}$, i.e $\tilde{\omega}_{\varphi_1}$ is a compatible extremal metric on Y.

$\square$

Proof of (1) $\Rightarrow$ (3). The proof is similar to the proof of (4) $\Rightarrow$ (3) above. As in the Step 1 of the latter, we consider the continuity path (3.33). We can assume without loss [21] that Y admits a $\mathbb{K}_Y$-invariant extremal Kähler metric in $[\tilde{\omega}_0]$. We want to solve the continuity path (3.33). The proof of openness is the same as it was in (4) $\Rightarrow$ (3) above.

We now focus on the closedness. By the results of of Chen–Cheng and He [25, 26, 47], the $\mathbb{K}$-relative Mabuchi energy is $\mathbb{K}^{\mathbb{C}}$-proper on $\mathcal{K}(Y, \tilde{\omega}_0)^{\mathbb{K}}$. By the same argument as in (4) $\Rightarrow$ (3), there exists a sequence $\tilde{\omega}_{\varphi_{t_i}}$ of solutions of (3.33) at $t = t_i$ and elements $\gamma_i \in \mathbb{K}_Y^{\mathbb{C}}$ such that $\gamma_i^*(\tilde{\omega}_{\varphi_{t_i}}) = \tilde{\omega}_{\varphi_{t_i}} + d_Y d_Y^c(\gamma \cdot \varphi_{t_i})$ converges smoothly to an extremal Kähler metric $\tilde{\omega}_{\varphi_1}$. The main difference with (4) $\Rightarrow$ (3) is that the convergence of $\tilde{\omega}_{\varphi_{t_i}}$ is obtained up to the action of $\mathbb{K}^{\mathbb{C}}$, which does not preserve $\mathcal{K}(X, \omega_0)^{\mathbb{T}}$. We thus need to modify slightly the argument in order to show that $\tilde{\omega}_{\varphi_1}$ still induces a (v, w)-cscK metric on any given fiber $X_b = \pi_B^{-1}(b) \subset Y$. We denote by $\omega_j(b) := (\tilde{\omega}_{\varphi_j})_{|X_b}$ and $\overline{\omega}_j(b) := (\gamma_j^*(\tilde{\omega}_{\varphi_j}))_{|X_b}$ the induced $\mathbb{T}$-invariant metrics on X_b. As $\tilde{\omega}_{\varphi_j}$ is bundle-compatible, Lemma 3.1.9 yields

$$Scal_{\mathrm{v}}(\omega_j(b)) = \left[\mathrm{v}(m_{\tilde{\omega}_{\varphi_j}})\left(Scal(\tilde{\omega}_{\varphi_j}) - \sum_{a=1}^{k}\frac{s_a}{\langle p_a, m_{\tilde{\omega}_{\varphi_j}}\rangle + c_a}\right)\right]_{|X_b}.$$

Using that $\gamma_j \in \mathbb{K}_Y^{\mathbb{C}}$ sends the fiber X_b to a fiber $X_{\gamma_j(b)}$ (this follows from the construction of $\mathbb{K}_Y$ in the proof of Lemma 3.1.10) the above equality holds true for the metrics $\overline{\omega}_j(b)$, where in the RHS we replace the metric $\tilde{\omega}_{\varphi_j}$ on Y with $\bar{\omega}_j := \gamma_j^*(\tilde{\omega}_{\varphi_j})$. It thus follows by the smooth convergence of $\overline{\omega}_j(b)$ to $\omega_1(b)$ that

$$\begin{aligned} Scal_{\mathrm{v}}(\omega_{\varphi_1}(b)) &= \left[\mathrm{v}(m_{\bar{\omega}_{\varphi_1}})\left(Scal(\bar{\omega}_{\varphi_1}) - \sum_{a=1}^{k}\frac{s_a}{\langle p_a, m_{\bar{\omega}_{\varphi_1}}\rangle + c_a}\right)\right]_{|X_b} \\ &= \left[\mathrm{v}(m_{\bar{\omega}_{\varphi_1}})\left(l_{\mathrm{ext}} - \sum_{a=1}^{k}\frac{s_a}{\langle p_a, m_{\bar{\omega}_{\varphi_1}}\rangle + c_a}\right)\right]_{|X_b} \end{aligned}$$

where for the equality on the second line we have used that the $\mathbb{K}_Y$-extremal function l_{ext} belongs to $\mathrm{Aff}(\mathfrak{t}_X^*)$ (see Lemma 3.1.10). Thus $\omega_1(b)$ is a (v,w)-cscK metric on X.

□

□

CHAPTER IV

CONCLUSION AND FUTURE DIRECTIONS

In this thesis, we investigated the Calabi problem on fibrations associated to principal torus bundles. When the fiber is toric, we obtained a Yau-Tian-Donaldson correspondence for the existence of extremal Kähler metrics, expressed in terms of a stability condition of the Delzant polytope of the fiber. We deduced numerous sufficient conditions for the stability of the polytope to hold and found new examples of extremal metrics. More generally, when the fiber is a compact Kähler manifold endowed with an isometric hamiltonian action of a torus, we showed that the existence of an extremal metric is characterised by the properness of the weighted Mabuchi energy of the fiber (for appropriate weight functions).

The main argument to established the correspondence consists of solving the continuity path of Chen in the space of bundle-compatible Kähler metrics. We hope that similar arguments could be adapted to a more general context. Some future directions are the following.

1. Solve Conjecture 1 by considering rigid semisimple fibration with blow-down and base which is only a local product. This will imply a Yau–Tian–Donaldson correspondence for projective bundles over a curve $\mathbb{P}(E) \to \Sigma_{\mathbf{g}}$, see [10].

2. Consider fibrations associated to a principal $\mathbb{T}$-bundle with a more general connection, i.e. remove the semisimplicity assumption (1.3).

3. Consider fibrations associated to a principal G-bundle where G is a compact connected group and the fiber is a compact Kähler G-manifold.

The equivalence established between the existence of a weighted cscK metric on the fiber of a semisimple principal fibration and the properness of the weighted Mabuchi energy can be seen as a weighted version of Chen-Cheng's theorem for special polynomial weights. We will consider the extension of this result to the case of general weights.

APPENDIX A

A PYTHON PROGRAM

We provide in this appendix an elementary Python program using SymPy which checks the sufficient condition from Corollary 2.2.5 for *simple* principal toric fibrations (that is, the case when the base has only one factor) with Fano toric fiber X of dimension one or two such that $[\omega_X]$ a multiple of $2\pi c_1(X)$.

The only data from the simple principal toric bundle construction needed are:

- from the base, the dimension $n \in \mathbb{Z}$ and scalar curvature $s \in \mathbb{Q}$
- from the Fano toric fiber of dimension $\ell \in \{1, 2\}$, the reflexive moment polytope $P \subset \mathbb{R}^\ell = \mathbb{Z}^\ell \otimes \mathbb{R}$, and the multiple $t = \frac{[\omega_X]}{2\pi c_1(X)} \in \mathbb{R}$
- An integer $p \in \mathbb{Z}$ if the fiber is one-dimensional, and with an element $p = (p_1, p_2) \in \mathbb{Z}^2$ if the fiber is of dimension two;
- a constant $c \in \mathbb{R}$ defining the admissible Kähler class.

We wish to compute the expression given by the right-hand side of (2.6)

$$\texttt{test} = 2(\ell + n + 1) + \frac{ts - 2nc}{p(x) + c} - tl_{\text{ext}}(x)$$

in order to check the condition. For this, it suffices to compute the extremal function l_{ext} by solving the linear system which defines it. Our short programs compute l_{ext}, then evaluate **test** at the vertices of P and returns the minimum if all the data are explicitly given. If the minimum returned by the program is non-negative, the data correspond to a simple principal toric fibration with an admissible Kähler class and $c > \frac{ts}{2n}$, then there exists an extremal Kähler metric. We may also let some of the data remain unknown and treat them as variables.

```
import sympy as sym
# variable on the line (here the fiber is one-dimensional)
x = sym.symbols('x')
# data of the simple principal toric fibration
p, c = sym.symbols('p,c')
n, s, t = 3, -6, 1
# weights
l = c+p*x
v, w0 = l**n, -s*l**(n-1) # for now, unknown l_ext is replaced
    with zero
# Donaldson-Futaki invariant with weights (v,w0)
def DF0(f):
    interior=sym.integrate(f*w0, (x, -t, t))
    facets=(f*v).subs(x,-t)+(f*v).subs(x,t)
    return(interior+facets)
# Compute the extremal function lext
X=sym.Matrix(2, 1, [1, x])
M=sym.Matrix(2, 2, lambda i,j:
    sym.integrate(X[i,0]*X[j,0]*v, (x, -t, t)))
V=sym.Matrix(2, 1, [DF0(1), DF0(x)])
Lext=M.LUsolve(V)
lext=((Lext.T)*X)[0,0]
# Compute expression test at the two vertices and print it
test=2*(1+1+n)+(t*s-2*n*c)/l-t*lext
print(sym.factor(test.subs(x,-t)))
print(sym.factor(test.subs(x,t)))
```

Program A.1 Rank one simple principal toric fibrations

Program A.1 prints the condition to check when c and p are variables, $n = 3$, $s = -6$ and $t = 1$, as used in Proposition 2.3.1. By modifying Line 5 and 6, one can obtain the conditions for an arbitrary simple principal $\mathbb{P}^1$-bundle.

```
1 import sympy as sym
2 # variables on the plane
3 x1, x2 = sym.symbols(’x1,x2’)
4 # data of toric fibration and admissible Kahler class
5 c, p1, p2, n, s, t = 12, 1, 2, 3, 18, 1
6 ## weights associated to the data
7 l=c+p1*x1+p2*x2
8 v=l**n
9 w0=-s*l**(n-1) # for now, unknown l_ext replaced with zero
10 # list of vertices of the polytope
11 vert= [[2*t,-t], [-t,-t], [-t,2*t]]
12 # Donaldson-Futaki invariant with weights (v,w0)
13 def DF0(f):
14   interior=sym.integrate(sym.integrate(f*w0,(x2,-t,t-x1)),(x1,-t,2*t))
15   facet1=sym.integrate((2*f*v).subs(x2,-t),(x1,-t,2*t))
16   facet2=sym.integrate((2*f*v).subs(x2,t-x1),(x1,-t,2*t))
17   facet3=sym.integrate((2*f*v).subs(x1,-t),(x2,-t,2*t))
18   return(interior+facet1+facet2+facet3)
19 # Compute the extremal function l_ext
20 X=sym.Matrix(3, 1, [1, x1, x2])
21 M=sym.Matrix(3, 3, lambda i,j:
22     sym.integrate(sym.integrate(X[i,0]*X[j,0]*v,(x2,-t,t-x1)),(x1,-t,2*t)))
23 V=sym.Matrix(3, 1, [DF0(1), DF0(x1), DF0(x2)])
24 Lext=M.LUsolve(V)
25 lext=((Lext.T)*X)[0,0]
26 # Compute and print the minimum of expression test on vertices
27 test=2*(1+2+n)+(t*s-2*n*c)/l-t*lext
28 test_vertices=test.subs(x1,vert[0][0]).subs(x2,vert[0][1])
```

```
29 for i in range(1,len(vert)):
30   test_vertices=sym.Min(test_vertices,
31                       test.subs(x1,vert[i][0]).subs(x2,vert[i][1]))
32 print("The minimum of expression test on vertices is ", test_vertices)
```

Program A.2 Simple principal $\mathbb{P}^2$ toric fibrations

Program A.2 computes the condition when all the data are given the fixed values $(c, p1, p2, n, s, t) = (12, 1, 2, 3, 18, 1)$. Changing the values on the right-hand side of Line 5 allows to check the sufficient condition for arbitrary fixed values. If one wants one or several of the above quantities to be treated as variables, for example c, p_1 and p_2, it suffices to remove these and the corresponding values on the right hand side of Line 5 and add the line

```
6 c, p1, p2 = sym.symbols('c,p1,p2')
```

Since the program will now compute values of `test` as symbolic expressions, it will no longer be able to determine the minimum. One should thus replace Lines 28–32 for example by

```
28 print(sym.separatevars(test.subs(x1,vert[2][0]).subs(x2,vert[2][1])))
```

to get the expressions from appendix B, to be used in the proof of Proposition 2.3.2.

Similarly, it is easy to modify the program to consider another Fano toric surface as a fiber (Recall that there are five smooth Fano toric surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$ and the equivariant blowups of $\mathbb{P}^2$ at up to three fixed points of the torus action). It suffices to modify Lines 10–18 according to the desired polytope. For example, if one wants to work with fiber the first Hirzebruch surface (i.e. the blowup of $\mathbb{P}^2$

at one point), then it suffices to replace Lines 10–18 with

```
10 # list of vertices of the polytope
11 vert= [[-t,-t], [t,-t], [t,0], [-t,2t]]
12 # Donaldson-Futaki invariant with weights (v,w0)
13 def DF0(f):
14   interior=sym.integrate(sym.integrate(f, (x2, -t, t-x1)), (x1, -t, t))
15   facet1=sym.integrate(f.subs(x2,-t), (x1, -t, t))
16   facet2=sym.integrate(f.subs(x2,t-x1), (x1, -t, t))
17   facet3=sym.integrate(f.subs(x1,-t), (x2, -t, 2t))
18   facet4=sym.integrate(f.subs(x1,t), (x2, -t, 0))
19   return(interior+facet1+facet2+facet3+facet4)
```

APPENDIX B

COMPLEMENT OF PROOF OF PROPOSITION 2.3.2

$$
\begin{aligned}
P(c,p_1,p_2) :=& 12250c^{10} + 24500c^9 p_1 - 39690c^8 p_1^2 + 18060c^7 p_1^3 - 22470c^6 p_1^4 \\
& - 31752c^5 p_1^5 - 53376c^4 p_1^6 + 22740c^3 p_1^7 - 57024c^2 p_1^8 \\
& + 1312p_1^1 0 - 49000c^9 p_2 + 34650c^7 p_1^2 p_2 + 286860c^6 p_1^3 p_2 \\
& + 152460c^5 p_1^4 p_2 + 360972c^4 p_1^5 p_2 - 59520c^3 p_1^6 p_2 + 230112c^2 p_1^7 p_2 \\
& + 18288cp_1^8 p_2 - 464p_1^9 p_2 - 127890c^8 p_2^2 - 212310c^7 p_1 p_2^2 \\
& - 615510c^6 p_1^2 p_2^2 - 373212c^5 p_1^3 p_2^2 - 921924c^4 p_1^4 p_2^2 - 425376c^2 p_1^6 p_2^2 \\
& - 160632cp_1^7 p_2^2 - 19296p_1^8 p_2^2 + 141540c^7 p_2^3 + 657300c^6 p_1 p_2^3 \\
& + 603288c^5 p_1^2 p_2^3 + 1408632c^4 p_1^3 p_2^3 + 390936c^3 p_1^4 p_2^3 + 571536c^2 p_1^5 p_2^3 \\
& + 349440cp_1^6 p_2^3 + 41376p_1^7 p_2^3 - 328650c^6 p_2^4 - 531720c^5 p_1 p_2^4 \\
& - 1421136c^4 p_1^2 p_2^4 - 806100c^3 p_1^3 p_2^4 - 829080c^2 p_1^4 p_2^4 - 497592cp_1^5 p_2^4 \\
& - 22416p_1^6 p_2^4 + 212688c^5 p_2^5 - 43812c^3 p_1^5 p_2^2 + 860184c^4 p_1 p_2^5 \\
& + 849456c^3 p_1^2 p_2^5 + 906192c^2 p_1^3 p_2^5 + 485712cp_1^4 p_2^5 - 7488p_1^5 p_2^5 \\
& - 286728c^4 p_2^6 - 527016c^3 p_1 p_2^6 - 725760c^2 p_1^2 p_2^6 - 329952cp_1^3 p_2^6 \\
& + 127890c^8 p_1 p_2 + 7352cp_1^9 + 22656p_1^4 p_2^6 + 150576c^3 p_2^7 \\
& + 363168c^2 p_1 p_2^7 + 156096cp_1^2 p_2^7 - 25728p_1^3 p_2^7 - 90792c^2 p_2^8
\end{aligned}
$$

$$- 46368cp_1p_2^8 + 16992p_1^2p_2^8 + 10304cp_2^9 - 7040p_1p_2^9 + 1408p_2^{10}$$
$$+ 132300c^7p_1^2 + 105840c^6p_1^3 - 11340c^5p_1^4 + 125496c^4p_1^5$$
$$+ 151200c^3p_1^6 - 79056c^2p_1^7 + 60048cp_1^8 - 12096p_1^9 - 396900c^7p_1p_2$$
$$- 449820c^6p_1^2p_2 - 260820c^5p_1^3p_2 - 374220c^4p_1^4p_2 + 323568p_1^3p_2^6$$
$$- 420336c^3p_1^5p_2 + 358992c^2p_1^6p_2 - 364176cp_1^7p_2 + 17712p_1^8p_2$$
$$+ 714420c^6p_1p_2^2 + 601020c^5p_1^2p_2^2 + 378756c^4p_1^3p_2^2 + 396900c^7p_2^2$$
$$+ 743904c^3p_1^4p_2^2 - 557280c^2p_1^5p_2^2 + 734832cp_1^6p_2^2 + 1436400cp_1^2p_2^6$$
$$+ 84240p_1^7p_2^2 - 476280c^6p_2^3 - 680400c^5p_1p_2^3 + 843696c^2p_1p_2^6$$
$$- 282744c^4p_1^2p_2^3 - 728784c^3p_1^3p_2^3 + 99792c^2p_1^4p_2^3 - 1073520cp_1^5p_2^3$$
$$- 287280p_1^6p_2^3 + 340200c^5p_2^4 + 45360c^4p_1p_2^4 + 568512c^3p_1^2p_2^4 + 829440c^2p_1^3p_2^4$$
$$+ 1551312cp_1^4p_2^4 - 244944c^3p_1p_2^5 - 241056c^2p_2^7 - 736128cp_1p_2^7 - 18144c^4p_2^5$$
$$+ 415152p_1^5p_2^4 - 279072p_1^2p_2^7 + 184032cp_2^8 + 139968p_1p_2^8 - 31104p_2^9$$
$$- 1175472c^2p_1^2p_2^5 - 1732752cp_1^3p_2^5 - 358992p_1^4p_2^5 + 81648c^3p_2^6$$

$$
\begin{aligned}
Q(c,p_1,p_2) :=\,& 6125c^9 + 2205c^7p_1^2 + 210c^6p_1^3 + 14175c^5p_1^4 - 7812c^4p_1^5 + 24c^3p_1^6 \\
&+ 9072c^2p_1^7 - 5004cp_1^8 + 688p_1^9 - 2205c^7p_1p_2 - 315c^6p_1^2p_2 \\
&- 31752c^2p_1^6p_2 + 20016cp_1^7p_2 - 3096p_1^8p_2 + 2205c^7p_2^2 - 315c^6p_1p_2^2 \\
&- 7812c^4p_1^3p_2^2 + 4356c^3p_1^4p_2^2 + 40824c^2p_1^5p_2^2 - 40320cp_1^6p_2^2 + 4464p_1^7p_2^2 \\
&+ 210c^6p_2^3 - 28350c^5p_1p_2^3 - 7812c^4p_1^2p_2^3 - 8592c^3p_1^3p_2^3 - 22680c^2p_1^4p_2^3 \\
&+ 50904cp_1^5p_2^3 - 1176p_1^6p_2^3 + 19530c^4p_1^4p_2 - 72c^3p_1^5p_2 \\
&+ 14175c^5p_2^4 + 19530c^4p_1p_2^4 + 4356c^3p_1^2p_2^4 - 22680c^2p_1^3p_2^4 \\
&- 1224p_1^5p_2^4 - 7812c^4p_2^5 - 72c^3p_1p_2^5 + 40824c^2p_1^2p_2^5 + 50904cp_1^3p_2^5 \\
&+ 24c^3p_2^6 - 31752c^2p_1p_2^6 - 40320cp_1^2p_2^6 - 1176p_1^3p_2^6 + 9072c^2p_2^7 \\
&+ 4464p_1^2p_2^7 - 5004cp_2^8 - 3096p_1p_2^8 + 688p_2^9 + 20016cp_1p_2^7 \\
&- 28350c^5p_1^3p_2 + 42525c^5p_1^2p_2^2 - 1224p_1^4p_2^5 - 56196cp_1^4p_2^4
\end{aligned}
$$

APPENDIX C

A PARTIAL CONVERSE OF LEMMA 3.1.13

Lemma C.0.1. *Suppose Y is a semisimple principal toric fibration with smooth toric fiber X over a product of positive Kähler–Einstein manifolds $\prod_{a=1}^{k}(B_a, J_a, \omega_a)$ of scalar curvatures s_a. Then Y is Fano iff the following conditions are satisfied:*

1. *X is a toric Fano manifold.*
2. *$\langle p_a, x\rangle + \frac{s_a}{2\dim(B_a)} > 0$ on P.*

In this case, if $\omega \in 2\pi c_1(X)$ is a toric Kähler metric on X, then the compatible Kähler metric $\tilde{\omega} = \omega + \sum_{a=1}^{k}(\langle p_a, x\rangle + \frac{s_a}{2\dim(B_a)})\omega_a + \langle dm_\omega \wedge \theta\rangle$ belongs to $2\pi c_1(Y)$.

Proof. Lemmas 3.1.13 and 3.1.4 show that (1) and (2) ensure that $\tilde{\omega}$ belongs to $2\pi c_1(Y)$ which, in turn, implies that Y is Fano (this is true without the assumption that X is toric).

We now deal with the converse direction. Suppose Y is a Fano manifold. Let ω be a $\mathbb{T}$-invariant Kähler metric on X. It follows from formula (3.32) that the 2-form

$$\begin{aligned}\tilde{\chi} :&= Ric(\omega) + \sum_{a=1}^{k} (\langle p_a, m_{Ric(\omega)} \rangle)\omega_a + Ric(\omega_a) + \langle dm_{Ric(\omega)} \wedge \theta \rangle \\ &= Ric(\omega) + \sum_{a=1}^{k} \left(\langle p_a, m_{Ric(\omega)} \rangle + \frac{s_a}{2\dim(B_a)} \rangle \right) \omega_a + \langle dm_{Ric(\omega)} \wedge \theta \rangle\end{aligned}$$

belongs to $2\pi c_1(Y)$. Restricting $\tilde{\chi}$ to a fiber $X \subset Y$, we get $c_1(Y)_{|X} = c_1(X) > 0$, i.e. X is Fano. We now take a Kähler metric $\omega_X \in 2\pi c_1(X)$ and we consider the 2-form

$$\omega_Y = \omega_X + \sum_{a=1}^{k} \left(\langle p_a, m_{\omega_X} \rangle + \frac{s_a}{2\dim(B_a)} \rangle \right) \omega_a + \langle dm_{\omega_X} \wedge \theta \rangle.$$

By Lemma 3.1.13, $\omega_Y \in 2\pi c_1(Y)$. Let x_0 be a vertex of P. Then $m_{\omega_X}^{-1}(x_0)$ defines a complex embedding of B in Y. Restricting ω_Y to the B_a-factor through this embedding, we get

$$\left(\langle p_a, m_{\omega_X}(x_0) \rangle + \frac{s_a}{2\dim(B_a)} \right) [\omega_a] > 0 \text{ on } B_a,$$

showing that $\langle p_a, x_0 \rangle + \frac{s_a}{2\dim(B_a)} > 0$. We can repeat the argument on each vertex of P and so $\langle p_a, x \rangle + c_a > 0$ on P.

□

APPENDIX D

WEIGHTED DIFFERENTIAL OPERATORS

Let $(X, \omega, \mathbb{T})$ be as in Section 3.1 and $\mathrm{v} > 0$ be a positive smooth weight function defined over the polytope P. We denote by ∇^ω the Levi–Civita connection of the Riemannian metric g_ω, and by δ_ω the formal adjoint of ∇^ω. We define the following weighted differential operators which are self-adjoint with respect to the volume form $v(m_\omega)\omega^{[m]}$ on X.

Definition D.0.1. *The* v-weighted Laplacian *of ψ is the second order operator acting of smooth functions defined by*

$$\Delta_{\omega,v}(\psi) = \frac{1}{v(m_\omega)}\delta_\omega(v(m_\omega)d\psi). \tag{D.1}$$

The v-weighted linear Lichnerowicz *operator is the forth-order operator given by*

$$\mathbb{L}_{\omega,v}(\psi) := \frac{\delta_\omega\delta_\omega\left(\mathrm{v}(m_\omega)(\nabla^\omega d\psi)^-\right)}{v(m_\omega)}, \tag{D.2}$$

where $(\nabla^\omega d\phi)^-$ stands for the $(0,2)$-symmetric tensor of type $(2,0)+(0,2)$ with respect to the complex structure of X. For any $\mathbb{T}$-invariant Kähler form χ on X, we define, similarly than the toric case (1.30)*, the second-order operator given by*

$$\mathbb{H}^\chi_{\omega,v}(\psi) := \langle\chi, dd^c\psi\rangle_\omega + \langle d\Lambda_\omega(\chi), d\psi\rangle_\omega + \frac{1}{v(m_\omega)}\langle\chi, d\mathrm{v}(m_\omega)\wedge d^c\psi\rangle_\omega, \tag{D.3}$$

where $\Lambda_\omega(\chi) := \left(\chi\wedge\omega^{[m-1]}\right)/\omega^{[m]} = \langle\chi,\omega\rangle_\omega$. The operator $\mathbb{H}^\xi_{\omega,\mathrm{v}}$ is a v-weighted version of the linear operator used in [46].

A straightforward computation shows that

Lemma D.0.2. *The v-weighted Lichnerowicz's operator can be written as*

$$\mathbb{L}_{\omega,v}(\psi) = \frac{1}{2}(\Delta_{\omega,v})^2(\psi) + \delta_{\omega,v}\left(\chi_{\omega,v}((d^c\psi)^\sharp)\right),$$

where $\delta_{\omega,v} := \frac{1}{v(m_\omega)}\delta_\omega v(m_\omega)$ is the formal adjoint of the exterior derivative d on functions with respect to the weighted volume form $v(m_\omega)\omega^{[m]}$, $\chi_{\omega,v} := \chi_\omega - \frac{1}{2}dd^c(\log v(m_\omega))$ is the Ricci form of the weighted volume form $v(m_\omega)\omega^{[m]}$, and $\sharp = g_\omega^{-1}$ stands for the riemannian duality between TM and T^*M by using the Kähler metric ω.

We now specialize to the case when $(Y, \tilde{\omega}, \mathbb{T}_Y)$ is a semisimple principal fibration with fiber $(X, \omega, \mathbb{T}_X)$ over B, as in Section 3.1. We then denote by $\Delta^Y_{\tilde{\omega}}$, $\mathbb{L}^Y_{\tilde{\omega}}$ and $(\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}})^Y$ the corresponding unweighted operators on $(Y, \tilde{\omega})$, where the Kähler form $\tilde{\chi}$ in the definition of $\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}}$ is bundle-compatible, i.e. given by (3.3) for a $\mathbb{T}_X$-invariant Kähler form χ on X. We further let $\Delta^{B_a}_{\omega_a}$ denote the Laplacian on (B_a, ω_a), and Δ^B_x and $\mathbb{L}^B_x$ respectively the Laplacian and Lichnerowicz operators on B with respect to the Kähler metric $\omega_B(x) := \sum_{a=1}^k(\langle p_a, m_\omega(x)\rangle + c_a)\omega_a$. We thus have the following result.

Lemma D.0.3. *Let ψ be a $\mathbb{T}_Y$-invariant smooth function on Y, seen as a $\mathbb{T}_X$-invariant function on $X \times B$ via (3.4), and $\tilde{\omega}$ a bundle-compatible $\mathbb{T}_Y$-invariant Kähler metric on Y associated to a $\mathbb{T}_X$-invariant Kähler metric ω on X. We then have*

$$\begin{aligned}
\Delta^Y_{\tilde{\omega}}\psi &= \Delta^X_{\omega,p}\psi_b + \Delta^B_x\psi_x,\\
\mathbb{L}^Y_{\tilde{\omega}}\psi &= \mathbb{L}^X_{\omega,p}\psi_b + \mathbb{L}^B_x\psi_x + \Delta^B_x\left(\Delta^X_{\omega,p}\psi_b)\right)_x + \Delta^X_{\omega,v}\left(\Delta^B_x\psi_x\right)_b\\
&\quad + \sum_{a=1}^k Q_a(x)\Delta^{B_a}_{\omega_a}\psi_x,\\
(\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega},1})^Y\psi &= (\mathbb{H}^{\chi}_{\omega,p})^X\psi_b + \sum_{a=1}^k P_a(x)\Delta^{B_a}_{\omega_a}\psi_x,
\end{aligned}$$

where $P_a(x), Q_a(x)$ *are smooth* $\mathbb{T}$*-invariant functions on* X*, and* ψ_x *and* ψ_b *are respectively the induced smooth functions on* B *and* X *via* (3.4)*.*

Proof. This first two equalities are established in [7] (see the proof of Lemma 8) in the special case when $(X, \omega, \mathbb{T}_X)$ is a toric variety whereas the third identity is proved in Proposition 1.3.6 (also in the case when $(X, \mathbb{T}_X)$ is toric). These computations extend to the general setting with no substantial additional difficulty (by using Lemma D.0.2 above for the second identity), but we include them below for the sake of self-containedness.

In the notation of Sect. 3.1

$$\begin{aligned} \Delta^Y_{\tilde{\omega}}(\psi) &= -\frac{d_Y d^c_Y \psi \wedge \tilde{\omega}^{[n+m-1]}}{\tilde{\omega}^{[n+m]}} \quad \text{(on } Y\text{)} \\ &= -\frac{d_Y d^c_Y \psi \wedge \tilde{\omega}^{[n+m-1]} \wedge \theta^{\wedge r}}{\tilde{\omega}^{[n+m]} \wedge \theta^{\wedge r}} \quad \text{(on } Z = X \times Q\text{)}, \end{aligned} \tag{D.4}$$

where $\theta^{\wedge r} := \bigwedge_{i=1}^r \theta_i$ with respect to any lattice basis $(\xi_i)_i$ of $\Lambda \subset \mathfrak{t}$. Viewing $d^c_{X\times B}\psi$ as a 1-form on Z, it admits the following decomposition with respect to (3.2)

$$d^c_{X\times B}\psi = (d^c_{X\times B}\psi)_{\mathcal{H}} + \sum_{i=1}^r (d^c_{X\times B}\psi)(\xi^Q_i - \xi^X_i)\theta_i = d^c_Y\psi - \langle d^c_X\psi, \theta\rangle. \tag{D.5}$$

We thus compute on Z:

$$\begin{aligned}(d_Y d^c_Y\psi)_{(x,b)} =& d_Z\left(d^c_X\psi + \sum_{j=1}^r d^c_X\psi(\xi_j^X)\theta_j + d^c_B\psi\right)\\ =& d_Z d^c_X\psi + \sum_{j=1}^r d_Z(d^c_X\psi(\xi_j^X))\theta_j\\ &+\sum_{j=1}^r d^c_X\psi_b(\xi_j^X)\left(\sum_{a=1}^k \xi^j(p_a)\pi_B^*\omega_a\right) + d_Z d^c_B\psi\\ =& d_X d^c_X\psi_b + d_B d^c_B\psi_x + \sum_{j=1}^r d_Z(d^c_X\psi(\xi_j^X))\wedge\theta_j + \sum_{a=1}^k d^c_X\psi(p_a^X)\pi_B^*\omega_a\\ &+ d_B d^c_X\psi + d_X d^c_B\psi,\end{aligned} \tag{D.6}$$

where for getting the third equality we used (3.1), as well as the identities $d_Q d^c_X\psi = d_B d^c_X\psi$ and $d_Q d^c_B\psi = d_B d^c_B\psi$ (which follow from the identification (3.4)). Using (3.6) and (3.24), we derive from (D.4) and (D.6)

$$\Delta^Y_{\tilde\omega}(\psi)(x,b) =(\Delta^X_\omega\psi_b)(x) + (\Delta^B_{\omega_B(x)}\psi_x)(b) - \sum_{a=1}^k \frac{n_a}{(\langle m_\omega, p_a\rangle + c_a)}(d^c_X\psi_b)(p_a^X),$$

where, we recall, for a fixed $x\in X$, we have set $\omega_B(x) := \sum_{a=1}^k(\langle p_a, m_\omega\rangle + c_a)\omega_a$, and p_a^X denotes the vector field field on X corresponding to $p_a\in\mathfrak{t}$. The first equality in the Lemma follows from the identity

$$\Delta^X_{\omega,\mathrm{v}}(\psi) := \frac{1}{\mathrm{v}(m_\omega)}\delta_\omega\Big(p(m_\omega)d\psi\Big) = \Delta^X_\omega(\psi) - \sum_{j=1}^r \frac{\mathrm{v}_{,j}(x_\omega)}{\mathrm{v}(x_\omega)} g_\omega(d\mu_\omega^{\xi_j}, d\psi),$$

taking in mind that for any smooth function on u on P and any $\mathbb{T}$-invariant smooth function ϕ on X, $g_\omega(d(u(m_\omega)), d\phi) = \sum_{i=1}^r u_{,i}(m_\omega)d^c\phi(\xi_i)$.

Now, we establish the expression of the corresponding Lichnerowicz operators.

Recall that (see e. g. [42])

$$\mathbb{L}^Y_{\tilde{\omega}}\psi := \frac{1}{2}(\Delta^Y_{\tilde{\omega}})^2(\psi) + \delta_{\tilde{\omega}}(\chi_{\tilde{\omega}}(d^c_Y\psi)). \tag{D.7}$$

Using the decomposition of $\Delta^Y_{\tilde{\omega}}$ we have just established, we have

$$(\Delta^Y_{\tilde{\omega}})^2(\psi) = (\Delta^X_{\omega,p})^2(\psi_b) + (\Delta^B_x)^2(\psi_x) + \Delta^X_{\omega,p}(\Delta^B_x(\psi_x)) + \Delta^B_x\left(\Delta^X_{\omega,p}(\psi_b)\right). \tag{D.8}$$

It remains to compute the Ricci term in (D.7). From (3.23), we have

$$\begin{aligned}\chi_{\tilde{\omega}} =& \chi_{\omega,p} + \pi^*_B\chi_{\omega_B} + \frac{1}{2}\sum_{a=1}^{k}\Delta^X_{\omega,p}(\langle m_\omega, p^X_a\rangle)\pi^*_B\omega_a \\ &+ \sum_{j=1}^{r} d_X\left(d^c_X\left(\kappa - \frac{1}{2}\log p(m_\omega)\right)(\xi^X_j)\right)\wedge\theta_j.\end{aligned} \tag{D.9}$$

where $\chi_{\omega,p} := \chi_\omega - \frac{1}{2}d_Xd^c_X\log p(m_\omega)$ is the Ricci form of the weighted volume form $p(m_\omega)\omega^{[m]}$. Using integration by parts, for any $\mathbb{T}_Y$-invariant smooth test function ϕ on Y, seen as a $\mathbb{T}_X$ and $\mathbb{T}_Q$-invariant function on $Z = X \times Q$ via (3.4), we have

$$\begin{aligned}&\int_Z \phi\delta_{\tilde{\omega}}(\chi_{\tilde{\omega}}(d^c_Y\psi))\tilde{\omega}^{[n+m]}\wedge\theta^{\wedge r} = -\int_Z \chi_{\tilde{\omega}}(d_Y\phi, d^c_Y\psi)\tilde{\omega}^{[n+m]}\wedge\theta^{\wedge r} \\ =& \int_Z \chi_{\tilde{\omega}}\wedge d_Y\phi\wedge d^c_Y\psi\wedge\tilde{\omega}^{[n+m-2]}\wedge\theta^{\wedge r} \\ &- \frac{1}{2}\int_Z Scal(\tilde{\omega})\tilde{g}_{\tilde{\omega}}(d_Y\phi, d_Y\psi)\tilde{\omega}^{[n+m]}\wedge\theta^{\wedge r} \\ =& \int_Z \chi_{\tilde{\omega}}\wedge d_Y\phi\wedge d^c_Y\psi\wedge\tilde{\omega}^{[n+m-2]}\wedge\theta^{\wedge r} \\ &- \frac{1}{2}\int_Z\left(\frac{Scal_p(\omega)}{p(m_\omega)} + q(m_\omega)\right)d_Y\phi\wedge d^c_Y\psi\wedge\tilde{\omega}^{[n+m-1]}\wedge\theta^{\wedge r}.\end{aligned} \tag{D.10}$$

From the above formula, using (3.24), (D.5) and (D.9), we compute (after some straightforward but long algebraic manipulations and integration by parts over X

and B)

$$\delta^{Y}_{\tilde{\omega}}(\chi_{\tilde{\omega}}(d^c_Y\psi)) = \delta^{X}_{\omega,p}\left(\chi_{\omega,p}(d^c_X\psi)\right) + \delta^{B}_{\omega_B(x)}\left(\chi_{\omega_B}(d^c_B\psi)\right)$$
$$+ \frac{1}{2}\sum_{a=1}^{k} \frac{q(m_\omega)}{(\langle m_\omega, p_a\rangle + c_a)} \Delta^{B}_{\omega_a}(\psi) + \frac{1}{2}\sum_{a=1}^{k} \frac{(n_a - 1)}{(\langle m_\omega, p_a\rangle + c_a)^2} \Delta^{X}_{\omega,p}(\langle m_\omega, p_a\rangle)\Delta^{B}_{\omega_a}(\psi_x)$$
$$+ \sum_{a,b=1}^{k} \frac{n_b}{(\langle m_\omega, p_a\rangle + c_a)(\langle m_\omega, p_b\rangle + c_b)} \Delta^{X}_{\omega,p}(\langle m_\omega, p_b\rangle)\Delta^{B}_{\omega_a}(\psi_x). \tag{D.11}$$

Combining (D.7), (D.8) and (D.11) yields the desired expression.

The expression for $(\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega},1})^{Y}(\psi)$ is obtained by similar arguments, using that

$$\begin{aligned}(\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega},1})^{Y}(\psi) =& \langle \tilde{\chi}, d_Y d^c_Y \psi\rangle_{\tilde{\omega}} + \langle d_Y \Lambda_{\tilde{\omega}}(\tilde{\chi}), d_Y\psi\rangle_{\tilde{\omega}} \\ =& -\Lambda_{\tilde{\omega}}(\tilde{\chi})\Delta^{Y}_{\tilde{\omega}}(\psi) - \frac{\tilde{\chi}\wedge d_Y d^c_Y\psi \wedge \tilde{\omega}^{[n+m-2]}}{\tilde{\omega}^{[n+m]}} \\ &+ \frac{d_Y\Lambda_{\tilde{\omega}}(\tilde{\chi})\wedge d^c_Y\psi\wedge\tilde{\omega}^{[n+m-1]}}{\tilde{\omega}^{[n+m]}}.\end{aligned}$$

□

BIBLIOGRAPHY

[1] Abreu, M. (1988). Kähler geometry of toric varieties and extremal metrics. *Internat. J. Math.*, *9*, 641–651.

[2] Abreu, M. (2003). Kähler geometry of toric manifolds in symplectic coordinates. *Symplectic and Contact Topology: Interactions and Perspectives" (eds. Y.Eliashberg, B.Khesin and F.Lalonde), Fields Institute Communications 35, American Mathematical Society*, *35*, 1–24.

[3] Apostolov, V. (2022). The Kähler geometry of toric manifolds. *arXiv:2208.12493*.

[4] Apostolov, V., Calderbank, D. & Gauduchon, P. (2006). Hamiltonian 2-forms in Kähler geometry, I General theory. *J. Differential Geom.*, *73*, 359–412.

[5] Apostolov, V., Calderbank, D., Gauduchon, P. & Tønnesen-Friedman, C. (2004). Hamiltonian 2-forms in Kähler geometry, II Global classification. *J. Differential Geom.*, *68*, 277–345.

[6] Apostolov, V., Calderbank, D., Gauduchon, P. & Tønnesen-Friedman, C. (2008). Hamiltonian 2-forms in Kähler geometry, III Extremal metrics and stability. *Invent. Math*, *173*, 547–601.

[7] Apostolov, V., Calderbank, D., Gauduchon, P. & Tønnesen-Friedman, C. (2011). Extremal Kähler metrics on projective bundles over a curve. *Adv. Math.*, *227*, 2385–2424.

[8] Apostolov, V., Calderbank, D. & Legendre, E. (2021a). Weighted K-stability of polarized varieties and extremality of Sasaki manifolds. *Adv. Math.*, *391*.

[9] Apostolov, V., Jubert, S. & Lahdili, A. (2021b). Weighted K-stability and coercivity with applications to extremal Kähler and Sasaki metrics. *to appear in Geometry and Topology, arXiv:2104.09709*.

[10] Apostolov, V. & Keller, J. (2019). Relative k-polystability of projective bundles over a curve. *Trans. Amer. Math. Soc.*, *372*, 233–266.

[11] Apostolov, V. & Maschler, G. (2019). Conformally Kähler, Einstein-Maxwell geometry. *J. Eur. Math. Soc.*, *21*, 1319–1360.

[12] Atiyah, M. F. (1982). Convexity and commuting hamiltonians. *Bull. London Math. Soc.*, *14*, 1–15.

[13] Aubin, T. (1976). Equations de type Monge-Ampère sur les variétés kählériennes compactes. *C. R. Acad. Sci. Paris*, *283*, 119–121.

[14] Aubin, T. (1978). Equations de type Monge-Ampère sur les variétés kählériennes compactes II. *Bull. Sci. Math.*, *102*, 63–95.

[15] Berman, R. J. & Berndtsson, B. (2013). Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. *Ann. Fac. Sci. Toulouse*, *6*, 649–711.

[16] Berman, R. J. & Berndtsson, B. (2017). Convexity of the K-energy on the space of Kähler metrics. *J. Amer. Math. Soc*, *30*, 1165–1196.

[17] Berman, R. J., Boucksom, S. & Jonsson, M. (2021). A variational approach to the Yau-Tian-Donaldson conjecture. *J. Amer. Math. Soc.*, *34*, 605–652.

[18] Berman, R. J. & Witt-Nyström, D. (2014). Complex optimal transport and the pluripotential theory of Kähler-Ricci solitons. *arXiv:1401.8264*.

[19] Blocki, Z. (2009). On geodesics in the space of Kähler metrics, proceedings of the "conference in geometry" dedicated to shing-tung Yau. *Advances in Geometric Analysis, ed. S. Janeczko, J. Li, D. Phong, Advanced Lectures in Mathematics*, *20*, 3–20.

[20] Calabi, E. (1982). Extremal Kähler metrics. *Seminar on Differential Geometry, Ann. of Math. Stud. , Princeton Univ. Press, Princeton,*, *102*, 259–290.

[21] Calabi, E. (1985). Extremal Kähler metrics, ii. *Differential Geometry and Complex Analysis, eds. I. Chavel and H. M. Farkas, Springer Verlag*, pp. 95–114.

[22] Chen, B., Li, A. & Sheng, L. (2014). Uniform K-stability for extremal metrics on toric varieties. *Differential Equations*, *257*, 1487–1500.

[23] Chen, X. (2000). The space of Kähler metrics. *J. Differential Geom.*, *56*, 259–290.

[24] Chen, X. (2017). On the existence of constant scalar curvature Kähler metric: a new perspective. *Ann. Math. du Québec*, *42*, 169–189.

[25] Chen, X. & Cheng, J. (2021a). On the constant scalar curvature Kähler metrics: apriori estimates. *J. Amer. Math. Soc.*, *34*, 909 – 936.

[26] Chen, X. & Cheng, J. (2021b). On the constant scalar curvature Kähler metrics II: Existence results. *J. Amer. Math. Soc.*, *34*, 937–1009.

[27] Chen, X., Donaldson, S. K. & Sun, S. (2015a). Kähler–Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. *J. Amer. Math. Soc.*, *28*, 183–197.

[28] Chen, X., Donaldson, S. K. & Sun, S. (2015b). Kähler–Einstein metrics on

Fano manifolds. II: Limits with cone angle less than 2π. *J. Amer. Math. Soc.*, *28*, 199–234.

[29] Chen, X., Donaldson, S. K. & Sun, S. (2015c). Kähler–Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π. *J. Amer. Math. Soc.*, *28*, 235–278.

[30] Chen, X., Paun, M. & Zeng, Y. (2015d). On deformation of extremal metrics. *arXiv:1506.01290v2*.

[31] Chu, J., Tosatti, V. & Weinkove, B. (2017). On the $C^{1,1}$ regularity of geodesics in the space of Kähler metrics. *Annals of Partial Differential Equations*, *3*, 2199–2576.

[32] Darvas, T. (2017). The Mabuchi completion of the space of Kähler potentials. *Amer. J. Math.*, *139*, 1275–1313.

[33] Darvas, T. & Rubinstein, Y. (2017). Tian's properness conjectures and Finsler geometry of the space of Kähler metrics. *J. Amer. Math. Soc.*, *30*, 347–387.

[34] Delcroix, T. (2020). K-Stability of Fano spherical varieties. *Ann. Sci. Ec. Norm. Super.*, *53*, 615–662.

[35] Delcroix, T. & Jubert, S. (2022). An effective weighted K-stability condition for polytopes and semisimple principaltoric fibratons. *accepted in Annales Henry Lesbegue, arXiv:2202.02996v2*.

[36] Delzant, T. (1988). Hamiltoniens périodiques et image convexe de l'application moment. *Bull. Soc. Math. France*, *116*, 315–339.

[37] Donaldson, S. K. (2002). Scalar curvature and stability of toric varieties. *J. Differential Geom.*, *62*, 289–349.

[38] Donaldson, S. K. (2008). Extremal metrics on toric surfaces: a continuity method. *J. Differential Geom.*, *79*, 389–432.

[39] Donaldson, S. K. (2009). Constant scalar curvature metrics on toric surfaces. *Geom. Funct. Anal.*, *19*, 83–136.

[40] Fujiki, A. (1992). Remarks on extremal Kähler metrics on ruled manifolds. *Nagoya Math. J*, *126*, 89–101.

[41] Futaki, A. & Mabuchi, T. (1995). Bilinear forms and extremal Kähler vector fields associated with Kähler classes. *Math. Ann.*, *301*, 199–210.

[42] Gauduchon, P. (2022). Calabi's extremal Kähler metrics: An elementary introduction. *Lecture Notes, available upon request.*

[43] Guan, D. (1999). On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles. *Math. Res. Lett.*, *6*, 547–555.

[44] Guillemin, V. (1994). Kähler structures on toric varieties. *J. Differential Geom.*, *40*, 285–309.

[45] Guillemin, V. & Sternberg, S. (1982). Convexity properties of the moment mapping. *Invent. Math.*, *67*, 491–513.

[46] Hashimoto, Y. (2015). Existence of twisted constant scalar curvature Kähler metrics with a large twist. *Math. Z.*, *292*, 791–803.

[47] He, W. (2019). On Calabi's extremal metrics and properness. *Trans. Amer. Math. Soc.*, *372*, 5595–5619.

[48] Hisamoto, T. (2016). Stability and coercivity for toric polarizations. *arXiv:1610.07998 v3.*

[49] Inoue, E. (2022). Constant μ-scalar curvature Kähler metric - formulation and foundational results. *J. Geom. Anal.*, *32*.

[50] Jubert, S. (2021). A Yau–Tian–Donaldson correspondence on semisimple principale toric fibrations. *arXiv:2108.12297, to appear in Annales de l'Institut Fourier*.

[51] Kobayashi, S. & Ochiai, T. (1973). Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, *13*, 31–47.

[52] Koebe, P. (1907). Über die uniformisierung reeller analytischer kurven. *Göttinger Nachrichten*, pp. 177–190.

[53] Lahdili, A. (2019). Kähler metrics with weighted constant scalar curvature and weighted K-stability. *Proc. London Math. Soc.*, *119*, 1065 – 114.

[54] Lahdili, A. (2020). Convexity of the weighted Mabuchi functional and the uniqueness of weighted extremal metrics. *arXiv:2007.01345*.

[55] Legendre, E. (2019). A note on extremal toric almost Kähler metrics. *Springer INdAM Ser.*, *31*, 53 – 74.

[56] Lejmi, M. (2010). Extremal almost-Kähler metrics. *Int. J. Math.*, *21*, 1639 – 1662.

[57] Levine, M. N. (1985). A remark on extremal Kähler metrics. *J. Differential Geom. 21 no. 1*, pp. 73–77.

[58] Li, A., Lian, Z. & Sheng, L. (2016). Some estimates for a generalized Abreu's equation. *Trans. Amer. Math. Soc.*, *48*, 87 – 103.

[59] Li, A., Lian, Z. & Sheng, L. (2017a). Differential inequalities on homogeneous toric bundles. *Journal of Geometry*, *108*, 775–790.

[60] Li, A., Lian, Z. & Sheng, L. (2017b). Interior regularity for a generalized Abreu equation. *Int. J. Math.*, *108*, 775 – 790.

[61] Li, C. (2020). Geodesic rays and stability in the cscK problem. *accepted by Annales Scientifiques de l'ENS, arXiv:2001.01366v3.*

[62] Li, C. & Han, J. (2022). On the Yau-Tian-Donaldson conjecture for generalized Kähler-Ricci soliton equations. *Comm. Pure Appl. Math.* https://doi.org/10.1002/cpa.22053.

[63] Li, C., Tian, G. & Wang, F. (2022). The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties. *Peking Math. J. 5*, *2*, 383–426.

[64] Lichnerowicz, A. (1958). Géométrie des groupes de transformation. *Travaux et Recherches Mathématiques*, *3*.

[65] Mabuchi, T. (1986). K-energy maps integrating Futaki invariants. *Tohoku Math. J.*, *38*, 575–593.

[66] Mabuchi, T. (1987). Some symplectic geometry on compact Kähler manifold (i). *Osaka J. Math.*, *24*, 227–252.

[67] Matsushima, Y. (1957). Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne. *Nagoya Math. J.*, *11*, 145–150.

[68] Nitta, Y. & Saito, S. (2021). A uniform version of the Yau-Tian-Donaldson correspondence for extremal Kähler metrics on polarized toric manifolds. *arXiv:2110.10386v1.*

[69] Podestà, F. & Spiro, A. (2010). Kähler-Ricci solitons on homogeneous toric bundless. *J. Reine Angew. Math.*, pp. 109–127.

[70] Poincaré, H. (1907). Sur l'uniformisation des fonctions analytiques. *Acta Mathematica*, *31*, 1–63.

[71] Schwarz, G. W. (1975). Smooth functions invariant under the action of a compact lie group. *Topology*, *14*, 63 – 68.

[72] Semmes, S. (1992). Complex Monge–Ampère and symplectic manifolds. *Amer. J. Math.*, *114*, 495–550.

[73] Sjöström-Dyrefelt, Z. (2017). K-semistability of cscK manifolds with transcendental cohomology class. *J. Geom. Anal.*, *28*, 2927 – 2960.

[74] Tian, G. (1987). On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(m) > 0$. *Invent. math.*, *89*, 225 – 246.

[75] Tian, G. (1990). On Calabi's conjecture for complex surfaces with positive first Chern class. *Invent. Math.*, *68*, 101–172.

[76] Tian, G. (1991). Kähler-Einstein on algebraic manifolds. Proceedings of the international congress of mathematicians, vol. i, ii. *Math. Soc. Japan*, pp. 143 – 186.

[77] Tian, G. (1997). Kähler-Einstein metrics with positive scalar curvature. *Invent. math.*, *137*, 1 – 37.

[78] Tian, G. (2015). K-stability and Kähler-Einstein metrics. *Communications on Pure and Applied Mathematics*, *68*, 1085–1156.

[79] Wang, X.-J. & Zhu, X. (2004). Kähler–Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.*, *188*, 87–103.

[80] Yau, S. (1997). Calabi's conjecture and some new results in algebraic geometry. *Proc. Natl. Acad. Sci. USA*, *74*, 1798–1799.

[81] Zhou, B. & Zhu, X. (2008). Relative K-stability and modified K-energy on toric manifolds. *Adv. Math.*, *219*, 1327–1362.