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VARIÉTÉS DE SHI ASSOCIÉES AUX GROUPES DE WEYL AFFINES

THÈSE

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RÉSUMÉ

Dans cette thèse nous étudions les groupes de Weyl affines avec un nouveau point de vue. Ces groupes sont connus pour donner lieu à une décomposition d'un espace euclidien sous forme d'alcôves. L'ensemble des alcôves est en bijection avec le groupe affine lui même. Il se trouve que ces alcôves transportent une grande quantité d'informations sur la géométrie du groupe ainsi que sa combinatoire.

Cette thèse décrit une nouvelle façon de voir les alcôves ; il faut les penser comme des points entiers dans un espace de plus grande dimension. Pour aboutir à ceci nous nous appuyons fortement sur les travaux de Jian-Yi Shi qui a caractérisé de façon combinatoire/géométrique les alcôves dans l'espace euclidien de “petite dimension”. Nous constatons par ailleurs que les points entiers associés aux alcôves ont une répartition cohérente dans ce nouvel espace. Nous expliquons ce phénomène, qui s'incarne par l'existence d'une variété affine dont les points entiers sont précisément ceux associés aux alcôves.

Nous étudions les composantes irréductibles de cette variété, et montrons en particulier que dans le cas du groupe symétrique affine, ces composantes sont en bijection avec les permutations circulaires du groupe symétrique sous-jacent.

Nous explicitons par la suite certains rapprochements entre les composantes irréductibles de la variété et le premier groupe de cohomologie du groupe cristallographique associé. Nous relions la matrice de Cartan avec les sections d'une certaine suite exacte et nous donnons de nouvelles obstructions cohomologiques. Finalement, nous donnons des équations dans le cas des groupes de types A, B, C, D, qui caractérisent entièrement ces obstructions.

ABSTRACT

In this thesis we investigate affine Weyl groups from a new point of view. These groups are known to give rise to a decomposition of Euclidean space into alcoves. The collection of alcoves is in bijection with the group. It turns out that these alcoves carry a great deal of combinatorial and geometrical data about the group. We are particularly interested in some specific features.

This thesis describes a new way to see alcoves; they need to be thought of as integral points in a space of bigger dimension. In order to do so, we build on the work of Jian-Yi Shi, who gave a combinatorial and geometrical characterization of alcoves in a Euclidean space of "small dimension". Further, we see that the distribution of these integral points is somehow coherent. We explain this result, which is embodied via the existence of an affine variety whose integral points are those associated to the alcoves.

The irreducible components of this variety are of particular interest to us. We study them further with several strategies. We establish in particular that in the context of the affine symmetric group, the components are in one-to-one correspondence with the circular permutations of the underlying symmetric group.

Finally, we make explicit a few links between the irreducible components and the first cohomology group of the associated crystallographic group. We connect the Cartan matrix with the set of sections of a specific short exact sequence, and we provide new cohomological obstructions. We conclude this investigation by giving in types A, B, C, D some equations that characterize entirely these obstructions.

INTRODUCTION

Soit V un espace euclidien de dimension finie et soit $\Phi \subset V$ un système de racines cristallographique. En prenant les hyperplans orthogonaux à ces racines, nous découpons l'espace V en régions de formes identiques. Ces régions sont appelées chambres de Weyl. En considérant le groupe orthogonal $O(V)$ et en prenant le sous-groupe engendré par les réflexions associées à ces hyperplans, nous obtenons un groupe W qui est appelé groupe de Weyl de Φ. Les chambres de Weyl sont en bijection avec les éléments de W, et il est d'usage de penser ces chambres comme les éléments du groupe lui même. Par exemple la figure ci-dessous représente les chambres de Weyl de $W(A_2)$, où le neutre est identifié à la chambre fondamentale D.

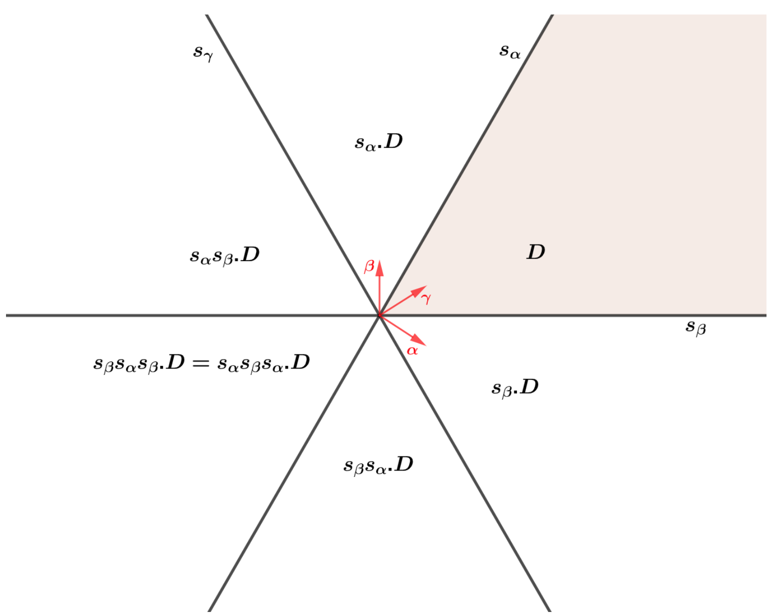

Figure 0.1 Chambres de $W(A_2)$, racines positives de A_2 et réflexions associées.

Les groupes de Weyl ont donné lieu à une vaste littérature durant le 20-ème siècle et continuent d'être étudiés. En particulier leur classification a été donnée en 1935 par H. S. M. Coxeter à l'aide de graphes, qui ont par la suite été appelés diagrammes de Coxeter. Il se trouve que ces groupes sont fortement connectés à la théorie de Lie et des groupes algébriques. En effet, les systèmes de racines permettent de classifier certains objets de théorie de Lie tels que :

- Les algèbres de Lie semi-simples, cf (Bourbaki, 1969), (Chevalley, 1948), (Humphreys, 1972) ou (Serre, 1985).

- Les groupes de Lie complexes simplement connexes qui sont simples modulo leur centre, cf (Bourbaki, 1969) ou (Humphreys, 1972).

- Les groupes de Lie compacts simples modulo leur centre, cf (Theodor et tom Dieck, 1985) ou (Bourbaki, 1969).

Trouver les sous-groupes de réflexions des groupes de Weyl a aussi été un enjeu majeur ces dernières années. Ceci a fourni en outre la détermination des classes de conjugaison de W (Carter, 1972), mais a aussi permis de comprendre l'action de W sur la cohomologie de la variété torique associée, cf (Lehrer, 2008) ou (Stembridge, 1994). Ces groupes ont également été étudiés de façon combinatoire, puisque entre autres, l'un d'entre eux est le groupe symétrique S_n. Enfin, les diagrammes de Dynkin, vus comme carquois, ont été utilisés dans de nombreux travaux concernant les représentations d'algèbres. Ce dernier sujet s'est considérablement développé durant les 50 dernières années, et fait partie aujourd'hui d'un champ de recherche extrêmement fécond.

Si nous translatons maintenant les hyperplans associés au groupe de Weyl W en utilisant le réseaux $\mathbb{Z}\Phi$, nous obtenons un arrangement d'hyperplans affines, noté $\mathcal{H}$, qui fournit une nouvelle décomposition de l'espace. Le groupe engendré par les réflexions associées à ces hyperplans est par définition le groupe de Weyl affine associé à W. Il est noté W_a. Les composantes connexes de $V \setminus \bigcup_{H \in \mathcal{H}} H$ sont appelées alcôves, et l'ensemble des alcôves est noté $\mathcal{A}$. De la même façon que pour les groupes de Weyl, les groupes de Weyl affines agissent géométriquement sur l'ensemble des alcôves. Il est par ailleurs connu, grâce à Tits, que cette action est simplement transitive. Dès lors, les éléments de W_a s'identifient bijectivement avec les éléments de $\mathcal{A}$. Il se trouve de plus que le groupe de Weyl W est canoniquement contenu dans W_a puisque $W_a \simeq \mathbb{Z}\Phi \rtimes W$. Dans la figure 0.2 nous donnons l'exemple du groupe de Weyl affine associé à A_2 où la partie colorée correspond à $W(A_2)$.

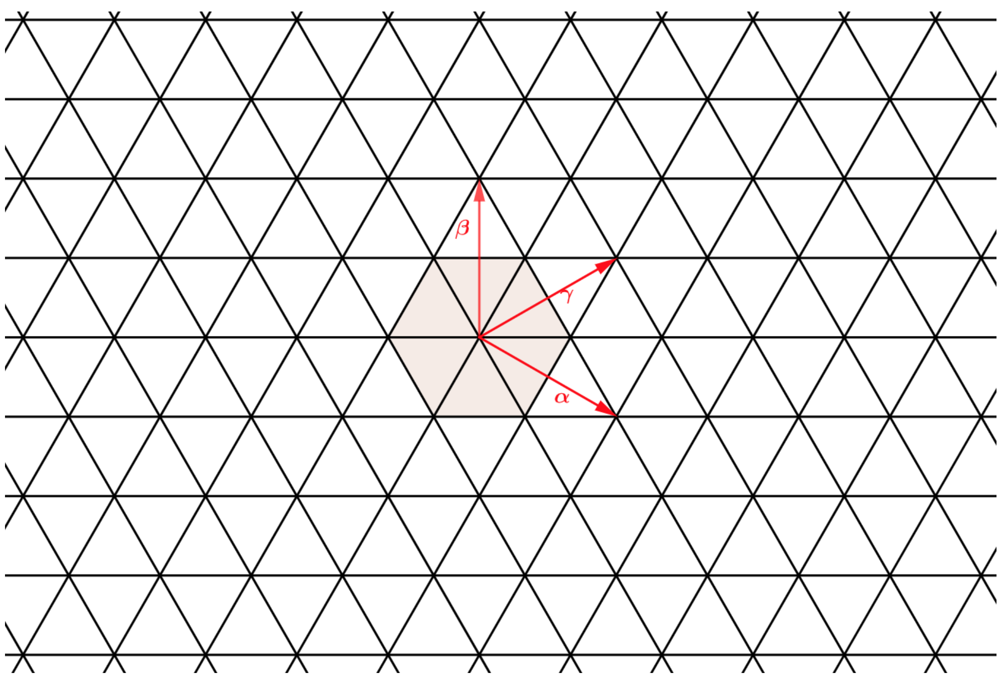

Figure 0.2 Alcôves du groupe de Weyl affine $W(\widetilde{A}_2)$. Les alcôves colorées sont celles associées à $W(A_2)$.

Les groupes de Weyl affines ont également donné lieu à de nombreuses recherches. Ils sont par exemple classifiés par les diagrammes de Dynkin étendus. Les systèmes de racines associés à ces groupes sont aussi d'un intérêt particulier. Ils ont, de la même façon que dans le cas des groupes de Weyl, été étudiés suivant leurs sous-groupes de réflexions, cf. (Dyer, 1990) ou (Dyer et Lehrer, 2011). Comprendre leur système de racines a également été le fruit de travaux conséquents, cf. (Dyer et Lehrer, 2011).

Un autre sujet fortement relié à ces groupes est celui des algèbres de Kac-Moody. Ces dernières font partie d'un champ de recherche important pour différentes raisons. Tout d'abord parce qu'elles généralisent les algèbres de Lie semi-simples, mais aussi parce qu'elles fournissent un cadre idéal pour l'étude des variétés de Schubert, cf. (Gutkin, 1986), (Mathieu, 1986) ou (Tits, 1989). De nombreuses connexions entre ces algèbres et les groupes de Weyl affines ont par la suite été établies, telles que : la classification de certaines algèbres de Kac-Moody via les diagrammes de Dynkin et Dynkin étendus, cf. (Kac, 1990), ou encore l'utilisation des groupes de Weyl affines dans la théorie algébrique des groupes de Chevalley et leurs algèbres de Lie, cf. (Verma, 1975).

Enfin, les groupes de Weyl affines ont fortement été étudiés par le biais de la théorie de Kazhdan-Lusztig. Cette théorie, née à la fin des années 70 dans le but de comprendre certaines représentations des groupes de Weyl provenant de groupes algébriques et agissant sur la cohomologie ℓ-adic, a inspiré un grand nombre de mathématiciens tels que : Beilinson, Bernstein, Deligne, Kazdhan, Luzstig, Shi, etc.

L'une des contributions de Jian-Yi Shi dans ce domaine provient notamment de deux articles publiés en 1987 et concernant les groupes de Weyl affines, cf. (Shi, 1987a) et (Shi, 1987b). Ces articles avaient pour but initial de donner une nouvelle décomposition de l'espace euclidien qui permettrait de mieux comprendre certaines cellules de Kazhdan-Lusztig. Un des résultats majeurs de ces articles fut entre autres de caractériser chaque alcôve par un Φ^+-uplet d'entiers $(k(w,\alpha))_{\alpha\in\Phi^+}$ soumis à certaines contraintes. Grâce à cette caractérisation, une nouvelle façon de penser les éléments de W_a émerge. La figure 0.3 ci-dessous illustre le cas du groupe affine $W(\widetilde{A}_2)$.

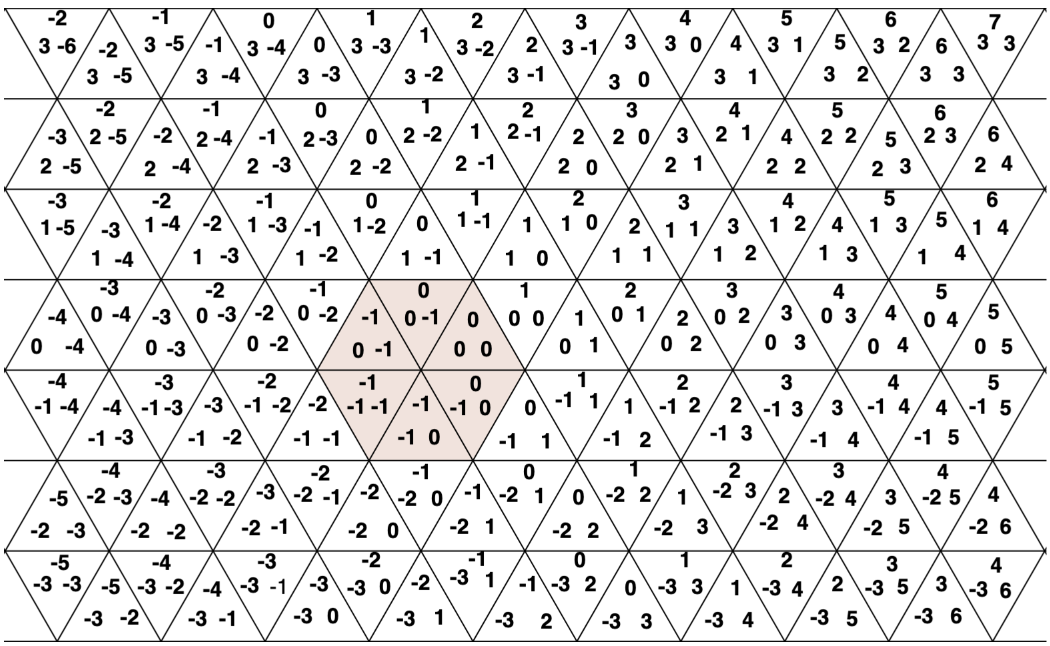

Figure 0.3 Coefficients $k(w,\alpha)$ du groupe $W(\widetilde{A}_2)$.

On se demande alors quels sont les avantages de cette nouvelle caractérisation ?

Elle a permis en particulier de décrire très facilement la longueur $\ell(w)$ de n'importe quel élément $w \in W_a$ via ces entiers :

$$\ell(w) = \sum_{\alpha\in\Phi^+} |k(w,\alpha)|. \tag{1}$$

La notion de longueur est un outil constamment utilisé en théorie des groupes de Coxeter, et donc en théorie des groupes de Weyl et Weyl affines. Rappelons que les groupes de Coxeter sont définis par générateurs et relations, ce qui entraine entre autres que tout élément du groupe s'obtient comme mot via une concaténation de générateurs. La longueur donne alors un critère numérique pour déterminer si une telle décomposition est optimale, c'est-à-dire avec le moins de générateurs possibles dans l'écriture du mot. Une écriture d'un élément est dite réduite si elle est optimale. La détermination des écritures réduites a été, durant le 20-ème siècle, source de nombreux travaux, publications, mais aussi collaborations entre mathématiciens et informaticiens. En outre, déterminer si deux écritures déterminent le même élément a été un sujet de recherche primordiale en théorie combinatoire des groupes de types finis. Cette question, usuellement appelée "le problème du mot", a été introduite pour la première fois en 1911 dans (Dehn, 1911). Il faudra attendre 1955 pour qu'un mathématicien russe, Piotr Novikov, fournisse un contre-exemple d'un groupe de type fini dont le problème du mot n'est pas résoluble, autrement dit indécidable. Il s'est alors avéré que cette question avait toute son importance, et a, à partir de cette période, suscité bien des tentatives. En particulier, le problème du mot dans le cas des groupes de Coxeter est résoluble, cf. (Tits, 1969). Il est à noter que cette question, même dans les groupes de Weyl affines, est de façon pratique assez fastidieuse. En effet, les relations au sein d'une décomposition sont cachées, et ne peuvent être apprivoisées, éventuellement, qu'avec beaucoup de pratique. C'est ici donc que l'importance des ordinateurs rentre en jeu. Fokko Du Cloux a développé pendant près de 15 ans le programme Coxeter, permettant l'implémentation de la combinatoire des groupes de Coxeter et du calcul des polynômes de Kazhdan-Lusztig. Ceci illustre en partie l'intérêt de détenir une formule telle que (1).

Une seconde motivation de cette nouvelle caractérisation fut la suivante. Comme nous l'avons dit plus haut, Jian-Yi Shi était dans les années 80 particulièrement intéressé par la théorie de Kazhdan-Lusztig. Il s'y intéressait d'une part via des concepts de natures algébriques, tels que les algèbres de Hecke, les algèbres enveloppantes d'algèbres de Lie semi-simples ou encore les représentations modulaires des groupes algébriques, mais aussi à partir d'objets de natures géométriques et combinatoires, tels que les variétés de Schubert, les complexes de faisceaux sur ces variétés, les cellules de Kazhdan-Lusztig, les doubles cellules de Kazhdan-Lusztig, etc. L'expérience de ces 30 dernières années a montré qu'il était extrêmement difficile de tirer de l'information sur la nature des cellules de Kazhdan-Lusztig. Encore aujourd'hui des personnes essayent de calculer dans des cas précis la forme de ces cellules ou celle des doubles cellules. Il n'y a pas de méthode générale pour les déterminer. Cependant les groupes de Weyl affines donnent certains éléments de réponse. En effet, dans (Shi, 1987b), Jian-Yi Shi introduisit la notion de signes admissibles. À partir de cette notion, la nouvelle décomposition de l'espace euclidien dont nous avons parlée plus haut apparaît. Les régions de cette décomposition sont communément appelées régions de Shi. On voudrait comprendre alors quel est le rapport entre les régions de Shi et les cellules de Kazhdan-lusztig ? Il se trouve que les régions de Shi et les doubles cellules sont relativement semblables d'un point de vue géométrique puisque certaines doubles cellules sont l'union de certaines régions de Shi. Nous donnons une liste non exhaustive de travaux liés à ce domaine :

- En 1983 Jian-Yi Shi donne une description de toutes les cellules de Kazhdan-Lusztig en type $\widetilde{A}_n$. Il utilise pour ce faire la notion de signes admissibles. Trois ans après il publie un livre parlant de tout ceci (et bien plus) : “The Kazhdan-Lusztig cells in certain affine Weyl groups”, cf. (Shi, 1986).

- En 1985/1986 Robert Bédard donne de nouveaux résultats sur certaines cellules dans les groupes affines et hyperboliques, cf. (Bédard, 1986).

- En 1988 Du Jie donne une description des cellules pour $W(\widetilde{B}_3)$, cf. (Jie, 1988).

- En 1988 Xi et Lusztig donnent de nouvelles relations parmi ces cellules dans les groupes affines, cf. (Lusztig et Xi, 1988).

- En 2010 Paul Gunnells donne certains résultats sur ces cellules via la théorie des automates, cf. (Gunnells, 2010).

Bien que la théorie de Kazhdan-Lusztig soit extrêmement riche nous ne l'avons pas particulièrement approfondie lors de cette thèse. Notre intérêt fut cependant dans la nature des coefficients $k(w,\alpha)$, qui avaient permis à Shi entre autres d'introduire la notion de signes admissibles (ou régions de Shi). C'est ici donc que commence notre recherche. Ce doctorat a différents objectifs, que nous allons maintenant décrire au travers des chapitres 1 et 2.

Chapitre 1. Nous introduisons et étudions dans cette thèse une variété affine $\widehat{X}_{W_a}$ associée à W_a. Cette variété, n'a a priori, pas de rapport avec les variétés associées aux groupes de Coxeter déjà connues, telles que : les variétés de Coxeter, les variétés de drapeaux, les variétés de Schubert, les variétés toriques, dont on peut trouver des références dans (Dudas, 2014), (Lehrer, 2008), (Milicevic *et al.*, 2020). La construction de cette variété est basée sur certains travaux de Jian-Yi Shi que l'on trouve dans (Shi, 1987a) et (Shi, 1987b).

Une des motivations initiales fut de comprendre comment se comporte la longueur d'un élément $w \in W_a$ lorsqu'on le multiplie à gauche par une réflexion $t \in W_a$. Plus précisément nous étions intéressés par les réflexions faisant descendre cette longueur de 1. Lors d'essais numériques nous avons constaté que la longueur se comportait étrangement bien par rapport au Φ^+-uplets $(k(w,\alpha))_{\alpha\in\Phi^+}$. Regardons par exemple la situation pour le groupe $W(\widetilde{A}_2)$ et intéressons-nous aux éléments de longueur 4 et 5. D'après la formule $\ell(w) = \sum\limits_{\alpha\in\Phi^+} |k(w,\alpha)|$ il est très simple de trouver ces éléments. Nous les représentons dans la figure 0.4 ci-dessous.

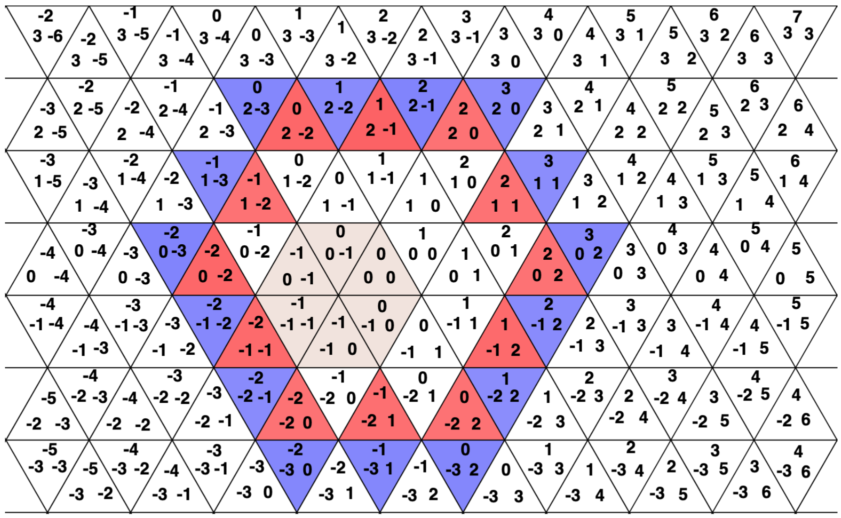

Figure 0.4 Alcôves de longueur 4 (rouge) et 5 (bleu) de $W(\widetilde{A}_2)$.

En pensant les alcôves comme des vecteurs que l'on voit plongés dans $\mathbb{R}^3$, nous voyons que les points semblent se positionner sur deux hyperplans parallèles. L'image de la figure 0.5 montre en particulier que les points de longueur 4 et de longueur 5 sont respectivement sur le même hyperplan.

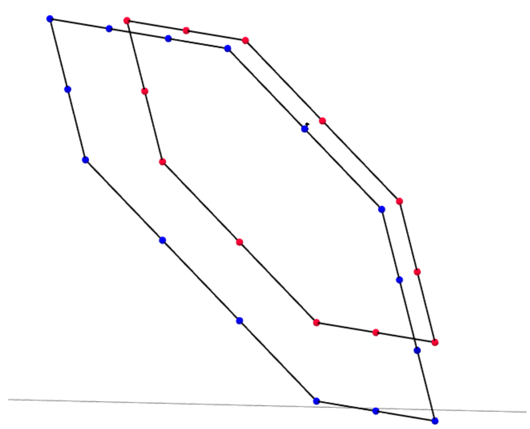

Figure 0.5 Éléments de longueur 4 (rouge) et 5 (bleu) de $W(\widetilde{A}_2)$.

Une question naturelle découlait alors de cette observation :

> Existe-t-il des équations régissant la répartition des éléments des groupes de Weyl affines ? Si oui, quelles sont-elles ?

Cette thèse a pour but de donner une réponse affirmative à cette question. Nous montrons que ce que nous avons observé sur cette image n'est pas le simple fruit du hasard. En particulier, dans l'exemple précédent les équations des hyperplans sont $X_{13} = X_{12} + X_{23}$ et $X_{13} = X_{12} + X_{23} + 1$. Nous verrons que ces équations sont entièrement reliées au système de racines A_2, et notamment à la décomposition de la seule racine positive et non simple dans la base des racines simples.

> Est-il possible de transférer toutes les opérations classiques du groupe affine afin de visualiser les éléments de W_a en tant qu'isométries de $\bigoplus_{\theta \in \Phi^+} \mathbb{R}e_\theta$?

Cette question nous conduira à définir la Φ^+-représentation au chapitre 1.4. Ce nouvel objet aura un rôle crucial dans la compréhension de l'action naturelle de W_a sur $\widehat{X}_{W_a}$. Cependant, il nous faudra avant tout généraliser certaines formules données dans (Shi, 1987a) et qui concernent les coefficients $k(w, \alpha)$. Ceci est fait à la proposition 1.4.6.

Un des objectifs centraux est par la suite de construire la variété $\widehat{X}_{W_a}$, dont les points entiers $\widehat{X}_{W_a}(\mathbb{Z})$ seront en bijection avec W_a. Ceci sera notre résultat central, cf. théorème 1.5.5. La première étape de cette construction est l'obtention d'un ensemble d'équations, qui définissent une variété affine notée X_{W_a}, et dont les Φ^+-uplets d'entiers associés à W_a sont solutions. Cependant, cette variété sera trop grande et nous n'aurons seulement qu'une injection de W_a dans $X_{W_a}(\mathbb{Z})$. Nous verrons donc comment épurer X_{W_a} afin d'en obtenir une bijection entre les points entiers d'une de ses sous-variétés, notée $\widehat{X}_{W_a}$, et W_a. Pour ce faire, nous introduisons les notions de vecteurs admissibles et vecteurs admis (voir définitions 5 et 7). À l'aide de ces deux définitions nous obtiendrons via la proposition 1.5.3 une décomposition de X_{W_a} comme suit :

$$X_{W_a} := \bigsqcup_{\lambda \text{ admissible}} X_{W_a}[\lambda].$$

La sous-variété $\widehat{X}_{W_a}$ sera elle donnée au théorème 1.5.5 par :

$$\widehat{X}_{W_a} := \bigsqcup_{\lambda \text{ admis}} X_{W_a}[\lambda].$$

Dans ces deux décompositions, les $X_{W_a}[\lambda]$ représentent les composantes irréductibles des deux variétés. Les nuances entre vecteurs admissibles et vecteurs admis nous mèneront à utiliser un certain polytope $P_{\mathcal{H}}$, auparavant étudié dans de nombreuses références comme (Bourbaki, 1968, Ch. VI, § 1.10), (Humphreys, 1990, Section 4.9), ou (Kane, 2001, pages 131-134). Ce polytope jouera un rôle déterminant par la suite puisque nous l'utiliserons pour déterminer le nombre de composantes irréductibles de $\widehat{X}_{W_a}$. Ce résultat sera également énoncé au théorème 1.5.5. Puisque cette variété est construite à partir de travaux concernant les régions de Shi, la question suivante s'est posée par elle même.

Existe-t-il un lien entre la variété $\widehat{X}_{W_a}$ et les régions de Shi ?

À partir de la notion d'admissibilité introduite dans (Shi, 1987b) nous donnons à la section 1.3.6 la définition de région signée (signed region), et nous montrons que les régions signées de W_a sont en bijection avec les régions de Shi de W_a. Nous expliquons par la suite que chaque région de Shi de W_a peut se voir comme l'intersection de $\widehat{X}_{W_a}$ avec un orthant de l'espace ambiant où réside la variété.

Après de nombreuses observations, il nous a semblé que l'ensemble des composantes irréductibles de $\widehat{X}_{W_a}$ était cohérent en un certain sens. Nous nous sommes alors posé la question :

> Existe-t-il une structure algébrique ou combinatoire sur l'ensemble des composantes irréductibles de $\widehat{X}_{W_a}$?

Au chapitre 1.6 nous établissons que l'ensemble des composantes irréductibles de $\widehat{X}_{W_a}$, noté $H^0(\widehat{X}_{W_a})$, admet une structure de treillis semi-distributif. Pour cela nous montrons que les éléments de $P_{\mathcal{H}}$ forment un intervalle dans l'ordre faible à droite de W_a. Cette démarche utilisera les notions de : cône de Tits et représentation géométrique de W_a. Cette perspective est expliquée à la section 1.3.4. Dans le cas du groupe symétrique, nous montrons que les composantes irréductibles de $\widehat{X}_{W(\widetilde{A}_n)}$ sont en bijection avec les permutations circulaires de $A_n = S_{n+1}$.

Puisqu'il existe une suite exacte courte reliant le groupe de Weyl affine, le groupe de Weyl fini, et le réseau $\mathbb{Z}\Phi$, la question suivante s'est imposée :

> Est-il possible d'interpréter géométriquement $H^1(W, \mathbb{Z}\Phi)$, le premier groupe de cohomologie du groupe de Weyl W, à travers la variété $\widehat{X}_{W_a}$? Si oui, y a-t-il une nouvelle façon de visualiser la notion de cobord de degré 1 ?

Dans le chapitre 1.7 nous relions ce groupe de cohomologie avec les composantes irréductibles de $\widehat{X}_{W_a}$. Nous étudions en particulier certaines orbites de points dans la variété et les relatons avec les sections de la suite exacte courte naturellement associée à W_a. Nous en déduisons de nouvelles obstructions cohomologiques, que nous relions à la matrice de Cartan du système de racines correspondant. Finalement, en manipulant les matrices de Cartan inverses pour les groupes de types A, B, C, D, nous parvenons à expliciter ces obstructions de façons concrètes via un ensemble d'équations modulaires.

Chapitre 2. Dans ce chapitre nous étudions davantage la variété $\widehat{X}_{W_a}$ lorsque $W_a = W(\widetilde{A}_n)$. D'aprés le théorème 1.5.5 du chapitre 1 nous savons que le nombre de composantes irréductibles de $\widehat{X}_{W(\widetilde{A}_n)}$ est $n!$. Ceci nous a suggéré qu'il devrait y avoir un lien entre ces composantes et les permutations circulaires de $W(A_n)$, elles aussi au nombre de $n!$. Nous explicitons par la suite une bijection entre ces deux ensembles à l'aide de la Φ^+-representation de $W(\widetilde{A}_n)$ et de l'action par conjugaison de $W(A_n)$ sur lui même (cf théorème 2.1.3). Nous étudions également l'action de $W(A_n)$ sur les composantes irréductibles de $\widehat{X}_{W(\widetilde{A}_n)}$ en en donnant des formules précises (cf proposition 2.3.2).

CHAPTER I

FIRST ARTICLE

Shi variety corresponding to an affine Weyl group

Nathan Chapelier

ABSTRACT

In this article we show that there exists a bijection between any affine Weyl group W_a and the integral points of an affine variety, denoted $\widehat{X}_{W_a}$, which we call the Shi variety of W_a. In order to do so, we use Jian-Yi Shi's characterization of alcoves in affine Weyl groups. We then study this variety further. We highlight combinatorial properties of irreducible components of $\widehat{X}_{W_a}$ and we show how they are related to a fundamental set $P_{\mathcal{H}}$ endowed with the right weak order of W_a. In the last section we exhibit a link between the first cohomology group of the underlying crystallographic group and some specific components of $\widehat{X}_{W_a}$.

1.1 Introduction

In this paper we introduce and investigate an affine variety $\widehat{X}_{W_a}$, called *the Shi variety* of W_a, associated to an affine Weyl group W_a. The construction of this variety is based on Jian-Yi Shi's work in (Shi, 1987a) and (Shi, 1987b). This development requires us to recall some basics about affine Weyl groups.

Let V be a Euclidean space, $\Phi \subset V$ be an irreducible crystallographic root system, and W be the corresponding Weyl group. Associated to Φ exists an infinite hyperplane arrangement, denoted $\mathcal{H}$, known as the affine Coxeter arrangement. This hyperplane arrangement cuts out V into simplices of the same volume which are called alcoves. The set of alcoves is denoted by $\mathcal{A}$.

The associated affine Weyl group W_a acts regularly on $\mathcal{A}$, implying in particular that there exists a one-to-one correspondence between W_a and $\mathcal{A}$ (see (Humphreys, 1990, ch.4) or Section 1.3 for more details). Let A_w be the corresponding alcove of $w \in W_a$.

In (Shi, 1987a) Jian-Yi Shi characterizes any alcove A_w by a Φ^+-tuple of integers $(k(w,\alpha))_{\alpha\in\Phi^+}$ subject to certain conditions. From this characterization he obtains in particular that the length $\ell(w)$ is given by $\ell(w) = \sum_{\alpha\in\Phi^+} |k(w,\alpha)|$.

The initial goal of this work was to understand how left multiplication by a reflection $t \in W_a$ affects the length of an element $w \in W_a$. Identifying which reflections decrease the length of w by exactly 1 was of particular interest to us. If we look at the example $W(\widetilde{A}_2)$ embedded in $\mathbb{R}^3$ together with its elements of length 4 and length 5, it suggests that these elements lie on two parallel hyperplanes (see Figure 1.1).

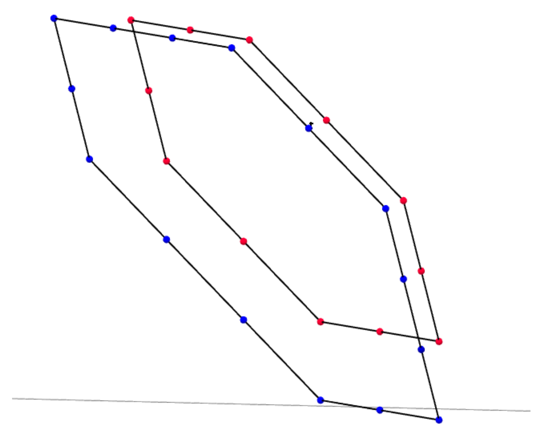

Figure 1.1 Elements of length 4 (red) and 5 (blue) in $W(\widetilde{A}_2)$.

We show in this article that the previous observation is exactly what occurs, and in the above example, the equations of the hyperplanes are $X_{13} = X_{12} + X_{23}$ and $X_{13} = X_{12} + X_{23} + 1$.

Plan of this work. The present paper has several goals that we describe now:

The preliminary step is to recall some standard definitions and terminologies related to Coxeter groups, root systems, affine Weyl groups, Tits cone, and Shi regions. We recall this material in Sections 1.2 and 1.3.

The first goal is to introduce the necessary tools in order to see elements of W_a as isometries in $\bigoplus_{\theta \in \Phi^+} \mathbb{R}e_\theta$. This leads us to define the Φ^+-representation in Section 4. This new object has an important role in the understanding of the natural action of W_a onto the variety $\widehat{X}_{W_a}$. To do so, we first generalize a few formulas of (Shi, 1987a) about the coefficients $k(w, \alpha)$. This is done in Section 1.4 and mainly in Proposition 1.4.6.

The second goal, achieved in Section 1.5, is to build the variety $\widehat{X}_{W_a}$, whose integral points $X_{W_a}(\mathbb{Z})$ are in bijection with W_a. This is the main result of this article (see Theorem 1.5.5). To do so, we first obtain a set of equations that cuts out an affine variety denoted X_{W_a}, and where the Φ^+-tuples of integers of W_a are solutions of these equations. However, this variety is too large and we only have an injection of W_a into $X_{W_a}(\mathbb{Z})$. We then see how to shrink X_{W_a} in order to get a one-to-one correspondence between the integers points of a subvariety and W_a. This is where we introduce the notion of admissible and admitted vectors (see Definitions 5 and 7). These definitions yield, inter alia, a decomposition of X_{W_a} as follows:

$$X_{W_a} := \bigsqcup_{\lambda \text{ admissible}} X_{W_a}[\lambda],$$

whereas the subvariety $\widehat{X}_{W_a}$ inherits the decomposition:

$$\widehat{X}_{W_a} := \bigsqcup_{\lambda \text{ admitted}} X_{W_a}[\lambda],$$

where the $X_{W_a}[\lambda]$ are the irreducible components of these two varieties. The difference between being admissible or admitted leads us to recover a polytope $P_{\mathcal{H}}$, which has already been well studied (see (Bourbaki, 1968) ch VI, § 1.10 or (Humphreys, 1990) 4.9 or (Kane, 2001) pages 131-134), and which will play a crucial role thereafter. Indeed, thanks to it, we will give among others the number of irreducible components of $\widehat{X}_{W_a}$ (see Theorem 1.5.5). In Section 1.5.4 we establish a link between the Shi regions of W_a and $\widehat{X}_{W_a}$. Finally, we end Section 1.5 by showing that the construction of $\widehat{X}_{W_a}$ is functorial in the sense that it only depends of the isomorphism class of Φ.

In Section 1.6, we show that the set of irreducible components of $\widehat{X}_{W_a}$, denoted $H^0(\widehat{X}_{W_a})$, has a structure of semidistributive lattice. For this purpose, we prove that the alcoves contained in $P_{\mathcal{H}}$ form a interval in the right weak order of W_a. This step will use the Tits cone $\mathcal{U}$ and the geometric representation of W_a, seen as a slice of $\mathcal{U}$. This perspective is explained beforehand in Section 2.

Our last goal is addressed in Section 7. The goal is to relate $H^1(W, \mathbb{Z}\Phi)$, the first cohomology group of W with coefficients in $\mathbb{Z}\Phi$, with the irreducible components of $\widehat{X}_{W_a}$ associated to the generators of W.

1.2 Generalities about Coxeter groups

Let (W, S) be a Coxeter system with e the identity element and S the set of Coxeter generators. For $s, t \in S$ we denote by m_{st} the order of st.

1.2.1 Geometrical representation

Let X be the $\mathbb{R}$-vector space with basis $\{e_s \mid s \in S\}$, and let B be the symmetric bilinear form on X defined by

$$B(e_s, e_t) = \begin{cases} -\cos(\frac{\pi}{m_{st}}) & \text{if } \ m_{st} < \infty \\ -1 & \text{if } \ m_{st} = \infty. \end{cases}$$

We denote by $O_B(X)$ the orthogonal group of X associated to B. For each $s \in S$ we define $\sigma_s : X \to X$ by $\sigma_s(x) = x - 2B(e_s, x)e_s$. The map $\sigma : W \hookrightarrow O_B(X)$ defined by $s \mapsto \sigma_s$ is called *the geometrical representation* of (W, S) (for more information the reader may refer to (Bourbaki, 1968) ch. V, § 4 or (Humphreys, 1990) ch 5.3). Through this representation we identify (W, S) with $(\sigma(W), \sigma(S))$.

Definition 1. Let us denote COS $:= \{-1\} \cup \{-\cos(\frac{\pi}{k}), k \in \mathbb{N}_{\geq 2}\}$. A simple system in (X, B) is a finite subset Γ in X such that:

i) Γ is linearly independent;

ii) for all $\alpha, \beta \in \Gamma$ distinct, $B(\alpha, \beta) \in$ COS;

iii) for all $\alpha \in \Gamma$, $B(\alpha, \alpha) = 1$.

We denote by $\Psi = W(\Gamma)$ the corresponding root system with basis Γ. Let us write $\Psi^+ := \Psi \cap \text{cone}(\Gamma)$ and $\Psi^- = -\Psi^+$. Then one has $\Psi = \Psi^- \sqcup \Psi^+$. If $\alpha \in \Psi$ we denote by s_α its corresponding reflection.

Let Γ be a simple system in (X, B). The group $W_\Gamma := \langle s_\alpha \mid \alpha \in \Gamma \rangle$ is a subgroup of W. Moreover it is a Coxeter group with set of generators $S_\Gamma = \{s_\alpha \mid \alpha \in \Gamma\}$ (We refer the reader to (Dyer, 1990) or (Dyer et Hohlweg, 2016) Section 2.5 for more details about subreflection groups and their root system). We say that Γ is a simple system for (W_Γ, S_Γ). In particular the set $\Delta := \{e_s \mid s \in S\}$ is a simple system for (W, S) and $S = S_\Delta$.

1.2.2 Length function and inversion set

The *length function* $\ell : W \longrightarrow \mathbb{N}^*$ is defined as follows: $\ell(w)$ is the smallest number r such that there exists an expression $w = s_{i_1} \dots s_{i_r}$ with $s_{i_k} \in S$. By convention, $\ell(e) = 0$. This function has been extensively studied and all basic information about it can be found in (Bourbaki, 1968) or (Humphreys, 1990). Let $w \in W$. An expression of w is called a *reduced expression* if it is a product of $\ell(w)$ generators. The *inversion set* of w is by definition

$$\begin{aligned} N(w) &:= \{\alpha \in \Psi^+ \mid \ell(s_\alpha w) < \ell(w)\} \\ &= \{\alpha \in \Psi^+ \mid w^{-1}(\alpha) \in \Psi^-\}. \end{aligned}$$

Moreover we have $|N(w)| = \ell(w)$. Let $I \subset S$. The *standard parabolic subgroup* W_I is defined to be the subgroup of W generated by all $s_\alpha \in I$. It is well-known that (W_I, I) is itself a Coxeter system. We denote Ψ_I its root system and Δ_I its simple system. A subgroup H of W is called *parabolic* if there exists $w \in W$ and $J \subset S$ such that $H = wW_Jw^{-1}$.

1.2.3 Contragredient representation and Tits cone

Since W acts on X via σ, it also acts on X^* with the contragredient action $\sigma^* : W \hookrightarrow GL(X^*)$ defined by $\sigma^*(w) = {}^t\sigma(w^{-1})$. The natural pairing of X^* with X is denoted by $\langle f, x\rangle$. Therefore the action of W on X^* is characterized by $\langle wf, wx\rangle = \langle f, x\rangle$ for $w \in W$, $f \in X^*$ and $x \in X$. If $s \in S$ and $x \in X$ we denote

$$\begin{aligned} Z_x &= \{f \in X^* \mid \langle f, x\rangle = 0\}, \\ A_s &= \{f \in X^* \mid \langle f, \alpha_s\rangle > 0\}. \end{aligned}$$

Finally, let C be the intersection of all A_s for $s \in S$ and let $D = \overline{C}$. The *Tits cone* $\mathcal{U}(W)$ of W is defined by $\mathcal{U}(W) := \bigcup_{w \in W} wD$. If $w \in W$, wC is called a *chamber* of

W. The action of W on $\{wC, w \in W\}$ is simply transitive (see (Bourbaki, 1968) ch. V, § 4. 4). We can then identify W with the set of chambers of $\mathcal{U}(W)$.

1.3 Affine Weyl groups and Shi parameterization

Let $(V, (-,-))$ be a Euclidean space. We denote $||x|| = \sqrt{(x,x)}$ for $x \in V$. Let Φ be an irreducible crystallographic root system in V with simple system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. We assume here that Φ is essential, that is, taking the lattice $\mathbb{Z}\Phi$, one has $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R} = V$. From now on, when we will say "root system" it will always mean irreducible essential crystallographic root system.

Let W be the *Weyl group* associated to $\mathbb{Z}\Phi$, that is the maximal (for inclusion) reflection subgroup of $O(V)$ admitting $\mathbb{Z}\Phi$ as a W-equivariant lattice. Let $\alpha \in \Phi$. We write

$$\begin{array}{cccccc} s_\alpha & : & V & \longrightarrow & V \\ & & x & \longmapsto & x - 2\frac{(\alpha,x)}{(\alpha,\alpha)}\alpha \end{array}.$$

It is known that the Coxeter group associated to Φ, i.e the subgroup of $O(V)$ generated by the reflections s_α, is actually the Weyl group W. The Weyl groups are classified by the Coxeter graphs of type A_n, $B_n = C_n$, D_n, E_6, E_7, E_8, F_4, and G_2 (see, for example,(Bourbaki, 1968), Chap. VI, §4, Theorem 1). We denote by $S := \{s_{\alpha_1}, .., s_{\alpha_n}\}$ a set of Coxeter generators of W.

Because of the classification of irreducible crystallographic root systems, we know that there are at most two possible root lengths in Φ. We call short root the shorter ones. To be consistent with (Shi, 1987a), we require $||\alpha|| = 1$ for any short root $\alpha \in \Phi^+$.

Let $\alpha \in \Phi$ such that $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ with $a_i \in \mathbb{Z}$. The height of α (with respect to Δ) is defined by the number $h(\alpha) = a_1 + \cdots + a_n$. Height gives us an organizational principle for making inductive proofs. Height also provides a preorder on Φ^+ defined as $\alpha \leq \beta$ if and only if $h(\alpha) \leq h(\beta)$. We also see that $h(\alpha+\beta) = h(\alpha) + h(\beta)$ and $h(-\alpha) = -h(\alpha)$ for all $\alpha, \beta, \alpha+\beta \in \Phi$. We denote by $-\alpha_0$ the *highest short root* of Φ^+.

1.3.1 Affine Weyl groups

From now on we will identify $\mathbb{Z}\Phi$ and the group of its associated translations. Let $k \in \mathbb{Z}$. Define the affine reflection as follows

$$\begin{array}{rcccc} s_{\alpha,k} & : & V & \longrightarrow & V \\ & & x & \longmapsto & x - (2\frac{(\alpha,x)}{(\alpha,\alpha)} - k)\alpha. \end{array}$$

Let us write $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. We consider the subgroups W_a and $W_a^\vee$ of $\mathrm{Aff}(V)$ defined as follows

$$W_a = \langle s_{\alpha,k} \mid \alpha \in \Phi,\ k \in \mathbb{Z} \rangle \quad \text{and} \quad W_a^\vee = \langle s_{\alpha^\vee,k} \mid \alpha \in \Phi,\ k \in \mathbb{Z} \rangle.$$

It is known that $W_a \simeq \mathbb{Z}\Phi \rtimes W$ (see (Humphreys, 1990), Ch 4). The group W_a is called the *affine Weyl group* associated to Φ. The Coxeter group structure on W induces a Coxeter group structure on W_a. The set $S_a := \{s_{\alpha_1}, \ldots, s_{\alpha_n}\} \cup \{s_{-\alpha_0,1}\}$ is a set of Coxeter generators of W_a. For short we will write $S_a = \{s_0, s_1, \ldots s_n\}$ where $s_0 := s_{-\alpha_0,1}$ and $s_i = s_{\alpha_i}$ for $i = 1, \ldots, n$. Throughout this article, affine Weyl group means irreducible affine Weyl group.

Let us make the following comment. In the literature about Weyl groups and affine Weyl groups it is more common to define the affine Weyl groups (associated to the root system Φ) with the reflections associated to the hyperplanes $\{x \in V \mid (x, \alpha) = k\}$. Therefore what we would call the affine Weyl group associated to Φ would commonly be described in the literature as the affine Weyl group associated to $\Phi^\vee$, that is $W_a^\vee$. Our choice to call W_a the affine Weyl group associated to Φ is because in (Shi, 1987a) the author gave the definition of W_a via the hyperplanes $\{x \in V \mid (x, \alpha^\vee) = k\}$, that is via the reflections $s_{\alpha,k}$, and since a lot of this article is based on Jian-Yi Shi's work it is natural to keep his conventions.

1.3.2 Shi parameterization

For any $\alpha \in \Phi$, any $k \in \mathbb{Z}$ and any $m \in \mathbb{R}$, we define the hyperplanes

$$\begin{aligned} H_{\alpha,k} &= \{x \in V \mid s_{\alpha,k}(x) = x\} \\ &= \{x \in V \mid (x, \alpha^\vee) = k\}, \end{aligned}$$

the half spaces

$$\begin{aligned} H^+_{\alpha,k} &= \{x \in V \mid k < (x, \alpha^\vee)\}, \\ H^-_{\alpha,k} &= \{x \in V \mid (x, \alpha^\vee) < k\}, \end{aligned}$$

and the strips

$$\begin{aligned} H^m_{\alpha,k} &= \{x \in V \mid k < (x, \alpha^\vee) < k + m\} \\ &= H^+_{\alpha,k} \cap H^-_{\alpha,k+m}. \end{aligned}$$

We denote by $\mathcal{H}$ the set of all the hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi^+$, $k \in \mathbb{Z}$, and by $\mathcal{H}^\vee$ the set of all the hyperplanes $H_{\alpha^\vee,k}$ with $\alpha \in \Phi^+$, $k \in \mathbb{Z}$. We have the relation $H_{-\alpha,k} = H_{\alpha,-k}$. Therefore we need only to consider the hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$. This last remark is quite convenient, because when we look at the collection of all the hyperplanes in V, we can think of them as being indexed either by elements of Φ or by elements of Φ^+.

Furthermore, if we have two hyperplanes $H_{\alpha,k}$ and $H_{\alpha,k'}$ the geometric space strictly contained between them is

$$H^{|k-k'|}_{\alpha,\min(k,k')}.$$

The lattice $\mathbb{Z}\Phi$ acts naturally on the strips and this action is given by

$$\tau_x H^m_{\alpha,k} = H^m_{\alpha,k+(\alpha^\vee,x)}.$$

Moreover W also acts on the strips as isometries of $\mathbb{R}^n$ and it follows that W_a acts also on the set of strips.

The connected components of

$$V \setminus \bigcup_{\substack{\alpha\in\Phi^+ \\ k\in\mathbb{Z}}} H_{\alpha,k}$$

are called *alcoves*. We denote $\mathcal{A}$ the set of all the alcoves and A_e the alcove defined as $A_e = \bigcap_{\alpha\in\Phi^+} H^1_{\alpha,0}$. Since W_a acts on the strips it also acts on $\mathcal{A}$. It turns out that this action on $\mathcal{A}$ is regular (see (Humphreys, 1990) chapter 4). Thus, there is a bijective correspondence between the elements of W_a and all the alcoves. This bijection is defined by $w \mapsto A_w$ where $A_w := wA_e$. We call A_w the corresponding alcove associated to $w \in W_a$. Any alcove of V can be written as an intersection of particular strips, that is there exists a Φ^+-tuple of integers $(k(w,\alpha))_{\alpha\in\Phi^+}$ such that

$$A_w = \bigcap_{\alpha\in\Phi^+} H^1_{\alpha,k(w,\alpha)}.$$

We use the same convention as in (Shi, 1987a; Shi, 1987b): for any $w \in W_a$ and any $\alpha \in \Phi$ we set

$$k(w,-\alpha) = -k(w,\alpha).$$

Definition 2. A point $x \in V$ is called special if $\mathrm{Stab}_{W_a}(x)$ is isomorphic to W. Intuitively this notion embodies the points in V that have the same geometry in their neighbourhood as the point 0.

Proposition 10.17 of (Abramenko et Brown, 2008) tells us that such points exist. Moreover, there exists a useful characterisation of these points:

Proposition 1.3.1 ((Abramenko et Brown, 2008), Proposition 10.19)**.** *A point $x \in V$ is special if and only if every hyperplane in $\mathcal{H}$ is parallel to a hyperplane passing through x.*

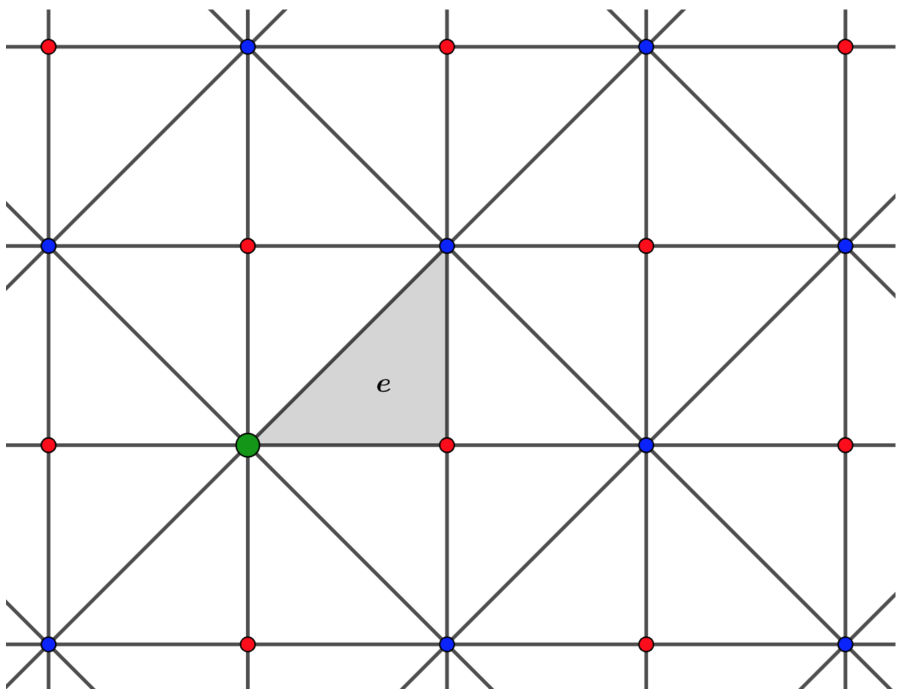

Figure 1.2 Example in $W(\widetilde{B}_2)$ of special points in blue and non-special points in red. The alcoves (or equivalently the chambers) pointed on the green point are the elements of W.

In the setting of affine Weyl groups the length of any element $w \in W_a$ is easy to compute via the coefficients $k(w, \alpha)$. Indeed, thanks to Proposition 4.3 in (Shi, 1987a) we have

$$\ell(w) = \sum_{\alpha \in \Phi^+} |k(w, \alpha)|.$$

In (Shi, 1987a) J.Y. Shi gave a characterization of the possible Φ^+-tuples $(k_\alpha)_{\alpha \in \Phi^+}$ over $\mathbb{Z}$ such that these tuples are the tuples of elements in W_a. In 1999 the same author gave an easier statement of this characterization. Here they are:

Theorem 1.3.2 (Theorem 5.2 of (Shi, 1987a)). *Let* $A = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha, k_\alpha}$ *with* $k_\alpha \in \mathbb{Z}$. *Then* A *is an alcove, if and only if, for all* α, $\beta \in \Phi^+$ *satisfying* $\alpha + \beta \in \Phi^+$, *we have the following inequality*

$$||\alpha||^2 k_\alpha + ||\beta||^2 k_\beta + 1 \leq ||\alpha+\beta||^2 (k_{\alpha+\beta}+1) \leq ||\alpha||^2 k_\alpha + ||\beta||^2 k_\beta + ||\alpha||^2 + ||\beta||^2 + ||\alpha+\beta||^2 - 1. \quad (1.1)$$

Theorem 1.3.3 (Theorem 1.1 of (Shi, 1999)). *Let* $A = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha, k_\alpha}$ *with* $k_\alpha \in \mathbb{Z}$. *Then* A *is an alcove, if and only if, for all* α, $\beta \in \Phi^+$ *satisfying* $\gamma := (\alpha^\vee + \beta^\vee)^\vee \in \Phi^+$ *the following inequality holds*

$$k_\alpha + k_\beta \leq k_\gamma \leq k_\alpha + k_\beta + 1. \quad (1.2)$$

Remark 1. We also want to warn the reader that compared to the conventions of J.Y. Shi in (Shi, 1987a) and (Shi, 1987b), we swap left and right multiplication.

The left multiplication is defined as follows: the alcove corresponding to $w'w$ is obtained by acting by the isometry w' on the alcove A_w.

The right multiplication is defined as follows: the alcove corresponding to ws is obtained by crossing the wall of A_w associated to the generator s. Right multiplication gives rise to a partial order on W_a (and W) known as the right weak order: we say that $w \leq w'$ in the right weak order if w is a prefix of w', that is if there exists $u \in W_a$ such that $w' = wu$ and $\ell(w') = \ell(w) + \ell(u)$.

Example 1. Take $W_a = W(\widetilde{B}_2)$ and set $\{\alpha_1, \alpha_2\}$ a simple system of B_2. Let $-\alpha_0 = \alpha_1 + \alpha_2$ be the highest short root of B_2. We denote $s_i := s_{\alpha_i}$ for $i = 1, 2$ and $s_0 := s_{-\alpha_0,1}$. We also identify the alcoves with the elements of W_a. We see for example in Figure 1.3 that the way to reach $s_0s_2s_1s_0s_2$ from the identity element e is by flipping the alcove A_e along its edges, which are associated with the generators of $W(\widetilde{B}_2)$.

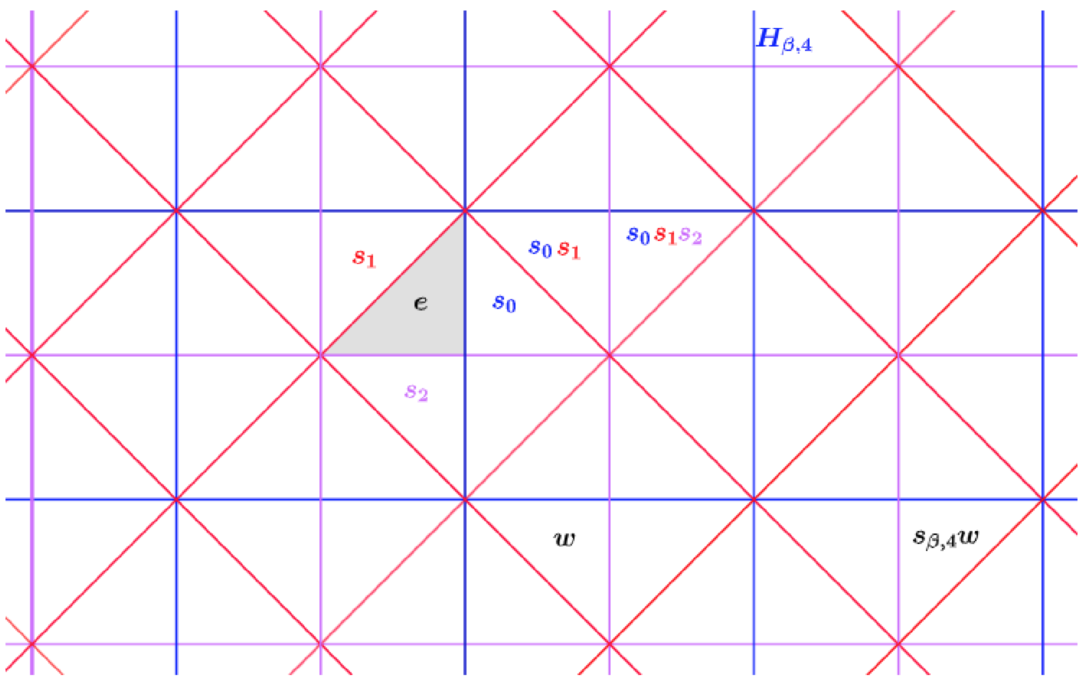

Figure 1.3 Flipping of the generators of $W(\widetilde{B}_2)$.

Example 2. For $W_a = \widetilde{A}_2$, the positive root system of A_2 is given by 3 roots, say $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$. Shi's parameterization in this case is shown in Figure 1.4 below.

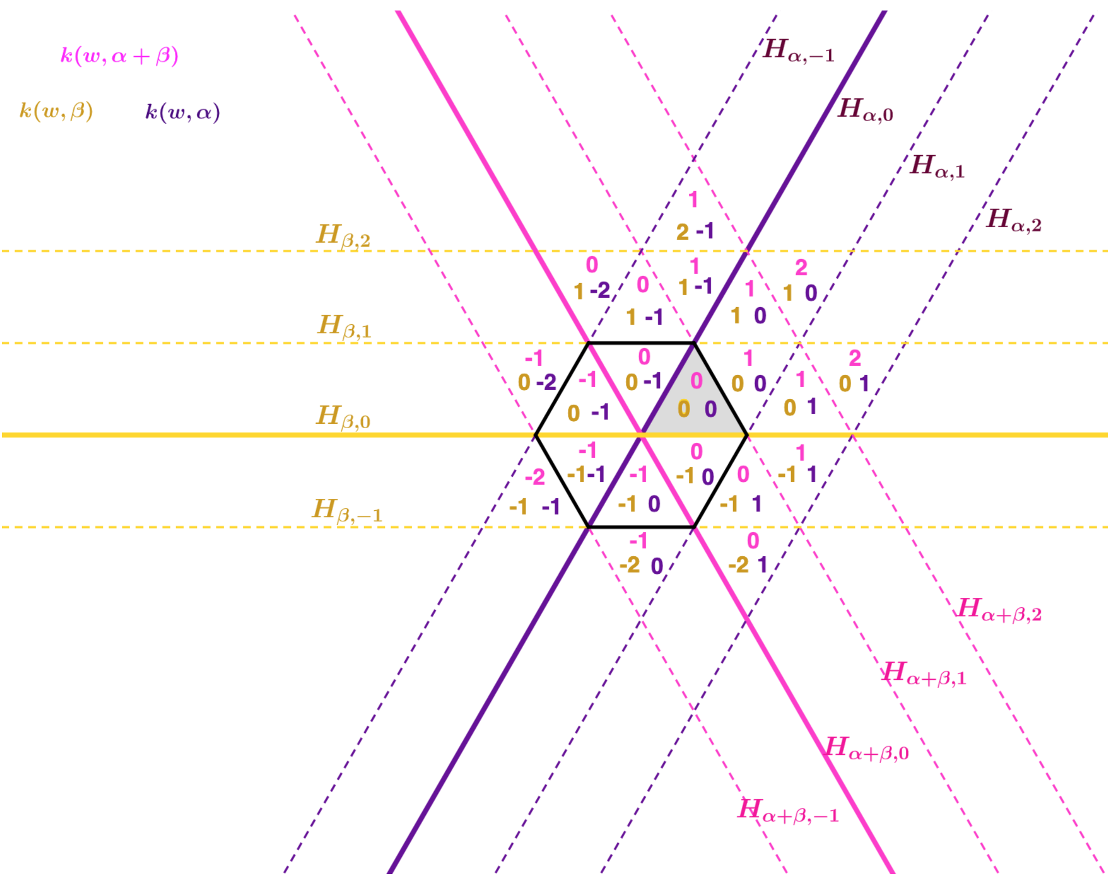

Figure 1.4 Shi parameterization of $W(\widetilde{A}_2)$.

1.3.3 Fundamental parallelepiped $P_{\mathcal{H}}$

Let $\mathbb{Z}\Phi^\vee$ be the coroot lattice and let us write $\mathbb{Z}\Phi^\vee = \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee$. We define its dual lattice $(\mathbb{Z}\Phi^\vee)^*$ as

$$(\mathbb{Z}\Phi^\vee)^* := \{x \in V \mid (x, y) \in \mathbb{Z}\ \forall y \in \mathbb{Z}\Phi^\vee\}.$$

The lattice $(\mathbb{Z}\Phi^\vee)^*$ is called the *weight lattice*. This lattice can be decomposed as $(\mathbb{Z}\Phi^\vee)^* = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$ where ω_i is such that $(\alpha_i^\vee, \omega_j) = \delta_{ij}$. The elements ω_i are called the *fundamental weights* (with respect to Δ). Notice that it is also possible to define the weight lattice by

$$(\mathbb{Z}\Phi^\vee)^* = \{x \in V \mid (x, y) \in \mathbb{Z}\ \forall y \in \Phi^\vee\}.$$

Since Φ is crystallographic, $\Phi^\vee$ is also crystallographic and the inclusion $\mathbb{Z}\Phi^\vee \subset (\mathbb{Z}\Phi^\vee)^*$ follows. Consequently, we are able to define the quotient group ${}^{(\mathbb{Z}\Phi^\vee)^*}/_{\mathbb{Z}\Phi^\vee}$. A good reference of these quotient groups is given in (Bourbaki, 1968). The *index of connection* of Φ is by definition the cardinality of this quotient group. We denote it as

$$f_\Phi := \left| {}^{(\mathbb{Z}\Phi^\vee)^*}/_{\mathbb{Z}\Phi^\vee} \right|.$$

We define $P_{\mathcal{H}} := \bigcap\limits_{\alpha\in\Delta} H^1_{\alpha,0}$ and $P_{\mathcal{H}^\vee} := \bigcap\limits_{\alpha\in\Delta} H^1_{\alpha^\vee,0}$. The fundamental weights ω_i are some of the vertices of $P_{\mathcal{H}}$ and we have $P_{\mathcal{H}} = \{\sum\limits_{i=1}^n c_i\omega_i \mid c_i \in [\![0, 1]\!]\}$. Since $(\omega_i, \omega_j) \geq 0$ for all i, j, the element of maximal norm in $P_{\mathcal{H}}$ is the vertex $\rho := \sum\limits_{i=1}^n \omega_i$. Moreover, if $z \in \operatorname{cone}(\Delta)$ we have $(z, \omega_i) \geq 0$ for all fundamental weight ω_i.

It is known (see (Bourbaki, 1968) ch. VI, § 1, exercice 7) that the index of connection of Φ is the determinant of the Cartan matrix associated to Φ. Moreover, the Cartan matrix associated to $\Phi^\vee$ is the transpose of the Cartan matrix associated to Φ. It follows that the indexes of connection of Φ and $\Phi^\vee$ are the same.

Finally, we define the two sets $\mathrm{Alc}(P_{\mathcal{H}}) := \{w \in W_a \mid A_w \subset P_{\mathcal{H}}\}$ and $\mathrm{Alc}(P_{\mathcal{H}^\vee}) := \{w \in W_a^\vee \mid A_w \subset P_{\mathcal{H}^\vee}\}$. It is well known (see (Kane, 2001), Section 11-6, Lemma C) that

$$|\mathrm{Alc}(P_{\mathcal{H}^\vee})| = \frac{|W(\Delta)|}{f_\Phi}.$$

Example 3. Let us take $W_a = W(\widetilde{B}_2)$ with simple system $\{\alpha_1, \alpha_2\}$. A short computation shows that $\omega_1 = \frac{1}{2}(2\alpha_1 + \alpha_2)$ and $\omega_2 = \alpha_1 + \alpha_2$.

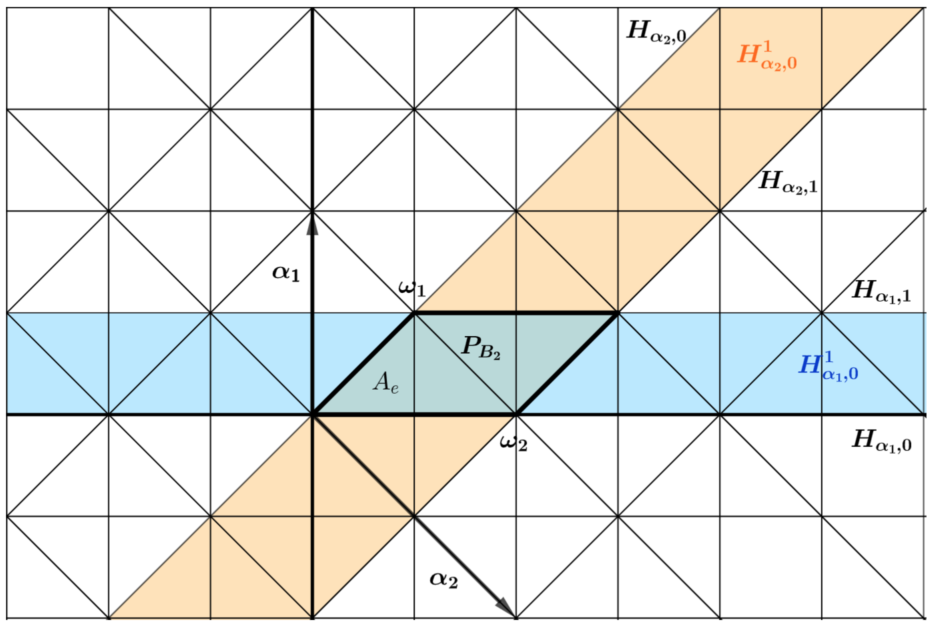

Figure 1.5 Fundamental parallelepiped P_{B_2}.

1.3.4 Geometrical representation of affine Weyl groups

Let $\widehat{V} = V \oplus \mathbb{R}\delta$ with δ an indeterminate. The inner product $(-,-)$ has a unique extension to a symmetric bilinear form on $\widehat{V}$ which is positive semidefinite and has a radical equal to the subspace $\mathbb{R}\delta$. This extension is also denoted $(-,-)$, and it turns out that the set of isotropic vectors associated to the form $(-,-)$ is exactly $\mathbb{R}\delta$. In particular for all $x, y \in V$ and for all $p, q \in \mathbb{Z}$ we have

$$(x + p\delta, y + q\delta) = (x, y). \tag{1.3}$$

The root system of W_a is denoted Φ_a and its simple system is denoted Δ_a. Using (Dyer et Lehrer, 2011) (Section 3.3 Definition 4 and Proposition 2) a concrete description of the affine (respectively, positive, simple) root system of W_a is provided by:

$$\begin{aligned}
\Phi_a &= \Phi^\vee + \mathbb{Z}\delta, \\
\Phi_a^+ &= ((\Phi^\vee)^+ + \mathbb{N}\delta) \sqcup ((\Phi^\vee)^- + \mathbb{N}^*\delta), \\
\Delta_a &= \Delta^\vee \cup \{\alpha_0^\vee + \delta\}.
\end{aligned}$$

The link between $\widehat{V}$ and the geometrical representation is as follows. Let $\Delta_a = \{\alpha_i^\vee \mid i = 1, \dots, n\} \cup \{\alpha_0^\vee + \delta\}$ be the simple system associated to W_a. To simplify the notations we denote $\lambda_i = \alpha_i^\vee$. We can now identify the X of Section 1.2.1 with $\widehat{V}$, by sending e_{s_0} to $\frac{\lambda_0+\delta}{||\lambda_0||}$ and e_{s_i} to $\frac{\lambda_i}{||\lambda_i||}$ for $s_i \in S$. Since δ is isotropic for $(-,-)$ we only consider the scalar products (λ_i, λ_j) for $i, j = 0, \dots, n$. It is well known that $(\lambda_i, \lambda_j) = ||\lambda_i|| \cdot ||\lambda_j|| \mathrm{cos}(\theta)$ where θ is the angle between λ_i and λ_j in the plane generated by these two vectors. Moreover, it is also well known that $\theta = \pi - \frac{\pi}{m_{ij}}$. It follows that

$$\begin{aligned}
(\lambda_i, \lambda_j) &= ||\lambda_i|| \cdot ||\lambda_j|| \mathrm{cos}(\pi - \frac{\pi}{m_{ij}}) = -||\lambda_i|| \cdot ||\lambda_j|| \mathrm{cos}(\frac{\pi}{m_{ij}}) \\
&= ||\lambda_i|| \cdot ||\lambda_j|| B(e_{s_i}, e_{s_j})
\end{aligned} \tag{1.4}$$

Furthermore we know that in the crystallographic root systems there are at most two root lengths. If λ_i is short we have set before that $||\lambda_i|| = 1$. Therefore in the simply laced cases we have $(\lambda_i, \lambda_j) = B(e_{s_i}, e_{s_j})$. When λ_i is longer than λ_j we have two situations to look at: if $m_{ij} = 4$ then $||\lambda_i|| = \sqrt{2}||\lambda_j|| = \sqrt{2}$, and in particular $(\lambda_i, \lambda_j) = \sqrt{2}B(e_{s_i}, e_{s_j})$. If $m_{ij} = 6$ then $||\lambda_i|| = \sqrt{3}||\lambda_j|| = \sqrt{3}$ and it follows that $(\lambda_i, \lambda_j) = \sqrt{3}B(e_{s_i}, e_{s_j})$.

The geometrical representation sends the reflection $s_{\alpha,k} \in W_a$ to the reflection

$s_{\alpha^\vee - k\delta}$ acting naturally in $\widehat{V}$. In particular one can think of the hyperplane $H_{\alpha,k}$ as the fixed points of $s_{\alpha^\vee - k\delta}$.

1.3.5 Tits cone and links with alcoves

Let us denote $\widehat{V}^* = \mathrm{Hom}(\widehat{V}, \mathbb{R})$ and $\widehat{V}^\perp = \{x \in \widehat{V} \mid (x, y) = 0 \ \forall y \in \widehat{V}\}$. Let $d : \widehat{V} \to \mathbb{R}$ be the map defined by $d(a + b\delta) = b$. We consider the following map

$$\begin{array}{ccccccc} i_k & : & V & \longrightarrow & \widehat{V}^* & & \\ & & v & \longmapsto & i_k(v) & : & \widehat{V} & \to & \mathbb{R} \\ & & & & & & x & \mapsto & (v, x) + kd(x). \end{array}$$

It is well known that the linear map

$$\begin{array}{ccccc} \Theta & : & \widehat{V} & \longrightarrow & \widehat{V}^* \\ & & v & \longmapsto & (v, -) \end{array}$$

is such that $\Theta(\widehat{V}) = \{f \in \widehat{V}^* \mid \langle f, \delta \rangle = 0\}$ and such that $ker(\Theta) = \widehat{V}^\perp$. Therefore Θ induces an isomorphism denoted $\overline{\Theta}$ between $\widehat{V}/\widehat{V}^\perp$ and $\{f \in \widehat{V}^* \mid \langle f, \delta \rangle = 0\}$ defined as $\overline{\Theta}(\overline{v}) := \Theta(v)$ for all $v \in \widehat{V}$.

Since $\widehat{V}^\perp = \mathbb{R}\delta$, one has $\widehat{V}/\widehat{V}^\perp \simeq V$ through the map φ defined as $\varphi(\overline{x + y\delta}) = x$. Thus, the map i_0 is nothing but $\overline{\Theta} \circ \varphi^{-1}$, while i_1 is injective and such that $i_1(V) = \{f \in \widehat{V}^* \mid \langle f, \delta \rangle = 1\}$. Let us define $\nu_1 \in \mathrm{Hom}(\widehat{V}^*, \widehat{V}^*)$ by $\nu_1(f) = f + d$.

Finally, one can summarize all of this via the following commutative diagram

$$\begin{tikzcd} \widehat{V}/\widehat{V}^\perp \arrow[rr, "\overline{\Theta}"] & & \{f \in \widehat{V}^* \mid \langle f, \delta \rangle = 0\} \arrow[d, "\nu_1"] \\ V \arrow[u, "\varphi^{-1}"] \arrow[urr, "i_0"] \arrow[rr, "i_1"'] & & \{f \in \widehat{V}^* \mid \langle f, \delta \rangle = 1\}. \end{tikzcd}$$

Our main goal is to see each alcove A_w of W_a, as the intersection of the chamber

wC and the hyperplane $\{f \in \widehat{V}^* \mid \langle f, \delta \rangle = 1\}$. Of course, alcoves and chambers don't live in the same space, but using the map i_1 we relate the notion of alcoves to the notion of chamber. More precisely, an alcove A_w (living in V) is sent via i_1 to $wC \cap \{f \in \widehat{V}^* | \langle f, \delta \rangle = 1\}$. An approximate idea is given in Figure 1.6.

The significant thing for us is to see that the hyperplane $H_{\alpha,k}$ in V is sent via i_1 to the affine subspace $Z_{-\alpha^\vee + k\delta} \cap \{f \in \widehat{V}^* | \langle f, \delta \rangle = 1\}$. Indeed for $v \in H_{\alpha,k}$ one has $i_1(v)(\delta) = (v, \delta) + d(\delta) = d(\delta) = 1$. Further, we also have

$$i_1(v)(-\alpha^\vee + k\delta) = -(v, \alpha^\vee) + d(-\alpha^\vee + k\delta) = -(v, \alpha^\vee) + k = 0$$

From now on we define $E(W_a)$, or E for short, to be the set:

$$E := \{f \in \widehat{V}^* \mid \langle f, \delta \rangle = 1\}.$$

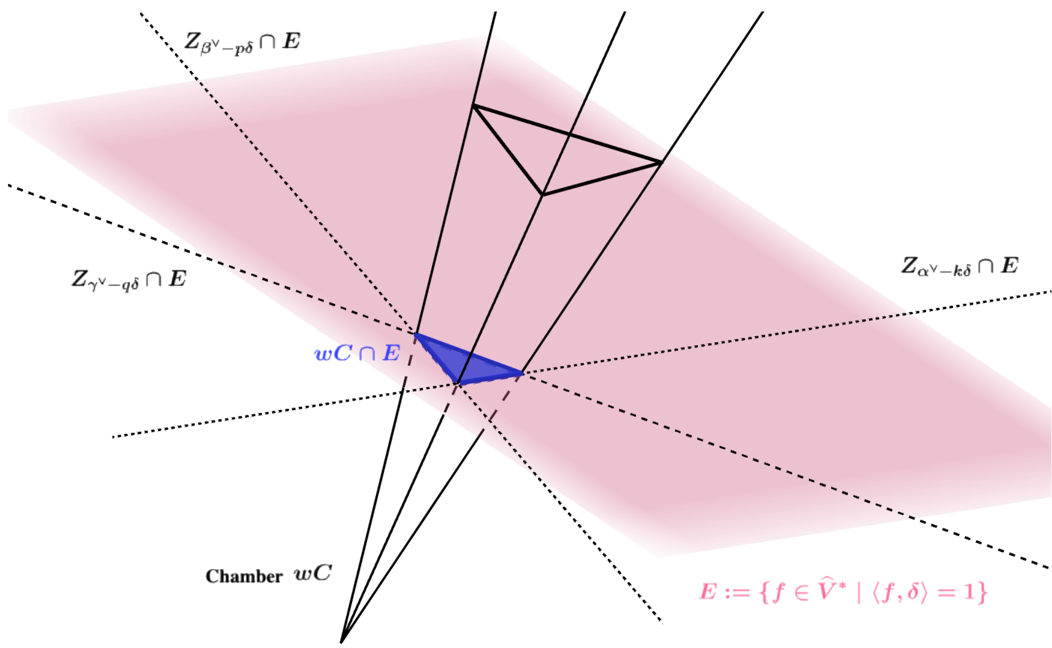

Figure 1.6 Chamber in the Tits cone of $W(\widetilde{B}_2)$ with its trace in E.

We end this section with the following fact:

Proposition 1.3.4. *Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$. Then we have for all $k, p \in \mathbb{Z}$ the following formula*

$$ord(s_\alpha s_\beta) = ord(s_{\alpha,k} s_{\beta,p}).$$

Proof. Let $x, y \in \Phi_a$. In order to prevent any confusion, we denote by t_x, t_y the reflections of $\widehat{V}$ associated to x and y, while $s_\alpha, s_\beta, s_{\alpha,k}, s_{\beta,p}$ are some reflections of V.

It turns out that $t_x t_y$ is a rotation of angle -2σ with σ the oriented angle between x and y. It implies in particular that $(t_x t_y)^q$ is a rotation of angle $-2q\sigma$. Consequently, $\mathrm{ord}(t_x t_y) = q$ if and only if $\sigma = \frac{b\pi}{q}$ for some $b \in \mathbb{Z}$. The main point is to see that the order is entirely controlled by the angle σ.

Therefore, when we apply this with $x := -\alpha^\vee$, $y := -\beta^\vee$ and $\sigma := \theta_1$, since $t_\alpha = t_{-\alpha^\vee}$ and $t_\beta = t_{-\beta^\vee}$, we have $\mathrm{ord}(t_\alpha t_\beta) = q$ if and only if $\theta_1 = \frac{b\pi}{q}$ for some $b \in \mathbb{Z}$. Let us set q to be the order of $t_\alpha t_\beta$. Without lost of generality, since the reflections t_x are vectorial they don't take account of the length of x, and then one can assume that $||x|| = 1$ for all $x \in \Phi_a$. Notice first of all that $\mathrm{ord}(s_\alpha s_\beta) = \mathrm{ord}(t_\alpha t_\beta)$ and $\mathrm{ord}(s_{\alpha,k} s_{\beta,p}) = \mathrm{ord}(t_{-\alpha^\vee+k\delta} t_{-\beta^\vee+p\delta})$. Furthermore, since δ is isotropic we have

$$(-\alpha^\vee + k\delta, -\beta^\vee + p\delta) = (\alpha^\vee, \beta^\vee) + kp(\delta, \delta) = (\alpha^\vee, \beta^\vee).$$

Moreover

$$(\alpha^\vee, \beta^\vee) = ||\alpha^\vee||.||\beta^\vee||\cos(\theta_1) = \cos(\theta_1),$$

and writing θ_2 to be the angle between $-\alpha^\vee + k\delta$ and $-\beta^\vee + p\delta$ we have

$$(-\alpha^\vee + k\delta, -\beta^\vee + p\delta) = ||-\alpha^\vee + k\delta||.||-\beta^\vee + p\delta||\cos(\theta_2) = \cos(\theta_2).$$

It follows then that $\cos(\theta_1) = \cos(\theta_2)$, which implies that $\theta_2 = \pm\theta_1 + 2r\pi$ for some $r \in \mathbb{Z}$. Then $\theta_2 = \frac{b\pi}{q} + 2r\pi = \frac{(b+2qr)\pi}{q}$ and it follows that $\mathrm{ord}(t_{-\alpha^\vee+k\delta} t_{-\beta^\vee+p\delta}) = q$. This ends the proof. $\square$

1.3.6 Shi regions

Introduced by Brink and Howlett the dominance order is the partial order $\preceq$ on Ψ^+ defined by

$$\alpha \preceq \beta \Longleftrightarrow \forall w \in W, \beta \in N(w) \Longrightarrow \alpha \in N(w).$$

For $\beta \in \Psi^+$, the *dominance set* of β is $\mathrm{Dom}(\beta) := \{\alpha \in \Psi^+ \mid \alpha \prec \beta\}$. The ∞-*depth* on Ψ^+ is defined by $\mathrm{dp}_\infty(\beta) = |\mathrm{Dom}(\beta)|$. We say that β is a *small root* if $\mathrm{dp}_\infty(\beta) = 0$. The set of small roots is denoted Σ. In the context of affine root systems the set of small roots is denoted Σ_a. The *Shi arrangement* of W is defined by

$$\mathcal{U}(W) \setminus \bigcup_{\alpha \in \Sigma} Z_\alpha.$$

The connected components of this arrangement are called the *Shi regions* of W (see (Dyer et Hohlweg, 2016) 3.1 and 3.2 for more details). We denote $Shi(W)$ the set of *Shi* regions of W. In particular when $X = \widehat{V}$, the Shi regions of W_a are the connected components of

$$\mathcal{U}(W_a) \setminus \bigcup_{\theta \in \Sigma_a} Z_\theta.$$

In (Shi, 1987b) J.Y. Shi introduced the notion of *signed regions*. For an integer k we define $sg(k) = 0$ if $k = 0$, $sg(k) = +$ if $k > 0$ and $sg(k) = -$ if $k < 0$. For a Φ^+-tuple of integers $(k_\alpha)_{\alpha \in \Phi^+}$ we define the map Sg as $Sg((k_\alpha)_{\alpha \in \Phi^+}) = (sg(k_\alpha))_{\alpha \in \Phi^+}$. We denote

$$Sg(w) := Sg((k(w, \alpha))_{\alpha \in \Phi^+}).$$

A *signed region* Γ of W_a is by definition a subset of W_a such that $Sg(w) = Sg(w')$ for all $w, w' \in \Gamma$. Therefore, being on the same signed region defines an equivalence relation on W_a. We denote $Sg(W_a)$ the set of signed regions of W_a.

Theorem 1.3.5. *The following map is a bijection*

$$\begin{array}{rcccc} \Lambda & : & Shi(W_a) & \longrightarrow & Sg(W_a) \\ & & \Omega & \longmapsto & i_1^{-1}(\Omega \cap E). \end{array}$$

Proof. Because of the definition of signed regions we know that the hyperplanes that separate these regions are exactly the $H_{\alpha,0}$ and $H_{\alpha,1}$ for $\alpha \in \Phi^+$. Thus, the signed regions of W_a are the connected components of $V \setminus \bigcup\limits_{\substack{\alpha\in\Phi^+\\ k\in\{0,1\}}} H_{\alpha,k}$. Moreover, it is known (see (Dyer et Hohlweg, 2016) Example 3.9) that for all $\alpha^\vee + k\delta \in \Phi_a^+$ we have the equality

$$dp_\infty(\alpha^\vee + k\delta) = \begin{cases} k & \text{if } \alpha^\vee \in (\Phi^\vee)^+ \text{ and } k \in \mathbb{N} \\ k-1 & \text{if } \alpha^\vee \in (\Phi^\vee)^- \text{ and } k \in \mathbb{N}^*. \end{cases}$$

Therefore, $dp_\infty(\alpha^\vee + k\delta) = 0$ if and only if $k \in \{0,1\}$. It follows that the set of hyperplanes that separate the Shi regions in $\mathcal{U}(W_a)$ is exactly

$$\{Z_{\alpha^\vee} \mid \alpha \in \Phi^+\} \sqcup \{Z_{-\alpha^\vee+\delta} \mid \alpha \in \Phi^+\}.$$

Hence the Shi regions of W_a are the connected components of

$$\mathcal{U}(W_a) \setminus \left[\bigcup_{\alpha\in\Phi^+} Z_{\alpha^\vee} \cup \bigcup_{\alpha\in\Phi^+} Z_{-\alpha^\vee+\delta}\right].$$

Moreover

$$i_1^{-1}(Z_{\alpha^\vee} \cap E) = H_{\alpha,0} \quad \text{and} \quad i_1^{-1}(Z_{-\alpha^\vee+\delta} \cap E) = H_{-\alpha,-1} = H_{\alpha,1}.$$

By continuity of i_1^{-1}, each Shi region intersected with E is sent to a signed region of W_a, and since $(i_1^{-1})_{|E}$ is a bijection, the map Λ is also a bijection. $\square$

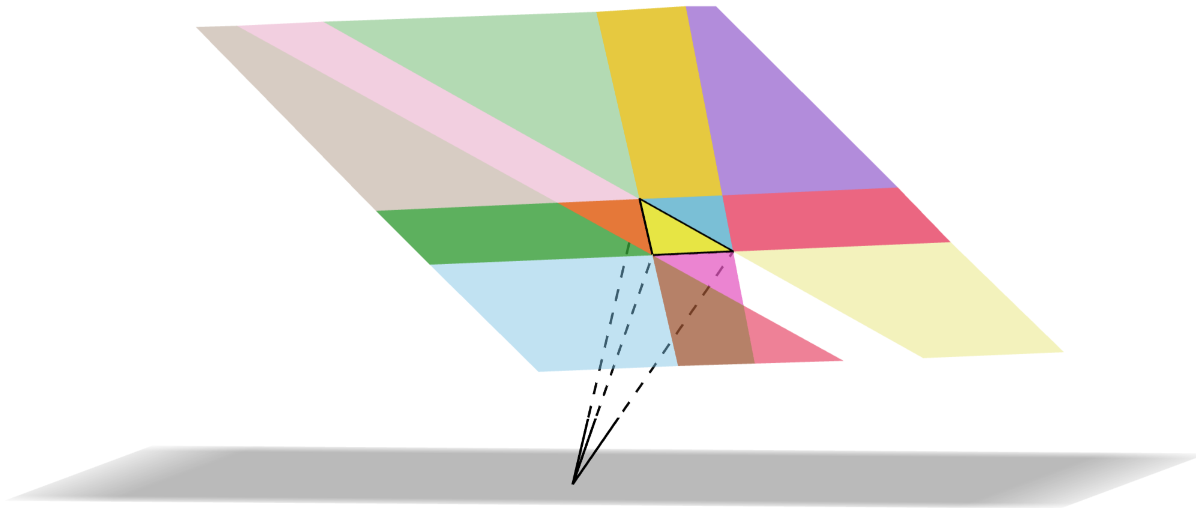

Figure 1.7 Shi regions of $W(\widetilde{A}_2)$ intersected with E.

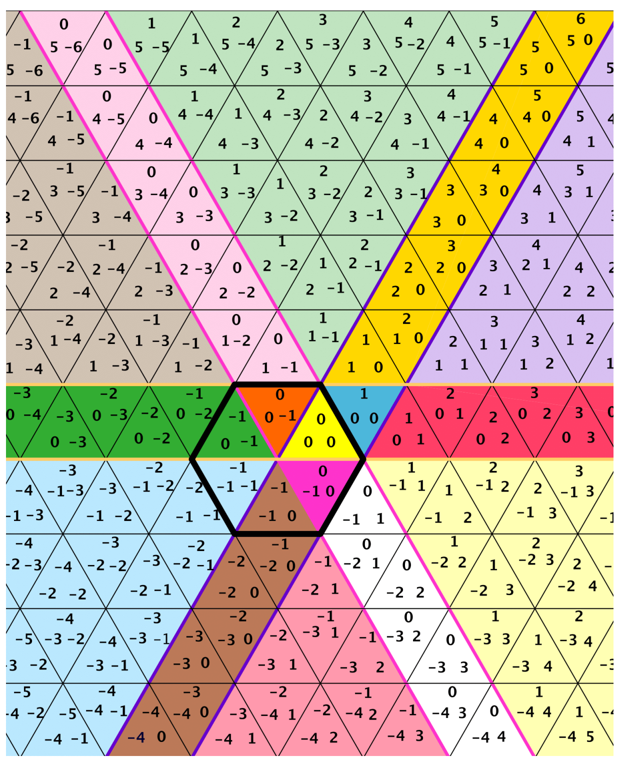

Figure 1.8 Signed regions of $W(\widetilde{A}_2)$.

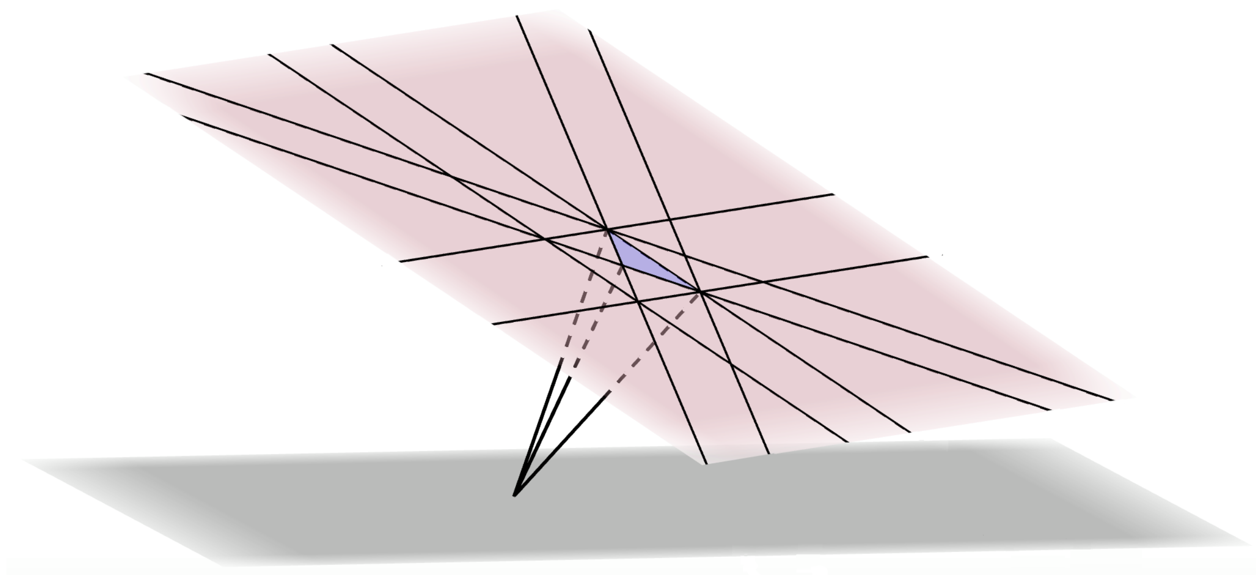

Figure 1.9 Shi regions of $W(\widetilde{B}_2)$ intersected with E.

1.4 Φ^+-representation

The idea of this section is to see the elements of W_a as affine isometries of a bigger space. Since any element $w \in W_a$ is characterized by a Φ^+-tuple of integers $(k(w,\alpha))_{\alpha\in\Phi^+}$ and since W_a acts geometrically on itself it is natural to want to see this action in $\mathbb{R}^{|\Phi^+|}$. For a Euclidean space E, we identify E with the space of its translations, and we denote by $\mathrm{Isom}(E)$ the space of its affine isometries, that is the elements of $E \rtimes GL(E)$ such that the euclidean scalar product is preserved. In other words $\mathrm{Isom}(E) = E \rtimes O(E)$.

Let us write $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_m\}$ with $m = |\Phi^+|$. We send Φ^+ to the canonical basis of $\mathbb{R}^m$ via the map ι. More specifically, since an element $w \in W_a$ is identified with the Φ^+-tuple $(k(w,\gamma))_{\gamma\in\Phi^+}$ we set $\iota(w) := (k(w,\gamma))_{\gamma\in\Phi^+}$. The natural action of $s_{\alpha,k} \in \mathrm{Isom}(\mathbb{R}^{|\Delta|})$ corresponds to acting on W_a by left multiplication, that is if $x \in A_w$ then $s_{\alpha,k}(x) \in A_{w'}$ where $w' = s_{\alpha,k}w$. We will denote the left multiplication by w as L_w. The main goal of this section is to prove the following theorem:

Theorem 1.4.1. *There exists an injective morphism $F : W_a \to Isom(\mathbb{R}^m)$ such that for any $w \in W_a$ the following diagram commutes. This morphism is called the Φ^+-representation of W_a.*

$$\begin{tikzcd} W_a \arrow[r, "L_w"] \arrow[d, hook, "\iota"'] & W_a \arrow[d, hook, "\iota"'] \\ \mathbb{R}^m \arrow[r, "F(w)"'] & \mathbb{R}^m. \end{tikzcd} \tag{1.5}$$

We recall first of all a few results of (Shi, 1987a) that we will use later. The following lemmas show how the coefficients $k(w,\alpha)$ behave when we act on w by the available operations in W_a.

Lemma 1.4.2 (Lemma 3.1 of (Shi, 1987a)). *Let w be an element of $W \subset W_a$ and $\alpha \in \Phi^+$. Then*

$$k(w,\alpha) = \begin{cases} 0 & \text{if } w^{-1}(\alpha) \in \Phi^+ \\ -1 & \text{if } w^{-1}(\alpha) \in \Phi^-. \end{cases}$$

Lemma 1.4.3 (Lemma 3.2 of (Shi, 1987a)). *Let w be an element of W and $x \in \mathbb{Z}\Phi$. Then for all $\alpha \in \Phi$ one has the following formula*

$$k(\tau_x w, \alpha) = k(w,\alpha) + (x, \alpha^\vee).$$

Corollary 1.4.4 (Formula (3.3.1) of (Shi, 1987a)). *Let $\alpha \in \Phi$. As defined in Section 1.3.2 we recall that α_0 is the negative of the highest short root of Φ and $s_0 := s_{-\alpha_0,1}$. Then for all $0 \leq i \leq n$ we have*

$$k(s_i,\alpha) = \begin{cases} 1 & \text{if } \alpha = -\alpha_i \\ 0 & \text{if } \alpha \neq \pm\alpha_i \\ -1 & \text{if } \alpha = \alpha_i. \end{cases}$$

Proposition 1.4.5 (Proposition 4.2 of (Shi, 1987a)[1]). *Let $w \in W_a$ and $s \in S_a$. Then for all $\alpha \in \Phi^+$ one has the following formula*

$$k(sw,\alpha) = k(w, \overline{s}(\alpha)) + k(s,\alpha).$$

Proposition 1.4.6. *Let $w \in W_a$, $\alpha, \beta \in \Phi^+$ and $p \in \mathbb{Z}$. Then we have*

1) $$k(s_\alpha w,\beta) = \begin{cases} k(w, s_\alpha(\beta)) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ k(w, s_\alpha(\beta)) - 1 & \text{if } s_\alpha(\beta) \in \Phi^- \end{cases}$$

2) $$k(s_{\alpha,p} w,\beta) = \begin{cases} k(w, s_\alpha(\beta)) - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ k(w, s_\alpha(\beta)) - 1 - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^- \end{cases}$$

[1]Proposition 1.4.5 is slightly different than Proposition 4.2 of (Shi, 1987a). In (Shi, 1987a) there is a typo noticed by M. Dyer in private communication.

Proof. 1) Let $s_\alpha = s_1 s_2 \dots s_n$ be a reduced expression of s_α. Since s_α is a reflection one has $s_1 s_2 \dots s_n = s_n \dots s_2 s_1$. Using Proposition 1.4.5 enough times we get that $k(s_\alpha w, \beta) = k(w, s_n \dots s_1(\beta)) + \sum_{i=1}^{n} k(s_i, s_{i-1} \dots s_1(\beta))$. When $i = 1$, the empty product on the righthand side is understood as the identity element. Further, by using again Proposition 1.4.5 enough times we get $k(s_\alpha, \beta) = \sum_{i=1}^{n} k(s_i, s_{i-1} \dots s_1(\beta))$. Thus we have

$$k(s_\alpha w, \beta) = k(w, s_n \dots s_1(\beta)) + \sum_{i=1}^{n} k(s_i, s_{i-1} \dots s_1(\beta)) = k(w, s_\alpha(\beta)) + k(s_\alpha, \beta).$$

Since $s_\alpha \in W$ and $\beta \in \Phi^+$ we have by Lemma 1.4.2 that $k(s_\alpha, \beta) = 0$ if $s_\alpha(\beta) \in \Phi^+$ and $k(s_\alpha, \beta) = -1$ if $s_\alpha(\beta) \in \Phi^-$. The result follows.

2) This is just a consequence of 1) and Lemma 1.4.3. Let us write $w = \tau_x \overline{w}$. Notice first of all that $s_\alpha(\beta)^\vee = s_\alpha(\beta^\vee)$ and

$$k(s_\alpha w, \beta) = k(s_\alpha \tau_x \overline{w}, \beta) = k(\tau_{s_\alpha(x)} s_\alpha \overline{w}, \beta) = k(s_\alpha \overline{w}, \beta) + (s_\alpha(x), \beta^\vee).$$

Thus it follows that

$$\begin{aligned} k(s_{\alpha,p} w, \beta) &= k(\tau_{p\alpha} s_\alpha \tau_x \overline{w}, \beta) = k(\tau_{p\alpha + s_\alpha(x)} s_\alpha \overline{w}, \beta) \\ &= k(s_\alpha \overline{w}, \beta) + (p\alpha + s_\alpha(x), \beta^\vee) \\ &= k(s_\alpha \overline{w}, \beta) + p(\alpha, \beta^\vee) + (s_\alpha(x), \beta^\vee) \\ &= k(s_\alpha \overline{w}, \beta) - p(\alpha, s_\alpha(\beta^\vee)) + (s_\alpha(x), \beta^\vee) \\ &= k(s_\alpha w, \beta) - p(\alpha, s_\alpha(\beta^\vee)) \\ &= \begin{cases} k(w, s_\alpha(\beta)) - p(\alpha, s_\alpha(\beta^\vee)) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ k(w, s_\alpha(\beta)) - 1 - p(\alpha, s_\alpha(\beta^\vee)) & \text{if } s_\alpha(\beta) \in \Phi^- \end{cases} \\ &= \begin{cases} k(w, s_\alpha(\beta)) - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ k(w, s_\alpha(\beta)) - 1 - p(\alpha, s_\alpha(\beta)^\vee) & \text{if } s_\alpha(\beta) \in \Phi^- \end{cases} \end{aligned}$$

□

Definition 3. Let $\alpha \in \Phi^+$ and $p \in \mathbb{Z}$.

1) We define $L_\alpha \in GL_m(\mathbb{R})$ via the matrix $(\ell_{i,j}(\alpha))_{i,j \in [\![1,m]\!]}$ where

$$\ell_{\beta_j,\beta_i}(\alpha) := \ell_{j,i}(\alpha) = \begin{cases} 1 & \text{if } \; s_\alpha(\beta_i) = \beta_j \\ 0 & \text{if } \; s_\alpha(\beta_i) \neq \pm\beta_j \\ -1 & \text{if } \; s_\alpha(\beta_i) = -\beta_j \end{cases}$$

2) We define the vector $v_{p,\alpha} \in \mathbb{R}^m$ as $v_{p,\alpha} = (v_{p,\alpha}(\gamma))_{\gamma \in \Phi^+}$ where

$$v_{p,\alpha}(\gamma) := \begin{cases} -p(\alpha, s_\alpha(\gamma)^\vee) & \text{if } \; s_\alpha(\gamma) \in \Phi^+ \\ -1 - p(\alpha, s_\alpha(\gamma)^\vee) & \text{if } \; s_\alpha(\gamma) \in \Phi^- \end{cases}.$$

For $\gamma \in \Phi^+$ we set the convention $v_{p,\alpha}(-\gamma) := -v_{p,\alpha}(\gamma)$.

3) We define $F(s_{\alpha,p})$ as the affine map such that for all $x \in \mathbb{R}^m$ one has

$$F(s_{\alpha,p})(x) := L_\alpha(x) + v_{p,\alpha}.$$

For short we will write $L_\alpha(w)$ instead of $L_\alpha(\iota(w))$. We denote $L_\alpha(w)[\theta]$ the coefficient in position θ of the vector $L_\alpha(w)$. This coefficient is called the θ-coefficient of $L_\alpha(w)$. Similarly, the coefficient in position θ of $F(s_{\alpha,p})(w)$ is denoted $F(s_{\alpha,p})(w)[\theta]$ and we have $F(s_{\alpha,p})(w)[\theta] = L_\alpha(w)[\theta] + v_{p,\alpha}(\theta)$. We call it the θ-coefficient of $F(s_{\alpha,p})(w)$.

Proposition 1.4.7. *Let $\alpha, \theta \in \Phi^+$ and $p \in \mathbb{Z}$. Then*

1) $L_\alpha^2 = id$ and $F(s_{\alpha,p}) \in Isom(\mathbb{R}^m)$.

2) $L_\alpha(w)[\theta] = k(w, s_\alpha(\theta))$.

3) The following diagram commutes

$$\begin{tikzcd} W_a \arrow[r, "L_{s_{\alpha,p}}"] \arrow[d, hook, "\iota"'] & W_a \arrow[d, hook, "\iota"'] \\ \mathbb{R}^m \arrow[r, "F(s_{\alpha,p})"'] & \mathbb{R}^m. \end{tikzcd}$$

Proof. 1) Since s_α is an involution, the way we defined L_α shows that $\ell_{i,j}(\alpha) = \ell_{j,i}(\alpha)$, that is $L_\alpha = L_\alpha^t$. Moreover each line and each column has only one $\ell_{i,j}(\alpha) \neq 0$. Then it follows that $L_\alpha^2 = id$ and for instance we have $L_\alpha \in GL(\mathbb{R}^m)$. In order to show that $F(s_{\alpha,p}) \in \mathrm{Isom}(\mathbb{R}^m)$ we just have to show that $L_\alpha \in O(\mathbb{R}^m)$. But $L_\alpha^t = L_\alpha = L_\alpha^{-1}$ which means that $L_\alpha \in O(\mathbb{R}^m)$.

2) From the way we defined L_α the θ-coefficient of $L_\alpha(w)$ has the value

$$\sum_{\gamma \in \Phi^+} \ell_{\theta,\gamma}(\alpha) k(w,\gamma).$$

Furthermore, we know that $\ell_{\theta,\gamma}(\alpha) \neq 0$ if and only if $\gamma = \pm s_\alpha(\theta)$. Therefore, if $\gamma = s_\alpha(\theta)$ we have $s_\alpha(\theta) \in \Phi^+$ and it follows $\ell_{\theta,\gamma}(\alpha) = 1$, which implies that the θ-coefficient of $L_\alpha(w)$ is $\ell_{\theta,\gamma}(\alpha)k(w,\gamma) = k(w, s_\alpha(\theta))$. If now we have $s_\alpha(\theta) = -\gamma$ then $s_\alpha(\theta) \in \Phi^-$, which implies that $\ell_{\theta,\gamma}(\alpha) = -1$. Thus, the θ-coefficient of $L_\alpha(w)$ is $\ell_{\theta,\gamma}(\alpha)k(w,\gamma) = (-1)k(w, -s_\alpha(\theta)) = k(w, s_\alpha(\theta))$. Hence, in all cases we have the θ-coefficient of $L_\alpha(w)$ equal to $k(w, s_\alpha(\theta))$.

3) Let $\theta \in \Phi^+$. Let us show that the θ-coefficient of the vector $F(s_{\alpha,p})(w)$ is the same as the θ-coefficient of the vector $\iota(s_{\alpha,p}w)$. By point 2) before we know that $L_\alpha(w)[\theta] = k(w, s_\alpha(\theta))$. Then we have

$$F(s_{\alpha,p})(w)[\theta] = L_\alpha(w)[\theta] + v_{p,\alpha}(\theta) = k(w, s_\alpha(\theta)) + v_{p,\alpha}(\theta).$$

With respect to $\iota(s_{\alpha,p}w)$, via the way we defined ι we know that the θ-coefficient of $\iota(s_{\alpha,p}w)$ is $k(s_{\alpha,p}w, \theta)$. However, because of Proposition 1.4.6 we know that

$$\begin{aligned} k(s_{\alpha,p}w,\theta) &= \begin{cases} k(w, s_\alpha(\theta)) - p(\alpha, s_\alpha(\theta)^\vee) & \text{if } \ s_\alpha(\theta) \in \Phi^+ \\ k(w, s_\alpha(\theta)) - 1 - p(\alpha, s_\alpha(\theta)^\vee) & \text{if } \ s_\alpha(\theta) \in \Phi^- \end{cases} \\ &= k(w, s_\alpha(\theta)) + \begin{cases} -p(\alpha, s_\alpha(\theta)^\vee) & \text{if } \ s_\alpha(\theta) \in \Phi^+ \\ -1 - p(\alpha, s_\alpha(\theta)^\vee) & \text{if } \ s_\alpha(\theta) \in \Phi^- \end{cases} \\ &= k(w, s_\alpha(\theta)) + v_{p,\alpha}(\theta). \end{aligned}$$

□

Example 4. Take $W_a = W(\widetilde{B}_2)$ with positive roots $\Phi^+ = \{e_1 - e_2, e_2, e_1, e_1 + e_2\}$ where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. An easy computation shows that

Table 1.1 Images of the positive roots by the reflections corresponding to the positive roots of $W(B_2)$.

	$e_1 - e_2$	e_2	e_1	$e_1 + e_2$
$s_{e_1-e_2}$	$-(e_1 - e_2)$	e_1	e_2	$e_1 + e_2$
s_{e_2}	$e_1 + e_2$	$-e_2$	e_1	$e_1 - e_2$
s_{e_1}	$-(e_1 + e_2)$	e_2	$-e_1$	$-(e_1 - e_2)$
$s_{e_1+e_2}$	$e_1 - e_2$	$-e_1$	$-e_2$	$-(e_1 + e_2)$

Table 1.2 Coordinates of the vectors $v_{p,\alpha}$ of $W(\widetilde{B}_2)$.

	$e_1 - e_2$	e_2	e_1	$e_1 + e_2$
v_{p,e_1-e_2}	$2p - 1$	$-2p$	$2p$	0
v_{p,e_2}	$-p$	$2p - 1$	0	p
v_{p,e_1}	$p - 1$	0	$2p - 1$	$p - 1$
v_{p,e_1+e_2}	0	$2p - 1$	$2p - 1$	$2p - 1$

Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. We give below the affine maps corresponding to the positive roots of B_2.

Table 1.3 Affine maps $F(s_{\alpha,p})$ for all positive roots $\alpha \in B_2$.

$$F(s_{e_1-e_2,p})(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 2p-1 \\ -2p \\ 2p \\ 0 \end{pmatrix}$$

$$F(s_{e_2,p})(x) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} -p \\ 2p-1 \\ 0 \\ p \end{pmatrix}$$

$$F(s_{e_1,p})(x) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} p-1 \\ 0 \\ 2p-1 \\ p-1 \end{pmatrix}$$

$$F(s_{e_1+e_2,p})(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 2p-1 \\ 2p-1 \\ 2p-1 \end{pmatrix}.$$

Proof of Theorem 1.4.1. Let $w \in W_a$. Since $W_a = \langle s_{\alpha,p}, \alpha \in \Phi^+, p \in \mathbb{Z} \rangle$ we can write $w = s_{\alpha_1,p_1} \dots s_{\alpha_q,p_q}$ with $\alpha_i \in \Phi^+$ and $p_i \in \mathbb{Z}$. We now want to extend to W_a the definition of F by $F(w) = F(s_{\alpha_1,p_1}) \dots F(s_{\alpha_q,p_q})$.

However we need to check that this definition doesn't depend on the choice of the expression of w. Let w_0 be the element of maximal length of W (the uniqueness of this element is well known, see for example (Humphreys, 1990) Section 1.8). This element is of length $|\Phi^+| = m$. Let $s_{i_1} s_{i_2} \dots s_{i_m}$ be a reduced expression of w_0 where $s_{i_j} \in S$. For all $k = 1, \dots, m$ and for all $\alpha \in \Phi^+$, Lemma 1.4.2 implies that $k(s_{i_1} s_{i_2} \dots s_{i_k}, \alpha)$ is either 0 or -1. Moreover, it is clear that in order to express

$\iota(s_{i_1}s_{i_2}\dots s_{i_{k+1}})$ in terms of $\iota(s_{i_1}s_{i_2}\dots s_{i_k})$ we just need to change only one 0 of $\iota(s_{i_1}s_{i_2}\dots s_{i_k})$ into a -1.

Let us order the coordinates of $\mathbb{R}^m$ such that

$$\begin{aligned}
\iota(s_{i_1}) &= (-1,0,\dots,0)\\
\iota(s_{i_1}s_{i_2}) &= (-1,-1,0,\dots,0)\\
&\vdots\\
\iota(s_{i_1}s_{i_2}\dots s_{i_{m-1}}) &= (-1,-1,\dots,-1,0)\\
\iota(s_{i_1}s_{i_2}\dots s_{i_m}) &= (-1,-1,\dots,-1,-1).
\end{aligned}$$

We see then that the set $\{\iota(s_{i_1}),\iota(s_{i_1}s_{i_2}),\dots,\iota(s_{i_1}s_{i_2}\dots s_{i_m})\}$ is a basis of $\mathbb{R}^m$. Adding $\iota(e)$ to this basis we get an affine basis of $\mathbb{R}^m$ that we denote by $\mathcal{B}$.

Let us assume now that we have two affine maps $\varphi_1(w),\varphi_2(w)\in \mathrm{Aff}(\mathbb{R}^m)$ making the diagram (1.5) commute. In particular for all $x\in W$ we have

$$\varphi_1(w)(\iota(x)) = (\iota\circ L_w)(x) = \varphi_2(w)(\iota(x)).$$

Thus, it follows that

$$\varphi_1(w)(u) = \varphi_2(w)(u) \quad \text{for all } u\in\mathcal{B}.$$

Hence $\varphi_1(w)=\varphi_2(w)$ and then there is at most one affine map making the diagram (1.5) commute. Using Proposition 1.4.7 3) we see that $F(s_{\alpha_1,p_1})\dots F(s_{\alpha_q,p_q})$ is such an affine map. Thus $F(w)$ is well defined and makes the diagram (1.5) commute.

Using Proposition 1.4.7 1) it is clear that $F(w)\in \mathrm{Isom}(\mathbb{R}^m)$, and by definition of

F it is also clear that the next diagram commutes for all $w_1, w_2 \in W_a$

$$\begin{tikzcd}
W_a \arrow[r, "L_{w_1}"] \arrow[rr, bend left, "L_{w_2 w_1}"] \arrow[d, hook, "\iota"'] & W_a \arrow[r, "L_{w_2}"] \arrow[d, hook, "\iota"'] & W_a \arrow[d, hook, "\iota"'] \\
\mathbb{R}^m \arrow[r, "F(w_1)"'] \arrow[rr, bend right, "F(w_2 w_1)"'] & \mathbb{R}^m \arrow[r, "F(w_2)"'] & \mathbb{R}^m .
\end{tikzcd}$$

Thus F is a morphism. This morphism is injective because $F(w_1) = F(w_2)$ implies for instance that $F(w_1)(\iota(e)) = F(w_2)(\iota(e))$ which means that $\iota \circ L_{w_1}(e) = \iota \circ L_{w_2}(e)$, that is $\iota(w_1) = \iota(w_2)$. Since $\iota(w_1) = (k(w_1, \alpha))_{\alpha \in \Phi^+}$ and $\iota(w_2) = (k(w_2, \alpha))_{\alpha \in \Phi^+}$ it follows that $w_1 = w_2$. $\square$

Example 5. Let us take again $W_a = W(\widetilde{B}_2)$. Because of the previous theorem we know that $F(s_{e_1-e_2} s_{e_2} s_{e_1-e_2}) = F(s_{e_1-e_2}) F(s_{e_2}) F(s_{e_1-e_2})$. When we make the matrix product of the right term we obtain

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We know that the matrix on the right is the one associated to s_{e_1}. Furthermore a direct computation shows that $s_{e_1-e_2} s_{e_2} s_{e_1-e_2} = s_{e_1}$.

1.5 Structure of affine variety on W_a

Let W_a be an affine Weyl group, Φ its crystallographic root system with $\Phi^+ = \{\beta_1, \dots, \beta_m\}$, and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ its simple system. The aim of this section is to understand, from a new geometric point of view, affine Coxeter groups based on their Φ^+-tuple of integers obtained in (Shi, 1987a). Shi gave in (Shi, 1987a) a characterization of elements in W_a via a collection of inequalities. He also gave an easier characterization of the Φ^+-tuples in Theorem 1.2. The goal here is to transform these inequalities into equations. From now on we will denote $A[X_\Delta] := A[X_{\alpha_1}, \dots, X_{\alpha_n}]$ and $A[X_{\Phi^+}] := A[X_{\beta_1}, \dots, X_{\beta_m}]$ where A is a commutative ring. In particular we have $A[X_{\alpha_1}, \dots, X_{\alpha_n}] \subset A[X_{\beta_1}, \dots, X_{\beta_m}]$. For $w \in W_a$ and $Q \in A[X_\Delta]$ we denote

$$Q(w) := Q(k(w, \alpha_1), \dots, k(w, \alpha_n)).$$

1.5.1 Decomposition of the coefficients $k(w, \alpha)$

Theorem 1.5.1. *Let $w \in W_a$. Then we have*

1) *For all $\theta \in \Phi^+$ there exists a linear polynomial $P_\theta \in \mathbb{Z}[X_\Delta]$ with positive coefficients and $\lambda_\theta(w) \in [\![0, h(\theta^\vee) - 1]\!]$ such that*

$$k(w, \theta) = P_\theta(w) + \lambda_\theta(w). \tag{1.6}$$

   Moreover, the polynomial P_θ is uniquely determined by the equation (1.6).

2) *$P_\theta = P_\alpha + P_\beta$ for all $\alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$.*

Proof. If $\theta \in \Delta$ then we set $P_\theta = X_\theta$ and $\lambda_\theta(w) = 0$. Assume now that $\theta \in \Phi^+ \setminus \Delta$. We proceed by induction on the height h of coroots. Let $\theta \in \Phi^+ \setminus \Delta$. Then $\theta^\vee \in (\Phi^\vee)^+$ and there exist $\alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$. Notice that it is always possible to write an element of $(\Phi^\vee)^+ \setminus \Delta^\vee$ as a sum of two others elements of $(\Phi^\vee)^+$. Thus we have $\theta = (\alpha^\vee + \beta^\vee)^\vee \in \Phi^+$ and because of Theorem 1.3.3 it follows $k(w,\alpha) + k(w,\beta) \leq k(w,\theta) \leq k(w,\alpha) + k(w,\beta) + 1$. Hence $k(w,\theta) = k(w,\alpha) + k(w,\beta) + \gamma_\theta(w)$ where $\gamma_\theta(w) \in [\![0,1]\!]$.

If $h(\theta^\vee) = 2$ one has necessarily $\alpha^\vee, \beta^\vee \in \Delta^\vee$. Therefore by setting $P_\theta := X_\alpha + X_\beta$ and $\lambda_\theta(w) := \gamma_\theta(w)$ we have shown the base case.

Assume now that $h(\theta^\vee) = d+1$ and assume that the formula (1.6) is true for all coroots of height less than $d+1$. We also assume that for all $\gamma \in \Phi^+$ satisfying $h(\gamma^\vee) < d+1$ the polynomial P_γ is linear and its coefficients are positive. Since $\alpha^\vee + \beta^\vee = \theta^\vee$, we have $h(\alpha^\vee) < h(\theta^\vee)$ and $h(\beta^\vee) < h(\theta^\vee)$. It follows that $k(w,\alpha) = P_\alpha(w) + \lambda_\alpha(w)$ and $k(w,\beta) = P_\beta(w) + \lambda_\beta(w)$ with $P_\alpha, P_\beta \in \mathbb{Z}[X_\Delta]$ both linear, $\lambda_\alpha(w) \in [\![0, h(\alpha^\vee) - 1]\!]$ and $\lambda_\beta(w) \in [\![0, h(\beta^\vee) - 1]\!]$. It follows that:

$$k(w,\theta) = P_\alpha(w) + \lambda_\alpha(w) + P_\beta(w) + \lambda_\beta(w) + \gamma_\theta(w).$$

By setting $P_\theta := P_\alpha + P_\beta$ and $\lambda_\theta(w) = \lambda_\alpha(w) + \lambda_\beta(w) + \gamma_\theta(w)$ we have P_θ linear, its coefficients are positive since those of P_α and P_β are positive, and finally since $h(\theta^\vee) = h(\alpha^\vee) + h(\beta^\vee)$ it results that $\lambda_\theta(w) \in [\![0, h(\theta^\vee) - 1]\!]$. This ends the induction and the proof of points 1) and 2). $\square$

Example 6. Let $\Delta(B_n) := \{\alpha_1, \dots, \alpha_n\}$ with its Dynkin diagram:

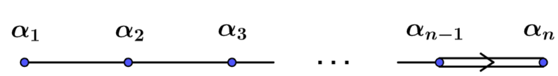

Figure 1.10 Dynkin diagram of type B_n.

It is well-known (see, for example, (Bourbaki, 1968)) that its positive root system decomposes as

$$\Phi^+(B_n) = I_1 \sqcup I_2 \sqcup I_3,$$

where $I_1 := \{\sum_{t=i}^{j-1} \alpha_t \mid 1 \leq i < j \leq n\}$, $I_2 := \{\sum_{t=i}^{n} \alpha_t \mid 1 \leq i \leq n\}$ and

$$I_3 := \{\sum_{t=i}^{j-1} \alpha_t + 2\sum_{t=j}^{n} \alpha_t \mid 1 \leq i < j \leq n\}.$$

It turns out that the short roots are those in I_2 and the long ones are those in $I_1 \sqcup I_3$. Moreover, the only simple root that is short is α_n. Let $\alpha \in \Phi^+(B_n)$ and $w \in W_a$. After computation we get:

i) If $\alpha = \sum_{t=i}^{j-1} \alpha_t \in I_1$ then there exists $\lambda_\alpha(w) \in [\![0, j-i-1]\!]$ such that $k(w,\alpha) = \sum_{t=i}^{j-1} k(w,\alpha_t) + \lambda_\alpha(w)$. Here $P_\alpha = \sum_{k=i}^{j-1} X_{\alpha_k}$ and it is easy to see that $j - i = h(\alpha) = h(\alpha^\vee)$.

ii) If $\alpha = \sum_{t=i}^{n} \alpha_t \in I_2$ then there exists $\lambda_\alpha(w) \in [\![0, 2(n-i)]\!]$ such that $k(w,\alpha) = 2\sum_{t=i}^{n-1} k(w,\alpha_t) + k(w,\alpha_n) + \lambda_\alpha(w)$. Here $P_\alpha := 2\sum_{k=i}^{n-1} X_{\alpha_k} + X_{\alpha_n}$ and it is easy to see that $h(\alpha^\vee) = 2(n-i) + 1$.

iii) If $\alpha = \sum_{t=i}^{j-1} \alpha_t + 2\sum_{t=j}^{n} \alpha_t \in I_3$ then there exists $\lambda_\alpha(w) \in [\![0, 2n-(i+j)]\!]$ such that

$$k(w,\alpha) = \begin{cases} \sum_{t=i}^{j-1} k(w,\alpha_t) + 2\sum_{t=j}^{n-1} k(w,\alpha_t) + k(w,\alpha_n) + \gamma_\alpha(w) & \text{if } j < n \\ \sum_{t=i}^{n} k(w,\alpha_t) + \gamma_\alpha(w) & \text{if } j = n. \end{cases}$$

Here we have

$$P_\alpha = \begin{cases} \sum_{k=i}^{j-1} X_{\alpha_k} + 2\sum_{k=j}^{n-1} X_{\alpha_k} + X_{\alpha_n} & \text{if } j < n \\ \sum_{k=i}^{n} X_{\alpha_k} & \text{if } j = n, \end{cases}$$

and it is easy to see that $h(\alpha^\vee) = 2n - (i+j) + 1$.

Before giving the main definitions of this section we want to discuss some terminologies and notations, particularly to clarify what we mean by "affine variety" in this thesis. During the 20th century the notion of (algebraic) variety evolved and many sophisticated and deep notions can be involved. For some people an affine variety V over a field k is just the set of solutions of polynomial equations with coefficients in k, which is sometimes called an algebraic set. For others it is the ringed space $(V, \mathcal{O}_V)$ where $\mathcal{O}_V$ is the sheaf of regular functions over V, or again for others it is a ringed space $(X, \mathcal{O}_X)$ isomorphic to $(V, \mathcal{O}_V)$.

In this thesis the term *affine variety* is employed with its simplest sense, that is as solutions of polynomial equations. We only use elementary notions about affine varieties and almost no background in algebraic geometry is required.

Let k be a field and $\mathcal{I}$ be an ideal of $k[X_1, \dots, X_p]$. We define

$$V(I) := \{x \in k^p \mid P(x) = 0 \ \forall P \in \mathcal{I}\},$$

and we say that $V(I)$ is the affine variety associated to $\mathcal{I}$.

Definition 4. Let $\theta \in \Phi^+$. Write $I_\theta := [\![0, h(\theta^\vee) - 1]\!]$. Notice that if θ is a simple root then $I_\theta = \{0\}$. For any root $\theta \in \Delta$ we set $P_\theta = X_\theta$ and $\lambda_\theta = 0$. We denote by $P_\theta[\lambda_\theta]$ the polynomial $P_\theta + \lambda_\theta - X_\theta \in \mathbb{Z}[X_{\Phi^+}]$. We define the ideal J_{W_a} of $\mathbb{Z}[X_{\Phi^+}]$ as

$$J_{W_a} := \sum_{\alpha \in \Phi^+} \langle \prod_{\lambda_\alpha \in I_\alpha} P_\alpha[\lambda_\alpha] \rangle.$$

We define then the affine variety X_{W_a} by

$$X_{W_a} := V(J_{W_a}).$$

Remark 2. The affine variety X_{W_a} is seen as an affine variety over the field $\mathbb{R}$, but could be also defined over the field $\mathbb{C}$. We will see later on that it is an union of affine subspaces of $\mathbb{R}^m$. Therefore, each notion involved about the components is elementary.

Definition 5. We say that $v = (v_\alpha)_{\alpha\in\Phi^+} \in \mathbb{N}^m$ is an *admissible vector* (or just *admissible*) if it satisfies the boundary conditions, that is if for all $\alpha \in \Phi^+$ one has $v_\alpha \in I_\alpha$. For instance, all the $\lambda := (\lambda_\alpha)_{\alpha\in\Phi^+}$ coming from the polynomials $P_\alpha[\lambda_\alpha]$ give rise to admissible vectors. Furthermore, each admissible vector arises this way. For short we will write λ instead of $(\lambda_\alpha)_{\alpha\in\Phi^+}$.

Definition 6. Let λ be an admissible vector. We denote

$$J_{W_a}[\lambda] := \sum_{\alpha\in\Phi^+} \langle P_\alpha[\lambda_\alpha]\rangle = \langle P_\alpha[\lambda_\alpha],\ \alpha\in\Phi^+\rangle,$$

$$X_{W_a}[\lambda] := V(J_{W_a}[\lambda]) \text{ and } X_{W_a}[0] := V(J_{W_a}[0_{\mathbb{R}^m}]).$$

Remark 3. We recall that our goal is to have a bijection between W_a and the integral points of a variety. X_{W_a} seems to be a good candidate because the way we defined it shows that $W_a \hookrightarrow X_{W_a}(\mathbb{Z})$. However, it turns out that X_{W_a} is too large. We explain this later on.

Example 7. Let us take $W_a = W(\widetilde{B}_2)$ with simple system $\Delta = \{\alpha,\beta\}$ and α the short root of Δ. The positive root system is $\Phi^+ = \{\alpha,\beta,\alpha+\beta,2\alpha+\beta\}$. Because of Example 6 we know that there exists $\lambda_{\alpha+\beta}(w) \in [\![0,2]\!]$ such that $k(w,\alpha+\beta) = k(w,\alpha) + 2k(w,\beta) + \lambda_{\alpha+\beta}(w)$, and there exists $\lambda_{2\alpha+\beta}(w) \in [\![0,1]\!]$ such that $k(w,2\alpha+\beta) = k(w,\alpha) + k(w,\beta) + \lambda_{\alpha+2\beta}(w)$. Hence we have 6 choices of $\lambda = (\lambda_\theta)_{\theta\in\Phi^+}$. However, if we set $k(w,\alpha) = 0$ and $k(w,\beta) = 0$ we see in Figure 1.12 that there are only 4 alcoves satisfying this and not 6. Those are $(0,0,0,0)$, $(0,0,1,0)$, $(0,0,1,1)$ and $(0,0,2,1)$ with the reading direction of Figure 1.11.

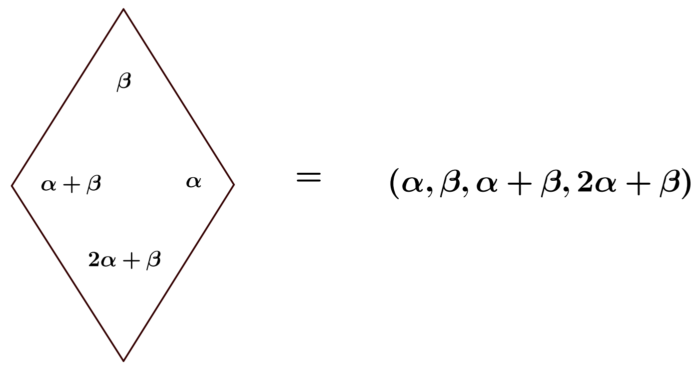

Figure 1.11 Reading direction of Φ^+-tuple in $W(\widetilde{B}_2)$.

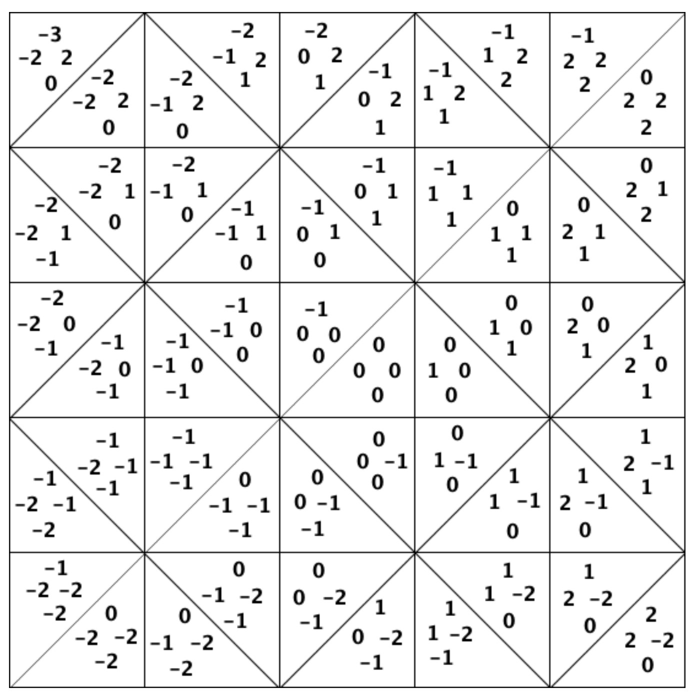

Figure 1.12 Alcoves in $W(\widetilde{B}_2)$.

1.5.2 Admissible, admitted vectors and irreducible components of $\widehat{X}_{W_a}$

Definition 7. We will denote by $S[W_a]$ the system of all the inequalities coming from Theorem 1.3.2 or equivalently Theorem 1.3.3. Let λ be an admissible vector. We say that λ is *admitted* if it satisfies the system $S[W_a]$.

Proposition 1.5.2. *Let $w \in W_a$. Let us write $k(w,\theta) = P_\theta(w) + \lambda_\theta(w)$ for all $\theta \in \Phi^+$. Then $(k(w,\theta))_{\theta\in\Phi^+}$ satisfies $S[W_a]$ if and only if $(\lambda_\theta(w))_{\theta\in\Phi^+}$ satisfies $S[W_a]$.*

Proof. Let $\theta, \alpha, \beta \in \Phi^+$ such that $\theta^\vee = \alpha^\vee + \beta^\vee$. Hence by Theorem 1.3.3 one has $k(w,\alpha)+k(w,\beta) \leq k(w,\theta) \leq k(w,\alpha)+k(w,\beta)+1$. Moreover by Theorem 1.5.1 we know that $k(w,\alpha) = P_\alpha(w) + \lambda_\alpha(w)$ and $k(w,\beta) = P_\beta(w) + \lambda_\beta(w)$. We also know by Theorem 1.5.1 that $P_\theta = P_\alpha + P_\beta$. Therefore we get $P_\theta(w) = P_\alpha(w) + P_\beta(w)$ and the previous inequalities become

$$P_\alpha(w)+\lambda_\alpha(w)+P_\beta(w)+\lambda_\beta(w) \leq P_\theta(w)+\lambda_\theta(w) \leq P_\alpha(w)+\lambda_\alpha(w)+P_\beta(w)+\lambda_\beta(w)+1.$$

These are the same as

$$\lambda_\alpha(w) + \lambda_\beta(w) \leq \lambda_\theta(w) \leq \lambda_\alpha(w) + \lambda_\beta(w) + 1.$$

Thus $(k(w,\alpha))_{\alpha\in\Phi^+} \in S[W_a]$ if and only if $(\lambda_\alpha)_{\alpha\in\Phi^+} \in S[W_a]$. □

Proposition 1.5.3. *The affine variety X_{W_a} decomposes as $X_{W_a} = \bigsqcup_{\lambda \text{ admissible}} X_{W_a}[\lambda]$. The irreducible components of X_{W_a} are the $X_{W_a}[\lambda]$ for λ admissible and we have $X_{W_a}[\lambda] = X_{W_a}[0] + \lambda$. Moreover, each irreducible component is of dimension $|\Delta| = n$.*

Proof. From the way we defined X_{W_a} it is obvious that $X_{W_a} = \bigsqcup_{\lambda \text{ admissible}} X_{W_a}[\lambda]$. Moreover, being an element of $X_{W_a}[\lambda]$, for λ admissible, is the same as being a solution of the system

$$\{P_\alpha + \lambda_\alpha - X_\alpha = 0 \mid \alpha \in \Phi^+\}.$$

But the solutions of this system are exactly the solutions of the system

$$\{P_\alpha - X_\alpha = 0 \mid \alpha \in \Phi^+\} + (\lambda_\alpha)_{\alpha\in\Phi^+},$$

that is the solutions of the system $X_{W_a}[0]+\lambda$. Since $X_{W_a}[0]+\lambda$ is an affine space it is clear that it is an irreducible component of X_{W_a}. As $X_{W_a}[\lambda] := V(J_{W_a}[\lambda])$ we see that this is just the intersection of the hyperplanes associated to the polynomials $P_\alpha[\lambda_\alpha]$ for all $\alpha \in \Phi^+$. Since there are $|\Phi^+| - |\Delta|$ such polynomials and since they are linearly independent as linear forms on $\mathbb{R}^{|\Phi^+|}$, it follows that the dimension of $X_{W_a}[\lambda]$ is $|\Delta|$. $\square$

Notation 1. Let $Y \subset \mathbb{R}^m$. We denote by $Y(\mathbb{Z})$ the set of integral points of Y.

Theorem 1.5.4. *Let γ be an admissible vector. The following points are equivalent*

i) γ is admitted.

ii) There is $x \in X_{W_a}[\gamma]$ such that there exists $w \in W_a$ satisfying $x = (k(w,\alpha))_{\alpha\in\Phi^+}$.

iii) All the integral points of $X_{W_a}[\gamma]$ arise as in ii).

Proof. From Theorem 1.3.3 we know that a Φ^+-tuple of integers $(k(w,\alpha))_{\alpha\in\Phi^+}$ associated to an element $w \in W_a$ is entirely characterized by the system of inequalities $S[W_a]$. We also know that a Φ^+-tuple $(k_\alpha)_{\alpha\in\Phi^+} \in \mathbb{Z}^m$ satisfying the system $S[W_a]$ provides an element $w \in W_a$ such that $(k(w,\alpha))_{\alpha\in\Phi^+} = (k_\alpha)_{\alpha\in\Phi^+}$.

Let $x = (x_\alpha)_{\alpha\in\Phi^+} \in X_{W_a}(\mathbb{Z})[\gamma]$. We can write x_α as $x_\alpha = P_\alpha(x)+\gamma_\alpha$. Then, thanks to Proposition 1.5.3 if γ is admitted we have x_α that satisfies $S[W_a]$. However, satisfying $S[W_a]$ implies being a Φ^+-tuple of an element of W_a. Thus, there exists $w \in W_a$ such that $x = (k(w,\alpha))_{\alpha\in\Phi^+}$. The direction $i) \Rightarrow ii)$ follows.

Conversely, if $x \in X_{W_a}[\gamma]$ such that $x = (k(w,\alpha))_{\alpha\in\Phi^+}$ for some $w \in W_a$, then we have again $k(w,\alpha) = P_\alpha(w) + \gamma_\alpha$ for all $\alpha \in \Phi^+$. Since $w \in W_a$, $(k(w,\alpha))_{\alpha\in\Phi^+}$ must satisfy $S[W_a]$, but once again because of Proposition 1.5.3 this is equivalent to say that $\gamma = (\gamma_\alpha)_{\alpha\in\Phi^+}$ is also a solution of $S[W_a]$, that is γ is admitted. The direction $ii) \Rightarrow i)$ follows.

If we have now a point $x' = (x'_\alpha)_{\alpha\in\Phi^+} \in X_{W_a}(\mathbb{Z})[\gamma]$ with $x' \neq x$ then x' decomposes as $x'_\alpha = P_\alpha(x') + \gamma_\alpha$ for all $\alpha \in \Phi^+$. However, since x is coming from an element of W_a we know that γ is admitted, which means that it is a solution of $S[W_a]$. By using Proposition 1.5.3 it follows that $(x'_\alpha)_{\alpha\in\Phi^+}$ satisfies $S[W_a]$. Thus, x' is coming from a point of W_a, that is there exists $w' \in W_a$ such that $x' = (k(w',\alpha))_{\alpha\in\Phi^+}$. This concludes the case $ii) \Rightarrow iii)$. The direction $iii) \Rightarrow ii)$ is clear. $\square$

Theorem 1.5.5. *1) The map $\iota : W_a \longrightarrow X_{W_a}(\mathbb{Z})$ defined by $w \longmapsto (k(w,\alpha))_{\alpha\in\Phi^+}$ induces by corestriction a bijective map from W_a to the integral points of a subvariety of X_{W_a}, denoted $\widehat{X}_{W_a}$, which we call the Shi variety of W_a. This subvariety is nothing but $\widehat{X}_{W_a} := \bigsqcup_{\lambda \text{ admitted}} X_{W_a}[\lambda]$. In other words, one has the following diagram:*

$$\begin{tikzcd}
W_a \arrow[r, hook, "\iota"] \arrow[dr, dashed, "\sim"'] & X_{W_a}(\mathbb{Z}) \arrow[d, phantom, "\circlearrowleft"', shift right=8] \\
& \widehat{X}_{W_a}(\mathbb{Z}). \arrow[u, hook]
\end{tikzcd}$$

2) Let $\widetilde{\alpha} = \sum_{i=1}^{n} c_i\alpha_i$ be the highest root in Φ^+. The number of irreducible components of $\widehat{X}_{W_a}$ is $n!\prod_{i=1}^{n} c_i$.

Proof. 1) From the way we defined X_{W_a} we know that each element of W_a, seen as Φ^+-tuple, belongs to $X_{W_a}(\mathbb{Z})$. That is ι is well defined. Moreover, we also know because of Proposition 5.1 in (Shi, 1987a) that ι is injective. Via Theorem 1.5.4, one has $\iota(w) \in X_{W_a}[\lambda]$ if and only if λ is admitted. Therefore, by deleting all the components $X_{W_a}[\gamma]$ with γ admissible but non-admitted, ι becomes bijective.

2) The number of irreducible components of $\widehat{X}_{W_a}$ is equal to the number of admitted vectors. Moreover, by Theorem 1.5.4 we know that for each admitted vector $\gamma = (\gamma_\alpha)_{\alpha\in\Phi^+}$ there exists a unique $w \in W_a$ such that $\gamma_\alpha = k(w, \alpha)$ for all $\alpha \in \Phi^+$. Since $\gamma_\alpha = 0$ for all $\alpha \in \Delta$, we have a bijection between the admitted vectors and the alcoves that belong to $P_{\mathcal{H}} = \bigcap\limits_{\alpha\in\Delta} H^1_{\alpha,0}$. Thanks to Section 1.3.3 we know that $|\mathrm{Alc}(P_{\mathcal{H}^\vee})| = \frac{|W(\Delta)|}{f_\Phi}$. Therefore, by duality it follows that $|\mathrm{Alc}(P_{\mathcal{H}})| = \frac{|W(\Delta^\vee)|}{f_{\Phi^\vee}}$. However it is clear that $|W(\Phi^\vee)| = |W(\Phi)|$, and since $f_\Phi = f_{\Phi^\vee}$ (see Section 1.3.3) it follows that $|\mathrm{Alc}(P_{\mathcal{H}})| = \frac{|W(\Delta)|}{f_\Phi}$. However the number $|W(\Delta)|$ is known and is equal to $f_\Phi n! \prod\limits_{i=1}^{n} c_i$ (see for example (Bourbaki, 1968), Ch. VI, § 2, prop. 7). The result follows. $\square$

Notation 2. For $\iota(w) \in X_{W_a}[\lambda]$ we denote $\lambda(w) := \lambda$. For instance $X_{W_a}[\lambda(s)]$ is the irreducible component having the Φ^+-tuple $(k(s,\alpha))_{\alpha\in\Phi^+}$ with $s \in S$.

Definition 8. We define $H^0(\widehat{X}_{W_a})$ to be the set of irreducible components of $\widehat{X}_{W_a}$. For $\lambda = (\lambda_\alpha)_{\alpha\in\Phi^+}$ an admitted vector we denote by w_λ the associated element of $\mathrm{Alc}(P_{\mathcal{H}})$, that is w_λ is such that $k(w_\lambda, \alpha) = \lambda_\alpha$ for all $\alpha \in \Phi^+$. We give in the next table the number of irreducible components of any affine Weyl group.

Table 1.4 Number of irreducible components of the Shi variety in any type.

Type	Coefficients of $\widetilde{\alpha}$	Index of connection	$\lvert W \rvert$	$\lvert H^0(\widehat{X}_{W_a}) \rvert$
A_n	1,1,...,1	$n+1$	$(n+1)!$	$n!$
B_n	1,2,2,...,2	2	$2^n n!$	$2^{n-1}n!$
C_n	2,2,...,2,1	2	$2^n n!$	$2^{n-1}n!$
D_n	1,2,...,2,1,1	4	$2^{n-1}n!$	$2^{n-3}n!$
E_6	1,2,2,3,2,1	3	$2^7 3^4 5$	$2^7 3^3 5$
E_7	2,2,3,4,3,2,1	2	$2^{10}3^4 5 \cdot 7$	$2^9 3^4 5 \cdot 7$
E_8	2,3,4,6,5,4,3,2	1	$2^{14}3^5 5^2 7$	$2^{14}3^5 5^2 7$
F_4	2,3,4,2	1	$2^7 3^2$	$2^7 3^2$
G_2	3,2	1	$2^2 3$	$2^2 3$

Example 8. Let us take the group $W(\widetilde{A}_2)$. $\widehat{X}_{W(\widetilde{A}_2)}$ has two components which are drawn in Figure 1.13 either in orange or in white. That is we can express this affine variety as $\widehat{X}_{W(\widetilde{A}_2)} = X_{W(\widetilde{A}_2)}[(0,0,0)] \sqcup X_{W(\widetilde{A}_2)}[(0,0,1)]$.

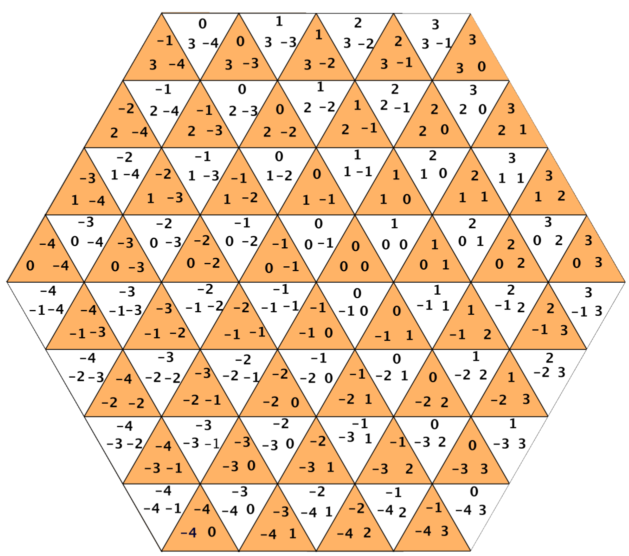

Figure 1.13 Irreducible components of $\widehat{X}_{W(\widetilde{A}_2)}$.

Example 9. Let us take the group $W(\widetilde{B}_2)$. In this group there are 4 components, given by the 4 colors below. Indeed one has the following splitting according to the admitted vectors

$$\widehat{X}_{W(\widetilde{B}_2)} = X_{W(\widetilde{B}_2)}[(0,0,0,0)] \sqcup X_{W(\widetilde{B}_2)}[(0,0,1,0)] \sqcup X_{W(\widetilde{B}_2)}[(0,0,1,1)] \sqcup X_{W(\widetilde{B}_2)}[(0,0,2,1)].$$

The first component corresponds with the pink, the second one with the yellow, the third one with the blue and the last one with the white. Moreover we can see that all the information is contained in the parallelogram where there are the 4 alcoves associated to the 4 admitted vectors. We also see that any reflection sends a component to another one. We will see in Proposition 1.5.8 that it is not a coincidence. It is also possible to see that the lattice leaves stable the components. For example if we pick up the alcove $A_w = (0,0,0,0)$ and if we take $x = e_1 + e_2$ then $A_{\tau_x w} = (0,2,2,2)$ and they are both of the same color pink.

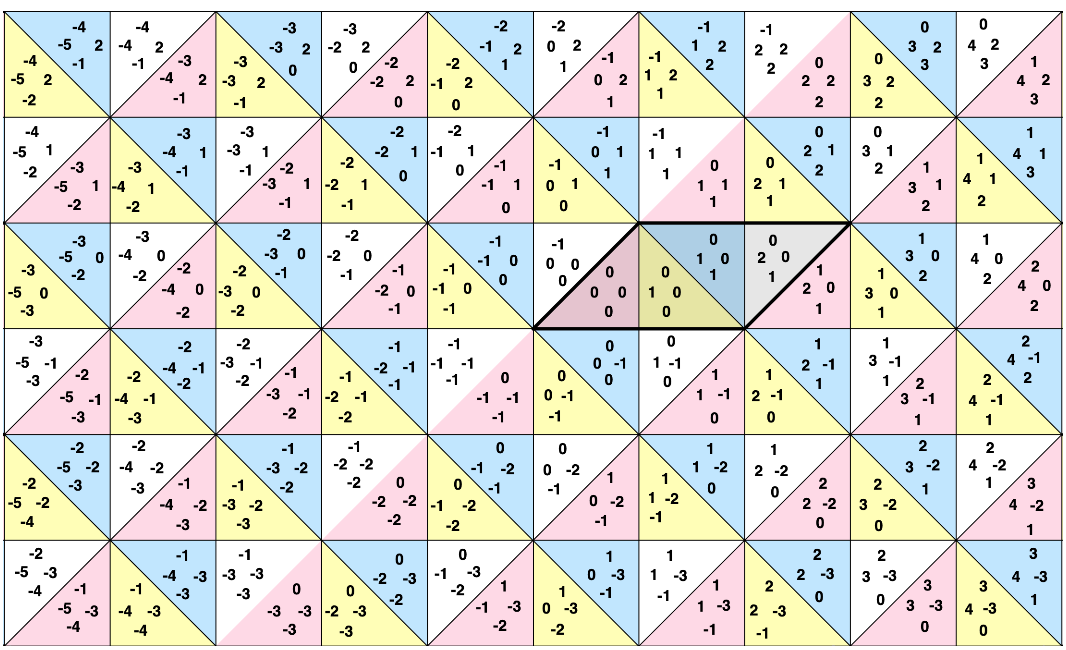

Figure 1.14 Irreducible components of $\widehat{X}_{W(\widetilde{B}_2)}$.

Example 10. Let us take the group $W(\widetilde{G}_2)$. We can see all the irreducible components in the polytope P_{G_2}. Here for example we have in Figure 1.15 the 12 irreducible components of $W(\widetilde{G}_2)$. If we would color any element of the finite part $W(G_2)$ according to the components in which they are, we would see that all the colors appear in $W(G_2)$. The reason is that $f_{G_2} = 1$.

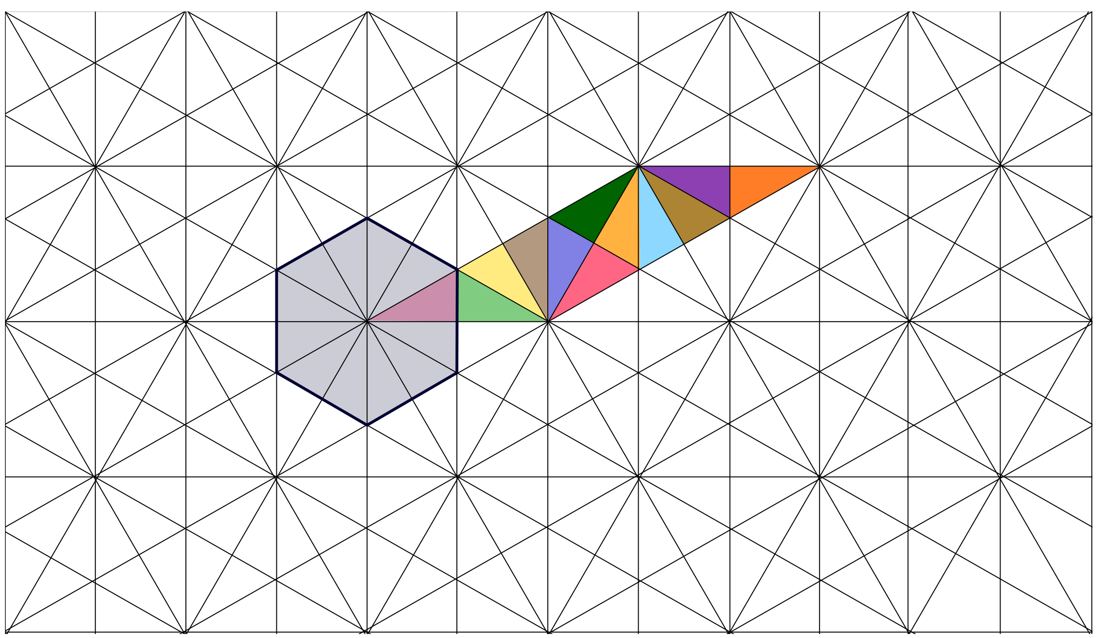

Figure 1.15 Polytope $P_{G_2}(A)$ seen as set of representatives of irreducible components of $\widehat{X}_{W(\widetilde{G}_2)}$.

1.5.3 Action on the components

Lemma 1.5.6. *Let θ be in Φ^+. Then $\theta^\vee = P_\theta(\alpha_1^\vee, \ldots, \alpha_n^\vee)$.*

Proof. Let us write $\theta^\vee = c_1\alpha_1^\vee + \cdots + c_n\alpha_n^\vee$ with $c_i \in \mathbb{N}$. Then there exists $i \in [\![1, n]\!]$ such that $\mu^\vee := c_1\alpha_1^\vee + \cdots + (c_i - 1)\alpha_i^\vee + \cdots + c_n\alpha_n^\vee \in (\Phi^\vee)^+$. Thus, by using Theorem 1.5.1 we have $P_\theta = P_\mu + P_{\alpha_i}$. Proceeding by induction the same way on $\mu^\vee$ we get that $P_\theta = c_1P_{\alpha_1} + \cdots + c_nP_{\alpha_n}$. However $P_{\alpha_i} = X_{\alpha_i}$ for all $i \in [\![1, n]\!]$. It follows that $P_\theta = c_1X_{\alpha_1} + \cdots + c_nX_{\alpha_n}$. □

Lemma 1.5.7. *Let $w = \tau_x\overline{w}$ be an element of W_a such that $x \in \mathbb{Z}\Phi$ and $\overline{w} \in W$. Let λ be an admitted vector. Assume that $(k(w, \alpha))_{\alpha\in\Phi^+} \in X_{W_a}[\lambda]$. Then $(k(\overline{w}, \alpha))_{\alpha\in\Phi^+} \in X_{W_a}[\lambda]$. In particular $X_{W_a}[\lambda]$ is stable under the action of $\mathbb{Z}\Phi$.*

Proof. We just have to show that for any $\alpha \in \Phi^+$ one has $\lambda_\alpha(w) = \lambda_\alpha(\overline{w})$. We know that $k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w)$. For all $\varepsilon \in \Delta$, thanks to Lemma 1.4.3, we have

$$k(w, \varepsilon) = k(\overline{w}, \varepsilon) + (x, \varepsilon^\vee) = P_\alpha(\overline{w}) + \lambda_\alpha(\overline{w}) + (x, \varepsilon^\vee).$$

However, we also have that $k(w, \alpha) = P_\alpha(w) + \lambda_\alpha(w) = P_\alpha(\{k(\overline{w}, \varepsilon) + (x, \varepsilon^\vee)\}_{\varepsilon\in\Delta}) + \lambda_\alpha(w)$. Since P_α is a linear polynomial we can split up the sum inside the variables and we get

$$\begin{aligned} k(w, \alpha) &= P_\alpha(\{k(\overline{w}, \varepsilon)\}_{\varepsilon\in\Delta}) + P_\alpha(\{(x, \varepsilon^\vee)\}_{\varepsilon\in\Delta}) + \lambda_\alpha(w) \\ &= k(\overline{w}, \alpha) - \lambda_\alpha(\overline{w}) + P_\alpha(\{(x, \varepsilon^\vee)\}_{\varepsilon\in\Delta}) + \lambda_\alpha(w) \\ &= k(\overline{w}, \alpha) - \lambda_\alpha(\overline{w}) + (x, P_\alpha(\{\varepsilon^\vee\}_{\varepsilon\in\Delta})) + \lambda_\alpha(w) \\ &= k(\overline{w}, \alpha) - \lambda_\alpha(\overline{w}) + (x, \alpha^\vee) + \lambda_\alpha(w). \end{aligned}$$

Finally it follows that $k(\overline{w}, \alpha) + (x, \alpha^\vee) = k(\overline{w}, \alpha) - \lambda_\alpha(\overline{w}) + (x, \alpha^\vee) + \lambda_\alpha(w)$ which is the same as $\lambda_\alpha(w) = \lambda_\alpha(\overline{w})$.

For the second statement let us take $g \in W_a$ such that $(k(g,\alpha))_{\alpha\in\Phi^+} \in X_{W_a}[\lambda]$ and let us take $y \in \mathbb{Z}\Phi$. What we have done just before implies that $(k(\overline{\tau_y g},\alpha))_{\alpha\in\Phi^+}$ and $(k(\tau_y g,\alpha))_{\alpha\in\Phi^+}$ belong to the same component. Since $\overline{\tau_y g} = \overline{g}$ we get that $(k(\overline{g},\alpha))_{\alpha\in\Phi^+}$ and $(k(\tau_y g,\alpha))_{\alpha\in\Phi^+}$ are in the same component. But once again we know that $(k(\overline{g},\alpha))_{\alpha\in\Phi^+}$ and $(k(g,\alpha))_{\alpha\in\Phi^+}$ are in the same component. It follows that $(k(g,\alpha))_{\alpha\in\Phi^+}$ and $(k(\tau_y g,\alpha))_{\alpha\in\Phi^+}$ are in the same component. Hence we have shown that $X_{W_a}[\lambda](\mathbb{Z})$ is stable under the action of $\mathbb{Z}\Phi$. It follows that $X_{W_a}[\lambda]$ is also stable under $\mathbb{Z}\Phi$. $\square$

Proposition 1.5.8. *Let $F : W_a \hookrightarrow$ Isom$(\mathbb{R}^n)$ be the Φ^+-representation of W_a. Then*

1) *W_a acts naturally on the irreducible components of $\widehat{X}_{W_a}$ via the action defined as $w \diamond X_{W_a}[\lambda] := F(w)(X_{W_a}[\lambda])$ for λ admitted. Furthermore if we assume that $w \in W_a$ decomposes as $w = \tau_x\overline{w}$, then $w \diamond X_{W_a}[\lambda] = \overline{w} \diamond X_{W_a}[\lambda]$. Finally this action is transitive.*

2) *The previous action induces an action on the admitted vectors by $w \diamond \lambda := \gamma$ such that $w \diamond X_{W_a}[\lambda] = X_{W_a}[\gamma]$. In other words we have $w \diamond X[\lambda] = X[w \diamond \lambda]$.*

Proof. 1) Let $x \in W_a$ such that $(k(x,\alpha))_{\alpha\in\Phi^+} \in X_{W_a}[\lambda]$. For any $w \in W_a$ there exists an admitted vector γ such that $(k(wx,\alpha))_{\alpha\in\Phi^+} \in X_{W_a}[\gamma]$. However, since we know that $F(w)(x) = (k(wx,\alpha))_{\alpha\in\Phi^+}$, it follows that $F(w)$ sends at least one element of $X_{W_a}[\lambda]$ to $X_{W_a}[\gamma]$. Moreover $F(w)$ is an isometry and then $F(w)(X_{W_a}[\lambda])$ must be an affine space of the same dimension as $X_{W_a}[\lambda]$.

Let us show now that $(F(w)(X_{W_a}[\lambda]))(\mathbb{Z}) = F(w)(X_{W_a}[\lambda](\mathbb{Z}))$. First of all the inclusion $F(w)(X_{W_a}[\lambda](\mathbb{Z})) \subset (F(w)(X_{W_a}[\lambda]))(\mathbb{Z})$ is clear. With respect to the other inclusion, let us take $z \in (F(w)(X_{W_a}[\lambda]))(\mathbb{Z})$. Then we have $F(w)^{-1}(z) \in X_{W_a}[\lambda]$ but $F(w)^{-1} = F(w^{-1})$ and the coefficients of the matrix associated to

$F(w^{-1})$ are integers together with the coordinates of the translation part. It follows that $F(w)^{-1}(z) \in X_{W_a}[\lambda](\mathbb{Z})$ and thus $z \in F(w)(X_{W_a}[\lambda](\mathbb{Z}))$. The equality follows.

Finally, $F(w)(X_{W_a}[\lambda])$ is an affine space of dimension $|\Delta|$ all of whose integer points are associated to elements of W_a. It follows that $F(w)(X_{W_a}[\lambda])$ must be an irreducible component of $\widehat{X}_{W_a}$. However we know that $(k(wx,\alpha))_{\alpha\in\Phi^+} \in X_{W_a}[\gamma]$, then since the components of X_{W_a} have no intersection we must have $F(w)(X_{W_a}[\lambda]) = X_{W_a}[\gamma]$. Thus, we have shown that the following map is well defined

$$\begin{array}{ccc} W_a \times H^0(\widehat{X}_{W_a}) & \longrightarrow & H^0(\widehat{X}_{W_a}) \\ (w, X_{W_a}[\lambda]) & \longmapsto & w \diamond X_{W_a}[\lambda]. \end{array}$$

Since F is a morphism we have $e \diamond X_{W_a}[\lambda] = F(e)(X_{W_a}[\lambda]) = id(X_{W_a}[\lambda]) = X_{W_a}[\lambda]$ and for all $w, w' \in W_a$ we also have

$$\begin{aligned} ww' \diamond X_{W_a}[\lambda] &= F(ww')(X_{W_a}[\lambda]) = F(w)F(w')(X_{W_a}[\lambda]) = F(w)(F(w')(X_{W_a}[\lambda])) \\ &= w \diamond (w' \diamond X_{W_a}[\lambda]). \end{aligned}$$

This concludes the first statement of 1). With respect to the second statement we just have to see that the action of $F(\tau_x)$ doesn't do anything on the components. By Lemma 1.5.7 we know that $X_{W_a}[\lambda]$ is stable under $\mathbb{Z}\Phi$. It follows that $F(\tau_x)(X_{W_a}[\lambda]) = X_{W_a}[\lambda]$.

To show the transitivity we just have to see that we can obtain any component of $H^0(\widehat{X}_{W_a})$ from the component $X_{W_a}[0]$. Let $X_{W_a}[\gamma]$ be such a component and $(k(g,\alpha))_{\alpha\in\Phi^+} \in X_{W_a}[\gamma]$. It follows that $(k(\overline{g},\alpha))_{\alpha\in\Phi^+} \in X_{W_a}[\gamma]$. As $\overline{g} = \overline{g}e$ and since $(k(e,\alpha))_{\alpha\in\Phi^+} \in X_{W_a}[0]$ it follows that $F(\overline{g})(X_{W_a}[0]) = X_{W_a}[\gamma]$, that is $\overline{g} \diamond X_{W_a}[0] = X_{W_a}[\gamma]$.

2) This is a straightforward consequence of the first point. $\square$

Proposition 1.5.9. *Let f_Φ be the index of connection of Φ and let λ be an admitted vector.*

1) In W there are exactly f_Φ elements belonging to $X_{W_a}[\lambda]$.

2) $X_{W_a}[\lambda]$ has exactly f_Φ orbits under the action of $\mathbb{Z}\Phi$.

3) Each orbit of $X_{W_a}[\lambda]$ has only one element of W.

Proof. 1) First of all we note that $W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z})) \neq \emptyset$. Indeed, for any $w = \tau_x\overline{w} \in \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))$ we know by Lemma 1.5.7 that $(k(\overline{w},\alpha))_{\alpha\in\Phi^+}$ is also in $X_{W_a}[\lambda]$, that is $\overline{w} \in W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))$. We need now to show that if we take another admitted vector γ then $|W\cap\iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))| = |W\cap\iota^{-1}(X_{W_a}[\gamma](\mathbb{Z}))|$. But this is true because any component can be sent to any other one via an element of W. In particular there exists $g \in W$ such that $g \diamond X_{W_a}[\lambda] = X_{W_a}[\gamma]$. Further, if $(k(w,\alpha))_{\alpha\in\phi^+} \in X_{W_a}[\lambda]$ with $w \in W$, then $F(g)((k(w,\alpha))_{\alpha\in\phi^+})$ is also associated to an element of W. The result follows since g is a bijection. By Theorem 1.5.5 we know that there are $n!\prod_{i=1}^{n} c_i$ irreducible components in $\widehat{X}_{W_a}$ where $\widetilde{\alpha} = \sum_{i=1}^{n} c_i\alpha_i$ is the longest root in Φ^+. However it is well known that $|W| = n!(\prod_{i=1}^{n} c_i)f_\Phi$. Since all the irreducible components have same number of elements in W, we deduce that this number is the quotient of $|W|$ by the number of irreducible components, which is exactly f_Φ.

2) This is a consequence of the point above. Let us take the f_Φ elements $\{w_1,\ldots,w_{f_\Phi}\}$ of $W\cap\iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))$. Then the $\mathbb{Z}\Phi$-orbits O_{w_i} and O_{w_j} are different for all $i,j \in \{1,\ldots,f_\Phi\}$. Indeed, if there was an element belonging to both then it would follow that there exists $x \in \mathbb{Z}\Phi$ such that $w_i = \tau_x w_j$, but this is impossible because w_i and w_j are in the finite group W. We therefore have f_Φ distinct orbits. We claim that there is no other orbit. Indeed, let $x \in X_{W_a}[\lambda]$. Then there exists $w \in W_a$ such that $x = \iota(w)$. Moreover, one can write $w = \tau_y\overline{w}$ with $y \in \mathbb{Z}\Phi$ and

$\overline{w} \in W$. This implies that $\iota(w)$ and $\iota(\overline{w})$ are in the same orbit O_x. However, since $\overline{w} \in W \cap \iota^{-1}(X_{W_a}[\lambda](\mathbb{Z}))$ it follows that $O_x = O_{w_i}$ for some $i = 1, \ldots, f_\Phi$.

3) Assume that we have an orbit in $X_{W_a}[\lambda]$ with at least two elements w_1, w_2 coming from W. Then there exists $x \in \mathbb{Z}\Phi$ such that $w_2 = \tau_x w_1$. Since w_1 and $w_2 \in W$ the previous equality cannot occur. It follows that each orbit has at most one element of W. Moreover, by the first point above we know that $X_{W_a}[\lambda]$ has f_ϕ elements of W. Since there are f_ϕ orbits in $X_{W_a}[\lambda]$ it follows that each orbit has a unique element of W. □

Proposition 1.5.10. *Let W_a be an affine Weyl group such that $W_a \neq W(\widetilde{A}_2)$. Let $s_i, s_j \in S$ such that $s_i \neq s_j$. Then $X_{W_a}[\lambda(s_i)] \neq X_{W_a}[\lambda(s_j)]$ (where the meaning of $X_{W_a}[\lambda(s_i)]$ is defined in Notation 2).*

Proof. For the B_2 $(= C_2)$ and G_2 cases one can directly see the result on Figures 1.16 and 1.15. We assume now that the rank of Φ is greater or equal than 3. Let us assume that s_i and s_j are in the same component $X_{W_a}[\lambda(s_i)]$. We claim that there exists a root $\theta \in \Phi^+ - \Delta$ such that $P_\theta(s_i) \neq P_\theta(s_j)$. Assuming this claim, because of Theorem 1.5.1 and Corollary 1.4.4 we have $k(s_i, \theta) = P_\theta(s_i) + \lambda_\theta(s_i) = 0$ and $k(s_j, \theta) = P_\theta(s_j) + \lambda_\theta(s_j) = 0$, which implies that $\lambda_\theta(s_i) \neq \lambda_\theta(s_j)$. However, since s_i and s_j are in $X_{W_a}[\lambda(s_i)]$, we must have $\lambda(s_i) = \lambda(s_j)$ and then $\lambda_\alpha(s_i) = \lambda_\alpha(s_j)$ for all $\alpha \in \Phi^+ - \Delta$. Therefore, s_i and s_j cannot lie in the same component.

Proof of the claim. Let Φ be a root system satisfying rank$(\Phi) \geq 3$. This is a general fact, that can checked case by case, that in any such a crystallographic root system there always exists a root $\theta \in \Phi^+ - \Delta$ such that either α_i is in the expression of θ (according to Δ) but α_j is not, or α_j is in the expression of θ (according to Δ) but α_i is not. We apply this statement on the dual root system of Φ.

Let $\theta \in \Phi^+ - \Delta$ such that $\alpha_i^\vee$ is in the expression of $\theta^\vee$ (according to $\Delta^\vee$) but $\alpha_j^\vee$ is not, the other case being symmetric. Thus, because of Lemma 1.5.6 we have X_{α_i} in the expression of P_θ, whereas X_{α_j} doesn't appear in this expression. Again because of Lemma 1.5.6 we know that all the coefficients of the polynomial P_θ are positive. Therefore, since $k(s_i, \alpha) = 0$ for all $\alpha \in \Phi^+ - \{\alpha_i\}$ and $k(s_i, \alpha_i) = -1$ we have $P_\theta(s_i) \leq -1$. Moreover, we also have $k(s_j, \alpha) = 0$ for all $\alpha \in \Phi^+ - \{\alpha_j\}$ and $k(s_j, \alpha_j) = -1$, which implies that $P_\theta(s_j) = 0$. $\square$

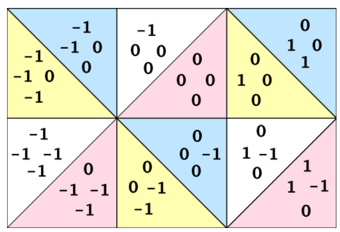

Figure 1.16 Generators $s_2 = (0, -1, 0, 0)$ in the white component and $s_1 = (-1, 0, 0, 0)$ in the blue component in type B_2.

Remark 4. The A_2 case fails because there are not enough roots. See Figure 1.13 where the generators $s_0 = (0, 0, 1)$, $s_1 = (-1, 0, 0)$ and $s_2 = (0, -1, 0)$ lie in the same component. The reading direction according to Figure 4 is as follows: $(\alpha, \beta, \alpha + \beta)$.

1.5.4 Link between Shi regions and $\widehat{X}_{W_a}$

This subsection is a short summary of our results that we embody through Shi regions. From Section 1.3.6 we know that there is a bijection between the Shi regions of W_a and the signed regions of W_a. Moreover, we have now a bijection between the integral points of an affine variety $\widehat{X}_{W_a}$ and the group W_a. The important fact is to see that this bijection respects the notion of signed regions.

To explain the previous point we need to introduce the notion of generalized orthant in $\mathbb{R}^m$. In the literature an orthant is a subset defined by constraining each Cartesian coordinate to be nonnegative or nonpositive, which gives 2^m such subspaces. We call generalized orthant a supspace of $\mathbb{R}^m$ defined by constraining each Cartesian coordinate to be strictly positive or strictly negative or zero.

Therefore, it is clear that any signed region of W_a is characterized in $\widehat{X}_{W_a}$ by the intersection of $\widehat{X}_{W_a}(\mathbb{Z})$ and a generalized orthant. We know from (Shi, 1987b) Theorem 8.1 that there are $(h+1)^n$ signed regions in W_a where h is the Coxeter number of Φ and n is the rank of Φ. Consquently, $\widehat{X}_{W_a}$ meets exactly $(h+1)^n$ generalized orthants in $\bigoplus_{\alpha\in\Phi^+}\mathbb{R}\alpha$. It would be interesting to see if we can recover this result by using the equations of the variety, namely if we would be able to determine which are the generalized orthants touched by the variety.

Finally we can state the following result:

Theorem 1.5.11. *Let us write $Ort(W_a)$ the set of generalized orthants in $\bigoplus_{\alpha\in\Phi^+}\mathbb{R}\alpha$ and $Ort(\widehat{X}_{W_a}) := \{\widehat{X}_{W_a}(\mathbb{Z})\cap\Upsilon \mid \Upsilon\in Ort(W_a)\}$. The following map is a bijection*

$$\begin{array}{ccc} Shi(W_a) & \longrightarrow & Ort(\widehat{X}_{W_a}) \\ \Omega & \longmapsto & \iota(i_1^{-1}(\Omega\cap E)). \end{array}$$

1.5.5 Invariance of $\widehat{X}_{W_a}$ up to isomorphism of root systems

Let $\Phi'\subset V'$ be another (irreducible essential crystallographic) root system where V' is a Euclidean space with inner product also denoted $(-,-)$.

We say that Φ and Φ' are isomorphic, denoted $\Phi\simeq\Phi'$, if there exists an isomorphism $f: V\to V'$ that sends Φ to Φ' and preserves the crystallography, that is $f(\Phi)=\Phi'$ and $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = 2\frac{(f(\alpha),f(\beta))}{(f(\alpha),f(\alpha))}$ for all $\alpha,\beta\in\Phi$.

Such an isomorphism preserves the angles between vectors and the ratio $||\alpha||/||\beta||$ for all $\alpha, \beta \in \Phi$ (see for example (Kane, 2001) Section 10.1). However, f needn't preserve inner products. For instance if $V' = V$ we don't necessary have $f \in O(V)$.

This notion gives rise to an equivalence relation over irreducible crystallographic root systems. A class for this relation is called an isomorphism class. We denote it by $\overline{\Phi}$.

Assume that $\Phi \simeq \Phi'$ via f. Such an isomorphism induces an isomorphism $\widetilde{f} : W(\Phi) \to W(\Phi')$, $w \mapsto f \circ w \circ f^{-1}$ between Weyl groups. This isomorphism extends to an isomorphism between the associated affine Weyl groups. We also denote it by $\widetilde{f} : W(\Phi_a) \to W(\Phi'_a)$.

In the notation of the affine variety $\widehat{X}_{W_a}$ we ignored the associated crystallographic root system underlying. From now on, if $W_a = W(\Phi_a)$ we set $\widehat{X}_\Phi := \widehat{X}_{W_a}$. If $\lambda \in \bigoplus\limits_{\alpha \in \Phi^+} \mathbb{R}\alpha$ is an admitted vector we will also write $\widehat{X}_\Phi[\lambda] := \widehat{X}_{W_a}[\lambda]$.

We show in this section that the Shi variety variety $\widehat{X}_{W_a}$ only depends on $\overline{\Phi}$ and not of the choice of Φ. Our goal is then to show that the construction of the variety $\widehat{X}_{W_a}$ is functorial, that is $\Phi \simeq \Phi'$ implies $\widehat{X}_\Phi \simeq \widehat{X}_{\Phi'}$.

Definition 9. Let $\alpha \in \Phi^+$. We denote:

$$P\Phi_\alpha := \{(x, y) \in \Phi^+ \times \Phi^+ \mid \alpha = (x^\vee + y^\vee)^\vee\},$$

$$P\Phi'_{f(\alpha)} := \{(x', y') \in \Phi'^+ \times \Phi'^+ \mid f(\alpha) = (x'^\vee + y'^\vee)^\vee\}.$$

Lemma 1.5.12. *Let $f : \Phi \to \Phi'$ be an isomorphism of root systems with simple systems $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and $\Delta' = f(\Delta)$. Let $\alpha, \theta \in \Phi^+$ and let $\{\nu_1, \dots, \nu_p\} \subset \Phi^+$. Assume that $\theta^\vee = b_1\nu_1^\vee + \cdots + b_p\nu_p^\vee$. Then we have*

1) $f(\theta)^\vee = b_1 f(\nu_1)^\vee + \cdots + b_p f(\nu_p)^\vee$.

2) $(x,y) \in P\Phi_\alpha \Longleftrightarrow (f(x), f(y)) \in P\Phi'_{f(\alpha)}$.

Proof. 1) Since $\theta^\vee = \frac{2\theta}{||\theta||^2}$, the assumption $\theta^\vee = b_1\nu_1^\vee + \cdots + b_p\nu_p^\vee$ is the same as

$$\theta = \frac{||\theta||^2}{||\nu_1||^2} b_1\nu_1 + \cdots + \frac{||\theta||^2}{||\nu_p||^2} b_p\nu_p.$$

It follows that

$$f(\theta) = \frac{||\theta||^2}{||\nu_1||^2} b_1 f(\nu_1) + \cdots + \frac{||\theta||^2}{||\nu_p||^2} b_p f(\nu_p).$$

However, since f is an isomorphism of root systems we know that it preserves the ratio of lengths $\frac{||\alpha||}{||\beta||}$ for all elements $\alpha, \beta \in \Phi$. Thus, $\frac{||\theta||^2}{||\nu_i||^2} = \frac{||f(\theta)||^2}{||f(\nu_i)||^2}$ for all $i \in [\![1,p]\!]$ and it follows that

$$f(\theta) = \frac{||f(\theta)||^2}{||f(\nu_1)||^2} b_1 f(\nu_1) + \cdots + \frac{||f(\theta)||^2}{||f(\nu_p)||^2} b_p f(\nu_p).$$

Therefore

$$f(\theta)^\vee = 2\frac{f(\theta)}{||f(\theta)||^2} = b_1 \frac{2f(\nu_1)}{||f(\nu_1)||^2} + \cdots + b_p \frac{2f(\nu_p)}{||f(\nu_p)||^2} = b_1 f(\nu_1)^\vee + \cdots + b_p f(\nu_p)^\vee.$$

2) By definition $(x,y) \in P\Phi_\alpha$ if and only if $\alpha^\vee = x^\vee + y^\vee$. Because of the point above this is equivalent to $f(\alpha)^\vee = f(x)^\vee + f(y)^\vee$, that is $(f(x), f(y)) \in P\Phi'_{f(\alpha)}$.

□

Let Ψ be an irreducible crystallographic root system. We set $\Psi^+ = \{\mu_1, \ldots, \mu_m\}$. We recall that $P_\alpha[\lambda_\alpha] := P_\alpha + \lambda_\alpha - X_\alpha$ for $\alpha \in \Psi^+$. Furthermore, J_{W_a} is the ideal of $\mathbb{R}[X_{\Psi^+}]$ defined by $J_{W_a} := \sum\limits_{\alpha \in \Psi^+} \langle \prod\limits_{\lambda_\alpha \in I_\alpha} P_\alpha[\lambda_\alpha] \rangle$. For λ an admissible vector we have $J_{W_a}[\lambda] := \sum\limits_{\alpha \in \Psi^+} \langle P_\alpha[\lambda_\alpha] \rangle$ together with $X_{W_a}[\lambda] := V(J_{W_a}[\lambda])$.

Let us define κ_Ψ to be the following algebras morphism

$$\begin{array}{cccc} \kappa_\Psi & : & \mathbb{R}[X_{\mu_1}, \ldots, X_{\mu_m}] & \longrightarrow & \mathbb{R}[X_1, \ldots, X_m] \\ & & X_{\mu_i} & \longmapsto & X_i. \end{array}$$

Definition 10. 1) We denote by Z_Ψ the affine variety living in $\bigoplus_{i=1}^{m} \mathbb{R}e_i$ and defined as follows

$$Z_\Psi := V(\kappa_\Psi(J_{W_a})).$$

2) Let λ be an admitted vector in $\bigoplus_{\alpha\in\Psi^+} \mathbb{R}\alpha$. Then $Z_\Psi[\lambda]$ is defined to be the irreducible component of Z_Ψ associated to λ, that is $Z_\Psi[\lambda] := V(\kappa_\Psi(J_{W_a}[\lambda]))$. It follows that

$$Z_\Psi = \bigsqcup_{\substack{\lambda \text{ admitted in} \\ \bigoplus_{\alpha\in\Psi^+} \mathbb{R}\alpha}} Z_\Psi[\lambda].$$

3) Assume that λ decomposes as $\lambda = \lambda_{\mu_1}\mu_1 + \cdots + \lambda_{\mu_m}\mu_m$. We define λ_e to be the vector $\lambda_e := \lambda_{\mu_1}e_1 + \cdots + \lambda_{\mu_m}e_m$.

Theorem 1.5.13. *Let assume that $\Phi \simeq \Phi'$ and let f be this isomorphism. We denote $f(\beta_i) = \beta_i'$ and $\Phi'^+ = \{\beta_1', \ldots, \beta_m'\}$. Then $Z_\Phi = Z_{\Phi'}$.*

Proof. We know that the affine varieties $\widehat{X}_\Phi$ and $\widehat{X}_{\Phi'}$ are defined by

$$\widehat{X}_\Phi = \bigsqcup_{\substack{\lambda \text{ admitted in} \\ \bigoplus_{\alpha\in\Phi^+} \mathbb{R}\alpha}} \widehat{X}_\Phi[\lambda] \quad \text{and} \quad \widehat{X}_{\Phi'} = \bigsqcup_{\substack{\gamma \text{ admitted in} \\ \bigoplus_{\beta\in\Phi'^+} \mathbb{R}\beta}} \widehat{X}_{\Phi'}[\gamma].$$

Because of Lemma 1.5.6, we know that the polynomials P_α, $\alpha \in \Phi^+$, only depend on the expression of $\alpha^\vee$ in the basis $\Delta^\vee$, that is of the expression of α in the basis Δ. Of course this also applies for P_β with $\beta \in \Phi'^+$. Let $\beta \in \Phi'^+$. Then there exists $\theta \in \Phi^+$ such that $\beta = f(\theta)$. Because of Lemma 1.5.6 we also know that if $\theta^\vee = b_1\alpha_1^\vee + \cdots + b_n\alpha_n^\vee$ then $P_\theta = b_1X_{\alpha_1} + \cdots + b_nX_{\alpha_n}$.

Because of Lemma 1.5.12, the equality $\theta^\vee = b_1\alpha_1^\vee + \cdots + b_n\alpha_n^\vee$ implies in particular that $\beta^\vee = b_1f(\alpha_1)^\vee + \cdots + b_nf(\alpha_n)^\vee$. Since $\{f(\alpha_i) \mid i = 1, \ldots, n\}$ is a simple system of Φ', the polynomial P_β decomposes as $P_\beta = b_1X_{f(\alpha_1)} + \cdots + b_nX_{f(\alpha_n)}$. It follows that $\kappa_\Phi(P_\alpha) = \kappa_{\Phi'}(P_{f(\alpha)})$ for all $\alpha \in \Phi^+$. Consequently, the components $Z_\Phi[0]$ and $Z_{\Phi'}[0]$ are equal. We claim now that

$$\{\lambda_e \mid \lambda \text{ admitted in } \bigoplus_{\alpha\in\Phi^+} \mathbb{R}\alpha\} = \{\gamma_e \mid \gamma \text{ admitted in } \bigoplus_{\beta\in\Phi'^+} \mathbb{R}\beta\}.$$

The key is to see that being an admitted vector in $\bigoplus_{\alpha\in\Phi^+} \mathbb{R}\alpha$ (respectively in $\bigoplus_{\beta\in\Phi'^+} \mathbb{R}\beta$) is the same thing as satisfying the system $S[W(\Phi_a)]$ (respectively $S[W(\Phi'_a)]$) and $\lambda_\alpha \in I_\alpha$ for all $\alpha \in \Phi^+$ (respectively $\gamma_\beta \in I_\beta$ for all $\beta \in \Phi'^+$).

First of all, let us show that λ admitted if and only if $f(\lambda)$ admitted. Then we will show further what we claim above.

•) Because of the definition of admitted vectors one has the equivalences:

$$\lambda \text{ admitted } \Leftrightarrow \begin{cases} \lambda_\alpha = \lambda_x + \lambda_y \text{ or } \lambda_\alpha = \lambda_x + \lambda_y + 1 \text{ for all } (x,y) \in P\Phi_\alpha,\ \alpha \in \Phi^+ \\ \lambda_\alpha \in I_\alpha \text{ for all } \alpha \in \Phi^+, \end{cases}$$

$f(\lambda)$ admitted $\Leftrightarrow$

$$\begin{cases} f(\lambda)_\beta = f(\lambda)_z + f(\lambda)_t \text{ or } f(\lambda)_\beta = f(\lambda)_z + f(\lambda)_t + 1 \ \forall\ (z,t) \in P\Phi'_\beta,\ \beta \in \Phi'^+ \\ f(\lambda)_\beta \in I_\beta \text{ for all } \beta \in \Phi'^+. \end{cases}$$

Let $x, y \in \Phi^+$. By Lemma 1.5.12 we have $(x,y) \in P\Phi_\alpha$ if and only if $(f(x), f(y)) \in P\Phi'_{f(\alpha)}$, and since f is linear we have $f(\lambda)_{f(\mu)} = \lambda_\mu$ for all $\mu \in \Phi^+$. Therefore it follows

$$\Big[\lambda_\alpha = \lambda_x + \lambda_y \text{ or } \lambda_\alpha = \lambda_x + \lambda_y + 1\Big]$$

$$\Longleftrightarrow$$

$$\Big[f(\lambda)_{f(\alpha)} = f(\lambda)_{f(x)} + f(\lambda)_{f(y)} \text{ or } f(\lambda)_{f(\alpha)} = f(\lambda)_{f(x)} + f(\lambda)_{f(y)} + 1\Big].$$

Lemma 1.5.12 implies that $h(\alpha) = h(f(\alpha)^\vee)$, and then $I_\alpha = I_{f(\alpha)}$. This proves the equivalence.

•) The set of admitted vectors of $\bigoplus_{\beta\in\Phi'^+}\mathbb{R}\beta$ is nothing but the image via f of the admitted vectors of $\bigoplus_{\alpha\in\Phi^+}\mathbb{R}\alpha$. Indeed, let $\gamma\in\bigoplus_{\beta\in\Phi'^+}\mathbb{R}\beta$ be an admitted vector. As f is bijective there exists $\xi\in\bigoplus_{\alpha\in\Phi^+}\mathbb{R}\alpha$ such that $f(\xi)=\gamma$. By applying our previous reasoning to f^{-1}, we obtain that γ admitted $\Leftrightarrow f^{-1}(\gamma)$ admitted $\Leftrightarrow \xi$ admitted. This means that β is the image of an admitted vector. □

1.6 Poset structure of $H^0(\widehat{X}_{W_a})$

Definition 11. The set $H^0(\widehat{X}_{W_a})$ has a natural poset structure. It is defined by $X_{W_a}[\lambda]\leq X_{W_a}[\gamma]$ if and only if $\lambda_\alpha\leq\gamma_\alpha$ for all $\alpha\in\Phi^+$. There is a minimal element in this poset which is the component corresponding to the admitted vector 0. We will write either $\lambda\leq\gamma$ or $X_{W_a}[\lambda]\leq X_{W_a}[\gamma]$. The cover relation of $\leq$ is denoted by $\lessdot$.

Example 11. For the Coxeter group $W(A_n)$, a way to express its root system is by taking the vectors in $\mathbb{R}^{n+1}$ defined by $\Phi=\{e_i-e_j\mid 1\leq i,j\leq n+1 \text{ and } i\neq j\}$ and consider the space $V:=\operatorname{span}(e_i-e_j\mid 1\leq i,j\leq n+1 \text{ and } i\neq j)$ so that Φ is essential. The positive roots are then $\Phi^+=\{e_i-e_j\mid 1\leq i<j\leq n+1\}$.

It is easy to see that in type A_n an admitted vector $\lambda=(\lambda_{i,j})_{1\leq i<j\leq n+1}$ is defined by the following conditions:

$$\begin{cases} \lambda_{i,j}+\lambda_{j,k}\leq\lambda_{i,k}\leq\lambda_{i,j}+\lambda_{j,k}+1 \quad \text{for all } i<j<k, \\ \lambda_{i,i+1}=0 \quad \text{for all } 1\leq i<n. \end{cases}$$

The natural way to express a vector $v\in\bigoplus_{\alpha\in A_n^+}\mathbb{R}\alpha$ is by putting its coordinates in a triangle ordered by the height. For example in type A_3 the vector $v=(v_{12},v_{13},v_{14},v_{23},v_{24},v_{34})$ will be presented as follows:

Figure 1.17 Presentation of a vector $v \in \bigoplus_{\alpha \in A_3^+} \mathbb{R}\alpha$ as a triangle. The base is the set of coordinates on the simple roots.

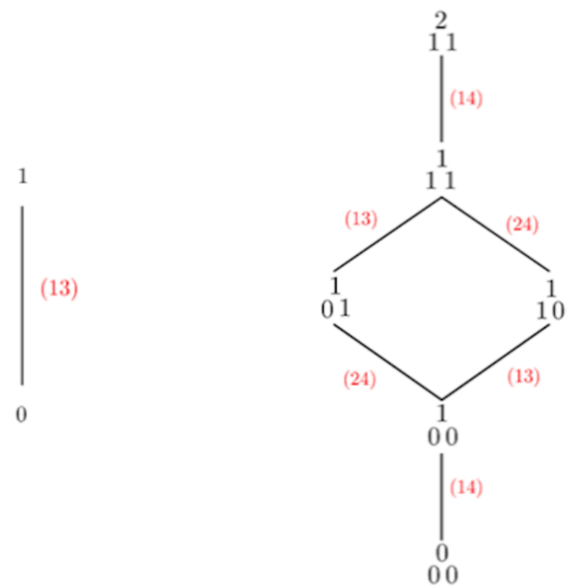

Figure 1.18 Posets associated to $\widehat{X}_{W(\widetilde{A}_2)}$ and $\widehat{X}_{W(\widetilde{A}_3)}$. The coordinates on the simple roots are erased since they are all equal to 0. The red labels represent the natural order on $\mathbb{Z}^6 = \bigoplus_{\alpha \in A_3^+} \mathbb{Z}\alpha$, that is the red label on an edge indicates the positive root whose coordinate is increased when going up the edge.

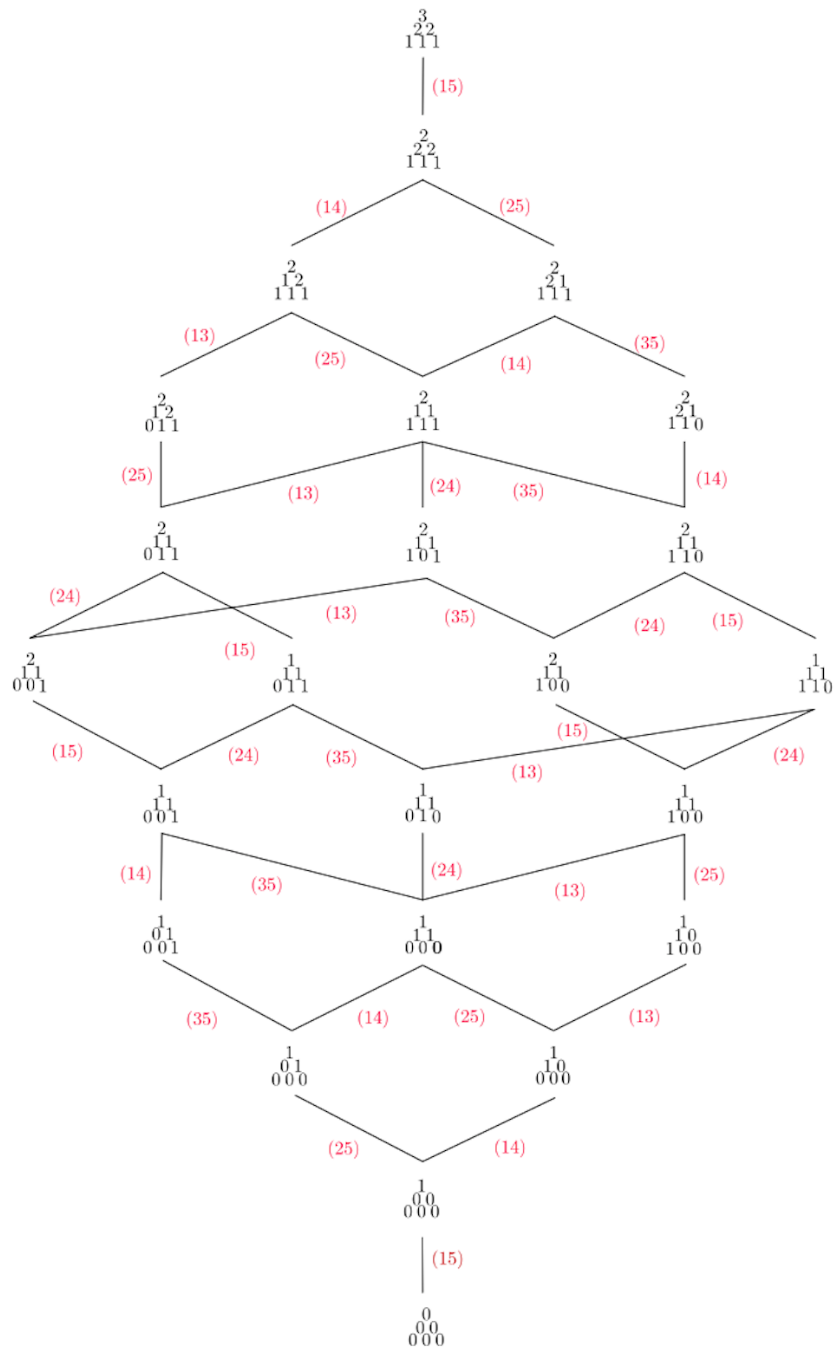

Figure 1.19 Poset associated to $\widehat{X}_{W(\widetilde{A}_4)}$. See Figure 1.18 for the comments.

Example 12. The polytope P_{B_2} is as follows:

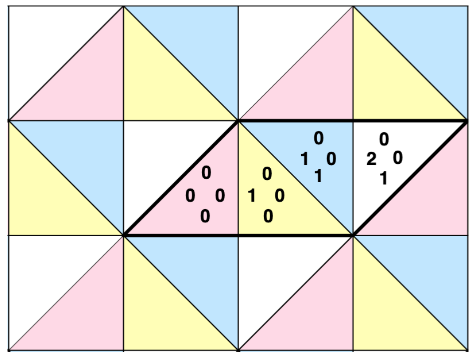

Figure 1.20 Polytope P_{B_2} seen as set of representatives of irreducible components of $\widehat{X}_{W(\widetilde{B}_2)}$.

Using Figure 1.11, we denote the admitted vectors by droping the two zeros corresponding to the simple roots, and by ordering the coordinates according to the height of the dual roots. Therefore, $H^0(\widehat{X}_{W(\widetilde{B}_2)})$ is as follows:

Figure 1.21 Poset associated to $\widehat{X}_{W(\widetilde{B}_2)}$.

Example 13. The positive roots of $B_3^\vee$ can be arranged according to their height into a shape looking like the temple of Kukulcan. Moreover the base is the set of dual simple roots. If λ is an admitted vector, its coordinates on the dual simple roots are 0, therefore we erase the base in this presentation of λ, and we obtain the following pictures:

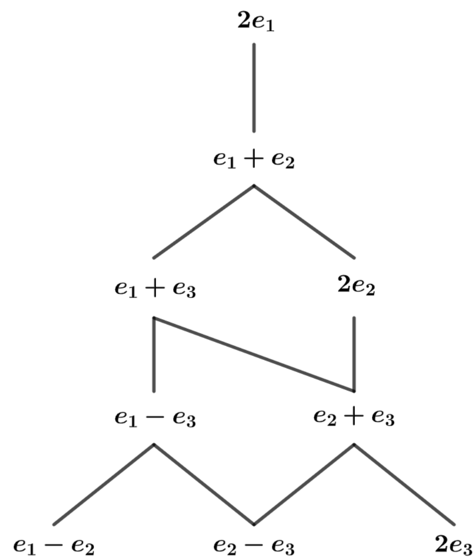

Figure 1.22 Positive roots of $B_3^\vee$.

$$\lambda = \begin{array}{ccccc} & & \lambda_1 & & \\ & & \lambda^{12} & & \\ & \lambda^{13} & & \lambda_2 & \\ & \lambda_{13} & & \lambda^{23} & \\ \lambda_{12} & & \lambda_{23} & & \lambda_3 \end{array} \longrightarrow \begin{array}{ccc} & \lambda_1 & \\ & \lambda^{12} & \\ \lambda^{13} & & \lambda_2 \\ \lambda_{13} & & \lambda^{23} \end{array}$$

Figure 1.23 Presentation of an admitted vector λ.

From this presentation of admitted vectors we can express the poset $H^0(\widehat{X}_{W(\widetilde{B}_3)})$ as follows:

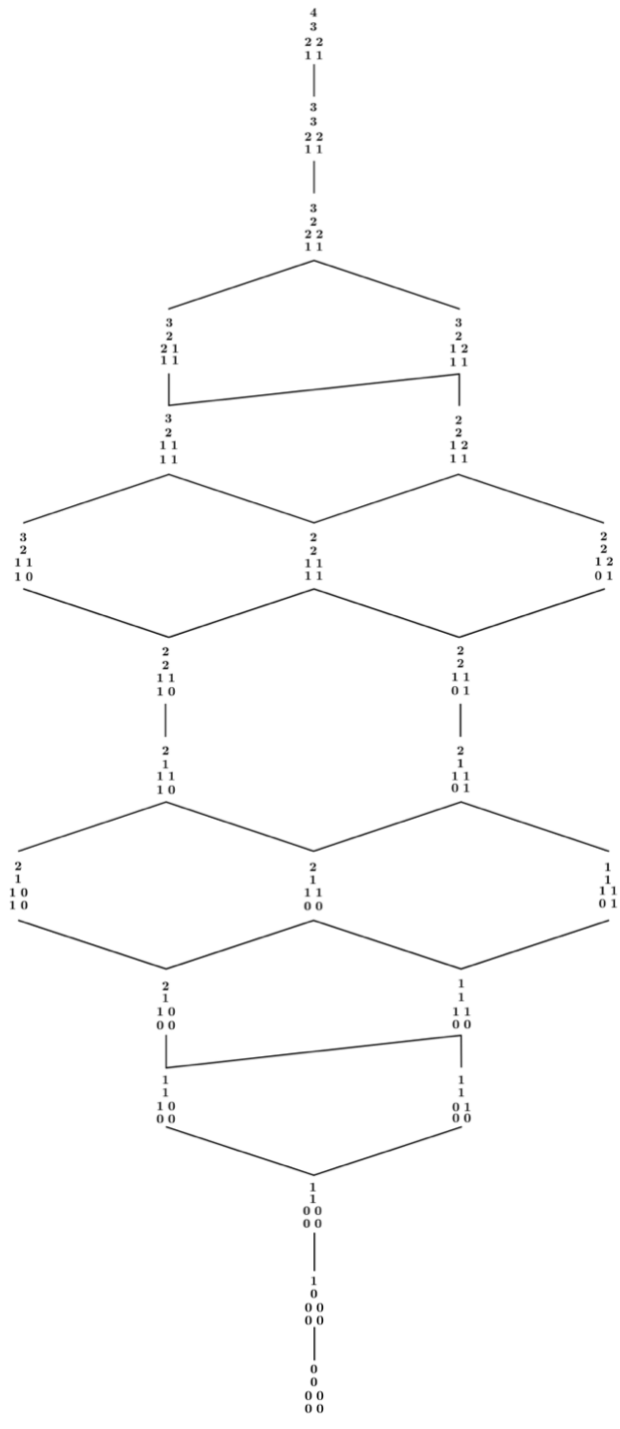

Figure 1.24 Poset associated to $\widehat{X}_{W(\widetilde{B}_3)}$.

Proposition 1.6.1. *Let λ and γ two admitted vectors. Then we have the equivalence between*

i) $\lambda \lessdot \gamma$.

ii) There exists a unique $\alpha \in \Phi^+$ such that $s_\alpha \diamond \lambda = \gamma$ and such that

$$\gamma_\beta = \begin{cases} \lambda_\beta + 1 & \text{if } \beta = \alpha \\ \lambda_\beta & \text{if } \beta \neq \alpha. \end{cases}$$

Proof. The direction ii) implies i) is obvious.

Let us prove direction i) implies ii). From the geometrical point of view, we know that two alcoves A_x and A_y share a common wall if and only if there exists a root $\eta \in \Phi^+$ satisfying the two following conditions

$$k(y, \eta) = k(x, \eta) + 1, \tag{C1}$$

$$k(x, \beta) = k(y, \beta) \text{ for all } \beta \in \Phi^+ - \{\eta\}. \tag{C2}$$

It turns out that if $A_x \subset P_{\mathcal{H}}$ and $A_y \subset P_{\mathcal{H}}$, we have $x \lessdot y$ if and only if (C1) and (C2) are satisfied. Indeed, each admitted vector corresponds to an alcove in the polytope $P_{\mathcal{H}}$. Therefore, assume that λ and γ are admitted vectors corresponding to adjacent alcoves A_λ and A_γ, with A_γ covering A_λ. Thus, there exists $k \in \mathbb{Z}$ such that $F(s_{\eta,k})(\lambda) = \gamma$ and it follows that $s_{\eta,k} \diamond \lambda = \gamma$. Since $s_{\eta,k} = \tau_{k\eta} s_\eta$ we have $s_{\eta,k} \diamond \lambda = \tau_{k\eta} s_\eta \diamond \lambda$. However, because of Proposition 1.5.8 we know that the irreducible components are invariant under translations. It follows that $\tau_{k\eta} s_\eta \diamond \lambda = s_\eta \diamond \lambda$. Finally, we have $s_\eta \diamond \lambda = \gamma$ with $k(w_\gamma, \eta) = k(w_\lambda, \eta) + 1$ and $k(w_\lambda, \beta) = k(w_\gamma, \beta)$ for all $\beta \in \Phi^+ - \{\eta\}$ which is exactly the condition ii). □

Proposition 1.6.2. *There exists a unique alcove A_w in $P_{\mathcal{H}}$ such that the point $x := \bigcap_{\alpha \in \Delta} H_{\alpha,1}$ is a vertex of A_w. Moreover, for $\alpha \in \Delta$ the hyperplanes $H_{\alpha,1}$ are some of the walls of A_w.*

Proof. Let $W_x := \langle s_{\alpha,1}, \alpha \in \Delta\rangle$, $\Delta_x := \{\alpha^\vee - \delta \mid \alpha \in \Delta\}$ and $\Phi_x := W_x(\Delta_x)$. The strategy consists to show two things: First the set Δ_x is a simple system of W_x and secondly $W_x = \mathrm{Stab}_{W_a}(x)$. Indeed, let us denote $\mathcal{D}_x$ to be the simplicial cone pointed in x, cut out by the hyperplanes $H_{\alpha,1}$ for $\alpha \in \Delta$ and containing the alcove A_e. If Δ_x is a simple system of W_x then $\mathcal{D}_x$ is the fundamental Weyl chamber of W_x, and if $W_x = \mathrm{Stab}_{W_a}(x)$ then there is no hyperplane going through x and $\mathcal{D}_x$. Thus, by setting A_w to be the alcove with vertex x and the $n-1$ walls $H_{\alpha,1}$ for $\alpha \in \Delta$ we have what we announced.

•) Since Δ is linearly independent it follows that Δ_x is also linearly independent. Because of Equation (1.3) we know that $(\alpha^\vee - \delta, \beta^\vee - \delta) = (\alpha^\vee, \beta^\vee)$ for all $\alpha, \beta \in \Delta$. Then, using Formula (1.4) we have $B(\alpha^\vee - \delta, \beta^\vee - \delta) = B(\alpha^\vee, \beta^\vee)$ for all $\alpha, \beta \in \Delta$. Therefore, Δ_x is a simple system (in the sense of Definition 1) for (W_x, S_x) where $S_x := \{s_{\alpha,1} \mid \alpha \in \Delta\}$.

•) First of all it is clear that $W_x \simeq W$. Therefore it follows that $|\Phi_x^+| = |\Phi^+|$ and then the number of hyperplanes passing through x is the same as the number of hyperplanes passing through 0. Moreover we know that each hyperplane of $\mathcal{H}$ (see Section 1.3.2) is parallel to a hyperplane passing through 0, that is parallel to a hyperplane $H_{\alpha,0}$ with $\alpha \in \Phi^+$. In particular each hyperplane passing through x is parallel to such a hyperplane. Therefore, it follows that each hyperplane of $\mathcal{H}$ is parallel to a hyperplane passing through x. Thus, Proposition 1.3.1 implies that x is a special point, that is $\mathrm{Stab}_{W_x}(x) \simeq W$. It follows then that $W_x \simeq \mathrm{Stab}_{W_a}(x)$. Finally, since W is finite, W_x is also finite and then, since $W_x \subset \mathrm{Stab}_{W_a}(x)$, it follows that $W_x = \mathrm{Stab}_{W_a}(x)$. $\square$

1.6.1 Lattice structure on $H^0(\widehat{X}_{W_a})$

In this paragraph we recall some basics about lattices. A *lattice* is a partially ordered set such that every pair x, y of elements has a meet (greatest lower bound) $x \wedge y$ and join (least upper bound) $x \vee y$. A lattice is distributive if the meet operation distributes over the join operation and the join distributes over the meet.

A lattice L is *join semidistributive* if whenever $x, y, z \in L$ satisfy $x \vee y = x \vee z$, they also satisfy $x \vee (y \wedge z) = x \vee y$. This is equivalent to the following condition: If X is a nonempty finite subset of L such that $x \vee y = z$ for all $x \in X$, then $(\bigwedge_{x \in X} x) \vee y = z$. The lattice is *meet semidistributive* if the dual condition $(x \wedge y = x \wedge z) \Rightarrow (x \wedge (y \vee z) = x \wedge y)$ holds. Equivalently, if X is a nonempty finite subset of L such that $x \wedge y = z$ for all $x \in X$, then $(\bigvee_{x \in X} x) \wedge y = z$. The lattice is *semidistributive* if it is both join semidistributive and meet semidistributive.

Theorem 1.6.3. *$H^0(\widehat{X}_{W_a})$ has a structure of semidistributive lattice.*

Proof. Let us begin by proving that $H^0(\widehat{X}_{W_a})$ is a lattice. The idea is to show that the admitted vectors, seen as alcoves in $P_{\mathcal{H}}$, define an interval in the right weak order of W_a. In order to do so, we first have to find a maximal and minimal element. Let λ be an admitted vector. Because of the way we defined it, we know that $0 \leq k(w_\lambda, \alpha)$ for all $\alpha \in \Phi^+$. Moreover we have the identity element which belongs to $P_{\mathcal{H}}$, and since its Φ^+-tuple is the vector $0_{\mathbb{R}^m}$ it follows that the admitted vector associated to the identity is lower than all the others admitted vectors in $P_{\mathcal{H}}$.

We need now to have a good candidate for the maximal element of $P_{\mathcal{H}}$. By Proposition 1.6.2 we know that there exists a unique element $w \in P_{\mathcal{H}}$ having $x := \bigcap_{\alpha \in \Delta} H_{\alpha,1}$ as vertex. We claim that w is greater (in the sense of Definition 11) than any other element in $P_{\mathcal{H}}$. If it wasn't the case we would have a hyperplane $H_{\alpha,k}$ with $\alpha \in \Phi^+ - \Delta,\ k \in \mathbb{N}$ that would cut $P_{\mathcal{H}}$ into two connected components such that A_w and A_e are in the same one and such that $x \notin H_{\alpha,k}$. Let $A_{w'}$ be an alcove in the connected component that does not contain A_e. It follows that $k(w,\alpha) < k(w',\alpha)$. Let y be a point of A_w and y' be a point of $A_{w'}$. Therefore, since y and $y' \in P_{\mathcal{H}}$ there exist $a_1,\dots,a_n$ and $b_1,\dots,b_n \in \mathbb{R}^+$ such that $y = a_1\alpha_1 + \cdots + a_n\alpha_n$ and $y' = b_1\alpha_1 + \cdots + b_n\alpha_n$.

We claim now that without lost of generality one can assume that $b_i \leq a_i$ for all $i \in [\![1,n]\!]$. Let us first explain that claim. Let us write y and y' in the basis of fundamental weights: $y = c_1\omega_1 + \cdots + c_n\omega_n$ and $y' = d_1\omega_1 + \cdots + d_n\omega_n$ with c_i and $d_i \in \mathbb{R}^+$. Since $y \in A_w$ and $y' \notin A_w$, and since x is a vertex of A_w, we can take y as close as we want to x. It follows here that there is no problem of assuming that $d_i \leq c_i$ for all i. Therefore we make the assumption that $d_i \leq c_i$ for all i. It turns out that the inverse of the Cartan matrix $C^{-1} = (h_{ij})_{i,j\in[\![1,n]\!]}$ of W is the change-of-basis matrix of the basis of simple roots to the basis of fundamental weights. Moreover, it is known (see (Lusztig et Tits, 1992) or (Yangjiang et Zou, 2017)) that all the coefficients of C^{-1} are positive. It follows that

$$C^{-1}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad C^{-1}\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Thus, the i-th coordinate of y is $\sum_{k=1}^{n} h_{ik}c_k$ and the i-th coordinate of y' is $\sum_{k=1}^{n} h_{ik}d_k$. Since $d_i \leq c_i$ for all i and since $h_{ik} \geq 0$ for all $k = 1,\dots,n$ it follows that $\sum_{k=1}^{n} h_{ik}d_k \leq \sum_{k=1}^{n} h_{ik}c_k$. However the i-th coordinate of y is nothing but a_i, and i-th coordinate of y' is b_i. Finally we have shown that $b_i \leq a_i$ for all i.

From the way that the coefficients $k(-,-)$ are defined, one has $k(w,\alpha) = \lfloor(\alpha^\vee, y)\rfloor$ and $k(w',\alpha) = \lfloor(\alpha^\vee, y')\rfloor$ with $\lfloor\ \rfloor$ the floor function. Via the expressions of y and

y' one has:

$$(\alpha^\vee, y) = (\alpha^\vee, a_1\alpha_1 + \cdots + a_n\alpha_n) = a_1(\alpha^\vee, \alpha_1) + \cdots + a_n(\alpha^\vee, \alpha_n),$$
$$(\alpha^\vee, y') = (\alpha^\vee, b_1\alpha_1 + \cdots + b_n\alpha_n) = b_1(\alpha^\vee, \alpha_1) + \cdots + b_n(\alpha^\vee, \alpha_n).$$

It follows that $0 \leq (\alpha^\vee, y') \leq (\alpha^\vee, y)$ and then $\lfloor(\alpha^\vee, y')\rfloor \leq \lfloor(\alpha^\vee, y)\rfloor$, which means that $k(w', \alpha) \leq k(w, \alpha)$. This is in contradiction with the above statement. Hence, w must be the maximal element of $P_{\mathcal{H}}$. Thus it follows that $\mathrm{Alc}(P_{\mathcal{H}}) \subset [e, w]$.

Let us show the other inclusion. The definition of $\mathrm{Alc}(P_{\mathcal{H}})$ in Section 1.3.3 shows that $h \in \mathrm{Alc}(P_{\mathcal{H}})$ if and only if $k(h, \alpha) \geq 0$ for all $\alpha \in \Phi^+$ and $k(h, \varepsilon) = 0$ for all $\varepsilon \in \Delta$. Let $g \in [e, w]$. Since $e \leq g \leq w$ it follows that $0 \leq k(g, \alpha) \leq k(w, \alpha)$ for all $\alpha \in \Phi^+$, and in particular $0 \leq k(g, \varepsilon) \leq k(w, \varepsilon)$ for all $\varepsilon \in \Delta$. It follows that $0 \leq k(g, \varepsilon) \leq 0$ for all $\varepsilon \in \Delta$, and then $k(g, \varepsilon) = 0$ for all $\varepsilon \in \Delta$. Hence $g \in \mathrm{Alc}(P_{\mathcal{H}})$.

Thanks to Theorem 8.1 of (Reading et Speyer, 2011), it is known that every interval in the right weak order of any infinite Coxeter group is a semidistributive lattice. Therefore, since the elements of $P_{\mathcal{H}}$ form an interval in the right weak order of W_a, $P_{\mathcal{H}}$ inherits a structure of semidistributive lattice. However, $P_{\mathcal{H}}$ is in bijection with the set of irreducible components of $\widehat{X}_{W_a}$: $H^0(\widehat{X}_{W_a})$. As the order relation on $P_{\mathcal{H}}$ is the same as the order relation on $H^0(\widehat{X}_{W_a})$, we finally showed that $H^0(\widehat{X}_{W_a})$ has a structure of semidistributive lattice. $\square$

1.7 Cohomological link with the irreducible components of $\widehat{X}_{W_a}$

In this section we show how the irreducible components obtained before can be used to express the first group of cohomology of any Weyl group W such that $W \neq W(A_2)$.

Recall our notations; Φ is an irreducible crystallographic root system associated to W and lying in a Euclidean space $(V,(-,-))$ with simple system $\Delta = \{\alpha_1,\ldots,\alpha_n\}$. The set of Coxeter generators of W associated to Δ is $S=\{s_1,\ldots,s_n\}$ and $m_{ij} := \mathrm{ord}(s_i s_j)$. We also keep the convention that $||\alpha|| = 1$ for all short roots α.

The Cartan matrix $C_\Phi = (h_{ij})_{i,j}$ associated to Φ is the matrix of $GL_n(\mathbb{R})$ defined by $h_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. It is known that C_Φ is the change-of-basis matrix of the basis of fundamental weights to the basis of simple roots. This implies in particular that C_Φ is invertible.

Let $w \in W_a$. For an admitted vector λ, $\lambda(w) = \lambda$ if and only if $\iota(w) \in \widehat{X}_{W_a}[\lambda]$. Finally, F is the Φ^+ representation of W_a and ι is the bijection between W_a and $\widehat{X}_{W_a}(\mathbb{Z})$.

We know that $W_a \simeq \mathbb{Z}\Phi \rtimes W$. Therefore we have the short exact sequence

$$1 \longrightarrow \mathbb{Z}\Phi \xrightarrow{i} \mathbb{Z}\Phi \rtimes W \xrightarrow{\pi} W \longrightarrow 1,$$

where i and π are the morphisms defined by $i(x) = \tau_x$ and $\pi(w) = \overline{w}$, where $w \in W_a$ decomposes as $w = \tau_y \overline{w}$.

1.7.1 Background about group cohomology

Let A be a G-module, that is A is an abelian group on which G acts compatibly with the abelian group structure on A, or equivalently a module over the ring $\mathbb{Z}[G]$. A morphism of G-modules is the same thing as a morphism of $\mathbb{Z}[G]$-modules. Let $\mathrm{Hom}_G(A,A')$ be the set of morphisms of G-modules between A and A'.

We define A^G to be the submodule of G-invariants, that is the subgroup of A defined by $A^G := \{a \in A \mid ga = a \text{ for all } g \in G\}$. The functor $F : A \mapsto A^G$ from the category of G-modules to the category of abelian groups is covariant and left-exact. Therefore, we can define its right derived functors R^iF and by definition the i-th cohomology group of G with coefficients in A is $H^i(G,A) := R^iF(A)$. Since the sets A^G and $\mathrm{Hom}_G(\mathbb{Z}, A)$ are canonically isomorphic, we can also define $H^i(G,A)$ via the functor Ext, that is $H^i(G,A) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A)$. For more details the reader may refer to (Harari, 2011).

Let B be an abelian group. Let us consider the following exact sequence:

$$1 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1.$$

A section of p is a morphism $\mathsf{s} : G \to E$ such that $p \circ \mathsf{s} = id_G$. We denote by $Sec_B(p)$ the set of sections of p. If $B = \mathbb{Z}\Phi$ we will just write $Sec(p)$. Two sections s and s' are called B-conjugated if there exists $b \in B$ such that for all $g \in G$ we have $\mathsf{s}'(g) = i(b)\mathsf{s}(g)i(b)^{-1}$. Let $\varphi : G \to Aut(B)$ be a morphism of groups. B inherits a structure of G-module where the action is given by $g.b := \varphi(g)(b)$. The following result is also well known and will be of particular interest to us:

Theorem 1.7.1 (Theorem 8.2 of (Castel, 2009))**.** *The set of sections up to B-conjugacy of the exact sequence*

$$1 \longrightarrow B \xrightarrow{i} B \rtimes_{\varphi} G \xrightarrow{p} G \longrightarrow 1$$

is in bijection with $H^1(G,B)$.

1.7.2 Link between $H^1(W, \mathbb{Z}\Phi)$ and the components $\widehat{X}_{W_a}[\lambda(s)]$

Definition 12. From Proposition 1.5.10 we know that two different Coxeter generators of W lie in different components. Let $O_{s_i} \subset \widehat{X}_{W_a}[\lambda(s_i)]$ be the corresponding orbit of $(k(s_i,\alpha))_{\alpha\in\Phi^+}$ under the action of $\mathbb{Z}\Phi$. The elements of O_{s_i} are those that can be written as $(k(\tau_x s_i,\alpha))_{\alpha\in\Phi^+}$ with $x \in \mathbb{Z}\Phi$. It is clear that O_{s_i}

has a structure of $\mathbb{Z}$-module (induced from the structure of $\mathbb{Z}\Phi$) where the scalar multiplication and the addition are defined as follows

$$d.(k(\tau_x s_i,\alpha))_{\alpha\in\Phi^+} := (k(\tau_{dx} s_i,\alpha))_{\alpha\in\Phi^+},$$

$$(k(\tau_x s_i,\alpha))_{\alpha\in\Phi^+} + (k(\tau_y s_i,\alpha))_{\alpha\in\Phi^+} := (k(\tau_{x+y} s_i,\alpha))_{\alpha\in\Phi^+}.$$

Definition 13. The structure of $\mathbb{Z}$-module on each orbit induces a structure of $\mathbb{Z}$-module on $\bigoplus\limits_{i=1}^{n} O_{s_i}$. Recall that ι is the map defined in Theorem 1.5.5. We put on the set $\bigoplus\limits_{i=1}^{n} O_{s_i}$ an equivalence relation defined as follows

$$\begin{aligned}&(\iota(\tau_{x_1}s_1),\dots,\iota(\tau_{x_n}s_n)) \sim (\iota(\tau_{y_1}s_1),\dots,\iota(\tau_{y_n}s_n))\\ \Longleftrightarrow\ &\exists z\in\mathbb{Z}\Phi \text{ such that } \forall\alpha_i\in\Delta \text{ one has: } x_i-y_i = 2\frac{(z,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i = (z,\alpha_i^\vee)\alpha_i.\end{aligned}$$

Example 14. In Figure 1.25 we have in blue the two components of $\widehat{X}_{W(\widetilde{B}_2)}$. Each different filling of the blue alcoves represents a $\mathbb{Z}B_2$-orbit in this component.

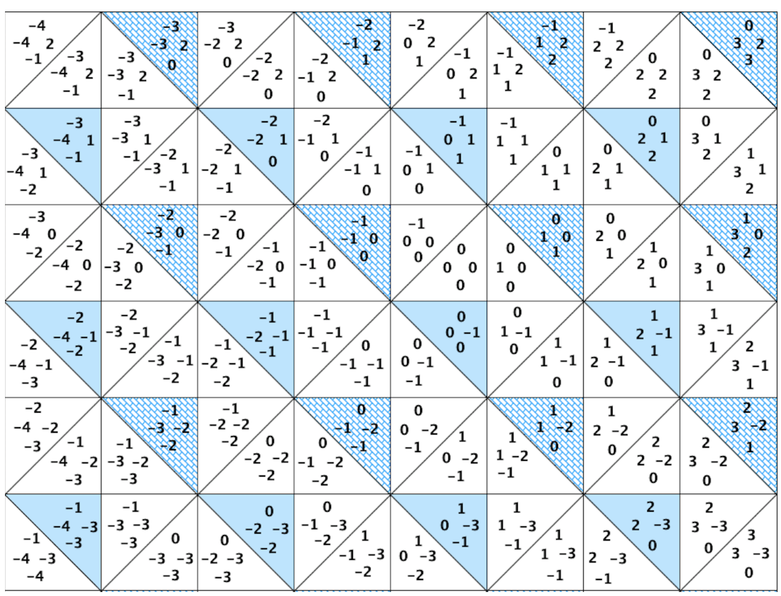

Figure 1.25 Orbits in $\widehat{X}_{W(\widetilde{B}_2)}[(-1,0,0,0)]$.

Definition 14. 1) Let $s_i \in S$. Let $L(s_i)$ be the subset of O_{s_i} defined by

$$L(s_i) := \{F(\tau_{x_i})(\iota(s_i)) \mid x_i \in \mathbb{Z}\alpha_i\} = \{\iota(s_i) + d\iota(\alpha_i) \mid d \in \mathbb{Z}\}.$$

The set $L(s_i)$ should be thought of as a dotted line in $\widehat{X}_{W_a}[\lambda(s_i)]$.

2) Let A be a ring such that $\mathbb{Z} \subset A \subset \mathbb{R}$. We extend the previous definition to A by

$$L_A(s_i) := \{\iota(s_i) + a\iota(\alpha_i) \mid a \in A\}.$$

3) The relation $\sim$ restricts naturally to $\prod_{i=1}^{n} L(s_i)$. We define the set $\widehat{X}^q$ by

$$\widehat{X}^q := \prod_{i=1}^{n} L(s_i)/\sim .$$

Notice that we can define a similar equivalence relation, also denoted $\sim$, on $\prod_{i=1}^{n} L_A(s_i)$ by saying that two elements $a, b \in \prod_{i=1}^{n} L_A(s_i)$ are equivalent if and only if there exists $t \in A\Phi$ such that $a_i - b_i = 2\frac{(t,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$ for all $i = 1, \ldots, n$. It is also possible to define the analog of $\widehat{X}^q$ by setting $\widehat{X}^q_A := \prod_{i=1}^{n} L_A(s_i)/\sim$.

Lemma 1.7.2. *Let A be a commutative unitary ring such that $\mathbb{Z} \subset A \subset \mathbb{R}$. Let s be a section of the exact sequence*

$$1 \longrightarrow A\Phi \xrightarrow{i} A\Phi \rtimes W \xrightarrow{\pi} W \longrightarrow 1,$$

defined by $\mathsf{s}(s_i) = \tau_{x_i}s_i$ for $s_i \in S$. Then the map ξ from $Sec_{A\Phi}(\pi)$ to $A\alpha_1 \times \cdots \times A\alpha_n$ defined by $\xi(\mathsf{s}) = (x_1, \ldots, x_n)$ is an isomorphism of A-modules. In particular $Sec_{A\Phi}(\pi)$ is in bijection with $L_A(s_1) \times \cdots \times L_A(s_n)$.

Proof. Let us first show that ξ is well defined, that is $x_i \in A\alpha_i$ for all i. We know that W is a group defined by generators and relations. The relations are given by $s_i^2 = e$ and some braid relations. The braid relations are either of type $st = ts$ or $sts = tst$ or $stst = tsts$ or $ststst = tststs$ for all generators $s, t \in S$. Since

s is a morphism, it must preserve all of these relations. Therefore, the relation $s_i^2 = e$ implies that $\tau_{x_i} s_i \tau_{x_i} s_i = e$, that is $\tau_{x_i + s_i(x_i)} s_i^2 = e$ and then $x_i + s_i(x_i) = 0$. Consequently we must have $s_i(x_i) = -x_i$, which means that $x_i \in H_{\alpha_i}^{\perp} = \mathbb{R}\alpha_i$. It follows that $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} A\alpha_i$. Thus, for all $i = 1, \ldots, n$ there exists $k_i \in A$ such that $x_i = k_i \alpha_i$. We claim that any choice of $(k_1, \ldots, k_n) \in A^n$ gives rise to a section. From this claim it is clear that the map ξ is a bijection. With respect to the morphism it is obvious that the structure of A-module on $Sec_{A\Phi}(\pi)$ is the same as the structure of $A\Phi$.

Let us now prove the claim. By Proposition 1.3.4 we know that $\mathrm{ord}(s_{\alpha_i,k_i} s_{\alpha_j,k_j}) = \mathrm{ord}(s_{\alpha_i} s_{\alpha_j})$. It follows that

$$\mathrm{ord}((s_{\alpha_i,k_i} s_{\alpha_j,k_j})^d) = \mathrm{ord}((s_{\alpha_i} s_{\alpha_j})^d)$$

for all $d \in \mathbb{N}$. In particular for $d = 2, 3, 4$ or 6, since $s_{\alpha_i,k_i} = \tau_{k_i\alpha_i} s_{\alpha_i} = \tau_{x_i} s_i$, we see that the braid relations are preserved under the section s. It follows that there is no constraint on the choices of the k_i. This ends the proof. $\square$

Theorem 1.7.3. *We have a bijection between $H^1(W, \mathbb{Z}\Phi)$ and $\widehat{X}^q$.*

Proof. $\mathbb{Z}\Phi$ is clearly a W-module and then because of Theorem 1.7.1 we know that $H^1(W, \mathbb{Z}\Phi)$ is in bijection with the sections, up to $\mathbb{Z}\Phi$-conjugacy, of the group extension

$$1 \longrightarrow \mathbb{Z}\Phi \xrightarrow{i} \mathbb{Z}\Phi \rtimes W \xrightarrow{\pi} W \longrightarrow 1.$$

Let s be a section of π. Since s is a morphism, the images of the generators characterize entirely s. Thus, the data of a section is the same thing as the data of n elements in $\mathbb{Z}\Phi \rtimes W$. However, because of Lemma 1.7.2 we know that these n elements must be written as $\tau_{x_1} s_1, \ldots, \tau_{x_n} s_n$ with $x_i \in \mathbb{Z}\alpha_i$. Therefore, the data of a section is the same thing as taking one element in each $L(s_i)$ for all $i = 1, \ldots, n$. There follows the bijection between $Sec(\pi)$ and $L(s_1) \times \cdots \times L(s_n)$.

Let us take s_1 and s_2 two different sections defined as $\mathsf{s}_1(s_i) = \tau_{x_i} s_i$ and $\mathsf{s}_2(s_i) = \tau_{y_i} s_i$ with $x_i, y_i \in \mathbb{Z}\alpha_i$ for all $i = 1, \ldots, n$. Since they are different there exists at least one index j such that $x_j \neq y_j$. Then, s_1 is Φ-conjugated to s_2 if and only if there exists $z \in \mathbb{Z}\Phi$ such that for all $w \in W$ one has $\mathsf{s}_2(w) = \tau_z \mathsf{s}_1(w) \tau_{-z}$. It turns out that $\tau_z \mathsf{s}_1(w) \tau_{-z} = \tau_{(id-w)(z)} \mathsf{s}_1(w)$, indeed since $\mathsf{s}_1(w) \in \mathbb{Z}\Phi \rtimes W$ there exists $u \in \mathbb{Z}\Phi$ such that $\mathsf{s}_1(w) = \tau_u w$. Therefore we have $\tau_z \mathsf{s}_1(w) \tau_{-z} = \tau_z \tau_u w \tau_{-z} = \tau_z \tau_u \tau_{-w(z)} w = \tau_z \tau_{-w(z)} \tau_u w = \tau_{(id-w)(z)} \mathsf{s}_1(w)$. Hence, it follows

$$\begin{aligned}
\mathsf{s}_1 \sim \mathsf{s}_2 \quad &\Longleftrightarrow \quad \exists z \in \mathbb{Z}\Phi \text{ such that } \forall\ w \in W,\ \mathsf{s}_2(w) = \tau_{(id-w)(z)} \mathsf{s}_1(w) \\
&\Longrightarrow \quad \exists z \in \mathbb{Z}\Phi \text{ such that } \forall\ s_i \in S,\ \mathsf{s}_2(s_i) = \tau_{(id-s_i)(z)} \mathsf{s}_1(s_i) \\
&\Longleftrightarrow \quad \exists z \in \mathbb{Z}\Phi \text{ such that } \forall\ s_i \in S,\ y_i - x_i = (id - s_i)(z).
\end{aligned}$$

It also turns out that we have the opposite direction. Indeed, let us write $w = t_1 t_2 \ldots t_p$ with $t_i \in S$ such that it is a reduced expression. We then have the following equalities

$$\mathsf{s}_1(w) = \mathsf{s}_1(t_1)\mathsf{s}_1(t_2)\ldots\mathsf{s}_1(t_p) = \tau_{x_1} t_1 \tau_{x_2} t_2 \ldots \tau_{x_p} t_p = \tau_{x_1 + t_1(x_2) + \cdots + t_1 t_2 \ldots t_{p-1}(x_p)} t_1 t_2 \ldots t_p,$$

$$\mathsf{s}_2(w) = \mathsf{s}_2(t_1)\mathsf{s}_2(t_2)\ldots\mathsf{s}_2(t_p) = \tau_{y_1} t_1 \tau_{y_2} t_2 \ldots \tau_{y_p} t_p = \tau_{y_1 + t_1(y_2) + \cdots + t_1 t_2 \ldots t_{p-1}(y_p)} t_1 t_2 \ldots t_p.$$

It follows that

$$\begin{aligned}
& \mathsf{s}_2(w) = \tau_{(id-w)(z)} \mathsf{s}_1(w) \\
\Longleftrightarrow\ & \tau_{y_1 + t_1(y_2) + \cdots + t_1 t_2 \ldots t_{p-1}(y_p)} t_1 t_2 \ldots t_p = \tau_{(id-w)(z)} \tau_{x_1 + t_1(x_2) + \cdots + t_1 t_2 \ldots t_{p-1}(x_p)} t_1 t_2 \ldots t_p \\
\Longleftrightarrow\ & y_1 + t_1(y_2) + \cdots + t_1 t_2 \ldots t_{p-1}(y_p) = z - w(z) + x_1 + t_1(x_2) + \cdots + t_1 t_2 \ldots t_{p-1}(x_p) \\
\Longleftrightarrow\ & z - w(z) = (y_1 - x_1) + t_1(y_2 - x_2) + \cdots + t_1 t_2 \ldots t_{p-1}(y_p - x_p).
\end{aligned}$$

However, with the assumption $y_i - x_i = (id - t_i)(z)$ for all $i = 1, \ldots, p$, it follows that

$$\begin{aligned}
z - w(z) &= z - t_1(z) + t_1(z) - t_1t_2(z) + t_1t_2(z) - \cdots - t_1t_2\ldots t_{p-1}(z) + t_1t_2\ldots t_{p-1}(z) - w(z) \\
&= z - t_1(z) + t_1(z - t_2(z)) + \cdots + t_1t_2\ldots t_{p-1}(z - t_p(z)) \\
&= y_1 - x_1 + t_1(y_2 - x_2) + \cdots + t_1t_2\ldots t_{p-1}(y_p - x_p).
\end{aligned}$$

Therefore we have the equivalence

$$\mathsf{s}_1 \sim \mathsf{s}_2 \iff \exists z \in \mathbb{Z}\Phi \text{ such that } \forall\ s_i \in S,\ y_i - x_i = (id - s_i)(z).$$

However $(id - s_i)(z) = z - s_i(z) = z - (z - 2\frac{(z,x_i)}{(x_i,x_i)}\alpha_i) = 2\frac{(z,x_i)}{(x_i,x_i)}\alpha_i$. Then we have

$$\mathsf{s}_1 \sim \mathsf{s}_2 \iff (\iota(\tau_{x_1}s_1), \ldots, \iota(\tau_{x_n}s_n)) \sim (\iota(\tau_{y_1}s_1), \ldots, \iota(\tau_{y_n}s_n)),$$

with ι the map defined in Theorem 1.5.5. This proves the bijection. □

1.7.3 Cohomology in degree 1 with coefficients in AΦ

This section is dedicated to the study of the cohomology in degree 1 when we change the lattice $\mathbb{Z}\Phi$. The main goal is to understand how the sections of π behave up to conjugacy when the space of coefficients $\mathbb{Z}\Phi$ is replaced by $A\Phi$. We will see for instance that the more we add points in $L(s_i)$ the smaller $H^1(W, A\Phi)$ becomes.

Consequently, it is of particular interest to find a (commutative unitary) ring A satisfying $\mathbb{Z} \subset A \subset \mathbb{R}$ and such that $H^1(W, A\Phi) = 0$. We also require to find A minimal. We will say that there is an $A\Phi$-obstruction between two sections s_1 and s_2 if they don't define the same class in $H^1(W, A\Phi)$.

We will see that the Cartan matrix of W has a critical role in the understanding of the obstructions between sections (see Proposition 1.7.5). Finally, our main result in this section will be the following one:

Theorem 1.7.4. *Let A be a commutative unitary ring such that $\mathbb{Z} \subset A \subset \mathbb{R}$. Let f_Φ be the index of connection of Φ and $\mathbb{Z}_{f_\phi} := \mathbb{Z}[\frac{1}{f_\phi}]$. Then the following points are equivalent*

i) $H^1(W, A\Phi) = 0$ *with A minimal.*

ii) $A = \mathbb{Z}_{f_\phi}$.

Example 15. Let us first illustrate this with an example. Let us take $W = W(B_2)$. The Cartan matrix of B_2 and its inverse are

$$C_{B_2} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad C_{B_2}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

Let s_1 and s_2 be two sections of $W(B_2)$. We know via Lemma 1.7.2 that we can identify s_1 with a point $(x_1, x_2) \in \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2$ and s_2 with a point $(y_1, y_2) \in \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2$. Denote $x_i = a_i\alpha_i$ and $y_i = b_i\alpha_i$. Moreover we know that

$$\mathsf{s}_1 \sim \mathsf{s}_2 \iff \exists z \in \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2 \text{ such that } \begin{cases} x_1 - y_1 = 2\frac{(z,\alpha_1)}{(\alpha_1,\alpha_1)}\alpha_1 \\ x_2 - y_2 = 2\frac{(z,\alpha_2)}{(\alpha_2,\alpha_2)}\alpha_2. \end{cases}$$

It follows that

$$\mathsf{s}_1 \sim \mathsf{s}_2 \iff \exists z \in \mathbb{Z}\alpha_1 \times \mathbb{Z}\alpha_2 \text{ such that } C_{B_2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}.$$

Thus, we must have z_1 and z_2 both integral, and such that

$$\begin{cases} 2z_1 = 2(a_1 - b_1) + 2(a_2 - b_2) \\ 2z_2 = a_1 - b_1 + 2(a_2 - b_2). \end{cases}$$

These arithmetical conditions show how $H^1(W(B_2), \mathbb{Z}\Phi)$ behaves compared to $L(s_1) \times L(s_2)$, and we see here that we do not have $H^1(W(B_2), \mathbb{Z}\Phi) = 0$. Indeed, by taking for example $a_2 = b_2 = 1$, $a_1 = 3$ and $b_1 = 2$ we have $2z_2 = 1$, which is impossible since z_2 is integral. We thus have an obstruction between s_1 and s_2, which implies that $H^1(W(B_2), \mathbb{Z}\Phi)$ is non-zero.

However, if we consider the equations in the ring $\mathbb{Z}_2 = \mathbb{Z}[\frac{1}{2}]$ instead of $\mathbb{Z}$, the previous obstruction no longer exists and it follows that $H^1(W(B_2), \mathbb{Z}_2\Phi) = 0$. Furthermore, the ring $\mathbb{Z}_2$ is the smallest one that satisfies the last equality.

Theorem 1.7.5. *Let A be a commutative unitary ring such that $\mathbb{Z} \subset A \subset \mathbb{R}$. Then the following points are equivalent:*

i) $H^1(W, A\Phi) = 0$.

ii) $C_\Phi \in GL_n(A)$.

Proof. Let us first show the direction ii) implies i). Let π be the morphism from $A\Phi \rtimes W$ to W defined by $\pi(\tau_x w) = w$ for all $x \in A\Phi$. We recall that $H^1(W, A\Phi)$ is in bijection with the set $Sec_{A\Phi}(\pi)$ up to $A\Phi$-conjugacy. Moreover, because of Lemma 1.7.2 we know that $Sec_{A\Phi}(\pi)$ is in bijection with $L_A(s_1) \times \cdots \times L_A(s_n)$ or again with $A\alpha_1 \times \cdots \times A\alpha_n$. Let $\mathsf{s}_1, \mathsf{s}_2 \in Sec_{A\Phi}(\pi)$ such that via the bijective correspondence above, s_1 is identified with the point $(x_1, \ldots, x_n) \in A\alpha_1 \times \cdots \times A\alpha_n$ and s_2 with the point $(y_1, \ldots, y_n)$. Let us denote $x_i = a_i\alpha_i$ and $y_i = b_i\alpha_i$ for all i. Therefore, the equivalence relation between these two sections becomes in $L_A(s_1) \times \cdots \times L_A(s_n)$ as follows:

$$\mathsf{s}_1 \sim \mathsf{s}_2 \Longleftrightarrow \exists z \in A\Phi \text{ s.t. } \begin{cases} x_1 - y_1 = 2\frac{(z,\alpha_1)}{(\alpha_1,\alpha_1)}\alpha_1 \\ \qquad\vdots \\ x_n - y_n = 2\frac{(z,\alpha_n)}{(\alpha_n,\alpha_n)}\alpha_n \end{cases}$$

$$\Longleftrightarrow \exists z \in A\Phi \text{ s.t. } \begin{cases} a_1 - b_1 = 2\frac{(z,\alpha_1)}{(\alpha_1,\alpha_1)} \\ \qquad\vdots \\ a_n - b_n = 2\frac{(z,\alpha_n)}{(\alpha_n,\alpha_n)}. \end{cases}$$

Writing $z = z_1\alpha_1 + \cdots + z_n\alpha_n$ it follows that

$$\mathsf{s}_1 \sim \mathsf{s}_2 \Longleftrightarrow \exists z \in A\Phi \text{ such that } \; C_\Phi \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix}.$$

With the assumption: $C_\Phi \in GL_n(A)$, we also have $C_\Phi^{-1} \in GL_n(A)$ and then there is no constraint for the choices of $z_1, \dots, z_n$ in A. Consequently, there is no obstruction between s_1 and s_2, which implies that $\bar{\mathsf{s}}_1 = \bar{\mathsf{s}}_2$ in $H^1(W, A\Phi)$. Hence, there is an only one class in $H^1(W, A\Phi)$, which implies the first direction.

Let us show now the other direction. Assume that $C_\Phi \notin GL_n(A)$. Since $C_\Phi \in GL_n(\mathbb{R})$ with its coefficients in $\mathbb{Z}$, the previous assumption implies in particular that $C_\Phi^{-1} \notin GL_n(A)$, and then there exists a coefficient q_{ij} of C_Φ^{-1} such that $q_{ij} \notin A$. Let s_1 be the section with corresponding point $x := (0, \dots, 1, \dots, 0) \in A\alpha_1 \times \cdots A\alpha_n$ where 1 is in position i, and s_2 be the trivial section. It turns out that $\mathsf{s}_1 \sim \mathsf{s}_2$, if and only if, there exists $z = z_1\alpha_1 + \cdots + z_n\alpha_n \in A\Phi$ such that $C_\Phi z = x - 0 = x$. Therefore, it follows that

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = C_\Phi^{-1} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} q_{i1} \\ \vdots \\ q_{ij} \\ \vdots \\ q_{in} \end{pmatrix}.$$

Since $z_i \in A$ and $q_{ij} \notin A$, the sections s_1 and s_2 define two different elements in $H^1(W, A\Phi)$. This is impossible since $H^1(W, A\Phi) = 0$. Therefore, we must have $C_\Phi \in GL_n(A)$. ◻

Proof of Theorem 1.7.4. It is well known that the determinant of C_Φ is f_Φ (see (Bourbaki, 1968) Ch. VI, § 1, exercice 7). Consequently $C_\Phi^{-1} = \frac{1}{det(C_\Phi)} D = \frac{1}{f_\Phi} D$

where $D \in M_n(\mathbb{Z})$. Therefore, because of Theorem 1.7.5, if $A = \mathbb{Z}_{f_\Phi}$ we have $C_\Phi \in GL_n(A)$ and it follows that $H^1(W, \mathbb{Z}_{f_\Phi}\Phi) = 0$.

Let us show now the direction i) implies ii). Since $H^1(W, A\Phi) = 0$, Theorem 1.7.4 tells us that $C_\Phi \in GL_n(A)$, as well as its inverse. Therefore, all the coefficients of C_Φ^{-1} are in A. However, all the coefficients of C_Φ^{-1} are of the form $\frac{d}{f_\Phi}$ with $d \in \mathbb{Z}$. We claim then that $\frac{1}{f_\Phi} \in A$. Indeed, a short observation (see (Yangjiang et Zou, 2017) together with its appendix for the exceptional cases) of the inverses of the Cartan matrices in type $A, B, C, E_6, E_7, E_8, F_4$ and G_2 shows that there are always a coefficient of the form $\frac{d}{f_\Phi}$ and another one of the form $\frac{d+1}{f_\Phi}$ in the same Cartan matrix. From this, the claim becomes obvious. In type D_n the previous statement fails but we have that $\frac{2(n-2)}{f_\Phi}$ and $\frac{n}{f_\Phi} \in A$. This implies in particular that $\frac{2}{f_\Phi} \in A$. Moreover in type D_n one has $f_\Phi = 4$. Hence $1/2 \in A$ and then $1/4$ as well. Therefore, in each situation $\mathbb{Z}[\frac{1}{f_\Phi}] \subset A$. Since A is supposed to be minimal and since $H^1(W, \mathbb{Z}_{f_\Phi}\Phi) = 0$, one has $A = \mathbb{Z}_{f_\Phi}$. □

1.7.4 Concrete realization of $H^1(W, \mathbb{Z}\Phi)$ into $\widehat{X}_{W_a}$

Because of Lemma 1.7.2 we know that the sections of π are in bijective correspondence with the points of $L(s_1) \times \cdots \times L(s_n)$. We also know that the elements of $H^1(W, \mathbb{Z}\Phi)$ are in bijection with the sections of π up to $\mathbb{Z}\Phi$-conjugacy. Furthermore, the condition of being $\mathbb{Z}\Phi$ conjugate, when considered as a condition on pairs of elements in $L(s_1) \times \cdots \times L(s_n)$, corresponds to the solvability of a system of linear equations defined by the Cartan matrix.

In this section we investigate in each Weyl group of type A, B, C, D how this equivalence relation behaves in $L(s_1) \times \cdots \times L(s_n)$. First of all, notice that when the index of connection is 1, the Cartan matrix is invertible in $\mathbb{Z}$, which implies via Theorem 1.7.4 that $H^1(W, \mathbb{Z}\Phi) = 0$. In other words, when $f_\Phi = 1$ there is no obstruction. Hence, if Φ is of type G_2, F_4 or E_8, the cohomology in degree 1 is trivial, and the only class of $H^1(W, \mathbb{Z}\Phi) = 0$ is respectively sent to $(\iota(s_1), \dots, \iota(s_n))$ with $n = 2$, 4 or 8.

Let s_1, $\mathsf{s}_2 \in Sec(\pi)$. Let $(x_1, \dots, x_n) \in \mathbb{Z}\alpha_1 \times \cdots \times \mathbb{Z}\alpha_n$ the corresponding point of s_1 and $(y_1, \dots, y_n)$ the corresponding point of s_2. Denote $x_i = a_i\alpha_i$ and $y_i = b_i\alpha_i$. We denote by $d_i(\mathsf{s}_1, \mathsf{s}_2)$, or just d_i if there is no confusion, the number $d_i := a_i - b_i$. We also define

$$d := d_1\alpha_1 + \cdots + d_n\alpha_n.$$

Type A_n

In this section $W = W(A_n)$ and $\Phi = A_n$. We denote by $C := C_{A_n}$ the Cartan matrix of A_n. The coefficients of C^{-1} are known (see (Yangjiang et Zou, 2017)) and are given by

$$(C^{-1})_{ij} = \frac{(n+1)\min(i,j) - ij}{n+1}.$$

Proposition 1.7.6. *The sections s_1 and s_2 define two different elements in $H^1(W, \mathbb{Z}\Phi)$ if and only if $\overline{d_1} + 2\overline{d_2} + \cdots + n\overline{d_n} \neq 0$ in $\mathbb{Z}/(n+1)\mathbb{Z}$.*

Proof. We know that $\mathsf{s}_1 \sim \mathsf{s}_2$ if and only if there exists $z \in \mathbb{Z}\alpha_1 \times \cdots \times \mathbb{Z}\alpha_n$ such that $Cz = d$, that is if and only if $z = C^{-1}d$. Denote $z = z_1\alpha_1 + \cdots + z_n\alpha_n$. We thus obtain the following system (S)

$$\begin{cases} (n+1)z_1 = nd_1 + (n-1)d_2 + \cdots + d_n \\ (n+1)z_2 = (n-1)d_1 + (2n-2)d_2 + \cdots + 2d_n \\ \qquad\qquad\vdots \\ (n+1)z_k = [(n+1)\min(k,1) - k]d_1 + [(n+1)\min(k,2) - 2k]d_2 + \cdots + \\ \qquad\qquad [(n+1)\min(k,n) - nk]d_n \\ \qquad\qquad\vdots \\ (n+1)z_n = d_1 + 2d_2 + \cdots + nd_n. \end{cases}$$

Therefore,if there exists a z satisfying this system, since all the z_i are integral, it must also satisfy for all $k = 1, \ldots, n$ the following condition

$$\sum_{j=1}^{n} \Big[(n+1)\min(k,j) - kj\Big] d_j \in (n+1)\mathbb{Z},$$

which is equivalent to the following equation in $\mathbb{Z}/(n+1)\mathbb{Z}$

$$\sum_{j=1}^{n} \Big[(n+1)\min(k,j) - kj\Big] \overline{d_j} = 0.$$

We claim now that if $d_1 + 2d_2 + \cdots + nd_n \in (n+1)\mathbb{Z}$ then $\sum_{j=1}^{n} \Big[(n+1)\min(k,j) - kj\Big] d_j \in (n+1)\mathbb{Z}$ for all others indexes k. Since $d_1 + 2d_2 + \cdots + nd_n \in (n+1)\mathbb{Z}$ there exists $r \in \mathbb{Z}$ such that $d_1 + 2d_2 + \cdots + nd_n = (n+1)r$, that is $d_1 = (n+1)r - 2d_2 - \cdots - nd_n$. Therefore

$$
\begin{aligned}
&\sum_{j=1}^{n}\Big[(n+1)\min(k,j)-kj\Big]d_j\\
&=(n+1-k)d_1+\sum_{j=2}^{n}\Big[(n+1)\min(k,j)-kj\Big]d_j\\
&=(n+1-k)((n+1)r-2d_2-\cdots-nd_n)+\sum_{j=2}^{n}\Big[(n+1)\min(k,j)-kj\Big]d_j\\
&=r(n+1-k)(n+1)+\sum_{j=2}^{n}\Big[(n+1)\min(k,j)-kj-j(n+1-k)\Big]d_j\\
&=r(n+1-k)(n+1)+\sum_{j=2}^{n}\Big[(n+1)(\min(k,j)-j)\Big]d_j\\
&=(n+1)\Big[r(n+1-k)+\sum_{j=2}^{n}\Big[\min(k,j)-j\Big]d_j\Big].
\end{aligned}
$$

To summarize, if there exists $z \in \mathbb{Z}\Phi$ such that (S) is satisfied then we must have $\sum_{j=1}^{n}\Big[(n+1)\min(k,j)-kj\Big]\overline{d_j}=0$ for all $k=1,\dots,n$, which is equivalent to the equation $\overline{d_1}+2\overline{d_2}+\cdots+n\overline{d_n}=0$. Therefore, if $\overline{d_1}+2\overline{d_2}+\cdots+n\overline{d_n}\neq 0$ there doesn't exist $z\in\mathbb{Z}\Phi$ such that $Cz=d$, whence s_1 and s_2 define two different classes in $H^1(W,\mathbb{Z}\Phi)$.

Conversely, if s_1 and s_2 define two different elements in $H^1(W,\mathbb{Z}\Phi)$ we don't have $z\in\mathbb{Z}\Phi$ satisfying (S). However, if $\overline{d_1}+2\overline{d_2}+\cdots+n\overline{d_n}=0$ it follows that $\sum_{j=1}^{n}\Big[(n+1)\min(k,j)-kj\Big]\overline{d_j}=0$ for all $k=1,\dots,n$. Thus, by setting $z_k := \frac{1}{n+1}\Big[\sum_{j=1}^{n}\Big[(n+1)\min(k,j)-kj\Big]d_j\Big]$ for all $k=1,\dots,n$ we have built an element in $\mathbb{Z}\Phi$ such that $Cz=d$, i.e. s_1 and s_2 are equivalent, which is impossible according to our assumption. $\square$

Corollary 1.7.7. *Assume that $n+1$ is a prime number. Then for each pair of sections s_1 and s_2 that satisfy $d_i(\mathsf{s}_1,\mathsf{s}_2)=d_j(\mathsf{s}_1,\mathsf{s}_2)$ for all i,j, we have $\bar{\mathsf{s}}_1=\bar{\mathsf{s}}_2$.*

Proof. Let s_1 and s_2 be two sections satisfying $d_i(\mathsf{s}_1,\mathsf{s}_2)=d_j(\mathsf{s}_1,\mathsf{s}_2)$ for all i,j. Since $n+1$ is a prime number, the polynomial $\overline{X}+2\overline{X}+\cdots+n\overline{X}$ admits all the elements of $\mathbb{Z}/(n+1)\mathbb{Z}$ as solutions. In particular, the equality $\overline{d_1}+2\overline{d_2}+\cdots+n\overline{d_n} = 0$ is true. Therefore, because of Proposition 1.7.6 these two sections define the same element in $H^1(W(A_n), \mathbb{Z}\Phi)$ and then $\bar{\mathsf{s}}_1 = \bar{\mathsf{s}}_2$. $\square$

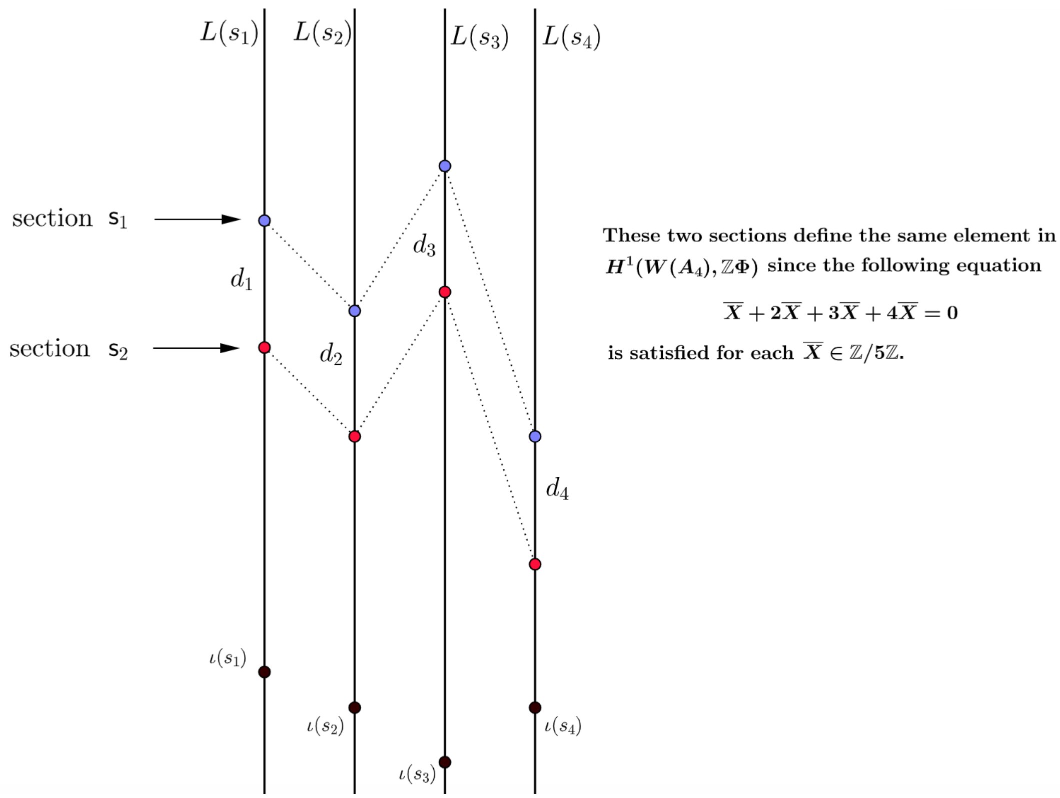

Figure 1.26 Two sections giving the same class in cohomology.

Remark 5. Note that if $n+1$ is not prime, corollary 7.7 does not necessarily hold. Indeed, assume that $n = 3$ and let us take s_1 that we identify with $(3, 4, 5)$ and s_2 with $(6, 7, 8)$. We have here $d_i(\mathsf{s}_1, \mathsf{s}_2) = 3$ for $i = 1, 2, 3$. Moreover $\overline{d_1} + 2\overline{d_2} + 3\overline{d_3} = \overline{2} \neq 0$. Hence $\overline{\mathsf{s}_1} \neq \overline{\mathsf{s}_2}$.

Type B_n

In this section $W = W(B_n)$ and $\Phi = B_n$. We denote by $C := C_{B_n}$ the Cartan matrix of B_n. The coefficients of C^{-1} are known (see (Yangjiang et Zou, 2017)) and are given by

$$(C^{-1})_{ij} = \frac{\min(i,j)}{1-\min(0,n-i-1)} = \begin{cases} \min(i,j) & \text{if } i < n \\ \frac{j}{2} & \text{if } i = n \end{cases} \quad 1 \leq i,j \leq n.$$

Proposition 1.7.8. *Let us write* $I_n := \{k \in [\![1,n]\!] \mid k \text{ is odd}\}$ *and* $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. *The sections* s_1 *and* s_2 *define two different elements in* $H^1(W, \mathbb{Z}\Phi)$ *if and only if* $\sum_{k \in I_n} \overline{d_k} = 1$ *in* $\mathbb{F}_2$.

Proof. The proof is almost the same as the A_n case above. The corresponding system (S) is in this situation as follows

$$\begin{cases} 2z_1 = 2d_1 + 2d_2 + \cdots + 2d_n \\ \qquad\vdots \\ 2z_k = 2\min(k,1)d_1 + 2\min(k,2)d_2 + \cdots + 2\min(k,n)d_n \\ \qquad\vdots \\ 2z_n = d_1 + 2d_2 + \cdots + nd_n. \end{cases}$$

It is obvious that $2\min(k,1)d_1 + \cdots + 2\min(k,n)d_n \in 2\mathbb{Z}$ for all $k = 1, \ldots, n-1$. Therefore, the only equation that matters in this system is the last one: $2z_n = d_1 + 2d_2 + \cdots + nd_n \in 2\mathbb{Z}$. This equation transferred in $\mathbb{F}_2$ becomes $\sum_{k \in I_n} k\overline{d_k} = 0$ and then $\sum_{k \in I_n} \overline{d_k} = 0$. The result follows. □

Type C_n

In this section $W = W(C_n)$ and $\Phi = C_n$. We denote by $C := C_{C_n}$ the Cartan matrix of C_n. The coefficients of C^{-1} are known (see (Yangjiang et Zou, 2017)) and are given by

$$(C^{-1})_{ij} = \frac{\min(i,j)}{1-\min(0,n-j-1)} = \begin{cases} \min(i,j) & \text{if } j < n \\ \frac{i}{2} & \text{if } j = n \end{cases} \quad 1 \leq i,j \leq n.$$

Proposition 1.7.9. *The sections* s_1 *and* s_2 *define two different elements in* $H^1(W, \mathbb{Z}\Phi)$ *if and only if* $\overline{d_n} = 1$ *in* $\mathbb{F}_2$*.*

Proof. The corresponding system (S) is in this situation as follows

$$\begin{cases} 2z_1 = 2d_1 + 2d_2 + \cdots + 2d_{n-1} + d_n \\ \qquad\qquad \vdots \\ 2z_k = 2\min(k,1)d_1 + 2\min(k,2)d_2 + \cdots + 2\min(k,n-1)d_{n-1} + kd_n \\ \qquad\qquad \vdots \\ 2z_n = 2d_1 + 4d_2 + \cdots + 2(n-1)d_{n-1} + nd_n. \end{cases}$$

Consequently, we must have in particular that $2d_1 + 2d_2 + \cdots + 2d_{n-1} + d_n \in 2\mathbb{Z}$, that is $d_n \in 2\mathbb{Z}$. Therefore, when $d_n \in 2\mathbb{Z}$ the system (S) is always satisfied, and it follows that $\bar{\mathsf{s}}_1 = \bar{\mathsf{s}}_2$. Since the converse direction is obvious, the result follows. $\square$

Type D_n

In this section $W = W(D_n)$ and $\Phi = D_n$. We denote by $C := C_{D_n}$ the Cartan matrix of D_n. It turns out that C is symmetric, this is why we just give the coefficients of C^{-1} with entries $1 \leq i \leq j \leq n$

$$(C^{-1})_{ij} = \begin{cases} i & \text{if } 1 \leq i \leq j \leq n-2 \\ \frac{i}{2} & \text{if } i < n-1, j = n-1 \text{ or } n \\ \frac{n-2}{2} & \text{if } i = n-1, j = n \\ \frac{n}{4} & \text{if } i = j = n-1 \text{ or } n. \end{cases}$$

A better visualization of this matrix is given by:

$$C^{-1} = \frac{1}{4}\left(\begin{array}{ccccc|cc} 4 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 4 & 8 & 8 & \cdots & 8 & 4 & 4 \\ 4 & 8 & 12 & \cdots & 12 & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 4 & 8 & 12 & \cdots & 4(n-2) & 2(n-2) & 2(n-2) \\ \hline 2 & 4 & 6 & \cdots & 2(n-2) & n & 2(n-2) \\ 2 & 4 & 6 & \cdots & 2(n-2) & 2(n-2) & n \end{array}\right).$$

Proposition 1.7.10. *Let us write $I_{n-2} := \{k \in [\![1, n-2]\!] \mid k \text{ is odd}\}$. The sections s_1 and s_2 define two different elements in $H^1(W, \mathbb{Z}\Phi)$ if and only if we have the following points*

i) $\overline{d_{n-1}} \neq \overline{d_n}$ *in* $\mathbb{F}_2$.

ii) $\sum_{k \in I_{n-2}} 2\overline{d_k} + n\overline{d_{n-1}} + (n-2)\overline{d_n} \neq 0$ *in* $\mathbb{Z}/4\mathbb{Z}$.

iii) $\sum_{k \in I_{n-2}} 2\overline{d_k} + (n-2)\overline{d_{n-1}} + n\overline{d_n} \neq 0$ *in* $\mathbb{Z}/4\mathbb{Z}$.

Proof. The first line of the corresponding system (S) is given by:

$$4z_1 = 4d_1 + 4d_2 + \cdots + 4d_{n-2} + 2d_{n-1} + 2d_n.$$

Thus, we must have $4d_1+4d_2+\cdots+4d_{n-2}+2d_{n-1}+2d_n \in 4\mathbb{Z}$, that is $2d_{n-1}+2d_n \in 4\mathbb{Z}$, which is equivalent to $\overline{d_{n-1}} = \overline{d_n}$ in $\mathbb{F}_2$. We see then that if $\overline{d_{n-1}} = \overline{d_n}$, the $(n-2)$ first equations of (S) can be satisfied.

The penultimate equation of (S) is

$$4z_{n-1} = \sum_{k=1}^{n-1} 2kd_k + nd_{n-1} + (n-2)d_n.$$

Therefore, this previous equality compels us to have $\sum_{k=1}^{n-1} 2kd_k + nd_{n-1} + (n-2)d_n \in 4\mathbb{Z}$, that is $\sum_{k=1}^{n-2} 2k\overline{d_k} + n\overline{d_{n-1}} + (n-2)\overline{d_n} = 0$ in $\mathbb{Z}/4\mathbb{Z}$. If k is even $2k\overline{d_k} = 0$, and if k is odd $2k\overline{d_k} = 2\overline{d_k}$. Hence we have $\sum_{k\in I_{n-2}} 2\overline{d_k} + n\overline{d_{n-1}} + (n-2)\overline{d_n} = 0$.

The reasoning is exactly the same for the last equation. This concludes the proof.

□

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CHAPTER II

SECOND ARTICLE

A symmetric group action on the irreducible components of the Shi variety associated to $W(\widetilde{A_n})$

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ABSTRACT

Let W_a be an affine Weyl group with corresponding finite root system Φ. In (Shi, 1987a) Jian-Yi Shi characterized each element $w \in W_a$ by a Φ^+-tuple of integers $(k(w,\alpha))_{\alpha\in\Phi^+}$ subject to certain conditions. In Chapter 1 a new interpretation of the coefficients $k(w,\alpha)$ is given. This description led us to define an affine variety $\widehat{X}_{W_a}$, called the Shi variety of W_a, whose integral points are in bijection with W_a. It turns out that this variety has more than one irreducible component, and the set of these components, denoted $H^0(\widehat{X}_{W_a})$, admits many interesting properties. In particular the group W_a acts on it. In this article we show that the irreducible components of $\widehat{X}_{W(\widetilde{A}_n)}$ are in bijection with the circular permutations of $W(A_n) = S_{n+1}$. We also compute the action of $W(A_n)$ on $H^0(\widehat{X}_{W(\widetilde{A}_n)})$.

2.1 Introduction

Let $(V, (-,-))$ be a Euclidean space. Let Φ be an irreducible crystallographic root system in V with simple system Δ. Let $m = |\Phi^+|$. From now on, when we will say "root system" it will always mean irreducible crystallographic root system.

Let W be the *Weyl group* associated to $\mathbb{Z}\Phi$, that is the maximal (for inclusion) reflection subgroup of $O(V)$ admitting $\mathbb{Z}\Phi$ as a W-equivariant lattice. We identify $\mathbb{Z}\Phi$ and the group of its associated translations and we denote by τ_x the translation corresponding to $x \in \mathbb{Z}\Phi$. Let $k \in \mathbb{Z}$ and $\alpha \in \Phi$. Define the affine reflection as follows

$$\begin{array}{rccc} s_{\alpha,k} & : & V & \longrightarrow & V \\ & & x & \longmapsto & x - (2\frac{(\alpha,x)}{(\alpha,\alpha)} - k)\alpha. \end{array}$$

We consider the subgroup W_a of $\mathrm{Aff}(V)$ generated by all affine reflections $s_{\alpha,k}$ with $\alpha \in \Phi$ and $k \in \mathbb{Z}$, that is

$$W_a = \langle s_{\alpha,k} \mid \alpha \in \Phi,\ k \in \mathbb{Z} \rangle.$$

The group W_a is called the *affine Weyl group* associated to Φ. The *inversion set* of $w \in W$ is by definition the set

$$N(w) := \{\xi \in \Phi^+ \mid w^{-1}(\xi) \in \Phi^-\}.$$

Moreover it is a well known fact that if $w = uv$ is a reduced expression of w then the inversion set of w decomposes as

$$N(w) = N(u) \sqcup uN(v). \tag{2.1}$$

It is also well known that $W_a = \mathbb{Z}\Phi \rtimes W$. Therefore, any element $w \in W_a$ decomposes as $w = \tau_x \overline{w}$ where $x \in \mathbb{Z}\Phi$ and $\overline{w} \in W$. The element $\overline{w}$ is called the *finite part* of w.

Let $\alpha \in \Phi$ and $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. For any $k \in \mathbb{Z}$ and any $m \in \mathbb{R}$, we set the hyperplanes

$$H_{\alpha,k} = \{x \in V \mid s_{\alpha,k}(x) = x\} = \{x \in V \mid (x, \alpha^\vee) = k\},$$

the half spaces

$$H^+_{\alpha,k} = \{x \in V \mid k < (x,\alpha^\vee)\} \quad \text{and} \quad H^-_{\alpha,k} = \{x \in V \mid (x,\alpha^\vee) < k\},$$

and the strips

$$\begin{aligned}H^m_{\alpha,k} &= \{x \in V \mid k < (x, \alpha^\vee) < k+m\}\\ &= H^+_{\alpha,k} \cap H^-_{\alpha,k+m}.\end{aligned}$$

The connected components of $V \setminus \bigcup_{\substack{\alpha\in\Phi^+\\ k\in\mathbb{Z}}} H_{\alpha,k}$ are called *alcoves*. We denote A_e the alcove defined as $A_e = \bigcap_{\alpha\in\Phi^+} H^1_{\alpha,0}$. It is well known that W_a acts regularly on the set of alcoves. It follows that there is a bijective correspondence between the elements of W_a and all the alcoves. This bijection is defined by $w \mapsto A_w$ where $A_w := wA_e$. We call A_w the corresponding alcove associated to $w \in W_a$. Any alcove of V can be written as an intersection of width-one strips, that is there exists a Φ^+-tuple of integers $(k(w, \alpha))_{\alpha\in\Phi^+}$ such that

$$A_w = \bigcap_{\alpha\in\Phi^+} H^1_{\alpha,k(w,\alpha)}.$$

In (Shi, 1987a) Jian-Yi Shi characterized the elements of any affine Weyl group $w \in W_a$ by the Φ^+-tuple of integers $(k(w, \alpha))_{\alpha\in\Phi^+}$ subject to certain conditions. Let $P_{\mathcal{H}}$ be the polytope:

$$P_{\mathcal{H}} := \bigcap_{\alpha\in\Delta} H^1_{\alpha,0},$$

and let $A_w \subset P_{\mathcal{H}}$. It is clear that $k(w, \alpha) = 0$ for all $\alpha \in \Delta$, and reciprocally, if $w' \in W_a$ is such that $k(w', \alpha) = 0$ for all $\alpha \in \Delta$ then $A_{w'} \subset P_{\mathcal{H}}$. The elements of this polytope seen as Φ^+-tuple of integers are called *admitted*, and more precisely a vector $\lambda \in \bigoplus_{\alpha\in\Phi^+} \mathbb{Z}\alpha$ is admitted if and only if there exists $w \in W_a$ such that $k(w, \alpha) = \lambda_\alpha$ for all $\alpha \in \Phi^+$ and such that $A_w \subset P_{\mathcal{H}}$.

Example 16. In type A_n, an admitted vector $\lambda = (\lambda_{i,j})_{1\le i<j\le n+1}$ is defined by the following conditions:

$$\begin{cases} \lambda_{i,j} + \lambda_{j,k} \le \lambda_{i,k} \le \lambda_{i,j} + \lambda_{j,k} + 1 & \text{for all } i < j < k,\\ \lambda_{i,i+1} = 0 \quad \text{for all } 1 \le i < n. \end{cases} \tag{2.2}$$

In Chapter 1 we obtained an affine variety $\widehat{X}_{W_a}$, called *the Shi variety* of W_a, whose integral points are in bijection with W_a. It turns out that the irreducible components of this variety are in bijection with the alcoves of the polytope $P_{\mathcal{H}}$. Writing $H^0(\widehat{X}_{W(\widetilde{A}_n)})$ to be the set of irreducible components of $\widehat{X}_{W_a}$, we have $H^0(\widehat{X}_{W(\widetilde{A}_n)}) = \{\widehat{X}_{W_a}[\lambda] \mid \lambda \text{ admitted}\}$ (see Section 1.5.2 for more details about the irreducible components $\widehat{X}_{W_a}[\lambda]$). From this bijection it follows a natural decomposition of $\widehat{X}_{W_a}$ as

$$\widehat{X}_{W_a} = \bigsqcup_{\lambda \text{ admitted}} X_{W_a}[\lambda].$$

Let $s_{\alpha,p} \in W_a$. In Chapter 1 we defined the affine map $F(s_{\alpha,p})$ as $F(s_{\alpha,p})(x) := L_\alpha(x) + v_{p,\alpha}$ for all $x \in \bigoplus_{\beta\in\Phi^+} \mathbb{R}\beta$, and with $L_\alpha \in GL_n(\mathbb{R})$ defined via the matrix $(\ell_{i,j}(\alpha))_{i,j\in[\![1,m]\!]}$ where

$$\ell_{\alpha_j,\alpha_i}(\alpha) := \ell_{j,i}(\alpha) = \begin{cases} 1 & \text{if } \ s_\alpha(\alpha_i) = \alpha_j \\ 0 & \text{if } \ s_\alpha(\alpha_i) \neq \pm\alpha_j \\ -1 & \text{if } \ s_\alpha(\alpha_i) = -\alpha_j, \end{cases} \tag{2.3}$$

and with $v_{p,\alpha} \in \bigoplus_{\beta\in\Phi^+} \mathbb{R}\beta$ the vector defined by $v_{p,\alpha} = (v_{p,\alpha}(\gamma))_{\gamma\in\Phi^+}$ where

$$v_{p,\alpha}(\gamma) := \begin{cases} -p(\alpha, s_\alpha(\gamma)^\vee) & \text{if } \ s_\alpha(\gamma) \in \Phi^+ \\ -1 - p(\alpha, s_\alpha(\gamma)^\vee) & \text{if } \ s_\alpha(\gamma) \in \Phi^-. \end{cases} \tag{2.4}$$

For $w \in W_a$ we denote L_w to be the left multiplication by w. In Chapter 1 we showed that F extends naturally to W_a. We also showed that F induces a geometrical action on the irreducible components. Those results are stated as follows:

Theorem 2.1.1 (Theorem 1.4.1 of Chapter 1)**.** *There exists an injective morphism $F : W_a \to Isom(\mathbb{R}^m)$ such that for any $w \in W_a$ the following diagram commutes. This morphism is called the Φ^+-representation of W_a, and the cor-*

responding action is called the Φ^+*-action of* W_a*.*

$$\begin{tikzcd} W_a \arrow[r, "L_w"] \arrow[d, hook, "\iota"'] & W_a \arrow[d, hook, "\iota"'] \\ \mathbb{R}^m \arrow[r, "F(w)"'] & \mathbb{R}^m. \end{tikzcd}$$

Proposition 2.1.2 (Proposition 1.5.8 of Chapter 1)**.** *Let* $F : W_a \hookrightarrow Isom(\mathbb{R}^n)$ *be the* Φ^+*-representation of* W_a*. Then*

1) W_a *acts naturally on the irreducible components of* $\widehat{X}_{W_a}$ *via the action defined as* $w \diamond X_{W_a}[\lambda] := F(w)(X_{W_a}[\lambda])$*. Furthermore if we assume that* $w \in W_a$ *decomposes as* $w = \tau_x \overline{w}$*, then* $w \diamond X_{W_a}[\lambda] = \overline{w} \diamond X_{W_a}[\lambda]$*. Finally this action is transitive.*

2) *The previous action induces an action on the admitted vectors by* $w \diamond \lambda := \gamma$ *such that* $w \diamond X_{W_a}[\lambda] = X_{W_a}[\gamma]$*. In other words we have* $w \diamond X[\lambda] = X[w \diamond \lambda]$*.*

In this article we thoroughly investigate $H^0(\widehat{X}_{W(\widetilde{A}_n)})$ from a combinatorial point of view. The main result of this article is Theorem 2.1.3. Let us first recall some basics about the symmetric group:

A circular permutation is an element of $\sigma \in S_{n+1} = W(A_n)$ such that σ can be written as $\sigma = (a_1\ a_2\ \cdots\ a_{n+1})$ with $a_i \in [\![1, n+1]\!]$, $a_i \neq a_j$ if $i \neq j$, $\sigma(a_i) = a_{i+1}$ for all $1 \leq i \leq n$, and $\sigma(a_{n+1}) = a_1$. One might also call it a $(n+1)$-cycle.

The action by conjugation of $W(A_n)$ on itself is defined for all $\sigma, \gamma \in W(A_n)$ by $\sigma.\gamma := \sigma\gamma\sigma^{-1}$. This action appears in a lot of areas and has been studied many times. In particular, understanding the orbits of the action, which are the conjugacy classes, yielded a lot of research work. For example, M.Geck and G.Pfeiffer used in (Geck et Pfeiffer, 2000) the technology of cuspidal class in order to express any conjugacy class in terms of these cuspidal classes.

We relate in this article the conjugacy class of $(1\ 2\ \cdots\ n+1)$ with the irreducible components of the Shi variety corresponding to $W(\widetilde{A}_n)$. Finally, the action by conjugation plays a crucial role in the following theorem:

Theorem 2.1.3. *There is a natural bijection between $H^0(\widehat{X}_{W(\widetilde{A}_n)})$ and the circular permutations of $W(A_n)$. In particular $|H^0(\widehat{X}_{W(\widetilde{A}_n)})| = n!$.*

In Theorem 2.1.3 the word "natural" is used because the bijection involved respects the action of $W(A_n)$ on two different sets, namely the action we have defined on $H^0(\widehat{X}_{W(\widetilde{A}_n)})$ and the conjugation action on the circular permutations in $W(A_n)$.

Because of Proposition 1.5.8 we know that the components are invariant under translations, that is only the finite part matters. More precisely, we will only look at the action of the finite part $W(A_n)$. Thus, the goal is to understand how the components behave when we apply $F(w)$ on them for any $w \in W(A_n)$.

Giving explicit formulas for the action of $W(A_n)$ on $H^0(\widehat{X}_{W(\widetilde{A}_n)})$ is of particular interest as well. These formulas are established in Proposition 2.3.2.

2.2 Bijection between $H^0(\widehat{X}_{W(\widetilde{A}_n)})$ and the set of circular permutations of $W(A_n)$

Let E be a $\mathbb{R}$-vector space and let G be a group that acts linearly on E. This action induces a linear action on the rth exterior power $\bigwedge^r(E)$ for all $r \in \mathbb{N}$. We call this action *the diagonal action* on $\bigwedge^r(E)$.

For instance, if $G = W(A_n)$ is the symmetric group and $E = \mathrm{span}(e_1, \cdots, e_{n+1})$ then G acts on E by permutation of the coordinates, that is $\sigma.e_i = e_{\sigma(i)}$. Then the diagonal action of G on $\bigwedge^2(E)$ is given by

$$\sigma \cdot (e_k \wedge e_\ell) = e_{\sigma(k)} \wedge e_{\sigma(\ell)}.$$

Let $\{e_1, \cdots, e_{n+1}\}$ be the canonical basis of $K := \mathbb{R}^{n+1}$. A way to represent the root system A_n in $\mathbb{R}^{n+1}$ is by $\Phi := \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\}$ with $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\}$ and with simple system $\Delta = \{e_i - e_{i+1} \mid 1 \leq i < j \leq n\}$. We recall that $m = |\Phi^+|$ and we set Y to be the $\mathbb{R}$-vector space with basis $e_{i,j}$ for $1 \leq i < j \leq n$. The elements $e_{i,j}$ are in obvious bijection with the positive roots $e_i - e_j$. We denote by $s_{k,\ell}$ the reflection $s_{e_k - e_\ell, 0}$, namely $s_{k,\ell}$ is the transposition of S_{n+1} that swaps k and ℓ. Finally, we will denote $L_{k,\ell} := L_{e_k - e_\ell}$.

Definition 15. We define the *affine diagonal action* of $W(A_n)$ on $\bigwedge^2(K)$ as follows. Write $W(A_n) = \langle s_1, \cdots, s_n \rangle$ where s_i is the adjacent transposition $(i, i+1)$. We define the operation $\odot$ on the generators s_i and on the basis of $\bigwedge^2(K)$, where for all $k < \ell$ we have

$$s_i \odot e_k \wedge e_\ell := (i, i+1) \cdot (e_k \wedge e_\ell) - e_i \wedge e_{i+1} = e_{s_i(k)} \wedge e_{s_i(\ell)} - e_i \wedge e_{i+1}, \quad (2.5)$$

and we extend it as follows

$$s_i \odot (\sum_{r<s} x_{r,s} e_r \wedge e_s) = (\sum_{r<s} x_{r,s} e_{s_i(r)} \wedge e_{s_i(s)}) - e_i \wedge e_{i+1}. \quad (2.6)$$

Proposition 2.2.1. *The operation $\odot : W(A_n) \times \bigwedge^2(K) \to \bigwedge^2(K)$ is an action.*

Proof. In order to prove this statement we just need to check that the relations of $W(A_n)$ are preserved under the operation $\odot$. It turns out that this is the case because

$$\begin{aligned}
&s_i \odot (s_{i+1} \odot (s_i \odot (e_k \wedge e_\ell))) \\
&= s_i \odot (s_{i+1} \odot (e_{s_i(k)} \wedge e_{s_i(\ell)} - e_i \wedge e_{i+1})) \\
&= s_i \odot (e_{s_{i+1}s_i(k)} \wedge e_{s_{i+1}s_i(\ell)} - e_{s_{i+1}(i)} \wedge e_{s_{i+1}(i+1)} - e_{i+1} \wedge e_{i+2}) \\
&= s_i \odot (e_{s_{i+1}s_i(k)} \wedge e_{s_{i+1}s_i(\ell)} - e_i \wedge e_{i+2} - e_{i+1} \wedge e_{i+2}) \\
&= e_{s_i s_{i+1} s_i(k)} \wedge e_{s_i s_{i+1} s_i(\ell)} - e_{s_i(i)} \wedge e_{s_i(i+2)} - e_{s_i(i+1)} \wedge e_{s_i(i+2)} - e_i \wedge e_{i+1} \\
&= e_{s_i s_{i+1} s_i(k)} \wedge e_{s_i s_{i+1} s_i(\ell)} - e_{i+1} \wedge e_{i+2} - e_i \wedge e_{i+2} - e_i \wedge e_{i+1}.
\end{aligned}$$

$$
\begin{aligned}
&s_{i+1} \odot (s_i \odot (s_{i+1} \odot (e_k \wedge e_\ell))) \\
&= s_{i+1} \odot (s_i \odot (e_{s_{i+1}(k)} \wedge e_{s_{i+1}(\ell)} - e_{i+1} \wedge e_{i+2})) \\
&= s_{i+1} \odot (e_{s_i s_{i+1}(k)} \wedge e_{s_i s_{i+1}(\ell)} - e_{s_i(i+1)} \wedge e_{s_i(i+2)} - e_i \wedge e_{i+1}) \\
&= s_{i+1} \odot (e_{s_i s_{i+1}(k)} \wedge e_{s_i s_{i+1}(\ell)} - e_i \wedge e_{i+2} - e_i \wedge e_{i+1}) \\
&= e_{s_{i+1} s_i s_{i+1}(k)} \wedge e_{s_{i+1} s_i s_{i+1}(\ell)} - e_{s_{i+1}(i)} \wedge e_{s_{i+1}(i+2)} - e_{s_{i+1}(i)} \wedge e_{s_{i+1}(i+1)} - e_{i+1} \wedge e_{i+2} \\
&= e_{s_i s_{i+1} s_i(k)} \wedge e_{s_i s_{i+1} s_i(\ell)} - e_i \wedge e_{i+1} - e_i \wedge e_{i+2} - e_{i+1} \wedge e_{i+2}.
\end{aligned}
$$

Thus the mesh relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ is preserved under this action. For $|i-j| \geq 2$ we know that $s_i s_j = s_j s_i$. Let us show that this relation is also preserved. This is the case because

$$
\begin{aligned}
s_i \odot (s_j \odot (e_k \wedge e_\ell)) &= s_i \odot (e_{s_j(k)} \wedge e_{s_j(\ell)} - e_j \wedge e_{j+1}) \\
&= e_{s_i s_j(k)} \wedge e_{s_i s_j(\ell)} - e_{s_i(j)} \wedge e_{s_i(j+1)} - e_i \wedge e_{i+1} \\
&= e_{s_i s_j(k)} \wedge e_{s_i s_j(\ell)} - e_j \wedge e_{j+1} - e_i \wedge e_{i+1},
\end{aligned}
$$

$$
\begin{aligned}
s_j \odot (s_i \odot (e_k \wedge e_\ell)) &= s_j \odot (e_{s_i(k)} \wedge e_{s_i(\ell)} - e_i \wedge e_{i+1}) \\
&= e_{s_j s_i(k)} \wedge e_{s_j s_i(\ell)} - e_{s_j(i)} \wedge e_{s_j(i+1)} - e_j \wedge e_{j+1} \\
&= e_{s_j s_i(k)} \wedge e_{s_j s_i(\ell)} - e_i \wedge e_{i+1} - e_j \wedge e_{j+1}.
\end{aligned}
$$

There is one type of relation left to check: those coming from the involutions.

$$
\begin{aligned}
s_i \odot (s_i \odot e_k \wedge e_\ell) &= s_i \odot (e_{s_i(k)} \wedge e_{s_i(\ell)} - e_i \wedge e_{i+1}) \\
&= e_{s_i s_i(k)} \wedge e_{s_i s_i(\ell)} - e_{s_i(i)} \wedge e_{s_i(i+1)} - e_i \wedge e_{i+1} \\
&= e_k \wedge e_\ell + e_i \wedge e_{i+1} - e_i \wedge e_{i+1} \\
&= e_k \wedge e_\ell.
\end{aligned}
$$

□

Remark 6. Notice that the action $\odot$ in Proposition 2.2.1 is not linear.

Definition 16. We define Θ from Y to $\bigwedge^2(K)$ to be the linear map sending the elements $e_{i,j}$ of the basis of Y to the elements $e_i \wedge e_j$ of the basis of $\bigwedge^2(K)$.

Proposition 2.2.2. *The map Θ is an isomorphism from the Φ^+-action of $W(A_n)$ on Y to the affine diagonal action of $W(A_n)$ on $\bigwedge^2(K)$.*

Proof. We need to show that the following diagram commutes for all $w \in W(A_n)$

$$\begin{tikzcd} Y \arrow[r, "F(w)"] \arrow[d, "\Theta"', "\wr"] & Y \arrow[d, "\Theta", "\wr"'] \\ \bigwedge^2(K) \arrow[r, "\varphi_w"'] & \bigwedge^2(K) \end{tikzcd}$$

where $\varphi_w(e_i \wedge e_j) := w \odot e_i \wedge e_j$. However, it is enough to only show the commutativity of this diagram for the generators $s_i \in S$. First of all, Θ is an isomorphism because the set $\{e_{i,j} \mid 1 \leq i < j \leq n+1\}$ is a basis of Y, and $\{e_i \wedge e_j \mid 1 \leq i < j \leq n+1\}$ is a basis of $\bigwedge^2(K)$. Let $1 \leq k < \ell \leq n+1$. With respect to the commutativity we have

$$\begin{aligned} \varphi_{s_i} \circ \Theta(e_{k,\ell}) &= \varphi_{s_i}(e_k \wedge e_\ell) \\ &= e_{s_i(k)} \wedge e_{s_i(\ell)} - e_i \wedge e_{i+1}. \end{aligned}$$

Moreover, it follows from (2.3) and (2.4) that $F(s_i)(e_{k,\ell}) = e_{s_i(k),s_i(\ell)} - e_{i,i+1}$. Therefore

$$\begin{aligned} \Theta \circ F(s_i)(e_{k,\ell}) &= \Theta(e_{s_i(k),s_i(\ell)} - e_{i,i+1}) \\ &= \Theta(e_{s_i(k),s_i(\ell)}) - \Theta(e_{i,i+1}) \\ &= e_{s_i(k)} \wedge e_{s_i(\ell)} - e_i \wedge e_{i+1}. \end{aligned}$$

$\square$

Corollary 2.2.3. *Let $w \in W(A_n)$ and $1 \leq k < \ell \leq n+1$. Then we have*

$$w \odot e_k \wedge e_\ell = e_{w(k)} \wedge e_{w(\ell)} - \sum_{e_r - e_s \in N(w)} e_r \wedge e_s.$$

Proof. We proceed by induction on the length. The base case follows from (2.5). Let $w := s_{i_1}s_{i_2}\cdots s_{i_p} := s_{i_1}w'$ be a reduced expression of w. Let us assume that $w' \odot e_k \wedge e_\ell = e_{w'(k)} \wedge e_{w'(\ell)} - \sum_{e_r - e_s \in N(w')} e_r \wedge e_s$. Because of (2.1) we know that

$$\begin{aligned} N(w) &= N(s_{i_1}) \sqcup s_{i_1}N(w') \\ &= \{e_{i_1} - e_{i_1+1}\} \sqcup s_{i_1}N(w'). \end{aligned} \tag{2.7}$$

Therefore, it follows that

$$\begin{aligned} s_{i_1}w' \odot e_k \wedge e_\ell &= s_{i_1} \odot (w' \odot e_k \wedge e_\ell) \\ &= s_{i_1} \odot (e_{w'(k)} \wedge e_{w'(\ell)} - \sum_{e_r - e_s \in N(w')} e_r \wedge e_s) \\ &\overset{(2.6)}{=} e_{s_{i_1}w'(k)} \wedge e_{s_{i_1}w'(\ell)} - \sum_{e_r - e_s \in N(w')} e_{s_{i_1}(r)} \wedge e_{s_{i_1}(s)} - e_{i_1} \wedge e_{i_1+1} \\ &\overset{(2.7)}{=} e_{w(k)} \wedge e_{w(\ell)} - \sum_{e_r - e_s \in N(w)} e_r \wedge e_s. \end{aligned}$$

□

Lemma 2.2.4. $(1\ 2\ \cdots\ n+1) \diamond X_{W(\widetilde{A}_n)}[0] = X_{W(\widetilde{A}_n)}[0]$.

Proof. Let us write $E_{i,j}$ for the equation $X_{i,j} - \sum_{r=i}^{j-1} X_{r,r+1} = 0$ for $1 \le i < j \le n+1$, and $E'_{i,j}$ for the equation $X_{1,j} - X_{1,i+1} - \sum_{r=i+1}^{n} X_{r,r+1} = 0$ for $1 < i < j \le n+1$. We know that $X_{W(\widetilde{A}_n)}[0]$ is the component of $\widehat{X}_{W(\widetilde{A}_n)}$ cut out by the equations $E_{i,j}$. We analyze the action of $(1\ 2\ \cdots\ n+1)$ on $X_{W(\widetilde{A}_n)}[0]$ by studying the action of $(1\ 2\ \cdots\ n+1)$ on the equations $E_{i,j}$, with the relations $X_{i,j} = -X_{j,i}$ for $i < j$. The equations that cut out $(1\ 2\ \cdots\ n+1) \diamond X_{W(\widetilde{A}_n)}[0]$ are exactly the $(1\ 2\ \cdots\ n+1) \cdot E_{i,j}$, and a short computation shows that

$$(1\ 2\ \cdots\ n+1) \cdot E_{i,j} = \begin{cases} E_{i+1,j+1} & \text{if } j \neq n+1 \\ E_{1,n+1} & \text{if } i = 1,\ j = n+1 \\ E'_{i,j} & \text{if } 1 < i,\ j = n+1. \end{cases} \tag{2.8}$$

Thus, some of the equations that determine the component $X_{W(\widetilde{A}_n)}[0]$ are just swapped, namely the $E_{i,j}$ with $j < n+1$, and $E_{1,n+1}$. The other ones are sent to some equations that are not in the set of equations cutting out $X_{W(\widetilde{A}_n)}[0]$.

However $(1\ 2\ \cdots\ n+1)$ acts as an homeomorphism on Y. Consequently, the set $(1\ 2\ \cdots\ n+1) \diamond X_{W(\widetilde{A}_n)}[0]$ must be an irreducible component of $X_{W(\widetilde{A}_n)}$. Thanks to (2.8) we see that $0 \in (1\ 2\ \cdots\ n+1) \diamond X_{W(\widetilde{A}_n)}[0]$. Since there is an only one component in $\widehat{X}_{W(\widetilde{A}_n)}$ that contains 0, namely $X_{W(\widetilde{A}_n)}[0]$, it follows that

$$(1\ 2\ \cdots\ n+1) \diamond X_{W(\widetilde{A}_n)}[0] = X_{W(\widetilde{A}_n)}[0].$$

□

Proof of Theorem 2.1.3. It is well known that in the symmetric group $W(A_n)$ the stabilizer of any circular permutation for the conjugation action is the subgroup generated by this permutation. More precisely, let σ be a circular permutation of $W(A_n)$. Then $Stab_{W(A_n)}(\sigma) = \langle\sigma\rangle$. It is also clear that this action acts transitively on the circular permutations. Let us consider now the action of $W(\widetilde{A}_n)$ on $H^0(\widehat{X}_{W(\widetilde{A}_n)})$. Since $W(\widetilde{A}_n) = \mathbb{Z}\Phi \rtimes W(A_n)$ we also have $W(A_n)$ that acts on $H^0(\widehat{X}_{W(\widetilde{A}_n)})$. Moreover, because of Lemma 2.2.4 we know that the circular permutation $(1\ 2\ \cdots\ n{+}1)$ of $W(A_n)$ is such that $(1\ 2\ \cdots\ n{+}1)\diamond X_{W(\widetilde{A}_n)}[0] = X_{W(\widetilde{A}_n)}[0]$. Then, the subgroup generated by $(1\ 2\ \cdots\ n+1)$ is included in $Stab_\diamond(X_{W(\widetilde{A}_n)}[0])$. Besides, by Proposition 1.5.8 we know that the action $\diamond$ is transitive. Theorem 1.5.5 of Chapter 1 tells us that $|H^0(\widehat{X}_{W(\widetilde{A}_n)})| = n!$. Thus it follows

$$\left| W(A_n) \big/ Stab_\diamond(X_{W(\widetilde{A}_n)}[0]) \right| = |H^0(\widehat{X}_{W(\widetilde{A}_n)})| = n!.$$

Then, one has $|Stab_\diamond(X_{W(\widetilde{A}_n)}[0])| = n+1$ and it follows that

$$Stab_\diamond(X_{W(\widetilde{A}_n)}[0]) = \langle(1\ 2\ \cdots\ n+1)\rangle.$$

Therefore, the map given by $(1\ 2\ \cdots\ n+1) \mapsto X_{W(\widetilde{A}_n)}[0]$ induces a bijective map defined by

$$\begin{array}{ccc} \{\text{Circular permutations of } W(A_n)\} & \longrightarrow & H^0(\widehat{X}_{W(\widetilde{A}_n)}) \\ \sigma(1\ 2\ \cdots\ n+1)\sigma^{-1} & \longmapsto & \sigma \diamond X_{W(\widetilde{A}_n)}[0]. \end{array}$$

□

In Figures 2.1, 2.2, 2.3 and 2.4 we describe the two posets respectively associated to $W(\widetilde{A}_3)$ and $W(\widetilde{A}_4)$.

When we express a vector $v \in \bigoplus_{\alpha \in A_n^+} \mathbb{R}\alpha$ as a row vector, or column vector, the way to proceed is by adding, from left to right and from bottom to top, the diagonals of the triangle. Therefore, the coefficient v_{ij} in the below triangle is the coordinate of v in position $\alpha = e_i - e_j$. For example in Figure 2.1 this expression is as follows: $v = (v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34})$.

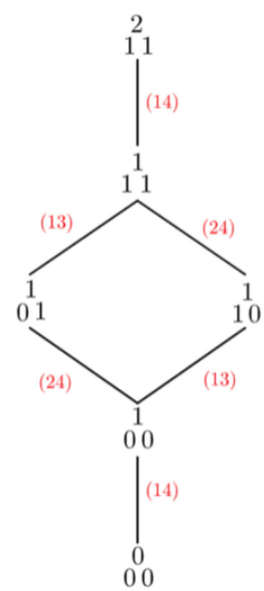

Figure 2.1 See Figure 1.18 for the comments.

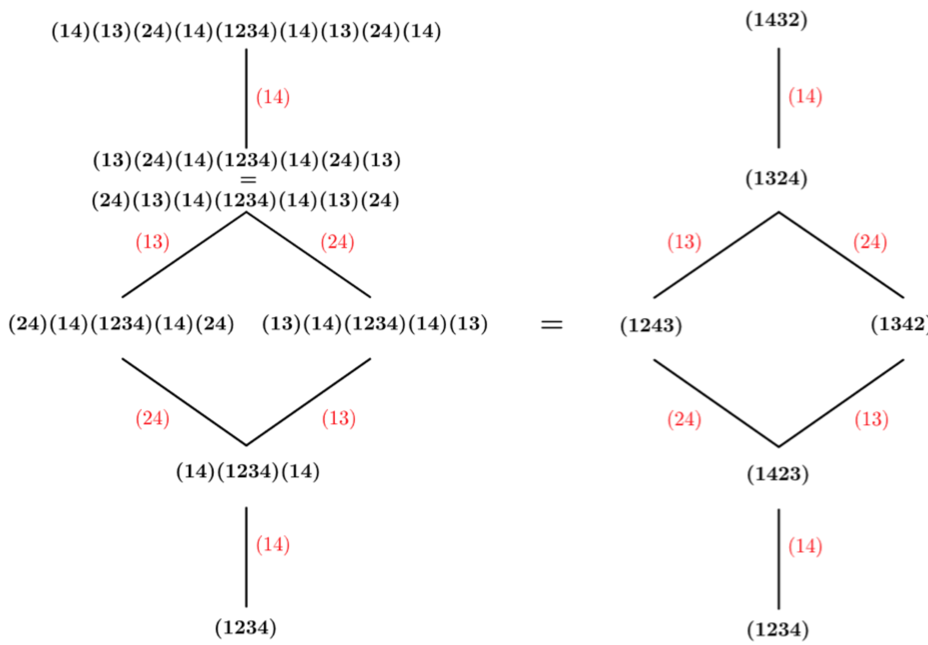

Figure 2.2 Poset of admitted vectors of $W(\widetilde{A}_3)$ on the left hand side and circular permutations of $W(A_3)$ on the right hand side; the red labels indicate the cover relation which is the conjugation action.

Remark 7. In (Abram *et al.*, 2020) we thoroughly investigated the structure of $H^0(\widehat{X}_{W(\widetilde{A}_n)})$ from another combinatorial point of view, and we gave a deeper explanation of the above bijection. We have shown in particular that this poset map is an isomophism of posets. We also showed that these posets are semidistributive lattices and we provided a way to compute the join of any pair of elements.

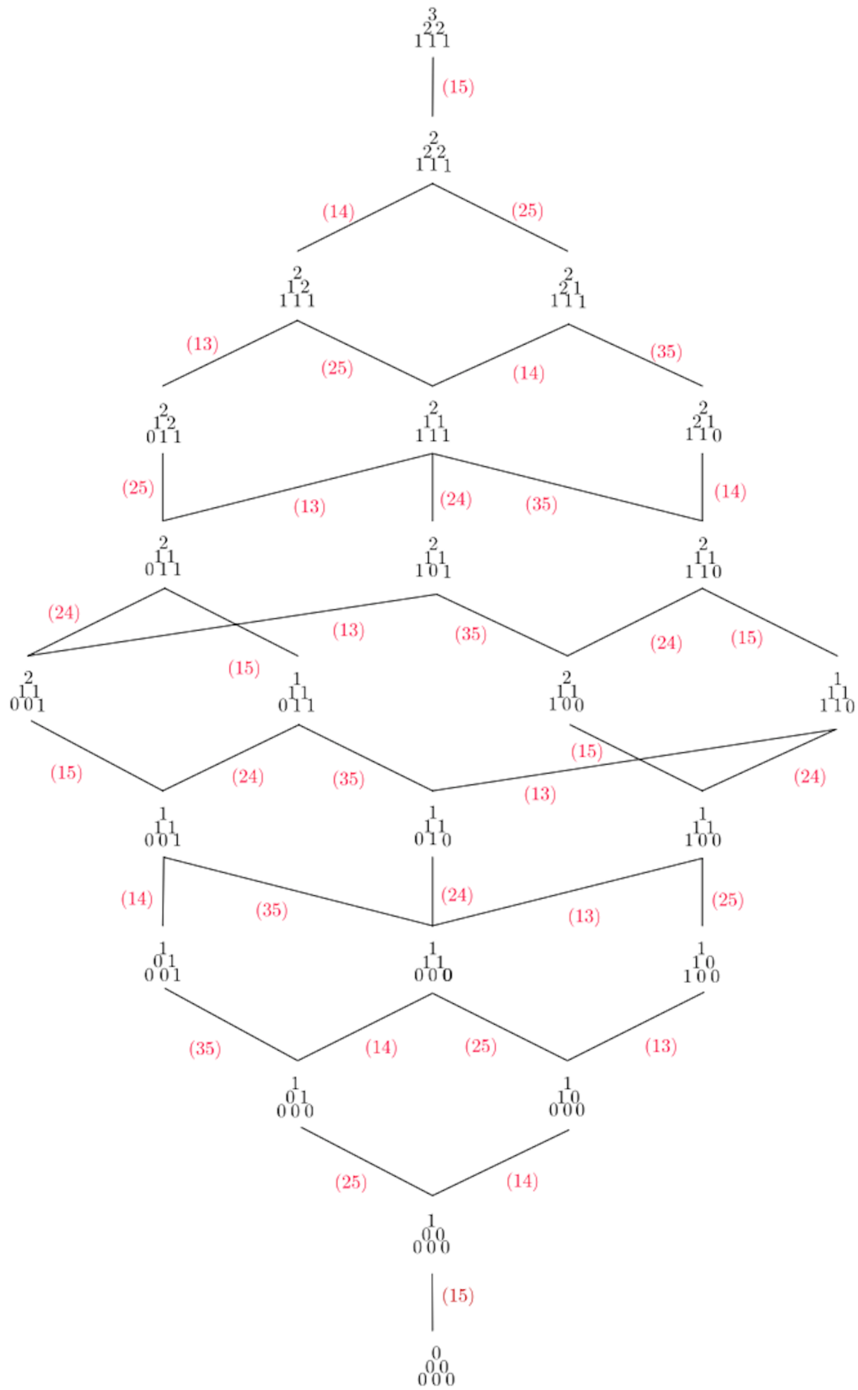

Figure 2.3 Poset of admitted vectors of $W(\widetilde{A}_4)$. See Figure 2.2 for the comments.

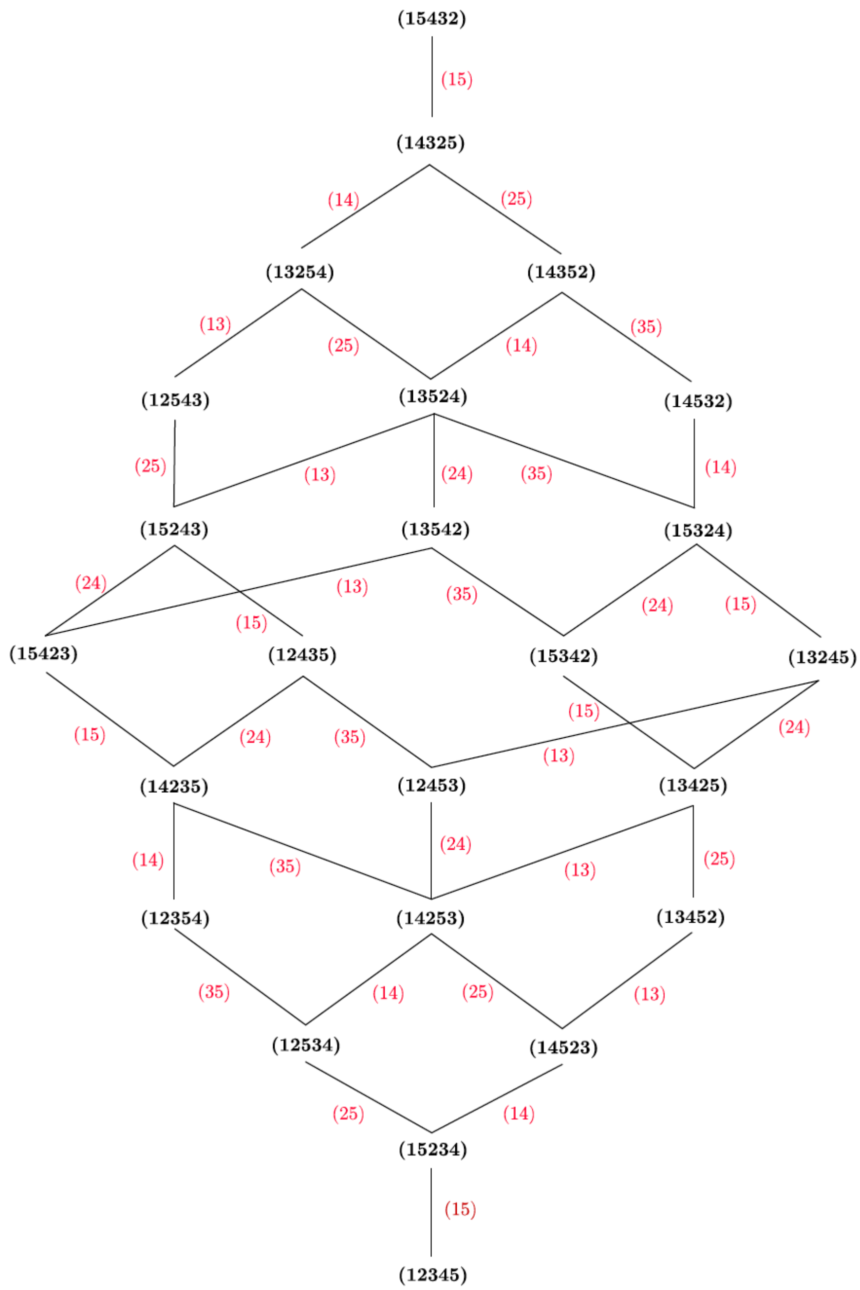

Figure 2.4 Poset of circular permutations of $W(\widetilde{A}_4)$. The red labels indicate the cover relation: this is the conjugation of the lower circular permutation by the red label.

2.3 Investigation of the $\diamond$ action on $H^0(\widehat{X}_{W(\widetilde{A}_n)})$

2.3.1 Motivation and example

In Chapter 1 a concrete description of the coefficients $k(w,\alpha)$ for $\alpha \in \Phi^+ - \Delta$ can be found in any type in terms of the coefficients $k(w,\delta)$ with $\delta \in \Delta$. We use here this description in type A.

Let $w \in W(\widetilde{A}_n)$ and let $1 \leq i < j \leq n+1$. We denote $k(w,\alpha) = k_{i,j}(w)$ where $\alpha = e_i - e_j$ is a positive root of A_n. According to Theorem 1.5.1 and Lemma 1.5.6 of Chapter 1, there exists a positive integer $\lambda_{i,j}$ such that the coefficient $k_{i,j}(w)$ is given by the formula:

$$k_{i,j}(w) = \sum_{r=i}^{j-1} k_{r,r+1}(w) + \lambda_{i,j}.$$

It is precisely from these formulas that the Shi variety $\widehat{X}_{W(\widetilde{A}_n)}$ is defined in Chapter 1. Therefore, if $\lambda = (\lambda_{i,j})$ is an admitted vector and if $x = (x_{i,j}) \in X_{W(\widetilde{A}_n)}[\lambda]$, because of the construction of the irreducible components we must have for all $1 \leq i < j \leq n+1$ that:

$$x_{i,j} = \sum_{r=i}^{j-1} x_{r,r+1} + \lambda_{i,j}. \tag{2.9}$$

Since $W(\widetilde{A}_n) \simeq \mathbb{Z}A_n \rtimes W(A_n)$, one can express w as $w = \tau_u \overline{w}$ where $u \in \mathbb{Z}A_n$ and $\overline{w} \in W(A_n)$. The goal of this section is to understand, in a practical way, the following action

$$\begin{array}{ccc} W(\widetilde{A}_n) \times H^0(\widehat{X}_{W(\widetilde{A}_n)}) & \longrightarrow & H^0(\widehat{X}_{W(\widetilde{A}_n)}) \\ (w, X_{W(\widetilde{A}_n)}[\lambda]) & \longmapsto & w \diamond X_{W(\widetilde{A}_n)}[\lambda]. \end{array}$$

It is known (see Chapter 1 Theorem 1.6.3) that $H^0(\widehat{X}_{W(\widetilde{A}_n)})$ is a lattice. It is of particular interest to provide convenient formulas for the action $\diamond$ in order to better understand this lattice structure, and more specially its cover relation.

We know that the irreducible components of $\widehat{X}_{W(\widetilde{A}_n)}$ are parameterized by the admitted vectors of $W(\widetilde{A}_n)$ (see Chapter 1 Theorem 1.5.5). Therefore it is equivalent to look at this action on the set of admitted vectors. Let λ be an admitted vector. Because of Theorem 1.5.8 we know that the action by translation has no effect, that is $w\diamond\lambda = \overline{w}\diamond\lambda$. Thus, it is enough to understand this action restricted to $W(A_n)$.

Let us then take $g \in W(A_n)$ and let us denote $\beta = g \diamond \lambda$. Our purpose is to express β in terms of λ. The way to proceed is as follows:

$W(A_n)$ acts on the integral points of $X_{W(\widetilde{A}_n)}[\lambda]$ via the Φ^+-representation. Hence, in order to determine the admitted vector β, we just have to find which component contains the element $F(g)(x)$.

Recall that $\{e_{i,j} \mid 1 \le i < j \le n+1\}$ is a basis of Y. Therefore one has

$$F(g)(x) = F(g)(\sum_{i<j} x_{i,j}e_{i,j}) = F(g)(\sum_{i<j}(\sum_{r=i}^{j-1} x_{r,r+1} + \lambda_{i,j})e_{i,j}).$$

Finally, the goal is to find the expression of β in terms of λ from the equation:

$$F(g)(\sum_{i<j}(\sum_{r=i}^{j-1} x_{r,r+1} + \lambda_{i,j})e_{i,j}) = \sum_{i<j}(\sum_{r=i}^{j-1} y_{r,r+1} + \beta_{i,j})e_{i,j}.$$

Example 17. Let us take the group $W(A_3)$. Because of Theorem 2.1.3 we know that $\widehat{X}_{W(\widetilde{A}_3)}$ has 6 irreducible components. We think of these components as admitted vectors and we delete the Δ-part since these coordinates are zero.

Using (2.2), a short computation shows that these vectors (with positions $[\lambda_{13}, \lambda_{14}, \lambda_{24}]$) are

$$\{[0,0,0], [0,1,0], [0,1,1], [1,1,1], [1,1,0], [1,2,1]\}.$$

Let $g = s_{1,2} \in W(A_3)$ be the reflection corresponding to the positive root $e_1 - e_2$, λ be an admitted vector and $x \in X_{W(\widetilde{A}_3)}[\lambda]$. Let $\beta = (\beta_{i,j})$ be the admitted vector such that $F(g)(x) \in X_{W(\widetilde{A}_3)}[\beta]$. Because of (2.3), (2.4) and (2.9) it is easy to see that the matrix representation of the affine map $F(g)$ is as follows:

$$F(g)(x) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{12}+x_{23}+\lambda_{13} \\ x_{12}+x_{23}+x_{34}+\lambda_{14} \\ x_{23} \\ x_{23}+x_{34}+\lambda_{24} \\ x_{34} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -x_{12}-1 \\ x_{23} \\ x_{23}+x_{34}+\lambda_{24} \\ x_{12}+x_{23}+\lambda_{13} \\ x_{12}+x_{23}+x_{34}+\lambda_{14} \\ x_{34} \end{pmatrix}.$$

Let us denote $F(g)(x) = y$. Once again because of (2.9) it follows that

$$\begin{pmatrix} -x_{12}-1 \\ x_{23} \\ x_{23}+x_{34}+\lambda_{24} \\ x_{12}+x_{23}+\lambda_{13} \\ x_{12}+x_{23}+x_{34}+\lambda_{14} \\ x_{34} \end{pmatrix} = \begin{pmatrix} y_{12} \\ y_{12}+y_{23}+\beta_{13} \\ y_{12}+y_{23}+y_{34}+\beta_{14} \\ y_{23} \\ y_{23}+y_{34}+\beta_{24} \\ y_{34} \end{pmatrix}.$$

Thus one has

$$\begin{cases} y_{12} = -x_{12}-1 \\ y_{23} = x_{12}+x_{23}+\lambda_{13} \\ y_{34} = x_{34} \end{cases} \quad \text{and} \quad \begin{cases} x_{23} = y_{12}+y_{23}+\beta_{13} \\ x_{23}+x_{34}+\lambda_{24} = y_{12}+y_{23}+y_{34}+\beta_{14} \\ x_{12}+x_{23}+x_{34}+\lambda_{14} = y_{23}+y_{34}+\beta_{24}. \end{cases}$$

It follows that

$$\begin{cases} \beta_{13} = x_{23}-(-x_{12}-1)-(x_{12}+x_{23}+\lambda_{13}) = -\lambda_{13}+1 \\ \beta_{14} = x_{23}+x_{34}+\lambda_{24}-(-x_{12}-1)-(x_{12}+x_{23}+\lambda_{13})-x_{34} = -\lambda_{13}+\lambda_{24}+1 \\ \beta_{24} = x_{12}+x_{23}+x_{34}+\lambda_{14}-(x_{12}+x_{23}+\lambda_{13})-x_{34} = -\lambda_{13}+\lambda_{14}. \end{cases}$$

Finally one has

$$g \diamond [\lambda_{13}, \lambda_{14}, \lambda_{24}] = [-\lambda_{13}+1, -\lambda_{13}+\lambda_{24}+1, -\lambda_{13}+\lambda_{14}].$$

Doing these computations for all the simple reflections we obtain the following table:

Table 2.1 Action of the simple reflections of $W(A_3)$ onto the set of admitted vectors.

	[0,0,0]	[0,1,0]	[0,1,1]	[1,1,1]	[1,1,0]	[1,2,1]
$s_{1,2}$	[1,1,0]	[1,1,1]	[1,2,1]	[0,1,0]	[0,0,0]	[0,1,1]
$s_{2,3}$	[1,1,1]	[1,2,1]	[1,1,0]	[0,0,0]	[0,1,1]	[0,1,0]
$s_{3,4}$	[0,1,1]	[1,1,1]	[0,0,0]	[0,1,0]	[1,2,1]	[1,1,0]

2.3.2 Computation of the $\diamond$ action

We keep the presentation of the root system A_n as a triangle with base Δ. We define the following sets in order to better understand how the action $\odot$ behaves.

Definition 17. Let $s_{k,\ell}$ be a reflection of $W(A_n)$. We denote

1) $E(k,\ell) = \{e_k - e_{k+1}, e_k - e_{k+2}, \cdots, e_k - e_\ell, e_{k+1} - e_\ell, e_{k+2} - e_\ell, \cdots, e_{\ell-1} - e_\ell\}$. It is easy to see that $E(k,\ell)$ is actually the inversion set of $s_{k,\ell}$.

2) $A_{i,j}(k,\ell) = \{p \in \{i,\cdots,j-1\} \mid e_p - e_{p+1} \notin E(k,\ell)\} = \{i,\cdots,j-1\} \setminus \{k,\ell-1\}$.

3) $B_{i,j}(k,\ell) = \{p \in \{i,\cdots,j-1\} \mid e_p - e_{p+1} \in E(k,\ell)\}$.

4) Let $\lambda \in Y$. We denote $\lambda_{k,\ell}$ the coordinate in position $e_k - e_\ell$ of λ. See for example Figure 2.5.

Definition 18. Let $w \in W(A_n)$. We define the function γ_w by

$$\begin{array}{rccc} \gamma_w & : & \widehat{X}_{W(\widetilde{A}_n)} \times [1, n+1] & \longrightarrow & \mathbb{Z} \\ & & (x, v) & \longmapsto & x_{w(\lfloor v \rfloor), w(\lfloor v+1 \rfloor)}. \end{array}$$

If $w = id$ we denote γ_w by γ.

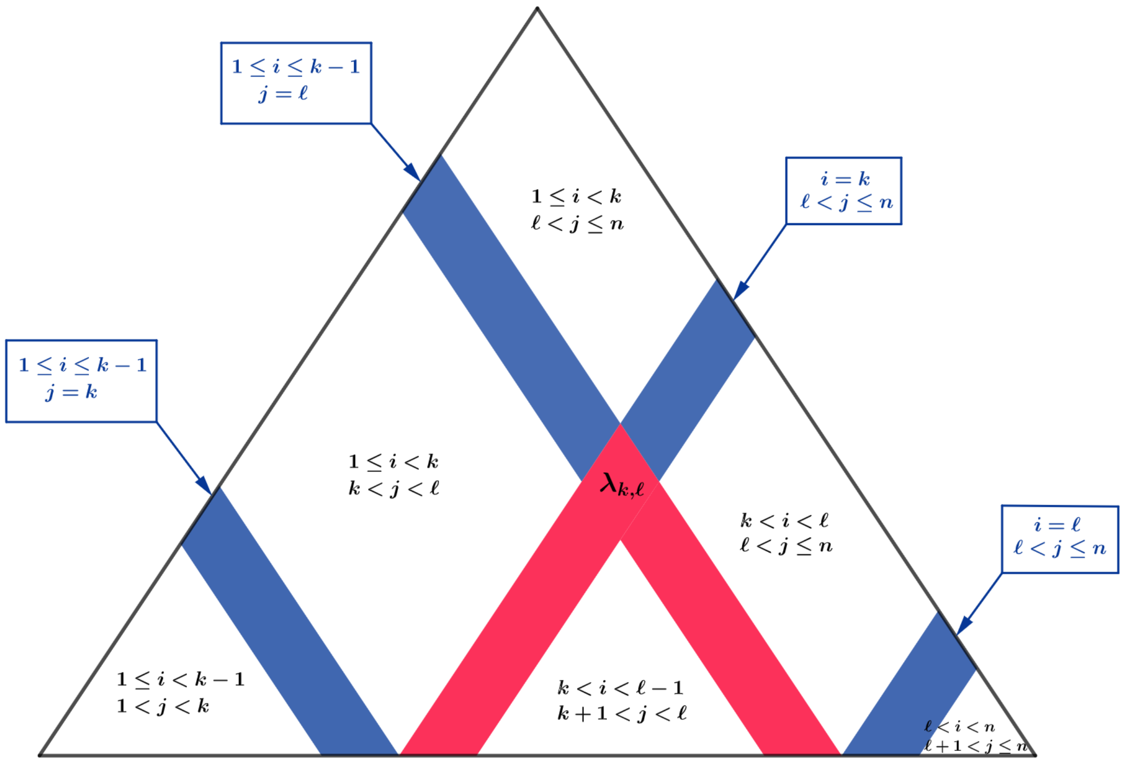

Figure 2.5 Those are the different regions of the root system A_{n-1} cut out by the action of $s_{k,\ell}$. The red part is $E(k,\ell)$, that is the inversion set of $s_{k,\ell}$. The blue part is the set of positive roots that are nontrivially permuted by $s_{k,\ell}$; the remaining positive roots are fixed.

Lemma 2.3.1. *Let* $x \in X_{W(\widetilde{A}_n)}[\lambda]$, $t := s_{k,\ell}$, $e_i - e_j \in \Phi^+$, *and let us denote* $y := F(t)(x)$. *Then we have the formula:*

$$\int\limits_{t(i)}^{t(j)} \gamma(x,v)dv - \int\limits_{i}^{j} \gamma(y,v)dv = -\int\limits_{i}^{j} \gamma_t(\lambda,v)dv + |B_{i,j}|.$$

Proof. From the definition of $F(t)$ it turns out that

$$y_{p,p+1} = \begin{cases} \sum\limits_{r=t(p)}^{t(p+1)-1} x_{r,r+1} + \lambda_{t(p),t(p+1)} & \text{if } \ p \in A_{i,j} \\ -\sum\limits_{r=t(p+1)}^{t(p)-1} x_{r,r+1} - \lambda_{t(p+1),t(p)} - 1 & \text{if } \ p \in B_{i,j}. \end{cases}$$

Therefore, using Definition 18 and the fact that $\lambda_{a,b} = -\lambda_{b,a}$, one can express $y_{p,p+1}$ as follows:

$$y_{p,p+1} = \begin{cases} \int\limits_{t(p)}^{t(p+1)} \gamma(x,v)dv + \lambda_{t(p),t(p+1)} & \text{if } \ p \in A_{i,j} \\ \int\limits_{t(p)}^{t(p+1)} \gamma(x,v)dv + \lambda_{t(p),t(p+1)} - 1 & \text{if } \ p \in B_{i,j}. \end{cases}$$

Consequently it follows that

$$\begin{aligned} &\int\limits_{t(i)}^{t(j)} \gamma(x,v)dv - \int\limits_{i}^{j} \gamma(y,v)dv \\ &= \int\limits_{t(i)}^{t(j)} \gamma(x,v)dv - \sum_{p=i}^{j-1} y_{p,p+1} \\ &= \int\limits_{t(i)}^{t(j)} \gamma(x,v)dv - \sum_{p\in A_{i,j}} (\int\limits_{t(p)}^{t(p+1)} \gamma(x,v)dv + \lambda_{t(p),t(p+1)}) - \sum_{p\in B_{i,j}} (\int\limits_{t(p)}^{t(p+1)} \gamma(x,v)dv + \lambda_{t(p),t(p+1)} - 1) \\ &= \int\limits_{t(i)}^{t(j)} \gamma(x,v)dv - \sum_{p\in A_{i,j}} \int\limits_{t(p)}^{t(p+1)} \gamma(x,v)dv - \sum_{p\in A_{i,j}} \lambda_{t(p),t(p+1)} - \sum_{p\in B_{i,j}} \int\limits_{t(p)}^{t(p+1)} \gamma(x,v)dv \qquad (2.10) \\ &\quad - \sum_{p\in B_{i,j}} \lambda_{t(p),t(p+1)} + |B_{i,j}|. \end{aligned}$$

However, since $A_{i,j} \sqcup B_{i,j} = \{1, \ldots, j-1\}$ it is clear that

$$\int\limits_{t(i)}^{t(j)} \gamma(x,v)dv = \sum_{p\in A_{i,j}} \int\limits_{t(p)}^{t(p+1)} \gamma(x,v)dv + \sum_{p\in B_{i,j}} \int\limits_{t(p)}^{t(p+1)} \gamma(x,v)dv. \qquad (2.11)$$

Therefore, using (2.11) in (2.10) it follows that

$$\int_{t(i)}^{t(j)} \gamma(x,v)dv - \int_{i}^{j} \gamma(y,v)dv = -\sum_{p\in A_{i,j}} \lambda_{t(p),t(p+1)} - \sum_{p\in B_{i,j}} \lambda_{t(p),t(p+1)} + |B_{i,j}|$$

$$= -\sum_{p=i}^{j-1} \lambda_{t(p),t(p+1)} + |B_{i,j}|$$

$$= -\int_{i}^{j} \gamma_t(\lambda,v)dv + |B_{i,j}|.$$

□

Proposition 2.3.2. *Let λ be an admitted vector, $\alpha := e_i - e_j$, $t := s_{k,\ell}$, $A_{i,j} := A_{i,j}(k,\ell)$, $B_{i,j} := B_{i,j}(k,\ell)$. Then we have the formulas:*

$$(t \diamond \lambda)_{i,j} = \begin{cases} \lambda_{t(i),t(j)} - \int_i^j \gamma_t(\lambda,v)dv + |B_{i,j}| & \textit{if } \ \alpha \notin N(t) \\ \lambda_{t(i),t(j)} - \int_i^j \gamma_t(\lambda,v)dv + |B_{i,j}| - 1 & \textit{if } \ \alpha \in N(t). \end{cases}$$

Proof. Let $x = (x_{i,j}) \in X_{W_a}[\lambda]$ with $\Theta(x) = \sum_{i<j} x_{i,j} e_i \wedge e_j$, $\Theta(\lambda) = \sum_{i<j} \lambda_{i,j} e_i \wedge e_j$, and $F(t)(x) := y$. Since W_a acts on the components, it is enough to see where goes x under this action. Let us denote by $\beta = (\beta_{i,j})$ the admitted vector such that $F(t)(x) \in X_{W(\widetilde{A}_n)}[\beta]$. The goal it then to express β in terms of λ.

This question is exactly the same as understanding the component $\Theta(X_{W(\widetilde{A}_n)}[\beta])$ in $\bigwedge^2(K)$. We will answer this question with the second point of view. We adopt the convention that for $j > i$, $x_{j,i} = -x_{i,j}$. These relations also apply to λ.

From Definition 15 and Corollary 2.2.3 we have

$$\begin{aligned}
\varphi_t(\sum_{i<j} x_{i,j} e_i \wedge e_j) &= t \odot (\sum_{i<j} x_{i,j} e_i \wedge e_j) \\
&= \sum_{i<j} x_{i,j} e_{t(i)} \wedge e_{t(j)} - \sum_{e_i - e_j \in N(t)} e_i \wedge e_j \\
&= \sum_{e_i - e_j \in N(t)} x_{i,j} e_{t(i)} \wedge e_{t(j)} + \sum_{e_i - e_j \notin N(t)} x_{i,j} e_{t(i)} \wedge e_{t(j)} - \sum_{e_i - e_j \in N(t)} e_i \wedge e_j \\
&= \sum_{e_i - e_j \in N(t)} x_{t(i),t(j)} e_i \wedge e_j + \sum_{e_i - e_j \notin N(t)} x_{t(i),t(j)} e_i \wedge e_j - \sum_{e_i - e_j \in N(t)} e_i \wedge e_j \\
&= \sum_{e_i - e_j \notin N(t)} x_{t(i),t(j)} e_i \wedge e_j + \sum_{e_i - e_j \in N(t)} (x_{t(i),t(j)} - 1) e_i \wedge e_j.
\end{aligned}$$

Since $\Theta \circ F(t) = \varphi_t \circ \Theta$ we obtain that

$$\begin{aligned}
&\Theta \circ F(t)(x) \\
&= \sum_{e_i - e_j \notin N(t)} x_{t(i),t(j)} e_i \wedge e_j + \sum_{e_i - e_j \in N(t)} (x_{t(i),t(j)} - 1) e_i \wedge e_j \\
&= \sum_{e_i - e_j \notin N(t)} (\sum_{r=t(i)}^{t(j)-1} x_{r,r+1} + \lambda_{t(i),t(j)}) e_i \wedge e_j - \sum_{e_i - e_j \in N(t)} (\sum_{r=t(j)}^{t(i)-1} x_{r,r+1} + \lambda_{t(j),t(i)} + 1) e_i \wedge e_j \\
&= \sum_{e_i - e_j \notin N(t)} (\int_{t(i)}^{t(j)} \gamma(x,v) dv + \lambda_{t(i),t(j)}) e_i \wedge e_j + \sum_{e_i - e_j \in N(t)} (\int_{t(i)}^{t(j)} \gamma(x,v) dv + \lambda_{t(i),t(j)} - 1) e_i \wedge e_j.
\end{aligned}$$

Moreover, we must have

$$\begin{aligned}
\Theta \circ F(t)(x) &= \sum_{e_i - e_j \notin N(t)} y_{i,j} e_i \wedge e_j + \sum_{e_i - e_j \in N(t)} y_{i,j} e_i \wedge e_j \\
&= \sum_{e_i - e_j \notin N(t)} (\sum_{r=i}^{j-1} y_{r,r+1} + \beta_{i,j}) e_i \wedge e_j + \sum_{e_i - e_j \in N(t)} (\sum_{r=i}^{j-1} y_{r,r+1} + \beta_{i,j}) e_i \wedge e_j \\
&= \sum_{e_i - e_j \notin N(t)} (\int_i^j \gamma(y,v) dv + \beta_{i,j}) e_i \wedge e_j + \sum_{e_i - e_j \in N(t)} (\int_i^j \gamma(y,v) dv + \beta_{i,j}) e_i \wedge e_j.
\end{aligned}$$

Hence we have

$$\int_i^j \gamma(y,v)dv + \beta_{i,j} = \begin{cases} \int_{t(i)}^{t(j)} \gamma(x,v)dv + \lambda_{t(i),t(j)} & \text{if } \; e_i - e_j \notin N(t) \\ \int_{t(i)}^{t(j)} \gamma(x,v)dv + \lambda_{t(i),t(j)} - 1 & \text{if } \; e_i - e_j \in N(t), \end{cases}$$

which is equivalent to

$$\beta_{i,j} = \begin{cases} \lambda_{t(i),t(j)} + \int_{t(i)}^{t(j)} \gamma(x,v)dv - \int_i^j \gamma(y,v)dv & \text{if } \; e_i - e_j \notin N(t) \\ \lambda_{t(i),t(j)} + \int_{t(i)}^{t(j)} \gamma(x,v)dv - \int_i^j \gamma(y,v)dv - 1 & \text{if } \; e_i - e_j \in N(t). \end{cases}$$

Moreover, because of Lemma 2.10 we know that

$$\int_{t(i)}^{t(j)} \gamma(x,v)dv - \int_i^j \gamma(y,v)dv = -\int_i^j \gamma_t(\lambda,v)dv + |B_{i,j}|.$$

Thus it follows that

$$\beta_{i,j} = \begin{cases} \lambda_{t(i),t(j)} - \int_i^j \gamma_t(\lambda,v)dv + |B_{i,j}| & \text{if } \; e_i - e_j \notin N(t) \\ \lambda_{t(i),t(j)} - \int_i^j \gamma_t(\lambda,v)dv + |B_{i,j}| - 1 & \text{if } \; e_i - e_j \in N(t). \end{cases}$$

This ends the proof since $\beta_{i,j} = (t \diamond \lambda)_{i,j}$.

$\square$

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CONCLUSION

Cette thèse est consacrée à l'étude des groupes de Weyl affines. Nous avons montré comment obtenir une variété affine dont les points entiers sont en bijection avec les éléments du groupe de Weyl affine associé. Nous avons par la suite étudié cette variété ainsi que ses propriétés. Cette étude nous a menés à regarder en détail ses composantes irréductibles et la combinatoire qu'elles contiennent.

Nous avons montré en particulier que l'ensemble de ces composantes irréductibles admettait une structure de treillis semi-distributif. Un des résultats qui fut très inattendu, fut de constater que dans le cas du groupe symétrique affine, les composantes irréductibles de sa variété sont en bijection avec les permutations circulaires du groupe symétrique sous-jacent.

Les principaux résultats de cette thèse se déclinent en plusieurs étapes :

1) Expression des coefficients $k(w, \alpha)$ sous formes polynomiales ;
2) Construction de la variété $\widehat{X}_{W_a}$ via les vecteurs admis ;
3) Étude de l'action naturelle de W_a sur sa variété $\widehat{X}_{W_a}$, grâce notamment à la Φ^+-représentation ;
4) Connexion entre les régions de Shi de W_a et certaines régions de $\widehat{X}_{W_a}$;
5) Description du premier groupe de cohomologie $H^1(W, \mathbb{Z}\Phi)$ via des orbites de points dans certaines composantes irréductibles de $\widehat{X}_{W_a}$;
6) Nouvelles obstructions cohomologiques qui ont relié la matrice de Cartan avec la notion de cobord en degré 1 ;

7) Expression de ces obstructions via des équations modulaires ;

8) Étude de l'action $\diamond$ de W_a sur l'ensemble des composantes irréductibles de $\widehat{X}_{W_a}$, avec en particulier dans le cas du groupe symétrique affine de nouveaux rapprochements entre les permutations circulaires et les composantes irréductibles.

Comme nous l'avons expliqué brièvement dans l'introduction, nous étions intéressés à comprendre les réflexions faisant descendre de 1 la longueur d'un élément $w \in W_a$. Plus précisément nous voulions comprendre l'ensemble

$$N^1(w) := \{t \in T, \ell(tw) = \ell(w) - 1\}$$

où w est l'élément minimal de sa région de Shi. Les raisons de notre intérêt pour cet ensemble sont entièrement reliées à la conjecture sur laquelle nous avons été invités à travailler au début de ce doctorat. Il s'agit de prouver que les régions de Shi sont en correspondance bijective avec certains éléments du groupe W_a, appelés éléments bas (low elements).

Cette conjecture est toujours une question ouverte et nous continuons de nous y intéresser. Cependant, de nouvelles avancées ont été faites. Nous avons réussi à montrer entre autres que résoudre cette conjecture, dans le cadre des groupes de Weyl affines, était équivalent à montrer pour w minimal que si $|k(w,\alpha)| > 1$ alors $s_{\alpha,k(w,\alpha)} \notin N^1(w)$. En particulier ceci suggère une nouvelle caractérisation des éléments minimaux. Ces avancées se basent sur un travail minutieux effectué avec Christophe Hohlweg, et concernant les p-ensembles d'inversions dans les groupes de réflexions généraux. Ce travail est en préparation, et c'est pourquoi nous avons décidé de ne pas l'incorporer au manuscrit de cette thèse.

Nous concluons en listant certaines perspectives que nous pensons envisageables :

1) Il serait intéressant d'étudier les régions de Shi à partir de la variété. Nous serions curieux de voir si le nombre de ces régions, nombre étant déjà connu et fourni par Jian-Yi Shi en tant que réponse à une conjecture de Carter (cf. (Shi, 1987b) théorème 8.1), pourrait se retrouver via la nature de l'arrangement des composantes irréductibles.

2) Puisque les régions de Shi peuvent se voir comme l'intersection de $\widehat{X}_{W_a}$ avec un certain orthant, il est probable que cette variété puisse fournir une meilleure compréhension des régions de Shi. Par ailleurs, il est tout autant probable que les notions de n-petites racines et n-petites inversions introduites dans (Dyer et Hohlweg, 2016), étant elles-mêmes fortement reliées aux régions de Shi, soient également connectées à la variété, et plus précisément à l'arrangement de ses composantes irréductibles.

3) Le paramétrage des composantes irréductibles de $\widehat{X}_{W_a}$ à l'aide des vecteurs admis, et donc à l'aide des éléments de $P_{\mathcal{H}}$, nous a suggéré à repenser $P_{\mathcal{H}}$. Bien sûr nous l'avons repensé comme les composantes irréductibles de $\widehat{X}_{W_a}$ ou comme les éléments d'un treillis. Mais il semblerait qu'une nouvelle facette de cet objet soit intéressante à explorer. Il s'agirait de le considérer comme un polytope dans l'espace où vit la variété. Plus précisément nous avons en tête de regarder l'enveloppe convexe des vecteurs admis. Nous serions curieux d'explorer les propriétés géométriques/combinatoires de ce nouvel objet, et notamment de comprendre les points entiers se trouvant sur sa frontière et dans sont intérieur. Nous serions également intéressés à déformer ce polytope via certaines opérations et voir ce qui en découle.

4) Il est naturel de penser à regarder ce qu'il se passe pour la cohomologie en degré deux. Nous n'avons que très peu réfléchi à ceci, et de ce fait, n'avons que peu d'intuition sur d'éventuels progrès.

5) Tout ce travail est en partie basé sur le paramétrage des alcôves par les Φ^+-uplets de Jian-Yi Shi. Il est normal de penser à généraliser ceci pour les groupes de Coxeter hyperboliques, bien qu'il semble y avoir à première vue de nombreuses complications. Dès lors, il semble naturel de se poser la question : y a-t-il une variété (affine ou autre) encodant une quelconque information sur le groupe hyperbolique ? Cette question est sans doute une question à long terme, car les difficultés sont nombreuses lorsque l'on passe de l'affine à l'hyperbolique.

6) Nous terminons cette conclusion par un sujet que nous pensons avoir un grand potentiel. Durant cette dernière année un objet a attiré notre attention, et bien que non intégré à cette thèse, nous lui avons porté très rapidement un grand intérêt. Il s'agit de la notion de *diagramme de lignes*.

   Cet objet de nature combinatoire et topologique a été introduit dans (Abram *et al.*, 2020) afin de comprendre de façon concrète la bijection entre les permutations circulaires et les vecteurs admis de type A_n.

   La construction de cet objet se fait soit à partir des vecteurs admis soit à partir des permutations circulaires. Elle nécessite pour cela des algorithmes utilisant des caractéristiques topologiques.

   Nous avons particulièrement été motivé par le fait que défaire un nœud (sous de bonnes conditions) dans un tel diagramme revient moralement à appliquer une inversion sur la permutation associée au diagramme. Ceci illustre entre autres la relation de couverture dans le poset des permutations circulaires associé.

   Il s'est avéré par la suite que ces diagrammes sont reliés aux transformations de Reidemeister, bien connues en théorie des noeuds. Ces transformations jouent un rôle dans l'algorithme de construction, ou plus exactement de déconstruction, du diagramme de lignes obtenu à partir d'une permutation circulaire.

Nous illustrons ci-dessous ces diagrammes que nous représentons sous forme circulaires ou cylindriques. Prenons par exemple la permutation circulaire $\sigma = (164235)$ dans $S_6 = W(A_5)$. Le diagramme de lignes associé à σ, noté D_σ, permet alors de trouver le vecteurs admis correspondant à σ.

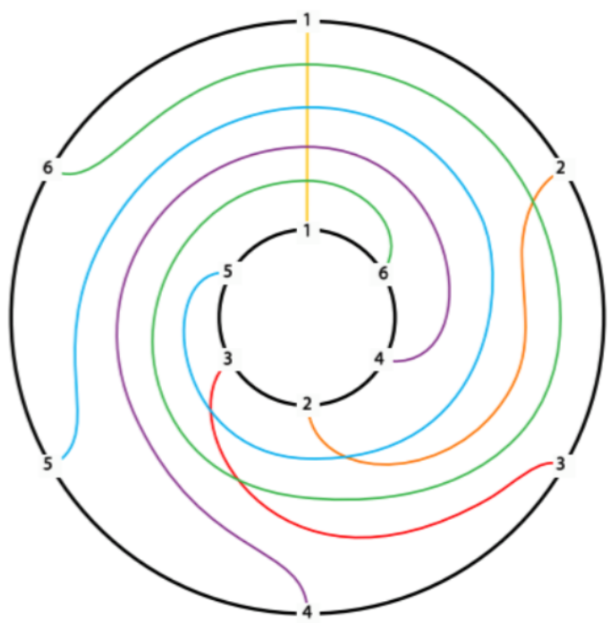

Figure 2.6 Diagramme de lignes D_σ sous forme circulaire.

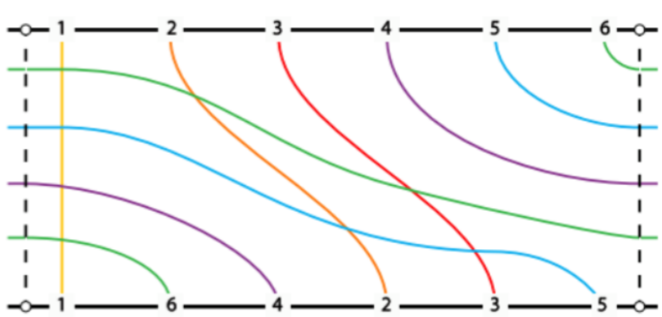

Figure 2.7 Diagramme de lignes D_σ sous forme cylindrique.

La façon de tracer ces diagrammes est la suivante. Prenons par exemple la représentation cylindrique. On place sur la ligne du haut de façon ordonnée les chiffres de 1 à 6. Sur la ligne du bas sont placés les chiffres de la même façon que le mot déterminant le cycle σ. On relie ensuite chaque chiffre du haut avec son chiffre du bas. Il y a quatre règles à respecter. Notons L_i la ligne reliant le i du haut avec le i du bas. Voici ces conditions :

i) L_i et L_{i+1} ne peuvent jamais se croiser.

ii) Chaque ligne se déplace dans le sens horaire (vision circulaire), c'est à dire de gauche à droite dans la représentation cylindrique.

iii) Si $2 < i+1 < j$ alors pour que L_j puisse croiser L_i elle doit avoir croisé L_1 (on peut penser L_1 comme un péage). Ce procédé est récursif au sens où si L_j veut recroiser L_i alors elle doit aussi recroiser L_1.

iv) Le point iii) ne traite pas la situation $i = 1$. Si L_j veut franchir L_1, elle ne peut le faire que si elle voit L_1 à sa gauche avant le croisement.

Le vecteur admis associé à σ est alors $\lambda_\sigma := (\lambda_{i,j})$ où $\lambda_{i,j}$ est les nombre de croisements entre les lignes L_i et L_j, c'est à dire :

$$\boldsymbol{\lambda_\sigma} = \begin{array}{ccccccccc} & & & & \mathbf{2} & & & & \\ & & & \mathbf{1} & & \mathbf{1} & & & \\ & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \\ & \mathbf{0} & & \mathbf{0} & & \mathbf{1} & & \mathbf{0} & \\ \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} \end{array}$$

Figure 2.8 Vecteur admis associé à la permutation circulaire $\sigma = (164235)$.

Il est à noter qu'il est aussi possible d'obtenir une permutation circulaire à partir de son vecteur admis en construisant un diagramme utilisant les coordonnés du vecteur, et qui sera équivalent, en un certain sens, à celui obtenu directement à partir de la permutation circulaire.

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